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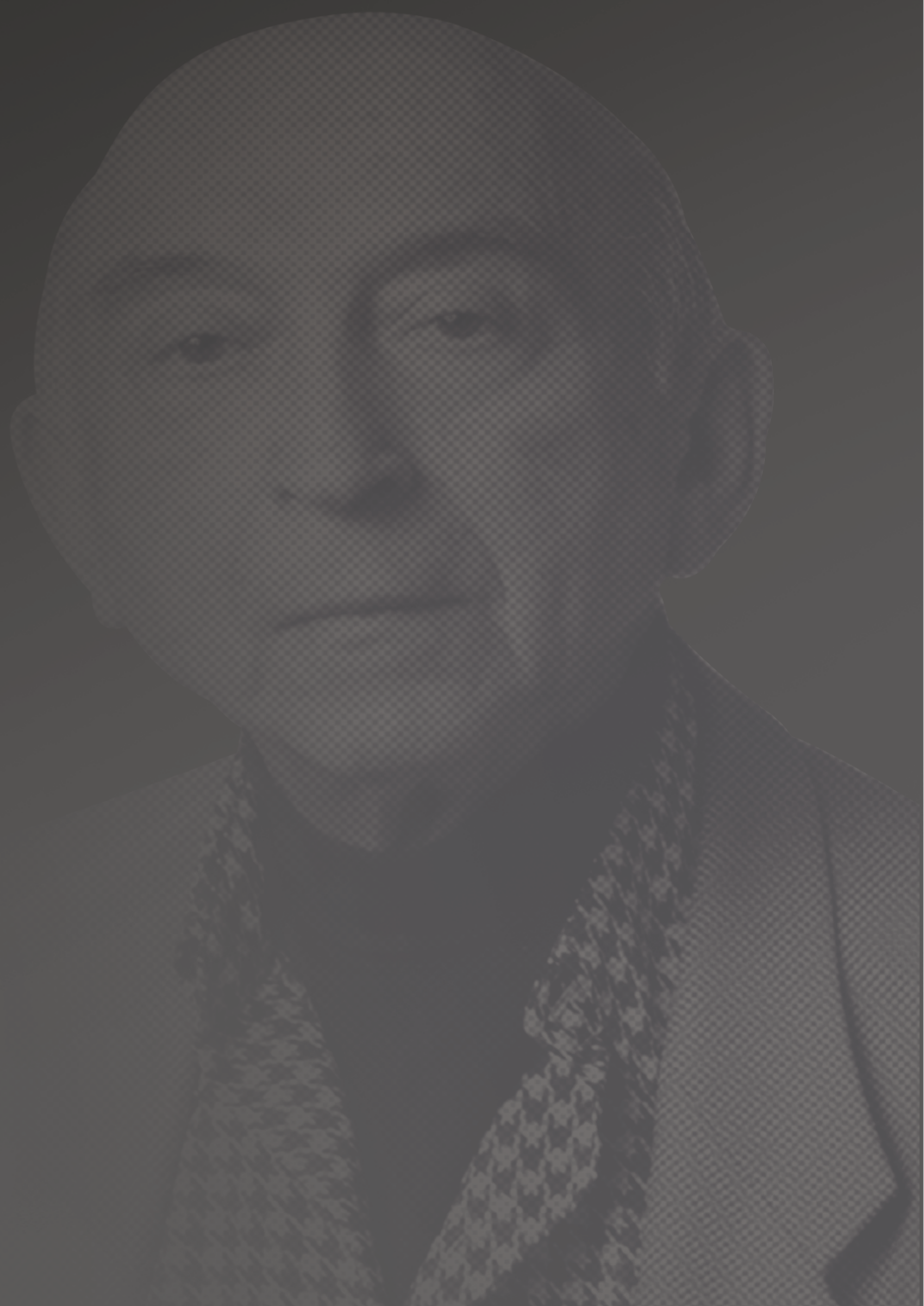
Transactions on Fuzzy Sets and Systems



Transactions on
Fuzzy Sets & Systems

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Transactions on Fuzzy Sets and Systems (TFSS) is an open access international scholarly journal. TFSS publishes new applied and pure articles related to fuzzy sets and systems as the semiannual journal and there is no charge for publishing an article in TFSS. All articles will be peer-review before publication. Manuscripts submitted to TFSS must be original and unpublished and not currently being considered for publication elsewhere.

The articles will be deposited immediately into the online repository, after the completion of the review processes. TFSS aims linking the ideas and techniques of fuzzy sets and systems with other disciplines to provide an international forum for refereed original research works in the theory and applications in all fields related to fuzzy science. Transactions on Fuzzy Sets and Systems (TFSS) is a semiannual international academic journal founded in 2022.

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Website: https://sanad.iau.ir/Journal/tfss, E-mails: tfssiauba@gmail.com; Online ISSN: 2821-0131

Transactions on Fuzzy Sets and Systems (Vol.4, Issue 1, May 2025)

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Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://sanad.iau.ir/journal/tfss/
Online ISSN: 2821-0131
Vol.4, No.1, (2025), 1-3
DOI: https://doi.org/10.71602/tfss.2025.1191816

Professor John N. Mordeson

Tofigh Allahviranloo

Professor John N. Mordeson is a distinguished mathematician and educator who has made significant contributions to the field of fuzzy logic and its applications. He is currently a Professor Emeritus of Mathematics at Creighton University. Dr. Mordeson earned his B.S., M.S., and Ph.D. degrees from Iowa State University. Throughout his career, he has authored twenty books and more than two hundred journal articles on fuzzy science, making remarkable advancements in the field. He also serves on the editorial boards of numerous academic journals, continuing to make valuable contributions to fuzzy science. John N. Mordeson was born

in the United States on April 22, 1934. His early fascination with mathematics and science was evident during his school years, where he consistently excelled. His intellectual curiosity led him to pursue higher education, embarking on a journey that would ultimately establish him as a prominent figure in mathematics and computer science. He received his B.S., M.S., and Ph.D. in mathematics from Iowa State University, Ames, IA, USA, in 1959, 1961, and 1963, respectively. Following the completion of his doctorate, Mordeson began his academic career as a professor of mathematics. His teaching style was renowned for its clarity and rigor, making complex mathematical concepts accessible to his students. Mordeson's research interests have been broad, but he is best known for his work in algebra and fuzzy mathematics, particularly in addressing global challenges such as climate change, the coronavirus pandemic, human trafficking, and biodiversity. Additionally, he has developed an extensive set of tools for applying fuzzy mathematics and graph theory to social issues, including human trafficking and illegal immigration.

Dr. Mordeson has made significant contributions to the field of fuzzy mathematics through his numerous books. Each work reflects his unique approach of merging theoretical advancements with practical applications. Among his most notable books are:

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How to cite: Allahviranloo T, Professor John N. Mordeson. *Transactions on Fuzzy Sets and Systems.* 2025; 4(1): 1-3. DOI: https://doi.org/10.71602/tfss.2025.1191816

"Fuzzy Automata and Languages: Theory and Applications" This book introduces fuzzy automata and languages, expanding classical automata theory by incorporating fuzziness into state transitions. It covers fundamental theory and applications in areas like pattern recognition and linguistics.

"Fuzzy Graphs and Fuzzy Hypergraphs" This work explores fuzzy extensions of graph theory, widely used in network analysis, computer science, and decision-making. It includes applications of fuzzy graphs in social networks, transportation networks, and communication systems.

"Fuzzy Mathematics in Medicine" This book discusses the role of fuzzy mathematics in medical contexts, particularly in diagnosis, prognosis, and decision-making under uncertainty. It demonstrates how fuzzy set theory can model medical scenarios with imprecise or incomplete information.

"Fuzzy Group Theory" Extending classical group theory with fuzzy set concepts, this book is aimed at researchers in algebra and offers insights into applying fuzzy sets to abstract algebraic structures like groups. "Fuzzy Decision Making in Modeling and Control" This book addresses decision-making processes in complex systems where uncertainty and ambiguity are present. It presents methods for using fuzzy logic to improve modeling and control in fields such as engineering and artificial intelligence.

"Fuzzy Set Theory and Fuzzy Controller Design" Mordeson explores fuzzy controllers, essential in industrial automation, and explains how fuzzy set theory principles can enhance controller design, particularly for systems challenging to model precisely.

"Fuzzy Mathematics: Approximation Theory" Focusing on approximation theory in fuzzy mathematics, this book explores how fuzzy set theory improves accuracy in mathematical function approximations, with applications in engineering, economics, and beyond.

"Interval-Valued Fuzzy Set Theory" Introducing interval-valued fuzzy sets, this book provides a more flexible representation of uncertainty, suitable for complex decision-making environments where each element has an interval of possible membership values.

"Applications of Fuzzy Sets and Fuzzy Logic" This text covers the practical uses of fuzzy sets and fuzzy logic across disciplines, from engineering and computer science to economics and social sciences, showcasing fuzzy logic's versatility in handling vagueness and imprecision. "Fuzzy Semigroups" A focus on semigroups in abstract algebra, this book extends classical semigroup theory into the fuzzy domain, modeling systems with partial or uncertain information, useful in algebra and computer science research.

Mordeson has authored and co-authored numerous research papers and books on fuzzy mathematics, making significant contributions to its development and dissemination. His work often bridged the gap between abstract mathematical theory and practical applications, making his research valuable to both academics and industry professionals. In addition to his research and teaching, Professor Mordeson has undertaken various leadership roles throughout his career. He has served as a department chair and participated in numerous academic committees, playing a key role in shaping the direction of research and education within his department. His influence extends beyond his institution through his active involvement in professional organizations, conferences, and the editorial boards of academic journals. He is a respected figure in the global mathematical community, known for his collaborations with other researchers and his mentorship of young mathematicians.

Throughout his career, Professor John N. Mordeson has received numerous awards and honors for his contributions to mathematics and education. His work has been widely cited, and his ideas have inspired generations of researchers. Mordesons lasting impact is evident not only in the mathematical theorems and concepts that bear his influence but also in the countless students and colleagues he has inspired over the years. His dedication to the pursuit of knowledge and his passion for teaching have left an indelible mark on the academic community. His passion for mathematics and education endures, and he often reflects on the importance of fostering curiosity and critical thinking in students. His work remains significant, and his influence is still evident in the fields of fuzzy mathematics and beyond.

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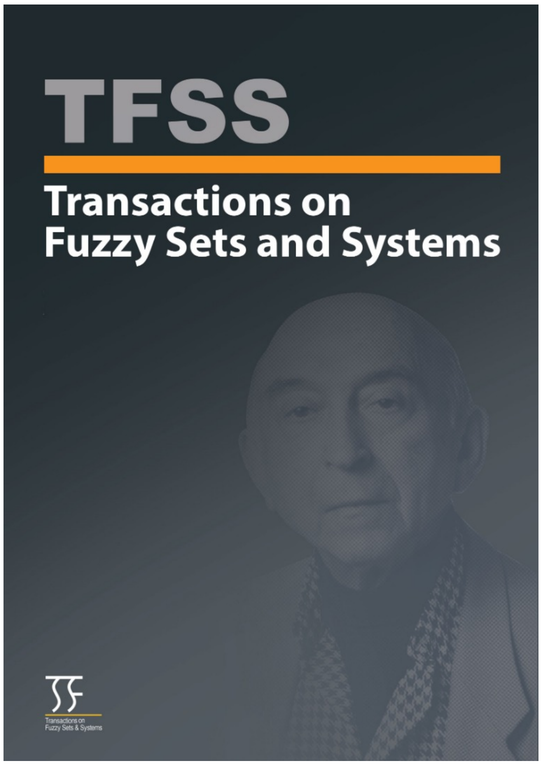

Transactions on Fuzzy Sets and Systems

ISSN: **2821-0131**

https://sanad.iau.ir/journal/tfss/

Some Orders in Groupoids and its Applications to Fuzzy Groupoids

Vol.4, No.1, (2025), 4-13. DOI: https://doi.org/10.71602/tfss.2025.1127399

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Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://sanad.iau.ir/journal/tfss/
Online ISSN: 2821-0131
Vol.4, No.1, (2025), 4-13
DOI: https://doi.org/10.71602/tfss.2025.1127399

Article Type: Original Research Article

Some Orders in Groupoids and its Applications to Fuzzy Groupoids

Akbar Rezaei, **Choonkil Park**, **Hee Sik Kim***

(This paper is dedicated to Professor "John N. Mordeson" on the occasion of his 91st birthday.)

Abstract. In this paper, we continue the investigation started in [1]. We obtain new results derived from novel concepts developed in analogy with others already established, e.g., the fact that leftoids $(X,*)$ for φ are super-transitive if and only if $\varphi(\varphi(x)) = \varphi(x)$ for all $x \in X$. In addition we apply fuzzy subsets in this context and we derive a number of results as consequences.

AMS Subject Classification 2020: 20N02; 03E72
Keywords and Phrases: Below, Above, Transitive, Fuzzy, Contractive, Contained, $(\alpha-, \beta-, \gamma_\alpha-, \gamma_\beta-)$order-preserving (reversing).

1 Introduction

In developing a general theory of groupoids one seeks to define concepts and obtain information that applies to as general a class of groupoids as possible. Thus, e.g., the observation that $(Bin(X), \Box)$ is a semigroup with identity is one such of this type. Another one is the description of the notions of order for all groupoids $(X,*)$. Here one does not expect there to be a "precise" answer of this type one often expects. Here we use the relations β (below: $x*y = y$) and α (above: $x*y = x$) which are then combined with $\leq$ $(x \leq y : x\beta y, y\alpha x)$ which compares with other definitions of $\leq$ made for certain classes of groupoids (e.g., BCK-algebras ([2, 3]), pogroupoids ([4, 5, 6]). The work done in [1] was convincing enough to suggest that a follow up paper might be in order, and that in this paper it might also be proper to open the door to introduce ideas that are both related to the material in [1] and to the general subject of "fuzzification" of crisp algebraic theories. Hopefully this effort has been successful.

Zadeh [7] introduced the notion of a fuzzy subset as a function from a set into unit interval, and Rosenfeld [8] applied this concept to the theory of groupoids and groups. Mordeson and Malik [9] published a book, *Fuzzy commutative algebra*, which are fuzzifications of several classical algebras, and Ahsan et al. [10] publisehed a book, *Fuzzy semirings with applications to automata theory.* Kim and Neggers [5] applied it to pogroupoids which are algebraic representations of partially ordered sets, and obtained an equivalent condition for some relation to be transitive for any fuzzy subset. Han et al. [11] discussed on linear fuzzifications of groupoids with special emphasis on BCK-algebras. Liu et al. [12] studied the notion of hyperfuzzy groupoids as a natural extension of the basic concepts of fuzzy groupoids.

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Received: 13 May 2024; **Revised**: 23 July 2024; **Accepted**: 25 July 2024; **Available Online**: 7 August 2024; **Published Online**: 7 May 2025.

How to cite: Rezaei A, Park C, Kim HS. Some orders in groupoids and its applications to fuzzy groupoids. *Transactions on Fuzzy Sets and Systems.* 2025; 4(1): 4-13. DOI: https://doi.org/10.71602/tfss.2025.1127399

We recall that the notion of the semigroup $(Bin(X), \square)$ was introduced by Kim and Neggers [13]. Shin et al. [14] introduced the notion of abelian fuzzy subsets on a groupoid, and discussed diagonal symmetric relations, convex sets, and fuzzy center on $Bin(X)$. Ahn et al. [1] studied fuzzy upper bounds in $Bin(X)$.

Allen et al. [15] studied several types of groupoids related to semigroups, i.e., twisted semigroups. Allen et al. [16] developed a theory of companion d-algebras, and they showed that if $(X, *, 0)$ is a d-algebra, then $(Bin(X), \oplus, \diamond_0)$ is also a d-algebra. Kim et al. [17] introduced the notions of generalized commutative laws in algebras, and investigated their relations by using Smarandache disjointness. Moreover, they showed that every pre-commutative BCK-algebra is bounded. Hwang et al. [18] generalized the notion of implicativity which was discussed in BCK-algebras, and applied it to several groupoids, BCK/BCI-algebras and their generalizations.

2 Preliminaries

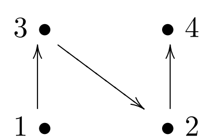
Let $(X, *)$ be a groupoid, i.e., a non-empty set X and a binary operation "$*$" on X, and let $x, y, z \in X$. x is said to be *below* y, denoted by $x\beta y$, if $x * y = y$; x is said to be *above* y, denoted by $x\alpha y$, if $x * y = x$. An element $z \in X$ is said to be *β-between x and y*, denoted by $z \in < x, y >_\beta$, if $x\beta z, z\beta y$; an element z is said to be *α-between x and y*, denoted by $z \in < x, y >_\alpha$, if $x\alpha z$ and $z\alpha y$.

We refer to [19] for basic concepts of the graph theory.

Example 2.1. [20] Let $D = (V, E)$ be a digraph and let $(V, *)$ be its associated groupoid, i.e., $*$ is a binary operation on V defined by

$$x * y := \begin{cases} x & \text{if } x \to y \notin E, \\ y & \text{otherwise.} \end{cases}$$

Let $D = (V, E)$ be a digraph with the following graph:



Then its associated groupoid $(V, *)$ has the following table:

$*$	1	2	3	4
1	1	1	3	1
2	2	2	2	4
3	3	2	3	3
4	4	4	4	4

It is easy to see that there are no elements $x, y \in V$ such that both $x\alpha y$ and $x\beta y$ hold simultaneously. Note that the relations α and β need not be transitive. In fact, $1 \to 3, 3 \to 2$ in E, but not $1 \to 2$ in E imply that $1\beta 3, 3\beta 2$, but not $1\beta 2$. Similarly, $1\alpha 4, 4\alpha 3$, but not $1\alpha 3$.

Remark 2.2. *In Example 2.1, $z \in < x, y >_\beta$ means that $x\beta z, z\beta y$, i.e., $x \to z \to y$ in E. Similarly, $z \in < x, y >_\alpha$ means that $x\alpha z, z\alpha y$, i.e., there is no arrow from x to z, and no arrow from z to y in E.*

Example 2.3. [20] Let $\mathbf{R}$ be the set of all real numbers and let $x, y \in \mathbf{R}$. If we define a binary operation "$*$" on $\mathbf{R}$ by $x * y := y^2$, then $(\mathbf{R}, *)$ is not a semigroup. In fact, $(x * y) * z = z^2$, while $x * (y * z) = z^4$. If $x\beta y$ and $y\beta z$, then $z = y * z = z^2$ and hence $z = 0$ or $z = 1$, which implies that $x * z = z$, i.e., $x\beta z$, proving that β is transitive.

Let $(X, *)$ be a groupoid and let $x, y \in X$. Define a binary relation "$\leq$" on X by $x \leq y \iff x\beta y, y\alpha x$. Then it is easy to see that $\leq$ is anti-symmetric.

Note that the order "$x \leq y$" defined by $x * y = 0$ in BCK-algebras is a partial order.

Let $(X, *)$ be a groupoid and let $x, y \in X$. We define an interval as follows:

$$[x, y] := \{q \in X \mid x \leq q \leq y\}.$$

Proposition 2.4. *[20] Let $(X, *)$ be a groupoid and let $x, y \in X$. Then $z \in [x, y]$ if and only if $z \in < x, y >_\beta$ and $z \in < y, x >_\alpha$.*

Given a set X and a function $\varphi : X \to X$, we consider a groupoid $(X, *, \varphi)$ where the multiplication is given by the formula

$$x * y = \varphi(x).$$

We call such a groupoid $(X, *, \varphi)$ a *leftoid* for φ. In particular, if $\varphi(x) = id_X(x) = x$, then $(X, *, id_X)$ has a multiplication

$$x * y = x$$

and the groupoid $(X, *)$ is referred to as a *left zero semigroup*. Similarly, we define a rightoid and right zero semigroup, i.e., $x * y := \varphi(y)$ for all $x, y \in X$.

Given a non-empty set X, we let $Bin(X)$ denote the collection of all groupoids $(X, *)$. Given groupoids $(X, *)$ and $(X, \bullet)$, we define a binary operation "$\square$" on $Bin(X)$ by

$$(X, \square) := (X, *) \,\square\, (X, \bullet)$$

where

$$x \,\square\, y = (x * y) \bullet (y * x)$$

for any $x, y \in X$. Using that notion, Kim and Neggers proved the following theorem.

Theorem 2.5. *[13] $(Bin(X), \square)$ is a semigroup, i.e., the operation "$\square$" is associative. Furthermore, the left-zero semigroup is the identity for this operation.*

3 Below and above in groupoids

Proposition 3.1. *Let $(X, *)$ be a leftoid for φ and let $x, y_1, y_2 \in X$. If $x\beta y_1$ and $x\beta y_2$, then $y_1 = y_2$.*

Proof. If $x\beta y_i$ $(i = 1, 2)$, then $\varphi(x) = x * y_1 = y_1$ and $\varphi(x) = x * y_2 = y_2$. Since φ is a mapping, we obtain $y_1 = y_2$. $\square$

In case of the rightoid $(X, *)$ for φ, if $x\beta y$, then $\varphi(y) = y$, i.e., there is no element $x \in X$ such that $x\beta y$ and $\varphi(y) \neq y$.

Proposition 3.2. *If $(X, *)$ is a leftoid for φ and $x\alpha y$, then x is a fixed point of φ.*

Proof. If $x\alpha y$, then $x = x * y = \varphi(x)$, i.e., x is a fixed point of φ. $\square$

Proposition 3.3. *If $(X, *)$ is a leftoid (rightoid) for φ and $x \leq y$, then $x \in \varphi^{-1}(y)$ and y is a fixed point of φ.*

Proof. The proof is straightforward. $\square$

Proposition 3.4. *Let $(X, *)$ be a leftoid for φ, and let β be transitive. If $x\beta z, z\beta y$, then $y = z$.*

Proof. Since β is transitive, if $x\beta z, z\beta y$, then $x\beta y$ and $x * z = z$. Since $(X, *)$ is a leftoid for φ, we obtain $z = \varphi(x)$ and hence $y = x * y = \varphi(x) = z$. □

The following property can be easily proved.

Proposition 3.5. *Let $(X, *)$ be a leftoid for φ. If β is transitive, then either $< x, y >_\beta = \emptyset$ or $< x, y >_\beta = \{y\}$ for all $x, y \in X$.*

Given a mapping $\varphi : X \to X$, we define a set by

$$\mathrm{Fix}(\varphi) := \{x \in X \mid \varphi(x) = x \}.$$

Theorem 3.6. *Let $(X, *)$ be a rightoid for φ and let $y \in \mathrm{Fix}(\varphi)$. If $x \in X$, then $< x, y >_\beta = \mathrm{Fix}(\varphi)$.*

Proof. If $z \in < x, y >_\beta$, then $x\beta z, z\beta y$. Since $(X, *)$ is a rightoid, we have $z = x * z = \varphi(z)$, and hence $z \in \mathrm{Fix}(\varphi)$. If $z \in \mathrm{Fix}(\varphi)$, then $z = \varphi(z)$. For any $x \in X$, since $(X, *)$ is a rightoid, we have $x * z = \varphi(z) = z$, i.e., $x\beta z$. Moreover, $z * y = \varphi(y) = y$, since $y \in \mathrm{Fix}(\varphi)$, i.e., $z\beta y$. This shows that $z \in < x, y >_\beta$ for all $x \in X$. □

Theorem 3.6 shows that if $(X, *)$ is a rightoid for φ, then $< x, y >_\beta = < x', y' >_\beta = \mathrm{Fix}(\varphi)$ for all $x, x' \in X$ and $y, y' \in \mathrm{Fix}(\varphi)$.

Note that if $x\alpha z, z\alpha y$, where $(X, *)$ is a rightoid for φ, then $x = x * z = \varphi(z)$ and $z = z * y = \varphi(y)$. This shows that x is uniquely determined by y under φ,

Theorem 3.7. *Let $(X, *)$ be a leftoid for φ and let $x, y \in X$. Then*

(i) $|[x, y]| \le 1$,

(ii) *if $z \in [x, y]$, then $y, z \in \mathrm{Fix}(\varphi)$,*

(iii) *if $x \in [x, y]$, then $x = y \in \mathrm{Fix}(\varphi)$.*

Proof. (i) Assume that there exist $z_1, z_2 \in [x, y]$. Then $x\beta z_1$ and $x\beta z_2$. This shows that $z_i = x * z_i = \varphi(x)$ where $i = 1, 2$. Since φ is a mapping, we obtain $z_1 = z_2$.
(ii) If $z \in [x, y]$, then $y\alpha z, z\alpha x$ and hence $y = y * z = \varphi(y)$ and $z = z * x = \varphi(z)$, proving that $y, z \in \mathrm{Fix}(\varphi)$.
(iii) If $x \in [x, y]$, then $x \in \mathrm{Fix}(\varphi)$ by (ii). We claim that $x = y$. Assume $x \ne y$. Since $x \le y$, we have $x\beta y, y\alpha x$. It follows that $y = x * y = \varphi(x)$. Since $\varphi(x) = x$, we have $x = y$, a contradiction. □

Proposition 3.8. *If $(X, *)$ is a rightoid for φ and $x, y \in X$, then $[x, y] \subseteq \mathrm{Fix}(\varphi)$, and $[x, y] = \{y\}$ when $[x, y] \ne \emptyset$.*

Proof. If $z \in [x, y]$, then $z\alpha x$ and hence $z = x * z = \varphi(z)$, proving that $z \in \mathrm{Fix}(\varphi)$. Now, $y\alpha z$ implies $y = y * z = \varphi(z) = z$, since $z \in \mathrm{Fix}(\varphi)$. □

4 Transitivity in groupoids

Given a groupoid $(X, *)$, the relation β (below) is given by $x\beta y$ iff $x * y = y$ ([20]). Now, if β is transitive, then $(x\beta y) * (y\beta z) = x\beta z$, i.e., $(x * y) * (y * z) = x * z$ when $x * y = y, y * z = z, x * z = z$. Thus, if $(x * y) * (y * z) = x * z$, then this identity reflects a transitivity-like property which in any case is more general than a transitivity in the β-relation. Of course, we can argue the same way for the α-relation (above) given by $x\alpha y$ iff $x * y = x$. Thus the condition $(x * y) * (y * z) = x * z$ also generalizes the α-relation in the same manner. Since α and β are definitely not the same, we shall consider the following.

A groupoid $(X,*)$ is said to be *super-transitive* if for all $x,y,z\in X$,

$$(x*y)*(y*z)=x*z.$$

Every left-zero semigroup as well as every right-zero semigroup is therefore super-transitive as well. Moreover, every Boolean group $(X,*)$ (i.e., $x^2=e_X$ for all $x\in X$) is super-transitive, since $(x*y)*(y*z)=x*(y*y)*z=x*z$ for all $x,y,z\in X$.

Theorem 4.1. *Let $(X,*)$ be a leftoid for φ. Then $(X,*)$ is super-transitive if and only $\varphi(\varphi(x))=\varphi(x)$ for all $x\in X$.*

Proof. If $(X,*)$ is super-transitive, then $(x*y)*(y*z)=x*z$ for all $x,y,z\in X$. Since $(X,*)$ is a leftoid, we have $(x*y)*(y*z)=\varphi(x)*\varphi(y)=\varphi(\varphi(x))$ and $x*z=\varphi(x)$. Assume $\varphi(\varphi(x))=x$ for all $x\in X$. Then $(x*y)*(y*z)=\varphi(x)*\varphi(y)=\varphi(\varphi(x))=\varphi(x)=x*z$, proving that $(X,*)$ is super-transitive. □

Corollary 4.2. *Let $(X,*)$ be a rightoid for φ. Then $(X,*)$ is super-transitive if and only $\varphi(\varphi(x))=x$ for all $x\in X$.*

Proof. The proof is similar to the proof of Theorem 4.1. □

Note that super-transitive groupoids with homomorphisms form a category, since super-transitivity is expressed in identity form.

A groupoid $(X,*)$ is said to be *α-transitive* if $x\alpha y, y\alpha z$ implies $x\alpha z$, and a groupoid $(X,*)$ is said to be *β-transitive* if $x\beta y, y\beta z$ implies $x\beta z$. A groupoid $(X,*)$ is *transitive* if it is both α-transitive and β-transitive.

Example 4.3. Let $X:=\{x,y,z\}$ be a set with the following table.

*	x	y	z
x	y	z	x
y	z	x	y
z	z	x	y

Then $(X,*)$ is trivially β-transitive, since $\beta=\{(u,v)|u*v=v\}=\emptyset$. But $\alpha=\{(x,z),(y,z),(z,x)\}$. This shows that $x\alpha z, z\alpha x$, but not $x\alpha x$, proving that $(X,*)$ is not α-transitive.

Proposition 4.4. *Every super-transitive groupoid is transitive.*

Proof. Let $(X,*)$ be a super-transitive groupoid. Assume that $x\alpha y, y\alpha z$. Then $x*y=x, y*z=y$, and hence $x*z=(x*y)*(y*z)=x*y=x$, i.e., $x\alpha z$, proving that $(X,*)$ is α-transitive.

Assume that $x\beta y, y\beta z$. Then $x*y=y, y*z=z$. Since $(X,*)$ is super-transitive, we obtain $x*z=(x*y)*(y*z)=y*z=z$, i.e., $x\beta z$, proving that $(X,*)$ is β-transitive. □

Corollary 4.5. *Let $(X,*)$ be a transitive groupoid. If $x\le y, y\le z$, then $x\le z$.*

Proof. The proof is straightforward. □

The converse of Proposition 4.4 need not be true in general.

Example 4.6. Let $X:=\{0,1,2,3\}$ be a set with the following table.

*	0	1	2	3
0	0	1	2	1
1	1	1	2	1
2	2	2	2	1
3	1	2	1	3

Then it is easy to see that $(X,*)$ is transitive, but it is not super-transitive, since $(2*1)*(1*3)=2\neq 1=2*3$.

5 Applications to fuzzy subgroupoids

In this section, we apply the concept of fuzzy subsets to groupoid theory mentioned in the above sections.

Let $(X, *) \in Bin(X)$. A mapping $\mu : X \to [0, 1]$ is said to be a *fuzzy subgroupoid* of X if, for all $x, y \in X$,

$$\mu(x * y) \geq \min\{\mu(x), \mu(y)\}.$$

A mapping $\mu : X \to [0, 1]$ is said to be a *contractive fuzzy subgroupoid* of X if, for all $x, y \in X$,

$$\mu(x * y) \leq \min\{\mu(x), \mu(y)\}.$$

Proposition 5.1. *Let $(X, *)$ be a left-zero semigroup.*

(i) *every fuzzy subset $\mu : X \to [0, 1]$ is a fuzzy subgroupoid of X,*

(ii) *if $\mu : X \to [0, 1]$ is contractive, then it is a constant mapping.*

Proof. (i) Given $x, y \in X$, since $(X, *)$ is a left-zero semigroup, we have $\mu(x * y) = \mu(x) \geq \min\{\mu(x), \mu(y)\}$, proving that μ is a fuzzy subgroupoid of X.
(ii) Assume that μ is contractive. Then $\mu(x * y) \leq \min\{\mu(x), u(y)\}$ for all $x, y \in X$. Since $(X, *)$ is a left-zero semigroup, we obtain that $\mu(x) \leq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$. This shows that $\mu(x) \leq \mu(y)$ for all $x, y \in X$, i.e., μ is a constant function. □

Let $(X, *) \in Bin(X)$. A mapping $\mu : X \to [0, 1]$ is said to be a *contained fuzzy subgroupoid* of X if, for all $x, y \in X$,

$$\mu(x * y) \leq \max\{\mu(x), \mu(y)\}.$$

Proposition 5.2. *Let $(X, *)$ be a left-zero semigroup. Then every mapping $\mu : X \to [0, 1]$ is a contained fuzzy subgroupoid of X.*

Proof. The proof is straightforward. □

Let $(X, *) \in Bin(X)$ and let $\mu : X \to [0, 1]$ be a mapping. A mapping $\mu^c : X \to [0, 1]$ is said to be a *complement* of μ if, for all $x \in X$, $\mu^c(x) := 1 - \mu(x)$.

Proposition 5.3. *Let $(X, *)$ be a groupoid. If $\mu : X \to [0, 1]$ is a contained fuzzy subgroupoid of X, then μ^c is a fuzzy subgroupoid of X.*

Proof. It follows from that $1 - \max\{\mu(x), \mu(y)\} = \min\{1 - \mu(x), 1 - \mu(y)\}$ for all $x, y \in X$. □

Let $(X, *) \in Bin(X)$. A mapping $\mu : X \to [0, 1]$ is said to be a *expansive fuzzy subgroupoid* of X if, for all $x, y \in X$,

$$\mu(x * y) \geq \max\{\mu(x), \mu(y)\}.$$

Note that every expansive fuzzy subgroupoid of X is also a fuzzy subgroupoid of X. Moreover, a fuzzy subset μ is an expansive fuzzy subgroupoid of X if and only if μ^c is a contractive fuzzy subgroupoid of X.

Example 5.4. Let $X := [0, \infty)$. Define a binary operation $x * y := x + y$ for all $x, y \in X$ where "+" is the usual addition of real numbers. Then every order-preserving mapping μ is expansive, since $\mu(x + y) \geq \mu(x)$ and $\mu(x + y) \geq \mu(y)$ for all $x, y \in X$.

Theorem 5.5. *Let $(X, *), (X, \bullet) \in Bin(X)$ and let $(X, \square) := (X, *)\square(X, \bullet)$. Then the following are hold:*

(i) *if μ is contractive fuzzy subgroupoid on $(X, *)$ and $(X, \bullet)$, then it is also contractive fuzzy subgroupoid on $(X, \square)$,*

(ii) *if μ is contained fuzzy subgroupoid on $(X,*)$ and $(X,\bullet)$, then it is also contained fuzzy subgroupoid on $(X,\square)$,*

(iii) *if μ is expansive fuzzy subgroupoid on $(X,*)$ and $(X,\bullet)$, then it is also expansive fuzzy subgroupoid on $(X,\square)$.*

Proof. (i) Given $x, y \in X$, we have

$$\begin{aligned}\mu(x\square y) &= \mu((x*y)\bullet(y*x)) \\ &\leq \min\{\mu(x*y),\mu(y*x)\} \\ &\leq \min\{\mu(x),\mu(y)\},\end{aligned}$$

proving that μ is a contractive fuzzy subgroupoid on $(X,\square)$. Others are similar to (i) and we omit the proofs. □

Proposition 5.6. *Let $(X,*) \in Bin(X)$. If $\mu : (X,*) \to [0,1]$ is both a contractive fuzzy subgroupoid of X and a fuzzy subgroupoid of X, then $\mu(x*y) = \min\{\mu(x),\mu(y)\}$ for all $x, y \in X$.*

Proof. It follows from that $\min\{\mu(x),\mu(y)\} \leq \mu(x*y) \leq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$. □

Let $(X,*) \in Bin(X)$. A mapping $\mu : X \to [0,1]$ is said to be *β-order-preserving* if $x\beta y$ implies $\mu(x) \leq \mu(y)$, and a mapping $\mu : X \to [0,1]$ is said to be *β-order-reversing* if $x\beta y$ implies $\mu(x) \geq \mu(y)$. A mapping $\mu : X \to [0,1]$ is said to be *expanding* if $\mu(x) \leq \mu(x*y)$ for all $x, y \in X$, and a mapping $\mu : X \to [0,1]$ is said to be *contracting* if $\mu(x) \geq \mu(x*y)$ for all $x, y \in X$.

Proposition 5.7. *Let $(X,*) \in Bin(X)$. Then every expanding (resp., contractive) fuzzy subset $\mu : X \to [0,1]$ is β-order-preserving (resp., reversing).*

Proof. Assume that $x\beta y$. Then $x*y = y$. Since μ is expanding, we obtain $\mu(x) \leq \mu(x*y) = \mu(y)$, proving that μ is β-order-preserving. The other part is similar, and we omit it. □

Theorem 5.8. *Let $(X,*) \in Bin(X)$ and let $a, b, a+b \in [0,1]$. Then the following conditions hold:*

(i) *if μ and ν are β-order-preserving, then $a\mu + b\nu$ is also β-order-preserving,*

(ii) *if μ, ν are expanding, then $a\mu + b\nu$ is also expanding,*

(iii) *if μ is β-order-preserving, then μ^c is β-order-reversing,*

(iv) *if μ is expanding, then μ^c is contracting.*

Proof. Let $\mu, \nu : X \to [0,1]$ be fuzzy subsets of X. Given $x \in X$, we have $(a\mu + b\nu)(x) = a\mu(x) + b\nu(x) \leq (a+b)\max\{\mu(x),\nu(x)\} \leq a+b \leq 1$.

We consider (i). If μ and ν are β-order-preserving and $x\beta y$, then $\mu(x) \leq \mu(y), \nu(x) \leq \nu(y)$. It follows that $(a\mu + b\nu)(x) = a\mu(x) + b\nu(x) \leq a\mu(y) + b\mu(y) = (a\mu + b\nu)(y)$. Other proofs can be shown easily, and we omit the proofs. □

Let $(X,*) \in Bin(X)$. A map $\mu : X \to [0,1]$ is said to be *γ_β-order-preserving* if $x\beta z, z\beta y$ implies $\mu(x) \leq \mu(z) \leq \mu(y)$.

Note that every β-order-preserving mapping μ of a groupoid $(X,*)$ is γ_β-order-preserving.

Proposition 5.9. *Let $(X,*)$ be a groupoid with the following property (P):*

$$x\beta z \implies \exists\, y \in X \text{ such that } z\beta y. \qquad (P)$$

If μ is a γ_β-order-preserving mapping on $(X,*)$, then it is a β-order-preserving mapping on $(X,*)$.

Proof. Let $x\beta z$. Since $(X,*)$ has the property (P), there exists $y \in X$ such that $z\beta y$. Since μ is a γ_β-order-preserving mapping, we obtain $\mu(x) \leq \mu(z) \leq \mu(y)$, which shows that μ is a β-order-preserving mapping. □

Example 5.10. Let $X := [0,\infty)$. Define a binary operation $*$ on X by $x * y := \max\{x, y\}$ for all $x, y \in X$. Assume $x\beta y$. Then $y = x * y = \max\{x, y\}$ and hence $x \leq y$. If we put $z := y + 1$, then $y * z = y * (y + 1) = \max\{y, y + 1\} = y + 1 = z$. Hence $(X,*)$ has the property (P).

Example 5.11. Let $X := [0,\infty)$. Define a binary operation $*$ on X by $x * y := x + y$ for all $x, y \in X$, where "+" is the usual addition of real numbers. Assume $x\beta 1$. Then $1 = x * 1 = x + 1$, and hence $x = 0$, i.e., $0\beta 1$. If we assume that there is $y \in X$ such that $1\beta y$, then $y = 1 * y = 1 + y$, which shows that $1 = 0$, a contradiction. Hence $(X,*)$ does not have the property (P).

Let $(X,*) \in Bin(X)$. A mapping $\mu : X \to [0,1]$ is said to be *α-order-preserving* if $x\alpha y$ implies $\mu(x) \geq \mu(y)$. A map $\mu : X \to [0,1]$ is said to be *γ_α-order-preserving* if $x\alpha z, z\alpha y$ implies $\mu(x) \geq \mu(z) \geq \mu(y)$.

Proposition 5.12. *Let $(X,*)$ be a groupoid with the following property (Q):*

$$x\alpha z \implies \exists\, y \in X \text{ such that } z\alpha y. \tag{Q}$$

If μ is a γ_α-order-preserving mapping on $(X,*)$, then it is a α-order-preserving mapping on $(X,*)$.

Proof. The proof is similar to the proof of Proposition 5.9. □

Let $(\mathbf{R},*)$ be a leftoid for φ, where $\varphi(x) := x^2$ for all $x \in \mathbf{R}$. Then $1 * 2 = \varphi(1) = 1$ and hence $a\alpha 2$. If we assume $(\mathbf{R},*)$ satisfies the condition (Q), then there exists $y \in \mathbf{R}$ such that $2\alpha y$. It follows that $2 = 2 * y = \varphi(2) = 4$, a contradiction. Hence such a groupoid $(\mathbf{R},*)$ does not satisfy the condition (Q).

Given a groupoid $(X,*)$, a map $\mu : X \to [0,1]$ is said to be a *super-symmetric fuzzy subset* of $(X,*)$ if $\mu((x * y) * (y * z)) \geq \mu(x * z)$ for all $x, y, z \in X$.

Thus, if $(X,*)$ is a left-zero semigroup, then every mapping $\mu : (X,*) \to [0,1]$ is a super-symmetric fuzzy subset of $(X,*)$, since $\mu((x * y) * (y * z)) = \mu(x) \geq \mu(x) = \mu(x * z)$ for all $x, y, z \in X$. Similarly, for any right zero semigroup, every mapping $\mu : (X,*) \to [0,1]$ is also a super-symmetric fuzzy subset of $(X,*)$.

Proposition 5.13. *Let $(X,*)$ be a leftoid for φ. If $\mu : X \to [0,1]$ is a map with $\mu(\varphi(x)) \geq \mu(x)$ for all $x \in X$, then μ is a super-symmetric fuzzy subset of $(X,*)$.*

Proof. Given $x, y, z \in X$, since $(X,*)$ is a leftoid for φ, we have $\mu((x * y) * (y * z)) = \mu(\varphi(x) * \varphi(y)) = \mu(\varphi(\varphi(x)) \geq \mu(\varphi(x)) = \mu(x * z)$, proving the proposition. □

It is a question of some interest to determine a super-symmetric fuzzy subset of a groupoid $(X,*)$ to be a fuzzy subgroupoid of $(X,*)$, i.e., $\mu(x*y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$. Clearly, every map $\mu : X \to [0,1]$ of a left-zero semigroup $(X,*)$ is also a fuzzy subgroupoid of $(X,*)$.

Proposition 5.14. *Let $(X,*)$ be a leftoid for φ. If $\mu : X \to [0,1]$ is a map with $\mu(\varphi(x)) \geq \mu(x)$ for all $x \in X$, then μ is a fuzzy subgroupoid of $(X,*)$.*

Proof. If $\mu(\varphi(x)) \geq \mu(x)$ for all $x \in X$, then $\mu(\varphi(\varphi(x)) \geq \mu(\varphi(x)) \geq \mu(x)$ and hence $\mu(x * y) = \mu(\varphi(x)) \geq \mu(x) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$, proving that μ is a fuzzy subgroupoid of $(X,*)$. □

Clearly, there is much more information waiting to be obtained here as well.

6 Conclusion

In this paper, we have continued the investigation started in [20] of what we may discover in the theory of groupoids (binary systems) by separating the concepts of below ($x\beta y$) and above ($x\alpha y$) in general groupoids, and then recombining them to obtain what looks to be a candidate for the best relation $\leq$ available in general. After doing so, we introduce the idea of super-transitivity in groupoids as a generalization of the notions β and α in identity form, $(x*y)*(y*z) = x*z$ which allows us to make claims about the class of groupoids for which this identity holds, i.e., that this yields a variety. Having done so we may then concern ourselves with introducing fuzzy subsets μ on groupoids $(X,*)$ which have certain properties of interest, e.g., being contracting on expanding which defined in the natural way provides new but not unexpected information. A bit stickier is the class of $\mu((x*y)*(y*z)) \geq \mu(x*z)$ for super-symmetric fuzzy subsets of $(X,*)$ introduced with a standard looking inequality and the problem being the determination of fuzzy subsets of this type which are also fuzzy subgroups and conversely. Certain problems look innocent enough but may yet prove not to be trivial as they are solved.

Conflict of Interest: The authors declare no conflict of interest.

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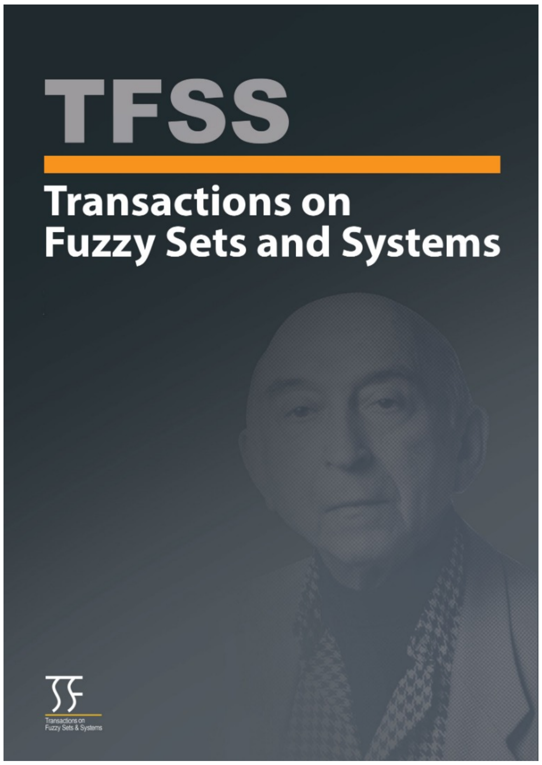

Transactions on Fuzzy Sets and Systems

ISSN: **2821-0131**

https://sanad.iau.ir/journal/tfss/

Inventory Problems and the Parametric Measure m_λ

Vol.4, No.1, (2025), 14-29. DOI: https://doi.org/10.71602/tfss.2025.1127863

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Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://sanad.iau.ir/journal/tfss/
Online ISSN: 2821-0131
Vol.4, No.1, (2025), 14-29
DOI: https://doi.org/10.71602/tfss.2025.1127863

Article Type: Original Research Article

Inventory Problems and the Parametric Measure m_λ

Irina Georgescu*

(This paper is dedicated to Professor "John N. Mordeson" on the occasion of his 91st birthday.)

Abstract. The credibility theory was introduced by B. Liu as a new way to describe the fuzzy uncertainty. The credibility measure is the fundamental notion of the credibility theory. Recently, L.Yang and K. Iwamura extended the credibility measure by defining the parametric measure m_λ (λ is a real parameter in the interval $[0, 1]$ and for $\lambda = 1/2$ we obtain as a particular case the notion of credibility measure).

By using the m_λ- measure, we studied in this paper a risk neutral multi-item inventory problem. Our construction generalizes the credibilistic inventory model developed by Y. Li and Y. Liu in 2019. In our model, the components of demand vector are fuzzy variables and the maximization problem is formulated by using the notion of m_λ–expected value.

We shall prove a general formula for the solution of optimization problem, from which we obtained effective formulas for computing the optimal solutions in the particular cases where the demands are trapezoidal and triangular fuzzy numbers. For $\lambda = 1/2$ we obtain as a particular case the computation formulas of the optimal solutions of the credibilistic inventory problem of Li and Liu. These computation formulas are applied for some m_λ- models obtained from numerical data.

AMS Subject Classification 2020: 03B52; 90B50
Keywords and Phrases: Fuzzy variables, Demand vectors, m_λ–measure, m_λ–inventory problem.

1 Introduction

Let us consider a company that produces several types of goods (items). It will be assumed that buyers can order in advance. An inventory problem is a mathematical model that describes the management of this company.

There are mathematical models in which the management activity of the company is carried out in a single period and models with several periods. The inventory models can also be classified according to the attitude towards risk of a decision-maker: there are models in which the decision maker has a risk-averse attitude and models in which his attitude is neutral.

The mathematical formulation of the inventory model starts from the following initial data (model parameters) : $c_1, \ldots, c_n$ are unit fixed costs per inventoried item, $d_1, \ldots, d_n$ are unit revenues per inventoried item and $h_1, \ldots, h_n$ are unit holding costs per inventoried item.

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Received: 30 May 2024; **Revised**: 26 June 2024; **Accepted**: 31 July 2024; **Available Online**: 19 August 2024; **Published Online**: 7 May 2025.

How to cite: Georgescu I. Inventory problems and the parametric measure m_λ. *Transactions on Fuzzy Sets and Systems.* 2025; 4(1): 14-29. DOI: https://doi.org/10.71602/tfss.2025.1127863

The demands are mathematically modeled by the variables $D_1, \cdots, D_n$; the order quantities will be the variables $x_1, \cdots, x_n$. The total profit from the sale of the n types of goods will have the following expression:: $\pi(\vec{x}, \vec{D}) = \sum_{i=1}^{n}(d_i x_i - c_i - \frac{h_i x_i^2}{2D_i})$ (see [1], [2]).

In a neutral inventory problem, one will determine those values of $x_1, \cdots, x_n$ for which the total profit $\pi(\vec{x}, \vec{D}) = \sum_{i=1}^{n}(d_i x_i - c_i - \frac{h_i x_i^2}{2D_i})$ is maximal. When making the decision the risk is taken into account, the values of $x_1, \cdots, x_n$ will be determined so that at the same time the maximum profit is achieved, and the risk (represented by various mathematical concepts) to be minimal.

The formulation of an inventory problem depends on how the demands $D_1, \cdots, D_n$ are modeled, as well as on how profit maximization and risk minimization are evaluated. The classical treatment of inventory problems is a probabilistic one: the demands $D_1, \cdots, D_n$ are random variables and, for risk - neutral models, the objective function of the maximization problem is the expected value of the total profit. In the case of a risk - averse attitude of the decision maker, several ways to describe the risk were proposed: in [3], [4] by means of mean-variance models and in [5] by using the value-at-risk (VaR) as a risk measure. In [6], the coherent risk measures [7] have been used in defining the objective function of an inventory problem. Using the multi-item inventory system introduced by Luciano et al. [5] (called, shortly, LCP-model), [8] developed several inventory problems, with decision-makers having various positions towards risk: from a neutral attitude to risk-averse attitude, corresponding to variance, mean-absolute deviation (MAD) and conditional value-at-risk (CVaR) as risk measures.

The credibility theory, specially developed by Liu in [9], is another way to model the fuzzy uncertainty. Its fundamental concept is the credibilistic measure [10] and its main indicators are the credibilistic expected value and the credibilistic variance (cf. [9], [10]). From the literature dedicated to the credibilistic treatment of inventory problems we mention the papers: [11], [12], [1], [13]. In this paper we will have as the starting point the papers [14], [15], [16] of Li and Liu: the first one concerns a multi - item inventory problem in which the decision - maker is neutral and the second one is a risk - averse inventory model. In both papers, the demands and the total profits are fuzzy variables and the expected profit is the credibilistic expected value of total profit. In [15] appears a risk evaluated by the notion of absolute semi - deviation.

In [17], Yang and Iwamura introduced a new measure m_λ as a convex linear combination of a possibility measure Pos and its associated necessity measure Nec (λ is a parameter in the interval $[0, 1]$). By using the measure m_λ, in [18] the notions of the expected value $E_\lambda(\xi)$ and the variance $Var_\lambda(\xi)$ of a fuzzy variable ξ are defined. These two indicators retain some algebraic properties of the possibilistic indicators corresponding to [9]. In this way, the credibility theory is enlarged to a new theory that models the fuzzy uncertainty (this will be named m_λ - theory). An issue that arises naturally is an m_λ–theory leading to the development of different economic and financial themes. Papers [19], [20], [21], [22] introduce new credibilistic real options models, which are based on the optimism-pessimism measure and interval–valued fuzzy numbers. The model outcomes are compared to the original credibilistic real options model through a numerical case example in a merger and acquisition context. Paper [18] applies m_λ–theory in the study of optimal portfolios when assets returns are described by triangular or trapezoidal fuzzy variables.

In this paper we shall study a multi - item risk neutral inventory problem in the framework of an m_λ - theory. We shall assume that the demands $D_1, \cdots, D_n$ are fuzzy variables and the criterion used in determination of the order quantities $x_1, \cdots, x_n$ is the maximization of the m_λ - expected value $\sum_{i=1}^{n}[d_i x_i - c_i - \frac{h_i x_i^2}{2} E_\lambda(\frac{1}{D_i})]$ of the total profit. We shall prove a general formula for computing the solution of optimization problem, of which we will then get formulas for effective computation of inventory problem solution whenever the demands are trapezoidal or triangular fuzzy numbers. For $\lambda = \frac{1}{2}$ we shall obtain as a particular case the credibilistic inventory problem of [14], as well as the form of its solution.

The paper is structured as follows. Section 2 contains introductory material on possibility and necessity measures, credibility measure and m_λ - measure, as well as on their relationship. In Section 3 we present the definition of the m_λ - expected value and some of its basic properties. Section 4 deals with the construction of a risk neutral inventory model whose objective function is defined by using the notion of m_λ - expected value. By using the linearity of m_λ–expected operator $E_\lambda(.)$, a general formula for the solution of the maximization problem is obtained. In Section 5 we proved some explicit formulas for this solution in the particular cases when the demands are trapezoidal and triangular fuzzy numbers. The proofs of these formulas are based on the form of m_λ–expected value $E_\lambda(A)$ of a trapezoidal fuzzy number A (see Proposition 3.4). Section 6 highlights how by applying the percentile method of Vercher et al. [23] we can build an inventory problem starting from a dataset. In this inventory problem, the components of a demand vector are trapezoidal fuzzy numbers, such that one can apply the formulas from Section 4 to compute the solution of the optimization problem.

2 Preliminaries

Let X be a universe whose elements can be individuals, objects, states, alternatives, etc. An events A is a subset of X: the set of events will be the family $P(X)$ of subsets of X. The complement of the event A will be denoted by A^c.

In this paper we shall assume that the elements of the universe X are real numbers ($X \subseteq \mathbb{R}$). A fuzzy variable will be an arbitrary function $\xi : X \to \mathbb{R}$.

The notions of possibility measure and necessity measure can be introduced both axiomatically and through a possibility distribution (cf. [24], [25], [26]).

A possibility measure on X is a function $Pos : \mathcal{P}(X) \to [0, 1]$ such that

$(Pos1)$ $Pos(\emptyset) = 0$; $Pos(X) = 1$;

$(Pos2)$ $Pos(\bigcup_{i\in I} A_i) = sup_{i\in I} Pos(A_i)$, for any family $(A_i)_{i\in I}$ of events.

A necessity measure on X is a function $Nec : \mathcal{P}(X) \to [0, 1]$ such that

$(Nec1)$ $Nec(\emptyset) = 0$; $Nec(X) = 1$;

$(Nec2)$ $Nec(\bigcap_{i\in I} A_i) = inf_{i\in I} Nec(A_i)$, for any family $(A_i)_{i\in I}$ of events.

The notions of possibility measure and necessity measure are dual: to each possibility measure Pos one can assign a necessity measure $Nec(A) = 1 - Pos(A^c)$ and, vice-versa, to each necessity measure Nec one can assign a possibility measure $Pos(A) = 1 - Nec(A^c)$.

Given a possibility measure Pos on the universe X, for any parameter $\lambda \in [0, 1]$ consider the function $m_\lambda : P(X) \to [0, 1]$ defined by

$$m_\lambda(A) = \lambda Pos(A) + (1 - \lambda)Nec(A), \tag{1}$$

for any event A;

(Nec is here the necessity measure associated with Pos).

This new measure was introduced by Yang and Iwamura in [17] as a convex linear combination of Pos and Nec by means of the weight λ. If $\lambda = \frac{1}{2}$ then one obtains the notion of credibility measure in the sense of Liu's monograph [9]:

$$Cred(A) = \frac{1}{2}(Pos(A) + Nec(A)), \tag{2}$$

for any event A.

A possibilistic distribution on X is a function $\mu : X \to [0, 1]$ such that $sup_{x\in X} = 1$; μ is normalized if $\mu(x) = 1$ for some $x \in X$.

Let us fix a possibility distribution $\mu : X \to [0,1]$. Then one can associate with μ a possibility measure Pos and a necessity measure Nec by taking

$$Pos(A) = sup_{x\in A}\mu(x) \tag{3}$$

for any event A;

$$Nec(A) = inf_{x\in A}\mu(x) \tag{4}$$

for any event A.

Then for each parameter $\lambda \in [0,1]$, the measure m_λ defined by (1) will have the following form:

$$m_\lambda(A) = \lambda sup_{x\in A}\mu(x) + (1-\lambda) inf_{x\in A}\mu(x), \tag{5}$$

for any event A.

According to [9], we say that the normalized possibility distribution μ is the membership function associated with a fuzzy variable ξ if for any event A we have

$$Pos(\xi \in A) = sup_{x\in A}\mu(x). \tag{6}$$

Then the following equalities hold:

$$Nec(\xi \in A) = inf_{x\in A}\mu(x); \tag{7}$$

$$m_\lambda(\xi \in A) = \lambda sup_{x\in A}\mu(x) + (1-\lambda) inf_{x\in A}\mu(x). \tag{8}$$

3 The Expected Value Associated with the Measure m_λ

We fix a parameter $\lambda \in [0,1]$ and assume that ξ is a fuzzy variable, μ is its membership function and m_λ is the measure defined in (5).

Following [18], the expected value of ξ w.r.t. the measure m_λ is defined by

$$E_\lambda(\xi) = \int_{-\infty}^{0} [m_\lambda(\xi \geq r) - 1]dr + \int_{0}^{\infty} m_\lambda(\xi \geq r)dr. \tag{9}$$

If $\lambda = \frac{1}{2}$ then one obtains the credibilistic expected value of ξ w.r.t. the credibility measure Cr defined in (2):

$$E_C(\xi) = \int_{0}^{\infty} Cr(\xi \geq r)dr - \int_{-\infty}^{0} Cr(\xi \leq r)dr. \tag{10}$$

The previous notion of credibilistic expected value was introduced by Liu and Liu in [10].

The following result shows that the expected operator $E_\lambda(\cdot)$ is linear.

Proposition 3.1. [18] *Let ξ_1, ξ_2 be two fuzzy variables such that $E_\lambda(\xi_1) < \infty$, $E_\lambda(\xi_2) < \infty$ and α, β are two non - negative real numbers. Then the following hold:*

$$E_\lambda(\xi_1 + \xi_2) = E_\lambda(\xi_1) + E_\lambda(\xi_2); \tag{11}$$

$$E_\lambda(\alpha\xi_1) = \alpha E_\lambda(\xi_1). \tag{12}$$

Lemma 3.2. *If $\xi > 0$ then $E_\lambda(\xi) = \int_0^\infty m_\lambda(\xi \geq r)dr$ and $E_\lambda(\xi) > 0$.*

According to [9], p.73, a trapezoidal fuzzy variable (= trapezoidal fuzzy number) $\xi = (r_1, r_2, r_3, r_4)$, with $r_1 \leq r_2 \leq r_3 \leq r_4$, is defined by the following membership function:

$$\mu_\xi(x) = \begin{cases} \frac{x-r_1}{r_2-r_1} & r_1 \leq x \leq r_2, \\ 1 & r_2 \leq x \leq r_3, \\ \frac{x-r_3}{r_4-r_3} & r_3 \leq x \leq r_4, \\ 0 & otherwise. \end{cases} \tag{13}$$

If $r_2 = r_3$ then one obtains the triangular fuzzy number $\xi = (r_1, r_2, r_4)$.

Lemma 3.3. [18] *For any trapezoidal fuzzy variable $\xi = (r_1, r_2, r_3, r_4)$ we have:*

$$m_\lambda(\xi \leq x) = \begin{cases} 1 & r_4 \leq x, \\ \frac{\lambda(r_4-x)+x-r_1}{r_4-r_3} & r_3 \leq x \leq r_4, \\ \lambda & r_2 \leq x \leq r_3, \\ \frac{\lambda(x-r_1)}{r_2-r_1} & r_1 \leq x \leq r_2, \\ 0 & x \leq r_1. \end{cases} \tag{14}$$

Proposition 3.4. [18] *For any trapezoidal fuzzy variable $\xi = (r_1, r_2, r_3, r_4)$ the expected value $E_\lambda(\xi)$ has the form*

$$E_\lambda(\xi) = (1-\lambda)\frac{r_1+r_2}{2} + \lambda\frac{r_3+r_4}{2}. \tag{15}$$

Corollary 3.5. *For any triangular fuzzy variable $\xi = (r_1, r_2, r_4)$ the expected value $E_\lambda(\xi)$ has the form*

$$E_\lambda(\xi) = (1-\lambda)\frac{r_1}{2} + \frac{r_2}{2} + \lambda\frac{r_4}{2}. \tag{16}$$

4 An Inventory Problem with Fuzzy Variables as Demands

This section concerns a risk - neutral multi - item inventory problem characterized by the following two hypotheses:

(I) the components of the demand vector are fuzzy variables;

(II) the objective function of the inventory model is defined by using the expected value operator $E_\lambda(\cdot)$ introduced in the previous section.

The inventory problem with n items has the following initial data:

- $c_1, \ldots, c_n$: unit fixed costs per inventoried item;
- $d_1, \ldots, d_n$: unit revenues per inventoried item;
- $h_1, \ldots, h_n$: unit holding costs per inventoried item;
- $\vec{D} = (D_1, \ldots, D_n)$: fuzzy demand vector in the inventory problem;
- $\vec{x} = (x_1, \ldots, x_n)$: order quantity vector in the inventory problem.

The components $D_1, \ldots, D_n$ of $\vec{D}$ are fuzzy variables. We shall assume that $c_i \geq 0$, $d_i \geq 0$ and $D_i > 0$, for all $i = 1, \ldots, n$.

Remark 4.1. *The initial data of the possibilistic inventory problem are similar to the probabilistic inventory problems from [5], [8], the credibilistic inventory problems from [14],[15] and the possibilistic inventory problems from [27].*

We will further observe that the essential difference between the three types of models lies in the way of choosing the objective function of the optimization problem: the models in [5], [8] use the probabilistic expected value, those in [14], [15] use Liu credibilistic expected value [9] and those in [27] use the possibilistic expected value from [28].

Starting from above input data we will formulate a risk-neutral problem. Similar with [14], p. 132, the quantity $d_i x_i$ is the total revenue of the i^{th} item and the fuzzy variables $\frac{h_i x_i^2}{2}\frac{1}{D_i}$ is the holding cost of the i^{th} item.

We fix a parameter $\lambda \in [0,1]$, so we can use the expected value operator $E_\lambda(\cdot)$ defined in (9). According to Lemma 3.2, we remark that $E_f(\frac{1}{D_i}) > 0$, for all $i = 1, \ldots, n$.

The profit function of item i has the following form:

$$\pi_i(x_i, D_i) = d_i x_i - c_i - \frac{h_i x_i^2}{2}\frac{1}{D_i} \tag{17}$$

The total profit function of the possibilistic inventory problem has the following form:

$$\pi(\vec{x}, \vec{D}) = \sum_{i=1}^{n} \pi_i(x_i, D_i) = \sum_{i=1}^{n} (d_i x_i - c_i - \frac{h_i x_i^2}{2}\frac{1}{D_i}) \tag{18}$$

Then the optimization problem associated with the previous inventory model has the following form:

$$\begin{cases} \max\limits_{\vec{x}} E_\lambda(\pi(\vec{x}, \vec{D})) \\ \vec{x} \geq 0 \end{cases} \tag{19}$$

Remark 4.2. *The objective function in the optimization problem (19) is the expected value $E_\lambda(\pi(\vec{x}, \vec{D}))$ of the fuzzy variable $\pi(\vec{x}, \vec{D})$ (w.r.t. the measure m_λ).*

Remark 4.3. *For $\lambda = \frac{1}{2}$ we obtain as a particular case the credibilistic inventory problem studied in [14]:*

$$\begin{cases} \max\limits_{\vec{x}} E_\lambda(\pi(\vec{x}, \vec{D})) \\ \vec{x} \geq 0 \end{cases} \tag{20}$$

By applying Proposition 3.1 to (18), the expected value $E_\lambda(\pi(\vec{x}, \vec{D}))$ can be written

$$E_\lambda(\pi(\vec{x}, \vec{D})) = \sum_{i=1}^{n} [d_i x_i - c_i - \frac{h_i x_i^2}{2} E_\lambda(\frac{1}{D_i})] \tag{21}$$

hence the optimization problem (19) becomes

$$\begin{cases} \max\limits_{x_1,\ldots,x_n} \sum\limits_{i=1}^{n} [d_i x_i - c_i - \frac{h_i x_i^2}{2} E_\lambda(\frac{1}{D_i})] \\ x_i \geq 0, i = 1, \ldots, n \end{cases} \tag{22}$$

The decision - maker aims to find the non - negative values $x_1, \ldots, x_n$ that maximize the expected total profit $E_\lambda(\pi(\vec{x}, \vec{D}))$.

In particular, setting $\lambda = \frac{1}{2}$ in (22) one obtains the credibilistic inventory problem from [14].

$$\begin{cases} \max\limits_{x_1,\ldots,x_n} \sum\limits_{i=1}^{n} [d_i x_i - c_i - \frac{h_i x_i^2}{2} E_C(\frac{1}{D_i})] \\ x_i \geq 0, i = 1, \ldots, n \end{cases} \tag{23}$$

Proposition 4.4. *The optimization problem (22) has the following solution:*

$$x_i^* = \frac{d_i}{h_i E_\lambda(\frac{1}{D_i})}, \tag{24}$$

for $i = 1, \ldots, n$

Proof. In order to find the solution of the optimization problem (22) we write the first - order condition

$$\frac{\partial}{\partial x_i} \sum_{i=1}^{n} (d_i x_i - c_i - \frac{h_i x_i^2}{2} E_\lambda(\frac{1}{D_i})) = 0,$$

for $i = 1, \ldots, n$,
therefore by a simple computation we obtain the equations

$$d_i - h_i E_\lambda(\frac{1}{D_i}) x_i = 0, \tag{25}$$

for $i = 1, \ldots, n$.
We remind that $E_\lambda(\frac{1}{D_i}) > 0$ for $i = 1, \ldots, n$. Thus the solution of the optimization problem (13) will have the following form

$$x_i^* = \frac{d_i}{h_i E_f(\frac{1}{A_i})},$$

for $i = 1, \ldots, n$.
□

5 Solution Form when the Demands are Trapezoidal Fuzzy Variables

According to Proposition 4.4, in order to compute the values $(x_1^*, \ldots, x_n^*)$ of the solution of inventory problem (22) we need to compute the expected values $E_\lambda(\frac{1}{D_1}), \ldots, E_\lambda(\frac{1}{D_n})$. The computation of these expected values depends on the form of the fuzzy variables $D_1, \cdots, D_n$ and in most cases this operation seems to be very difficult. In this section we solve this problem whenever the demands $D_1, \cdots, D_n$ are trapezoidal or triangular fuzzy numbers. The formulas obtained for the computation of the optimal solutions $x_1^*, \ldots, x_n^*$ have simple algebraic forms which makes them very suitable from a computational point of view.

We will fix the parameter $\lambda \in [0, 1]$. The following proposition is a key result of this section: the application of the formula (26) will lead us to find the form of optimal solutions $x_1^*, \ldots, x_n^*$.

Proposition 5.1. *Let D be a trapezoidal fuzzy number $D = (r_1, r_2, r_3, r_4)$ such that $0 < r_1 \le r_2 \le r_3 \le r_4$ then the expected value $E_\lambda(\frac{1}{D})$ has the following form*

$$E_\lambda(\frac{1}{D}) = \frac{\lambda}{r_2 - r_1} ln \frac{r_2}{r_1} + \frac{1 - \lambda}{r_4 - r_3} ln \frac{r_4}{r_3} \tag{26}$$

Proof. Firstly we observe that the condition $0 < r_1$ means $D > 0$, hence one obtains $\frac{1}{D} > 0$. By using Lemma 3.3 we get the following equalities:

$$m_\lambda(\frac{1}{D} \ge r) = m_\lambda(D \le \frac{1}{r}) = \begin{cases} 1 & r_4 \le \frac{1}{r}, \\ \frac{\lambda(r_4 - \frac{1}{r}) + \frac{1}{r} - r_3}{r_4 - r_3} & r_3 \le \frac{1}{r} \le r_4, \\ \lambda & r_2 \le \frac{1}{r} \le r_3, \\ \frac{\lambda(\frac{1}{r} - r_1}{r_2 - r_1} & r_1 \le \frac{1}{r} \le r_2, \\ 0 & \frac{1}{r} \le r_1. \end{cases}$$

which can be written as follows:

$$m_\lambda(\frac{1}{D} \geq r) = \begin{cases} 1 & r \leq \frac{1}{r_4}, \\ \frac{1}{r_4-r_3}[(1-\lambda)\frac{1}{r} + \lambda r_4 - r_3] & \frac{1}{r_4} \leq r \leq \frac{1}{r_3}, \\ \lambda & \frac{1}{r_3} \leq r \leq \frac{1}{r_2}, \\ \frac{\lambda}{r_2-r_1}[\frac{1}{r} - r_1] & \frac{1}{r_2} \leq r \leq \frac{1}{r_1}, \\ 0 & \frac{1}{r} \leq 0. \end{cases} \tag{27}$$

According to Lemma 3.2 we obtain

$$E_\lambda(\frac{1}{D}) = \int_0^\infty m_\lambda(\frac{1}{D} \geq r)dr = I_1 + I_2 + I_3 + I_4 \tag{28}$$

where I_1, I_2, I_3, I_4 have the following expressions:

$$I_1 = \int_0^{\frac{1}{r_4}} dr = \frac{1}{r_4}$$

$$I_2 = \frac{1}{r_4 - r_3}\int_{\frac{1}{r_4}}^{\frac{1}{r_3}} [\lambda(r_4 - \frac{1}{r}) + \frac{1}{r} - r_3]dr = \frac{1}{r_4 - r_3}[(1-\lambda)\ln\frac{r_4}{r_3} + (\lambda r_4 - r_3)(\frac{1}{r_3} - \frac{1}{r_4})]$$

$$I_3 = \lambda\int_{\frac{1}{r_3}}^{\frac{1}{r_2}} dr = \lambda(\frac{1}{r_2} - \frac{1}{r_3})$$

$$I_4 = \frac{\lambda}{r_2 - r_1}\int_{\frac{1}{r_2}}^{\frac{1}{r_1}} [\frac{1}{r} - r_1]dr = \frac{\lambda}{r_2 - r_1}[ln\frac{r_2}{r_1} - r_1(\frac{1}{r_1} - \frac{1}{r_2})]$$

Substituting in (28) these values of I_1, I_2, I_3, I_4 we get the formula (26).
□

Corollary 5.2. [14] *Let D be a trapezoidal fuzzy number $D = (r_1, r_2, r_3, r_4)$ such that $0 < r_1 \leq r_2 \leq r_3 \leq r_4$ then the credibilistic expected value $E_C(\frac{1}{D})$ has the following form*

$$E_C(\frac{1}{D}) = \frac{1}{2(r_2 - r_1)}ln\frac{r_2}{r_1} + \frac{1}{2(r_4 - r_3)}ln\frac{r_4}{r_3} \tag{29}$$

Proof. If we take $\lambda = \frac{1}{2}$ in (26) then we obtain the formula (29). □

Remark 5.3. *If in formula (26) one takes $r_2 = r_3$ then D is the triangular fuzzy number $D = (r_1, r_2, r_4)$ and the expected value $E_\lambda(\frac{1}{D})$ has the following form*

$$E_\lambda(\frac{1}{D}) = \frac{\lambda}{r_2 - r_1}ln\frac{r_2}{r_1} + \frac{1-\lambda}{r_4 - r_3}ln\frac{r_4}{r_2} \tag{30}$$

If in (30), we set $\lambda = \frac{1}{2}$ then we get the formula of the credibilistic expected value $E_\lambda(\frac{1}{D})$ from Theorem 2 of [14]:

$$E_C(\frac{1}{D}) = \frac{1}{2(r_2 - r_1)}ln\frac{r_2}{r_1} + \frac{1}{2(r_4 - r_2)}ln\frac{r_4}{r_2} \tag{31}$$

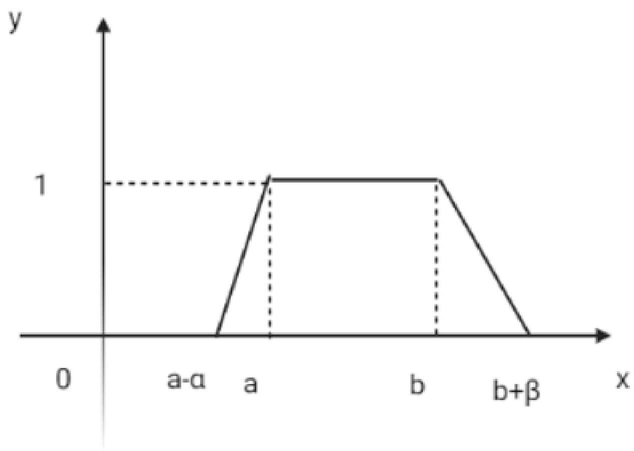

Figure 1: Trapezoidal fuzzy number

Remark 5.4. *Often in literature a trapezoidal fuzzy number D is given under the form $D = (a-\alpha, a, b, b+\beta)$, with $a, b \in \mathbb{R}$ and $\alpha, \beta \geq 0$ (Figure 1). Thus its membership μ_D has the form:*

$$\mu_D(x) = \begin{cases} 1 - \frac{a-x}{\alpha} & a-\alpha \leq x \leq a, \\ 1 & a \leq x \leq b, \\ 1 - \frac{x-b}{\beta} & b \leq x \leq b+\beta, \\ 0 & otherwise. \end{cases}$$

Assuming that $0 < a - \alpha$ we have $D > 0$ and the formula (26) becomes

$$E_\lambda(\frac{1}{D}) = \frac{\lambda}{\alpha} ln \frac{a}{a-\alpha} + \frac{1-\lambda}{\beta} ln \frac{b+\beta}{b} \quad (32)$$

Remark 5.5. *Assume that a triangular fuzzy number D is written under the form $D = (a-\alpha, a, a+\beta)$, with $a \in \mathbb{R}$ and $\alpha, \beta \geq 0$. If $0 < a - \alpha$ the formula (30) becomes*

$$E_\lambda(\frac{1}{D}) = \frac{\lambda}{\alpha} ln \frac{a}{a-\alpha} + \frac{1-\lambda}{\beta} ln \frac{a+\beta}{a} \quad (33)$$

The previous formulas (26), (30), (32) and (33) provide very computable expressions for the expected value $E_\lambda(\frac{1}{D})$ for the particular cases when D is a trapezoidal or a triangular fuzzy number.

By using these formulas we are now able to compute the solution $x_1^*, \cdots, x_n^*$ of the optimization problem (22) whenever the components $D_1, \ldots, D_n$ of demand vector are trapezoidal fuzzy numbers, respectively triangular fuzzy numbers.

Theorem 5.6. *Assume that the components $A_1, \ldots, A_n$ of demand vector $\vec{A}$ are trapezoidal fuzzy numbers $D_i = (a_i - \alpha_i, a_i, b_i, b_i + \beta_i)$, $i = 1, \ldots, n$, where $0 < a_i - \alpha_i \leq a_i \leq b_i \leq b_i + \beta_i$, for $i = 1, \ldots, n$. Then the solution of the optimization problem (13) has the following form*

$$x_i^* = \frac{d_i}{h_i[\frac{\lambda}{\alpha_i} ln \frac{a_i}{a_i - \alpha_i} + \frac{1-\lambda}{\beta_i} ln \frac{b_i+\beta_i}{b_i}]}, \quad (34)$$

for all $i = 1, \ldots, n$.

Proof. By (32), for each $i = 1, \ldots, n$ we have

$$E_\lambda(\frac{1}{D_i}) = \frac{\lambda}{\alpha_i} ln \frac{a_i}{a_i - \alpha_i} + \frac{1-\lambda}{\beta_i} ln \frac{b_i + \beta}{b_i}$$

If we substitute these values of $E_\lambda(\frac{1}{D_1}), \cdots, E_\lambda(\frac{1}{D_n})$ in (24) then we get the desired formula (34). □

Corollary 5.7. *If $D_1, \ldots, D_n$ are the triangular fuzzy numbers $D_i = (a_i - \alpha_i, a_i, a_i + \beta_i)$, $i = 1, \ldots, n$, where $0 < a_i - \alpha_i \leq a_i \leq b_i + \beta_i$, for $i = 1, \ldots, n$ then the solution of optimization problem (19) has the form*

$$x_i^* = \frac{d_i}{h_i[\frac{\lambda}{\alpha_i} ln \frac{a_i}{a_i - \alpha_i} + \frac{1-\lambda}{\beta_i} ln \frac{a_i + \beta_i}{a_i}]}, \tag{35}$$

for all $i = 1, \ldots, n$.

Proof. If in (34) one sets $b = a$, then the formula (35) follows immediately. □

Now we shall write the formula (34) for the following particular values of λ:

(a) $\lambda = 1/3$ (the pessimistic case)

$$x_i^* = \frac{3d_i}{h_i[\frac{1}{\alpha_i} ln \frac{a_i}{a_i - \alpha_i} + \frac{2}{\beta_i} ln \frac{b_i + \beta_i}{b_i}]}, \tag{36}$$

for all $i = 1, \ldots, n$.

(b) $\lambda = 1/2$ (the credibilistic case [14])

$$x_i^* = \frac{2d_i}{h_i[\frac{1}{\alpha_i} ln \frac{a_i}{a_i - \alpha_i} + \frac{1}{\beta_i} ln \frac{b_i + \beta_i}{b_i}]}, \tag{37}$$

for all $i = 1, \ldots, n$.

(c) $\lambda = 2/3$ (the optimistic case)

$$x_i^* = \frac{3d_i}{h_i[\frac{2}{\alpha_i} ln \frac{a_i}{a_i - \alpha_i} + \frac{1}{\beta_i} ln \frac{b_i + \beta_i}{b_i}]}, \tag{38}$$

for all $i = 1, \ldots, n$.

6 A Numerical Example

In order to solve the optimization problems associated with some inventory models we should know the form of the variables $D_1, \cdots, D_n$ and of the (probabilistic, credibilistic, possibilistic, etc.) indicators that appear in models. In the examples of credibilistic inventory problems from [14], [15] the expressions of $D_1, \cdots, D_n$ are assumed to be trapezoidal fuzzy numbers.

In general, the mathematical expressions of $D_1, \cdots, D_n$ are not known, but through measurements can be found different values of them. In the numerical example of possibilistic inventory problem from [27] it started from a data table, then the method of Vercher et al. [23] was applied to determine the concrete form of fuzzy numbers $D_1, \cdots, D_n$.

In this section we will present the solution of an m_λ-inventory problem in which the initial information on the variables $D_1, \cdots, D_n$ (which in our case are trapezoidal numbers) is given in the form of a numerical table. In order to obtain the trapezoidal numbers that describe the demands $D_1, \cdots, D_n$ we will apply the sample percentile method of Vercher et al. [23].

Table 1: Data on demand vector

Item1	Item2	Item3	Item4	Item5	Item6	Item7	Item8	Item9	Item10
35	20	30	25	28	33	18	18	31	20
30	30	50	25	32	37	27	28	33	27
15	35	28	36	25	20	33	17	25	31
25	35	40	35	35	40	35	20	30	37
25	28	25	32	50	37	28	19	35	35
28	25	42	27	45	28	28	37	35	37
31	27	36	35	43	35	35	27	22	35
30	24	39	28	27	35	24	37	27	25
44	33	44	28	32	22	39	30	29	25
37	34	37	44	44	32	47	36	28	37
23	17	22	33	32	29	31	31	45	19

Table 2: Trapezoidal fuzzy numbers

A_1	A_2	A_3	A_4	A_5
(28,30,9,10.5)	(27,30,8.5,5)	(36,39,12.5, 5)	(28,32,3,4)	(32,35,6,10)
A_6	A_7	A_8	A_9	A_{10}
(32,35,11,2)	(28,33,7,6)	(27,30,9.5,7)	(29,31,5.5,4)	(27,35,7.5,2)

We continue with the presentation of the values of basic parameters d_i, c_i, d_i, so the inventory problem is entirely defined. Finally, we apply the formulas (36)- (38) in order to obtain the optimal solutions of the model.

Our inventory problem has a demand vector of size 10. Table 1 contains the data we have on demand vector.

In column i of Table 1 are placed the known values of item i. In a probabilistic inventory model, the above columns will contain values of random variables. In this case the maximization problem of the model will be obtained by usual statistical methods.

Under the hypothesis that the 10 items are modeled by trapezoidal fuzzy numbers, one has to convert the data from the above table in 10 such fuzzy numbers (each column is assigned to a trapezoidal fuzzy number).

Let's present shortly the percentile method of Vercher et al. [23], by which to a data set of real numbers $x_1, \ldots, x_m$ one assigns a trapezoidal fuzzy number $A = (a, b, \alpha, \beta)$.

Let us denote by P_k the k-the percentile of the sample $x_1, \ldots, x_m$. Then the trapezoidal fuzzy number $A = (a, b, \alpha, \beta)$ will be determined by the formulas:

$$a = P_{40}, b = P_{60}, \alpha = P_{40} - P_5, \beta = P_{95} - P_{60} \tag{39}$$

By applying Vercher et al.'s method [23] to each of the columns of Table 1 obtains the trapezoidal fuzzy numbers in Table 2.

The trapezoidal fuzzy numbers $A_1, \ldots, A_{10}$ obtained from Table 1 will be the components of the demand vector of a risk neutral multi–item inventory problem. This inventory problem will be defined by the data in the first five columns of Table 3:

Columns two, three and four of Table 3 contain the unit fixed costs, unit revenues and holding costs of the model. The trapezoidal fuzzy numbers from the fifth column make up the demand vector in the m_λ–inventory

Table 3: The elements of the inventory problem

Item	d_i	c_i	h_i	$A_i = (a_i, b_i, \alpha_i, \beta_i)$	$x_i^*(\lambda = 1/3)$	$x_i^*(\lambda = 1/2)$	$x_i^*(\lambda = 2/3)$
1	12	2	0.5	(28,30,9,10.5)	718.21	668.76	627.44
2	11	1	0.6	(27,30,8.5,5)	518.19	486.88	459.14
3	14	3	0.5	(36,39,12.5, 5)	1019.75	961.42	909.4
4	10	4	0.8	(28,32,3,4)	387.92	371.9	357.14
5	11	5	0.9	(32,35,6,10)	432.03	409.19	388.64
6	10	3	0.9	(32,35,11,2)	355.13	336.3	319.37
7	12	2	0.5	(28,33,7,6)	743.93	696.25	654.32
8	15	1	0.6	(27,30,9.5,7)	710.45	661.32	618.54
9	13	3	0.7	(29,31,5.5,4)	563.24	541.63	521.61
10	13	4	0.9	(27,35,7.5,2)	437.88	405.88	378.24

problem. In fact, for distinct parameters $\lambda \in [0, 1]$ we obtain distinct inventory problems. We consider the three inventory models (a)-(c) corresponding to the parameters 1/3, 1/2 and 2/3. By applying the formulas (36)-(38) we obtain the solutions of the three optimization problems. These solutions are placed in the last three columns of Table 3.

Remark 6.1. Regarding the last three columns of Table 3, it is noticed that with the increase of the parameter λ ($\frac{1}{3} < \frac{1}{2} < \frac{2}{3}$) the solution values of the optimization problem decrease. The theoretical argument of this fact is given by Proposition 8.2 in the Appendix.

7 Conclusion

In the work we studied a new inventory model whose construction is based on the parametric measure m_λ (introduced by Yang and Iwamura in [17]) and on the notion of m_λ–expected value (introduced by Dzouche et al. in [18]). More precisely, in this inventory model, the demands and the total profit are fuzzy variables and the objective function of the optimization problem is the m_λ–expected value of total profit. It was found the general form of the solution of the optimization problem and when the demands are trapezoidal or triangular fuzzy variables computationally efficient forms of the solution have been found.

An open problem is finding the calculation formulas for the optimal solutions also when the demands are represented by other types of fuzzy variables: discrete repartitions, Erlang fuzzy variables, etc.

The inventory model in the paper is risk–neutral. Another open problem is the study of risk–averse inventory models in the framework of m_λ–theory. It would also be interesting to treat some mean–value inventory model, in which besides maximizing the m_λ–expected value of the total profit to be required to minimize the m_λ–variance of the total profit (the notion of m_λ–variance has been defined in [18]). Defining a notion of mean–absolute deviation in the context of an m_λ–theory would lead to an inventory model in which the risk is eventually represented by this indicator.

Continuing the research line from [14], [15], in paper [16] is investigated an inventory problem in which the components of the demand vector are type-2 fuzzy variables. This model is studied with the techniques of Liu's credibility theory [9]. It arises naturally a question of extending this model to m_λ–theory, so that giving the parameter λ the value $\frac{1}{2}$ to obtain as a particular case some results of [16].

The newsvendor problem is a core concept in inventory management dealing with stochastic demand. Traditionally, it centers on a single goal: either minimizing expected costs or maximizing expected profits.

A mean–variance model for the newsvendor problem is presented in paper [29]. A newsvendor problem is studied in which the maximization of expected profit and the minimization of risk, expressed by the profit variance, are required. It would be interesting to formulate and study a newsvendor problem in which the expected profit is expressed by m_λ–expected value and the risk of profit by m_λ–variance (according to Definition 2 of [18]).

8 Appendix

One asks the question of how the solutions of the optimization problem (20) vary depending on the parameter λ. We will give a solution to this problem in case when the demands $D_1, \ldots, D_n$ are trapezoidal fuzzy numbers.

Lemma 8.1. *Assume that ξ is a trapezoidal fuzzy variable. If $\lambda_1 \leq \lambda_2$ then $E_{\lambda_1}(\xi) \leq E_{\lambda_2}(\xi)$.*

Proof. See Proposition 1 of [18]. □

Let λ_1, λ_2 be two parameters in the interval $[0, 1]$. We consider the two inventory problems with the same input data, but with different objective functions of the optimization problems, defined by the expected operators $E_{\lambda_1}(\xi)$ and $E_{\lambda_2}(\xi)$, respectively.

We denote by $x_1^*, \ldots, x_n^*$ the solution of the optimization problem corresponding to $E_{\lambda_1}(\xi)$ and with $y_1^*, \ldots, y_n^*$ the solution of the optimization problem corresponding to $E_{\lambda_2}(\xi)$.

Proposition 8.2. *Assume that the demands $D_1, \ldots, D_n$ are trapezoidal fuzzy variables. If $\lambda_1 \leq \lambda_2$ then $x_i^* \geq y_i^*$ for any $i = 1, \ldots, n$.*

Proof. Assume that $\lambda_1 \leq \lambda_2$. By Proposition 4.4, the solutions $x_1^*, \ldots, x_n^*$ and $y_1^*, \ldots, y_n^*$ are written in the following form:

$$x_i^* = \frac{d_i}{h_i E_{\lambda_1}(\frac{1}{D_i})} \tag{40}$$

for $i = 1, \ldots, n$.

$$y_i^* = \frac{d_i}{h_i E_{\lambda_2}(\frac{1}{D_i})} \tag{41}$$

for $i = 1, \ldots, n$.

Applying Lemma 8.1 for any $i = 1, \ldots, n$ the following implications hold:

$$\lambda_1 \leq \lambda_2 \Rightarrow E_{\lambda_1}(\frac{1}{D_i}) \leq E_{\lambda_2}(\frac{1}{D_i}) \Rightarrow \frac{1}{E_{\lambda_2}(\frac{1}{D_i})} \leq \frac{1}{E_{\lambda_1}(\frac{1}{D_i})} \tag{42}$$

By (40)-(42) for any $i = 1, \ldots, n$ we will have:

$$x_i^* - y_i^* = \frac{d_i}{h_i}(\frac{1}{E_{\lambda_2}(\frac{1}{D_i})} - \frac{1}{E_{\lambda_1}(\frac{1}{D_i})}) \geq 0.$$

We conclude that $x_i^* \geq y_i^*$ for any $i = 1, \ldots, n$. □

Acknowledgements: The author would like to express her gratitude to the editor and the reviewers for their valuable suggestions in obtaining the final version of the paper.

Conflict of Interest: The author declares no conflict of interest.

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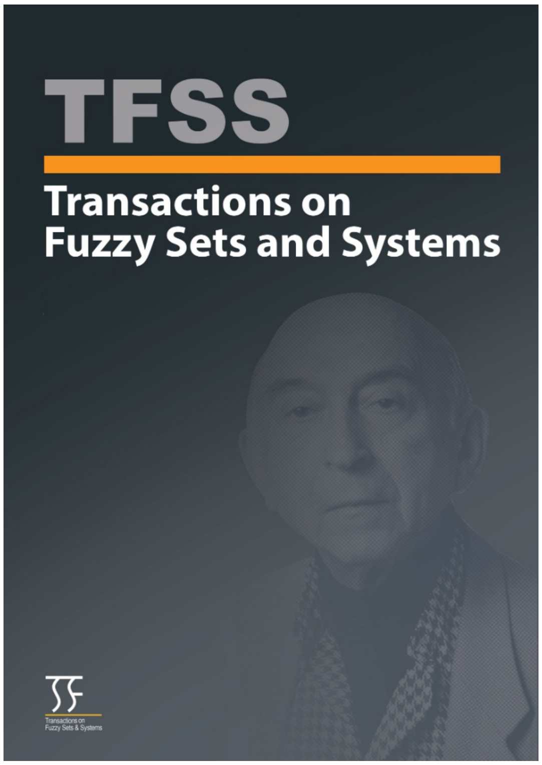

Transactions on Fuzzy Sets and Systems

ISSN: **2821-0131**

https://sanad.iau.ir/journal/tfss/

Fuzzy Filters of Pre-ordered Residuated Systems

Vol.4, No.1, (2025), 30-41. DOI: https://doi.org/10.71602/tfss.2025.1129288

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Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://sanad.iau.ir/journal/tfss/
Online ISSN: 2821-0131
Vol.4, No.1, (2025), 30-41
DOI: https://doi.org/10.71602/tfss.2025.1129288

Article Type: Original Research Article

Fuzzy Filters of Pre-ordered Residuated Systems

Young Bae Jun*

(This paper is dedicated to Professor "John N. Mordeson" on the occasion of his 91st birthday.)

Abstract. Using the concept of fuzzy points, the notion of fuzzy filters in pre-ordered residuated systems is introduced, and their relevant properties are investigated and analyzed. Characterization of fuzzy filters are displayed. Fuzzy filters are formed using filters. The concepts of positive set, $\in_t$-set, (extended) q_t-set are defined and the conditions under which they become filters are explored. The concept of fuzzy filter with thresholds is introduced and related properties are investigated.

AMS Subject Classification 2020: 08A02; 06A11; 06B75; 08A72
Keywords and Phrases: Residuated relational system, Pre-ordered residuated system, Filter, $\in_t$-set, (Extended) q_t-set, Positive set.

1 Introduction

The concept of a residuated relational system introduced by Bonzio et al. [1] is a mathematical structure used in the study of ordered algebraic systems, particularly in the fields of logic, lattice theory, and category theory. These systems generalize certain aspects of algebraic structures like lattices and posets, with a focus on the relationship between operations and their adjoints, often in the context of residuation. They developed the concept of a pre-ordered residuated system, which is nothing but a residuated relational system whose relation is pre-order, i.e., reflexive and transitive. Their work like this is based on generalizing the concept of residuated poset, by replacing the usual partial order to a pre-order. Romano [2, 3, 4, 5] called the pre-ordered residuated system a quasi-ordered residuated system. He introduced and analyzed the notion of filters in pre-ordered residuated systems. The purpose of this paper is to study the filter of a pre-ordered residuated system using the fuzzy set theory. For this, we will use the concept of fuzzy points. We introduce the concept of fuzzy filters in a pre-ordered residuated system, and investigate their relevant properties. We consider characterizations of fuzzy filter. We construct $\in_t$-set, (extended) q_t-set, positive set, etc., and explore the conditions under which these can be filters.

2 Preliminaries

Definition 2.1 ([1])**.** Let $(X, \odot, \to, 1)$ be an algebra of type (2,2,0) and let R be a binary operation on X. A structure $\mathbb{X} := (X, \odot, \to, 1, R)$ is called a *residuated relational system* if the following three conditions are valid.

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Received: 14 June 2024; **Revised**: 14 August 2024; **Accepted**: 14 August 2024; **Available Online**: 25 August 2024; **Published Online**: 7 May 2025.

How to cite: Jun YB. Fuzzy filters of pre-ordered residuated systems. *Transactions on Fuzzy Sets and Systems.* 2025; 4(1): 30-41. DOI: https://doi.org/10.71602/tfss.2025.1129288

(i) $(X, \odot, 1)$ is a commutative monoid,

(ii) $(\forall \mathfrak{a} \in X)\ ((\mathfrak{a}, 1) \in R)$,

(iii) $(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X)\ ((\mathfrak{a} \odot \mathfrak{b}, \mathfrak{c}) \in R) \Leftrightarrow (\mathfrak{a}, \mathfrak{b} \to \mathfrak{c}) \in R)$.

Let $\mathbb{X} := (X,\ \odot,\ \to,\ 1,\ R)$ be a residuated relational system. For every element y of X, consider the following two mappings:

$$f_y : X \to X,\ x \mapsto x \odot y \text{ and } g_y : X \to X,\ x \mapsto y \to x.$$

Proposition 2.2 ([1]). *Every residuated relational system* $\mathbb{X} := (X,\ \odot,\ \to,\ 1,\ R)$ *satisfies:*

$$(\forall x, y \in X)(g_x(y) = 1 \ \Rightarrow\ (x, y) \in R). \tag{1}$$
$$(\forall x \in X)((x, g_1(1)) \in R). \tag{2}$$
$$(\forall x \in X)((1, g_x(1))) \in R). \tag{3}$$
$$(\forall x, y, z \in X)(g_x(y) = 1 \ \Rightarrow\ (f_x(z), y) \in R). \tag{4}$$
$$(\forall x, y \in X)((x, g_y(1)) \in R). \tag{5}$$

Moreover, if R is reflexive, then

$$(\forall x \in X)((1, g_x(x)) \in R). \tag{6}$$
$$(\forall x, y \in X)(f_x((g_x(y)), y) \in R). \tag{7}$$
$$(\forall x, y \in X)((x, g_y(f_y(x))) \in R). \tag{8}$$
$$(\forall x, y \in X)((x, g_1(x)) \in R,\ (g_1(x), x) \in R). \tag{9}$$
$$(\forall x, y \in X)(x, g_{g_x(y)}(y)) \in R). \tag{10}$$

Also, if R is antisymmetric, then

$$(\forall x, y \in X)((x, y) \in R \ \Leftrightarrow\ g_x(y) = 1). \tag{11}$$
$$\textit{If } R \textit{ is also reflexive, then } (f_y(x), x) \in R \textit{ and } (f_y(x), y) \in R. \tag{12}$$

Recall that a binary relation "R" on a set X is said to be *pre-order* if it is reflexive and transitive. Note that the pre-order relation is sometimes called the quasi-order relation.

Definition 2.3 ([1, 2]). A residuated relational system $\mathbb{X} := (X,\ \odot,\ \to,\ 1,\ R)$ is called a *pre-ordered residuated system* if R is a pre-order relation on X.

The pre-ordered residuated system $\mathbb{X} := (X,\ \odot,\ \to,\ 1,\ R)$ will be denoted by $\mathbb{X} := (X,\ \odot,\ \to,\ 1,\ \lesssim)$.

Definition 2.4 ([2]). Let $\mathbb{X} := (X,\ \odot,\ \to,\ 1,\ \lesssim)$ be a pre-ordered residuated system. A subset F of X is called a *filter* of $\mathbb{X}$ if it satisfies:

$$(x, y \in X)(x \in F,\ x \lesssim y \ \Rightarrow\ y \in F), \tag{13}$$
$$(x, y \in X)(x \in F,\ g_x(y) \in F \ \Rightarrow\ y \in F). \tag{14}$$

A fuzzy set $\eth$ in a set X of the form

$$\eth(\mathfrak{b}) := \begin{cases} t \in (0, 1] & \text{if } \mathfrak{b} = \mathfrak{a}, \\ 0 & \text{if } \mathfrak{b} \neq \mathfrak{a}, \end{cases}$$

is said to be a *fuzzy point* with support $\mathfrak{a}$ and value t and is denoted by $\langle \mathfrak{a}_t \rangle$.

For a fuzzy set $\eth$ in a set X, we say that a fuzzy point $\langle \mathfrak{a}_t \rangle$ is

(i) *contained* in $\eth$, denoted by $\langle \mathfrak{a}_t \rangle \in \eth$, (see [6]) if $\eth(\mathfrak{a}) \geq t$.

(ii) *quasi-coincident* with $\eth$, denoted by $\langle \mathfrak{a}_t \rangle\, q\, \eth$, (see [6]) if $\eth(\mathfrak{a}) + t > 1$.

If a fuzzy point $\langle \mathfrak{a}_t \rangle$ is contained in $\eth$ or is quasi-coincident with $\eth$, we denote it $\langle \mathfrak{a}_t \rangle \in\!\vee q\, \eth$. If a fuzzy point $\langle \mathfrak{a}_t \rangle$ is contained in $\eth$ and is quasi-coincident with $\eth$, we denote it $\langle \mathfrak{a}_t \rangle \in\!\wedge q\, \eth$. If $\langle \mathfrak{a}_t \rangle\, \alpha\, \eth$ is not established for $\alpha \in \{\in, q, \in\!\vee q, \in\!\wedge q\}$, it is denoted by $\langle \mathfrak{a}_t \rangle\, \overline{\alpha}\, \eth$.

Given $t \in (0,1]$ and a fuzzy set $\eth$ in a set X, consider the following sets

$$(\eth, t)_{\in} := \{\mathfrak{a} \in X \mid \langle \mathfrak{a}_t \rangle \in \eth\} \text{ and } (\eth, t)_q := \{\mathfrak{a} \in X \mid \langle \mathfrak{a}_t \rangle\, q\, \eth\}$$

which are called an $\in_t$-*set* and a q_t-*set* of $\eth$, respectively, in X. Also, we consider the set

$$(\eth, t)_{\in\vee q} := \{\mathfrak{a} \in X \mid \langle \mathfrak{a}_t \rangle \in\!\vee q\, \eth\}$$

which is called the $t(\in\!\vee q)$-*set* of $\eth$.

It is clear that $(\eth, t)_{\in\vee q} = (\eth, t)_{\in} \cup (\eth, t)_q$.

3 Fuzzy Filters

In what follows, let $\mathbb{X} := (X, \odot, \rightarrow, 1, \lesssim)$ denote a pre-ordered residuated system, and it will be simply written by $\mathbb{X}$ only.

First, we introduce a central concept that will be used throughout the paper.

Definition 3.1. A fuzzy set $\eth$ in X is called a *fuzzy filter* of $\mathbb{X}$ if its nonempty $\in_t$-set $(\eth, t)_{\in}$ is a filter of $\mathbb{X}$ for all $t \in (0,1]$.

Example 3.2. Let $X := (-\infty, 1] \subset \mathbb{R}$ (the set of real numbers). If we define two binary operations "$\odot$" and "$\rightarrow$" on X as follows:

$$x \odot y = \min\{x, y\} \text{ and } x \rightarrow y = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{if } x > y, \end{cases}$$

for all $x, y \in X$, then $\mathbb{X} := (X, \odot, \rightarrow, 1, \leq)$ is a pre-ordered residuated system (see [5]). Let $\eth$ be a fuzzy set in X given by

$$\eth : X \rightarrow [0,1],\ x \mapsto \begin{cases} 0.78 & \text{if } x \in (0,1], \\ 0.62 & \text{if } x \in (-3,0], \\ 0.37 & \text{otherwise.} \end{cases}$$

Then $\eth$ is a fuzzy filter of $\mathbb{X} := (X, \odot, \rightarrow, 1, \leq)$.

Example 3.3. Let $X = \{b_1, b_2, b_3, b_4\}$ be a set and two binary operations "$\odot$" and "$\rightarrow$" on X are given as follows:

$\odot$	b_1	b_2	b_3	b_4
b_1	b_1	b_2	b_3	b_4
b_2	b_2	b_2	b_2	b_4
b_3	b_3	b_2	b_2	b_4
b_4	b_4	b_4	b_4	b_4

$\rightarrow$	b_1	b_2	b_3	b_4
b_1	b_1	b_2	b_3	b_4
b_2	b_1	b_1	b_1	b_1
b_3	b_1	b_2	b_1	b_4
b_4	b_1	b_2	b_3	b_1

We give a binary relation "$\lesssim$" as follows:

$$\lesssim:= \{(b_1,b_1),(b_2,b_2),(b_3,b_3),(b_4,b_4),(b_4,b_1),(b_3,b_1),(b_2,b_1),(b_2,b_3),(b_2,b_4)\}.$$

Then $\mathbb{X} := (X, \odot, \to, b_1, \lesssim)$ is a pre-ordered residuated system (see [7]). Let $\eth$ be a fuzzy set in X given by

$$\eth : X \to [0,1],\ x \mapsto \begin{cases} \frac{1}{n} & \text{if } x = b_1, \\ \frac{1}{4n} & \text{if } x = b_2, \\ \frac{1}{2n} & \text{if } x = b_3, \\ \frac{1}{3n} & \text{if } x = b_4, \end{cases}$$

where n is a natural number. Then $\eth$ is a fuzzy filter of $\mathbb{X} := (X, \odot, \to, b_1, \leq)$.

We discuss the characterization of fuzzy filters.

Theorem 3.4. *A fuzzy set $\eth$ in X is a fuzzy filter of $\mathbb{X}$ if and only if it satisfies:*

$$(\forall x,y \in X)(x \lesssim y \ \Rightarrow\ \eth(x) \leq \eth(y)), \tag{15}$$

$$(\forall x,y \in X)(\eth(y) \geq \min\{\eth(x), \eth(g_x(y))\}). \tag{16}$$

Proof. Assume that $\eth$ is a fuzzy filter of $\mathbb{X}$. Then its nonempty $\in_t$-set $(\eth,t)_\in$ is a filter of $\mathbb{X}$ for all $t \in (0,1]$. If (15) is not valid, then there exists $\mathfrak{a},\mathfrak{b} \in X$ such that $\mathfrak{a} \lesssim \mathfrak{b}$ and $\eth(\mathfrak{a}) > \eth(\mathfrak{b})$. Then $\mathfrak{a} \in (\eth,\eth(\mathfrak{a}))_\in$, but $\mathfrak{b} \notin (\eth,\eth(\mathfrak{a}))_\in$ which is a contradiction. Hence $\eth(x) \leq \eth(y)$ for all $x,y \in X$ with $x \lesssim y$. Suppose that (16) is false. Then $\eth(\mathfrak{b}) < \min\{\eth(\mathfrak{a}), \eth(g_\mathfrak{a}(\mathfrak{b}))\}$ for some $\mathfrak{a},\mathfrak{b} \in X$. If we take $t := \min\{\eth(\mathfrak{a}), \eth(g_\mathfrak{a}(\mathfrak{b}))\}$, then $\mathfrak{a} \in (\eth,t)_\in$, $g_\mathfrak{a}(\mathfrak{b}) \in (\eth,t)_\in$ and $\mathfrak{b} \notin (\eth,t)_\in$. This is a contradiction, and thus $\eth(y) \geq \min\{\eth(x), \eth(g_x(y))\}$ for all $x,y \in X$.

Conversely, let $\eth$ be a fuzzy set in X that satisfies (15) and (16). Let $x,y \in X$. If $x \in (\eth,t)_\in$ and $x \lesssim y$, then $t \leq \eth(x) \leq \eth(y)$ by (15), i.e., $\langle y_t \rangle \in \eth$. Thus $y \in (\eth,t)_\in$. If $x \in (\eth,t)_\in$ and $g_x(y) \in (\eth,t)_\in$, then

$$\eth(y) \geq \min\{\eth(x), \eth(g_x(y))\} \geq t$$

by (16) and so $y \in (\eth,t)_\in$. Hence $(\eth,t)_\in$ is a filter of $\mathbb{X}$ for all $t \in (0,1]$, and therefore $\eth$ is a fuzzy filter of $\mathbb{X}$. $\square$

Theorem 3.5. *In $\mathbb{X}$, a fuzzy set $\eth$ in X satisfies (15) if and only if the following assertion is valid.*

$$(\forall x,y,z \in X)(\forall t \in (0,1])(f_y(x) \in (\eth,t)_\in,\ x \lesssim g_y(z) \ \Rightarrow\ z \in (\eth,t)_\in). \tag{17}$$

Proof. Assume that $\eth$ satisfies (15) and let $x,y,z \in X$ be such that $x \lesssim g_y(z)$ and $f_y(x) \in (\eth,t)_\in$ for all $t \in (0,1]$. Then $f_y(x) \lesssim z$ by Definition 2.1(iii), and so $\eth(z) \geq \eth(f_y(x)) \geq t$ by (15). Hence $z \in (\eth,t)_\in$.

Conversely, let $\eth$ be a fuzzy set in X that satisfies (17). In the proof of Theorem 3.4, we can observe that $\eth$ satisfies (15) if and only if $\eth$ satisfies:

$$(\forall x,y \in X)(\forall t \in (0,1])(x \in (\eth,t)_\in,\ x \lesssim y \ \Rightarrow\ y \in (\eth,t)_\in).$$

Let $x,y \in X$ be such that $x \lesssim y$ and $x \in (\eth,t)_\in$ for all $t \in (0,1]$. Then $f_1(x) = x \in (\eth,t)_\in$ and $x \lesssim g_1(y)$. It follows from (17) that $y \in (\eth,t)_\in$. Therefore $\eth$ satisfies (15). $\square$

Proposition 3.6. *In $\mathbb{X}$, if a fuzzy set $\eth$ in X satisfies (15), then*

$$(\forall x \in X)(\forall t \in (0,1])(x \in (\eth,t)_\in \ \Leftrightarrow\ g_1(x) \in (\eth,t)_\in), \tag{18}$$

or equivalently,

$$(\forall x \in X)(\forall t \in (0,1])(\eth(x) \geq t \ \Leftrightarrow\ \eth(g_1(x)) \geq t). \tag{19}$$

Proof. Let $\eth$ be a fuzzy set in X that satisfies (15). Then by the proof process of Theorem 3.4, we know the following:

$$x \in (\eth, t)_\in,\ x \lesssim y \ \Rightarrow\ y \in (\eth, t)_\in$$

for all $x, y \in X$ and $t \in (0, 1]$. Using (9) leads to $x \lesssim g_1(x)$ and $g_1(x) \lesssim x$. It follows that if $x \in (\eth, t)_\in$ (resp. $g_1(x) \in (\eth, t)_\in$), then $g_1(x) \in (\eth, t)_\in$ (resp., $x \in (\eth, t)_\in$). Hence (18) is valid. □

Corollary 3.7. *Every fuzzy filter $\eth$ of $\mathbb{X}$ satisfies the condition* (18).

Lemma 3.8 ([1]). *Every pre-ordered residuated system $\mathbb{X} := (X, \odot, \to, 1, \lesssim)$ satisfies:*

$$(\forall x, y \in X)(f_y(x) \lesssim x,\ f_y(x) \lesssim y). \tag{20}$$

Proposition 3.9. *In $\mathbb{X}$, if a fuzzy set $\eth$ in X satisfies* (15), *then*

$$(\forall t \in (0, 1])((\eth, t)_\in \neq \emptyset \ \Rightarrow\ 1 \in (\eth, t)_\in). \tag{21}$$

$$(\forall x, y \in X)(\forall t \in (0, 1])(f_y(x) \in (\eth, t)_\in \ \Rightarrow\ x \in (\eth, t)_\in,\ y \in (\eth, t)_\in). \tag{22}$$

$$(\forall x, y \in X)(\forall t \in (0, 1])(x, y \in (\eth, t)_\in,\ x \lesssim y \ \Rightarrow\ g_x(y) \in (\eth, t)_\in). \tag{23}$$

$$(\forall x, y \in X)(\forall t \in (0, 1])(1 \in (\eth, t)_\in,\ x \lesssim y \ \Rightarrow\ g_x(y) \in (\eth, t)_\in). \tag{24}$$

Proof. Assume that $(\eth, t)_\in \neq \emptyset$ for all $t \in (0, 1]$, and let $x \in (\eth, t)_\in$. Since $x \lesssim 1$ by Definition 2.1(ii), it follows from (15) that $\eth(1) \geq \eth(x) \geq t$. Hence $1 \in (\eth, t)_\in$. Let $x, y \in X$ and $t \in (0, 1]$ be such that $f_y(x) \in (\eth, t)_\in$. Then $\eth(f_y(x)) \geq t$. Since $f_y(x) \lesssim x$ and $f_y(x) \lesssim y$ by (20), it follows from (15) that $\eth(x) \geq \eth(f_y(x)) \geq t$ and $\eth(y) \geq \eth(f_y(x)) \geq t$, that is, $\langle x_t \rangle \in \eth$ and $\langle y_t \rangle \in \eth$. Hence $x \in (\eth, t)_\in$ and $y \in (\eth, t)_\in$, and so (22) is valid. Let $x, y \in X$ and $t \in (0, 1]$ be such that $x, y \in (\eth, t)_\in$ and $x \lesssim y$. Since $f_x(x) \lesssim x$ by (20), we have $f_x(x) \lesssim y$ by the transitivity of $\lesssim$. Thus $x \lesssim g_x(y)$ by Definition 2.1(iii), which implies from (15) that $\eth(g_x(y)) \geq \eth(x) \geq t$. Hence $g_x(y) \in (\eth, t)_\in$. Suppose that $1 \in (\eth, t)_\in$ for all $t \in (0, 1]$ and let $x, y \in X$ be such that $x \lesssim y$. Then $f_x(1) = x \lesssim y$, and so $1 \lesssim g_x(y)$ by Definition 2.1(iii). Using (15) leads to $\eth(g_x(y)) \geq \eth(1) \geq t$, and so $g_x(y) \in (\eth, t)_\in$. □

Corollary 3.10. *Every fuzzy filter $\eth$ of $\mathbb{X}$ satisfies the four conditions* (21), (22), (23) *and* (24).

Theorem 3.11. *For every nonempty subset F of X, consider a fuzzy set $\eth_F$ in X which is defined by*

$$\eth_F : X \to [0, 1],\ x \mapsto \begin{cases} s_1 & \text{if } x \in F, \\ s_2 & \text{otherwise} \end{cases}$$

where $s_1 > s_2$ in $[0, 1]$. Then $\eth_F$ is a fuzzy filter of $\mathbb{X}$ if and only if F is a filter of $\mathbb{X}$.

Proof. Assume that $\eth_F$ is a fuzzy filter of $\mathbb{X}$. Let $x, y \in X$. If $x \in F$ and $x \lesssim y$, then $\eth_F(y) \geq \eth_F(x) = s_1$ by (15), and so $\eth_F(y) = s_1$. Thus $y \in F$. If $x \in F$ and $g_x(y) \in F$, then $\eth_F(x) = s_1$ and $\eth_F(g_x(y)) = s_1$. Using (16) leads to $\eth_F(y) \geq \min\{\eth_F(x), \eth_F(g_x(y))\} = s_1$, and so $\eth_F(y) = s_1$. Thus $y \in F$. Therefore F is a filter of $\mathbb{X}$.

Conversely, suppose that F is a filter of $\mathbb{X}$. For every $x, y \in X$ with $x \lesssim y$, if $x \in F$, then $y \in F$ and so $\eth_F(y) = s_1 = \eth_F(x)$. If $x \notin F$, then $\eth_F(x) = s_2 < \eth_F(y)$. Let $x, y \in X$. If $x \in F$ and $g_x(y) \in F$, then $y \in F$ and thus $\eth_F(y) = s_1 = \min\{\eth_F(x), g_x(y)\}$. If $x \notin F$ or $g_x(y) \notin F$, then $\eth_F(x) = s_2$ or $\eth_F(g_x(y)) = s_2$. Hence $\eth_F(y) \geq s_2 = \min\{\eth_F(x), \eth_F(g_x(y))\}$. Therefore $\eth_F$ is a fuzzy filter of $\mathbb{X}$ by Theorem 3.4 □

Let $\eth$ be a non-constant fuzzy set in X and we construct the next set called *positive set*.

$$X_0 := \{x \in X \mid \eth(x) \neq 0\}. \tag{25}$$

It is clear that $X_0 \neq \emptyset$. We explore conditions for the positive set of $\eth$ to be a filter.

Theorem 3.12. *If $\eth$ is a non-constant fuzzy filter of $\mathbb{X}$, then its positive set is a filter of $\mathbb{X}$.*

Proof. Let $\eth$ be a non-constant fuzzy filter of $\mathbb{X}$. Let $x, y \in X$. If $x \in X_0$ and $x \lesssim y$, then $\eth(y) \geq \eth(x) \neq 0$ by (15), and so $y \in X_0$. If $x \in X_0$ and $g_x(y) \in X_0$, then $\eth(y) \geq \min\{\eth(x), \eth(g_x(y))\} \neq 0$ by (16). Hence $y \in X_0$, and therefore X_0 is a filter of $\mathbb{X}$. □

In the following example, we can see that the converse of Theorem 3.12 is not true in general.

Example 3.13. Consider the pre-ordered residuated system $\mathbb{X} := (X, \odot, \rightarrow, b_1, \lesssim)$ in Example 3.3. Let $\eth$ be a fuzzy set in X given by

$$\eth : X \to [0,1],\ x \mapsto \begin{cases} \frac{0.2}{k} & \text{if } x = b_1, \\ \frac{0}{k} & \text{if } x = b_2, \\ \frac{0.8}{k} & \text{if } x = b_3, \\ \frac{0.6}{k} & \text{if } x = b_4, \end{cases}$$

where k is a natural number. Then $X_0 = \{b_1, b_3, b_4\}$ is filter of $\mathbb{X}$. If we take $t := \frac{0.5}{k}$, then $(\eth, t)_\in = \{b_3, b_4\}$. We can observe that $b_4 \lesssim b_1$ and $b_1 \notin (\eth, t)_\in$. Hence $\eth$ is not a fuzzy filter of $\mathbb{X}$.

Theorem 3.14. *If a non-constant fuzzy set $\eth$ in X satisfies the following conditions:*

$$(\forall x, y \in X)(\forall t \in (0,1])(x \in (\eth, t)_\in,\ x \lesssim y\ \Rightarrow\ y \in (\eth, t)_q), \tag{26}$$

$$(\forall x, y \in X)(\forall t \in (0,1])(x \in (\eth, t)_\in,\ g_x(y) \in (\eth, t)_\in\ \Rightarrow\ y \in (\eth, t)_q), \tag{27}$$

then the positive set of $\eth$ is a filter of $\mathbb{X}$.

Proof. Let $x, y \in X$ be such that $x \in X_0$ and $x \lesssim y$. Since $x \in (\eth, \eth(x))_\in$, it follows from (26) that $y \in (\eth, \eth(x))_q$. If $y \notin X_0$, then $\eth(y) = 0$ and so $\langle y_{\eth(x)} \rangle\, \overline{q}\, \eth$, i.e., $y \notin (\eth, \eth(x))_q$. This is a contradiction, and thus $y \in X_0$. Let $x \in X_0$ and $g_x(y) \in X_0$. If we take $t := \min\{\eth(x), \eth(g_x(y))\}$, then $x \in (\eth, t)_\in$ and $g_x(y) \in (\eth, t)_\in$. Using (27) leads to $y \in (\eth, t)_q$. Hence $\eth(y) + t > 1$, and so $\eth(y) \neq 0$, i.e., $y \in X_0$. Therefore X_0 is a filter of $\mathbb{X}$. □

Theorem 3.15. *If a non-constant fuzzy set $\eth$ in X satisfies the following conditions:*

$$(\forall x, y \in X)(\forall t \in (0,1])(x \in (\eth, t)_q,\ x \lesssim y\ \Rightarrow\ y \in (\eth, t)_\in), \tag{28}$$

$$(\forall x, y \in X)(\forall t \in (0,1])(x \in (\eth, t)_q,\ g_x(y) \in (\eth, t)_q\ \Rightarrow\ y \in (\eth, t)_\in), \tag{29}$$

then the positive set of $\eth$ is a filter of $\mathbb{X}$.

Proof. Let $x, y \in X$ be such that $x \in X_0$ and $x \lesssim y$. Then $\eth(x) \neq 0$, and so $\eth(x) + 1 > 1$, i.e., $x \in (\eth, 1)_q$. Thus $y \in (\eth, 1)_\in$ by (28), which shows that $y \in X_0$. Let $x \in X_0$ and $g_x(y) \in X_0$. Then $\eth(x) \neq 0 \neq \eth(g_x(y))$, and hence $\eth(x) + 1 > 1$ and $\eth(g_x(y)) + 1 > 1$, that is, $x \in (\eth, 1)_q$ and $g_x(y) \in (\eth, 1)_q$. It follows from (29) that $y \in (\eth, 1)_\in$. Thus $y \in X_0$ and therefore X_0 is a filter of $\mathbb{X}$. □

Theorem 3.16. *If a non-constant fuzzy set $\eth$ in X satisfies the following conditions:*

$$(\forall x, y \in X)(\forall t \in (0,1])(x \in (\eth, t)_q,\ x \lesssim y\ \Rightarrow\ y \in (\eth, t)_q), \tag{30}$$

$$(\forall x, y \in X)(\forall t \in (0,1])(x \in (\eth, t)_q,\ g_x(y) \in (\eth, t)_q\ \Rightarrow\ y \in (\eth, t)_q), \tag{31}$$

then the positive set of $\eth$ is a filter of $\mathbb{X}$.

Proof. Let $x, y \in X$ be such that $x \in X_0$ and $x \lesssim y$. Then $ð(x) \neq 0$, and so $ð(x) + 1 > 1$, i.e., $x \in (ð, 1)_q$. Thus $y \in (ð, 1)_q$ by (30), which implies $ð(y) + 1 > 1$. Hence $ð(y) \neq 0$, and so $y \in X_0$. Let $x \in X_0$ and $g_x(y) \in X_0$. Then $ð(x) \neq 0 \neq ð(g_x(y))$, and hence $ð(x) + 1 > 1$ and $ð(g_x(y)) + 1 > 1$, that is, $x \in (ð, 1)_q$ and $g_x(y) \in (ð, 1)_q$. Using (31) induces $y \in (ð, 1)_q$. If $y \notin X_0$, then $ð(y) = 0$ and thus $y \notin (ð, 1)_q$ which is a contradiction. Hence $y \in X_0$ and therefore X_0 is a filter of $\mathbb{X}$. □

We establish conditions for the q_t-set $(ð, t)_q$ to be a filter of $\mathbb{X}$.

Theorem 3.17. *If ð is a fuzzy filter of $\mathbb{X}$, then its nonempty q_t-set is a filter of $\mathbb{X}$ for all $t \in (0, 1]$.*

Proof. Assume that $(ð, t)_q \neq \emptyset$ for all $t \in (0, 1]$. Let $x, y \in X$ be such that $x \in (ð, t)_q$ and $x \lesssim y$. Then $ð(x) + t > 1$ and so $ð(y) \geq ð(x) > 1 - t$ by (15). Hence $y \in (ð, t)_q$. Let $x \in (ð, t)_q$ and $g_x(y) \in (ð, t)_q$. Then $ð(x) + t > 1$ and $ð(g_x(y)) + t > 1$. It follows from (16) that $ð(y) \geq \min\{ð(x), ð(g_x(y))\} > 1 - t$. Thus $y \in (ð, t)_q$, and therefore $(ð, t)_q$ is a filter of $\mathbb{X}$ for all $t \in (0, 1]$. □

Proposition 3.18. *Given a fuzzy set ð in X, if its q_t-set is a filter of $\mathbb{X}$ for all $t \leq 0.5$, then the following conditions are established.*

$$(\forall x, y \in X)(\forall t \in (0, 0.5])(x \in (ð, t)_q,\ x \lesssim y \ \Rightarrow\ y \in (ð, t)_\in), \tag{32}$$

$$(\forall x, y \in X)(\forall t_1, t_2 \in (0, 0.5]) \left(\begin{array}{c} x \in (ð, t_1)_q,\ g_x(y) \in (ð, t_2)_q \\ \Rightarrow\ y \in (ð, \max\{t_1, t_2\})_\in \end{array} \right). \tag{33}$$

Proof. Let $t \in (0, 0.5]$ and suppose that $(ð, t)_q$ is a filter of $\mathbb{X}$. Let $x, y \in X$ be such that $x \in (ð, t)_q$ and $x \lesssim y$. Then $y \in (ð, t)_q$ by (13), and so $ð(y) > 1 - t \geq t$. Hence $y \in (ð, t)_\in$. Let $x, y \in X$ and $t_1, t_2 \in (0, 0.5]$ be such that $x \in (ð, t_1)_q$ and $g_x(y) \in (ð, t_2)_q$. Then $x \in (ð, \max\{t_1, t_2\})_q$ and $g_x(y) \in (ð, \max\{t_1, t_2\})_q$. It follows from (14) that $y \in (ð, \max\{t_1, t_2\})_q$. Hence

$$ð(y) > 1 - \max\{t_1, t_2\} \geq \max\{t_1, t_2\},$$

and so $y \in (ð, \max\{t_1, t_2\})_\in$. □

Proposition 3.19. *Given a fuzzy set ð in X, if its q_t-set is a filter of $\mathbb{X}$ for all $t \geq 0.5$, then the following conditions are established.*

$$(\forall x, y \in X)(\forall t \in (0.5, 1])(x \in (ð, t)_\in,\ x \lesssim y \ \Rightarrow\ y \in (ð, t)_q), \tag{34}$$

$$(\forall x, y \in X)(\forall t_1, t_2 \in (0.5, 1]) \left(\begin{array}{c} x \in (ð, t_1)_\in,\ g_x(y) \in (ð, t_2)_\in \\ \Rightarrow\ y \in (ð, \max\{t_1, t_2\})_q \end{array} \right). \tag{35}$$

Proof. Let $t \in (0.5, 1]$ and suppose that $(ð, t)_q$ is a filter of $\mathbb{X}$. Let $x, y \in X$. If $x \in (ð, t)_\in$ and $x \lesssim y$, then $ð(x) \geq t > 1 - t$, i.e., $x \in (ð, t)_q$. Hence $y \in (ð, t)_q$ by (13). If $x \in (ð, t_1)_\in$ and $g_x(y) \in (ð, t_2)_\in$, then $ð(x) \geq t_1 > 1 - t_1 \geq 1 - \max\{t_1, t_2\}$ and $ð(g_x(y)) \geq t_2 > 1 - t_2 \geq 1 - \max\{t_1, t_2\}$, that is, $x \in (ð, \max\{t_1, t_2\})_q$ and $g_x(y) \in (ð, \max\{t_1, t_2\})_q$. It follows from (14) that $y \in (ð, \max\{t_1, t_2\})_q$. □

Corollary 3.20. *Every fuzzy filter ð of $\mathbb{X}$ satisfies (32), (33), (34) and (35).*

Theorem 3.21. *If a fuzzy set ð in X satisfies the following conditions:*

$$(\forall x, y \in X)(\forall t \in (0.5, 1])(x \in (ð, t)_q,\ x \lesssim y \ \Rightarrow\ y \in (ð, t)_{\in \vee q}), \tag{36}$$

$$(\forall x, y \in X)(\forall t \in (0.5, 1])(x \in (ð, t)_q,\ g_x(y) \in (ð, t)_q \ \Rightarrow\ y \in (ð, t)_{\in \vee q}), \tag{37}$$

then the nonempty q_t-set is a filter of $\mathbb{X}$ for all $t \in (0.5, 1]$.

Proof. Let $x, y \in X$ and $t \in (0.5, 1]$, and assume that $(\eth, t)_q$ is nonempty. If $x \in (\eth, t)_q$ and $x \lesssim y$, then $y \in (\eth, t)_{\in\vee q}$ by (36). It follows that $y \in (\eth, t)_\in$ or $y \in (\eth, t)_q$. If $y \in (\eth, t)_\in$, then $\eth(y) \geq t > 1 - t$ and so $y \in (\eth, t)_q$. Let $x \in (\eth, t)_q$ and $g_x(y) \in (\eth, t)_q$. Then $y \in (\eth, t)_{\in\vee q}$ by (37), and thus $y \in (\eth, t)_\in$ or $y \in (\eth, t)_q$. If $y \in (\eth, t)_\in$, then $\eth(y) \geq t > 1 - t$ and so $y \in (\eth, t)_q$. Consequently, $(\eth, t)_q$ is a filter of $\mathbb{X}$. □

Proposition 3.22. *Given a filter F of $\mathbb{X}$, if we define a fuzzy set ∂_F in X as follows:*

$$\partial_F : X \to [0,1],\ x \mapsto \begin{cases} s_1 & \text{if } x \in F, \\ s_2 & \text{otherwise} \end{cases}$$

where $s_1 \geq 0.5 > s_2 = 0$, then the following assertions hold.

$$(\forall x, y \in X)(\forall t \in (0,1])(x \in (\partial_F, t)_q,\ x \lesssim y \ \Rightarrow\ y \in (\partial_F, t)_{\in\vee q}). \tag{38}$$

$$(\forall x, y \in X)(\forall t_1, t_2 \in (0,1]) \left(\begin{array}{c} x \in (\partial_F, t_1)_q,\ g_x(y) \in (\partial_F, t_2)_q \\ \Rightarrow\ y \in (\partial_F, \min\{t_1, t_2\})_{\in\vee q}. \end{array} \right). \tag{39}$$

Proof. Let $x, y \in X$ and $t \in (0, 1]$ be such that $x \in (\partial_F, t)_q$ and $x \lesssim y$. Then $\partial_F(x) + t > 1$. If $x \notin F$, then $\partial_F(x) = s_2 = 0$ and so $t > 1$ a contradiction. Thus $x \in F$ and hence $y \in F$ since F is a filter of $\mathbb{X}$. Hence $\partial_F(y) = s_1 \geq 0.5$. If $t \leq 0.5$, then $\partial_F(y) \geq 0.5 \geq t$ and so $y \in (\partial_F, t)_\in$. If $t > 0.5$, then $\partial_F(y) + t > 0.5 + 0,5 = 1$ which means $y \in (\partial_F, t)_q$. Thus $y \in (\partial_F, t)_{\in\vee q}$. Let $x, y \in X$ and $t_1, t_2 \in (0, 1]$ be such that $x \in (\partial_F, t_1)_q$ and $g_x(y) \in (\partial_F, t_2)_q$, that is, $\partial_F(x) + t_1 > 1$ and $\partial_F(g_x(y)) + t_2 > 1$. If $x \notin F$ or $g_x(y) \notin F$, then $\partial_F(x) = s_2 = 0$ or $\partial_F(g_x(y)) = s_2 = 0$. Hence $t_1 > 1$ or $t_2 > 1$ which is a contradiction. Hence $x \in F$ and $g_x(y) \in F$. Since F is a filter of $\mathbb{X}$, we have $y \in X$ and thus $\eth_F(y) = s_1 \geq 0.5$. If $t_1 \leq 0.5$ or $t_2 \leq 0.5$, then $\eth_F(y) \geq 0.5 \geq \min\{t_1, t_2\}$. Thus $y \in (\partial_F, \min\{t_1, t_2\})_\in$. If $t_1 > 0.5$ and $t_2 > 0.5$, then $\partial_F(y) + \min\{t_1, t_2\} > 0.5 + 0.5 = 1$, i.e., $y \in (\partial_F, \min\{t_1, t_2\})_q$. Therefore, $y \in (\partial_F, \min\{t_1, t_2\})_{\in\vee q}$. □

Theorem 3.23. *If ∂_F is the fuzzy set in X which is described in Proposition 3.22, then its q_t-set $(\partial_F, t)_q$ is a filter of $\mathbb{X}$ for all $t \in (0.5, 1]$*

Proof. Let $x, y \in X$ and $t \in (0.5, 1]$. If $x \in (\partial_F, t)_q$ and $x \lesssim y$, then $\partial_F(x) + t > 1$, and so $x \in F$ because if not, then $\partial_F(x) = s_2 = 0$ and thus $t > 1$ a contradiction. Since F is a filter of $\mathbb{X}$, we get $y \in F$. So $\partial_F(y) = s_1 \geq 0.5$. Since $t \in (0.5, 1]$, it follows that $\partial_F(y) + t = s_1 + t > 0.5 + 0.5 = 1$, i.e., $y \in (\partial_F, t)_q$. Suppose that $x \in (\partial_F, t)_q$ and $g_x(y) \in (\partial_F, t)_q$. Then $\partial_F(x) + t > 1$ and $\partial_F(g_x(y)) + t > 1$. If $x \notin F$ or $g_x(y) \notin F$, then $\partial_F(x) = s_2 = 0$ or $\partial_F(g_x(y)) = s_2 = 0$. Hence $t = \partial_F(x) + t > 1$ or $t = \partial_F(g_x(y)) + t > 1$, a contradiction. Thus $x \in F$ and $g_x(y) \in F$, which induces $y \in F$. So $\partial_F(y) = s_1 \geq 0.5$. Since $t \in (0.5, 1]$, it follows that $\partial_F(y) + t = s_1 + t > 0.5 + 0.5 = 1$, i.e., $y \in (\partial_F, t)_q$. Therefore $(\partial_F, t)_q$ is a filter of $\mathbb{X}$. □

Definition 3.24. A fuzzy set $\eth$ in X is called a $(0.5, 1]$-*fuzzy filter* of $\mathbb{X}$ if its nonempty $\in_t$-set $(\eth, t)_\in$ is a filter of $\mathbb{X}$ for all $t \in (0.5, 1]$.

Example 3.25. Consider the pre-ordered residuated system $\mathbb{X} := (X, \odot, \to, b_1, \lesssim)$ in Example 3.3. Let ∂ be a fuzzy set in X given by

$$\partial : X \to [0,1],\ x \mapsto \begin{cases} 0.9 & \text{if } x = b_1, \\ 0.4 & \text{if } x = b_2, \\ 0.7 & \text{if } x = b_3, \\ 0.3 & \text{if } x = b_4. \end{cases}$$

Then ∂ is a $(0.5, 1]$-fuzzy filter of $\mathbb{X}$.

It is clear that every fuzzy filter is a $(0.5, 1]$-fuzzy filter. But the converse may not be true. In fact, the $(0.5, 1]$-fuzzy filter ∂ in Example 3.25 is not a fuzzy filter of $\mathbb{X}$ since $b_2 \lesssim b_4$ and $\partial(b_2) = 0.4 \not\leq 0.3 = \partial(b_4)$.

We now discuss the characterization of $(0.5, 1]$-fuzzy filters.

Theorem 3.26. *A fuzzy set $\eth$ in X is a $(0.5, 1]$-fuzzy filter of $\mathbb{X}$ if and only if it satisfies:*

$$(\forall x, y \in X)(x \lesssim y \;\Rightarrow\; \eth(x) \leq \max\{\eth(y), 0.5\}), \tag{40}$$

$$(\forall x, y \in X)(\max\{\eth(y), 0.5\} \geq \min\{\eth(x), \eth(g_x(y))\}). \tag{41}$$

Proof. Assume that $\eth$ in X is a $(0.5, 1]$-fuzzy filter of $\mathbb{X}$. Then the nonempty $\in_t$-set $(\eth, t)_\in$ is a filter of $\mathbb{X}$ for all $t \in (0.5, 1]$. If the condition (40) is not valid, then there exists $\mathfrak{a}, \mathfrak{b} \in X$ such that $\mathfrak{a} \lesssim \mathfrak{b}$ and $\eth(\mathfrak{a}) > \max\{\eth(\mathfrak{b}), 0.5\}$. Hence $t := \eth(\mathfrak{a}) \in (0.5, 1]$ and $\mathfrak{a} \in (\eth, t)_\in$. But $\mathfrak{b} \notin (\eth, t)_\in$, a contradiction. Hence $\eth(x) \leq \max\{\eth(y), 0.5\}$ for all $x, y \in X$ with $x \lesssim y$. Suppose that the condition (41) is not establish. Then $\max\{\eth(\mathfrak{b}), 0.5\} < \min\{\eth(\mathfrak{a}), \eth(g_{\mathfrak{a}}(\mathfrak{b}))\}$ for some $\mathfrak{a}, \mathfrak{b} \in X$. If we take $s := \min\{\eth(\mathfrak{a}), \eth(g_{\mathfrak{a}}(\mathfrak{b}))\}$, then $s \in (0.5, 1]$, $\mathfrak{a} \in (\eth, s)_\in$ and $g_{\mathfrak{a}}(\mathfrak{b}) \in (\eth, s)_\in$. But $\max\{\eth(\mathfrak{b}), 0.5\} < s$ leads to $\mathfrak{b} \notin (\eth, s)_\in$, which is a contradiction. Therefore $\max\{\eth(y), 0.5\} \geq \min\{\eth(x), \eth(g_x(y))\}$ for all $x, y \in X$.

Conversely, let $\eth$ be a fuzzy set in X that satisfies two conditions (40) and (41). Let $t \in (0.5, 1]$ be such that $(\eth, t)_\in \neq \emptyset$. If $x \in (\eth, t)_\in$ and $x \lesssim y$, then $\max\{\eth(y), 0.5\} \geq \eth(x) \geq t > 0.5$ by (40). Thus $\eth(y) \geq t$, i.e., $y \in (\eth, t)_\in$. If $x \in (\eth, t)_\in$ and $g_x(y) \in (\eth, t)_\in$, then $\eth(x) \geq t$ and $\eth(g_x(y)) \geq t$. It follows from (41) that $\max\{\eth(y), 0.5\} \geq \min\{\eth(x), \eth(g_x(y))\} \geq t$. Since $t > 0.5$, we get $\eth(y) \geq t$ and so $y \in (\eth, t)_\in$. Therefore $\eth$ is a $(0.5, 1]$-fuzzy filter of $\mathbb{X}$. □

Corollary 3.27. *Every fuzzy filter $\eth$ of $\mathbb{X}$ satisfies the two conditions (40) and (41).*

Theorem 3.28. *If $\eth$ is a $(0.5, 1]$-fuzzy filter of $\mathbb{X}$, then its nonempty q_t-set is a filter of $\mathbb{X}$ for all $t \in (0, 0.5)$.*

Proof. Assume that $\eth$ is a $(0.5, 1]$-fuzzy filter of $\mathbb{X}$. Let $t \in (0, 0.5)$ be such that $(\eth, t)_q \neq \emptyset$. If $x \in (\eth, t)_q$ and $x \lesssim y$, then

$$\max\{\eth(y), 0.5\} \geq \eth(x) > 1 - t > 0.5$$

by (40), and so $\eth(y) > 1 - t$. Hence $y \in (\eth, t)_q$. Let $x, y \in X$ be such that $x \in (\eth, t)_q$ and $g_x(y) \in (\eth, t)_q$. Then $\eth(x) > 1 - t$ and $\eth(g_x(y)) > 1 - t$. It follows from (41) that

$$\max\{\eth(y), 0.5\} \geq \min\{\eth(x), \eth(g_x(y))\} > 1 - t > 0.5.$$

Hence $\eth(y) > 1 - t$ and thus $y \in (\eth, t)_q$. Therefore $(\eth, t)_q$ is a filter of $\mathbb{X}$. □

Now let's think about a more generalized form of Definition 3.1 and Definition 3.24.

Let $\eth$ be a fuzzy set in X. Then the $\in_t$-set $(\eth, t)_\in$ is a filter of $\mathbb{X}$ for some $t \in (0, 1]$, but can not be a filter of $\mathbb{X}$ for other $t \in (0, 1]$. Let

$$J_X := \{t \in (0, 1] \mid (\eth, t)_\in \text{ is a filter of } \mathbb{X}\}.$$

If $J_X = (0, 1]$, then $\eth$ is a fuzzy filter of $\mathbb{X}$. If $J_X = (0.5, 1]$, then $\eth$ is a $(0.5, 1]$-fuzzy filter of $\mathbb{X}$. However, in general, the question arises as to what the form of the fuzzy filter is if J_X is a non-empty subset of $(0, 1]$, for example $J_X = (0, 0.5]$ or $J_X = (\delta, \varepsilon]$ for $\delta, \varepsilon \in (0, 1]$ with $\delta < \varepsilon$. Based on this question, we consider the following definition.

Definition 3.29. Let $\delta < \varepsilon$ in $[0, 1]$. A fuzzy set $\eth$ in X is called a *fuzzy filter with thresholds* δ and ε (briefly, $(\delta, \varepsilon]$-*fuzzy filter*) of $\mathbb{X}$ if its nonempty $\in_t$-set $(\eth, t)_\in$ is a filter of $\mathbb{X}$ for all $t \in (\delta, \varepsilon]$.

It is clear that if a fuzzy set $\eth$ in X satisfies $\eth(x) \leq \delta < \varepsilon$ for all $x \in X$, then $\eth$ is a $(\delta, \varepsilon]$-fuzzy filter of $\mathbb{X}$, and every fuzzy filter is a $(\delta, \varepsilon]$-fuzzy filter for every $\delta, \varepsilon \in (0, 1]$ with $\delta < \varepsilon$.

Example 3.30. Consider the pre-ordered residuated system $\mathbb{X} := (X, \odot, \rightarrow, b_1, \lesssim)$ in Example 3.3. Let $\eth$ be a fuzzy set in X given by

$$\eth : X \to [0,1], \ x \mapsto \begin{cases} 0.6 & \text{if } x = b_1, \\ 0.3 & \text{if } x = b_2, \\ 0.8 & \text{if } x = b_3, \\ 0.5 & \text{if } x = b_4. \end{cases}$$

Then $\eth$ is a $(0.27, 0.58]$-fuzzy filter of $\mathbb{X}$ since

$$(\eth, t)_\in = \begin{cases} \{b_1, b_3\} & \text{if } 0.5 < t \leq 0.58, \\ \{b_1, b_3, b_4\} & \text{if } 0.3 < t \leq 0.5, \\ X & \text{if } 0.27 < t \leq 0.3, \end{cases}$$

is a filter of $\mathbb{X}$ for all $t \in (0.27, 0.58]$. But it is not a $(0.27, 0.65]$-fuzzy filter of $\mathbb{X}$ because if we take $t := 0.63 \in (0.27, 0.65]$, then $(\eth, t)_\in = \{b_3\}$ is not a filter of $\mathbb{X}$.

It is obvious that if $(\delta, \varepsilon_1] \subseteq (\delta, \varepsilon_2]$, then every $(\delta, \varepsilon_2]$-fuzzy filter is a $(\delta, \varepsilon_1]$-fuzzy filter, but the converse may not be true as shown in Example 3.30.

Theorem 3.31. *A fuzzy set $\eth$ in X is a $(\delta, \varepsilon]$-fuzzy filter of $\mathbb{X}$ if and only if the following conditions hold.*

$$(\forall x, y \in X)(x \lesssim y \ \Rightarrow \ \min\{\eth(x), \varepsilon\} \leq \max\{\eth(y), \delta\}), \tag{42}$$

$$(\forall x, y \in X)(\max\{\eth(y), \delta\} \geq \min\{\eth(x), \eth(g_x(y)), \varepsilon\}). \tag{43}$$

Proof. Suppose that $\eth$ is a $(\delta, \varepsilon]$-fuzzy filter of $\mathbb{X}$. Let $x \lesssim y$ in $\mathbb{X}$. If $\min\{\eth(x), \varepsilon\} > \max\{\eth(y), \delta\}$, then there exists $t \in (0, 1]$ such that

$$\min\{\eth(x), \varepsilon\} \geq t > \max\{\eth(y), \delta\}.$$

Then $\eth(y) < t$ and $\eth(x) \geq t$, that is, $y \notin (\eth, t)_\in$ and $x \in (\eth, t)_\in$, and $t \in (\delta, \varepsilon]$. This is a contradiction, and so $\min\{\eth(x), \varepsilon\} \leq \max\{\eth(y), \delta\}$. If (43) is not established, then

$$\max\{\eth(\mathfrak{b}), \delta\} < t \leq \min\{\eth(\mathfrak{a}), \eth(g_{\mathfrak{a}}(\mathfrak{b})), \varepsilon\}$$

for some $\mathfrak{a}, \mathfrak{b} \in X$ and $t \in (0, 1]$. It follows that $t \in (\delta, \varepsilon]$, $\mathfrak{b} \notin (\eth, t)_\in$, $\mathfrak{a} \in (\eth, t)_\in$ and $g_{\mathfrak{a}}(\mathfrak{b}) \in (\eth, t)_\in$. This is a contradiction, and thus $\eth$ satisfies the condition (43).

Conversely, we assume that $\eth$ satisfies the two conditions (42) and (43). Let $x, y \in X$ and $t \in (\delta, \varepsilon]$. If $x \lesssim y$ and $x \in (\eth, t)_\in$, then

$$\max\{\eth(y), \delta\} \geq \min\{\eth(x), \varepsilon\} \geq t > \delta$$

by (42). Hence $\eth(y) \geq t$, i.e., $y \in (\eth, t)_\in$. If $x \in (\eth, t)_\in$ and $g_x(y) \in (\eth, t)_\in$, then

$$\max\{\eth(y), \delta\} \geq \min\{\eth(x), \eth(g_x(y)), \varepsilon\} \geq t > \delta$$

by (43), and so $\eth(y) \geq t$, i.e., $y \in (\eth, t)_\in$. Consequently, $(\eth, t)_\in$ is a filter of $\mathbb{X}$ for all $t \in (\delta, \varepsilon]$. Therefore $\eth$ is a $(\delta, \varepsilon]$-fuzzy filter of $\mathbb{X}$. □

Given a fuzzy set $\eth$ in X, we say the set

$$(\eth, t)^*_q := \{x \in X \mid \eth(x) + t \geq 1\}$$

is an *extended q_t-set* of $\eth$.

Theorem 3.32. *If* $\eth$ *is a* $(\delta, \varepsilon]$*-fuzzy filter of* $\mathbb{X}$ *and* $\delta < 0.5$*, then its nonempty extended* q_t*-set is a filter of* $\mathbb{X}$ *for all* $t \in (0, \delta] \cap [1 - \varepsilon, 1]$.

Proof. Assume that $\eth$ is a $(\delta, \varepsilon]$-fuzzy filter of $\mathbb{X}$ and $\delta < 0.5$. Let $t \in (0, \delta] \cap [1-\varepsilon, 1]$ be such that $(\eth, t)^*_q \neq \emptyset$. If $x \in (\eth, t)^*_q$ and $x \lesssim y$, then

$$\max\{\eth(y), \delta\} \geq \min\{\eth(x), \varepsilon\} \geq \min\{1 - t, \varepsilon\} = 1 - t \geq 1 - \delta > \delta$$

by (42). Hence $\eth(y) \geq 1 - t$, that is, $y \in (\eth, t)^*_q$. Let $x, y \in X$ be such that $x \in (\eth, t)^*_q$ and $g_x(y) \in (\eth, t)^*_q$. Then $\eth(x) + t \geq 1$ and $\eth(g_x(y)) + t \geq 1$. It follows from (43) that

$$\max\{\eth(y), \delta\} \geq \min\{\eth(x), \eth(g_x(y)), \varepsilon\} \geq \min\{1 - t, \varepsilon\} = 1 - t \geq 1 - \delta > \delta.$$

Thus $\eth(y) \geq 1 - t$, that is, $y \in (\eth, t)^*_q$. Consequently, $(\eth, t)^*_q$ is a filter of $\mathbb{X}$. □

4 Conclusion

As a mathematical structure, a residuated relational system has been introduced by S. Bonzio and I. Chajda in 2018, and it combines elements of algebra, order theory, and relational calculus. They also extended the residuated relational system by introducing pre-ordered residuated systems using pre-order relation, and further studied the various properties involved. D. A. Romano [2, 3, 4, 5] introduced and analyzed the concept of (weak implicative, shipt, implicative, comparative) filters in pre-ordered residuated systems. With the purpose of this paper in the study of filters in pre-ordered residuated systems using the concept of fuzzy points, we introduced fuzzy filter and identified various properties. Based on the ideas of this paper and the results obtained, we will study various fuzzy versions for different types of filters, for example, (weak implicative, shipt, implicative, comparative) filters, in the future.

Acknowledgements: The author appreciates the valuable suggestions of anonymous reviewers.

Conflict of Interest: The author declares no conflict of interest.

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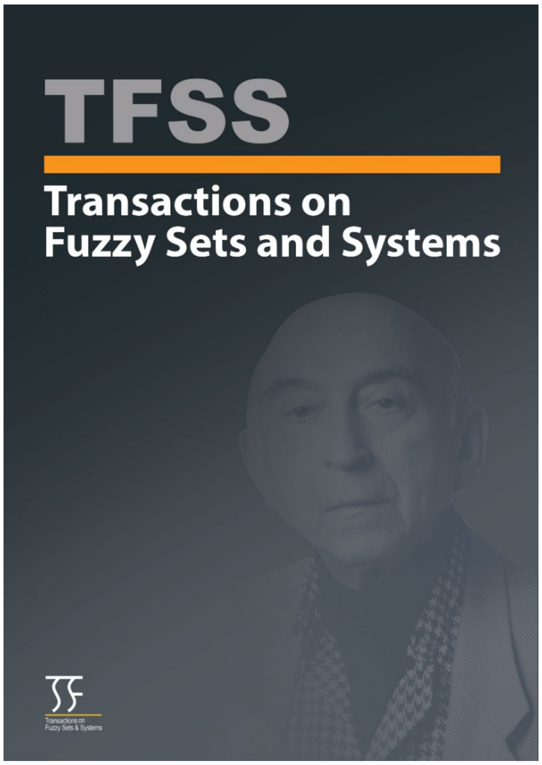

Transactions on Fuzzy Sets and Systems

ISSN: **2821-0131**

https://sanad.iau.ir/journal/tfss/

A Novel Method of Decision-Making Based on Intuitionistic Fuzzy Set Theory

Vol.4, No.1, (2025), 42-53. DOI: https://doi.org/10.71602/tfss.2025.1183369

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Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://sanad.iau.ir/journal/tfss/
Online ISSN: 2821-0131
Vol.4, No.1, (2025), 42-53
DOI: https://doi.org/10.71602/tfss.2025.1183369

Article Type: Original Research Article

A Novel Method of Decision-Making Based on Intuitionistic Fuzzy Set Theory

Jaydip Bhattacharya*

(This paper is dedicated to Professor "John N. Mordeson" on the occasion of his 91st birthday.)

Abstract. Atanassov's intuitionistic fuzzy set is more adept at representing and managing uncertainty. Within intuitionistic fuzzy set theory, intuitionistic fuzzy measure is a significant field of study. In order to address decision making, we present a novel similarity metric between intuitionistic fuzzy sets in this study. First, based on the minimum and maximum levels of similarity, we suggest a new similarity metric between intuitionistic fuzzy values. It is capable of overcoming the limitations of current approaches to gauging the degree of resemblance between fuzzy intuitionistic sets. It is also possible to show some aspects of the suggested similarity measure between intuitionistic fuzzy sets by taking into account the modal operators and their different extensions. Finally, we apply the proposed similarity measure between intuitionistic fuzzy sets to deal with a real life problem. The suggested action can provide a precise outcome. The application section examines a real-world issue of choosing the best course of action among n options based on m criteria. A fictitious case study is created along with the method's algorithm.

AMS Subject Classification 2020: 90C70; 03F55
Keywords and Phrases: Intuitionistic fuzzy sets, Modal operators, Measure of similarity, Decision making, Optimal solution.

1 Introduction

In 1965, L.A. Zadeh [1] created and introduced the idea of a fuzzy set. Eighteen years later, in 1983, Atanassov [2] introduced the concept of intuitionistic fuzzy sets as an extension of fuzzy sets. The fundamental distinction between these two ideas is that, in intuitionistic fuzzy set theory, hesitation margin is taken into account in addition to both membership function and non-membership function. In fuzzy set theory, only the membership function is taken into account. Scholars and researchers [3, 4, 5, 6, 7, 8] are exerting great effort to advance and refine this field.
The notion of modal operators were first introduced by Atanassov [9] in 1986. Modal operators ($\Box, \Diamond$) defined over the set of all intuitionistic fuzzy sets that convert every intuitionistic fuzzy set into a fuzzy set. Atanassov [9] also introduced the operators ($\boxplus, \boxtimes$) in intuitionistic fuzzy set. More relations and properties on these operators are regorously studied in [10, 11, 12, 3, 4, 5]. The second extension of the operators $\boxplus$ and $\boxtimes$ are introduced by K. Dencheva [13].
There are circumstances in which fuzzy set theory is not the best fit and should be replaced with intuitionistic fuzzy set theory. intuitionistic fuzzy set theory has been researched as a helpful resource for decision-making

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Received: 4 August 2024; **Revised**: 1 September 2024; **Accepted**: 4 September 2024; **Available Online**: 18 September 2024; **Published Online**: 7 May 2025.

How to cite: Bhattacharya j. A novel method of decision-making based on intuitionistic fuzzy set theory. *Transactions on Fuzzy Sets and Systems.* 2025; 4(1): 42-53. DOI: https://doi.org/10.71602/tfss.2025.1183369

issues, logic programming, etc. In this work, we establish a similarity measure between two intuitionistic fuzzy sets A and B of a set E and apply it to a problem involving decision-making. The issue under consideration is choosing the best course of action from n options based on m criteria in cases when the information at hand is intuitionistic fuzzy.

Recently, there has been a lot of focus on measures of similarity between Intuitionistic Fuzzy Sets as a crucial tool for image processing, machine learning, pattern detection, and decision making [14, 15]. Numerous measurements of similarity have been put forth. Some of them are derived from the widely used distance measures.

The first study was carried out by Szmidt and Kacprzyk [16] extending the well-known distances measures, such as the Hamming distance and the Euclidian distance, to IFS environment and comparing them with the approaches used for ordinary fuzzy sets. Therefore, several new distance measures were proposed and applied to pattern recognition. Grzegorzewski [17] also extended the Hamming distance, the Euclidean distance, and their normalized counterparts to IFS environment. Hung and Yang [18] extended the Hausdorff distance to Intuitionistic Fuzzy Sets and proposed three similarity measures.

On the other hand, instead of extending the well-known measures, some studies defined new similarity measures for Intuitionistic Fuzzy Sets. Dengfeng and Chuntian [19] suggested a new similarity measure for IFSs based on the membership degree and the nonmembership degree. Ye [15] conducted a similar comparative study of the existing similarity measures between Intuitionistic Fuzzy Sets and proposed a cosine similarity measure and a weighted cosine similarity measure. Xu and Chen [20] introduced a series of distance and similarity measures, which are various combinations and generalizations of the weighted Hamming distance, the weighted Euclidean distance, and the weighted Hausdorff distance. Xu and Yager [21] developed a similarity measure between Intuitionistic Fuzzy Sets and applied the developed similarity measure for consensus analysis in group decision making based on intuitionistic fuzzy preference relations.

Zeng and Guo [22] investigated the relationship among the normalized distance, the similarity measure, the inclusion measure, and the entropy of interval-valued fuzzy sets. It was also showed that the similarity measure, the inclusion measure, and the entropy of interval-valued fuzzy sets could be induced by the normalized distance of interval-valued fuzzy sets based on their axiomatic definitions. Moreover, Zhang and Yu [23] presented a new distance (or similarity) measure based on interval comparison, where the Intuitionistic Fuzzy Sets were, respectively, transformed into the symmetric triangular fuzzy numbers. Comparison with the widely used methods indicated that the proposed method contained more information, with much less loss of information. Li et al. [24] introduced an axiomatic definition of the similarity measure of Intuitionistic Fuzzy Sets. The relationship between the entropy and the similarity measure of IFS was investigated in detail. It was proved that the similarity measure and the entropy of IFS can be transformed into each other based on their axiomatic definitions.

Several writers have recently discussed the use of various similarity measures in image processing, pattern recognition, medical diagnosis, and decision making. Song et al.[25] presented some applications to pattern recognition and presented a new similarity metric for intuitionistic fuzzy sets. Ejegwa et al.[26] represented Thao et al.'s correlation coefficient of Intuitionistic fuzzy sets for medical diagnostic analysis on some selected patients. Based on Spearman's correlation coefficient, Ejegwa et al. [27] identified medical emergencies in 2024 using novel intuitionistic fuzzy correlation measurements. Recently tendency coefficient based on weighted distance measure for intuitionistic fuzzy sets was discussed by Anum et al. [28]. Additionally, Ejegwa et al. [29] presented a novel approach to calculating the distance between intuitionistic fuzzy sets and discussed about how to use it in the admissions process. Zhou et al.[30] provided a detailed discussion of the generalised similarity operator for intuitionistic fuzzy sets and how to apply it using the multiple criteria decision making technique and the recognition principle. In a paper pertaining to the intuitionistic fuzzy sets approach, Nwokoro et al.[31] also made predictions regarding maternal outcomes.

Therefore, we propose a novel method for decision-making based on intuitionistic fuzzy set theory. The proposed similarity measure depends on membership degree, and hesitation margin. This paper proves that the proposed measures satisfy the properties of the axiomatic definition for similarity measures. In addition, several numerical examples are provided to establish some relations. The final section presents the suggested similarity measure's use for decision-making.

2 Preliminary Concepts

Throughout this study, intuitionistic fuzzy set and fuzzy set are denoted by IFS and FS respectively.

Definition 2.1. [9] *Let X be a nonempty set. An intuitionistic fuzzy set A in X is an object having the form $A=\{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$,where the functions $\mu_A, \nu_A : x \to [0,1]$define respectively, the degree of membership and degree of non-membership of the element $x \in X$ to the set A, which is a subset of X, and for every element $x \in X$, $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.*
Furthermore, we have $\pi_A(x)= 1\text{-}\mu_A(x)\text{-}\nu_A(x)$ called the intuitionistic fuzzy set index or hesitation margin of x in A. $\pi_A(x)$ is the degree of indeterminacy of $x \in X$ to the IFS A and $\pi_A(x) \in [0,1]$ that is $\pi_A : x \to [0,1]$ and $0 \leq \pi_A(x) \leq 1$ for every $x \in X$.
$\pi_A(x)$ expresses the lack of knowledge of whether x belongs to IFS A or not.

Definition 2.2. [9] *Let X be a nonempty set. If A is an IFS drawn from X, then the modal operators which are also termed as necessity and possibility operators can be defined as*

1. $\Box A= \{\langle x, \mu_A(x), 1-\mu_A(x)\rangle : x \in X\}$

2. $\Diamond A= \{\langle x, 1-\nu_A(x), \nu_A(x)\rangle : x \in X\}$

For a proper IFS, $\Box A \subset A \subset \Diamond A$ and $\Box A \neq A \neq \Diamond A$.

Definition 2.3. [9] *Let X be a nonempty set. If A is an IFS drawn from X, then,*

1. $\boxplus A= \{\langle x, \frac{\mu_A(x)}{2}, \frac{\nu_A(x)+1}{2}\rangle : x \in X\}$

2. $\boxtimes A= \{\langle x, \frac{\mu_A(x)+1}{2}, \frac{\nu_A(x)}{2}\rangle : x \in X\}$

For a proper IFS, $\boxplus A \subset A \subset \boxtimes A$ and $\boxplus A \neq A \neq \boxtimes A$.

Definition 2.4. [32] *Let $\alpha \in [0,1]$ and let A be an IFS. Then the first extension of the operators $\boxplus$ and $\boxtimes$ can be defined as*

1. $\boxplus_\alpha A= \{\langle x, \alpha\mu_A(x), \alpha\nu_A(x) + 1 - \alpha\rangle : x \in X\}$

2. $\boxtimes_\alpha A= \{\langle x, \alpha\mu_A(x) + 1 - \alpha, \alpha\nu_A(x)\rangle : x \in X\}$.

Definition 2.5. [13] *Let $\alpha, \beta, \alpha+\beta \in [0,1]$ and let A be an IFS. Then the second extension of the operators $\boxplus$ and $\boxtimes$ can be defined as*

1. $\boxplus_{\alpha,\beta} A = \{\langle x, \alpha\mu_A(x), \alpha\nu_A(x)+\beta\rangle : x \in X\}$

2. $\boxtimes_{\alpha,\beta} A = \{\langle x, \alpha\mu_A(x)+\beta, \alpha\nu_A(x)\rangle : x \in X\}$.

Definition 2.6. [33] *Let us consider two IFSs A and B of a fixed set E. The similarity measure between A and B denoted by $s(A,B)$ is defind by an interval $[e_{AB}, e'_{AB}]$, where*

$$e_{AB} = \max \min_{x \in E}\{\mu_A(x), \mu_B(x)\}$$

$$e'_{AB} = \max \min_{x \in E}\{\mu_A(x)+\pi_A(x), \mu_B(x)+\pi_B(x)\}$$

Here e_{AB} indicates the minimum amount of similarity and e'_{AB} indicates the maximum amount of similarity between A and B.
It can be noted that

1. $s(A,B) \subseteq [0,1]$.

2. $s(A,B) = s(B,A)$.

3. *If $\pi_A(x) = 0$ and $\pi_B(x) = 0, \forall x \in E$, then $e_{AB} = e'_{AB}$.*

Moreover it may be mentioned that $e_{AB} \neq e'_{AB}$ for $A = B$.
.

Proposition 2.7. [33] *Let A and B be two IFSs and $s(A,B) = [e_{AB}, e'_{AB}]$, then*

1. $s(\Box A, \Box B) = e_{AB}$,
2. $s(\Diamond A, \Diamond B) = e'_{AB}$.

3 Measure of Similarity between Intuitionistic Fuzzy Sets

This section provides an example-based explanation of Definition 2.6, leading to some intriguing findings.

Example 3.1. Consider two IFSs A and B of $E = \{x_1, x_2, x_3, x_4\}$ given by the following table:

x	μ_A	ν_A	μ_B	ν_B
x_1	0.65	0.26	0.72	0.18
x_2	0.32	0.46	0.56	0.38
x_3	0.80	0.12	0.48	0.42
x_4	0.70	0.25	0.83	0.12

Using Definition 2.6, we have $e_{AB} = 0.70$, $e'_{AB} = 0.75$ and hence similarity measure between A and B is [0.70, 0.75].

Theorem 3.2. *Let A and B be two IFSs and $s(A, B) = [e_{AB}, e'_{AB}]$, then*

1. $s(\boxplus A, \boxplus B) = [\frac{1}{2}e_{AB},\ \frac{1}{2}e'_{AB}\ ]$,
2. $s(\boxtimes A, \boxtimes B) = [\frac{1}{2}e_{AB} + \frac{1}{2},\ \frac{1}{2}e'_{AB} + \frac{1}{2}]$.

Proof. 1. L.H.S $= \max\min_{x\in E}\{\frac{\mu_A(x)}{2}, \frac{\mu_B(x)}{2}\}, \max\min_{x\in E}\{\frac{\mu_A(x)}{2} + \frac{\pi_A(x)}{2}, \frac{\mu_B(x)}{2} + \frac{\pi_B(x)}{2}\}$
$= \max\min_{x\in E} \frac{1}{2}\{\mu_A(x), \mu_B(x)\}, \max\min_{x\in E} \frac{1}{2}\{\mu_A(x) + \pi_A(x), \mu_B(x) + \pi_B(x)\}$
$= \frac{1}{2}\max\min_{x\in E}\{\mu_A(x), \mu_B(x)\}, \frac{1}{2}\max\min_{x\in E}\{\mu_A(x) + \pi_A(x), \mu_B(x) + \pi_B(x)\}$
$= [e_{AB}, e'_{AB}]$
Similarly the other statement can be proved.

□

Theorem 3.3. *Let $\alpha \in [0, 1]$ and let A & B be two IFSs. If $s(A, B) = [e_{AB}, e'_{AB}]$, then*

1. $s(\boxplus_\alpha A, \boxplus_\alpha B) = [\alpha e_{AB},\ \alpha e'_{AB}]$,
2. $s(\boxtimes_\alpha A, \boxtimes_\alpha B) = [\alpha e_{AB} + 1 - \alpha,\ \alpha e'_{AB} + 1 - \alpha\ ]$.

Proof. 1. L.H.S $= \max\min_{x\in E}\{\alpha\mu_A(x), \alpha\mu_B(x)\}, \max\min_{x\in E}\{\alpha\mu_A(x) + \alpha\pi_A(x), \alpha\mu_B(x) + \alpha\pi_B(x)\}$
$= \alpha\max\min_{x\in E}\{\mu_A(x), \mu_B(x)\}, \alpha\max\min_{x\in E}\{\mu_A(x) + \pi_A(x), \mu_B(x) + \pi_B(x)\}$
$= [\alpha e_{AB}, \alpha e'_{AB}]$
Similarly the other statement can be proved.
□

Theorem 3.4. *Let A & B be two IFSs with $\alpha, \beta \in [0, 1]$ and $\alpha + \beta = 1$. If $s(A, B) = [e_{AB}, e'_{AB}]$, then*

1. $s(\boxplus_{\alpha,\beta} A, \boxplus_{\alpha,\beta} B) = [\alpha e_{AB},\ \alpha e'_{AB}]$,
2. $s(\boxtimes_{\alpha,\beta} A, \boxtimes_{\alpha,\beta} B) = [\alpha e_{AB} + \beta,\ \alpha e'_{AB} + \beta\ ]$.

Proof. Similar to the Theorem 3.3 □
The above theorem is not true for $\alpha, \beta \in [0, 1]$ and $\alpha + \beta < 1$.
If we consider the example 3.1 with $\alpha = 0.7$ and $\beta = 0.1$ then it is found that $s(\boxplus_{\alpha,\beta} A, \boxplus_{\alpha,\beta} B) = [0.49, 0.725]$ $\neq [\alpha e_{AB},\ \alpha e'_{AB}]$ and $s(\boxtimes_{\alpha,\beta} A, \boxtimes_{\alpha,\beta} B) = [0.59, 0.825] \neq [\alpha e_{AB} + \beta,\ \alpha e'_{AB} + \beta\ ]$.

Example 3.5. Consider the IFSs A and B of E as in example 3.1. To find $s(\Box A, \Box B)$ and $s(\Diamond A, \Diamond B)$ we have to construct the new tables as

x	μ_A	$1 - \mu_A$	μ_B	$1 - \mu_B$
x_1	0.65	0.35	0.72	0.28
x_2	0.32	0.68	0.56	0.44
x_3	0.80	0.20	0.48	0.52
x_4	0.70	0.30	0.83	0.17

Hence $s(\Box A, \Box B) = 0.70 = e_{AB}$.
And

x	$1-\nu_A$	ν_A	$1-\nu_B$	ν_B
x_1	0.74	0.26	0.82	0.18
x_2	0.54	0.46	0.62	0.38
x_3	0.88	0.12	0.58	0.42
x_4	0.75	0.25	0.88	0.12

Hence $s(\Diamond A, \Diamond B) = 0.75 = e'_{AB}$.

Example 3.6. Consider the IFSs A and B of E as in example 3.1. To find $s(\boxplus A, \boxplus B)$ and $s(\boxtimes A, \boxtimes B)$ we have to construct the new tables as

x	$\frac{\mu_A(x)}{2}$	$\frac{\nu_A(x)+1}{2}$	$\frac{\mu_B(x)}{2}$	$\frac{\nu_B(x)+1}{2}$
x_1	0.325	0.63	0.36	0.59
x_2	0.16	0.73	0.28	0.69
x_3	0.40	0.56	0.24	0.71
x_4	0.35	0.625	0.415	0.56

Hence $s(\boxplus A, \boxplus B) = [0.35,\ 0.375] = [\frac{e_{AB}}{2}, \frac{e'_{AB}}{2}]$.
And

x	$\frac{\mu_A(x)+1}{2}$	$\frac{\nu_A(x)}{2}$	$\frac{\mu_B(x)+1}{2}$	$\frac{\nu_B(x)}{2}$
x_1	0.825	0.13	0.86	0.09
x_2	0.66	0.23	0.78	0.19
x_3	0.90	0.06	0.74	0.21
x_4	0.85	0.125	0.915	0.06

Hence $s(\boxtimes A, \boxtimes B) = [0.85,\ 0.875] = [\frac{e_{AB}}{2} + \frac{1}{2}, \frac{e'_{AB}}{2} + \frac{1}{2}]$.

Example 3.7. Consider the IFSs A and B of E as in example 3.1. To find $s(\boxplus_\alpha A, \boxplus_\alpha B)$ we construct the table with $\alpha = 0.7$.

x	$\alpha\mu_A(x)$	$\alpha\nu_A(x)+1-\alpha$	$\alpha\mu_B(x)$	$\alpha\nu_B(x)+1-\alpha$
x_1	0.455	0.482	0.504	0.426
x_2	0.224	0.622	0.392	0.566
x_3	0.56	0.384	0.336	0.594
x_4	0.49	0.475	0.581	0.384

Hence $s(\boxplus_\alpha A, \boxplus_\alpha B) = [0.49,\ 0.525] = [\alpha e_{AB}, \alpha e'_{AB}]$.
In a similar manner, we create the table that follows to locate $s(\boxtimes_\alpha A, \boxtimes_\alpha B)$.

x	$\alpha\mu_A(x)+1-\alpha$	$\alpha\nu_A(x)$	$\alpha\mu_B(x)+1-\alpha$	$\alpha\nu_B(x)$
x_1	0.755	0.182	0.804	0.126
x_2	0.524	0.322	0.692	0.266
x_3	0.86	0.084	0.636	0.294
x_4	0.79	0.175	0.881	0.084

Hence $s(\boxtimes_\alpha A, \boxtimes_\alpha B) = [0.79,\ 0.825] = [\alpha e_{AB} + 1 - \alpha, \alpha e'_{AB} + 1 - \alpha]$.

Example 3.8. Consider the IFSs A and B of E as in example 3.1. To find $s(\boxplus_{\alpha,\beta}A, \boxplus_{\alpha,\beta}B)$ we construct the table taking $\alpha = 0.7$ and $\beta = 0.3$ with $\alpha, \beta \in [0,1]$ and $\alpha + \beta = 1$.

x	$\alpha\mu_A(x)$	$\alpha\nu_A(x) + \beta$	$\alpha\mu_B(x)$	$\alpha\nu_B(x) + \beta$
x_1	0.455	0.482	0.504	0.426
x_2	0.224	0.622	0.392	0.566
x_3	0.56	0.384	0.336	0.594
x_4	0.49	0.475	0.581	0.384

Hence $s(\boxplus_{\alpha,\beta}A, \boxplus_{\alpha,\beta}B) = [0.49,\ 0.525] = [\alpha e_{AB}, \alpha e'_{AB}]$.
In a similar manner, we create the table that follows to locate $s(\boxtimes_{\alpha,\beta}A, \boxtimes_{\alpha,\beta}B)$.

x	$\alpha\mu_A(x) + \beta$	$\alpha\nu_A(x)$	$\alpha\mu_B(x) + \beta$	$\alpha\nu_B(x)$
x_1	0.755	0.182	0.804	0.126
x_2	0.524	0.322	0.692	0.266
x_3	0.86	0.084	0.636	0.294
x_4	0.79	0.175	0.881	0.084

Hence $s(\boxtimes_{\alpha,\beta}A, \boxtimes_{\alpha,\beta}B) = [0.79,\ 0.825] = [\alpha e_{AB} + \beta, \alpha e'_{AB} + \beta]$.
The measure of similarity has been thoroughly explored and defined in intuitionistic fuzzy set theory by numerous authors [33, 34, 35].
Chen [36] defined a similarity measure between two fuzzy sets A and B of X using the vector approach as follows:

$$s(A, B) = \frac{\overline{A}.\overline{B}}{\overline{A}^2 \vee \overline{B}^2} \tag{1}$$

Where, $\overline{A}$ is the vector $\langle \mu_A(x_1), \mu_A(x_2), ...\rangle$, $\overline{B}$ is the vector $\langle \mu_B(x_1), \mu_B(x_2), ...\rangle$ and $X = \{x_1, x_2, x_3, ...\}$, the symbol "." stands for scalar product of two vectors.
De et al.[33] also provide an analogous definition for the similarity measurement between two IFSs A and B of E.

$$s(A, B) = \frac{\sum_{x \in E} \overline{A}_x.\overline{B}_x}{\sum_{x \in E}(\overline{A}_x^2) \vee \sum_{x \in E}(\overline{B}_x^2)} \tag{2}$$

Where $\overline{A}_x$ is the vector $[\mu_A(x), \pi_A(x)]$ and $\overline{B}_x$ is the vector $[\mu_B(x), \pi_B(x)] \forall x \in E$.
Clearly,

1. $s(A, B) \in [0, 1]$.
2. $s(A, B) = s(B, A)$.
3. $e_{AB} = e'_{AB}$ if $A = B$.
4. If $\pi_A(x) = 0$ and $\pi_B(x) = 0, \forall x \in E$, then $s(A, B)$ becomes equal to the measure of similarity defined by Chen [4].

In this section, a new kind of similarity measure between two intuitionistic fuzzy sets are defined.

Definition 3.9. *Let us consider two IFSs A and B of a fixed set E. Similarity measure $s(A,B)$ between A and B is defined by*

$$s(A,B) = \frac{e_{AB}}{e'_{AB}} = \frac{\max\min_{x\in E}\{\mu_A(x), \mu_B(x)\}}{\max\min_{x\in E}\{\mu_A(x)+\pi_A(x), \mu_B(x)+\pi_B(x)\}} \tag{3}$$

The larger the value of $s(A,B)$, the more the similarity between the intuitionistic fuzzy sets.
Now let's look at example 3.1. It may be demonstrated that, for equation (2), the value of similarity measure $s(A,B) = 0.9254$, while, by Definition 3.9, similarity measure $s(A,B) = 0.9333$. Therefore, Definition 3.9 is more suited to offer the optimal solution.

Theorem 3.10. *For any two IFSs A and B of a fixed set E, the following statements are true:*

1. $0 \leq s(A,B) \leq 1$.

2. $s(A,B) = s(B,A)$.

3. *If $\pi_A(x) = 0$ and $\pi_B(x) = 0, \forall x \in E$, then $s(A,B)$ becomes equal to 1.*

Proof. Obvious. □
In the above theorem, $e_{AB} \neq e'_{AB}$ if $A = B$.

4 Application for Decision Making

This section describes a procedure for determining, given n possibilities, the most efficient course of action based on m criteria. Suppose that there are n actions A, B, C,...where each action depends upon all of the m criteria x_1, x_2, x_3,... .
A criterion-value $\langle\mu_A, \nu_A\rangle$ consists of the membership value and the non-membership value of the alternative A. The indeterministic or hesitation part is the remaining amount $\pi_A = 1 - \mu_A - \nu_A$. Here $\langle\mu_A, \nu_A\rangle$ are the IFSs of the set A under all criteria.
For two IFSs A and B of E, A is said to dominate B if $s(S,A) \geq s(S,B)$. It is clear that the super IFS S dominates all.

4.1 Algorithm

The steps of algorithm of this method are as follows:
First step: Construct the criteria-matrix using the standard and available alternatives.
Second step: Calculate $s(S,X) = \frac{e_{SX}}{e'_{SX}}$.
Third step: Find all the similarity measures like $s(S,X)$, where $X = A, B, C, D$ and E.
Fourth step: If $s(S,X)$ has more than one value, choose that one corresponding to which the indeterministic part is greatest.
Fifth step: Choose the optimal action.

4.2 A Case-Study

Here, we look at how a student might be selected for a desirable engineering branch based on a few different factors. Let S be the standard alternative and A, B, C, D, and E, are the available alternatives or the desirable engineering branches as Computer Science, Electronics, Biotechnology, Chemical and Mechanical Engineering. Moreover, the criteria are

1. Cut-off marks in entrance test (x_1),
2. Students' choice (x_2),
3. Availability of subjects or branches (x_3),
4. Availability of seats (x_4).

Here, we create a case study using hypothetical information. The criteria-matrix is displayed as follows.

x	S $\langle \mu_S, \nu_S \rangle$	A $\langle \mu_A, \nu_A \rangle$	B $\langle \mu_B, \nu_B \rangle$	C $\langle \mu_C, \nu_C \rangle$	D $\langle \mu_D, \nu_D \rangle$	E $\langle \mu_E, \nu_E \rangle$
x_1	$\langle 0.9, 0.05 \rangle$	$\langle 0.7, 0.2 \rangle$	$\langle 0.76, 0.2 \rangle$	$\langle 0.86, 0.1 \rangle$	$\langle 0.9, 0.02 \rangle$	$\langle 0.75, 0.2 \rangle$
x_2	$\langle 0.8, 0.1 \rangle$	$\langle 0.75, 0.22 \rangle$	$\langle 0.83, 0.14 \rangle$	$\langle 0.78, 0.18 \rangle$	$\langle 0.79, 0.15 \rangle$	$\langle 0.79, 0.15 \rangle$
x_3	$\langle 0.85, 0.05 \rangle$	$\langle 0.81, 0.12 \rangle$	$\langle 0.8, 0.1 \rangle$	$\langle 0.7, 0.2 \rangle$	$\langle 0.81, 0.14 \rangle$	$\langle 0.83, 0.13 \rangle$
x_4	$\langle 0.88, 0.05 \rangle$	$\langle 0.65, 0.25 \rangle$	$\langle 0.61, 0.24 \rangle$	$\langle 0.68, 0.3 \rangle$	$\langle 0.57, 0.28 \rangle$	$\langle 0.67, 0.28 \rangle$

Hence we get,

$s(S, A) = \frac{e_{SA}}{e'_{SA}} = \frac{\max\{0.70,0.75,0.81,0.65\}}{\max\{0.80,0.78,0.88,0.75\}} = \frac{0.81}{0.88} = 0.92045.$

$s(S, B) = \frac{e_{SB}}{e'_{SB}} = \frac{\max\{0.76,0.80,0.80,0.61\}}{\max\{0.80,0.86,0.90,0.76\}} = \frac{0.80}{0.90} = 0.88889.$

$s(S, C) = \frac{e_{SC}}{e'_{SC}} = \frac{\max\{0.86,0.78,0.70,0.68\}}{\max\{0.90,0.82,0.80,0.70\}} = \frac{0.86}{0.90} = \mathbf{0.95556}.$

$s(S, D) = \frac{e_{SD}}{e'_{SD}} = \frac{\max\{0.90,0.79,0.81,0.57\}}{\max\{0.95,0.85,0.86,0.72\}} = \frac{0.90}{0.95} = 0.94737.$

$s(S, E) = \frac{e_{SE}}{e'_{SE}} = \frac{\max\{0.75,0.79,0.83,0.67\}}{\max\{0.80,0.85,0.87,0.72\}} = \frac{0.83}{0.87} = 0.95402.$

This indicates that the best alternative is C i.e., Biotechnology is the optimal solution.

5 Conclusion

In order to determine the similarity measure between intuitionistic fuzzy sets, we describe a model or method for intuitionistic fuzzy sets in this study. The primary characteristic of this model is that the hesitation margin has also been taken into account and computed.We looked at a multi-criteria decision-making problem where the data were intuitionistic fuzzy rather than crisp. We accomplish this by comparing each of the criterion value sets with the super intuitionistic fuzzy set S. The best effective course of action is determined to be the criteria value set that most closely resembles S. The similarity measuring method is the name of the procedure. In addition to determining the best course of action, the method assists in creating a panel that reveals the second, third, and so on ideal actions. The proposed similarity measure shows great capacity for determining intuitionistic fuzzy sets. It has been illustrated that the proposed similarity measure performs as well as or better than previous measures. Further research will be focused on its applications in other practical fields.

Acknowledgements: The author is grateful to the anonymous reviewers for their very valuable comments.

Conflict of Interest: The author declares no conflict of interest.

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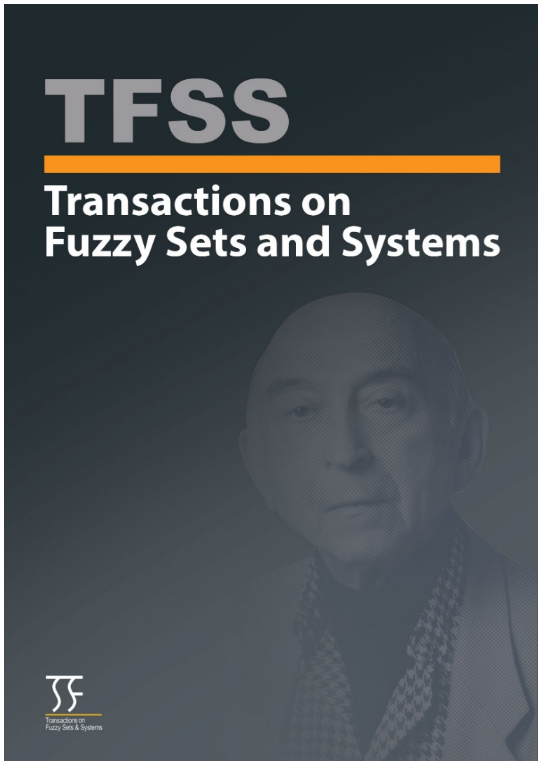
Transactions on Fuzzy Sets and Systems



Transactions on Fuzzy Sets and Systems

ISSN: **2821-0131**

https://sanad.iau.ir/journal/tfss/

Best States For Women To Work and Women's Peace and Security

Vol.4, No.1, (2025), 54-63. DOI: https://doi.org/10.71602/tfss.2025.1130212

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Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://sanad.iau.ir/journal/tfss/
Online ISSN: 2821-0131
Vol.4, No.1, (2025), 54-63
DOI: https://doi.org/10.71602/tfss.2025.1130212

Article Type: Original Research Article

Best States For Women To Work and Women's Peace and Security

John N Mordeson , **Sunil Mathew*** , **Davender S Malik**

(This paper is dedicated to Professor "John N. Mordeson" on the occasion of his 91st birthday.)

Abstract. In [1], states are ranked with respect to the best states to work. In [2], states are ranked with respect to the peace and security for women. We determine the fuzzy similarity measure of these to rankings. We find the similarity to be high for one of the measures and very high for the other. We then break the United States into regions and determine the fuzzy similarity measure of these two rankings for each region. The fuzzy similarity here is medium for one measure and high for the other. Similarity plays a role in many fields. There exists many special definitions of similarity which have been used in different areas. We choose to use fuzzy similarity measures which seem appropriate in rankings. In fact, we develop some new measures.

AMS Subject Classification 2020: 03B52; 03E72
Keywords and Phrases: Women, Work, Peace and security, State rankings, Fuzzy similarity measures, Distance functions.

1 Introduction

It is stated in [3] that states have had to step up for workers and their families in the past few decades, as Congress has stalled on taking action. For example, while the federal minimum wage has been stuck at $7.25 an hour for 14 years, most states have mandated higher wages. In [3], The Best States to Work Index provides how the states rank overall and by policy area.

In [1], it is stated that since women make up the majority of the workforce-and-many are supporting families-this dimension considers how far the tipped minimum wage goes to cover the cost of living for a family of three (one wage earner and two children). In [1], The Best States for Working Women Index provides how the states rank overall and by policy area.

The U. S. Women, Peace and Security Index (WPSI) is a measurement of women's rights and opportunities in the United states. It examines how women's legal protections vary by state, and how their rights and opportunities vary based on their race. The index incorporates three basic dimensions of women's well-being: inclusion, justice, and security. Inclusion includes economic, social, and political aspects, justice includes formal laws and informal discrimination, and security includes the family, community, and societal levels.

In [1], states are ranked with respect to the best states to work. In [2], states are ranked with respect to the peace and security for women. The rankings can be found in Tables 1 - 6. We determine the fuzzy similarity measure of these two rankings. We find the similarity to be high. We then break the United States into regions and determine the fuzzy similarity measure of these two rankings for each region. Similarity plays

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Received: 27 August 2024; **Revised**: 6 September 2024; **Accepted**: 7 September 2024; **Available Online**: 5 October 2024; **Published Online**: 7 May 2025.

How to cite: Mordeson JN, Mathew S, Malik DS. Best states for women to work and women's peace and security. *Transactions on Fuzzy Sets and Systems.* 2025; 4(1): 54-63. DOI: https://doi.org/10.71602/tfss.2025.1130212

a role in many fields. There exists many special definitions of similarity which have been used in different areas. We choose to use fuzzy similarity measures which seem appropriate in rankings. In particular, we use the t-norm algebraic product and the t-conorm, algebraic sum.

Let X be a set with n elements. We let $\mathcal{FP}(X)$ denote the fuzzy power set of X. We let $\wedge$ denote minimum and $\vee$ maximum. For two fuzzy subsets μ, ν of X, we write $\mu \subseteq \nu$ if $\mu(x) \leq \nu(x)$ for all $x \epsilon X$. If μ is a fuzzy subset of X, we let μ^c denote the complement of μ, i.e., $\mu^c(x) = 1 - \mu(x)$ for all $x \epsilon X$.

Let A be a one-to-function of X onto $\{1, 2, ..., n\}$. Then A is called a **ranking** of X. Define the fuzzy subset μ_A of X by for all $x \epsilon X, \mu_A(x) = \frac{A(x)}{n}$. Then μ_A is called the **fuzzy subset associated** with A. For A a ranking of X, we have $\sum_{x \epsilon X} A(x) = \frac{n(n+1)}{2}$ and $\sum_{x \epsilon X} \mu_A(x) = \frac{n+1}{2}$ since $\sum_{x \epsilon X} A(x) = 1 + 2 + ... + n$.

Throughout the paper, A and B will denote rankings of a set X with n elements.

2 Distance Functions and Fuzzy Similarity Measures

Let $\mathcal{T}$ be a t-norm and $\mathcal{S}_T$ a t-conorm. Then $\mathcal{T}$ and $\mathcal{S}_T$ are called **dual** if for all $a, b \epsilon [0, 1], \mathcal{T}(a, b) = 1 - \mathcal{S}_T(1 - a, 1 - b)$. Clearly, $\wedge$ are $\vee$ dual.

Definition 2.1. [4] Let $\mathcal{T}$ and $\mathcal{S}$ be a t-norm and t -conorm, respectively. Define the function $d : [0, 1] \times [0, 1] \to [0, 1]$ by $\forall a, b \epsilon [0, 1]$,

$$d(a, b) = \begin{cases} \mathcal{S}(a, b) - \mathcal{T}(a, b) \text{ if } a \neq b, \\ 0 \text{ if } a = b. \end{cases}$$

Consider (4) in the following result. Suppose $a \leq b \leq c$. We show $\mathcal{S}(a, c) - \mathcal{T}(a, c) \leq \mathcal{S}(a, b) - \mathcal{T}(a, b) + \mathcal{S}(b, c) - \mathcal{T}(b, c)$. This is equivalent to $\mathcal{S}(a, c) + \mathcal{T}(a, b) + \mathcal{T}(b, c) \leq \mathcal{S}(a, b) + \mathcal{S}(b, c) + \mathcal{T}(a, c)$. Now $\mathcal{S}(a, c) \leq \mathcal{S}(b, c)$ and $\mathcal{T}(a, b) \leq \mathcal{T}(a, c)$. Also, $\mathcal{T}(b, c) \leq b \wedge c \leq b \leq a \vee b \leq \mathcal{S}(a, b)$.

Theorem 2.2. *[4] Let $\mathcal{T}$ and $\mathcal{S}$ be a t-norm and t -conorm, respectively. Let d be defined as in Definition 2.1. Then d satisfies the following properties: $\forall a, b, c \epsilon [0, 1]$,*

(1) $0 \leq d(a, b) \leq 1$;

(2) $d(a, b) = 0$ *if and only if* $a = b$;

(3) $d(a, b) = d(b, a)$;

(4) $d(a, c) \leq d(a, b) + d(b, c)$ *if* $b \wedge c \leq b \leq a \vee b$.

Let $\mathcal{T}$ and $\mathcal{S}$ be a given t-norm and t-norm, respectively. Let d be defined as in Definition 2.1. Define $D : \mathcal{FP}(X) \times \mathcal{FP}(X) \to [0, 1]$ by all $(\mu, \nu) \epsilon \mathcal{FP}(X) \times \mathcal{FP}(X), D(\mu, \nu) = \sum_{x \epsilon X} d(\mu(x), \nu(x))$.

Define $S : \mathcal{FP}(X) \times \mathcal{FP}(X) \to [0, 1]$ as follows: $\forall (\mu, \nu) \epsilon \mathcal{FP}(X) \times \mathcal{FP}(X), S(\mu, \nu) = 1 - D(\mu, \nu)$.Then $S(\mu, \rho) = 1 - D(\mu, \rho) \geq 1 - D(\mu, \nu) - D(\nu, \rho) = S(\mu, \nu) - D(\nu, \rho) = S(\nu, \rho) - D(\mu, \nu)$. Thus $S(\mu, \rho) \leq S(\mu, \nu)$ and $S(\mu, \rho) \leq S(\nu, \rho)$ if $\mu \subseteq \nu \subseteq \rho$.

We have that $D_H(\mu, \nu) = \frac{1}{n} \sum_{i=1}^{n} |\mu(x_i) - \nu(x_i)| = \frac{1}{n} \sum_{i=1}^{n} ((\mu(x_i) \vee \nu(x_i) - \mu(x_i) \wedge \nu(x_i))$. This motivates the consideration of the following definition. Let $f(x_i) = (\mu(x_i) \oplus \nu(x_i) - \mu(x_i) \otimes \nu(x_i)$ if $\mu(x_i) \neq \nu(x_i)$ and $f(x_i) = 0$ if $\mu(x_i) = \nu(x_i)$.

For all $\mu, \nu \epsilon \mathcal{FP}(X)$, define $D_{\otimes}(\mu, \nu) = \frac{1}{n} \sum_{i=1}^{n} f(x_i)$. Define $D_{\otimes}^{+}(\mu, \nu) = \frac{1}{n} \sum_{i=1}^{n} ((\mu(x_i) \oplus \nu(x_i) - \mu(x_i) \otimes \nu(x_i))$. Then $D_{\otimes}^{+}(\mu, \nu) = D_{\otimes}(\mu, \nu) + \sum_{x \epsilon X^{+}} ((\mu(x) \oplus \nu(x) - \mu(x) \otimes \nu(x))$, where $X^{+} = \{x \epsilon X | \mu(x) = \nu(x)\}$. We note that $D_{\otimes}^{+}(\mu, \nu)$ does not satisfy (2) of Theorem 2.2.

Define $S_{\otimes}(\mu, \nu) = 1 - D_{\otimes}(\mu, \nu)$ and $S_{\otimes}^{+}(\mu, \nu) = 1 - D_{\otimes}^{+}(\mu, \nu)$.

We first wish to determine the smallest value $S^+_\otimes(\mu_A,\mu_B)$ can be for a given X. The smallest value $S^+_\otimes(\mu_A,\mu_B)$ can be determined from the largest value $D^+_\otimes(\mu_A,\mu_B)$ can be. Now $\sum_{i=1}^n(\mu_A(x_i)+\mu_B(x_i))$ is the fixed value $n+1$.Hence the largest value for $D^+_\otimes(\mu_A,\mu_B)$ is determined from the smallest $\sum_{i=1}^n \mu_A(x_i)\mu_B(x_i)$ since $\sum_{i=1}^n(\mu_A(x_i)\oplus\mu_B(x_i)-\mu_A(x_i)\otimes\mu_B(x_i))=\sum_{i=1}^n(\mu_A(x_i)+\mu_A(x_i)-\mu_A(x_i)\mu_B(x_i)-\mu_A(x_i)\mu_B(x_i))$.

The rankings $A:1,...,i,...,n$ and $B:n,...,n-i+1,...1$ yield the smallest value for $\sum_{i=1}^n \mu_A(x_i)\mu_B(x_i)$. We have

$$\begin{aligned}\sum_{i=1}^n A(x_i)B(x_i) &= \sum_{i=1}^n i(n-i+1)\\ &= (n+1)\sum_{i=1}^n i-\sum_{i=1}^n i^2\\ &= \frac{(n+1)n(n+1)}{2}-\frac{n(n+1)(2n+1)}{6}\\ &= n[\frac{n^2+2n+1}{2}-\frac{2n^2+3n+1}{6}\\ &= n[\frac{1}{6}n^2+\frac{1}{2}n+\frac{1}{3}].\end{aligned}$$

Thus

$$\begin{aligned}\sum_{i=1}^n \mu_A(x_i)\mu_B(x_i) &= \frac{1}{n^2}n[\frac{1}{6}n^2+\frac{1}{2}n+\frac{1}{3}]\\ &= \frac{1}{6}n+\frac{1}{2}+\frac{1}{3n}.\end{aligned}$$

Hence

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^n(\mu_A(x_i)+\mu_B(x_i)-2\mu_A(x_i)\mu_B(x_i)) &= \frac{1}{n}\sum_{i=1}^n(\mu_A(x_i)+\mu_B(x_i)\\ &\quad -2\frac{1}{n}\sum_{i=1}^n\mu_A(x_i)\mu_B(x_i))\\ &= \frac{1}{n}[n+1-2(\frac{1}{6}n+\frac{1}{2}+\frac{1}{3n})]\\ &= 1+\frac{1}{n}-\frac{1}{3}-\frac{1}{n}-\frac{2}{3n^2}\\ &= \frac{2}{3}-\frac{2}{3n^2}.\end{aligned}$$

Thus the smallest value $S^+_\otimes(\mu_A,\mu_B)$ can be is $1-(\frac{2}{3}-\frac{2}{3n^2})=\frac{1}{3}+\frac{2}{3n^2}$.

We have just proved the following result.

Theorem 2.3. *Thus the smallest value* $S^+_\otimes(\mu_A,\mu_B)$ *can be is* $1-(\frac{2}{3}-\frac{2}{3n^2})=\frac{1}{3}+\frac{2}{3n^2}$.

Theorem 2.4 ([5], Theorem 3.5)**.** *If* n *is even, the smallest value* $S_H(\mu_A,\mu_B)$ *can be is* $\frac{1}{2}$.
If n *is odd, the smallest value* $S_H(\mu_A,\mu_B)$ *can be is* $\frac{1}{2}+\frac{1}{2n^2}$.

Example 2.5. Let $n = 3$. Consider the rankings $A : 1, 2, 3$ and $B : 3, 2, 1$. Then $S^+_\otimes(\mu_A, \mu_B) = 1 - \frac{1}{3}(\frac{6+6}{3} - 2\frac{3+4+3}{9}) = 1 - \frac{1}{3}(4 - \frac{20}{9}) = \frac{11}{27}$. Using the above result, $S^+_\otimes(\mu_A, \mu_B) = \frac{1}{3} + \frac{2}{3n^2}$, we obtain $\frac{1}{3} + \frac{2}{27} = \frac{11}{27}$.

Theorem 2.6. *[4] Let $\mathcal{T}$ and $\mathcal{S}_T$ be a dual t-norm and t -conorm, respectively. Let d be defined as in Definition 2.1. Then (4) of Theorem 2.2 holds.*

Recall that $X^+ = \{x\epsilon X | \mu_A(x) = \mu_B(x)\}$ for given μ_A, μ_B.

Theorem 2.7. *Let $s^+_\otimes$ be the smallest value $S^+_\otimes(\mu_A, \mu_B)$ can be. Then $s_\otimes = s^+_\otimes + \sum_{x\epsilon X^+}((\mu_A(x) \oplus_B (x) - \mu_A(x) \otimes \mu_B(x))$ is the smallest value $S_\otimes(\mu_A, \mu_B)$ can be, where $S_\otimes = 1 - D_\otimes$.*

Proof. Recall $D^+_\otimes(\mu_A, \mu_B) = D_\otimes(\mu_A, \mu_B) + \sum_{x\epsilon X^+}((\mu_A(x) \oplus \mu_B(x) - \mu_A(x) \otimes \mu_B(x))$. Now $S^+_\otimes = 1 - D^+_\otimes$ and $S_\otimes = 1 - D_\otimes$. Let $s_\otimes$ be the smallest value $S_\otimes(\mu_A, \mu_B)$ can be. Now $S^+_\otimes(\mu_A, \mu_B) = 1 - D^+_\otimes(\mu_A, \mu_B) = 1 - (D_\otimes(\mu_A, \mu_B) + \sum_{x\epsilon X^+}((\mu_A(x) \oplus \mu_B(x) - \mu_A(x) \otimes \mu_B(x)) = S_\otimes(\mu_A, \mu_B) + \sum_{x\epsilon X^+}((\mu_A(x) \oplus \mu_B(x) - \mu_A(x) \otimes \mu_B(x))$. Now $s^+_\otimes = s + \sum_{x\epsilon X^+}((\mu_A(x) \oplus \mu_B(x) - \mu_A(x) \otimes \mu_B(x))$ for some s determine by $D_\otimes$. Then $s \geq s_\otimes$. Suppose $s > s_\otimes$. Then $s^+_\otimes = s + \sum_{x\epsilon X^+}((\mu_A(x) \oplus \mu_B(x) - \mu_A(x) \otimes \mu_B(x) > s_\otimes + \sum_{x\epsilon X^+}((\mu_A(x) \oplus \mu_B(x) - \mu_A(x) \otimes \mu_B(x))$, a contradiction. Thus $s = s_\otimes$. Hence $s^+_\otimes = s_\otimes + \sum_{x\epsilon X^+}((\mu_A(x) \oplus \mu_B(x) - \mu_A(x) \otimes \mu_B(x))$. □

Theorem 2.8. *The largest value $S^+_\otimes(\mu_A, \mu_B)$ can be is $\frac{2}{3} + \frac{1}{3n^2}$.*

Proof. We first find the smallest $D^+_\otimes(\mu_A, \mu_B)$ can be. This value is determined from the rankings $A : 1, 2, ..., n$ and $B : 1, 2, ..., n$. We have

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n}(\frac{i}{n} + \frac{i}{n} - 2\frac{i}{n}\frac{i}{n}) &= \frac{1}{n}\sum_{i=1}^{n}\frac{2i}{n} - \frac{2}{n}\sum_{i=1}^{n}\frac{i^2}{n^2} \\
&= \frac{2}{n^2}\sum_{i=1}^{n} i - \frac{2}{n^3}\sum_{i=1}^{n} i^2 \\
&= \frac{2}{n^2}(\frac{n(n+1)}{2} - \frac{2}{n^3}(\frac{n(n=1)(2n+1)}{6} \\
&= \frac{n+1}{n} - \frac{1}{n^2}\frac{(n+1)(2n+1)}{3} \\
&= \frac{n+1}{n} - \frac{1}{3n^2}(2n^2 + 2n + 1) \\
&= 1 + \frac{1}{n} - \frac{2}{3} - \frac{1}{n} - \frac{1}{3n^2} \\
&= \frac{1}{3} - \frac{1}{3n^2}.
\end{aligned}$$

Thus the largest value $S^+_\otimes(\mu_A, \mu_B)$ can be is $1- (\frac{1}{3} - \frac{1}{3n^2}) = \frac{2}{3} + \frac{1}{3n^2}$. □

Consider Theorems 2.4, 2.7, and 2.8. Suppose that s denotes the smallest value for some fuzzy similarity measure S and l the largest. Define

$$\widehat{S}(\mu_A, \mu_B) = \frac{S(\mu_A, \mu_B) - s}{l - s}.$$

Then $\widehat{S}(\mu_A, \mu_B)$ varies between 0 and 1. For values between 0 and 0.2, we say that the fuzzy similarity is very low, between 0.2 and 0.4 low, between 0.4 and 0.6 medium, between 0.6 and 0.8 high, and between 0.8 and 1 very high. Some related work can be seen in [6].

3 United States

We determine fuzzy similarity measures for the rankings, best states for women and the peace and security index for the United States.

:

Table 1: United States

State	Women	WPSI	State	Women	WPSI
Oregon	1	18	Florida	26	30
California	2	15	Michigan	27	21
New York	3	8	Missouri	28	38
Washington	4	24	South Dakota	29	29
Connecticut	5	2	Indiana	30	34
Massachusetts	6	1	Ohio	31	25
New Jersey	7	11	Iowa	32	23
Nevada	8	35	Idaho	33	39
Colorado	9	14	Pennsylvania	34	17
Hawaii	10	10	Kentucky	35	47
Puerto Rico			Oklahoma	36	42
Illinois	11	13	Wisconsin	37	16
District of Columbia	12	3	North Dakota	38	20
Vermont	13	4	Kansas	39	26
Maine	14	9	Arizona	40	31
Rhode Island	15	5	Louisiana	41	51
New Mexico	16	40	Arkansas	42	49
Minnesota	17	12	West Virginia	43	46
Maryland	18	7	Utah	44	36
Virginia	19	27	Wyoming	45	43
Delaware	20	22	South Carolina	46	44
Alaska	21	28	Texas	47	41
Nebraska	22	19	Mississippi	48	50
Montana	23	32	Alabama	49	48
Tennessee	24	45	Georgia	50	37
New Hampshire	25	6	North Carolina	51	33

We consider $D_H(\mu_A, \mu_B) = \frac{1}{n}\sum_{x\epsilon X} |\mu_A(x) - \mu_B(x)|$. Here $n = 51$. We find $D_H(\mu_A, \mu_B) = \frac{1}{51}\,\frac{456}{51} = \frac{456}{2601} = 0.1753$. Thus $S_H(\mu_A, \mu_B) = 1 -\ D_H(\mu_A, \mu_B) = 0.8247$.

By Theorem 2.4, the smallest $S_H(\mu_A, \mu_B)$ can be is $\frac{1}{2} + \frac{1}{2n^2} = \frac{1}{2} + \frac{1}{5202} = 0.5002$. Thus $\widehat{S_H}(\mu_A, \mu_B) = \frac{0.8247-0.5002}{1-0.5002} = \frac{0.3245}{0.4998} = 0.6495$. The fuzzy similarity measure is high.

We now consider $D_{\otimes}(\mu_A, \mu_B) = \frac{1}{n}\sum_{x\epsilon X^+}(\mu_A(x) \oplus \mu_B(x) - \mu_A(x) \otimes \mu_B(x))$. We first see that $\mu_A(\text{Hawaii}) = \mu_B(\text{Hawaii})$ and $\mu_A(\text{South Dakota}) = \mu_B(\text{South Dakota})$. We find

$$\begin{aligned} D_{\otimes}(\mu_A, \mu_B) &= \frac{1}{51}\left(\frac{2574}{51} - 2\frac{41497}{51^2}\right) \\ &= \frac{2574}{2601} - \frac{82994}{132651} = 0.9896 - 0.6257 = 0.3639. \end{aligned}$$

Thus $S_{\otimes}(\mu_A, \mu_B) = 1 - 0.3639 = 0.6361$.

By Theorem 2.3, the smallest $S^+_\otimes(\mu_A,\mu_B)$ can be is $\frac{1}{3}+\frac{2}{3n^2}=\frac{1}{3}+\frac{2}{7803}=0.3333+.00003=0.3336$. Thus the smallest $S_\otimes(\mu_A,\mu_B)$ can be is $0,3336+0.0062+0.0202=0.3600$.

By Theorem 2.8, the largest $S^+_\otimes(\mu_A,\mu_B)$ can be is $\frac{2}{3}+\frac{2}{3n^2}=\frac{2}{3}+\frac{2}{7803}=0.6667+0.0003=0.6670$. Hence the largest $S_\otimes(\mu_A,\mu_B)$ can be is $0,6670+0.0062+0.0202=0.6734$.

Thus $\widehat{S_\otimes}=\frac{06361-0.3600}{0.6734-0.3600}=\frac{0.2761}{0.3134}=0.8810$. The fuzzy similarity measure is very high.

4 Regions

Suppose $\mu_A(x)=\mu_B(x)=1$ for some $x\epsilon X$. Then $\mu_A(x)\oplus\mu_B(x)-\ \mu_A(x)\otimes\mu_B(x)=\mu_A(x)+\mu_B(x)-2_A(x)\mu_B(x)=0$. Thus $S_\otimes(\mu_a,\mu_B)=S^+_\otimes(\mu_a,\mu_B)$ if this is the only x in X such that $\mu_A(x)=\mu_B(x)$. Thus we have $S_\otimes(\mu_A,\mu_B)=S^+_\otimes(\mu_A,\mu_B)$ for the following region.

Table 2: West

State	Women	WPSI
Oregon	1	4
California	2	3
Montana	3	7
Washington	4	5
Nevada	5	8
Colorado	6	2
Hawaii	7	1
Alaska	8	6
Idaho	9	10
Utah	10	9
Wyoming	11	11

Here $n=11$. $S_H(\mu_A,\mu_B)=1-\frac{26}{121}=1-0.2149=0.7851$. The smallest $S_H(\mu_A,\mu_B)$ can be is $\frac{1}{2}+\frac{1}{2n^2}=0.5+\frac{1}{242}=0.5041$. Thus $\widehat{S_H}(\mu_A,\mu_B)=\frac{0.7851-0.5041}{1-0.5041}=\frac{0.2810}{0.4959}=0.5666$. The fuzzy similarity measure is medium.

We first note that $\mu_A(\text{Wyoming})=\mu_B(\text{Wyoming})$. We have that

$$\begin{aligned} D_\otimes(\mu_A,\mu_B) &= \frac{1}{11}(\frac{110}{11}-2\frac{338}{121}) \\ &= \frac{110}{121}-\frac{676}{1331}=0.9091-0.5079=0.4012. \end{aligned}$$

Thus $S_\otimes(\mu_A,\mu_B)=1-0.4012=0.5988$. By Theorem 2.3, the smallest $S_\otimes(\mu_A,\mu_B)$ can be is $\frac{1}{3}+\frac{2}{3n^2}=\frac{1}{3}+\frac{2}{363}=0.3333+.00055=0.3355$.

By Theorem 2.8, the largest $S(\mu_A,\mu_B)$ can be is $\frac{2}{3}+\frac{2}{3n^2}=\frac{2}{3}+\frac{2}{363}=0.6667+0.0055=0.6814=0.6612$. Thus $\widehat{S_\otimes}=\frac{0.5988-0.3355}{0.6612-0.335}=\frac{0.2233}{0.3257}=0.6856$. The fuzzy similarity measure is high.

Table 3: Southwest

State	Women	WPSI
New Mexico	1	2
Oklahoma	2	4
Arizona	3	1
Texas	4	3

Here $n = 4$. $S_H = 1 - \frac{6}{16} = 1 - 0.3750 = 0.6250$. The smallest S_H can be is $\frac{1}{2} = 0$. Thus $\widehat{S_H} = \frac{0.6250-0.500}{1-0.5000} = \frac{0.1250}{0.5000} = 0.2500$. The fuzzy similarity measure is low.

We have that $D_{\otimes}(\mu_A, \mu_B) = \frac{1}{4}(\frac{20}{4} - 2\frac{25}{16}) = \frac{20}{16} - \frac{50}{64} = 1.25 - 0.7812 = 0.4688$. Hence $S_{\otimes}(\mu_A, \mu_B) = S_{\otimes}^{+}(\mu_A, \mu_B) = 1 - 0.4688 = 0.5412$. By Theorem 2.3, the smallest $S_{\otimes}^{+}(\mu_A, \mu_B)$ can be is $\frac{1}{3} + \frac{2}{3n^2} = \frac{1}{3} + \frac{2}{48} = 0.3333 + .0147 = 0.3480$

By Theorem 2.8, the largest $S(\mu_A, \mu_B)$ can be is $\frac{2}{3} + \frac{2}{3n^2} = \frac{2}{3} + \frac{2}{48} = 0.6667 + 0.0417 = 0.6814$.

Thus $\widehat{S_{\otimes}}(\mu_A, \mu_B) = \frac{0.5412-0.3480}{0.6814-0.3480} = \frac{0.1932}{0.3334} = 0.5895$. The fuzzy similarity measure is medium.

Table 4: Midwest

State	Women	WPSI
Illinois	1	2
Minnesota	2	1
Nebraska	3	4
Michigan	4	6
Missouri	5	12
South Dakota	6	10
Indiana	7	11
Ohio	8	8
Iowa	9	7
Wisconsin	10	3
North Dakota	11	5
Kansas	12	9

Here $n = 12$. $S_H(\mu_A, \mu_B) = 1 - \frac{38}{144} = 1 - 0.2639 = 0.7361$. The smallest S_H can be is $\frac{1}{2} = 0.5000$. Thus $\widehat{S_H}(\mu_A, \mu_B) = \frac{0.7361-0.5000}{1-0.5000} = \frac{0.2361}{0.5000} = 0.4722$. The fuzzy similarity measure is medium.

We first note that $\mu_A(\text{Ohio}) = \mu_B(\text{Ohio})$. We have that

$$\begin{aligned} D_{\otimes}(\mu_A, \mu_B) &= \frac{1}{12}(\frac{140}{12} - 2\frac{493}{144}) \\ &= \frac{140}{144} - \frac{986}{1728} \\ &= 0.9722 - 5706 \\ &= 0.4016. \end{aligned}$$

Thus $S_{\otimes}(\mu_A, \nu_B) = 1 - 0.4016 = 0.5984$. By Theorem 2.3, the smallest $S_{\otimes}(\mu_A, \mu_B)$ can be is $s_{\otimes}^{+} + \sum_{x \epsilon X^+}((\mu_A(x) \oplus \mu_B(x) - \mu_A(x) \otimes \mu_B(x)) = \frac{1}{3} + \frac{2}{3n^2} + 0.00043 + 0.0371 = \frac{1}{3} + \frac{2}{432} + 0.0371 = \frac{1}{3} + 0.0036 + 0.0371 = 0.3333 = 0.0046 = 0.371 = 0.3750$, where $s_{\otimes}^{+}$ is the smallest $S_{\otimes}^{+}(\mu_A, \mu_B)$ can be.

By Theorem 2.8, the largest $S_{\otimes}(\mu_A, \mu_B)$ can be is $l_{\otimes}^{+} + \sum_{x \epsilon X^+}((_A(x) \oplus \mu_B(x) - \mu_A(x) \otimes \mu_B(x)) = \frac{2}{3} + \frac{2}{3n^2} + 0.0370 + 0.0210 = \frac{2}{3} + \frac{2}{432} + 0.0307 = 0.6667 + 0.0046 + 0.0370 = 0.7083$, where $l_{\otimes}^{+}$ is the largest $S_{\otimes}^{+}(\mu_A, \mu_B)$ can be.

Thus $\widehat{S_{\otimes}}(\mu_A, \mu_B) = \frac{0.5984-0.3750}{0.7983-0.3750} = \frac{0.2234}{0.3333} = 0.6703$. The fuzzy similarity measure is high.

Table 5: Southeast

State	Women	WPSI
Puerto Rico		
Washington D. C.	1	1
Virginia	2	2
Tennessee	3	7
Florida	4	3
Kentucky	5	9
Louisiana	6	13
Arkansas	7	11
West Virginia	8	8
South Carolina	9	6
Mississippi	10	12
Alabama	11	10
Georgia	12	5
North Carolina	13	4

Here $n = 13$. $S_H(\mu_A, \mu_B) = 1 - \frac{42}{169} = 1 - 0.2485 = 0.7515$. The smallest $S_H(\mu_A, \mu_B)$ can be is $\frac{1}{2} + \frac{1}{2n^2} = 0.5 + \frac{1}{338} = 0.5030$. Thus $\widehat{S_H}(\mu_A, \mu_B) = \frac{0.7515-0.5030}{1-0.5030} = \frac{0.2485}{0.4970} = 0.5000$. The fuzzy similarity measure is medium.

We first note that μ_A(Washington D. C.) $= \mu_B$(Washington D. C.), μ_A(Virginia) $= \mu_B$(Virginia), and μ_A(West Virginia) $= \mu_B$(West Virginia). We have that

$$
\begin{aligned}
D_{\otimes}(\mu_A, \mu_B) &= \frac{1}{14}(\frac{140}{14} - 2\frac{629}{196}) \\
&= \frac{140}{196} - \frac{1258}{2744} \\
&= 0.7143 - 4585 \\
&= 0.2558.
\end{aligned}
$$

Thus $S_{\otimes}(\mu_A, \mu_B) = 1 - 0.2558 = 0.7442$. By Theorem 2.3, the smallest $S_{\otimes}(\mu_A, \nu_B)$ can be is $s^+_{\otimes} + \sum_{x\epsilon X^+}((\mu_A(x) \oplus \mu_B(x) - \mu_A(x) \otimes \mu_B(x)) = \frac{1}{3} + \frac{2}{3n^2} + 0.00043 + 0.0210 = \frac{1}{3} + \frac{2}{507} + 0.0253 = \frac{1}{3} + 0.0039 + 0.0252 = 0.3333 + 0.3625$, where $s^+_{\otimes}$ is the smallest $S^+_{\otimes}(\mu_A, \mu_B)$ can be.

By Theorem 2.8, the largest $S_{\otimes}(\mu_A, \mu_B)$ can be is $l^+_{\otimes}(\mu_A, \mu_B) + \sum_{x\epsilon X^+}((\mu_A(x) \oplus \mu_B(x) - \mu_a(x) \otimes \mu_B(x)) = \frac{2}{3} + \frac{2}{3n^2} + 0.0043 + 0.0210 = \frac{2}{3} + \frac{2}{507} + 0.0253 = 0.6667 + 0.0039 + 0.0253 = 0.6958$, where $l^+_{\otimes}$ is the largest $S^+_{\otimes}(\mu_A, \mu_B)$ can be.

Thus $\widehat{S_{\otimes}}(\mu_A, \mu_B) = \frac{0.6422-0.3625}{0.6958-0.3625} = \frac{0.2797}{0.3333} = 0.8392$. The fuzzy similarity measure is very high.

Table 6: Northeast

State	Women	WPSI
New York	1	7
Connecticut	2	2
Massachusetts	3	1
New Jersey	4	9
Vermont	5	3
Maine	6	8
Rhode Island	7	4
Maryland	8	6
Delaware	9	11
New Hampshire	10	5
Pennsylvania	11	10

Here $n = 11$. $S_H(\mu_A, \mu_B) = 1 - \frac{35}{121} = 1 - 0.2893 = 0.7107$. The smallest $S_H(\mu_A, \mu_B)$ can be is $\frac{1}{2} + \frac{1}{2n^2} = 0.5 + \frac{1}{242} = 0.5041$. Thus $\widehat{S_H}(\mu_A, \mu_B) = \frac{0.7107-0.5041}{1-0.5041} = \frac{0.2066}{0.4959} = 0.4166$. The fuzzy similarity measure is medium.

We first note that $\mu_A(\text{Connecticut}) = \mu_B(\text{Connecticut})$. We have that

$$\begin{aligned} D_\otimes(\mu_A, \mu_B) &= \frac{1}{11}(\frac{128}{11} - 2\frac{444}{121}) \\ &= \frac{128}{121} - \frac{888}{1331} \\ &= 1.0579 - 0.6672 \\ &= 0.3907. \end{aligned}$$

Thus $S_\otimes(\mu_A, \mu_B) = 1 - 0.3907 = 0.6093$. By Theorem 2.3, the smallest $S_\otimes(\mu_A, \mu_B)$ can be is $s_\otimes^+ + \sum_{x\epsilon X^+}((\mu_A(x) \oplus \mu_B(x) - \mu_A(x) \otimes \mu_B(x)) = \frac{1}{3} + \frac{2}{3n^2} + 0.0271 = \frac{1}{3} + \frac{2}{363} + 0.0271 = \frac{1}{3} + 0.0055 + 0.0271 = 0.3659$, where $s_\otimes^+$ is the smallest $S_\otimes^+(\mu_A, \mu_B)$ can be.

By Theorem 2.8, the largest $S_\otimes(\mu_A, \mu_B)$ can be is $l_\otimes^+ + \sum_{x\epsilon X^+}((\mu_A(x) \oplus \mu_B(x) - \mu_A(x) \otimes \mu_B(x)) = \frac{2}{3} + \frac{2}{3n^2} + 0.0271 = 0.6667 + 0.0033 + 0.0271 = 0.6993$, where $l_\otimes^+$ is the largest $S_\otimes^+(\mu_A, \mu_B)$ can be.

Thus $\widehat{S_\otimes}(\mu_A, \mu_B) = \frac{0.6093-0.3659}{0.6993-0.3659} = \frac{0.2434}{0.3334} = 0.7301$.The fuzzy similarity measure is high.

5 Conclusion

In this paper, we used two fuzzy similarity measures of the rankings best states for women to work and the peace and security of women. We accomplished this for the United States in general and for various regions of the U. S We found the similarity to be medium to high for one fuzzy similarity measures and high to very high for another. Additional results on the best places for women to work can be found in [5].

Conflict of Interest: The authors declare no conflict of interest.

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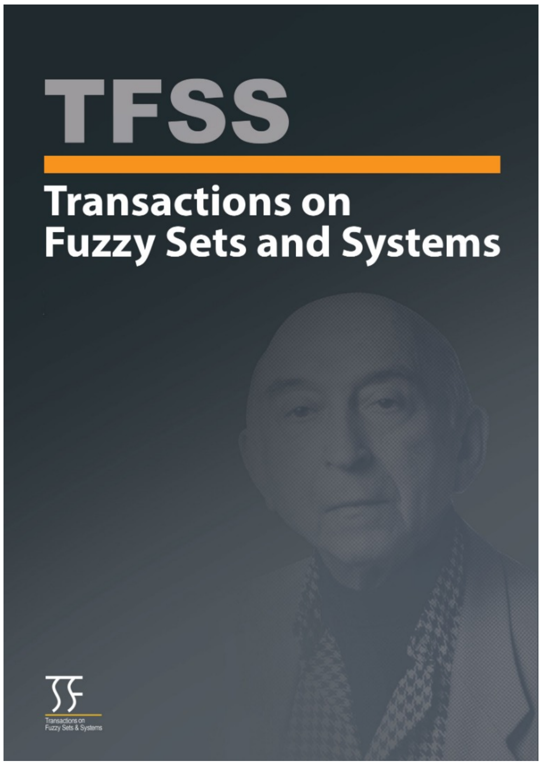

Transactions on Fuzzy Sets and Systems

ISSN: **2821-0131**

https://sanad.iau.ir/journal/tfss/

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Vol.4, No.1, (2025), 64-107. DOI: https://doi.org/10.71602/tfss.2025.1126750

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Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://sanad.iau.ir/journal/tfss/
Online ISSN: 2821-0131
Vol.4, No.1, (2025), 64-107
DOI: https://doi.org/10.71602/tfss.2025.1126750

Article Type: Original Research Article

A Theoretical Development of Linear Diophantine Fuzzy Graph Structures

Saba Ayub*, **Muhammad Shabir**

(This paper is dedicated to Professor "John N. Mordeson" on the occasion of his 91st birthday.)

Abstract. Graph structure (GS) is an advancement of the graph concept which effectively represents intricate situations with various connections, frequently used in computer science and mathematics to illustrate relationships among objects and extensively researched in fuzzy sets (FS), intuitionistic fuzzy set (IFS), pythagorean fuzzy set (PFS) and q-rung orthopair fuzzy set (q-ROFS). Meanwhile, a linear Diophantine fuzzy set (LDFS) is a remarkable extension of the existing notions of a FS, IFS, PFS and q-ROFS by comporting reference parameters that removed all the limitations related to membership degree (MD) and non-membership degree (NMD). According to the best of our knowledge, there is a lack of elegantly proposed GS extension for LDFSs in the current literature. As a result, this research focuses on introducing first linear Diophantine fuzzy graph structure (LDFGS) concept which extends the existing notions of GS in various contexts of FSs. Several key concepts in LDFGSs are presented, such as $\breve{\rho}_i$-edge, $\breve{\rho}_i$-path, strength of $\breve{\rho}_i$-path, $\breve{\rho}_i$-strength of connectedness, $\breve{\rho}_i$-degree of a vertex, vertex degree, total $\breve{\rho}_i$-degree of a vertex, and total vertex degree in an LDFGS. In addition, we introduce the $\breve{\rho}_i$-size, size, and order of an LDFGS. Moreover, this article presents the ideas of the maximal product of two LDFGSs, strong LDFGS, degree and $\breve{\rho}_i$-degree of the maximal product, $\breve{\rho}_i$-regular and regular LDFGSs, along with examples for clarification. Certain significant results related to the proposed concepts also demonstrated with explanatory examples such as the maximal product of two strong LDFGSs is also a strong LDFGS, the maximal product of two connected LDFGSs is also a connected LDFGS but the maximal product of two regular LDFGS may not be a regular LDGS. Moreover, many interesting and alternative formulas for calculating $\breve{\rho}_i$-degrees of an LDFGS in various situations are proved with examples. LDFGSs are highly beneficial for solving numerous combinatorial problems involving multiple relations, and they surpass existing concepts of GSs within the FS context due to their flexibility in selecting MD and NMD alongside their reference parameters.

AMS Subject Classification 2020: 03B52; 03E72; 28E10; 18B35
Keywords and Phrases: Linear Diophantine fuzzy sets, Graph structure, Maximal product, Degree of a vertex, Total degree of a vertex.

1 Introduction

Incorporating uncertainties into real-world applications has become essential for addressing a variety of practical issues such as data analysis, computational intelligence, and sustainability. In 1965, Zadeh [1] poineered the concept of FS and fuzzy logic for modelling uncertain situations by assigning the MD to each object rather than absolute membership and absolute non-membership. Since then, FS theory have been studied by

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Received: 19 July 2024; **Revised**: 21 September 2024; **Accepted**: 27 September 2024; **Available Online**: 15 October 2024; **Published Online**: 7 May 2025.

How to cite: Ayub S, Shabir M. A theoretical development of linear Diophantine fuzzy graph structures. *Transactions on Fuzzy Sets and Systems.* 2025; 4(1): 64-107. DOI: https://doi.org/10.71602/tfss.2025.1126750

scholars and scientists in a wide range of practical fields including artificial intelligence, medical science, computational sciences and decision analysis [2, 3, 4]. Since MD is not sufficient to describe many real situations, there is a need for NMD such as educated and uneducated, perfection and imperfection, sick and healthy, etc. In order to deal with such situations, Attanassov [5, 6] proposed the idea of IFS with the addition of NMD such that the sum of MD and NMD is not greater than one. Due to the large space of MD and NMD, IFSs were studied enormously in various fields of applications [7, 8]. However, there are still many real-life problems where the condition of IFS is not satisfied. For instance, a professional is asked to comment on the viability of a strategy to invest in the real estate industry. Imagine that the expert rates this investment plan's degree of feasibility at 0.8 and its degree of impossibility at 0.6. Since $0.8 + 0.6 > 1$, the IFS cannot be utilized to appropriately express this information. As a result, Yager [9] presented the idea of PFS, which meets the requirement that the sum of squares of the MD and the NMD is less than equal to 1 for each element. But if the decision-maker expresses his view as 0.9 for agree and 0.8 for not agree, we can see that $0.9^2 + 0.8^2 > 1$. To deal with the situations, Yager [10] investigated q-ROFS as a more generic version of IFS and PFS. In q-ROFSs, the total of the q-powers for truthfulness and falsehood grades is kept within a unit interval. This indicates that q-ROFSs provide additional data storage to characterize ambiguous or unclear facts. Researchers have given the PFS theory and q-ROFS theory a lot of attention over the past five years, and numerous insightful theoretical and practical findings have been made in a variety of fields. For instance, Yager [11] presented a multi-attribute decision making technique for PFS. Khan et al. [12] developed a new ranking technique for q-ROFSs based on entropy function and hesitancy index with a detailed critical analysis of the previously ranking methods. Liu and Wang [13] proposed some q-ROF aggregation operators and utilized them to solve multi-attribute decision making problems.

Although MD and NMDs are subject to certain restraints under the theories mentioned earlier of IFS, PFS, and q-ROFS. To overcome all these restrictions associated with MD and NMD, Riaz and Hashmi [14] introduced an augmented generalized form of FS known as LDFS with the inclusion of reference parameters. Due to the inclusion of reference or control parameters, LDFSs have a wide space of MD and NMD, in contrast to the commonly used ongoing conceptions, made this theory more advanced, trustworthy and easy to model uncertainties. Due to the advancement of LDFSs and its freedom regarding MD and NMD, various scientists have started to create fresh theories about this emerging and sophisticated concept. For instance, Almagrabi et al. [15] established the concept of q-linear Diophantine fuzzy set (q-LDFS) and its application in emergency decision support system for COVID19. Ayub et al. [16] introduced the notion of linear Diophantine fuzzy relations (LDFRs) and studied their algebraic structures with an application in decision making. Further Ayub et al. [17, 18] studied the roughness of a crisp set by using the level sets of an LDFR and by $\big(\langle s,t\rangle, \langle u,v\rangle\big)$-indiscernibility of an LDFR over dual universes, respectively. A comprehensive details on the study of rough approximations of an LDFS via an LDFR, inituitionistic fuzzy relation (IFR) and fuzzy relation (FR) together with their applications in the field of decision making, respectively, have presented in [19, 20, 21]. Gül and Aydoğdu [22] proposed linear Diophantine fuzzy TOPSIS (LDF-TOPSIS) based on some novel distance and entropy definitions for LDFSs. Iampan et al. [23] presented linear Diophantine fuzzy Einstien aggregation operators for multi-criteria decision-making problems. Inan et al. [24] established a multiple attribute decision model to compare the firms occupational health and safety management perspectives. Riaz et al. [25] introduced linear Diophantine fuzzy soft rough sets with a practical application to select the sustainable material handling equipment. Kamaci [26, 27] studied linear Diophantine fuzzy algebraic structures and introduced the concept of complex linear Diophantine fuzzy sets with their applications using cosine similarity measures, respectively. Further Riaz et al. [28] proposed the concept of spherical linear Diophantine fuzzy sets and presented their applications in modeling uncertainties in MCDM.

The concept of graph theory (GT) started with finding a walk linking seven bridges in Konigsberg. Subsequently, it has developed enormously in all the domains of sciences and humanities with wide applications in the field of operations research, economics and system analysis. A graph is used to represent mathematical networks that define the association between vertices and edges. A vertex can be used to symbolize a workstation, while the edges denote the association between stations. However, graphs often do not reflect many physical processes appropriately due to the obvious complexity of various properties of the structures. Many real-world phenomena have been emphasized to define the concept of fuzzy graphs (FGs). In 1973, Kauffman [29] introduced the concept of the fuzzy graph (FG) based on Zadehs fuzzy relations (FR) [30]. Mordeson [4] have further studied FGs and fuzzy hypergraphs. Fuzzy graph theory (FGT) has many applications in various areas, including, data mining, networking, image segmentation, clustering, communication, planning, image capturing, and scheduling. A detailed study on FGs has presented in [4, 31]. Karunambigai and Parvathi [32] utilized IFS to describe an intuitionistic fuzzy graph (IFG). Shannon and Atanassov [33], and Parvathi et al. [34] utilized IFS to describe intuitionistic fuzzy graphs (IFGs) and their basic operations via intuitionistic fuzzy relation (IFR) [35]. Verma et al. [36] established the concept of pythagorean fuzzy graph (PFG) by first coining the idea of pythagorean fuzzy relation (PFR). Akram et al. [37, 38] studied certain PFS-graphs and q-ROF graphs (q-ROFGs) under Hamacher operators. Hanif et al. [39] presented the concept of an LDF graph (LDFG) by using the idea of an LDF relation (LDFR) which was introduced by Ayub et al. [16].

Since a graph is a pair of set of vertices $\mathscr{V}$ and one relation $\mathscr{E}$ on $\mathscr{V}$, which is capable of describing abundant real-life phenomenons. However, in many real life situations that concern more than one type of relations, GT cannot work efficiently. In order to deal such situations, Sampathkumar [40] generalized the notion of graphs and introduced the concept of graph structures (GSs). GS has n mutually disjoint, symmetric and irreflexive relations. Ramakrishnan and Dinesh [41, 42, 43] introduced fuzzy graph structures (FGSs) and investigated some related properties. Later on, Akram and Sitara [44, 45] and Akram et al. [46] investigated degree, total degree and few properties of semi-strong min product, maximal product and residue product of FGSs. Sharma and Bansal [47, 48] introduced the concept of IF-graph structure (IFGS). Further, Sharma et al. [49] presented the notion of regular IFGSs with a detailed study of their important consequences and useful examples for illustration. Sitara et al. [50] studied the concept of q-rung picture fuzzy graph structure (q-RPFGS).

1.1 Research Gaps and Motivations

The following subsection will summarize the main objectives and areas of knowledge lacking in the theories discussed earlier.

1. GSs are commonly employed in analyzing various structures, such as graphs, signed graphs, semigraphs, edge-colored graphs, and edge-labled graphs. GSs play a crucial role in researching various areas within computer science and computational intelligence. FGSs are more beneficial compared to GS due to their ability to address the uncertainty and ambiguity commonly found in various real-world phenomena.

2. The latest extension of FS theory, called LDFS introduced by Riaz and Hashmi [14], eliminates constraints related to MD and NMD found in previous concepts like FS, IFS, PyFS, and q-ROFS by adding reference parameters. It allows the decision maker greater freedom in their judgment when facing any decision-making issue. Indeed, reference parameters play a significant role in determining the optimal solution in decision analysis.

3. Recently, Hanif et al. [39] proposed the concept of LDF-graph (LDFG) with some fundamental operations and properties. LDFGs are more beneficial than FG, IFG, PFGS, and q-ROFG because they have a broader range of MD and NMD.

4. Since GSs are more valuable than graphs due to their ability to handle multiple relationship issues effectively. By viewing existing literature, it appears that there is a lack of investigation on LDF graph structures (LDFGS).

5. To address this research gap, we explore GS within LDFSs and introduce the concept of LDFGS, which eliminates specific restrictions on MD and NMD found in current FGSs.

6. Several key concepts of LDFGSs are introduced with demonstrative examples. Certain significant and fascinating results are proved using different scenarios along with concrete examples. LDFGSs are certainly better than the current concepts of FGSs, IFGSs, and q-RPFGS because of the expanded scope of MD and NMD. LDFGSs are a valuable resource in addressing issues involving numerous connections within the context of LDFSs.

1.2 Aim of the Proposed Study

The main purposes of this research paper are:

- To establish a detailed study on GS in the context of LDFSs and hence introduce the concept of LDFGS.
- To define key notions such as $\breve{\rho}_i$-edge, $\breve{\rho}_i$-path, strength of $\breve{\rho}_i$-path, $\breve{\rho}_i$-strength of connectedness, $\breve{\rho}_i$-degree of a vertex, degree of a vertex, total $\breve{\rho}_i$-degree of a vertex, and total degree of a vertex in an LDFGS, $\breve{\rho}_i$-size of an LDFGS, size, and the order of an LDFGS.
- To introduce the notion of the maximal product of two LDFGSs, strong LDFGS, degree and $\breve{\rho}_i$-degree of the maximal product.
- To present the concept of $\breve{\rho}_i$-regular and regular LDFGS.
- To develop their important consequences with illustrative examples.

1.3 Organization of the Paper

Our remaining part of this paper is organized in the following manners:

Certain basic notions related to FS, IFS, PFS, q-ROFS, LDFS, FR, IFR, LDFR, GS, FGS, and IFGS are presented in Section 2. In Section 3, the concept of LDFGS is introduced with an explanatory example. Furthermore, some fundamental concepts in LDFGS such as $\breve{\rho}_i$-edge, $\breve{\rho}_i$-path, strength of $\breve{\rho}_i$-path, $\breve{\rho}_i$-strength of connectedness, $\breve{\rho}_i$-degree of a vertex, degree of a vertex, total $\breve{\rho}_i$-degree of a vertex, and total degree of a vertex in an LDFGS, $\breve{\rho}_i$-size of an LDFGS, size of an LDFGS, and the order of an LDFGS are introduced with constructive examples. In Section 4, the notion of the maximal product of two LDFGSs, strong LDFGS, degree and $\breve{\rho}_i$-degree of the maximal product are introduced. Some important results related to these concepts are also proved with illustrative examples. Section 5 presents the concept of $\breve{\rho}_i$-regular and regular LDFGS with some related consequences and examples. Finally, section 6 consists of some concluding remarks of this research article and some future research directions related to the novel born ideas in this research article.

2 Preliminaries

In this section, some fundamental notions of FS, IFS, PFS, q-ROFS, LDFS, FR, IFR, LDFR, GS, FGS and IFGS are given which are indispensable to understanding the contributions of this paper. For more details, we refer the reader to study [16, 41, 42, 40, 14]. Throughout this research manuscript, $\mathscr{V}, \mathscr{V}_1,$ and $\mathscr{V}_2$ are denoted as universal sets, unless otherwise stated.

Definition 2.1. [1] A FS on $\mathscr{V}$ is defined by $\mathscr{F} = \left\{ \left\langle \mathbf{x}, \varkappa^m_{\mathscr{F}}(\mathbf{x}) \right\rangle : \mathbf{x} \in \mathscr{V} \right\}$, where $\varkappa^m_{\mathscr{F}} : \mathscr{V} \longrightarrow [0,1]$ is a membership function (MF) which assigns the MD to each object $\mathbf{x} \in \mathscr{V}$.

Definition 2.2. [5] An IFS $\mathscr{I}$ on $\mathscr{V}$ is a set of triplets of the form:

$$\mathscr{I} = \Big\{ \Big(\mathbf{x}, \big\langle \varkappa^m_{\mathscr{I}}(\mathbf{x}), \varkappa^n_{\mathscr{I}}(\mathbf{x}) \big\rangle \Big) : \mathbf{x} \in \mathscr{V} \Big\}, \tag{1}$$

where $\varkappa^m_{\mathscr{I}}, \varkappa^n_{\mathscr{I}} : \mathscr{V} \longrightarrow [0,1]$ specify the MD and NMD, respectively, which satisfy $0 \leq \varkappa^m_{\mathscr{I}}(\mathbf{x}) + \varkappa^n_{\mathscr{I}}(\mathbf{x}) \leq 1$, for all $\mathbf{x} \in \mathscr{V}$.

Definition 2.3. [9] A PFS $\mathscr{P}$ on $\mathscr{V}$ is an object of the form:

$$\mathscr{P} = \Big\{ \Big(\mathbf{x}, \big\langle \varkappa^m_{\mathscr{P}}(\mathbf{x}), \varkappa^n_{\mathscr{P}}(\mathbf{x}) \big\rangle \Big) : \mathbf{x} \in \mathscr{V} \Big\}, \tag{2}$$

where $\varkappa^m_{\mathscr{P}}, \varkappa^n_{\mathscr{P}} : \mathscr{V} \longrightarrow [0,1]$ specify for the MD and NMD, respectively, fulfilling $0 \leq \big(\varkappa^m_{\mathscr{P}}(\mathbf{x})\big)^2 + \big(\varkappa^n_{\mathscr{P}}(\mathbf{x})\big)^2 \leq 1$, for all $\mathbf{x} \in \mathscr{V}$.

Definition 2.4. [11, 10] A q-ROFS $\mathscr{Q}$ on $\mathscr{V}$ is an object of the form:

$$\mathscr{Q} = \Big\{ \Big(\mathbf{x}, \big\langle \varkappa^m_{\mathscr{Q}}(\mathbf{x}), \varkappa^n_{\mathscr{Q}}(\mathbf{x}) \big\rangle \Big) : \mathbf{x} \in \mathscr{V} \Big\}, \tag{3}$$

where $\varkappa^m_{\mathscr{Q}}, \varkappa^n_{\mathscr{Q}} : \mathscr{V} \longrightarrow [0,1]$ are used for the MD and NMD, respectively such that $0 \leq \big(\varkappa^m_{\mathscr{Q}}(\mathbf{x})\big)^q + \big(\varkappa^n_{\mathscr{Q}}(\mathbf{x})\big)^q \leq 1$, for all $\mathbf{x} \in \mathscr{V}$, where $q \in [1, \infty)$.

Definition 2.5. [14] An LDFS $\mathfrak{L}$ over $\mathscr{V}$ is an expression of the following form :

$$\mathfrak{L} = \Big\{ \Big(\mathbf{x}, \big\langle \varkappa^m_{\mathfrak{L}}(\mathbf{x}), \varkappa^n_{\mathfrak{L}}(\mathbf{x}) \big\rangle, \big\langle \alpha_{\mathfrak{L}}(\mathbf{x}), \beta_{\mathfrak{L}}(\mathbf{x}) \big\rangle \Big) : \mathbf{x} \in \mathscr{V} \Big\}, \tag{4}$$

where $\varkappa^m_{\mathfrak{L}}, \varkappa^n_{\mathfrak{L}} : \mathscr{V} \longrightarrow [0,1]$ are MD and NMD, and $\alpha_{\mathfrak{L}}(\mathbf{x}), \beta_{\mathfrak{L}}(\mathbf{x}) \in [0,1]$ are corresponding reference parameters, respectively, with $0 \leq \alpha_{\mathfrak{L}}(\mathbf{x}) + \beta_{\mathfrak{L}}(x) \leq 1$ and $0 \leq \alpha_{\mathfrak{L}}(\mathbf{x})\varkappa^m_{\mathfrak{L}}(\mathbf{x}) + \beta_{\mathfrak{L}}(\mathbf{x})\varkappa^n_{\mathfrak{L}}(\mathbf{x}) \leq 1$, for all $\mathbf{x} \in \mathscr{V}$. The degree of hesitation of any $\mathbf{x} \in \mathscr{V}$ is denoted and defined as $\boxtimes_{\mathfrak{L}}(\mathbf{x}) = 1 - \big(\alpha_{\mathfrak{L}}(\mathbf{x})\varkappa^m_{\mathfrak{L}}(\mathbf{x}) + \beta_{\mathfrak{L}}(\mathbf{x})\varkappa^n_{\mathfrak{L}}(\mathbf{x})\big)$, for all $\mathbf{x} \in \mathscr{V}$.

From now onward, we will use **LDFS**($\mathscr{V}$) for the set of all LDFSs over $\mathscr{V}$. For simplicity, we will use $\mathfrak{L} = \big(\langle \varkappa^m_{\mathfrak{L}}(\mathbf{x}), \varkappa^n_{\mathfrak{L}}(\mathbf{x})\rangle, \langle \alpha_{\mathfrak{L}}(\mathbf{x}), \beta_{\mathfrak{L}}(\mathbf{x})\rangle\big)$ for an LDFS over $\mathscr{V}$.

Definition 2.6. [14] Let $\mathfrak{L}_1 = \Big(\big\langle \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}), \varkappa^n_{\mathfrak{L}_1}(\mathbf{x})\big\rangle, \big\langle \alpha_{\mathfrak{L}_1}(\mathbf{x}), \beta_{\mathfrak{L}_1}(\mathbf{x})\big\rangle\Big)$ and $\mathfrak{L}_2 = \Big(\big\langle \varkappa^m_{\mathfrak{L}_2}(\mathbf{x}), \varkappa^n_{\mathfrak{L}_2}(\mathbf{x})\big\rangle, \big\langle \alpha_{\mathfrak{L}_2}(\mathbf{x}), \beta_{\mathfrak{L}_2}(\mathbf{x})\big\rangle\Big)$ be two LDFSs on $\mathscr{V}$. Then, for all $x \in \mathscr{V}$,

(1) $\mathfrak{L}_1 \subseteq \mathfrak{L}_2$ if and only if $\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}) \leq \varkappa^m_{\mathfrak{L}_2}(\mathbf{x}), \varkappa^n_{\mathfrak{L}_1}(\mathbf{x}) \geq \varkappa^n_{\mathfrak{L}_2}(\mathbf{x}),$ and $\alpha_{\mathfrak{L}_1}(\mathbf{x}) \leq \alpha_{\mathfrak{L}_2}(\mathbf{x}), \beta_{\mathfrak{L}_1}(\mathbf{x}) \geq \beta_{\mathfrak{L}_2}(\mathbf{x})$;

(2) $\mathfrak{L}_1 \cup \mathfrak{L}_2 = \Big(\big\langle \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{x}), \varkappa^n_{\mathfrak{L}_1}(\mathbf{x}) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{x})\big\rangle, \big\langle \alpha_{\mathfrak{L}_1}(\mathbf{x}) \vee \alpha_{\mathfrak{L}_2}(\mathbf{x}), \beta_{\mathfrak{L}_1}(\mathbf{x}) \wedge \beta_{\mathfrak{L}_2}(\mathbf{x})\big\rangle\Big)$;

(3) $\mathfrak{L}_1 \cap \mathfrak{L}_2 = \Big(\big\langle \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}) \wedge \varkappa^m_{\mathfrak{L}_2}(\mathbf{x}), \varkappa^n_{\mathfrak{L}_1}(\mathbf{x}) \vee \varkappa^n_{\mathfrak{L}_2}(\mathbf{x})\big\rangle, \big\langle \alpha_{\mathfrak{L}_1}(\mathbf{x}) \wedge \alpha_{\mathfrak{L}_2}(\mathbf{x}), \beta_{\mathfrak{L}_1}(\mathbf{x}) \vee \beta_{\mathfrak{L}_2}(\mathbf{x})\big\rangle\Big)$;

(4) $\mathfrak{L}_1^c = \Big(\big\langle \varkappa^n_{\mathfrak{L}_1}(\mathbf{x}), \varkappa^m_{\mathfrak{L}_1}(\mathbf{x})\big\rangle, \big\langle \beta_{\mathfrak{L}_1}(\mathbf{x}), \alpha_{\mathfrak{L}_1}(\mathbf{x})\big\rangle\Big)$;

A subset $\mathscr{E}$ of the cartesian product $\mathscr{V}_1 \times \mathscr{V}_2$ is a binary relation from $\mathscr{V}_1$ to $\mathscr{V}_2$ which is basically the set of edges from $\mathscr{V}_1$ to $\mathscr{V}_2$.

Definition 2.7. [30] A FR ρ on $\mathscr{V}_1 \times \mathscr{V}_2$ is defined as:

$$\rho = \Big\{ \big\langle (\mathbf{x}_1, \mathbf{x}_2), \varkappa_\rho^m(\mathbf{x}_1, \mathbf{x}_2) \big\rangle, (\mathbf{x}_1, \mathbf{x}_2) \in \mathscr{V}_1 \times \mathscr{V}_2 \Big\}, \tag{5}$$

where $\varkappa_\rho^m : \mathscr{V}_1 \times \mathscr{V}_2 \longrightarrow [0, 1]$ is a MF which specifies the grade of membership to which the objects $\mathbf{x}_1 \in \mathscr{V}_1$ and $\mathbf{x}_2 \in \mathscr{V}_2$ are connected to each other.

Definition 2.8. [35] An IFR $\dot{\rho}$ from $\mathscr{V}_1$ to $\mathscr{V}_2$ is an object of the form:

$$\dot{\rho} = \Big\{ \Big((\mathbf{x}_1, \mathbf{x}_2), \big\langle \varkappa_{\dot{\rho}}^m(\mathbf{x}_1, \mathbf{x}_2), \varkappa_{\dot{\rho}}^n(\mathbf{x}_1, \mathbf{x}_2) \big\rangle \Big) : \mathbf{x}_1 \in \mathscr{V}_1, \mathbf{x}_2 \in \mathscr{V}_2 \Big\}, \tag{6}$$

where $\varkappa_{\dot{\rho}}^m, \varkappa_{\dot{\rho}}^n : \mathscr{V}_1 \times \mathscr{V}_2 \longrightarrow [0, 1]$ indicate the MD and NMD from $\mathscr{V}_1$ to $\mathscr{V}_2$, respectively with $0 \leq \varkappa_{\dot{\rho}}^m(\mathbf{x}_1, \mathbf{x}_2) + \varkappa_{\dot{\rho}}^n(\mathbf{x}_1, \mathbf{x}_2) \leq 1$, for all $(\mathbf{x}_1, \mathbf{x}_2) \in \mathscr{V}_1 \times \mathscr{V}_2$.

Definition 2.9. [16] An LDFR $\check{\rho}$ from $\mathscr{V}_1$ to $\mathscr{V}_2$ is an expression having the form:

$$\check{\rho} = \Big\{ \Big((\mathbf{x}_1, \mathbf{x}_2), \big\langle \varkappa_{\check{\rho}}^m(\mathbf{x}_1, \mathbf{x}_2), \varkappa_{\check{\rho}}^n(\mathbf{x}_1, \mathbf{x}_2) \big\rangle, \big\langle \alpha_{\check{\rho}}(\mathbf{x}_1, \mathbf{x}_2), \beta_{\check{\rho}}(\mathbf{x}_1, \mathbf{x}_2) \big\rangle \Big) : \mathbf{x}_1 \in \mathscr{V}_1, \mathbf{x}_2 \in \mathscr{V}_2 \Big\}, \tag{7}$$

where $\varkappa_{\check{\rho}}^m, \varkappa_{\check{\rho}}^n : \mathscr{V}_1 \times \mathscr{V}_2 \longrightarrow [0, 1]$ denotes the MD and NMD among the entities of $\mathscr{V}_1$ and $\mathscr{V}_2$, and $\alpha_{\check{\rho}}(\mathbf{x}_1, \mathbf{x}_2)$, $\beta_{\check{\rho}}(\mathbf{x}_1, \mathbf{x}_2) \in [0, 1]$ are the corresponding reference parameters to $\varkappa_{\check{\rho}}^m(\mathbf{x}_1, \mathbf{x}_2)$ and $\varkappa_{\check{\rho}}^n(\mathbf{x}_1, \mathbf{x}_2)$, respectively. These MD and NMD obey the constraint $0 \leq \alpha_{\check{\rho}}(\mathbf{x}_1, \mathbf{x}_2)\varkappa_{\check{\rho}}^m(\mathbf{x}_1, \mathbf{x}_2) + \beta_{\check{\rho}}(\mathbf{x}_1, \mathbf{x}_2)\varkappa_{\check{\rho}}^n(\mathbf{x}_1, \mathbf{x}_2) \leq 1$ for all $(\mathbf{x}_1, \mathbf{x}_2) \in \mathscr{V}_1 \times \mathscr{V}_2$ with $0 \leq \alpha_{\check{\rho}}(\mathbf{x}_1, \mathbf{x}_2) + \beta_{\check{\rho}}(\mathbf{x}_1, \mathbf{x}_2) \leq 1$. The degree of hesitation can be calculated as:

$$\gamma \boxtimes_{\check{\rho}} (\mathbf{x}_1, \mathbf{x}_2) = 1 - \Big(\alpha_{\check{\rho}}(\mathbf{x}_1, \mathbf{x}_2)\varkappa_{\check{\rho}}^m(\mathbf{x}_1, \mathbf{x}_2) + \beta_{\check{\rho}}(\mathbf{x}_1, \mathbf{x}_2)\varkappa_{\check{\rho}}^n(\mathbf{x}_1, \mathbf{x}_2) \Big), \tag{8}$$

where γ is the corresponding reference parameter of indeterminacy. For simplicity, we shall use $\check{\rho} = \Big(\big\langle \varkappa_{\check{\rho}}^m(\mathbf{x}_1, \mathbf{x}_2), \varkappa_{\check{\rho}}^n(\mathbf{x}_1, \mathbf{x}_2) \big\rangle, \big\langle \alpha_{\check{\rho}}(\mathbf{x}_1, \mathbf{x}_2), \beta_{\check{\rho}}(\mathbf{x}_1, \mathbf{x}_2) \big\rangle \Big)$ for an LDFR from $\mathscr{V}_1$ to $\mathscr{V}_2$. The collection of all LDFRs from $\mathscr{V}_1$ to $\mathscr{V}_2$ by $\mathbf{LDFS}(\mathscr{V}_1 \times \mathscr{V}_2)$.

Definition 2.10. [16] Let $\check{\rho}_1 = \Big(\big\langle \varkappa_{\check{\rho}_1}^m(\mathbf{x}_1, \mathbf{x}_2), \varkappa_{\check{\rho}_1}^n(\mathbf{x}_1, \mathbf{x}_2) \big\rangle, \big\langle \alpha_{\check{\rho}_1}(\mathbf{x}_1, \mathbf{x}_2), \beta_{\check{\rho}_1}(\mathbf{x}_1, \mathbf{x}_2) \big\rangle \Big)$ be an LDFR from $\mathscr{V}_1$ to $\mathscr{V}_2$ and $\check{\rho}_2 = \Big(\big\langle \varkappa_{\check{\rho}_2}^m(\mathbf{x}_2, \mathbf{x}_3), \varkappa_{\check{\rho}_2}^n(\mathbf{x}_2, \mathbf{x}_3) \big\rangle, \big\langle \alpha_{\check{\rho}_2}(\mathbf{x}_2, \mathbf{x}_3), \beta_{\check{\rho}_2}(\mathbf{x}_2, \mathbf{x}_3) \big\rangle \Big)$ be an LDFR from $\mathscr{V}_2$ to $\mathscr{V}_3$. Then, their composition is denoted and defined by :

$$\check{\rho}_1 \circ \check{\rho}_2 = \Big(\big\langle \big(\varkappa_{\check{\rho}_1}^m \circ \varkappa_{\check{\rho}_2}^m \big)(\mathbf{x}_1, \mathbf{x}_3), \big(\varkappa_{\check{\rho}_1}^n \circ \varkappa_{\check{\rho}_2}^n \big)(\mathbf{x}_1, \mathbf{x}_3) \big\rangle, \big\langle \big(\alpha_{\check{\rho}_1} \circ \alpha_{\check{\rho}_2} \big)(\mathbf{x}_1, \mathbf{x}_3), \big(\beta_{\check{\rho}_1} \circ \beta_{\check{\rho}_2} \big)(\mathbf{x}_1, \mathbf{x}_3) \big\rangle \Big) \tag{9}$$

where

$$\big(\varkappa_{\check{\rho}_1}^m \circ \varkappa_{\check{\rho}_2}^m \big)(\mathbf{x}_1, \mathbf{x}_3) = \bigvee_{\mathbf{x}_2 \in \mathscr{V}_2} \Big(\varkappa_{\check{\rho}_1}^m(\mathbf{x}_1, \mathbf{x}_2) \wedge \varkappa_{\check{\rho}_2}^m(\mathbf{x}_2, \mathbf{x}_3) \Big), \tag{10}$$

$$\big(\varkappa_{\check{\rho}_1}^n \circ \varkappa_{\check{\rho}_2}^n \big)(\mathbf{x}_1, \mathbf{x}_3) = \bigwedge_{\mathbf{x}_2 \in \mathscr{V}_2} \Big(\varkappa_{\check{\rho}_1}^n(\mathbf{x}_1, \mathbf{x}_2) \vee \varkappa_{\check{\rho}_2}^n(\mathbf{x}_2, \mathbf{x}_3) \Big), \tag{11}$$

$$\big(\alpha_{\check{\rho}_1} \circ \alpha_{\check{\rho}_2} \big)(\mathbf{x}_1, \mathbf{x}_3) = \bigvee_{\mathbf{x}_2 \in \mathscr{V}_2} \Big(\alpha_{\check{\rho}_1}(\mathbf{x}_1, \mathbf{x}_2) \wedge \alpha_{\check{\rho}_2}(\mathbf{x}_2, \mathbf{x}_3) \Big), \tag{12}$$

$$\big(\beta_{\check{\rho}_1} \circ \beta_{\check{\rho}_2} \big)(\mathbf{x}_1, \mathbf{x}_3) = \bigwedge_{\mathbf{x}_2 \in \mathscr{V}_2} \Big(\beta_{\check{\rho}_1}(\mathbf{x}_1, \mathbf{x}_2) \vee \beta_{\check{\rho}_2}(\mathbf{x}_2, \mathbf{x}_3) \Big), \tag{13}$$

for all $\mathbf{x}_1 \in \mathscr{V}_1, \mathbf{x}_2 \in \mathscr{V}_2, \mathbf{x}_3 \in \mathscr{V}_3$.

Definition 2.11. Let $\breve{\rho} = \Big(\big\langle \varkappa^m_{\breve{\rho}}(\mathbf{x}_1, \mathbf{x}_2), \varkappa^n_{\breve{\rho}}(\mathbf{x}_1, \mathbf{x}_2)\big\rangle, \big\langle \alpha_{\breve{\rho}}(\mathbf{x}_1, \mathbf{x}_2), \beta_{\breve{\rho}}(\mathbf{x}_1, \mathbf{x}_2)\big\rangle\Big)$ be an LDFR from $\mathscr{V}_1$ to $\mathscr{V}_2$. Then, the set

$$Supp(\breve{\rho}) = \big\{(\mathbf{x}_1, \mathbf{x}_2) : \varkappa^m_{\breve{\rho}}(\mathbf{x}_1, \mathbf{x}_2) > 0, \varkappa^n_{\breve{\rho}}(\mathbf{x}_1, \mathbf{x}_2) > 0\rangle, \langle \alpha_{\breve{\rho}}(\mathbf{x}_1, \mathbf{x}_2) > 0, \beta_{\breve{\rho}}(\mathbf{x}_1, \mathbf{x}_2) > 0\big\} \tag{14}$$

is called the support of $\breve{\rho}$.

Definition 2.12. [40] Let $\mathscr{V}$ be any non-empty set known as the vertex set and $\mathscr{E}_1, \mathscr{E}_2, \cdots, \mathscr{E}_k$ be mutually disjoint relations (sets of edges) of $\mathscr{V}$ such that each $\mathscr{E}_i$, $1 \leq i \leq k$ is symmetric and irreflexive. Then, $\mathscr{G} = \big(\mathscr{V}, \mathscr{E}_1, \mathscr{E}_2, \cdots, \mathscr{E}_k\big)$ is called a graph structure (GS).

Definition 2.13. [41, 42] Let $\mathscr{G} = \big(\mathscr{V}, \mathscr{E}_1, \mathscr{E}_2, \cdots, \mathscr{E}_k\big)$ be a GS. Then, $\hat{\mathscr{G}} = \big(\mathscr{F}, \rho_1, \rho_2, \cdots, \rho_k\big)$ is called fuzzy graph structure (FGS) of GS $\mathscr{G}$, where $\mathscr{F}$ is a FS on $\mathscr{V}$ and ρ_i are irreflexive, symmetric and mutually exclusive FRs on $\mathscr{V}$, for all $1 \leq i \leq k$, if $0 \leq \varkappa^m_{\rho_i}(\mathbf{x}, \mathbf{y}) \leq \varkappa^m_{\mathscr{F}}(\mathbf{x}) \wedge \varkappa^m_{\mathscr{F}}(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathscr{V}$, $i = 1, 2, \cdots, k$.

Definition 2.14. [47] Let $\mathscr{G} = \big(\mathscr{V}, \mathscr{E}_1, \mathscr{E}_2, ..., \mathscr{E}_k\big)$ be a GS, $\mathscr{I} = \big\langle \varkappa^m_{\mathscr{I}}(\mathbf{x}), \varkappa^n_{\mathscr{I}}(\mathbf{x})\big\rangle$ be an IFS on $\mathscr{V}$ and $\dot{\rho}_i = \big\langle \varkappa^m_{\dot{\rho}_i}(\mathbf{x}_1, \mathbf{x}_2), \varkappa^n_{\dot{\rho}_i}(\mathbf{x}_1, \mathbf{x}_2)\big\rangle$ be irreflexive, symmetric and mutually disjoint IFRs on $\mathscr{V}$, $i = 1, 2, ..., n$, where $\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2 \in \mathscr{V}$. Then, $\dot{\mathscr{G}} = \big(\mathscr{I}, \dot{\rho}_1, \dot{\rho}_2, ..., \dot{\rho}_n\big)$ is called intuitionistic fuzzy graph structure (IFGS) of $\mathscr{G}$, if

$$\varkappa^m_{\dot{\rho}_i}(\mathbf{x}_1, \mathbf{x}_2) \leq \varkappa^m_{\mathscr{I}}(\mathbf{x}) \wedge \varkappa^n_{\mathscr{I}}(\mathbf{x}_2), \text{ and } \varkappa^n_{\dot{\rho}_i}(\mathbf{x}_1, \mathbf{x}_2) \geq \varkappa^n_{\mathscr{I}}(\mathbf{x}_1) \vee \varkappa^n_{\mathscr{I}}(\mathbf{x}_2),$$

for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathscr{V}$, $i = 1, 2, ..., n$.

3 Linear Diophantine Fuzzy Graph Structures (LDFGS)

In this section, we introduce the idea of LDFGS and some basic notions in LDFGSs containing $\breve{\rho}_i$-edge, $\breve{\rho}_i$-path, strength of $\breve{\rho}_i$-path, $\breve{\rho}_i$-strength of connectedness, $\breve{\rho}_i$-degree of a vertex, degree of a vertex, total $\breve{\rho}_i$-degree of a vertex, and total degree of a vertex in an LDFGS, $\breve{\rho}_i$-size of an LDFGS, size of an LDFGS, and the order of an LDFGS are introduced with constructive examples. Throughout this section, we will use simply $\mathscr{G}$ for a GS $\mathscr{G} = \big(\mathscr{V}, \mathscr{E}_1, \mathscr{E}_2, ..., \mathscr{E}_n\big)$ (see Definition 2.12).

Definition 3.1. Let $\mathfrak{L} = \Big(\big\langle \varkappa^m_{\mathfrak{L}}(\mathbf{x}), \varkappa^n_{\mathfrak{L}}(\mathbf{x})\big\rangle, \big\langle \alpha_{\mathfrak{L}}(\mathbf{x}), \beta_{\mathfrak{L}}(\mathbf{x})\big\rangle\Big)$ be an LDFS over $\mathscr{V}$, $\mathscr{G}$ be a GS and $\breve{\rho}_i \in$ **LDFS**$(\mathscr{E}_i)$, $i \in \{1, 2, \cdots, k\}$. Then, $\breve{\mathscr{G}} = \big(\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2, ..., \breve{\rho}_k\big)$ is called an LDFGS of $\mathscr{G}$, if for all $\mathbf{x}, \mathbf{y} \in \mathscr{V}$:

$$\left.\begin{aligned} \varkappa^m_{\breve{\rho}_i}(\mathbf{x}, \mathbf{y}) &\leq \varkappa^m_{\mathfrak{L}}(\mathbf{x}) \wedge \varkappa^m_{\mathfrak{L}}(\mathbf{y}), \\ \varkappa^n_{\breve{\rho}_i}(\mathbf{x}, \mathbf{y}) &\geq \varkappa^n_{\mathfrak{L}}(\mathbf{x}) \vee \varkappa^n_{\mathfrak{L}}(\mathbf{y}), \\ \alpha_{\breve{\rho}_i}(\mathbf{x}, \mathbf{y}) &\leq \alpha_{\mathfrak{L}}(\mathbf{x}) \wedge \alpha_{\mathfrak{L}}(\mathbf{y}), \\ \beta_{\breve{\rho}_i}(\mathbf{x}, \mathbf{y}) &\geq \beta_{\mathfrak{L}}(\mathbf{x}) \vee \beta_{\mathfrak{L}}(\mathbf{y}). \end{aligned}\right\} \tag{15}$$

Example 3.2. Let $\mathscr{V} = \big\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\big\}$, $\mathscr{E}_1 = \big\{(\mathbf{x}_1, \mathbf{x}_2), (\mathbf{x}_1, \mathbf{x}_3), (\mathbf{x}_3, \mathbf{x}_4)\big\}$, and $\mathscr{E}_2 = \big\{(\mathbf{x}_1, \mathbf{x}_4), (\mathbf{x}_2, \mathbf{x}_3), (\mathbf{x}_2, \mathbf{x}_4)\big\}$. Then, $\mathscr{G} = \big(\mathscr{V}, \mathscr{E}_1, \mathscr{E}_2\big)$ is the GS. Define an LDFS $\mathfrak{L} \in$ **LDFS**$(\mathscr{V})$ exhibited in TABLE 1.

Consider two LDFRs $\breve{\rho}_1, \breve{\rho}_2$ over $\mathscr{E}_1, \mathscr{E}_2$, respectively which are shown in TABLES 2 and 3, respectively.

By simple calculations, we can easily see that $\breve{\mathscr{G}} = \big(\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2\big)$ is an LDFGS of GS $\mathscr{G} = \big(\mathscr{V}, \mathscr{E}_1, \mathscr{E}_2\big)$ shown in Figure 1.

Table 1: Tabular representation of LDFS $\mathfrak{L}$

$\mathscr{V}$	$\Big(\big\langle \varkappa^m_{\mathfrak{L}}(\mathbf{x}), \varkappa^n_{\mathfrak{L}}(\mathbf{x})\big\rangle, \big\langle \alpha_{\mathfrak{L}}(\mathbf{x}), \beta_{\mathfrak{L}}(\mathbf{x})\big\rangle\Big)$
$\mathbf{x}_1$	$\big(\langle 0.4, 0.3\rangle, \langle 0.2, 0.1\rangle\big)$
$\mathbf{x}_2$	$\big(\langle 0.6, 0.2\rangle, \langle 0.3, 0.2\rangle\big)$
$\mathbf{x}_3$	$\big(\langle 0.4, 0.5\rangle, \langle 0.4, 0.2\rangle\big)$
$\mathbf{x}_4$	$\big(\langle 0.7, 0.3\rangle, \langle 0.6, 0.2\rangle\big)$

Table 2: $\breve{\rho}_1$

$\breve{\rho}_1$	$\Big(\big\langle \varkappa^m_{\breve{\rho}_1}(\mathbf{x},\mathbf{y}), \varkappa^n_{\breve{\rho}_1}(\mathbf{x},\mathbf{y})\big\rangle, \big\langle \alpha_{\breve{\rho}_1}(\mathbf{x},\mathbf{y}), \beta_{\breve{\rho}_1}(\mathbf{x},\mathbf{y})\big\rangle\Big)$
$(\mathbf{x}_1, \mathbf{x}_2)$	$\big(\langle 0.4, 0.4\rangle, \langle 0.2, 0.3\rangle\big)$
$(\mathbf{x}_1, \mathbf{x}_3)$	$\big(\langle 0.3, 0.6\rangle, \langle 0.2, 0.3\rangle\big)$
$(\mathbf{x}_3, \mathbf{x}_4)$	$\big(\langle 0.4, 0.5\rangle, \langle 0.4, 0.2\rangle\big)$

Table 3: $\breve{\rho}_2$

$\breve{\rho}_2$	$\Big(\big\langle \varkappa^m_{\breve{\rho}_2}(\mathbf{x},\mathbf{y}), \varkappa^n_{\breve{\rho}_2}(\mathbf{x},\mathbf{y})\big\rangle, \big\langle \alpha_{\breve{\rho}_2}(\mathbf{x},\mathbf{y}), \beta_{\breve{\rho}_2}(\mathbf{x},\mathbf{y})\big\rangle\Big)$
$(\mathbf{x}_1, \mathbf{x}_4)$	$\big(\langle 0.4, 0.3\rangle, \langle 0.2, 0.3\rangle\big)$
$(\mathbf{x}_2, \mathbf{x}_3)$	$\big(\langle 0.3, 0.5\rangle, \langle 0.2, 0.3\rangle\big)$
$(\mathbf{x}_2, \mathbf{x}_4)$	$\big(\langle 0.6, 0.4\rangle, \langle 0.3, 0.4\rangle\big)$

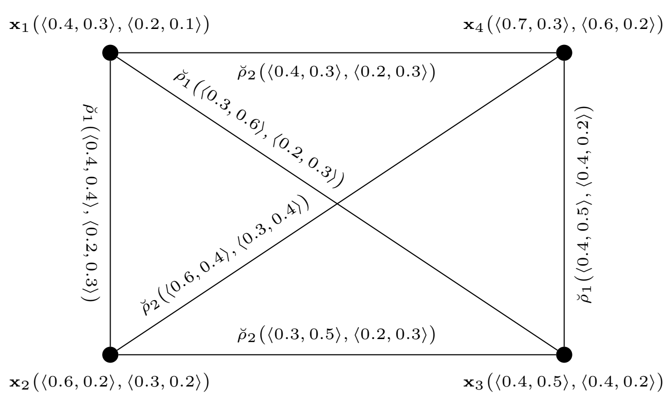

Figure 1: $\breve{\mathscr{G}} = \big(\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2\big)$

Definition 3.3. Let $\breve{\mathscr{G}} = \big(\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2, ..., \breve{\rho}_k\big)$ be an LDFGS with underlying GS $\mathscr{G}$. If $(\mathbf{x}, \mathbf{y}) \in Supp(\breve{\rho}_i)$, then $(\mathbf{x}, \mathbf{y})$ is called $\breve{\rho}_i$-edge of $\breve{\mathscr{G}}$.

Example 3.4. In Example 3.2, $(\mathbf{x}_1, \mathbf{x}_4)$, $(\mathbf{x}_2, \mathbf{x}_3)$, $(\mathbf{x}_2, \mathbf{x}_4)$ are $\breve{\rho}_2$-edges since $Supp(\breve{\rho}_2) = \big\{(\mathbf{x}_1, \mathbf{x}_4), (\mathbf{x}_2, \mathbf{x}_3), (\mathbf{x}_2, \mathbf{x}_4)\big\}$ and $(\mathbf{x}_1, \mathbf{x}_2)$, $(\mathbf{x}_1, \mathbf{x}_3)$, $(\mathbf{x}_3, \mathbf{x}_4)$ are $\breve{\rho}_1$-edges since $Supp(\breve{\rho}_1) = \big\{(\mathbf{x}_1, \mathbf{x}_2), (\mathbf{x}_1, \mathbf{x}_3), (\mathbf{x}_3, \mathbf{x}_4)\big\}$.

Definition 3.5. Let $\breve{\mathscr{G}} = \big(\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2, ..., \breve{\rho}_k\big)$ be an LDFGS with underlying GS $\mathscr{G}$. A $\breve{\rho}_i$-path of $\breve{\mathscr{G}}$ is a sequence of vertices $(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_l)$ which are distinct except possibly $\mathbf{x}_0 = \mathbf{x}_l$, such that $(\mathbf{x}_{j-1}, \mathbf{x}_j)$ is a $\breve{\rho}_i$-edge for all $j = 1, 2, 3, ..., l$.

Example 3.6. In Example 3.2, $(\mathbf{x}_3, \mathbf{x}_1, \mathbf{x}_2)$, and $(\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_4)$ are $\breve{\rho}_1$-paths. And, $(\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)$, $(\mathbf{x}_1, \mathbf{x}_4, \mathbf{x}_2)$, and $(\mathbf{x}_1, \mathbf{x}_4, \mathbf{x}_2, \mathbf{x}_3)$ are $\breve{\rho}_2$-paths.

Definition 3.7. In an LDFGS $\breve{\mathscr{G}} = (\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2, ..., \breve{\rho}_k)$ with underlying GS $\mathscr{G}$, two vertices $\mathbf{x}, \mathbf{y}$ of $\breve{\mathscr{G}}$ are said to be $\breve{\rho}_i$-connected, if they are joined by a $\breve{\rho}_i$-path.

Example 3.8. In Example 3.2, all vertices $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, $\mathbf{x}_4$ are $\breve{\rho}_1$- and $\breve{\rho}_2$-connected according to the Example 3.6 since they are joined by both $\breve{\rho}_1$- and $\breve{\rho}_2$- paths. Since for all $\mathbf{x}, \mathbf{y} \in \mathscr{V}$ they are connected by $\breve{\rho}_i$ for all $i = 1, 2$, so $\breve{\mathscr{G}}$ is connected LDFGS because $\breve{\rho}_1(\mathbf{x}_1, \mathbf{x}_3) > 0$, $\breve{\rho}_1(\mathbf{x}_1, \mathbf{x}_2) > 0$, and $\breve{\rho}_1(\mathbf{x}_3, \mathbf{x}_4) > 0$ so, $\mathbf{x}_1$, $\mathbf{x}_3$ are $\breve{\rho}_1$-connected, $\mathbf{x}_1$, $\mathbf{x}_2$ are $\breve{\rho}_1$-connected, and $\mathbf{x}_3$, $\mathbf{x}_4$ are $\breve{\rho}_1$-connected, respectively. Similarly, $\mathbf{x}_2$, $\mathbf{x}_3$ are $\breve{\rho}_2$-connected, $\mathbf{x}_2$, $\mathbf{x}_4$ are $\breve{\rho}_2$-connected, and $\mathbf{x}_1$, $\mathbf{x}_4$ are $\breve{\rho}_2$-connected.

Definition 3.9. Let $\mathfrak{P} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_l)$ be a $\breve{\rho}_i$-path of an LDFGS $\hat{\mathscr{G}} = (\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2, ..., \breve{\rho}_k)$ with underlying GS $\mathscr{G}$. Then, the strength of the $\breve{\rho}_i$-path $\mathfrak{P}$, is denoted and defined as:

$$St(\mathfrak{P}) = \Big(\big\langle \varkappa^m_{St(\mathfrak{P})}, \varkappa^n_{St(\mathfrak{P})}\big\rangle, \big\langle \alpha_{St(\mathfrak{P})}, \beta_{St(\mathfrak{P})}\big\rangle\Big), \tag{16}$$

where

$$\left.\begin{aligned} \varkappa^m_{St(\mathfrak{P})} &= \bigwedge_{j=1}^{k} \varkappa^m_{\breve{\rho}_i}(\mathbf{x}_{j-1}, \mathbf{x}_j), \varkappa^n_{St(\mathfrak{P})} = \bigvee_{j=1}^{k} \varkappa^n_{\breve{\rho}_i}(\mathbf{x}_{j-1}, \mathbf{x}_j) \\ \alpha_{St(\mathfrak{P})} &= \bigwedge_{j=1}^{k} \alpha_{\breve{\rho}_i}(\mathbf{x}_{j-1}, \mathbf{x}_j), \beta_{St(\mathfrak{P})} = \bigvee_{j=1}^{k} \beta_{\breve{\rho}_i}(\mathbf{x}_{j-1}, \mathbf{x}_j) \end{aligned}\right\} \tag{17}$$

for $i = 1, 2, ..., k$.

Example 3.10. (Continued from Example 3.6) We have seen that $(\mathbf{x}_3, \mathbf{x}_1, \mathbf{x}_2)$, and $(\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_4)$ are $\breve{\rho}_1$-paths. And, $(\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)$, $(\mathbf{x}_1, \mathbf{x}_4, \mathbf{x}_2)$, and $(\mathbf{x}_1, \mathbf{x}_4, \mathbf{x}_2, \mathbf{x}_3)$ are $\breve{\rho}_2$-paths. We can calculate their strengths as follows:

Strength of $\breve{\rho}_1$-path $\mathfrak{P}_1 = (\mathbf{x}_3, \mathbf{x}_1, \mathbf{x}_2)$:

$$\begin{aligned} \varkappa^m_{St(\mathfrak{P}_1)} &= \wedge_{j=2}^{3} \varkappa^m_{\breve{\rho}_1}(\mathbf{x}_{j-1}, \mathbf{x}_j) = \varkappa^m_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_2) \wedge \varkappa^m_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_3) = 0.4 \wedge 0.3 = 0.3 \\ \varkappa^n_{St(\mathfrak{P}_1)} &= \vee_{j=2}^{3} \varkappa^n_{\breve{\rho}_1}(\mathbf{x}_{j-1}, \mathbf{x}_j) = \varkappa^n_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_2) \vee \varkappa^n_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_3) = 0.4 \vee 0.6 = 0.6 \\ \alpha_{St(\mathfrak{P}_1)} &= \wedge_{j=2}^{3} \alpha_{\breve{\rho}_1}(\mathbf{x}_{j-1}, \mathbf{x}_j) = \alpha_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_2) \wedge \alpha_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_3) = 0.2 \wedge 0.2 = 0.2 \\ \beta_{St(\mathfrak{P}_1)} &= \vee_{j=2}^{3} \beta_{\breve{\rho}_1}(\mathbf{x}_{j-1}, \mathbf{x}_j) = \beta_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_2) \vee \beta_{\breve{\rho}_1}(\mathbf{x}_1, \mathbf{x}_3) = 0.3 \vee 0.3 = 0.3 \end{aligned}$$

So, $St(\mathfrak{P}_1) = (\langle 0.3, 0.6\rangle, \langle 0.2, 0.3\rangle)$. Similarly, we can calculate strength of $\breve{\rho}_1$-path $\mathfrak{P}_2 = (\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_4)$ which is given by $St(\mathfrak{P}_2) = (\langle 0.3, 0.6\rangle, \langle 0.2, 0.3\rangle)$, strength of $\breve{\rho}_2$-path $\mathfrak{P}_3 = (\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)$ is $St(\mathfrak{P}_3) = (\langle 0.3, 0.5\rangle, \langle 0.2, 0.4\rangle)$ and strength of $\breve{\rho}_2$-path $\mathfrak{P}_3 = (\mathbf{x}_1, \mathbf{x}_4, \mathbf{x}_2, \mathbf{x}_3)$ is $St(\mathfrak{P}_4) = (\langle 0.3, 0.5\rangle, \langle 0.2, 0.4\rangle)$.

Definition 3.11. Let $\breve{\mathscr{G}} = (\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2, ..., \breve{\rho}_n)$ be an LDFGS of GS $\mathscr{G}$. Then, $\breve{\rho}_i$-strength of connectedness of any two vertices $\mathbf{x}_1, \mathbf{x}_2$ is denoted and defined as:

$$\big(\breve{\rho}_i\big)^{\infty}(\mathbf{x}_1, \mathbf{x}_2) = \bigg(\Big\langle \big(\varkappa^m_{\breve{\rho}_i}\big)^{\infty}(\mathbf{x}_1, \mathbf{x}_2), \big(\varkappa^n_{\breve{\rho}_i}\big)^{\infty}(\mathbf{x}_1, \mathbf{x}_2)\Big\rangle, \Big\langle \big(\alpha_{\breve{\rho}_i}\big)^{\infty}(\mathbf{x}_1, \mathbf{x}_2), \big(\beta_{\breve{\rho}_i}\big)^{\infty}(\mathbf{x}_1, \mathbf{x}_2)\Big\rangle\bigg), \tag{18}$$

where

$$\left.\begin{aligned}\left(\varkappa^m_{\breve{\rho}_i}\right)^\infty(\mathbf{x}_1,\mathbf{x}_2) = \bigvee_{j=1}^{\infty}\left(\varkappa^m_{\breve{\rho}_i}\right)^j(\mathbf{x}_1,\mathbf{x}_2), \text{ and } \left(\varkappa^n_{\breve{\rho}_i}\right)^\infty(\mathbf{x}_1,\mathbf{x}_2) = \bigwedge_{j=1}^{\infty}\left(\varkappa^n_{\breve{\rho}_i}\right)^j(\mathbf{x}_1,\mathbf{x}_2).\\ \left(\alpha_{\breve{\rho}_i}\right)^\infty(\mathbf{x}_1,\mathbf{x}_2) = \bigvee_{j=1}^{\infty}\left(\alpha_{\breve{\rho}_i}\right)^j(\mathbf{x}_1,\mathbf{x}_2), \text{ and } \left(\beta_{\breve{\rho}_i}\right)^\infty(\mathbf{x}_1,\mathbf{x}_2) = \bigwedge_{j=1}^{\infty}\left(\beta_{\breve{\rho}_i}\right)^j(\mathbf{x}_1,\mathbf{x}_2).\end{aligned}\right\} \tag{19}$$

Here, $\left(\breve{\rho}_i\right)^j(\mathbf{x}_1,\mathbf{x}_2) = \left(\left\langle\left(\varkappa^m_{\breve{\rho}_i}\right)^j(\mathbf{x}_1,\mathbf{x}_2),\left(\varkappa^n_{\breve{\rho}_i}\right)^j(\mathbf{x}_1,\mathbf{x}_2)\right\rangle,\left\langle\left(\alpha_{\breve{\rho}_i}\right)^j(\mathbf{x}_1,\mathbf{x}_2),\left(\beta_{\breve{\rho}_i}\right)^j(\mathbf{x}_1,\mathbf{x}_2)\right\rangle\right) =$ $\left((\breve{\rho}_i)^{j-1}\circ\breve{\rho}_i\right)(\mathbf{x}_1,\mathbf{x}_2)$, and the composition $\circ$ among any two LDFRs is provided in Definition 2.10.

Example 3.12. In Example 3.2, we can evaluate the terms as defined in above definition as follows:

$$\left(\varkappa^m_{\breve{\rho}_2}\right)^\infty(\mathbf{x}_1,\mathbf{x}_2) = \bigvee_z\left\{\varkappa^m_{\breve{\rho}_2}(\mathbf{x}_1,\mathbf{z})\wedge\varkappa^m_{\breve{\rho}_2}(\mathbf{z},\mathbf{x}_2)\right\} = \vee\{\varkappa^m_{\breve{\rho}_2}(\mathbf{x}_1,\mathbf{x}_4)\wedge\varkappa^m_{\breve{\rho}_2}(\mathbf{x}_4,\mathbf{x}_2)\} = 0.4\wedge 0.6 = 0.4$$

$$\left(\varkappa^n_{\breve{\rho}_2}\right)^\infty(\mathbf{x}_1,\mathbf{x}_2) = \bigwedge_z\{\varkappa^n_{\breve{\rho}_2}(\mathbf{x}_1,\mathbf{z})\vee\varkappa^n_{\breve{\rho}_2}(\mathbf{z},\mathbf{x}_2)\} = \wedge\{\varkappa^n_{\breve{\rho}_2}(\mathbf{x}_1,\mathbf{x}_4)\vee\varkappa^n_{\breve{\rho}_2}(\mathbf{x}_4,\mathbf{x}_2)\} = 0.3\vee 0.4 = 0.4$$

$$\left(\alpha_{\breve{\rho}_2}\right)^\infty(\mathbf{x}_1,\mathbf{x}_2) = \bigvee_z\{\alpha_{\breve{\rho}_2}(\mathbf{x}_1,\mathbf{z})\wedge\alpha_{\breve{\rho}_2}(\mathbf{z},\mathbf{x}_2)\} = \vee\{\alpha_{\breve{\rho}_2}(\mathbf{x}_1,\mathbf{x}_4)\wedge\alpha_{\breve{\rho}_2}(\mathbf{x}_4,\mathbf{x}_2)\} = 0.2\wedge 0.3 = 0.2$$

$$\left(\beta_{\breve{\rho}_2}\right)^\infty(\mathbf{x}_1,\mathbf{x}_2) = \bigwedge_z\{\beta_{\breve{\rho}_2}(\mathbf{x}_1,\mathbf{z})\vee\beta_{\breve{\rho}_2}(\mathbf{z},\mathbf{x}_2)\} = \wedge\{\beta_{\breve{\rho}_2}(\mathbf{x}_1,\mathbf{x}_4)\vee\beta_{\breve{\rho}_2}(\mathbf{x}_4,\mathbf{x}_2)\} = 0.3\vee 0.4 = 0.4$$

So, $\left(\breve{\rho}_2\right)^\infty(\mathbf{x}_1,\mathbf{x}_2) = \left(\langle 0.4,0.4\rangle,\langle 0.2,0.4\rangle\right)$. Similarly, we can find $\left(\breve{\rho}_2\right)^\infty(\mathbf{x}_1,\mathbf{x}_3) = \left(\langle 0.3,0.5\rangle,\langle 0.2,0.4\rangle\right)$ and $\left(\breve{\rho}_1\right)^\infty(\mathbf{x}_2,\mathbf{x}_3) = \left(\langle 0.3,0.6\rangle,\langle 0.2,0.3\rangle\right)$.

Definition 3.13. Let $\breve{\mathscr{G}} = \left(\mathfrak{L},\breve{\rho}_1,\breve{\rho}_2,...,\breve{\rho}_n\right)$ be an LDFGS of $\mathscr{G}$. Then, $\breve{\mathscr{G}}$ is called connected LDFGS, if each of its two vertices $\mathbf{x}_1,\mathbf{x}_2$ are $\breve{\rho}_i$-connected, that is, $\left(\breve{\rho}_i\right)^\infty(\mathbf{x},\mathbf{y}) > 0$ for any $\mathbf{x},\mathbf{y}\in\mathscr{V}$, and $i = 1,2,\ldots,n$.

Example 3.14. In Example 3.12, it can be easily observed that $\left(\breve{\rho}_i\right)^\infty(\mathbf{x}_j,\mathbf{x}_k) > 0$ for all $i = 1,2$, and $j,k = 1,2,3,4$. Hence, this LDFGS is connected.

Definition 3.15. Let $\breve{\mathscr{G}} = \left(\mathfrak{L},\breve{\rho}_1,\breve{\rho}_2,...,\breve{\rho}_n\right)$ be an LDFGS with underlying GS $\mathscr{G}$. Then $\breve{\rho}_i$-degree of a vertex $\mathbf{x}\in\mathscr{V}$ is denoted and defined by

$$\mathbb{D}_{\breve{\rho}_i}(\mathbf{x}) = \left(\left\langle\varkappa^m_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}),\varkappa^n_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x})\right\rangle,\left\langle\alpha_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}),\beta_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x})\right\rangle\right), \tag{20}$$

where

$$\left.\begin{aligned}\varkappa^m_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}) = \sum_{i=1,\mathbf{x}\neq\mathbf{y},(\mathbf{x},\mathbf{y})\in\mathscr{E}_i}^{k}\varkappa^m_{\breve{\rho}_i}(\mathbf{x},\mathbf{y}), \varkappa^n_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}) = \sum_{i=1,\mathbf{x}\neq\mathbf{y},(\mathbf{x},\mathbf{y})\in\mathscr{E}_i}^{k}\varkappa^n_{\breve{\rho}_i}(\mathbf{x},\mathbf{y}),\\ \alpha_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}) = \sum_{i=1,\mathbf{x}\neq\mathbf{y},(\mathbf{x},\mathbf{y})\in\mathscr{E}_i}^{k}\alpha_{\breve{\rho}_i}(\mathbf{x},\mathbf{y}), \beta_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}) = \sum_{i=1,\mathbf{x}\neq\mathbf{y},(\mathbf{x},\mathbf{y})\in\mathscr{E}_i}^{k}\beta_{\breve{\rho}_i}(\mathbf{x},\mathbf{y}).\end{aligned}\right\} \tag{21}$$

Definition 3.16. Let $\breve{\mathscr{G}} = \left(\mathfrak{L},\breve{\rho}_1,\breve{\rho}_2,...,\breve{\rho}_n\right)$ be an LDFGS with underlying GS $\mathscr{G}$. Then the degree of the vertex $\mathbf{x}\in\mathscr{V}$ is denoted and characterized as:

$$\mathbb{D}(\mathbf{x}) = \sum_{i=1}^{k}\mathbb{D}_{\breve{\rho}_i}(\mathbf{x}) = \left(\left\langle\varkappa^m_{\mathbb{D}}(\mathbf{x}),\varkappa^n_{\mathbb{D}}(\mathbf{x})\right\rangle,\left\langle\alpha_{\mathbb{D}}(\mathbf{x}),\beta_{\mathbb{D}}(\mathbf{x})\right\rangle\right), \tag{22}$$

where

$$\varkappa^m_{\mathbb{D}}(\mathbf{x}) = \sum_{i=1}^{k}\varkappa^m_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}), \varkappa^n_{\mathbb{D}}(\mathbf{x}) = \sum_{i=1}^{k}\varkappa^n_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}), \alpha_{\mathbb{D}}(\mathbf{x}) = \sum_{i=1}^{k}\alpha_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}), \beta_{\mathbb{D}}(\mathbf{x}) = \sum_{i=1}^{k}\beta_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}). \tag{23}$$

Example 3.17. If we revisit Example 3.2, then according to Definition 3.15, $\breve{\rho}_i$-degrees of vertices can be calculate as follows:

$$\varkappa^m_{\mathbb{D}_{\breve{\rho}_1}}(\mathbf{x}_1) = \sum_{\mathbf{x}_1\neq\mathbf{y},(\mathbf{x}_1,\mathbf{y})\in\mathscr{E}_1} \varkappa^m_{\breve{\rho}_1}(\mathbf{x}_1,\mathbf{y}) = \varkappa^m_{\breve{\rho}_1}(\mathbf{x}_1,\mathbf{x}_3) + \varkappa^m_{\breve{\rho}_1}(\mathbf{x}_1,\mathbf{x}_2) = 0.3+0.4 = 0.7$$
$$\varkappa^n_{\mathbb{D}_{\breve{\rho}_1}}(\mathbf{x}_1) = \sum_{\mathbf{x}_1\neq\mathbf{y},(\mathbf{x}_1,\mathbf{y})\in\mathscr{E}_1} \varkappa^n_{\breve{\rho}_1}(\mathbf{x}_1,\mathbf{y}) = \varkappa^n_{\breve{\rho}_1}(\mathbf{x}_1,\mathbf{x}_3) + \varkappa^n_{\breve{\rho}_1}(\mathbf{x}_1,\mathbf{x}_2) = 0.6+0.4 = 1$$
$$\alpha_{\mathbb{D}_{\breve{\rho}_1}}(\mathbf{x}_1) = \sum_{\mathbf{x}_1\neq\mathbf{y},(\mathbf{x}_1,\mathbf{y})\in\mathscr{E}_1} \alpha_{\breve{\rho}_1}(\mathbf{x}_1,\mathbf{y}) = \alpha_{\breve{\rho}_1}(\mathbf{x}_1,\mathbf{x}_3) + \alpha_{\breve{\rho}_1}(\mathbf{x}_1,\mathbf{x}_2) = 0.2+0.2 = 0.4$$
$$\beta_{\mathbb{D}_{\breve{\rho}_1}}(\mathbf{x}_1) = \sum_{\mathbf{x}_1\neq\mathbf{y},(\mathbf{x}_1,\mathbf{y})\in\mathscr{E}_1} \beta_{\breve{\rho}_1}(\mathbf{x}_1,\mathbf{y}) = \beta_{\breve{\rho}_1}(\mathbf{x}_1,\mathbf{x}_3) + \beta_{\breve{\rho}_1}(\mathbf{x}_1,\mathbf{x}_2) = 0.3+0.3 = 0.6$$

So, $\mathbb{D}_{\breve{\rho}_1}(\mathbf{x}_1) = \big(\langle 0.7, 1\rangle, \langle 0.4, 0.6\rangle\big)$. Similarly, we can evaluate $\breve{\rho}_1$- and $\breve{\rho}_2$-degrees of all $\mathbf{x} \in \mathscr{V}$ which are displayed in TABLES 4 and 5, respectively.

Table 4: $\mathbb{D}_{\breve{\rho}_1}$

$\mathscr{V}$	$\Big(\big\langle \varkappa^m_{\mathbb{D}_{\breve{\rho}_1}}(\mathbf{x}), \varkappa^n_{\mathbb{D}_{\breve{\rho}_1}}(\mathbf{x})\big\rangle, \big\langle \alpha_{\mathbb{D}_{\breve{\rho}_1}}(\mathbf{x}), \beta_{\mathbb{D}_{\breve{\rho}_1}}(\mathbf{x})\big\rangle\Big)$
$\mathbf{x}_1$	$\big(\langle 0.7, 1\rangle, \langle 0.4, 0.6\rangle\big)$
$\mathbf{x}_2$	$\big(\langle 0.4, 0.4\rangle, \langle 0.2, 0.3\rangle\big)$
$\mathbf{x}_3$	$\big(\langle 0.7, 1\rangle, \langle 0.6, 0.5\rangle\big)$
$\mathbf{x}_4$	$\big(\langle 0.4, 0.5\rangle, \langle 0.4, 0.6\rangle\big)$

Table 5: $\mathbb{D}_{\breve{\rho}_2}$

$\mathscr{V}$	$\Big(\big\langle \varkappa^m_{\mathbb{D}_{\breve{\rho}_2}}(\mathbf{x}), \varkappa^n_{\mathbb{D}_{\breve{\rho}_2}}(\mathbf{x})\big\rangle, \big\langle \alpha_{\mathbb{D}_{\breve{\rho}_2}}(\mathbf{x}), \beta_{\mathbb{D}_{\breve{\rho}_2}}(\mathbf{x})\big\rangle\Big)$
$\mathbf{x}_1$	$\big(\langle 0.9, 0.9\rangle, \langle 0.5, 0.7\rangle\big)$
$\mathbf{x}_2$	$\big(\langle 0.9, 0.9\rangle, \langle 0.5, 0.7\rangle\big)$
$\mathbf{x}_3$	$\big(\langle 0.3, 0.5\rangle, \langle 0.2, 0.3\rangle\big)$
$\mathbf{x}_4$	$\big(\langle 1, 0.7\rangle, \langle 0.5, 0.7\rangle\big)$

Now, in the light of Definition 3.16, we calculate the degrees $\mathbb{D}(\mathbf{x}) = \sum_{i=1}^{k} \mathbb{D}_{\breve{\rho}_i}(\mathbf{x})$ as follows:

$$\mathbb{D}(\mathbf{x}_1) = \mathbb{D}_{\breve{\rho}_1}(\mathbf{x}_1) + \mathbb{D}_{\breve{\rho}_2}(\mathbf{x}_1) = \big(\langle 0.7, 1\rangle, \langle 0.4, 0.6\rangle\big) + \big(\langle 0.9, 0.9\rangle, \langle 0.5, 0.7\rangle\big) = \big(\langle 1.6, 1.9\rangle, \langle 0.9, 1.3\rangle\big)$$
$$\mathbb{D}(\mathbf{x}_2) = \mathbb{D}_{\breve{\rho}_1}(\mathbf{x}_2) + \mathbb{D}_{\breve{\rho}_2}(\mathbf{x}_2) = \big(\langle 0.4, 0.4\rangle, \langle 0.2, 0.3\rangle\big) + \big(\langle 0.9, 0.9\rangle, \langle 0.5, 0.7\rangle\big) = \big(\langle 1.3, 1.3\rangle, \langle 0.7, 1\rangle\big)$$
$$\mathbb{D}(\mathbf{x}_3) = \mathbb{D}_{\breve{\rho}_1}(\mathbf{x}_3) + \mathbb{D}_{\breve{\rho}_2}(\mathbf{x}_3) = \big(\langle 0.7, 1\rangle, \langle 0.6, 0.5\rangle\big) + \big(\langle 0.3, 0.5\rangle, \langle 0.2, 0.3\rangle\big) = \big(\langle 1, 1.6\rangle, \langle 0.8, 0.8\rangle\big)$$
$$\mathbb{D}(\mathbf{x}_4) = \mathbb{D}_{\breve{\rho}_1}(\mathbf{x}_4) + \mathbb{D}_{\breve{\rho}_2}(\mathbf{x}_4) = \big(\langle 0.4, 0.5\rangle, \langle 0.4, 0.6\rangle\big) + \big(\langle 1, 0.7\rangle, \langle 0.5, 0.7\rangle\big) = \big(\langle 1.4, 1.2\rangle, \langle 0.9, 0.9\rangle\big)$$

which can be also be seen in TABLE 6.

Table 6: $\mathbb{D}(\mathbf{x})$

$\mathscr{V}$	$\big(\langle \varkappa^m_{\mathbb{D}}(\mathbf{x}), \varkappa^n_{\mathbb{D}}(\mathbf{x})\rangle, \langle \alpha_{\mathbb{D}}(\mathbf{x}), \beta_{\mathbb{D}}(\mathbf{x})\rangle\big)$
$\mathbf{x}_1$	$\big(\langle 1.6, 1.9\rangle, \langle 0.9, 1.3\rangle\big)$
$\mathbf{x}_2$	$\big(\langle 1.3, 1.3\rangle, \langle 0.7, 1\rangle\big)$
$\mathbf{x}_3$	$\big(\langle 1, 1.6\rangle, \langle 0.8, 0.8\rangle\big)$
$\mathbf{x}_4$	$\big(\langle 1.4, 1.2\rangle, \langle 0.9, 0.9\rangle\big)$

Definition 3.18. Let $\breve{\mathscr{G}} = \big(\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2, ..., \breve{\rho}_n\big)$ be an LDFGS with underlying GS $\mathscr{G}$. Then total $\breve{\rho}_i$-degree of a vertex $\mathbf{x} \in \mathscr{V}$ is denoted and defined as:

$$\mathbb{TD}_{\breve{\rho}_i}(\mathbf{x}) = \mathbb{D}_{\breve{\rho}_i}(\mathbf{x}) + \mathfrak{L}(\mathbf{x}) = \Big(\big\langle \varkappa^m_{\mathbb{TD}_{\breve{\rho}_i}}(\mathbf{x}), \varkappa^n_{\mathbb{TD}_{\breve{\rho}_i}}(\mathbf{x})\big\rangle, \big\langle \alpha_{\mathbb{TD}_{\breve{\rho}_i}}(\mathbf{x}), \beta_{\mathbb{TD}_{\breve{\rho}_i}}(\mathbf{x})\big\rangle\Big), \tag{24}$$

where

$$\left.\begin{aligned} \varkappa^m_{\mathbb{TD}_{\breve{\rho}_i}}(\mathbf{x}) = \varkappa^m_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}) + \varkappa^m_{\mathfrak{L}}(\mathbf{x}), \varkappa^n_{\mathbb{TD}_{\breve{\rho}_i}}(\mathbf{x}) = \varkappa^n_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}) + \varkappa^n_{\mathfrak{L}}(\mathbf{x}), \\ \alpha_{\mathbb{TD}_{\breve{\rho}_i}}(\mathbf{x}) = \alpha_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}) + \alpha_{\mathfrak{L}}(\mathbf{x}), \beta_{\mathbb{TD}_{\breve{\rho}_i}}(\mathbf{x}) = \beta_{\mathbb{D}_{\breve{\rho}_i}}(\mathbf{x}) + \beta_{\mathfrak{L}}(\mathbf{x}). \end{aligned}\right\} \tag{25}$$

Definition 3.19. Let $\breve{\mathscr{G}} = \big(\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2, ..., \breve{\rho}_n\big)$ be an LDFGS with underlying GS $\mathscr{G}$. Then the total degree of the vertex $\mathbf{x} \in \mathscr{V}$ is denoted and defined as:

$$\mathbb{TD}(\mathbf{x}) = \sum_{i=1}^{k} \mathbb{TD}_{\breve{\rho}_i}(\mathbf{x}) = \Big(\big\langle \varkappa^m_{\mathbb{TD}}(\mathbf{x}), \varkappa^n_{\mathbb{TD}}(\mathbf{x})\big\rangle, \big\langle \alpha_{\mathbb{TD}}(\mathbf{x}), \beta_{\mathbb{TD}}(\mathbf{x})\big\rangle\Big), \tag{26}$$

where

$$\varkappa^m_{\mathbb{TD}}(\mathbf{x}) = \sum_{i=1}^{k} \varkappa^m_{\mathbb{TD}_{\breve{\rho}_i}}(\mathbf{x}), \varkappa^n_{\mathbb{TD}}(\mathbf{x}) = \sum_{i=1}^{k} \varkappa^n_{\mathbb{TD}_{\breve{\rho}_i}}(\mathbf{x}), \alpha_{\mathbb{TD}}(\mathbf{x}) = \sum_{i=1}^{k} \alpha_{\mathbb{TD}_{\breve{\rho}_i}}(\mathbf{x}), \beta_{\mathbb{TD}}(\mathbf{x}) = \sum_{i=1}^{k} \beta_{\mathbb{TD}_{\breve{\rho}_i}}(\mathbf{x}). \tag{27}$$

Example 3.20. (Continued from Examples 3.2 and 3.17) We can calculate the $\breve{\rho}_i$-degrees for each vertex $\mathbf{x} \in \mathscr{V}$ by using Definition 3.18 as follows:

$$\begin{aligned}
\mathbb{TD}_{\breve{\rho}_1}(\mathbf{x}_1) = \mathbb{D}_{\breve{\rho}_1}(\mathbf{x}_1) + \mathfrak{L}(\mathbf{x}_1) = \big(\langle 0.7, 1\rangle, \langle 0.4, 0.6\rangle\big) + \big(\langle 0.4, 0.3\rangle, \langle 0.2, 0.1\rangle\big) = \big(\langle 1.1, 1.3\rangle, \langle 0.6, 0.7\rangle\big), \\
\mathbb{TD}_{\breve{\rho}_1}(\mathbf{x}_2) = \mathbb{D}_{\breve{\rho}_1}(\mathbf{x}_2) + \mathfrak{L}(\mathbf{x}_2) = \big(\langle 0.4, 0.4\rangle, \langle 0.2, 0.3\rangle\big) + \big(\langle 0.6, 0.2\rangle, \langle 0.3, 0.2\rangle\big) = \big(\langle 1, 0.6\rangle, \langle 0.5, 0.5\rangle\big), \\
\mathbb{TD}_{\breve{\rho}_1}(\mathbf{x}_3) = \mathbb{D}_{\breve{\rho}_1}(\mathbf{x}_3) + \mathfrak{L}(\mathbf{x}_3) = \big(\langle 0.7, 1.1\rangle, \langle 0.6, 0.5\rangle\big) + \big(\langle 0.4, 0.5\rangle, \langle 0.4, 0.2\rangle\big) = \big(\langle 1.1, 1.6\rangle, \langle 1, 0.7\rangle\big), \\
\mathbb{TD}_{\breve{\rho}_1}(\mathbf{x}_4) = \mathbb{D}_{\breve{\rho}_1}(\mathbf{x}_4) + \mathfrak{L}(\mathbf{x}_4) = \big(\langle 0.4, 0.5\rangle, \langle 0.4, 0.2\rangle\big) + \big(\langle 0.7, 0.3\rangle, \langle 0.6, 0.2\rangle\big) = \big(\langle 1.1, 0.8\rangle, \langle 1, 0.4\rangle\big),
\end{aligned}$$

which is also demonstrated in TABLE 7. Also, $\breve{\rho}_2$-degrees for each vertex $\mathbf{x} \in \mathscr{V}$ are calculated in TABLE 8. Now, according of Definition 3.19, $\mathbb{TD}(\mathbf{x}) = \sum_{i=1}^{k} \mathbb{TD}_{\breve{\rho}_i}(\mathbf{x})$ are calculated as follows:

$$\begin{aligned}
\mathbb{TD}(\mathbf{x}_1) = \mathbb{TD}_{\breve{\rho}_1}(\mathbf{x}_1) + \mathbb{TD}_{\breve{\rho}_2}(\mathbf{x}_1) = \big(\langle 1.1, 1.3\rangle, \langle 0.6, 0.7\rangle\big) + \big(\langle 1.3, 1.2\rangle, \langle 0.7, 0.8\rangle\big) = \big(\langle 2.4, 2.5\rangle, \langle 1.3, 1.5\rangle\big), \\
\mathbb{TD}(\mathbf{x}_2) = \mathbb{TD}_{\breve{\rho}_1}(\mathbf{x}_2) + \mathbb{TD}_{\breve{\rho}_2}(\mathbf{x}_2) = \big(\langle 1, 0.6\rangle, \langle 0.5, 0.5\rangle\big) + \big(\langle 1.5, 1.1\rangle, \langle 0.8, 0.9\rangle\big) = \big(\langle 2.5, 1.7\rangle, \langle 1.3, 1.4\rangle\big), \\
\mathbb{TD}(\mathbf{x}_3) = \mathbb{TD}_{\breve{\rho}_1}(\mathbf{x}_3) + \mathbb{TD}_{\breve{\rho}_2}(\mathbf{x}_3) = \big(\langle 1.1, 1.6\rangle, \langle 1, 0.7\rangle\big) + \big(\langle 0.7, 1\rangle, \langle 0.6, 0.5\rangle\big) = \big(\langle 1.8, 2.6\rangle, \langle 1.6, 1.2\rangle\big), \\
\mathbb{TD}(\mathbf{x}_4) = \mathbb{TD}_{\breve{\rho}_1}(\mathbf{x}_4) + \mathbb{TD}_{\breve{\rho}_2}(\mathbf{x}_4) = \big(\langle 1.1, 1.6\rangle, \langle 1, 0.7\rangle\big) + \big(\langle 1.7, 1\rangle, \langle 1.1, 0.9\rangle\big) = \big(\langle 2.8, 2.6\rangle, \langle 2.1, 1.6\rangle\big),
\end{aligned}$$

which is also shown in TABLE 9.

Table 7: $\mathbb{TD}_{\breve{\rho}_1}$

$\mathscr{V}$	$\left(\left\langle \varkappa^m_{\mathbb{TD}_{\breve{\rho}_1}}(\mathbf{x}), \varkappa^n_{\mathbb{TD}_{\breve{\rho}_1}}(\mathbf{x})\right\rangle, \left\langle \alpha_{\mathbb{TD}_{\breve{\rho}_1}}(\mathbf{x}), \beta_{\mathbb{TD}_{\breve{\rho}_1}}(\mathbf{x})\right\rangle\right)$
$\mathbf{x}_1$	$(\langle 1.1, 1.3\rangle, \langle 0.6, 0.7\rangle)$
$\mathbf{x}_2$	$(\langle 1, 0.6\rangle, \langle 0.5, 0.5\rangle)$
$\mathbf{x}_3$	$(\langle 1.1, 1.6\rangle, \langle 1, 0.7\rangle)$
$\mathbf{x}_4$	$(\langle 1.1, 0.8\rangle, \langle 1, 0.4\rangle)$

Table 8: $\mathbb{TD}_{\breve{\rho}_2}$

$\mathscr{V}$	$\left(\left\langle \varkappa^m_{\mathbb{TD}_{\breve{\rho}_2}}(\mathbf{x}), \varkappa^n_{\mathbb{TD}_{\breve{\rho}_2}}(\mathbf{x})\right\rangle, \left\langle \alpha_{\mathbb{TD}_{\breve{\rho}_2}}(\mathbf{x}), \beta_{\mathbb{TD}_{\breve{\rho}_2}}(\mathbf{x})\right\rangle\right)$
$\mathbf{x}_1$	$(\langle 1.3, 1.2\rangle, \langle 0.7, 0.8\rangle)$
$\mathbf{x}_2$	$(\langle 1.5, 1.1\rangle, \langle 0.8, 0.9\rangle)$
$\mathbf{x}_3$	$(\langle 0.7, 1\rangle, \langle 0.6, 0.5\rangle)$
$\mathbf{x}_4$	$(\langle 1.7, 1\rangle, \langle 1.1, 0.9\rangle)$

Table 9: $\mathbb{TD}$

$\mathscr{V}$	$\left(\left\langle \varkappa^m_{\mathbb{TD}}(\mathbf{x}), \varkappa^n_{\mathbb{TD}}(\mathbf{x})\right\rangle, \left\langle \alpha_{\mathbb{TD}}(\mathbf{x}), \beta_{\mathbb{TD}}(\mathbf{x})\right\rangle\right)$
$\mathbf{x}_1$	$(\langle 2.4, 2.5\rangle, \langle 1.3, 1.5\rangle)$
$\mathbf{x}_2$	$(\langle 2.5, 1.7\rangle, \langle 1.3, 1.4\rangle)$
$\mathbf{x}_3$	$(\langle 1.8, 2.6\rangle, \langle 1.6, 1.2\rangle)$
$\mathbf{x}_4$	$(\langle 2.8, 2.6\rangle, \langle 2.1, 1.6\rangle)$

Definition 3.21. Let $\breve{\mathscr{G}} = (\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2, ..., \breve{\rho}_n)$ be an LDFGS with underlying GS $\mathscr{G}$. Then order of $\breve{\mathscr{G}}$ is denoted and described as follows:

$$\mathbb{O}(\breve{\mathscr{G}}) = \left(\left\langle \sum_{\mathbf{x}\in\mathscr{V}} \varkappa^m_{\mathfrak{L}}(\mathbf{x}), \sum_{\mathbf{x}\in\mathscr{V}} \varkappa^n_{\mathfrak{L}}(\mathbf{x})\right\rangle, \left\langle \sum_{\mathbf{x}\in\mathscr{V}} \alpha_{\mathfrak{L}}(\mathbf{x}), \sum_{\mathbf{x}\in\mathscr{V}} \beta_{\mathfrak{L}}(\mathbf{x})\right\rangle\right). \tag{28}$$

Example 3.22. If we consider the Example 3.2, then we can find $\mathbb{O}(\breve{\mathscr{G}})$ as follows:

$$\sum_{\mathbf{x}\in\mathscr{V}} \varkappa^m_{\mathfrak{L}}(\mathbf{x}) = 0.4 + 0.6 + 0.4 + 0.7 = 2,$$

$$\sum_{\mathbf{x}\in\mathscr{V}} \varkappa^n_{\mathfrak{L}}(\mathbf{x}) = 0.3 + 0.2 + 0.5 + 0.3 = 1.3,$$

$$\sum_{\mathbf{x}\in\mathscr{V}} \alpha_{\mathfrak{L}}(\mathbf{x}) = 0.2 + 0.3 + 0.4 + 0.6 = 1.5,$$

$$\sum_{\mathbf{x}\in\mathscr{V}} \beta_{\mathfrak{L}}(\mathbf{x}) = 0.1 + 0.2 + 0.2 + 0.2 = 0.7.$$

Hence, $\mathbb{O}(\breve{\mathscr{G}}) = \big(\langle 2, 1.3\rangle, \langle 1.5, 0.7\rangle\big)$.

Definition 3.23. Let $\breve{\mathscr{G}} = \big(\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2, ..., \breve{\rho}_n\big)$ be an LDFGS with underlying GS $\mathscr{G}$. The $\breve{\rho}_i$-size of $\breve{\mathscr{G}}$ is denoted and postulated as:

$$\mathbb{S}_{\breve{\rho}_i}(\breve{\mathscr{G}}) = \Big(\big\langle \varkappa^m_{\mathbb{S}_{\breve{\rho}_i}}, \varkappa^n_{\mathbb{S}_{\breve{\rho}_i}}\big\rangle, \big\langle \alpha_{\mathbb{S}_{\breve{\rho}_i}}, \beta_{\mathbb{S}_{\breve{\rho}_i}}\big\rangle\Big), \tag{29}$$

where

$$\varkappa^m_{\mathbb{S}_{\breve{\rho}_i}} = \sum_{(\mathbf{x},\mathbf{y})\in\mathscr{E}_i} \varkappa^m_{\breve{\rho}_i}(\mathbf{x},\mathbf{y}), \varkappa^n_{\mathbb{S}_{\breve{\rho}_i}} = \sum_{(\mathbf{x},\mathbf{y})\in\mathscr{E}_i} \varkappa^n_{\breve{\rho}_i}(\mathbf{x},\mathbf{y}), \alpha_{\mathbb{S}_{\breve{\rho}_i}} = \sum_{(\mathbf{x},\mathbf{y})\in\mathscr{E}_i} \alpha_{\breve{\rho}_i}(\mathbf{x},\mathbf{y}), \beta_{\mathbb{S}_{\breve{\rho}_i}} = \sum_{(\mathbf{x},\mathbf{y})\in\mathscr{E}_i} \beta_{\breve{\rho}_i}(\mathbf{x},\mathbf{y}). \tag{30}$$

Moreover, the size of $\breve{\mathscr{G}}$ is denoted and characterized as:

$$\mathbb{S}(\breve{\mathscr{G}}) = \sum_{i=1}^{n} \mathbb{S}_{\breve{\rho}_i}(\breve{\mathscr{G}}). \tag{31}$$

Example 3.24. If we revisit Example 3.2, we have

$$\varkappa^m_{\mathbb{S}_{\breve{\rho}_1}} = \sum_{(\mathbf{x},\mathbf{y})\in\mathscr{E}_1} \varkappa^m_{\breve{\rho}_1}(\mathbf{x},\mathbf{y}) = 0.4 + 0.3 + 0.4 = 1.1,$$
$$\varkappa^n_{\mathbb{S}_{\breve{\rho}_1}} = \sum_{(\mathbf{x},\mathbf{y})\in\mathscr{E}_1} \varkappa^n_{\breve{\rho}_1}(\mathbf{x},\mathbf{y}) = 0.4 + 0.6 + 0.5 = 1.5,$$
$$\alpha_{\mathbb{S}_{\breve{\rho}_1}} = \sum_{(\mathbf{x},\mathbf{y})\in\mathscr{E}_1} \alpha_{\breve{\rho}_1}(\mathbf{x},\mathbf{y}) = 0.2 + 0.2 + 0.4 = 0.8,$$
$$\beta_{\mathbb{S}_{\breve{\rho}_1}} = \sum_{(\mathbf{x},\mathbf{y})\in\mathscr{E}_1} \beta_{\breve{\rho}_1}(\mathbf{x},\mathbf{y}) = 0.3 + 0.3 + 0.2 = 0.8.$$

Thus, $\mathbb{S}_{\breve{\rho}_1}(\breve{\mathscr{G}}) = \big(\langle \varkappa^m_{\mathbb{S}_{\breve{\rho}_1}}, \varkappa^n_{\mathbb{S}_{\breve{\rho}_1}}\rangle, \langle \alpha_{\mathbb{S}_{\breve{\rho}_1}}, \beta_{\mathbb{S}_{\breve{\rho}_1}}\rangle\big) = \big(\langle 1.1, 1.5\rangle, \langle 0.8, 0.8\rangle\big)$. Similarly, $\mathbb{S}_{\breve{\rho}_2}(\breve{\mathscr{G}}) = \big(\langle 1.3, 1.2\rangle, \langle 0.7, 1\rangle\big)$.

Further, the size of $\breve{\mathscr{G}}$ is calculated as:

$$\mathbb{S}(\breve{\mathscr{G}}) = \mathbb{S}_{\breve{\rho}_1}(\breve{\mathscr{G}}) + \mathbb{S}_{\breve{\rho}_2}(\breve{\mathscr{G}}) = \big(\langle 1.1, 1.5\rangle, \langle 0.8, 0.8\rangle\big) + \big(\langle 1.3, 1.2\rangle, \langle 0.7, 1\rangle\big) = \big(\langle 2.4, 2.7\rangle, \langle 1.8, 1.5\rangle\big).$$

4 Maximal Product of Two Linear Diophantine Fuzzy Graph Structures

In this section, we introduce the notions of maximal product of two LDFGSs, strong LDFGS, degree and $\breve{\rho}_i$-degree of a vertex in maximal product. Furthermore, certain consequences related to these concepts are proved with some useful examples.

Definition 4.1. Let $\breve{\mathscr{G}}_1 = \big(\mathfrak{L}_1, \breve{\rho'}_1, \breve{\rho'}_2, \cdots, \breve{\rho'}_n\big)$ and $\breve{\mathscr{G}}_2 = \big(\mathfrak{L}_2, \breve{\rho''}_1, \breve{\rho''}_2, \cdots, \breve{\rho''}_n\big)$ be two LDFGSs of the GSs $\mathscr{G}_1 = \big(\mathscr{V}_1, \mathscr{E}'_1, \mathscr{E}'_2, \cdots, \mathscr{E}'_n\big)$ and $\mathscr{G}_2 = \big(\mathscr{V}_2, \mathscr{E}''_1, \mathscr{E}''_2, \cdots, \mathscr{E}''_n\big)$, respectively. Then, $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2 = \big(\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2, \cdots, \breve{\rho}_n\big)$ is called maximal LDFGS with underlying crisp GS $\mathscr{G} = \big(\mathscr{V}, \mathscr{E}_1, \mathscr{E}_2, \cdots, \mathscr{E}_n\big)$, where $\mathscr{V} = \mathscr{V}_1 \times \mathscr{V}_2$ and $\mathscr{E}_i = \big\{\big((\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x_2}, \mathbf{y_2})\big) : \mathbf{x}_1 = \mathbf{x}_2, (\mathbf{y}_1, \mathbf{y}_2) \in \mathscr{E}''_i \text{ or } \mathbf{y}_1 = \mathbf{y}_2, (\mathbf{x}_1, \mathbf{x}_2) \in \mathscr{E}'_i\big\}$. LDF vertex set $\mathfrak{L}$ and LDFRs $\breve{\rho}_i$ in maximal product $\breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ are defined as :

$$\begin{aligned}\mathfrak{L} &= \mathfrak{L}_1 * \mathfrak{L}_2\\ &= \Big(\big\langle \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}), \varkappa^n_{\mathfrak{L}_1}(\mathbf{x})\big\rangle, \big\langle \alpha_{\mathfrak{L}_1}(\mathbf{x}), \beta_{\mathfrak{L}_1}(\mathbf{x})\big\rangle\Big) * \Big(\big\langle \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}), \varkappa^n_{\mathfrak{L}_2}(\mathbf{y})\big\rangle, \big\langle \alpha_{\mathfrak{L}_2}(\mathbf{y}), \beta_{\mathfrak{L}_2}(\mathbf{y})\big\rangle\Big)\\ &= \Big(\big\langle \big(\varkappa^m_{\mathfrak{L}_1} * \varkappa^m_{\mathfrak{L}_2}\big)(\mathbf{x},\mathbf{y}), \big(\varkappa^n_{\mathfrak{L}_1} * \varkappa^n_{\mathfrak{L}_2}\big)(\mathbf{x},\mathbf{y})\big\rangle, \big\langle \big(\alpha_{\mathfrak{L}_1} * \alpha_{\mathfrak{L}_2}\big)(\mathbf{x},\mathbf{y}), \big(\beta_{\mathfrak{L}_1} * \beta_{\mathfrak{L}_2}\big)(\mathbf{x},\mathbf{y})\big\rangle\Big)\\ &= \Big(\big\langle \varkappa^m_{\mathfrak{L}}(\mathbf{x},\mathbf{y}), \varkappa^n_{\mathfrak{L}}(\mathbf{x},\mathbf{y})\big\rangle, \big\langle \alpha_{\mathfrak{L}}(\mathbf{x},\mathbf{y}), \beta_{\mathfrak{L}}(\mathbf{x},\mathbf{y})\big\rangle\Big), \end{aligned} \tag{32}$$

where

$$\left.\begin{aligned}\varkappa^m_{\mathfrak{L}}(\mathbf{x},\mathbf{y}) &= \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}),\\ \varkappa^n_{\mathfrak{L}}(\mathbf{x},\mathbf{y}) &= \varkappa^n_{\mathfrak{L}_1}(\mathbf{x}) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{y}),\\ \alpha_{\mathfrak{L}}(\mathbf{x},\mathbf{y}) &= \alpha_{\mathfrak{L}_1}(\mathbf{x}) \vee \alpha_{\mathfrak{L}_2}(\mathbf{y}),\\ \beta_{\mathfrak{L}}(\mathbf{x},\mathbf{y}) &= \beta_{\mathfrak{L}_1}(\mathbf{x}) \wedge \beta_{\mathfrak{L}_2}(\mathbf{y}),\end{aligned}\right\} \tag{33}$$

for all $(\mathbf{x},\mathbf{y}) \in \mathscr{V} = \mathscr{V}_1 \times \mathscr{V}_2$ and $\breve{\rho}_i = \breve{\rho}'_i * \breve{\rho}''_i$ are defined as :

$$\begin{aligned}\breve{\rho}_i &= \breve{\rho}'_i * \breve{\rho}''_i\\ &= \Big(\big\langle \varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_1,\mathbf{y}_1), \varkappa^n_{\breve{\rho}'_i}(\mathbf{x}_1,\mathbf{y}_1)\big\rangle, \big\langle \alpha_{\breve{\rho}'_i}(\mathbf{x}_1,\mathbf{y}_1), \beta_{\breve{\rho}'_i}(\mathbf{x}_1,\mathbf{y}_1)\big\rangle\Big) * \Big(\big\langle \varkappa^m_{\breve{\rho}''_i}(\mathbf{x}_2,\mathbf{y}_2), \varkappa^n_{\breve{\rho}''_i}(\mathbf{x}_2,\mathbf{y}_2)\big\rangle, \big\langle \alpha_{\breve{\rho}''_i}(\mathbf{x}_2,\mathbf{y}_2), \beta_{\breve{\rho}''_i}(\mathbf{x}_2,\mathbf{y}_2)\big\rangle\Big)\\ &= \Big(\big\langle \big(\varkappa^m_{\breve{\rho}'_i} * \varkappa^m_{\breve{\rho}''_i}\big)(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2), \big(\varkappa^n_{\breve{\rho}'_i} * \varkappa^n_{\breve{\rho}''_i}\big)(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2)\big\rangle, \big\langle \big(\alpha_{\breve{\rho}'_i} * \alpha_{\breve{\rho}''_i}\big)(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2), \big(\beta_{\breve{\rho}'_i} * \beta_{\breve{\rho}''_i}\big)(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2)\big\rangle\Big)\\ &= \Big(\big\langle \varkappa^m_{\breve{\rho}_i}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2), \varkappa^n_{\breve{\rho}_i}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2)\big\rangle, \big\langle \alpha_{\breve{\rho}_i}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2), \beta_{\breve{\rho}_i}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2)\big\rangle\Big), \end{aligned} \tag{34}$$

where

$$\varkappa^m_{\breve{\rho}_i}\big((\mathbf{x}_1,\mathbf{y}_1),(\mathbf{x}_2,\mathbf{y}_2)\big) = \begin{cases}\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_1) \vee \varkappa^m_{\breve{\rho}''_i}(\mathbf{y}_1,\mathbf{y}_2), & \text{if } \mathbf{x}_1 = \mathbf{x}_2, (\mathbf{y}_1,\mathbf{y}_2) \in \mathscr{E}''_i\\ \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_1) \vee \varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_1,\mathbf{x}_2), & \text{if } \mathbf{y}_1 = \mathbf{y}_2, (\mathbf{x}_1,\mathbf{x}_2) \in \mathscr{E}'_i\end{cases} \tag{35}$$

$$\varkappa^n_{\breve{\rho}_i}\big((\mathbf{x}_1,\mathbf{y}_1),(\mathbf{x}_2,\mathbf{y}_2)\big) = \begin{cases}\varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_1) \wedge \varkappa^n_{\breve{\rho}''_i}(\mathbf{y}_1,\mathbf{y}_2), & \text{if } \mathbf{x}_1 = \mathbf{x}_2, (\mathbf{y}_1,\mathbf{y}_2) \in \mathscr{E}''_i\\ \varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_1) \wedge \varkappa^n_{\breve{\rho}'_i}(\mathbf{x}_1,\mathbf{x}_2), & \text{if } \mathbf{y}_1 = \mathbf{y}_2, (\mathbf{x}_1,\mathbf{x}_2) \in \mathscr{E}'_i\end{cases} \tag{36}$$

$$\alpha_{\breve{\rho}_i}\big((\mathbf{x}_1,\mathbf{y}_1),(\mathbf{x}_2,\mathbf{y}_2)\big) = \begin{cases}\alpha_{\mathfrak{L}_1}(\mathbf{x}_1) \vee \alpha_{\breve{\rho}''_i}(\mathbf{y}_1,\mathbf{y}_2), & \text{if } \mathbf{x}_1 = \mathbf{x}_2, (\mathbf{y}_1,\mathbf{y}_2) \in \mathscr{E}''_i\\ \alpha_{\mathfrak{L}_2}(\mathbf{y}_1) \vee \alpha_{\breve{\rho}'_i}(\mathbf{x}_1,\mathbf{x}_2), & \text{if } \mathbf{y}_1 = \mathbf{y}_2, (\mathbf{x}_1,\mathbf{x}_2) \in \mathscr{E}'_i\end{cases} \tag{37}$$

$$\beta_{\breve{\rho}_i}\big((\mathbf{x}_1,\mathbf{y}_1),(\mathbf{x}_2,\mathbf{y}_2)\big) = \begin{cases}\beta_{\mathfrak{L}_1}(\mathbf{x}_1) \wedge \beta_{\breve{\rho}''_i}(\mathbf{y}_1,\mathbf{y}_2), & \text{if } \mathbf{x}_1 = \mathbf{x}_2, (\mathbf{y}_1,\mathbf{y}_2) \in \mathscr{E}''_i\\ \beta_{\mathfrak{L}_2}(\mathbf{y}_1) \wedge \beta_{\breve{\rho}'_i}(\mathbf{x}_1,\mathbf{x}_2), & \text{if } \mathbf{y}_1 = \mathbf{y}_2, (\mathbf{x}_1,\mathbf{x}_2) \in \mathscr{E}'_i\end{cases} \tag{38}$$

$i = 1, 2, \cdots, n$.

Example 4.2. Consider two LDFGSs $\breve{\mathscr{G}}_1 = \big(\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \breve{\rho}'_3\big)$ and $\breve{\mathscr{G}}_2 = \big(\mathfrak{L}_2, \breve{\rho}''_1\big)$, which is depicted in Figure 2 with underlying GSs $\mathscr{G}_1 = (\mathscr{V}_1, \mathscr{E}'_1, \mathscr{E}'_2, \mathscr{E}'_3)$ and $\mathscr{G}_2 = (\mathscr{V}_2, \mathscr{E}''_1)$, respectively, where $\mathscr{V}_1 = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $\mathscr{V}_2 = \{\mathbf{v}_1, \mathbf{v}_2\}$ are two sets of vertices and $\mathscr{E}'_1 = \{(\mathbf{u}_1, \mathbf{u}_3)\}$, $\mathscr{E}'_2 = \{(\mathbf{u}_1, \mathbf{u}_2)\}$, and $\mathscr{E}'_3 = \{(\mathbf{u}_2, \mathbf{u}_3)\}$ are the set of edges on $\mathscr{V}_1$, and $\mathscr{E}''_1 = \{(\mathbf{v}_1, \mathbf{v}_2)\}$ is the edges set on $\mathscr{V}_2$ such that $\mathscr{E}'_i$ and $\mathscr{E}''_i$ are irreflexive and symmetric binary relations on $\mathscr{V}_1$ and $\mathscr{V}_2$, respectively. The LDFSs $\mathfrak{L}_1$ on $\mathscr{V}_1$ and $\mathfrak{L}_2$ on $\mathscr{V}_2$ are given in the TABLES 10 and 11, respectively. The LDFRs $\breve{\rho}'_1, \breve{\rho}'_2, \breve{\rho}'_3$ over the $\mathscr{E}'_1, \mathscr{E}'_2, \mathscr{E}'_3$, and $\breve{\rho}''_1$ over $\mathscr{E}''_1$ given in TABLES 12, 13, 14 and 15 respectively. By using Definition 4.1, we obtain the following LDFS $\mathfrak{L} = \mathfrak{L}_1 * \mathfrak{L}_2$ illustrated in FIGURE 3 and shown in TABLE 16 and LDFRs $\breve{\rho}_i = \breve{\rho}'_i * \breve{\rho}''_i$ for $i = 1, 2, 3$ shown in TABLE 17, 18, 19, respectively.

Table 10: LDFS $\mathfrak{L}_1$

$\mathscr{V}_1$	$\big(\langle\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}),\varkappa^n_{\mathfrak{L}_1}(\mathbf{x})\rangle,\langle\alpha_{\mathfrak{L}_1}(\mathbf{x}),\beta_{\mathfrak{L}_1}(\mathbf{x})\rangle\big)$
$\mathbf{u}_1$	$\big(\langle 0.6,0.5\rangle,\langle 0.4,0.3\rangle\big)$
$\mathbf{u}_2$	$\big(\langle 0.4,0.3\rangle,\langle 0.5,0.4\rangle\big)$
$\mathbf{u}_3$	$\big(\langle 0.8,0.9\rangle,\langle 0.6,0.3\rangle\big)$

Table 11: LDFS $\mathfrak{L}_2$

$\mathscr{V}_2$	$\big(\langle\varkappa^m_{\mathfrak{L}_2}(\mathbf{x}),\varkappa^n_{\mathfrak{L}_2}(\mathbf{x})\rangle,\langle\alpha_{\mathfrak{L}_2}(\mathbf{x}),\beta_{\mathfrak{L}_2}(\mathbf{x})\rangle\big)$
$\mathbf{v}_1$	$\big(\langle 0.7,0.4\rangle,\langle 0.3,0.2\rangle\big)$
$\mathbf{v}_2$	$\big(\langle 0.3,0.2\rangle,\langle 0.4,0.1\rangle\big)$

Table 12: $\breve{\rho}'_1$

$\mathscr{E}'_1$	$\big(\langle\varkappa^m_{\breve{\rho}'_1}(\mathbf{x},\mathbf{y}),\varkappa^n_{\breve{\rho}'_1}(\mathbf{x},\mathbf{y})\rangle,\langle\alpha_{\breve{\rho}'_1}(\mathbf{x},\mathbf{y}),\beta_{\breve{\rho}'_1}(\mathbf{x},\mathbf{y})\rangle\big)$
$(\mathbf{u}_1,\mathbf{u}_3)$	$\big(\langle 0.6,0.9\rangle,\langle 0.4,0.5\rangle\big)$

Table 13: $\breve{\rho}'_2$

$\mathscr{E}'_2$	$\big(\langle\varkappa^m_{\breve{\rho}'_2}(\mathbf{x},\mathbf{y}),\varkappa^n_{\breve{\rho}'_2}(\mathbf{x},\mathbf{y})\rangle,\langle\alpha_{\breve{\rho}'_2}(\mathbf{x},\mathbf{y}),\beta_{\breve{\rho}'_2}(\mathbf{x},\mathbf{y})\rangle\big)$
$(\mathbf{u}_1,\mathbf{u}_2)$	$\big(\langle 0.4,0.5\rangle,\langle 0.3,0.4\rangle\big)$

Table 14: $\breve{\rho}'_3$

$\mathscr{E}'_3$	$\big(\langle\varkappa^m_{\breve{\rho}'_3}(\mathbf{x},\mathbf{y}),\varkappa^n_{\breve{\rho}'_3}(\mathbf{x},\mathbf{y})\rangle,\langle\alpha_{\breve{\rho}'_3}(\mathbf{x},\mathbf{y}),\beta_{\breve{\rho}'_3}(\mathbf{x},\mathbf{y})\rangle\big)$
$(\mathbf{u}_2,\mathbf{u}_3)$	$\big(\langle 0.4,0.9\rangle,\langle 0.5,0.4\rangle\big)$

Table 15: $\breve{\rho}''_1$

$\mathscr{E}''_1$	$\big(\langle\varkappa^m_{\breve{\rho}''_1}(\mathbf{x},\mathbf{y}),\varkappa^n_{\breve{\rho}''_1}(\mathbf{x},\mathbf{y})\rangle,\langle\alpha_{\breve{\rho}''_1}(\mathbf{x},\mathbf{y}),\beta_{\breve{\rho}''_1}(\mathbf{x},\mathbf{y})\rangle\big)$
$(\mathbf{v}_1,\mathbf{v}_2)$	$\big(\langle 0.3,0.5\rangle,\langle 0.2,0.3\rangle\big)$

Definition 4.3. An LDFGS $\breve{\mathscr{G}}=\big(\mathfrak{L},\breve{\rho}_1,\breve{\rho}_2,\cdots,\breve{\rho}_n\big)$ is called $\breve{\rho}_i$-strong, if

$$\left.\begin{aligned}\varkappa^m_{\breve{\rho}_i}(\mathbf{x},\mathbf{y})&=\varkappa^m_{\mathfrak{L}}(\mathbf{x})\wedge\varkappa^m_{\mathfrak{L}}(\mathbf{y}),\\ \varkappa^n_{\breve{\rho}_i}(\mathbf{x},\mathbf{y})&=\varkappa^n_{\mathfrak{L}}(\mathbf{x})\vee\varkappa^n_{\mathfrak{L}}(\mathbf{y}),\\ \alpha_{\breve{\rho}_i}(\mathbf{x},\mathbf{y})&=\alpha_{\mathfrak{L}}(\mathbf{x})\wedge\alpha_{\mathfrak{L}}(\mathbf{y}),\\ \beta_{\breve{\rho}_i}(\mathbf{x},\mathbf{y})&=\beta_{\mathfrak{L}}(\mathbf{x})\vee\beta_{\mathfrak{L}}(\mathbf{y}),\end{aligned}\right\}\tag{39}$$

for all $\mathbf{x},\mathbf{y}\in\mathscr{V}$. If $\breve{\mathscr{G}}$ is $\breve{\rho}_i$-strong for all $i=1,2,\cdots,n$, then $\breve{\mathscr{G}}$ is called strong LDFGS.

Theorem 4.4. *Maximal product of two strong LDFGSs is also a strong LDFGS.*

Table 16: $\mathfrak{L} = \mathfrak{L}_1 * \mathfrak{L}_2$

$\mathscr{V}$	$\big(\langle \varkappa^m_{\mathfrak{L}}(\mathbf{x},\mathbf{y}), \varkappa^n_{\mathfrak{L}}(\mathbf{x},\mathbf{y})\rangle, \langle \alpha_{\mathfrak{L}}(\mathbf{x},\mathbf{y}), \beta_{\mathfrak{L}}(\mathbf{x},\mathbf{y})\rangle\big)$
$(\mathbf{u}_1,\mathbf{v}_1)$	$\big(\langle 0.7, 0.4\rangle, \langle 0.4, 0.2\rangle\big)$
$(\mathbf{u}_1,\mathbf{v}_2)$	$\big(\langle 0.6, 0.2\rangle, \langle 0.4, 0.1\rangle\big)$
$(\mathbf{u}_2,\mathbf{v}_1)$	$\big(\langle 0.7, 0.3\rangle, \langle 0.5, 0.2\rangle\big)$
$(\mathbf{u}_2,\mathbf{v}_2)$	$\big(\langle 0.4, 0.2\rangle, \langle 0.5, 0.1\rangle\big)$
$(\mathbf{u}_3,\mathbf{v}_1)$	$\big(\langle 0.8, 0.4\rangle, \langle 0.6, 0.2\rangle\big)$
$(\mathbf{u}_3,\mathbf{v}_2)$	$\big(\langle 0.8, 0.2\rangle, \langle 0.6, 0.1\rangle\big)$

Table 17: $\breve{\rho}_1$

$\mathscr{E}_1$	$\big(\langle \varkappa^m_{\breve{\rho}_1}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2), \varkappa^n_{\breve{\rho}_1}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2)\rangle, \langle \alpha_{\breve{\rho}_1}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2), \beta_{\breve{\rho}_1}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2)\rangle\big)$
$(\mathbf{u}_1\mathbf{v}_1,\mathbf{u}_1\mathbf{v}_2)$	$\big(\langle 0.6, 0.5\rangle, \langle 0.4, 0.3\rangle\big)$
$(\mathbf{u}_1\mathbf{v}_1,\mathbf{u}_2\mathbf{v}_1)$	$\big(\langle 0.7, 0.4\rangle, \langle 0.3, 0.2\rangle\big)$
$(\mathbf{u}_2\mathbf{v}_1,\mathbf{u}_2\mathbf{v}_2)$	$\big(\langle 0.4, 0.3\rangle, \langle 0.5, 0.4\rangle\big)$
$(\mathbf{u}_3\mathbf{v}_1,\mathbf{u}_3\mathbf{v}_2)$	$\big(\langle 0.3, 0.5\rangle, \langle 0.6, 0.3\rangle\big)$
$(\mathbf{u}_1\mathbf{v}_2,\mathbf{u}_2\mathbf{v}_2)$	$\big(\langle 0.6, 0.2\rangle, \langle 0.4, 0.1\rangle\big)$

Table 18: $\breve{\rho}_2$

$\mathscr{E}_2$	$\big(\langle \varkappa^m_{\breve{\rho}_2}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2), \varkappa^n_{\breve{\rho}_2}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2)\rangle, \langle \alpha_{\breve{\rho}_2}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2), \beta_{\breve{\rho}_2}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2)\rangle\big)$
$(\mathbf{u}_2\mathbf{v}_1,\mathbf{u}_3\mathbf{v}_1)$	$\big(\langle 0.7, 0.4\rangle, \langle 0.5, 0.2\rangle\big)$
$(\mathbf{u}_2\mathbf{v}_2,\mathbf{u}_3\mathbf{v}_2)$	$\big(\langle 0.4, 0.2\rangle, \langle 0.5, 0.1\rangle\big)$

Table 19: $\breve{\rho}_3$

$\mathscr{E}_3$	$\big(\langle \varkappa^m_{\breve{\rho}_3}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2), \varkappa^n_{\breve{\rho}_3}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2)\rangle, \langle \alpha_{\breve{\rho}_3}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2), \beta_{\breve{\rho}_3}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2)\rangle\big)$
$(\mathbf{u}_1\mathbf{v}_1,\mathbf{u}_3\mathbf{v}_1)$	$\big(\langle 0.7, 0.4\rangle, \langle 0.4, 0.2\rangle\big)$
$(\mathbf{u}_1\mathbf{v}_2,\mathbf{u}_3\mathbf{v}_2)$	$\big(\langle 0.6, 0.2\rangle, \langle 0.4, 0.2\rangle\big)$

Proof. Let $\breve{\mathscr{G}}_1 = \big(\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \cdots, \breve{\rho}'_n\big)$ and $\breve{\mathscr{G}}_2 = \big(\mathfrak{L}_2, \breve{\rho}''_1, \breve{\rho}''_2, \cdots, \breve{\rho}''_n\big)$ be two strong LDFGSs. Then, according to the Definition 4.1, we have the following cases:

Case i: When $\mathbf{x}_1 = \mathbf{x}_2$ and $(\mathbf{y}_1, \mathbf{y}_2) \in \mathscr{E}''_i$. Then,

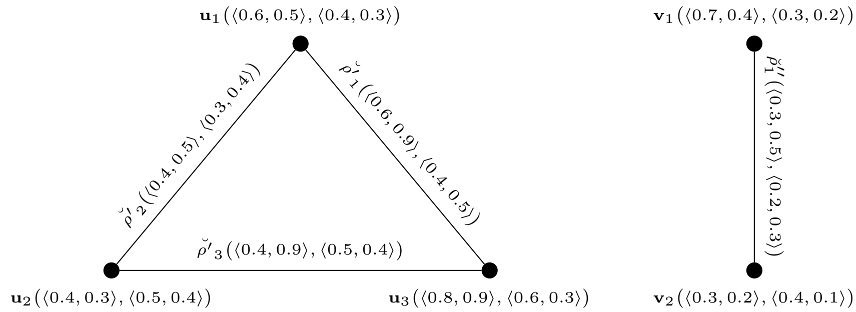

Figure 2: LDFGSs $\breve{\mathscr{G}}_1 = \left(\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \breve{\rho}'_3\right)$ and $\breve{\mathscr{G}}_2 = \left(\mathfrak{L}_2, \breve{\rho}''_1\right)$

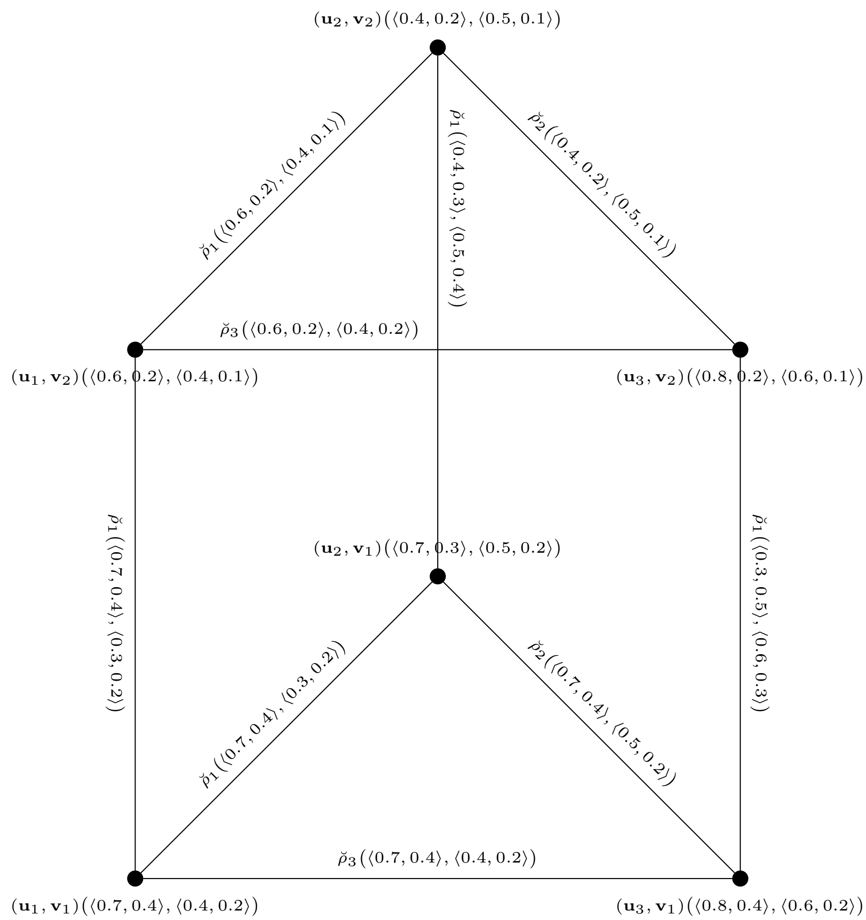

Figure 3: Maximal product $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$

$$
\begin{aligned}
\varkappa^m_{\breve{\rho}_i}\big((\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x_2}, \mathbf{y}_2)\big) &= \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_1) \vee \varkappa^m_{\rho''_i}(\mathbf{y}_1, \mathbf{y}_2) \\
&= \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_1) \vee \left[\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_1) \wedge \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_2)\right] \\
&= \left[\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_1) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_1)\right] \wedge \left[\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_1) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_2)\right] \\
&= \left[\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_1) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_1)\right] \wedge \left[\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_2) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_2)\right] \\
&= \varkappa^m_{\mathfrak{L}}(\mathbf{x}_1, \mathbf{y}_1) \wedge \varkappa^m_{\mathfrak{L}}(\mathbf{x}_2, \mathbf{y}_2).
\end{aligned}
$$

Similarly we can show that $\varkappa^n_{\breve{\rho}_i}\big((\mathbf{x}_1,\mathbf{y}_1),(\mathbf{x_2},\mathbf{y}_2)\big) = \varkappa^n_{\mathfrak{L}}(\mathbf{x}_1,\mathbf{y}_1)\vee\varkappa^n_{\mathfrak{L}}(\mathbf{x}_2,\mathbf{y}_2)$, $\alpha_{\breve{\rho}_i}\big((\mathbf{x}_1,\mathbf{y}_1),(\mathbf{x_2},\mathbf{y}_2)\big) = \alpha_{\mathfrak{L}}(\mathbf{x}_1,\mathbf{y}_1)\wedge\alpha_{\mathfrak{L}}(\mathbf{x}_2,\mathbf{y}_2)$, and $\beta_{\breve{\rho}_i}\big((\mathbf{x}_1,\mathbf{y}_1),(\mathbf{x_2},\mathbf{y}_2)\big) = \beta_{\mathfrak{L}}(\mathbf{x}_1,\mathbf{y}_1)\vee\beta_{\mathfrak{L}}(\mathbf{x}_2,\mathbf{y}_2)$.

Case ii: When $\mathbf{y}_1=\mathbf{y}_2$ and $(\mathbf{x}_1,\mathbf{x}_2)\in\mathscr{E}'_i$. Then,

$$\begin{aligned}
\varkappa^m_{\breve{\rho}_i}\big((\mathbf{x}_1,\mathbf{y}_1),(\mathbf{x_2},\mathbf{y}_2)\big) &= \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_1)\vee\varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_1,\mathbf{x}_2)\\
&= \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_1)\vee\big[\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_1)\wedge\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_2)\big]\\
&= \big[\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_1)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_1)\big]\wedge\big[\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_1)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_2)\big]\\
&= \big[\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_1)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_1)\big]\wedge\big[\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_2)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_2)\big]\\
&= \varkappa^m_{\mathfrak{L}}(\mathbf{x}_1,\mathbf{y}_1)\wedge\varkappa^m_{\mathfrak{L}}(\mathbf{x}_2,\mathbf{y}_2).
\end{aligned}$$

In the same way, we can prove that $\varkappa^n_{\breve{\rho}_i}\big((\mathbf{x}_1,\mathbf{y}_1),(\mathbf{x_2},\mathbf{y}_2)\big) = \varkappa^n_{\mathfrak{L}}(\mathbf{x}_1,\mathbf{y}_1)\vee\varkappa^n_{\mathfrak{L}}(\mathbf{x}_2,\mathbf{y}_2)$, $\alpha_{\breve{\rho}_i}\big((\mathbf{x}_1,\mathbf{y}_1),(\mathbf{x_2},\mathbf{y}_2)\big) = \alpha_{\mathfrak{L}}(\mathbf{x}_1,\mathbf{y}_1)\wedge\alpha_{\mathfrak{L}}(\mathbf{x}_2,\mathbf{y}_2)$, and $\beta_{\breve{\rho}_i}\big((\mathbf{x}_1,\mathbf{y}_1),(\mathbf{x_2},\mathbf{y}_2)\big) = \beta_{\mathfrak{L}}(\mathbf{x}_1,\mathbf{y}_1)\vee\beta_{\mathfrak{L}}(\mathbf{x}_2,\mathbf{y}_2)$. Thus, $\breve{\mathscr{G}}=\breve{\mathscr{G}}_1*\breve{\mathscr{G}}_2$ is a strong LDFGS. □

Theorem 4.5. *The maximal product of two connected LDFGSs is a connected LDFGS.*

Proof. Let $\breve{\mathscr{G}}_1=\big(\mathfrak{L}_1,\breve{\rho}'_1,\breve{\rho}'_2,\cdots,\breve{\rho}'_n\big)$ and $\breve{\mathscr{G}}_2=\big(\mathfrak{L}_2,\breve{\rho}''_1,\breve{\rho}''_2,\cdots,\breve{\rho}''_n\big)$ be two connected LDFGSs with underlying GSs $\mathscr{G}_1=\big(\mathscr{V}_1,\mathscr{E}'_1,\mathscr{E}'_2,\cdots,\mathscr{E}'_n\big)$ and $\mathscr{G}_2=\big(\mathscr{V}_2,\mathscr{E}''_1,\mathscr{E}''_2,\cdots,\mathscr{E}''_n\big)$, respectively. Let $\mathscr{V}_1=\{\mathbf{x}_1,\mathbf{x}_2,\cdots,\mathbf{x}_m\}$ and $\mathscr{V}_2=\{\mathbf{y}_1,\mathbf{y}_2,\cdots,\mathbf{y}_q\}$. Then, according to the Definition 3.13,

$$\begin{gathered}
(\varkappa^m_{\breve{\rho}'_i})^\infty(\mathbf{x}_i,\mathbf{x}_j)>0 \text{ and } (\varkappa^m_{\breve{\rho}''_i})^\infty(\mathbf{y}_i,\mathbf{y}_j)>0;\\
(\varkappa^n_{\breve{\rho}'_i})^\infty(\mathbf{x}_i,\mathbf{x}_j)>0 \text{ and } (\varkappa^n_{\breve{\rho}''_i})^\infty(\mathbf{y}_i,\mathbf{y}_j)>0;\\
(\alpha_{\breve{\rho}'_i})^\infty(\mathbf{x}_i,\mathbf{x}_j)>0 \text{ and } (\alpha_{\breve{\rho}''_i})^\infty(\mathbf{y}_i,\mathbf{y}_j)>0;\\
(\beta_{\breve{\rho}'_i})^\infty(\mathbf{x}_i,\mathbf{x}_j)>0 \text{ and } (\beta_{\breve{\rho}''_i})^\infty(\mathbf{y}_i,\mathbf{y}_j)>0,
\end{gathered}$$

for all $\mathbf{x}_i,\mathbf{x}_j\in\mathscr{V}_1$ and $\mathbf{y}_i,\mathbf{y}_j\in\mathscr{V}_2$. Consider m subgraphs of $\mathscr{G}$ with the vertex sets $\big\{(\mathbf{x}_i,\mathbf{y}_1),(\mathbf{x}_i,\mathbf{y}_2),\cdots,(\mathbf{x}_i,\mathbf{y}_q)\big\}$ for $i=1,2,\cdots,m$. Each of these subgraphs of $\mathscr{G}$ is connected since $\mathbf{x}_i$'s are the same and $\mathscr{G}_2$ is connected, each $\mathbf{y}_i$ is adjacent to at least one of the vertices in $\mathscr{V}_2$. Since $\mathscr{G}_1$ is connected, each $\mathbf{x}_i$ is also adjacent to at least one of the vertices in $\mathscr{V}_1$. Therefore, there exists one edge between any pair of the above m subgraphs. Thus, we have

$$\begin{aligned}
&(\varkappa^m_{\breve{\rho}_i})^\infty\big((\mathbf{x}_i,\mathbf{y}_j),(\mathbf{x}_k,\mathbf{y}_l)\big)>0,(\varkappa^n_{\breve{\rho}_i})^\infty\big((\mathbf{x}_i,\mathbf{y}_j),(\mathbf{x}_k,\mathbf{y}_l)\big)>0, \text{ and}\\
&\qquad(\alpha_{\breve{\rho}_i})^\infty\big((\mathbf{x}_i,\mathbf{y}_j),(\mathbf{x}_k,\mathbf{y}_l)\big)>0,(\beta_{\breve{\rho}'_i})^\infty\big((\mathbf{x}_i,\mathbf{y}_j),(\mathbf{x}_k,\mathbf{y}_l)\big)>0,
\end{aligned}$$

for all $((\mathbf{x}_i,\mathbf{y}_j),(\mathbf{x}_k,\mathbf{y}_l))\in\mathscr{E}_i$. Hence, $\breve{\mathscr{G}}$ is connected LDFGS. □

Definition 4.6. Let $\breve{\mathscr{G}}=\breve{\mathscr{G}}_1*\breve{\mathscr{G}}_2=\big(\mathfrak{L},\breve{\rho}_1,\breve{\rho}_2,\cdots,\breve{\rho}_n\big)$ be the maximal product of LDFGSs $\breve{\mathscr{G}}_1=\big(\mathfrak{L}_1,\breve{\rho}'_1,\breve{\rho}'_2,\cdots,\breve{\rho}'_n\big)$ and $\breve{\mathscr{G}}_2=\big(\mathfrak{L}_2,\breve{\rho}''_1,\breve{\rho}''_2,\cdots,\breve{\rho}''_n\big)$. Then, the degree of a vertex in $\breve{\mathscr{G}}$ is postulated as follows:

$$\mathbb{D}_{\breve{\mathscr{G}}}(\mathbf{x}_i,\mathbf{y}_j)=\Big(\big\langle\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j),\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle,\big\langle\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j),\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle\Big), \tag{40}$$

where

$$\left.\begin{aligned}
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i)\\
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l)\wedge\varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i)\\
\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \alpha_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k)\vee\alpha_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \alpha_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l)\vee\alpha_{\mathfrak{L}_1}(\mathbf{x}_i)\\
\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \beta_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k)\wedge\beta_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \beta_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l)\wedge\beta_{\mathfrak{L}_1}(\mathbf{x}_i)
\end{aligned}\right\} \quad (41)$$

Also, $\breve{\rho}_i - \mathbb{D}_{\breve{\mathscr{G}}}(\mathbf{x}_i,\mathbf{y}_j)$ of a vertex $(\mathbf{x}_i,\mathbf{y}_j)$ of maximal product $\breve{\mathscr{G}}$ is defined as follows:

$$\breve{\rho}_i - \mathbb{D}_{\breve{\mathscr{G}}}(\mathbf{x}_i,\mathbf{y}_j) = \Big(\big\langle\varkappa^m_i - \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j), \varkappa^n_i - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle, \big\langle\alpha_i - \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j), \beta_i - \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle\Big), \quad (42)$$

where

$$\left.\begin{aligned}
\varkappa^m_i - \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_i,\mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\rho''_i}(\mathbf{y}_j,\mathbf{y}_l)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i)\\
\varkappa^n_i - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_i,\mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\breve{\rho}''_i}(\mathbf{y}_j,\mathbf{y}_l)\wedge\varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i)\\
\alpha_i - \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \alpha_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k)\vee\alpha_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_i,\mathbf{x}_i=\mathbf{x}_k} \alpha_{\breve{\rho}''_i}(\mathbf{y}_j,\mathbf{y}_l)\vee\alpha_{\mathfrak{L}_1}(\mathbf{x}_i)\\
\beta_i - \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \beta_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k)\wedge\beta_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_i,\mathbf{x}_i=\mathbf{x}_k} \beta_{\mathscr{E}''_i}(\mathbf{y}_j,\mathbf{y}_l)\wedge\beta_{\mathfrak{L}_1}(\mathbf{x}_i)
\end{aligned}\right\} \quad (43)$$

Example 4.7. (Continued from Example 4.2) With the same LDFGSs $\breve{\mathscr{G}}_1$, $\breve{\mathscr{G}}_2$ and their maximal product $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ with underlying GSs $\mathscr{G}_1$, $\mathscr{G}_2$ and their maximal product $\mathscr{G} = \mathscr{G}_1 * \mathscr{G}_2$. According to Definition 4.6, the degrees of vertices in $\breve{\mathscr{G}}$ are calculated as follows:

$$\begin{aligned}
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_1) &= \varkappa^m_{\breve{\rho}'_1}(\mathbf{u}_1,\mathbf{u}_2)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1) + \varkappa^m_{\breve{\rho}'_3}(\mathbf{u}_1,\mathbf{u}_3)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1) + \varkappa^m_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_1)\\
&= 0.4\vee 0.7 + 0.6\vee 0.7 + 0.3\vee 0.6 = 2\\
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_2) &= \varkappa^m_{\breve{\rho}'_1}(\mathbf{u}_1,\mathbf{u}_2)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2) + \varkappa^m_{\breve{\rho}'_3}(\mathbf{u}_1,\mathbf{u}_3)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2) + \varkappa^m_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_1)\\
&= 0.4\vee 0.3 + 0.6\vee 0.3 + 0.3\vee 0.6 = 1.6\\
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_1) &= \varkappa^m_{\breve{\rho}'_1}(\mathbf{u}_2,\mathbf{u}_1)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1) + \varkappa^m_{\breve{\rho}'_2}(\mathbf{u}_2,\mathbf{u}_3)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1) + \varkappa^m_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_2)\\
&= 0.4\vee 0.7 + 0.4\vee 0.7 + 0.3\vee 0.4 = 1.8\\
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_2) &= \varkappa^m_{\breve{\rho}'_1}(\mathbf{u}_2,\mathbf{u}_1)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2) + \varkappa^m_{\breve{\rho}'_2}(\mathbf{u}_2,\mathbf{u}_3)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2) + \varkappa^m_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_2)\\
&= 0.4\vee 0.3 + 0.4\vee 0.3 + 0.3\vee 0.4 = 1.2\\
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3,\mathbf{v}_1) &= \varkappa^m_{\breve{\rho}'_2}(\mathbf{u}_3,\mathbf{u}_2)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1) + \varkappa^m_{\breve{\rho}'_3}(\mathbf{u}_3,\mathbf{u}_1)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1) + \varkappa^m_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_3)\\
&= 0.4\vee 0.7 + 0.6\vee 0.7 + 0.3\vee 0.8 = 2.2\\
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3,\mathbf{v}_2) &= \varkappa^m_{\breve{\rho}'_2}(\mathbf{u}_3,\mathbf{u}_2)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2) + \varkappa^m_{\breve{\rho}'_3}(\mathbf{u}_3,\mathbf{u}_1)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2) + \varkappa^m_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_3)\\
&= 0.4\vee 0.3 + 0.6\vee 0.3 + 0.3\vee 0.8 = 1.8
\end{aligned}$$

similarly,

$$\begin{aligned}
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_1) &= \varkappa^n_{\breve{\rho}'_1}(\mathbf{u}_1,\mathbf{u}_2)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1)+\varkappa^n_{\breve{\rho}'_3}(\mathbf{u}_1,\mathbf{u}_3)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1)+\varkappa^n_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\wedge\varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_1)\\
&= 0.5\wedge 0.4+0.9\wedge 0.4+0.5\wedge 0.5=1.3\\
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_2) &= \varkappa^n_{\breve{\rho}'_1}(\mathbf{u}_1,\mathbf{u}_2)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2)+\varkappa^n_{\breve{\rho}'_3}(\mathbf{u}_1,\mathbf{u}_3)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2)+\varkappa^n_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\wedge\varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_1)\\
&= 0.5\wedge 0.2+0.9\wedge 0.2+0.5\wedge 0.5=0.9\\
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_1) &= \varkappa^n_{\breve{\rho}'_1}(\mathbf{u}_2,\mathbf{u}_1)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1)+\varkappa^n_{\breve{\rho}'_2}(\mathbf{u}_2,\mathbf{u}_3)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1)+\varkappa^n_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\wedge\varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_2)\\
&= 0.5\wedge 0.4+0.9\wedge 0.4+0.5\wedge 0.3=1.1\\
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_2) &= \varkappa^n_{\breve{\rho}'_1}(\mathbf{u}_2,\mathbf{u}_1)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2)+\varkappa^n_{\breve{\rho}'_2}(\mathbf{u}_2,\mathbf{u}_3)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2)+\varkappa^n_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\wedge\varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_2)\\
&= 0.5\wedge 0.2+0.9\wedge 0.2+0.5\wedge 0.3=0.7\\
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3,\mathbf{v}_1) &= \varkappa^n_{\breve{\rho}'_2}(\mathbf{u}_3,\mathbf{u}_2)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1)+\varkappa^n_{\breve{\rho}'_3}(\mathbf{u}_3,\mathbf{u}_1)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1)+\varkappa^n_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\wedge\varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_3)\\
&= 0.9\wedge 0.4+0.9\wedge 0.4+0.5\wedge 0.9=1.3\\
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3,\mathbf{v}_2) &= \varkappa^n_{\breve{\rho}'_2}(\mathbf{u}_3,\mathbf{u}_2)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2)+\varkappa^n_{\breve{\rho}'_3}(\mathbf{u}_3,\mathbf{u}_1)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2)+\varkappa^n_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\wedge\varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_3)\\
&= 0.9\wedge 0.2+0.9\wedge 0.2+0.5\wedge 0.9=0.9
\end{aligned}$$

In the similar way, $\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)$ and $\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)$ are calculated for all $\mathbf{x}_i\in\mathscr{V}_1$ and $\mathbf{y}_j\in\mathscr{V}_2$, shown in TABLE 20.

Table 20: $\mathbb{D}_{\breve{\mathscr{G}}}$

$\mathscr{V}$	$\Big(\big\langle\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j),\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle,\big\langle\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j),\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle\Big)$
$(\mathbf{u}_1,\mathbf{v}_1)$	$\big(\langle 2,0.9\rangle,\langle 1.1,0.7\rangle\big)$
$(\mathbf{u}_1,\mathbf{v}_2)$	$\big(\langle 1.6,1.1\rangle,\langle 1.2,0.5\rangle\big)$
$(\mathbf{u}_2,\mathbf{v}_1)$	$\big(\langle 1.8,0.7\rangle,\langle 1.3,0.7\rangle\big)$
$(\mathbf{u}_2,\mathbf{v}_2)$	$\big(\langle 1.2,1.3\rangle,\langle 1.4,0.5\rangle\big)$
$(\mathbf{u}_3,\mathbf{v}_1)$	$\big(\langle 2.2,0.9\rangle,\langle 1.5,0.7\rangle\big)$
$(\mathbf{u}_3,\mathbf{v}_2)$	$\big(\langle 0.8,1.3\rangle,\langle 1.5,0.5\rangle\big)$

Now, we calculate $\breve{\rho}_i-\mathbb{D}_{\breve{\mathscr{G}}}(\mathbf{x}_i,\mathbf{y}_j)$ for all $i=1,2,3$ as follows:

$$\begin{aligned}
\varkappa^m_1-\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_1) &= \varkappa^m_{\breve{\rho}'_1}(\mathbf{u}_1,\mathbf{u}_2)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1)+\varkappa^m_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_1)=0.4\vee 0.7+0.3\vee 0.6=1.3\\
\varkappa^m_1-\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_2) &= \varkappa^m_{\breve{\rho}'_1}(\mathbf{u}_1,\mathbf{u}_2)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2)+\varkappa^m_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_1)=0.4\vee 0.3+0.3\vee 0.6=1\\
\varkappa^m_1-\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_1) &= \varkappa^m_{\breve{\rho}'_1}(\mathbf{u}_2,\mathbf{u}_1)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1)+\varkappa^m_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_2)=0.4\vee 0.7+0.3\vee 0.4=1.1\\
\varkappa^m_1-\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_2) &= \varkappa^m_{\breve{\rho}'_1}(\mathbf{u}_2,\mathbf{u}_1)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2)+\varkappa^m_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_2)=0.4\vee 0.3+0.3\vee 0.4=0.8\\
\varkappa^m_1-\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3,\mathbf{v}_1) &= \varkappa^m_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_3)=0.3\vee 0.8=0.8\\
\varkappa^m_1-\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3,\mathbf{v}_2) &= \varkappa^m_{\breve{\rho}''_1}(\mathbf{v}_1,\mathbf{v}_2)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_3)=0.3\vee 0.8=0.8
\end{aligned}$$

Similarly, $\varkappa_1^n - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)$ can be calculated as:

$$\varkappa_1^n - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1, \mathbf{v}_1) = \varkappa^n_{\breve{\rho}'_1}(\mathbf{u}_1, \mathbf{u}_2) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1) + \varkappa^n_{\breve{\rho}''_1}(\mathbf{v}_1, \mathbf{v}_2) \wedge \varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_1) = 0.5 \wedge 0.4 + 0.5 \wedge 0.5 = 0.9$$
$$\varkappa_1^n - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1, \mathbf{v}_2) = \varkappa^n_{\breve{\rho}'_1}(\mathbf{u}_1, \mathbf{u}_2) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2) + \varkappa^n_{\breve{\rho}''_1}(\mathbf{v}_1, \mathbf{v}_2) \wedge \varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_1) = 0.5 \wedge 0.2 + 0.5 \wedge 0.5 = 0.7$$
$$\varkappa_1^n - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2, \mathbf{v}_1) = \varkappa^n_{\breve{\rho}'_1}(\mathbf{u}_2, \mathbf{u}_1) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1) + \varkappa^n_{\breve{\rho}''_1}(\mathbf{v}_1, \mathbf{v}_2) \wedge \varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_2) = 0.5 \wedge 0.4 + 0.5 \wedge 0.3 = 0.7$$
$$\varkappa_1^n - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2, \mathbf{v}_2) = \varkappa^n_{\breve{\rho}'_1}(\mathbf{u}_2, \mathbf{u}_1) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2) + \varkappa^n_{\breve{\rho}''_1}(\mathbf{v}_1, \mathbf{v}_2) \wedge \varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_2) = 0.5 \wedge 0.2 + 0.5 \wedge 0.3 = 0.5$$
$$\varkappa_1^n - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3, \mathbf{v}_1) = \varkappa^n_{\breve{\rho}''_1}(\mathbf{v}_1, \mathbf{v}_2) \wedge \varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_3) = 0.9 \wedge 0.4 = 0.4$$
$$\varkappa_1^n - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3, \mathbf{v}_2) = \varkappa^n_{\breve{\rho}''_1}(\mathbf{v}_1, \mathbf{v}_2) \wedge \varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_3) = 0.9 \wedge 0.5 = 0.5$$

Moreover, $\alpha_1 - \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)$ and $\beta_1 - \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)$ are evaluated by following the same steps, which are given in TABLE 21.

Table 21: $\breve{\rho}_1 - \mathbb{D}_{\breve{\mathscr{G}}}$

$\mathscr{V}$	$\big(\langle \varkappa_1^m - \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \varkappa_1^n - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\rangle, \langle \alpha_1 - \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \beta_1 - \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\rangle\big)$
$(\mathbf{u}_1, \mathbf{v}_1)$	$(\langle 1.3, 0.9\rangle, \langle 0.7, 0.5\rangle)$
$(\mathbf{u}_1, \mathbf{v}_2)$	$(\langle 1, 0.7\rangle, \langle 0.8, 0.4\rangle)$
$(\mathbf{u}_2, \mathbf{v}_1)$	$(\langle 1.1, 0.7\rangle, \langle 0.8, 0.5\rangle)$
$(\mathbf{u}_2, \mathbf{v}_2)$	$(\langle 0.8, 0.5\rangle, \langle 0.9, 0.4\rangle)$
$(\mathbf{u}_3, \mathbf{v}_1)$	$(\langle 0.8, 0.5\rangle, \langle 0.6, 0.3\rangle)$
$(\mathbf{u}_3, \mathbf{v}_2)$	$(\langle 0.8, 0.5\rangle, \langle 0.6, 0.3\rangle)$

Now, we calculate $\breve{\rho}_2 - \mathbb{D}_{\breve{\mathscr{G}}}(\mathbf{x}_i, \mathbf{y}_j)$ as:

$$\varkappa_2^m - \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2, \mathbf{v}_1) = \varkappa^m_{\breve{\rho}'_2}(\mathbf{u}_2, \mathbf{u}_3) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1) = 0.4 \vee 0.7 = 0.7$$
$$\varkappa_2^m - \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2, \mathbf{v}_2) = \varkappa^m_{\breve{\rho}'_2}(\mathbf{u}_2, \mathbf{u}_3) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2) = 0.4 \vee 0.3 = 0.4$$
$$\varkappa_2^m - \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3, \mathbf{v}_1) = \varkappa^m_{\breve{\rho}'_2}(\mathbf{u}_2, \mathbf{u}_3) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1) = 0.4 \vee 0.7 = 0.7$$
$$\varkappa_2^m - \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3, \mathbf{v}_2) = \varkappa^m_{\breve{\rho}'_2}(\mathbf{u}_2, \mathbf{u}_3) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2) = 0.4 \vee 0.3 = 0.4$$
$$\varkappa_2^n - \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2, \mathbf{v}_1) = \varkappa^n_{\breve{\rho}'_2}(\mathbf{u}_2, \mathbf{u}_3) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1) = 0.9 \wedge 0.4 = 0.4$$
$$\varkappa_2^n - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2, \mathbf{v}_2) = \varkappa^n_{\breve{\rho}'_2}(\mathbf{u}_2, \mathbf{u}_3) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2) = 0.9 \wedge 0.2 = 0.2$$
$$\varkappa_2^n - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3, \mathbf{v}_1) = \varkappa^n_{\breve{\rho}'_2}(\mathbf{u}_2, \mathbf{u}_3) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1) = 0.9 \wedge 0.4 = 0.4$$
$$\varkappa_2^n - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3, \mathbf{v}_2) = \varkappa^n_{\rho'_2}(\mathbf{u}_2, \mathbf{u}_3) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2) = 0.9 \wedge 0.2 = 0.2$$

In the similar manners, we have calculated $\alpha_2 - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)$ and $\beta_2 - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)$ for all $\mathbf{x}_i \in \mathscr{V}_1$, and $\mathbf{y}_j \in \mathscr{V}_2$ which presented in TABLE 22. By following the similar methodology as above, $\breve{\rho}_3 - \mathbb{D}_{\breve{\mathscr{G}}}(\mathbf{x}_i, \mathbf{y}_j)$ are evaluated for all $\mathbf{x}_i \in \mathscr{V}_1$, and $\mathbf{y}_j \in \mathscr{V}_2$ which are shown in TABLE 23.

Table 22: $\breve{\rho}_2 - \mathbb{D}_{\breve{\mathscr{G}}}$

$\mathscr{V}$	$\Big(\big\langle \varkappa_2^m - \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j), \varkappa_2^n - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle, \big\langle \alpha_2 - \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j), \beta_2 - \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle\Big)$
$(\mathbf{u}_2,\mathbf{v}_1)$	$\big(\langle 0.7, 0.4\rangle, \langle 0.5, 0.2\rangle\big)$
$(\mathbf{u}_2,\mathbf{v}_2)$	$\big(\langle 0.4, 0.2\rangle, \langle 0.5, 0.1\rangle\big)$
$(\mathbf{u}_3,\mathbf{v}_1)$	$\big(\langle 0.7, 0.4\rangle, \langle 0.5, 0.2\rangle\big)$
$(\mathbf{u}_3,\mathbf{v}_2)$	$\big(\langle 0.4, 0.2\rangle, \langle 0.5, 0.1\rangle\big)$

Table 23: $\breve{\rho}_3 - \mathbb{D}_{\breve{\mathscr{G}}}$

$\mathscr{V}$	$\Big(\big\langle \varkappa_3^m - \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j), \varkappa_3^n - \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle, \big\langle \alpha_3 - \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j), \beta_3 - \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle\Big)$
$(\mathbf{u}_1,\mathbf{v}_1)$	$\big(\langle 0.7, 0.4\rangle, \langle 0.4, 0.2\rangle\big)$
$(\mathbf{u}_1,\mathbf{v}_2)$	$\big(\langle 0.6, 0.2\rangle, \langle 0.4, 0.1\rangle\big)$
$(\mathbf{u}_3,\mathbf{v}_1)$	$\big(\langle 0.7, 0.4\rangle, \langle 0.4, 0.2\rangle\big)$
$(\mathbf{u}_3,\mathbf{v}_2)$	$\big(\langle 0.7, 0.2\rangle, \langle 0.4, 0.1\rangle\big)$

Theorem 4.8. *If* $\breve{\mathscr{G}}_1 = \big(\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \cdots, \breve{\rho}'_k\big)$ *and* $\breve{\mathscr{G}}_2 = \big(\mathfrak{L}_2, \breve{\rho}''_1, \breve{\rho}''_2, \cdots, \breve{\rho}''_k\big)$ *are two LDFGSs such that* $\mathfrak{L}_1 \subseteq \breve{\rho}''_i$, $i = 1, 2, \cdots, k$, *then the degree of any vertex in maximal product* $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2 = \big(\mathfrak{L}, \breve{\rho}_1, \breve{\rho}_2, \cdots, \breve{\rho}_k\big)$ *is given by:*

$$\mathbb{D}_{\breve{\mathscr{G}}}(\mathbf{x}_i,\mathbf{y}_j) = \Big(\big\langle \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j), \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle, \big\langle \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j), \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle\Big), \tag{44}$$

where

$$\left.\begin{aligned}
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j),\\
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j),\\
\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\alpha_{\mathfrak{L}_2}(\mathbf{y}_j) + \alpha_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j),\\
\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\beta_{\mathfrak{L}_2}(\mathbf{y}_j) + \beta_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j).
\end{aligned}\right\} \tag{45}$$

Proof. Let $\breve{\mathscr{G}}_1 = \big(\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \cdots, \breve{\rho}'_k\big)$ and $\breve{\mathscr{G}}_2 = \big(\mathfrak{L}_2, \breve{\rho}''_1, \breve{\rho}''_2, \cdots, \breve{\rho}''_k\big)$ be two LDFGSs such that $\mathfrak{L}_1 \subseteq \breve{\rho}''_i$, then $\breve{\rho}'_i \subseteq \mathfrak{L}_2$, $i = 1, 2, \cdots, k$. Thus,

$$\begin{aligned}
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l) \vee \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i)\\
&= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l)\\
&= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j);
\end{aligned}$$

Also,

$$\begin{aligned}\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l)\wedge\varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i)\\ &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l)\\ &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j)+\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j)\end{aligned}$$

By adopting the procedure, we can show that

$$\mathbb{D}_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)=\mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\alpha_{\mathfrak{L}_2}(\mathbf{y}_j)+\alpha_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j) \text{ and } \mathbb{D}_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)=\mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\beta_{\mathfrak{L}_2}(\mathbf{y}_j)+\beta_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j).$$

□

Theorem 4.9. *If* $\breve{\mathscr{G}}_1=\left(\mathfrak{L}_1,\breve{\rho}'_1,\breve{\rho}'_2,\cdots,\breve{\rho}'_k\right)$ *and* $\breve{\mathscr{G}}_2=\left(\mathfrak{L}_2,\breve{\rho}''_1,\breve{\rho}''_2,\cdots,\breve{\rho}''_k\right)$ *are LDFGSs such that* $\mathfrak{L}_1\subseteq\breve{\rho}''_i$, $i=1,2,\cdots,k$, *and* $\mathfrak{L}_2$ *is constant LDFS of LDF value* $\left(\langle a,b\rangle,\langle c,d\rangle\right)$, *where* $a,b,c,d\in[0,1]$ *are fixed, then the degree of any vertex in maximal product* $\breve{\mathscr{G}}=\breve{\mathscr{G}}_1*\breve{\mathscr{G}}_2$ *is given as:*

$$\mathbb{D}_{\breve{\mathscr{G}}}(\mathbf{x}_i,\mathbf{y}_j)=\left(\left\langle\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j),\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\right\rangle,\left\langle\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j),\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\right\rangle\right), \tag{46}$$

where

$$\left.\begin{aligned}\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)&=\mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)a+\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j),\\ \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)&=\mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)b+\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j),\\ \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)&=\mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)c+\alpha_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j),\\ \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)&=\mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)d+\beta_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j).\end{aligned}\right\} \tag{47}$$

Proof. Let $\breve{\mathscr{G}}_1=\left(\mathfrak{L}_1,\breve{\rho}'_1,\breve{\rho}'_2,\cdots,\breve{\rho}'_k\right)$ and $\breve{\mathscr{G}}_2=\left(\mathfrak{L}_2,\breve{\rho}''_1,\breve{\rho}''_2,\cdots,\breve{\rho}''_k\right)$ be two LDFGSs such that $\mathfrak{L}_1\subseteq\breve{\rho}''_i$, then $\breve{\rho}'_i\subseteq\mathfrak{L}_2$, $i=1,2,\cdots,k$ and $\mathfrak{L}_2$ is a constant LDFS. Therefore,

$$\begin{aligned}\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k)\vee\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l)\vee\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i)\\ &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l)\\ &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)a+\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j).\end{aligned}$$

Also,

$$\begin{aligned}\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k)\wedge\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l)\wedge\varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i)\\ &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l)\\ &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)b+\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j).\end{aligned}$$

Similarly, we can show that

$$\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)=\mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)c+\alpha_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j) \text{ and } \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)=\mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\beta_{\mathfrak{L}_2}(\mathbf{y}_j)+\beta_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j).$$

□

Theorem 4.10. *If $\breve{\mathscr{G}}_1 = \left(\mathfrak{L}_1, \breve{\rho'}_1, \breve{\rho'}_2, \cdots, \breve{\rho'}_k\right)$ and $\breve{\mathscr{G}}_2 = \left(\mathfrak{L}_2, \breve{\rho''}_1, \breve{\rho''}_2, \cdots, \breve{\rho''}_k\right)$ are two LDFGSs such that $\mathfrak{L}_2 \subseteq \breve{\rho'}_i$, $i = 1, 2, \cdots, k$, then the degree of any vertex in maximal product $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ is given by:*

$$\mathbb{D}_{\breve{\mathscr{G}}}(\mathbf{x}_i, \mathbf{y}_j) = \Big(\big\langle \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\big\rangle, \big\langle \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\big\rangle\Big), \tag{48}$$

where

$$\left.\begin{aligned}
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i),\\
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i),\\
\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \alpha_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_1}(\mathbf{y}_j)\alpha_{\mathfrak{L}_1}(\mathbf{x}_i),\\
\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \beta_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\beta_{\mathfrak{L}_1}(\mathbf{x}_i).
\end{aligned}\right\} \tag{49}$$

Proof. Let $\breve{\mathscr{G}}_1 = \left(\mathfrak{L}_1, \breve{\rho'}_1, \breve{\rho'}_2, \cdots, \breve{\rho'}_k\right)$ and $\breve{\mathscr{G}}_2 = \left(\mathfrak{L}_2, \breve{\rho''}_1, \breve{\rho''}_2, \cdots, \breve{\rho''}_k\right)$ be two LDFGSs such that $\mathfrak{L}_2 \subseteq \breve{\rho'}_i$, then $\breve{\rho''}_i \subseteq \mathfrak{L}_1$, $i = 1, 2, \cdots, k$. So,

$$\begin{aligned}
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_i, \mathbf{x}_k) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\breve{\rho}''_j}(\mathbf{y}_j, \mathbf{y}_l) \vee \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i)\\
&= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_i, \mathbf{x}_k) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i)\\
&= \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i).
\end{aligned}$$

Also,

$$\begin{aligned}
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\breve{\rho}'_i}(\mathbf{x}_i, \mathbf{x}_k) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\breve{\rho}''_j}(\mathbf{y}_j, \mathbf{y}_l) \wedge \varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i)\\
&= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\breve{\rho}'_i}(\mathbf{x}_i, \mathbf{x}_k) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i)\\
&= \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i).
\end{aligned}$$

Similarly, we can show that $\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) = \alpha_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\alpha_{\mathfrak{L}_1}(\mathbf{x}_i)$ and $\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) = \beta_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)$ $\beta_{\mathfrak{L}_1}(\mathbf{x}_i)$. □

Theorem 4.11. *If $\breve{\mathscr{G}}_1 = \left(\mathfrak{L}_1, \breve{\rho'}_1, \breve{\rho'}_2, \cdots, \breve{\rho'}_k\right)$ and $\breve{\mathscr{G}}_2 = \left(\mathfrak{L}_2, \breve{\rho''}_1, \breve{\rho''}_2, \cdots, \breve{\rho''}_k\right)$ are two LDFGSs such that $\mathfrak{L}_2 \subseteq \breve{\rho'}_i$, $i = 1, 2, \cdots, k$, and $\mathfrak{L}_1$ is constant LDFS of LDF value $\left(\langle a, b\rangle, \langle c, d\rangle\right)$, where $a, b, c, d \in [0, 1]$ are fixed, then the degree of any vertex in maximal product $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ is given by:*

$$\mathbb{D}_{\breve{\mathscr{G}}}(\mathbf{x}_i, \mathbf{y}_j) = \Big(\big\langle \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\big\rangle, \big\langle \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\big\rangle\Big), \tag{50}$$

where

$$\left.\begin{aligned}
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)a,\\
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)b,\\
\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \alpha_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)c,\\
\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \beta_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)d.
\end{aligned}\right\} \tag{51}$$

Proof. Let $\breve{\mathscr{G}}_1 = \left(\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \cdots, \breve{\rho}'_k\right)$ and $\breve{\mathscr{G}}_2 = \left(\mathfrak{L}_2, \breve{\rho}''_1, \breve{\rho}''_2, \cdots, \breve{\rho}''_k\right)$ be two LDFGSs such that $\mathfrak{L}_2 \subseteq \breve{\rho}'_i$, $i = 1, 2, \cdots, k$, and $\mathfrak{L}_1$ is constant LDFS of LDF value $\left(\langle a, b\rangle, \langle c, d\rangle\right)$. Therefore,

$$\begin{aligned}\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_i, \mathbf{x}_k) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\breve{\rho}''_j}(\mathbf{y}_j, \mathbf{y}_l) \vee \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i)\\ &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_i, \mathbf{x}_k) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i)\\ &= \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)a.\end{aligned}$$

Also,

$$\begin{aligned}\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\breve{\rho}'_i}(\mathbf{x}_i, \mathbf{x}_k) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\breve{\rho}''_j}(\mathbf{y}_j, \mathbf{y}_l) \wedge \varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i)\\ &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\breve{\rho}'_i}(\mathbf{x}_i, \mathbf{x}_k) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i)\\ &= \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)b.\end{aligned}$$

Similarly, it can be shown that $\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) = \alpha_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)c$ and $\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) = \beta_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)d$.
□

Theorem 4.12. *If* $\breve{\mathscr{G}}_1 = \left(\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \cdots, \breve{\rho}'_k\right)$ *and* $\breve{\mathscr{G}}_2 = \left(\mathfrak{L}_2, \breve{\rho}''_1, \breve{\rho}''_2, \cdots, \breve{\rho}''_k\right)$ *are two LDFGSs such that* $\breve{\rho}''_i \subseteq \mathfrak{L}_1$ *and* $\breve{\rho}'_i \subseteq \mathfrak{L}_2$, $i = 1, 2, \cdots, k$, *then the degree of any vertex in maximal product* $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ *is characterized as:*

$$\mathbb{D}_{\breve{\mathscr{G}}}(\mathbf{x}_i, \mathbf{y}_j) = \left(\left\langle \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\right\rangle, \left\langle \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\right\rangle\right), \tag{52}$$

where

$$\left.\begin{aligned}\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i),\\ \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i),\\ \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\alpha_{\mathfrak{L}_2}(\mathbf{y}_j) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\alpha_{\mathfrak{L}_1}(\mathbf{x}_i),\\ \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\beta_{\mathfrak{L}_2}(\mathbf{y}_j) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\beta_{\mathfrak{L}_1}(\mathbf{x}_i).\end{aligned}\right\} \tag{53}$$

Proof. Let $\breve{\mathscr{G}}_1 = \left(\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \cdots, \breve{\rho}'_k\right)$ and $\breve{\mathscr{G}}_2 = \left(\mathfrak{L}_2, \breve{\rho}''_1, \breve{\rho}''_2, \cdots, \breve{\rho}''_k\right)$ be two LDFGSs such that such that $\breve{\rho}''_i \subseteq \mathfrak{L}_1$ and $\breve{\rho}'_i \subseteq \mathfrak{L}_2$, $i = 1, 2, \cdots, k$. Then,

$$\begin{aligned}\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_i, \mathbf{x}_k) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\breve{\rho}''_j}(\mathbf{y}_j, \mathbf{y}_l) \vee \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i)\\ &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i)\\ &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i).\end{aligned}$$

Also,

$$\begin{aligned}\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\breve{\rho}'_i}(\mathbf{x}_i, \mathbf{x}_k) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\breve{\rho}''_j}(\mathbf{y}_j, \mathbf{y}_l) \wedge \varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i)\\ &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i, \mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j, \mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i)\\ &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i).\end{aligned}$$

Similarly, we can show that $\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)=\mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\alpha_{\mathfrak{L}_2}(\mathbf{y}_j)+\mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\alpha_{\mathfrak{L}_1}(\mathbf{x}_i)$ and $\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)=\mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\beta_{\mathfrak{L}_2}(\mathbf{y}_j)$
$+\mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\beta_{\mathfrak{L}_1}(\mathbf{x}_i)$. □

Example 4.13. Consider two LDFGSs $\breve{\mathscr{G}}_1=\left(\mathfrak{L}_1,\breve{\rho}'_1,\breve{\rho}'_2,\breve{\rho}'_3\right)$ and $\breve{\mathscr{G}}_2=\left(\mathfrak{L}_2,\breve{\rho}''_1\right)$, which is depicted in Figure 4 with underlying GSs $\mathscr{G}_1=(\mathscr{V}_1,\mathscr{E}'_1,\mathscr{E}'_2,\mathscr{E}'_3)$ and $\mathscr{G}_2=(\mathscr{V}_2,\mathscr{E}''_1)$, respectively, same as in Example 4.2. The LDFSs $\mathfrak{L}_1$ on $\mathscr{V}_1$ and $\mathfrak{L}_2$ on $\mathscr{V}_2$ are given in the TABLES 24 and 25, respectively. The LDFRs $\breve{\rho}'_1,\breve{\rho}'_2,\breve{\rho}'_3$ over the $\mathscr{E}'_1,\mathscr{E}'_2,\mathscr{E}'_3$, and $\breve{\rho}''_1$ over $\mathscr{E}''_1$ given in TABLES 26, 27, 28 and 29 respectively with $\breve{\rho}'_i\subseteq\mathfrak{L}_2$ and $\breve{\rho}''_i\subseteq\mathfrak{L}_1$, for $i=1,2,3$. By using Definition 4.1, the LDFS $\mathfrak{L}=\mathfrak{L}_1*\mathfrak{L}_2$ is shown in TABLE 30 and LDFRs $\breve{\rho}_i=\breve{\rho}'_i*\breve{\rho}''_i$ for $i=1,2,3$ shown in TABLE 31, 32, 33, respectively. The resulting LDFGS $\breve{\mathscr{G}}=\breve{\mathscr{G}}_1*\breve{\mathscr{G}}_2=\left(\mathfrak{L},\breve{\rho}_1,\breve{\rho}_2,\breve{\rho}_3\right)$ is illustrated in FIGURE 5.

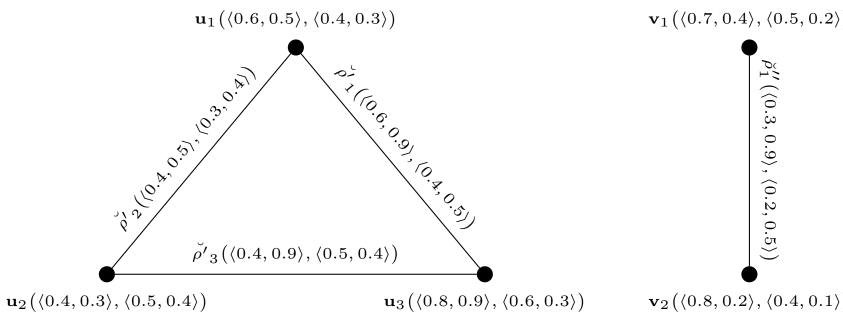

Figure 4: LDFGSs $\breve{\mathscr{G}}_1=\left(\mathfrak{L}_1,\breve{\rho}'_1,\breve{\rho}'_2,\breve{\rho}'_3\right)$ and $\breve{\mathscr{G}}_2=\left(\mathfrak{L}_2,\breve{\rho}''_1\right)$

Table 24: LDFS $\mathfrak{L}_1$

$\mathscr{V}_1$	$\left(\langle\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}),\varkappa^n_{\mathfrak{L}_1}(\mathbf{x})\rangle,\langle\alpha_{\mathfrak{L}_1}(\mathbf{x}),\beta_{\mathfrak{L}_1}(\mathbf{x})\rangle\right)$
$\mathbf{u}_1$	$\left(\langle 0.6,0.5\rangle,\langle 0.4,0.3\rangle\right)$
$\mathbf{u}_2$	$\left(\langle 0.4,0.3\rangle,\langle 0.5,0.4\rangle\right)$
$\mathbf{u}_3$	$\left(\langle 0.8,0.9\rangle,\langle 0.6,0.3\rangle\right)$

Table 25: LDFS $\mathfrak{L}_2$

$\mathscr{V}_2$	$\left(\langle\varkappa^m_{\mathfrak{L}_2}(\mathbf{x}),\varkappa^n_{\mathfrak{L}_2}(\mathbf{x})\rangle,\langle\alpha_{\mathfrak{L}_2}(\mathbf{x}),\beta_{\mathfrak{L}_2}(\mathbf{x})\rangle\right)$
$\mathbf{v}_1$	$\left(\langle 0.7,0.4\rangle,\langle 0.5,0.2\rangle\right)$
$\mathbf{v}_2$	$\left(\langle 0.8,0.2\rangle,\langle 0.4,0.1\rangle\right)$

Table 26: $\breve{\rho'}_1$

$\mathscr{E}'_1$	$\left(\langle\varkappa^m_{\breve{\rho'}_1}(\mathbf{x},\mathbf{y}),\varkappa^n_{\breve{\rho'}_1}(\mathbf{x},\mathbf{y})\rangle,\langle\alpha_{\breve{\rho'}_1}(\mathbf{x},\mathbf{y}),\beta_{\breve{\rho'}_1}(\mathbf{x},\mathbf{y})\rangle\right)$
$(\mathbf{u}_1,\mathbf{u}_3)$	$\left(\langle 0.6,0.9\rangle,\langle 0.4,0.5\rangle\right)$

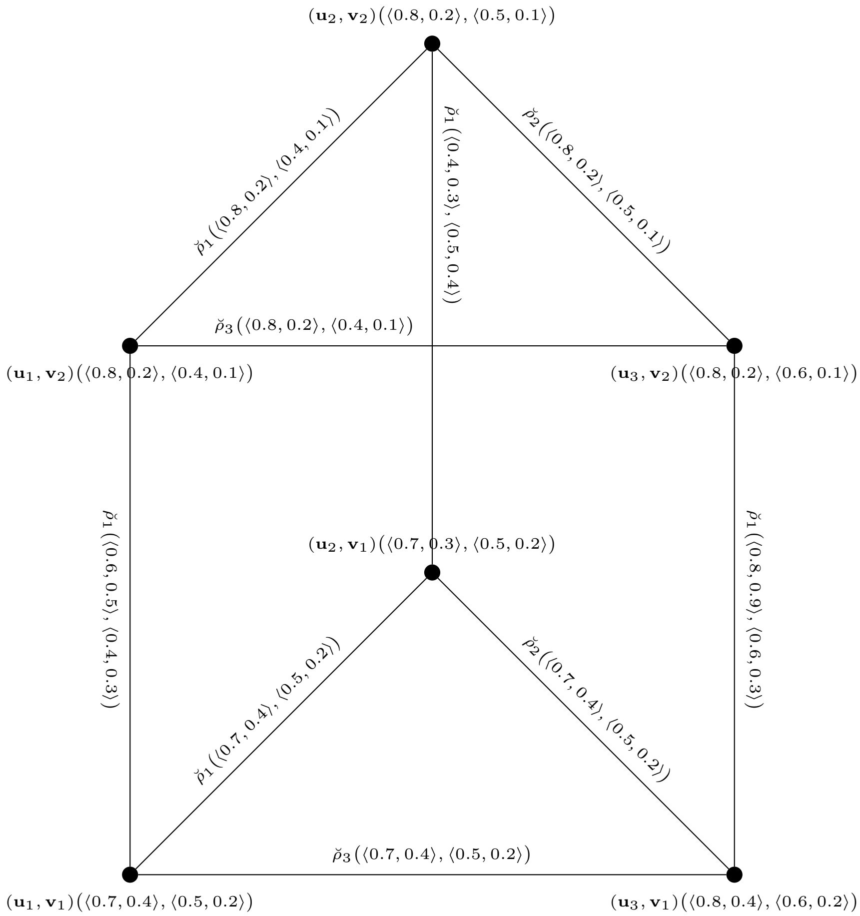

Figure 5: Maximal product $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$

Table 27: $\breve{\rho'}_2$

$\mathscr{E}'_2$	$\big(\langle \varkappa^m_{\breve{\rho'}_2}(\mathbf{x},\mathbf{y}), \varkappa^n_{\breve{\rho'}_2}(\mathbf{x},\mathbf{y})\rangle, \langle \alpha_{\breve{\rho'}_2}(\mathbf{x},\mathbf{y}), \beta_{\breve{\rho'}_2}(\mathbf{x},\mathbf{y})\rangle\big)$
$(\mathbf{u}_1,\mathbf{u}_2)$	$\big(\langle 0.4,0.5\rangle,\langle 0.3,0.4\rangle\big)$

Table 28: $\breve{\rho'}_3$

$\mathscr{E}'_3$	$\big(\langle \varkappa^m_{\breve{\rho'}_3}(\mathbf{x},\mathbf{y}), \varkappa^n_{\breve{\rho'}_3}(\mathbf{x},\mathbf{y})\rangle, \langle \alpha_{\breve{\rho'}_3}(\mathbf{x},\mathbf{y}), \beta_{\breve{\rho'}_3}(\mathbf{x},\mathbf{y})\rangle\big)$
$(\mathbf{u}_2,\mathbf{u}_3)$	$\big(\langle 0.4,0.9\rangle,\langle 0.5,0.4\rangle\big)$

Table 29: $\breve{\rho''}_1$

$\mathscr{E}''_1$	$\big(\langle \varkappa^m_{\breve{\rho''}_1}(\mathbf{x},\mathbf{y}), \varkappa^n_{\breve{\rho''}_1}(\mathbf{x},\mathbf{y})\rangle, \langle \alpha_{\breve{\rho''}_1}(\mathbf{x},\mathbf{y}), \beta_{\breve{\rho''}_1}(\mathbf{x},\mathbf{y})\rangle\big)$
$(\mathbf{v}_1,\mathbf{v}_2)$	$\big(\langle 0.3,0.9\rangle,\langle 0.2,0.5\rangle\big)$

Table 30: $\mathfrak{L} = \mathfrak{L}_1 * \mathfrak{L}_2$

$\mathscr{V}$	$\big(\langle \varkappa^m_{\mathfrak{L}}(\mathbf{x},\mathbf{y}), \varkappa^n_{\mathfrak{L}}(\mathbf{x},\mathbf{y})\rangle, \langle \alpha_{\mathfrak{L}}(\mathbf{x},\mathbf{y}), \beta_{\mathfrak{L}}(\mathbf{x},\mathbf{y})\rangle\big)$
$(\mathbf{u}_1,\mathbf{v}_1)$	$\big(\langle 0.7, 0.4\rangle, \langle 0.5, 0.2\rangle\big)$
$(\mathbf{u}_1,\mathbf{v}_2)$	$\big(\langle 0.8, 0.2\rangle, \langle 0.4, 0.1\rangle\big)$
$(\mathbf{u}_2,\mathbf{v}_1)$	$\big(\langle 0.7, 0.3\rangle, \langle 0.5, 0.2\rangle\big)$
$(\mathbf{u}_2,\mathbf{v}_2)$	$\big(\langle 0.8, 0.2\rangle, \langle 0.5, 0.1\rangle\big)$
$(\mathbf{u}_3,\mathbf{v}_1)$	$\big(\langle 0.8, 0.4\rangle, \langle 0.6, 0.2\rangle\big)$
$(\mathbf{u}_3,\mathbf{v}_2)$	$\big(\langle 0.8, 0.2\rangle, \langle 0.6, 0.1\rangle\big)$

Table 31: $\breve{\rho}_1$

$\mathscr{E}_1$	$\big(\langle \varkappa^m_{\breve{\rho}_1}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2), \varkappa^n_{\breve{\rho}_1}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2)\rangle, \langle \alpha_{\breve{\rho}_1}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2), \beta_{\breve{\rho}_1}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2)\rangle\big)$
$(\mathbf{u}_1\mathbf{v}_1,\mathbf{u}_1\mathbf{v}_2)$	$\big(\langle 0.6, 0.5\rangle, \langle 0.4, 0.3\rangle\big)$
$(\mathbf{u}_1\mathbf{v}_1,\mathbf{u}_2\mathbf{v}_1)$	$\big(\langle 0.7, 0.4\rangle, \langle 0.5, 0.2\rangle\big)$
$(\mathbf{u}_2\mathbf{v}_1,\mathbf{u}_2\mathbf{v}_2)$	$\big(\langle 0.4, 0.3\rangle, \langle 0.5, 0.4\rangle\big)$
$(\mathbf{u}_3\mathbf{v}_1,\mathbf{u}_3\mathbf{v}_2)$	$\big(\langle 0.8, 0.9\rangle, \langle 0.6, 0.3\rangle\big)$
$(\mathbf{u}_1\mathbf{v}_2,\mathbf{u}_2\mathbf{v}_2)$	$\big(\langle 0.8, 0.2\rangle, \langle 0.4, 0.1\rangle\big)$

Table 32: $\breve{\rho}_2$

$\mathscr{E}_2$	$\big(\langle \varkappa^m_{\breve{\rho}_2}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2), \varkappa^n_{\breve{\rho}_2}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2)\rangle, \langle \alpha_{\breve{\rho}_2}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2), \beta_{\breve{\rho}_2}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2)\rangle\big)$
$(\mathbf{u}_2\mathbf{v}_1,\mathbf{u}_3\mathbf{v}_1)$	$\big(\langle 0.7, 0.4\rangle, \langle 0.5, 0.2\rangle\big)$
$(\mathbf{u}_2\mathbf{v}_2,\mathbf{u}_3\mathbf{v}_2)$	$\big(\langle 0.8, 0.2\rangle, \langle 0.5, 0.1\rangle\big)$

Table 33: $\breve{\rho}_3$

$\mathscr{E}_3$	$\big(\langle \varkappa^m_{\breve{\rho}_3}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2), \varkappa^n_{\breve{\rho}_3}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2)\rangle, \langle \alpha_{\breve{\rho}_3}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2), \beta_{\breve{\rho}_3}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2)\rangle\big)$
$(\mathbf{u}_1\mathbf{v}_1,\mathbf{u}_3\mathbf{v}_1)$	$\big(\langle 0.7, 0.4\rangle, \langle 0.5, 0.2\rangle\big)$
$(\mathbf{u}_1\mathbf{v}_2,\mathbf{u}_3\mathbf{v}_2)$	$\big(\langle 0.8, 0.2\rangle, \langle 0.4, 0.1\rangle\big)$

Then, using the formula given in Theorem 4.12, we calculate the degrees of the vertices in the maximal product as follows:

$$\begin{aligned}
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_1) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_1)\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1)+\mathbb{D}_{\mathscr{G}_2}(\mathbf{v}_1)\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_1) = (2)(0.7)+(1)(0.6) = 2\\
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_2) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_1)\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2)+\mathbb{D}_{\mathscr{G}_2}(\mathbf{v}_2)\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_1) = (2)(0.8)+(1)(0.6) = 2.2\\
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_1) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_2)\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1)+\mathbb{D}_{\mathscr{G}_2}(\mathbf{v}_1)\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_2) = (2)(0.7)+(1)(0.4) = 1.8\\
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_2) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_2)\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2)+\mathbb{D}_{\mathscr{G}_2}(\mathbf{v}_2)\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_2) = (2)(0.8)+(1)(0.4) = 2\\
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3,\mathbf{v}_1) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_3)\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1)+\mathbb{D}_{\mathscr{G}_2}(\mathbf{v}_1)\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_3) = (2)(0.7)+(1)(0.8) = 2.2\\
\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3,\mathbf{v}_2) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_3)\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2)+\mathbb{D}_{\mathscr{G}_2}(\mathbf{v}_2)\varkappa^m_{\mathfrak{L}_1}(\mathbf{u}_3) = (2)(0.8)+(1)(0.8) = 2.4
\end{aligned}$$

And,

$$\begin{aligned}
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_1) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_1)\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1)+\mathbb{D}_{\mathscr{G}_2}(\mathbf{v}_1)\varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_1) = (2)(0.4)+(1)(0.5) = 1.3\\
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_2) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_1)\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2)+\mathbb{D}_{\mathscr{G}_2}(\mathbf{v}_2)\varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_1) = (2)(0.2)+(1)(0.5) = 0.9\\
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_1) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_2)\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1)+\mathbb{D}_{\mathscr{G}_2}(\mathbf{v}_1)\varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_2) = (2)(0.4)+(1)(0.3) = 1.1\\
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_2) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_2)\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2)+\mathbb{D}_{\mathscr{G}_2}(\mathbf{v}_2)\varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_2) = (2)(0.2)+(1)(0.3) = 0.7\\
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3,\mathbf{v}_1) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_3)\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1)+\mathbb{D}_{\mathscr{G}_2}(\mathbf{v}_1)\varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_3) = (2)(0.4)+(1)(0.9) = 1.7\\
\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{u}_3,\mathbf{v}_2) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_3)\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2)+\mathbb{D}_{\mathscr{G}_2}(\mathbf{v}_2)\varkappa^n_{\mathfrak{L}_1}(\mathbf{u}_3) = (2)(0.6)+(1)(0.9) = 1.3
\end{aligned}$$

In the similar way, we get $\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)$ and $\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)$ for all $\mathbf{x}_i \in \mathscr{V}_1$ and $\mathbf{y}_i \in \mathscr{V}_2$, which are shown in TABLE 34.

Table 34: $\mathbb{D}_{\breve{\mathscr{G}}}$

$\mathscr{V}$	$\Big(\big\langle \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j), \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle, \big\langle \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j), \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle\Big)$
$(\mathbf{u}_1,\mathbf{v}_1)$	$(\langle 2, 1.3\rangle, \langle 1.4, 0.7\rangle)$
$(\mathbf{u}_1,\mathbf{v}_2)$	$(\langle 2.2, 0.9\rangle, \langle 1.2, 0.5\rangle)$
$(\mathbf{u}_2,\mathbf{v}_1)$	$(\langle 1.8, 1.1\rangle, \langle 1.5, 0.8\rangle)$
$(\mathbf{u}_2,\mathbf{v}_2)$	$(\langle 2, 0.7\rangle, \langle 1.3, 0.6\rangle)$
$(\mathbf{u}_3,\mathbf{v}_1)$	$(\langle 2.2, 1.7\rangle, \langle 1.6, 0.7\rangle)$
$(\mathbf{u}_3,\mathbf{v}_2)$	$(\langle 2.4, 1.3\rangle, \langle 1.4, 0.5\rangle)$

Theorem 4.14. *If $\breve{\mathscr{G}}_1 = (\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \cdots, \breve{\rho}'_k)$ and $\breve{\mathscr{G}}_2 = (\mathfrak{L}_2, \breve{\rho}''_1, \breve{\rho}''_2, \cdots, \breve{\rho}''_k)$ are two LDFGSs, such that $\breve{\rho}''_i \supseteq \mathfrak{L}_1$, $i = 1, 2, \cdots, k$, then the total degree of any vertex in maximal product $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ is described as:*

$$\mathbb{TD}_{\breve{\mathscr{G}}}(\mathbf{x}_i,\mathbf{y}_j) = \Big(\big\langle \varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j), \varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle, \big\langle \alpha_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j), \beta_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle\Big), \tag{54}$$

where

$$\left.\begin{aligned} \varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j),\\ \varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{y}_j),\\ \alpha_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\alpha_{\mathfrak{L}_2}(\mathbf{y}_j) + \alpha_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j),\\ \beta_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\beta_{\mathfrak{L}_2}(\mathbf{y}_j) + \beta_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j). \end{aligned}\right\} \tag{55}$$

Proof. Let $\breve{\mathscr{G}}_1 = \left(\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \cdots, \breve{\rho}'_k\right)$ and $\breve{\mathscr{G}}_2 = \left(\mathfrak{L}_2, \breve{\rho}''_1, \breve{\rho}''_2, \cdots, \breve{\rho}''_k\right)$ be two LDFGSs such that such that $\breve{\rho}''_i \supseteq \mathfrak{L}_1$, then $\breve{\rho}'_i \supseteq \mathfrak{L}_2$ and $\mathfrak{L}_1 \subseteq \mathfrak{L}_2$ $i = 1, 2, \cdots, k$. We have,

$$\begin{aligned} \varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l) \vee \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i) + \varkappa^m_{\mathfrak{L}}(\mathbf{x}_i,\mathbf{y}_j)\\ &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \varkappa^m_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l) + \left[\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i) \vee \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j)\right]\\ &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \left(\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j) + \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j)\right)\\ &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j). \end{aligned}$$

Also,

$$\begin{aligned} \varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\breve{\rho}'_i}(\mathbf{x}_i,\mathbf{x}_k) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l) \wedge \varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i) + \varkappa^n_{\mathfrak{L}}(\mathbf{x}_i,\mathbf{y}_j)\\ &= \sum_{(\mathbf{x}_i,\mathbf{x}_k)\in\mathscr{E}'_i,\mathbf{y}_j=\mathbf{y}_l} \varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \sum_{(\mathbf{y}_j,\mathbf{y}_l)\in\mathscr{E}''_j,\mathbf{x}_i=\mathbf{x}_k} \varkappa^n_{\breve{\rho}''_j}(\mathbf{y}_j,\mathbf{y}_l) + \left[\varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i) \wedge \varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j)\right]\\ &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \left(\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j) + \varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j)\right)\\ &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j). \end{aligned}$$

Similarly, we can show that $\alpha_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\alpha_{\mathfrak{L}_2}(\mathbf{y}_j)+\alpha_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j)$ and $\beta_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\beta_{\mathfrak{L}_2}(\mathbf{y}_j)+\beta_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j)$. □

Example 4.15. Let $\mathscr{G}_1 = (\mathscr{V}_1, \mathscr{E}'_1)$ and $\mathscr{G}_2 = (\mathscr{V}_2, \mathscr{E}'_2)$ be GSs with $\mathscr{V}_1 = \{\mathbf{u}_1, \mathbf{u}_2\}$, $\mathscr{V}_2 = \{\mathbf{v}_1, \mathbf{v}_2\}$, $\mathscr{E}'_1 = \{(\mathbf{u}_1, \mathbf{u}_2)\}$ and $\mathscr{E}''_1 = \{(\mathbf{v}_1, \mathbf{v}_2)\}$. The LDFGSs $\breve{\mathscr{G}}_1 = (\mathfrak{L}_1, \breve{\rho}'_1)$ and $\breve{\mathscr{G}}_2 = (\mathfrak{L}_2, \breve{\rho}''_1)$ with underlying GSs $\mathscr{G}_1$ and $\mathscr{G}_2$, respectively are shown in FIGURE 6, where $\mathfrak{L}_1$ on $\mathscr{V}_1$ and $\mathfrak{L}_2$ on $\mathscr{V}_2$ are given in TABLES 35 and 36, respectively, and LDFRs $\breve{\rho}'_1$ and $\breve{\rho}''_1$ presented in TABLES 37 and 38, respectively with the condition $\mathfrak{L}_1 \subseteq \breve{\rho}''_1$. By using the Definition 4.1, we obtain the maximal LDFGS $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2 = \left(\mathfrak{L}, \breve{\rho}\right)$ is portrayed in FIGURE 7, where $\mathfrak{L} = \mathfrak{L}_1 * \mathfrak{L}_2$ given in TABLE 39 on $\mathscr{V} = \mathscr{V}_1 \times \mathscr{V}_2 = \left\{(\mathbf{u}_1, \mathbf{v}_1), (\mathbf{u}_1, \mathbf{v}_2), (\mathbf{u}_2, \mathbf{v}_1), (\mathbf{u}_2, \mathbf{v}_2)\right\}$ and LDFR $\breve{\rho}_1 = \breve{\rho}'_1 * \breve{\rho}''_1$ on $\mathscr{E}_1 = \mathscr{E}'_1 \times \mathscr{E}''_1 = \left\{(\mathbf{u}_1\mathbf{v}_1, \mathbf{u}_1\mathbf{v}_2), (\mathbf{u}_1\mathbf{v}_1, \mathbf{u}_2\mathbf{v}_1), (\mathbf{u}_1\mathbf{v}_2, \mathbf{u}_2\mathbf{v}_2), (\mathbf{u}_2\mathbf{v}_1, \mathbf{u}_2\mathbf{v}_2)\right\}$ presented in TABLE 40.

Table 35: $\mathfrak{L}_1$

$\mathscr{V}_1$	$\left(\langle\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}), \varkappa^n_{\mathfrak{L}_1}(\mathbf{x})\rangle, \langle\alpha_{\mathfrak{L}_1}(\mathbf{x}), \beta_{\mathfrak{L}_1}(\mathbf{x})\rangle\right)$
$\mathbf{u}_1$	$\left(\langle 0.6, 0.5\rangle, \langle 0.4, 0.2\rangle\right)$
$\mathbf{u}_2$	$\left(\langle 0.5, 0.7\rangle, \langle 0.3, 0.5\rangle\right)$

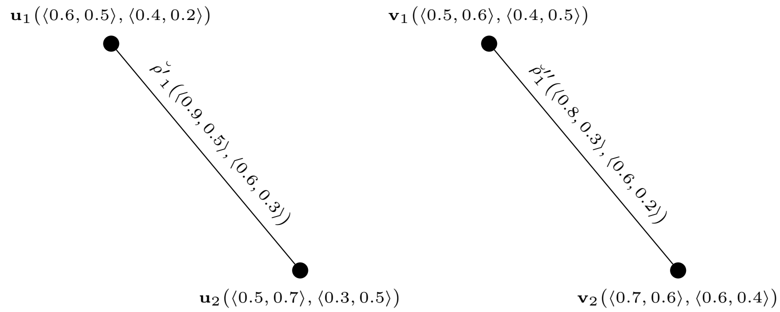

Figure 6: LDFGSs $\breve{\mathscr{G}}_1 = \left(\mathfrak{L}_1, \breve{\rho}'_1\right)$ and $\breve{\mathscr{G}}_2 = \left(\mathfrak{L}_2, \breve{\rho}''_1\right)$

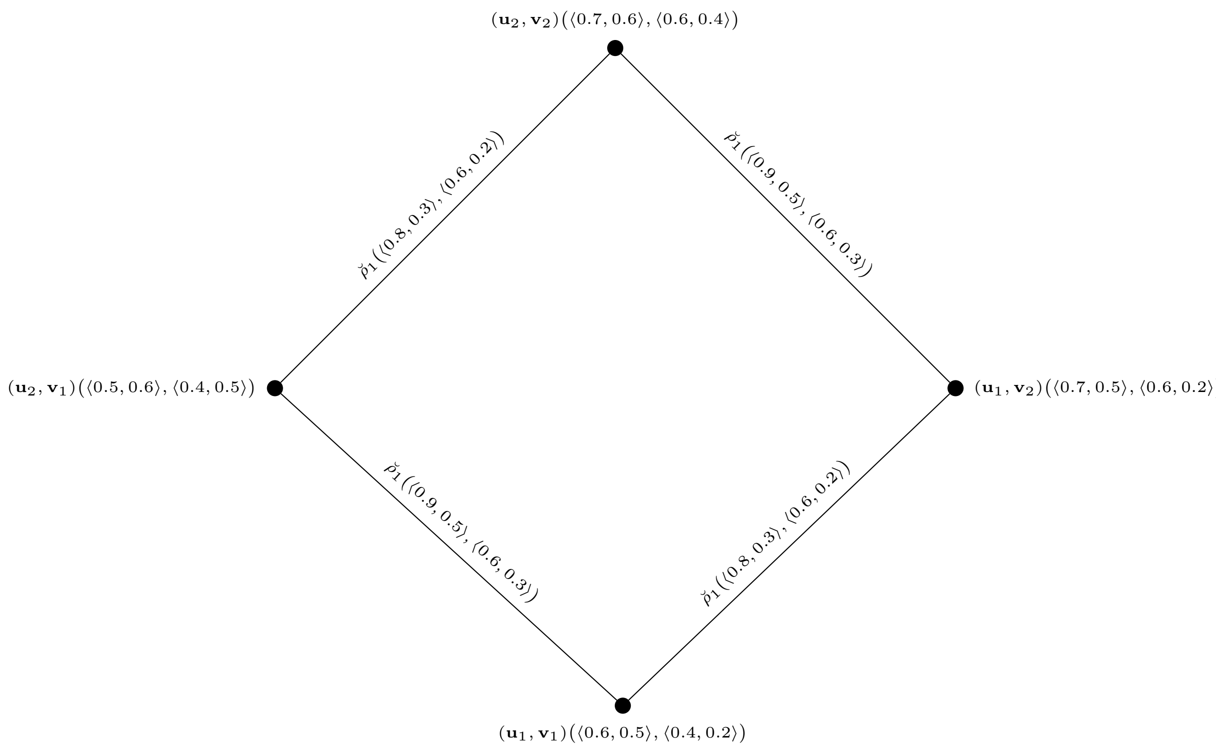

Figure 7: The maximal LDFGS $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$

Table 36: $\mathfrak{L}_2$

$\mathscr{V}_2$	$\left(\langle \varkappa^m_{\mathfrak{L}_2}(\mathbf{x}), \varkappa^n_{\mathfrak{L}_2}(\mathbf{x})\rangle, \langle \alpha_{\mathfrak{L}_2}(\mathbf{x}), \beta_{\mathfrak{L}_2}(\mathbf{x})\rangle\right)$
$\mathbf{v}_1$	$\left(\langle 0.5, 0.6\rangle, \langle 0.4, 0.5\rangle\right)$
$\mathbf{v}_2$	$\left(\langle 0.7, 0.6\rangle, \langle 0.6, 0.4\rangle\right)$

Using the Formula given Theorem 4.14, we calculate the total degrees of all the vertices of the maximal product in the sequel:

Table 37: $\breve{\rho'}_1$

$\mathscr{E}_1'$	$\big(\langle \varkappa^m_{\breve{\rho'}_1}(\mathbf{x}), \varkappa^n_{\breve{\rho'}_1}(\mathbf{x})\rangle, \langle \alpha_{\breve{\rho'}_1}(\mathbf{x}), \beta_{\breve{\rho'}_1}(\mathbf{x})\rangle\big)$
$(\mathbf{u}_1, \mathbf{u}_2)$	$\big(\langle 0.9, 0.5\rangle, \langle 0.6, 0.3\rangle\big)$

Table 38: $\breve{\rho}_1''$

$\mathscr{E}_1''$	$\big(\langle \varkappa^m_{\breve{\rho}_1''}(\mathbf{x}), \varkappa^n_{\breve{\rho}_1''}(\mathbf{x})\rangle, \langle \alpha_{\breve{\rho}_1''}(\mathbf{x}), \beta_{\breve{\rho}_1''}(\mathbf{x})\rangle\big)$
$(\mathbf{v}_1, \mathbf{v}_2)$	$\big(\langle 0.8, 0.3\rangle, \langle 0.6, 0.2\rangle\big)$

Table 39: $\mathfrak{L} = \mathfrak{L}_1 * \mathfrak{L}_2$

$\mathscr{V}$	$\big(\langle \varkappa^m_{\mathfrak{L}}(\mathbf{x},\mathbf{y}), \varkappa^n_{\mathfrak{L}}(\mathbf{x},\mathbf{y})\rangle, \langle \alpha_{\mathfrak{L}}(\mathbf{x},\mathbf{y}), \beta_{\mathfrak{L}}(\mathbf{x},\mathbf{y})\rangle\big)$
$(\mathbf{u}_1, \mathbf{v}_1)$	$\big(\langle 0.6, 0.5\rangle, \langle 0.4, 0.2\rangle\big)$
$(\mathbf{u}_1, \mathbf{v}_2)$	$\big(\langle 0.7, 0.5\rangle, \langle 0.6, 0.2\rangle\big)$
$(\mathbf{u}_2, \mathbf{v}_1)$	$\big(\langle 0.5, 0.6\rangle, \langle 0.4, 0.5\rangle\big)$
$(\mathbf{u}_2, \mathbf{v}_2)$	$\big(\langle 0.7, 0.6\rangle, \langle 0.6, 0.4\rangle\big)$

Table 40: $\breve{\rho}_1 = \breve{\rho}_1' \times \breve{\rho}_1''$

$\mathscr{E}_1$	$\big(\langle \varkappa^m_{\breve{\rho}_1}(\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2), \varkappa^n_{\breve{\rho}_1}(\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2)\rangle, \langle \alpha_{\breve{\rho}_1}(\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2), \beta_{\breve{\rho}_1}(\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_2)\rangle\big)$
$(\mathbf{u}_1\mathbf{v}_1, \mathbf{u}_1\mathbf{v}_2)$	$\big(\langle 0.8, 0.3\rangle, \langle 0.6, 0.2\rangle\big)$
$(\mathbf{u}_1\mathbf{v}_1, \mathbf{u}_2\mathbf{v}_1)$	$\big(\langle 0.9, 0.5\rangle, \langle 0.6, 0.3\rangle\big)$
$(\mathbf{u}_1\mathbf{v}_2, \mathbf{u}_2\mathbf{v}_2)$	$\big(\langle 0.9, 0.5\rangle, \langle 0.6, 0.3\rangle\big)$
$(\mathbf{u}_2\mathbf{v}_1, \mathbf{u}_2\mathbf{v}_2)$	$\big(\langle 0.8, 0.3\rangle, \langle 0.6, 0.2\rangle\big)$

$$\varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{u}_1, \mathbf{v}_1) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_1)\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1) + \varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{v}_1) = (1)(0.5) + (0.8 + 0.5) = 1.8$$
$$\varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{u}_1, \mathbf{v}_2) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_1)\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2) + \varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{v}_2) = (1)(0.7) + (0.8 + 0.7) = 2.2$$
$$\varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{u}_2, \mathbf{v}_1) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_2)\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_1) + \varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{v}_1) = (1)(0.5) + (0.8 + 0.5) = 1.8$$
$$\varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{u}_2, \mathbf{v}_2) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_2)\varkappa^m_{\mathfrak{L}_2}(\mathbf{v}_2) + \varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{v}_2) = (1)(0.7) + (0.8 + 0.7) = 2.2$$

Also,

$$\varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_1)=\mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_1)\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1)+\varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{v}_1)=(1)(0.6)+(0.3+0.6)=1.5$$
$$\varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{u}_1,\mathbf{v}_2)=\mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_1)\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2)+\varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{v}_2)=(1)(0.6)+(0.3+0.6)=1.5$$
$$\varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_1)=\mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_2)\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_1)+\varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{v}_1)=(1)(0.6)+(0.3+0.6)=1.5$$
$$\varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{u}_2,\mathbf{v}_2)=\mathbb{D}_{\mathscr{G}_1}(\mathbf{u}_2)\varkappa^n_{\mathfrak{L}_2}(\mathbf{v}_2)+\varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{v}_2)=(1)(0.6)+(0.3+0.6)=1.5$$

In the similar way, we've calculated $\alpha_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)=\mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\alpha_{\mathfrak{L}_2}(\mathbf{y}_j)+\alpha_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j)$, and $\beta_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)=\mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\beta_{\mathfrak{L}_2}(\mathbf{y}_j)+\beta_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j)$ for all $\mathbf{x}_i\in\mathscr{V}_1$, and $\mathbf{y}_j\in\mathscr{V}_2$, which are listed in the TABLE 41.

Table 41: $\mathbb{TD}_{\breve{\mathscr{G}}}(\mathbf{x}_i,\mathbf{y}_j)$

$\mathscr{V}$	$\big(\langle\varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j),\varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\rangle,\langle\alpha_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j),\beta_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\rangle\big)$
$(\mathbf{u}_1,\mathbf{v}_1)$	$\big(\langle 1.8,1.5\rangle,\langle 1.4,1.2\rangle\big)$
$(\mathbf{u}_1,\mathbf{v}_2)$	$\big(\langle 2.2,1.5\rangle,\langle 1.8,1.1\rangle\big)$
$(\mathbf{u}_2,\mathbf{v}_1)$	$\big(\langle 1.8,1.5\rangle,\langle 1.4,1.2\rangle\big)$
$(\mathbf{u}_2,\mathbf{v}_2)$	$\big(\langle 2.2,1.5\rangle,\langle 1.8,1.1\rangle\big)$

Theorem 4.16. *If $\breve{\mathscr{G}}_1=\big(\mathfrak{L}_1,\breve{\rho}'_1,\breve{\rho}'_2,\cdots,\breve{\rho}'_k\big)$ and $\breve{\mathscr{G}}_2=\big(\mathfrak{L}_2,\breve{\rho}''_1,\breve{\rho}''_2,\cdots,\breve{\rho}''_k\big)$ are two LDFGSs, such that $\breve{\rho}''_i\supseteq\mathfrak{L}_1$, $i=1,2,\cdots,k$, and $\mathfrak{L}_2$ is constant LDFS of LDF value $\big(\langle a,b\rangle,\langle c,d\rangle\big)$, where $a,b,c,d\in[0,1]$ are fixed, then the total degree of any vertex in maximal product $\breve{\mathscr{G}}=\breve{\mathscr{G}}_1*\breve{\mathscr{G}}_2$ is characterized as:*

$$\mathbb{TD}_{\breve{\mathscr{G}}}(\mathbf{x}_i,\mathbf{y}_j)=\Big(\big\langle\varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j),\varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle,\big\langle\alpha_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j),\beta_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle\Big),\tag{56}$$

where

$$\left.\begin{aligned}\varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)&=\varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j)+\mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)a,\\ \varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)&=\varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j)+\mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)b,\\ \alpha_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)&=\alpha_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j)+\mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)c,\\ \beta_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)&=\alpha_{\mathbb{TD}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j)+\mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)d.\end{aligned}\right\}\tag{57}$$

Proof. Analogous to the proof of Theorems 4.9 and 4.14. □

Theorem 4.17. *If $\breve{\mathscr{G}}_1=\big(\mathfrak{L}_1,\breve{\rho}'_1,\breve{\rho}'_2,\cdots,\breve{\rho}'_k\big)$ and $\breve{\mathscr{G}}_2=\big(\mathfrak{L}_2,\breve{\rho}''_1,\breve{\rho}''_2,\cdots,\breve{\rho}''_k\big)$ are two LDFGSs, such that $\breve{\rho}'_i\supseteq\mathfrak{L}_2$, $i=1,2,\cdots,k$, then the total degree of any vertex in maximal product $\breve{\mathscr{G}}=\breve{\mathscr{G}}_1*\breve{\mathscr{G}}_2$ is postulated as:*

$$\mathbb{TD}_{\breve{\mathscr{G}}}(\mathbf{x}_i,\mathbf{y}_j)=\Big(\big\langle\varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j),\varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle,\big\langle\alpha_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j),\beta_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle\Big),\tag{58}$$

where

$$\left.\begin{aligned}\varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)&=\mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i)+\varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i),\\ \varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)&=\mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i)+\varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i),\\ \alpha_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)&=\mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\alpha_{\mathfrak{L}_1}(\mathbf{x}_i)+\alpha_{\mathbb{TD}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i),\\ \beta_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)&=\mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\beta_{\mathfrak{L}_1}(\mathbf{x}_i)+\beta_{\mathbb{TD}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i).\end{aligned}\right\}\tag{59}$$

Proof. Identical to the proof of Theorems 4.10 and 4.14. □

Theorem 4.18. *If $\breve{\mathscr{G}}_1 = \big(\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, \cdots, \breve{\rho}'_k\big)$ and $\breve{\mathscr{G}}_2 = \big(\mathfrak{L}_2, \breve{\rho}''_1, \breve{\rho}''_2, \cdots, \breve{\rho}''_k\big)$ are two LDFGSs, such that $\breve{\rho}'_i \supseteq \mathfrak{L}_2$, $i = 1, 2, \cdots, k$, and $\mathfrak{L}_1$ is constant LDFS of LDF value $\big(\langle a, b\rangle, \langle c, d\rangle\big)$, where $a, b, c, d \in [0, 1]$ are fixed, then the total degree of any vertex in maximal product $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ is given by:*

$$\mathbb{TD}_{\breve{\mathscr{G}}}(\mathbf{x}_i, \mathbf{y}_j) = \Big(\big\langle \varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\big\rangle, \big\langle \alpha_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\big\rangle\Big), \tag{60}$$

where

$$\left.\begin{aligned}
\varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{y}_j)a + \varkappa^m_{\mathbb{TD}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i),\\
\varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{y}_j)b + \varkappa^n_{\mathbb{TD}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i),\\
\alpha_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{y}_j)c + \alpha_{\mathbb{TD}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i),\\
\beta_{\mathbb{TD}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{y}_j)d + \beta_{\mathbb{TD}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i).
\end{aligned}\right\} \tag{61}$$

Proof. Analogous to the proof of Theorems 4.11 and 4.14. □

5 Regular Linear Diophantine Fuzzy Graph Structures

In this section, we have defined the notions of $\breve{\rho}_i$-regular and regular LDFGSs. Some fascinating consequences are also proved with illustrative examples.

Definition 5.1. An LDFGS $\breve{\mathscr{G}}$ is said to be $\big(\langle \mathbf{a}, \mathbf{b}\rangle, \langle \mathbf{c}, \mathbf{d}\rangle\big)$-$\breve{\rho}_i$ regular, if $\mathbb{D}_{\breve{\rho}_i}(\mathbf{x}) = \big(\langle \mathbf{a}, \mathbf{b}\rangle, \langle \mathbf{c}, \mathbf{d}\rangle\big)$, for all $\mathbf{x} \in \mathscr{V}$. Moreover, $\breve{\mathscr{G}}$ is called $\big(\langle \mathbf{a}, \mathbf{b}\rangle, \langle \mathbf{c}, \mathbf{d}\rangle\big)$-regular, if $\mathbb{D}(\mathbf{x}) = \big(\langle \mathbf{a}, \mathbf{b}\rangle, \langle \mathbf{c}, \mathbf{d}\rangle\big)$, for all $\mathbf{x} \in \mathscr{V}$.

Example 5.2. From Example 3.2, we can easily see that $\breve{\mathscr{G}}$ is neither $\breve{\rho}_1$ nor $\breve{\rho}_2$ regular. Also, not regular LDFGS.

Remark 5.3. The maximal product of two regular LDFGSs may not be regular, which can justified through Example 5.4.

Example 5.4. Let $\mathscr{V}_1 = \{\mathbf{u}_1, \mathbf{u}_2\}$, $\mathscr{V}_2 = \{\mathbf{v}_1, \mathbf{v}_2\}$, $\mathscr{E}'_1 = \{(\mathbf{u}_1, \mathbf{u}_2)\}$ and $\mathscr{E}''_1 = \{(\mathbf{v}_1, \mathbf{v}_2)\}$. Then, $\mathscr{G}_1 = (\mathscr{V}_1, \mathscr{E}'_1)$ and $\mathscr{G}_2 = (\mathscr{V}_2, \mathscr{E}'_2)$ are GSs.

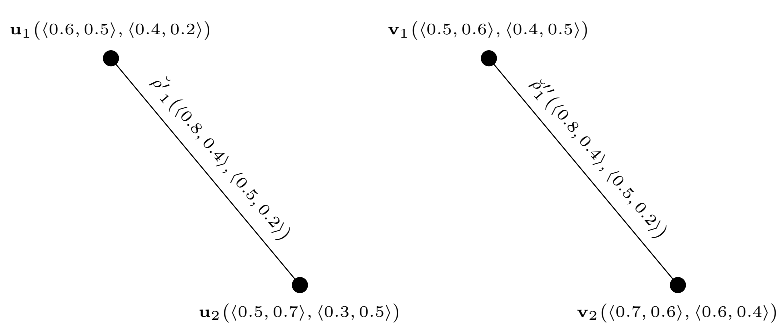

Figure 8: LDFGSs $\breve{\mathscr{G}}_1 = \big(\mathfrak{L}_1, \breve{\rho}'_1\big)$ and $\breve{\mathscr{G}}_2 = \big(\mathfrak{L}_2, \breve{\rho}''_1\big)$

Consider LDFSs $\mathfrak{L}_1$ on $\mathscr{V}_1$ and $\mathfrak{L}_2$ on $\mathscr{V}_2$ which are given in TABLES 42 and 43, respectively. LDFRs $\breve{\rho}'_1$ and $\breve{\rho}''_1$ are exhibited in TABLES 44 and 45, respectively.

Table 42: $\mathfrak{L}_1$

$\mathscr{V}_1$	$\big(\langle \varkappa^m_{\mathfrak{L}_1}(\mathbf{x}), \varkappa^n_{\mathfrak{L}_1}(\mathbf{x})\rangle, \langle \alpha_{\mathfrak{L}_1}(\mathbf{x}), \beta_{\mathfrak{L}_1}(\mathbf{x})\rangle\big)$
$\mathbf{u}_1$	$\big(\langle 0.6, 0.5\rangle, \langle 0.4, 0.2\rangle\big)$
$\mathbf{u}_2$	$\big(\langle 0.5, 0.7\rangle, \langle 0.3, 0.5\rangle\big)$

Table 43: $\mathfrak{L}_2$

$\mathscr{V}_2$	$\big(\langle \varkappa^m_{\mathfrak{L}_2}(\mathbf{x}), \varkappa^n_{\mathfrak{L}_2}(\mathbf{x})\rangle, \langle \alpha_{\mathfrak{L}_2}(\mathbf{x}), \beta_{\mathfrak{L}_2}(\mathbf{x})\rangle\big)$
$\mathbf{v}_1$	$\big(\langle 0.5, 0.6\rangle, \langle 0.4, 0.5\rangle\big)$
$\mathbf{v}_2$	$\big(\langle 0.7, 0.6\rangle, \langle 0.6, 0.4\rangle\big)$

Table 44: $\breve{\rho'}_1$

$\mathscr{E}'_1$	$\big(\langle \varkappa^m_{\breve{\rho'}_1}(\mathbf{x}), \varkappa^n_{\breve{\rho'}_1}(\mathbf{x})\rangle, \langle \alpha_{\breve{\rho'}_1}(\mathbf{x}), \beta_{\breve{\rho'}_1}(\mathbf{x})\rangle\big)$
$(\mathbf{u}_1, \mathbf{u}_2)$	$\big(\langle 0.8, 0.4\rangle, \langle 0.5, 0.2\rangle\big)$

Table 45: $\breve{\rho}''_1$

$\mathscr{E}''_1$	$\big(\langle \varkappa^m_{\breve{\rho}''_1}(\mathbf{x}), \varkappa^n_{\breve{\rho}''_1}(\mathbf{x})\rangle, \langle \alpha_{\breve{\rho}''_1}(\mathbf{x}), \beta_{\breve{\rho}''_1}(\mathbf{x})\rangle\big)$
$(\mathbf{v}_1, \mathbf{v}_2)$	$\big(\langle 0.8, 0.4\rangle, \langle 0.5, 0.2\rangle\big)$

It becomes evident that $\breve{\mathscr{G}}_1 = (\mathscr{V}_1, \breve{\rho'}_1)$ and $\breve{\mathscr{G}}_2 = (\mathscr{V}_2, \breve{\rho''}_1)$ are LDFGSs which are depicted in FIGURE 8 and they are $\big(\langle 0.8, 0.4\rangle, \langle 0.5, 0.2\rangle\big)$-regular.

By employing Definition 4.1, we obtain the following LDFS $\mathfrak{L} = \mathfrak{L}_1 * \mathfrak{L}_2$ given in TABLE 16 on $\mathscr{V} = \mathscr{V}_1 \times \mathscr{V}_2 = \big\{(\mathbf{u}_1, \mathbf{v}_1), (\mathbf{u}_1, \mathbf{v}_2), (\mathbf{u}_2, \mathbf{v}_1), (\mathbf{u}_2, \mathbf{v}_2)\big\}$ and LDFR $\breve{\rho}_1 = \breve{\rho}'_1 * \breve{\rho}''_1$ shown in $\mathfrak{L}$ on $\mathscr{V}$ is calculated in TABLE 46 on $\mathscr{E}_1 = \mathscr{E}'_1 \times \mathscr{E}''_1 = \big\{(\mathbf{u}_1\mathbf{v}_1, \mathbf{u}_1\mathbf{v}_2), (\mathbf{u}_1\mathbf{v}_1, \mathbf{u}_2\mathbf{v}_1), (\mathbf{u}_1\mathbf{v}_2, \mathbf{u}_2\mathbf{v}_2), (\mathbf{u}_2\mathbf{v}_1, \mathbf{u}_2\mathbf{v}_2)\big\}$. LDFR $\breve{\rho}_1 = \breve{\rho}'_1 \times \breve{\rho}''_1$ is calculated in Table 47.

Then the maximal LDFGS $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2 = \big(\mathfrak{L}, \breve{\rho}_1\big)$ is portrayed in FIGURE 9.

Table 46: $\mathfrak{L} = \mathfrak{L}_1 * \mathfrak{L}_2$

$\mathscr{V}$	$\big(\langle \varkappa^m_{\mathfrak{L}}(\mathbf{x}, \mathbf{y}), \varkappa^n_{\mathfrak{L}}(\mathbf{x}, \mathbf{y})\rangle, \langle \alpha_{\mathfrak{L}}(\mathbf{x}, \mathbf{y}), \beta_{\mathfrak{L}}(\mathbf{x}, \mathbf{y})\rangle\big)$
$(\mathbf{u}_1, \mathbf{v}_1)$	$\big(\langle 0.6, 0.5\rangle, \langle 0.4, 0.2\rangle\big)$
$(\mathbf{u}_1, \mathbf{v}_2)$	$\big(\langle 0.7, 0.5\rangle, \langle 0.6, 0.2\rangle\big)$
$(\mathbf{u}_2, \mathbf{v}_1)$	$\big(\langle 0.5, 0.6\rangle, \langle 0.4, 0.5\rangle\big)$
$(\mathbf{u}_2, \mathbf{v}_2)$	$\big(\langle 0.7, 0.6\rangle, \langle 0.6, 0.4\rangle\big)$

From Definition 4.6, we can calculate the $\breve{\rho}_1$-degrees of each vertex of $\mathfrak{L}$ as follows:

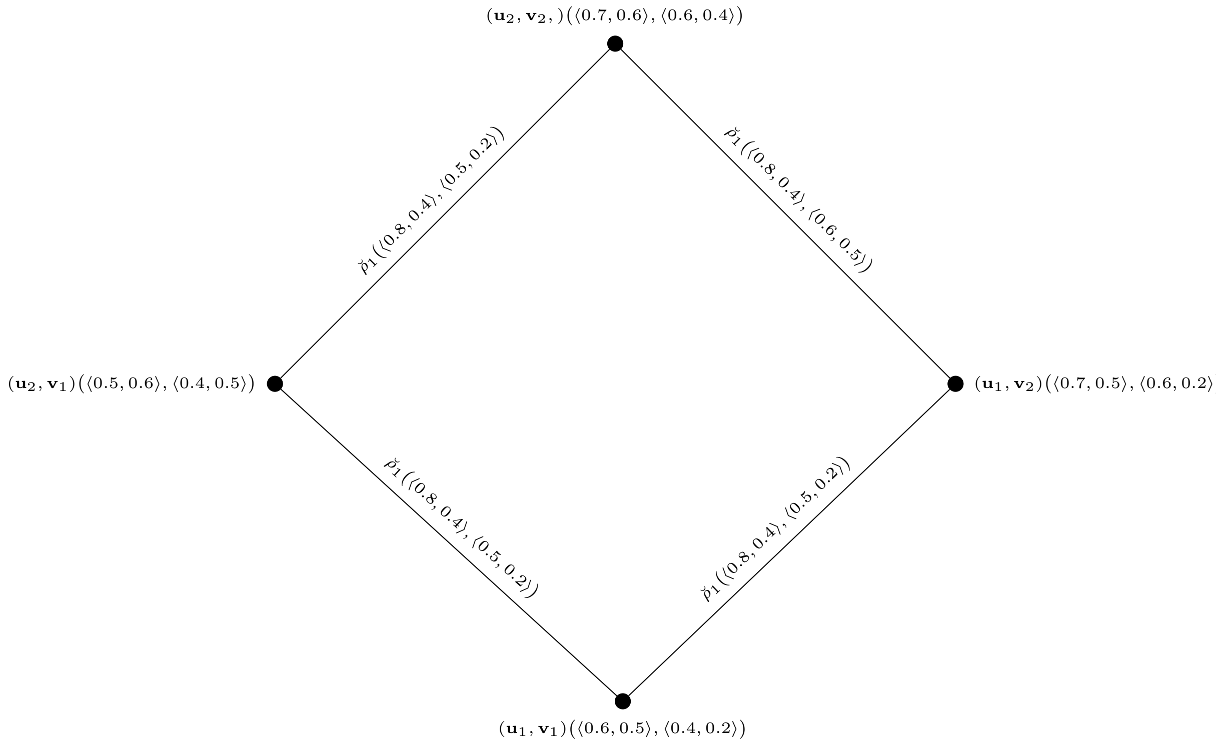

Figure 9: The maximal LDFGS $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$

Table 47: $\breve{\rho}_1 = \breve{\rho}_1' \times \breve{\rho}_1''$

$\mathscr{E}_1$	$\big(\langle \varkappa^m_{\breve{\rho}_1}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2), \varkappa^n_{\breve{\rho}_1}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2)\rangle, \langle \alpha_{\breve{\rho}_1}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2), \beta_{\breve{\rho}_1}(\mathbf{x}_1\mathbf{y}_1,\mathbf{x}_2\mathbf{y}_2)\rangle\big)$
$(\mathbf{u}_1\mathbf{v}_1,\mathbf{u}_1\mathbf{v}_2)$	$\big(\langle 0.8,0.4\rangle,\langle 0.5,0.2\rangle\big)$
$(\mathbf{u}_1\mathbf{v}_1,\mathbf{u}_2\mathbf{v}_1)$	$\big(\langle 0.8,0.4\rangle,\langle 0.5,0.2\rangle\big)$
$(\mathbf{u}_1\mathbf{v}_2,\mathbf{u}_2\mathbf{v}_2)$	$\big(\langle 0.8,0.4\rangle,\langle 0.6,0.5\rangle\big)$
$(\mathbf{u}_2\mathbf{v}_1,\mathbf{u}_2\mathbf{v}_2)$	$\big(\langle 0.8,0.4\rangle,\langle 0.5,0.2\rangle\big)$

$$\begin{aligned}
\mathbb{D}_{\breve{\rho}_1}(\mathbf{u}_1,\mathbf{v}_1) &= \big(\langle \varkappa^m_{\breve{\rho}_1}(\mathbf{u}_1\mathbf{v}_1,\mathbf{u}_1\mathbf{v}_2) + \varkappa^m_{\breve{\rho}_1}(\mathbf{u}_1\mathbf{v}_1,\mathbf{u}_2\mathbf{v}_1), \varkappa^n_{\breve{\rho}_1}(\mathbf{u}_1\mathbf{v}_1,\mathbf{u}_1\mathbf{v}_2) + \varkappa^n_{\breve{\rho}_1}(\mathbf{u}_1\mathbf{v}_1,\mathbf{u}_2\mathbf{v}_1)\rangle, \\
&\quad \langle \alpha_{\breve{\rho}_1}(\mathbf{u}_1\mathbf{v}_1,\mathbf{u}_1\mathbf{v}_2) + \alpha_{\breve{\rho}_1}(\mathbf{u}_1\mathbf{v}_1,\mathbf{u}_2\mathbf{v}_1), \beta_{\breve{\rho}_1}(\mathbf{u}_1\mathbf{v}_1,\mathbf{u}_1\mathbf{v}_2) + \beta_{\breve{\rho}_1}(\mathbf{u}_1\mathbf{v}_1,\mathbf{u}_2\mathbf{v}_1)\rangle\big) \\
&= \big(\langle 0.8,0.4\rangle,\langle 0.5,0.2\rangle\big) + \big(\langle 0.8,0.4\rangle,\langle 0.5,0.2\rangle\big) \\
&= \big(\langle 1.6,0.8\rangle,\langle 1,0.4\rangle\big)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbb{D}_{\breve{\rho}_1}(\mathbf{u}_1,\mathbf{v}_2) &= \big(\langle 0.8,0.4\rangle,\langle 0.5,0.2\rangle\big) + \big(\langle 0.8,0.4\rangle,\langle 0.6,0.2\rangle\big) = \big(\langle 1.6,0.8\rangle,\langle 1.1,0.4\rangle\big) \\
\mathbb{D}_{\breve{\rho}_1}(\mathbf{u}_2,\mathbf{v}_1) &= \big(\langle 0.8,0.4\rangle,\langle 0.5,0.2\rangle\big) + \big(\langle 0.8,0.4\rangle,\langle 0.5,0.2\rangle\big) = \big(\langle 1.6,0.8\rangle,\langle 1,0.4\rangle\big) \\
\mathbb{D}_{\breve{\rho}_1}(\mathbf{u}_2,\mathbf{v}_2) &= \big(\langle 0.8,0.4\rangle,\langle 0.5,0.2\rangle\big) + \big(\langle 0.8,0.4\rangle,\langle 0.5,0.2\rangle\big) = \big(\langle 1.6,0.8\rangle,\langle 1,0.4\rangle\big)
\end{aligned}$$

Clearly, $\breve{\mathscr{G}}$ is not regular since $\mathbb{D}_{\breve{\rho}_1}(\mathbf{u}_1,\mathbf{v}_1)=\big(\langle 1.6,0.8\rangle,\langle 1,0.4\rangle\big)\neq\big(\langle 1.6,0.8\rangle,\langle 1.1,0.4\rangle\big)=\mathbb{D}_{\breve{\rho}_1}(\mathbf{u}_1,\mathbf{v}_2)$.

Theorem 5.5. *If $\breve{\mathscr{G}}_1=\big(\mathfrak{L}_1,\breve{\rho'}_1,\breve{\rho'}_2,...,\breve{\rho'}_k\big)$ is $\big(\langle\mathbf{r},\mathbf{s}\rangle,\langle\mathbf{s},\mathbf{t}\rangle\big)$-regular LDFGS and $\breve{\mathscr{G}}_2=\big(\mathfrak{L}_2,\breve{\rho''}_1,\breve{\rho''}_2,...,\breve{\rho''}_k\big)$ is an LDFGS, such that $\breve{\rho''}_i\supseteq\mathfrak{L}_1$, $i=1,2,...,k$, and $\mathfrak{L}_2$ is constant LDFS of LDF value $\big(\langle a,b\rangle,\langle c,d\rangle\big)$, where $a,b,c,d\in[0,1]$ are fixed, then maximal product $\breve{\mathscr{G}}=\breve{\mathscr{G}}_1*\breve{\mathscr{G}}_2$ is regular if and only if $\breve{\mathscr{G}}_2$ is regular.*

Proof. Let $\breve{\mathscr{G}}_1=\big(\mathfrak{L}_1,\breve{\rho'}_1,\breve{\rho'}_2,...,\breve{\rho'}_k\big)$ be partially regular LDFGS and $\breve{\mathscr{G}}_2=\big(\mathfrak{L}_2,\breve{\rho''}_1,\breve{\rho''}_2,...,\breve{\rho''}_k\big)$ be an LDFGS, such that $\breve{\rho''}_i\supseteq\mathfrak{L}_1$, $i=1,2,...,k$, and $\mathfrak{L}_2=\big(\langle a,b\rangle,\langle c,d\rangle\big)$ be a constant LDFGS. Then,

$$\mathbb{D}_{\breve{\mathscr{G}}}(\mathbf{x}_i,\mathbf{y}_j)=\Big(\big\langle\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j),\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle,\big\langle\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j),\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle\Big)$$

where

$$\begin{aligned}\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)&=\mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)a+\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j);\\ \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)&=\mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)b+\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j);\\ \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)&=\mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)c+\alpha_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j);\\ \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)&=\mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)d+\beta_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_j).\end{aligned}$$

This holds for all vertices of $\mathscr{V}=\mathscr{V}_1\times\mathscr{V}_2$. Hence, maximal product $\breve{\mathscr{G}}=\breve{\mathscr{G}}_1*\breve{\mathscr{G}}_2$ is regular.
Conversely, suppose that maximal product $\breve{\mathscr{G}}=\breve{\mathscr{G}}_1*\breve{\mathscr{G}}_2$ is regular. Then, for any two vertices of $\mathscr{V}=\mathscr{V}_1\times\mathscr{V}_2$,

$$\begin{aligned}&\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_1,\mathbf{y}_1)=\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_2,\mathbf{y}_2)\\ &\Rightarrow\mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_1)a+\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_1)=\mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_2)a+\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_2)\\ &\Rightarrow\mathbf{r}a+\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_1)=\mathbf{r}a+\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_2)\\ &\Rightarrow\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_1)=\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_2)\end{aligned}$$

Similarly, $\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_1,\mathbf{y}_1)=\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_2,\mathbf{y}_2)$ implies that $\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_1)=\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_2)$; $\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_1,\mathbf{y}_1)=\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_2,\mathbf{y}_2)$ implies that $\alpha_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_1)=\alpha_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_2)$; $\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_1,\mathbf{y}_1)=\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_2,\mathbf{y}_2)$ implies that $\beta_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_1)=\beta_{\mathbb{D}_{\breve{\mathscr{G}}_2}}(\mathbf{y}_2)$. This holds for all vertices of $\breve{\mathscr{G}}_2$. Hence, $\breve{\mathscr{G}}_2$ is regular LDFGS. □

Theorem 5.6. *If $\breve{\mathscr{G}}_1=\big(\mathfrak{L}_1,\breve{\rho'}_1,\breve{\rho'}_2,...,\breve{\rho'}_k\big)$ is partially regular LDFGS and $\breve{\mathscr{G}}_2=\big(\mathfrak{L}_2,\breve{\rho''}_1,\breve{\rho''}_2,...,\breve{\rho''}_k\big)$ is an LDFGS, such that $\breve{\rho'}_i\supseteq\mathfrak{L}_2$, $i=1,2,...,k$, and $\mathfrak{L}_2$ is constant LDFS of LDF value $\big(\langle a,b\rangle,\langle c,d\rangle\big)$, where $a,b,c,d\in[0,1]$ are fixed, then maximal product $\breve{\mathscr{G}}=\breve{\mathscr{G}}_1*\breve{\mathscr{G}}_2$ is regular if and only if $\breve{\mathscr{G}}_1$ is regular.*

Proof. Suppose with the given assumptions, we have from Theorem 4.11,

$$\mathbb{D}_{\breve{\mathscr{G}}}(\mathbf{x}_i,\mathbf{y}_j)=\Big(\big\langle\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j),\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle,\big\langle\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j),\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)\big\rangle\Big),$$

where

$$\begin{aligned}\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)&=\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i)+\mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)a;\\ \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)&=\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i)+\mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)b;\\ \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)&=\alpha_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i)+\mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)c;\\ \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i,\mathbf{y}_j)&=\beta_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_i)+\mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)d.\end{aligned}$$

which holds for all vertices of $\mathscr{V} = \mathscr{V}_1 \times \mathscr{V}_2$. Hence, maximal product $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ is regular.
Conversely, assume that maximal product $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ is regular. Then for any two vertices of $\mathscr{V} = \mathscr{V}_1 \times \mathscr{V}_2$, we have:

$$\begin{aligned} \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_1, \mathbf{y}_1) &= \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_2, \mathbf{y}_2) \\ \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_1) + \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{y}_1)\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_1) &= \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_2) + \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{y}_2)\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_2) \\ \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_1) + \mathbf{r}_2\mathbf{a} &= \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_2) + \mathbf{r}_2\mathbf{a} \\ \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_1) &= \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_2) \end{aligned}$$

Similarly, $\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_1, \mathbf{y}_1) = \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_2, \mathbf{y}_2)$ implies that $\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_1) = \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_2)$; $\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_1, \mathbf{y}_1) = \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_2, \mathbf{y}_2)$ implies that $\alpha_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_1) = \alpha_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_2)$; $\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_1, \mathbf{y}_1) = \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_2, \mathbf{y}_2)$ implies that $\beta_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_1) = \beta_{\mathbb{D}_{\breve{\mathscr{G}}_1}}(\mathbf{x}_2)$. This proves that $\breve{\mathscr{G}}_1$ regular LDFGS. □

Theorem 5.7. *If $\breve{\mathscr{G}}_1 = \big(\mathfrak{L}_1, \breve{\rho}'_1, \breve{\rho}'_2, ..., \breve{\rho}'_k\big)$ and $\breve{\mathscr{G}}_2 = \big(\mathfrak{L}_2, \breve{\rho}''_1, \breve{\rho}''_2, ..., \breve{\rho}''_k\big)$ are two $\big(\langle r_1, s_1\rangle, \langle t_1, u_1\rangle\big)$-regular and $\big(\langle r_2, s_2\rangle, \langle t_2, u_2\rangle\big)$-regular LDFGSs, respectively, such that $\breve{\rho}''_i \subseteq \mathfrak{L}_1$ and $\breve{\rho}''_i \subseteq \mathfrak{L}_2$, $i = 1, 2, ..., k$ and $\mathfrak{L}_2$ is a constant LDFS of LDF value $\big(\langle a, b\rangle, \langle c, d\rangle\big)$, where $a, b, c, d \in [0, 1]$ are fixed, then maximal product $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ is regular if and only if $\mathfrak{L}_1$ is a constant LDFS of LDF value $\big(\langle a', b'\rangle, \langle c', d'\rangle\big)$, where $a', b', c', d' \in [0, 1]$ are fixed.*

Proof. With the given assumptions, we have from Theorem 4.12,

$$\mathbb{D}_{\breve{\mathscr{G}}}(\mathbf{x}_i, \mathbf{y}_j) = \Big(\big\langle \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\big\rangle, \big\langle \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j), \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j)\big\rangle\Big),$$

where

$$\begin{aligned} \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_j) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_i) = r_1 a + r_2 a'; \\ \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_j) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\varkappa^n_{\mathfrak{L}_1}(\mathbf{x}_i) = s_1 b + s_2 b'; \\ \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\alpha_{\mathfrak{L}_2}(\mathbf{y}_j) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\alpha_{\mathfrak{L}_1}(\mathbf{x}_i) = t_1 c + t_2 c'; \\ \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_i, \mathbf{y}_j) &= \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_i)\beta_{\mathfrak{L}_2}(\mathbf{y}_j) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_j)\beta_{\mathfrak{L}_1}(\mathbf{x}_i) = u_1 d + u_2 d'; \end{aligned}$$

which holds for all vertices of $\mathscr{V} = \mathscr{V}_1 \times \mathscr{V}_2$. Hence, $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ is regular.
Conversely, assume that $\breve{\mathscr{G}} = \breve{\mathscr{G}}_1 * \breve{\mathscr{G}}_2$ is regular. For any two vertices of $\mathscr{V} = \mathscr{V}_1 \times \mathscr{V}_2$, we have:

$$\begin{aligned} &\varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_1, \mathbf{y}_1) = \varkappa^m_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_2, \mathbf{y}_2) \\ &\mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_1)\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_1) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_1)\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_1) = \mathbb{D}_{\mathscr{G}_1}(\mathbf{x}_2)\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_2) + \mathbb{D}_{\mathscr{G}_2}(\mathbf{y}_2)\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_2) \\ &r_1\varkappa^m_{\mathfrak{L}_2}(\mathbf{x}_1) + r_2\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_1) = r_1\varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_1) + r_2\varkappa^m_{\mathfrak{L}_1}(\mathbf{x}_1) \\ &\varkappa^m_{\mathfrak{L}_2}(\mathbf{x}_1) = \varkappa^m_{\mathfrak{L}_2}(\mathbf{y}_1) \end{aligned}$$

Similarly, $\varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_1, \mathbf{y}_1) = \varkappa^n_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_2, \mathbf{y}_2)$ implies $\varkappa^n_{\mathfrak{L}_2}(\mathbf{x}_1) = \varkappa^n_{\mathfrak{L}_2}(\mathbf{y}_1)$; $\alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_1, \mathbf{y}_1) = \alpha_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_2, \mathbf{y}_2)$ implies $\alpha_{\mathfrak{L}_2}(\mathbf{x}_1) = \alpha_{\mathfrak{L}_2}(\mathbf{y}_1)$; $\beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_1, \mathbf{y}_1) = \beta_{\mathbb{D}_{\breve{\mathscr{G}}}}(\mathbf{x}_2, \mathbf{y}_2)$ implies $\beta_{\mathfrak{L}_2}(\mathbf{x}_1) = \beta_{\mathfrak{L}_2}(\mathbf{y}_1)$, which holds for all vertices of $\mathscr{G}_1$. Hence, $\mathfrak{L}_1$ is constant LDFS. □

6 Conclusion

Graphs are used in various applications such as social networks, recommendation systems, routing algorithms, and many more. A GS has n mutually disjoint, symmetric and irreflexive relations. Understanding these structures and their properties is key to leveraging graphs effectively in solving real-world problems. However, in certain scenarios, several features of GT might be uncertain. FGSs have many advantages to cope with vagueness and uncertainty. FGSs are more advantageous to circumvent uncertainty. In this research study, we have applied the notion of LDFSs to GSs and introduced a novel concept LDFGS. We have defined $\breve{\rho}_i$-edge, $\breve{\rho}_i$-path, strength of $\breve{\rho}_i$-path, $\breve{\rho}_i$-strength of connectedness, $\breve{\rho}_i$-degree of a vertex, vertex degree, total $\breve{\rho}_i$-degree of a vertex, and total vertex degree in an LDFGS. Also, we have introduced the $\breve{\rho}_i$-size, size, and order of an LDFGS. Moreover, the ideas of the maximal product of two LDFGSs, strong LDFGS, degree and $\breve{\rho}_i$-degree of the maximal product, $\breve{\rho}_i$-regular and regular LDFGS are introduced, along with examples for clarification. Certain significant results related to the proposed concepts also demonstrated with explanatory examples such as the maximal product of two strong LDFGSs is also a strong LDFGS, the maximal product of two connected LDFGSs is also a connected LDFGS but the maximal product of two regular LDFGS may not be a regular LDGS. Moreover, many interesting and alternative formulas for calculating $\breve{\rho}_i$-degrees of an LDFGS in various situations are proved with examples. LDFGSs are highly beneficial for solving numerous combinatorial problems involving multiple relations than the existing GSs in the context of FS, IFS, PFS and q-ROFS. LDFGSs as an extension of IFGS and LDFG to GSs deals the graph theoretical aspects in more appropriate way due to their flexibility in selecting MD and NMD alongside their reference parameters.

In the future, we aim to extend our approach to (1) rough linear Diophantine fuzzy graph structures, (2) rough linear Diophantine fuzzy soft graph structures, (3) linear Diophantine fuzzy soft graph structures, and (4) Spherical linear Diophantine fuzzy graph structures.

Acknowledgements: The authors are very grateful to the editor and the reviewers for their insightful and constructive comments and suggestions towards improving our manuscript.

Conflict of Interest: The authors declare no conflict of interest.

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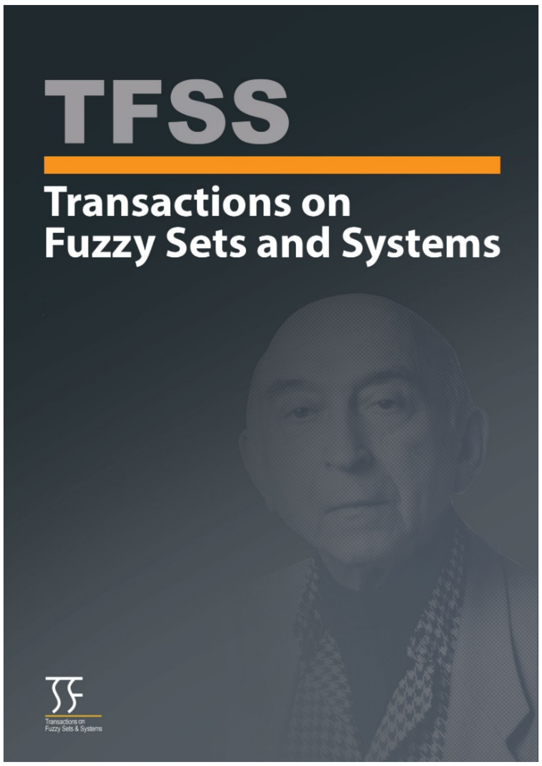

Transactions on Fuzzy Sets and Systems

ISSN: **2821-0131**

https://sanad.iau.ir/journal/tfss/

A Modified Pythagorean Fuzzy Similarity Operator with Application in Questionnaire Analysis

Vol.4, No.1, (2025), 108-128. DOI: https://doi.org/10.71602/tfss.2025.1183142

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Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://sanad.iau.ir/journal/tfss/
Online ISSN: 2821-0131
Vol.4, No.1, (2025), 108-128
DOI: https://doi.org/10.71602/tfss.2025.1183142

Article Type: Original Research Article

A Modified Pythagorean Fuzzy Similarity Operator with Application in Questionnaire Analysis

Paul Augustine Ejegwa

(This paper is dedicated to Professor "John N. Mordeson" on the occasion of his 91st birthday.)

Abstract. This work presents a modified Pythagorean fuzzy similarity operator and utilizes its potential in the analysis of questionnaire. Similarity operator is a formidable methodology for decision-making under uncertain domains. Pythagorean fuzzy set is an extended form of intuitionistic fuzzy set with a better accuracy in complex real-world applications. Lots of discussions bordering on the uses of Pythagorean fuzzy sets have been explored based on Pythagorean fuzzy similarity operators. Among the extant Pythagorean fuzzy similarity operators, the work of Zhang et al. is significant but it contains some flaws which need to be corrected/modified to enhance reliable interpretation. To this end, this work explicates the Zhang et al.'s techniques of Pythagorean fuzzy similarity operator by pinpointing their drawbacks to develop an enhanced Pythagorean fuzzy similarity operator, which appropriately satisfies the similarity conditions and yields consistent results in comparison to the Zhang et al.'s techniques. Succinctly speaking, the aim of the work is to correct the flaws in Zhang et al.'s techniques via modifications. To theoretically validate the enhanced Pythagorean fuzzy similarity operator, we discuss it properties and find out that the similarity conditions are well satisfied. In addition, the enhanced PFSO and the Zhang et al.'s PFSOs are compared in the context of precision, and it is verified that the enhanced Pythagorean fuzzy similarity operator can successfully measure the similarity between vastly related but inconsistent PFSs and as well yields a very reasonable results. Furthermore, the enhanced Pythagorean fuzzy similarity operator is applied to the analysis of questionnaire on virtual library to ascertain the extent of awareness and effects of virtual library on students' academic performance via real data collected from fieldwork. Finally, it is certified that the enhanced Pythagorean fuzzy similarity operator can handle diverse everyday problems more precisely than the Zhang et al.'s Pythagorean fuzzy similarity operators.

AMS Subject Classification 2020: 03E72; 94D05; 03B52; 28E10
Keywords and Phrases: Decision making under uncertainty, Intuitionistic fuzzy set, Questionnaire analysis, Pythagorean fuzzy set, Similarity operator.

1 Introduction

The occurrence of vagueness and uncertainty in decision-making (DM) is a common experience witnessed by decision-makers. Due to this, fuzzy set (FS) [1] was introduced to curbed uncertainty but imprecision could not be tackled by FS. To resolve the problem of imprecision, intuitionistic fuzzy set (IFS) was developed [2], and it has been widely used to discuss practical DM problems. IFS is described by membership degree (MD) and non-membership degree (ND), where their sum cannot exceeds one. Several practical problems

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Received: 6 September 2024; **Revised**: 28 November 2024; **Accepted**: 29 November 2024; **Available Online**: 23 December 2024; **Published Online**: 7 May 2025.

How to cite: Ejegwa PA. A modified Pythagorean fuzzy similarity operator with application in questionnaire Analysis. *Transactions on Fuzzy Sets and Systems.* 2025; 4(1): 108-128. DOI: https://doi.org/10.71602/tfss.2025.1183142

have been solved via IFS in real-life problems using distance operators [3–5], aggregation operators [6], and partial correlation coefficient operator [7]. In addition, other applications of IFSs have been discussed in medical emergency [8], selection of artificial intelligence [9], admission process [10], and decision-making [11]. The similarity metric is a vital research aspect in FS and its generalizations, and it is useful in determining the similarity index between two objects. Several techniques of similarity operator between IFS have been developed and gainfully applied in many fields, like pattern recognition [12], disaster control [13] and medical diagnostic problems [14]. From the ongoing, it is clear that similarity operators of IFSs have been effectively used in sundry fields, but there are some cases where IFSs cannot be utilized. For instance, if a decision maker has MD as 0.7 and ND as 0.5, then the IFS model cannot be applicable.

By extending the spatial scope of IFS, the term "IFS of type 2" or Pythagorean fuzzy set (PFS) was developed [15, 16]. In PFS, the sum of MD and ND may exceeds one but the square sum of MD and ND is at most one. PFS has a wider dimension of utilizations compare to IFS. In a way to discuss the usefulness of PFSs, a number of aggregation operators were discussed like Einstein operators, interactive power averaging operator, geometric aggregation operators using Einstein t-conorm and t-norm [17–19] to illustrate some DM problems. In the same vein, PFSs are pretty applicable in DM problems based on correlation coefficient operators [20–23] and distance operators [24–31]. Moreover, Hemalatha and Venkateswarlu [32] used PFSs to discuss transportation problem using mean square approach, Li et al. [33] presented an analysis of football activities using Pythagorean fuzzy approach, and various applications of PFSs have been discussed in decision-making [34–37].

In a clear term, PFS is a special case of IFS, which is fashioned to deal with some problems in which IFS is inadmissible. For that reason, the application of similarity operators on PFSs is of great important. The studies on similarity operators on PFS are carried out by modifying similar studies under IFSs. Zeng et al. [31] presented some methods of similarity operators between PFSs using some distance operators since both similarity operator and distance operator are dual in nature. Peng et al. [38] constructed some similarity operators for PFSs and used same in clustering analysis, medical diagnosis and pattern recognition. To compute the similarity between PFSs, Zhang [39] developed a similarity operator on PFSs and used it to discuss multi-criteria decision-making (MCDM) problems. Wei and Wei [40] constructed several similarity operators on PFSs through cosine function with applications in health science and pattern recognition. Recently, Zhang et al. [41] developed four methods of Pythagorean fuzzy similarity operator (PFSO), which were utilized to discuss pattern recognition problems. While the first two methods in [41] discarded the hesitation margins, the other two took into account the whole parameters of PFSs for e reliable outcomes. Nonetheless, the methods produce identical value other than one whenever the PFSs are equal, which is a violation of the similarity axioms and thus render the methods unreliable.

The interest of this work is to provide corrections to the four similarity operators between PFSs constructed in [41] by providing a new similarity operator between PFSs, which is the product of the hybridization of the four similarity operators. For emphasis, similarity operators in [41] have the following setbacks: (1) they fail to fulfill the similarity conditions if the PFSs are equal; (2) they yield similarity values that are not defined within the similarity value range, and thus lack practical interpretation. To this end, this paper proposes a hybridized similarity operator that corrects the work of Zhang et al. [41], and proves that the corrected version can successfully solve the mentioned setbacks observed in [41] via comparative examples using real collected data. This work contributes to the study of similarity operator under uncertain environments, soft computing, questionnaire analysis, and decision-making procedures.

The article is structured as follows: Section 2 recaps certain properties of PFSs; Section 3 discusses the Zhang et al.'s PFSOs and their setbacks; Section 4 provides solution to the setbacks in Zhang et al.'s PFSOs and discusses the properties of the modified PFSO; Section 5 discusses the application of the corrected Zhang et al.'s PFSOs in the analysis of questionnaire, and as well as, presents a comparative analysis to express the advantage of the corrected versions; and Section 6 concludes the paper with suggestions for future inquiries.

2 Preliminaries

This section discusses properties of PFSs and the Zhang et al's similarity functions. For clarity sake, assume A to be the universe of discourse, $\wp$ as IFS, and ℓ as PFS.

Definition 2.1. [2]*An IFS* $\wp$ *in* A *is defined by* $\wp = \{(a, M_\wp(a), N_\wp(a)) : a \in A\}$, *where* $M_\wp\colon A \to [0,1]$ *and* $N_\wp\colon A \to [0,1]$ *are MD and NMD of* $a \in A$ *in which*

$$0 \leq M_\wp(a) + N_\wp(a) \leq 1.$$

In addition, HM of $\wp$ *in* A *is defined by* $H_\wp(a) = 1 - M_\wp(a) - N_\wp(a)$.

Definition 2.2. [16] *A PFS* ℓ *in* A *is defined by* $\ell = \{(a, M_\ell(a), N_\ell(a)) : a \in A\}$, *where* $M_\ell\colon A \to [0,1]$ *and* $N_\ell\colon A \to [0,1]$ *are MD and NMD of* $a \in A$ *in which*

$$0 \leq M_\ell^2(a) + N_\ell^2(a) \leq 1.$$

In addition, HM of ℓ *in* A *is defined by* $H_\ell(a) = \sqrt{1 - M_\ell^2(a) - N_\ell^2(a)}$.

Now, we present some operations on PFSs as follows:

Definition 2.3. [16] *If* ℓ, ℓ_1 *and* ℓ_2 *are PFSs in* A, *then*

(i) $\ell_1 \preceq \ell_2$ *iff* $M_{\ell_1}(a) \preceq M_{\ell_2}(a)$ *and* $N_{\ell_1}(a) \preceq N_{\ell_2}(a)$ $\forall a \in A$.

(ii) $\ell_1 = \ell_2$ *iff* $M_{\ell_1}(a) = M_{\ell_2}(a)$ *and* $N_{\ell_1}(a) = N_{\ell_2}(a)$ $\forall a \in A$.

(iii) $\ell_1 \subseteq \ell_2$ *iff* $M_{\ell_1}(a) \leq M_{\ell_2}(a)$ *and* $N_{\ell_1}(a) \geq N_{\ell_2}(a)$ $\forall a \in A$.

(iv) $\overline{\ell} = \{(a, N_\ell(a), M_\ell(a)) : a \in A\}$.

(v) $\ell_1 \cap \ell_2 = \{(a, \min\{M_{\ell_1}(a), M_{\ell_2}(a)\}, \max\{N_{\ell_1}(a), N_{\ell_2}(a)\}) : a \in A\}$.

(vi) $\ell_1 \cup \ell_2 = \{(a, \max\{M_{\ell_1}(a), N_{\ell_2}(a)\}, \min\{N_{\ell_1}(a), N_{\ell_2}(a)\}) : a \in A\}$.

One of the means to estimate the similarity between PFSs is via similarity measure between them.

Definition 2.4. [1] *If* ℓ, ℓ_1 *and* ℓ_2 *are PFSs in* $A = \{a_1, a_2, \cdots, a_Q\}$, *then the similarity metric between* ℓ_1 *and* ℓ_2 *represented by* $\Gamma(\ell_1, \ell_2)$ *is a function,* $\Gamma\colon PFS \times PFS \to [0,1]$ *such that:*

(i) $\Gamma(\ell_1, \ell_1) = 1$, $\Gamma(\ell_2, \ell_2) = 1$,

(ii) $\Gamma(\ell_1, \ell_2) = 1 \Leftrightarrow \ell_1 = \ell_2$,

(iii) $0 \leq \Gamma(\ell_1, \ell_2) \leq 1$,

(iv) $\Gamma(\ell_1, \ell_2) = \Gamma(\ell_2, \ell_1)$,

(v) $\Gamma(\ell_1, \ell) \leq \Gamma(\ell_1, \ell_2) + \Gamma(\ell_2, \ell)$.

In short, $\Gamma(\ell_1, \ell_2) \approx 1$ implies there is high similarity between ℓ_1 and ℓ_2, and $\Gamma(\ell_1, \ell_2) \approx 0$ implies there is a negligible similarity between ℓ_1 and ℓ_2.

3 Zhang et al.'s PFSOs and Numerical Illustrations

The exponential-based techniques of similarity operators under PFSs were presented by Zhang et al. [41] because of the failures of some existing approaches of PFSOs. Zhang et al. developed four exponential based-similarity operators, enumerated as follows:

$$\Gamma_1(\ell_1,\ell_2)=\frac{1}{Q}\Sigma_{j=1}^{Q}\Big[2^{1-\max\{|M_{\ell_1}^2(a_j)-M_{\ell_2}^2(a_j)|,|N_{\ell_1}^2(a_j)-N_{\ell_2}^2(a_j)|\}}-1\Big], \tag{1}$$

$$\Gamma_2(\ell_1,\ell_2)=\frac{1}{Q}\Sigma_{j=1}^{Q}\Big[2^{1-\frac{|M_{\ell_1}^2(a_j)-M_{\ell_2}^2(a_j)|+|N_{\ell_1}^2(a_j)-N_{\ell_2}^2(a_j)|}{2}}-1\Big], \tag{2}$$

$$\Gamma_3(\ell_1,\ell_2)=\frac{1}{Q}\Sigma_{j=1}^{Q}\Big[2^{1-\max\{|M_{\ell_1}^2(a_j)-M_{\ell_2}^2(a_j)|,|N_{\ell_1}^2(a_j)-N_{\ell_2}^2(a_j)|,|H_{\ell_1}^2(a_j)-H_{\ell_2}^2(a_j)|\}}-1\Big], \tag{3}$$

$$\Gamma_3(\ell_1,\ell_2)=\frac{1}{Q}\Sigma_{j=1}^{Q}\Big[2^{1-\frac{|M_{\ell_1}^2(a_j)-M_{\ell_2}^2(a_j)|+|N_{\ell_1}^2(a_j)-N_{\ell_2}^2(a_j)|+|H_{\ell_1}^2(a_j)-H_{\ell_2}^2(a_j)|}{2}}-1\Big], \tag{4}$$

where ℓ_1 and ℓ_2 are the PFSs defined in A, $a_j \in A$ and $|A| = Q$. The similarity methods in (1) and (2) excluded the hesitation margins, which makes the approaches defective. Nonetheless, (1) and (2) were enhanced as (3) and (4), respectively, to yield reliable results. Howbeit, all these methods yield similar results, most especially, as the hesitation margins become small and for smaller Q. We show the defectiveness of these methods in the following examples:

Example 3.1. Suppose we have two PFSs ℓ_1 and ℓ_2 defined in $A = \{a_1, a_2, a_3\}$ as follows:

$$\ell_1 = \{(a_1, 0.5, 0.4), (a_2, 0.8, 0.1), (a_3, 0.7, 0.2)\} = \ell_2$$

This is a case of equal PFSs, and we are expected to have $\Gamma_1(\ell_1,\ell_2) = \Gamma_2(\ell_1,\ell_2) = \Gamma_3(\ell_1,\ell_2) = \Gamma_4(\ell_1,\ell_2) = 1$. Then, by applying (1)–(4) we get

$$\left.\begin{aligned}\Gamma_1(\ell_1,\ell_2)&=\frac{2^{1-0}-1}{3}=0.3333\\ \Gamma_2(\ell_1,\ell_2)&=\frac{2^{1-0}-1}{3}=0.3333\\ \Gamma_3(\ell_1,\ell_2)&=\frac{2^{1-0}-1}{3}=0.3333\\ \Gamma_4(\ell_1,\ell_2)&=\frac{2^{1-0}-1}{3}=0.3333,\end{aligned}\right\},$$

which violate a similarity condition, i.e., $\Gamma(\ell_1,\ell_2) = 1 \Leftrightarrow \ell_1 = \ell_2$. Hence, these approaches need to be corrected to satisfy the condition.

Again, we observe that these approaches sometimes produce results that are not within $0 \le \Gamma(\ell_1,\ell_2) \le 1$, as seen in Example 3.2.

Example 3.2. Suppose that

$$\ell_1 = \{(a_1, 1, 0), (a_2, 0.8, 0), (a_3, 0.7, 0.1)\},$$
$$\ell_2 = \{(a_1, 0.8, 0.1), (a_2, 1, 0), (0.9, 0.1)\},$$
$$\ell_3 = \{(a_1, 0.6, 0.2), (a_2, 0.8, 0), (1, 0)\}$$

are PFSs in $A = \{a_1, a_2, a_3\}$. In case there is another PFS defined by

$$\ell = \{(a_1, 0.5, 0.3), (a_2, 0.8, 0.2), (1, 0)\}.$$

Now, we apply the approaches to find the similarities between ℓ with each of ℓ_1, ℓ_2, and ℓ_3, respectively, and get the following results:

$$\left.\begin{aligned}\Gamma_1(\ell_j,\ell) &= -0.0626, 0.0142, 0.2675\\ \Gamma_2(\ell_j,\ell) &= 0.077, 0.1268, 0.2887\\ \Gamma_3(\ell_j,\ell) &= -0.055, 0.0142, 0.2844\\ \Gamma_4(\ell_j,\ell) &= -0.0626, 0.0142, 0.2675\end{aligned}\right\},$$

for $j = 1, 2, 3$. The negative similarity values proof the failure of the PFSOs. To solve these defectiveness, the Zhang et al.'s techniques are corrected as follows:

4 Corrections to Zhang et al.'s PFSOs

Because of the problems associated with Zhang et al.'s methods, it is necessary to correct the methods to enhance reliability, precision, and the satisfication of similarity conditions.

Definition 4.1. Suppose $\ell_1 = \{(a_j, M_{\ell_1}(a_j), N_{\ell_1}(a_j)) : a_j \in A\}$ and $\ell_2 = \{(a_j, M_{\ell_2}(a_j), N_{\ell_2}(a_j)) : a_j \in A\}$ are PFSs for $A = \{a_1, a_2, \cdots, a_Q\}$, then the new similarity operator between ℓ_1 and ℓ_2, which corrects the Zhang et al.'s PFSOs is defined by:

$$\tilde{\Gamma}_*(\ell_1,\ell_2) = \sum_{j=i}^{Q}\left[2^{1-\frac{1}{3Q}\left(\left|M^2_{\ell_1}(a_j)-M^2_{\ell_2}(a_j)\right|+\left|N^2_{\ell_1}(a_j)-N^2_{\ell_2}(a_j)\right|+\left|H^2_{\ell_1}(a_j)-H^2_{\ell_2}(a_j)\right|\right)}-1\right]. \tag{5}$$

By incorporating the influence of weight of the elements of A, we have:

$$\tilde{\Gamma}(\ell_1,\ell_2) = \sum_{j=i}^{Q}\left[2^{1-\frac{1}{3}\omega_j\left(\left|M^2_{\ell_1}(a_j)-M^2_{\ell_2}(a_j)\right|+\left|N^2_{\ell_1}(a_j)-N^2_{\ell_2}(a_j)\right|+\left|H^2_{\ell_1}(a_j)-H^2_{\ell_2}(a_j)\right|\right)}-1\right], \tag{6}$$

where $\omega_j \in [0, 1]$ and $\sum_{j=1}^{Q}\omega_j = 1$.

If $\omega_j = \left(\frac{1}{Q}, \frac{1}{Q}, \cdots, \frac{1}{Q}\right)^T$, then (6) becomes (5). The first advantage of this corrected version is that, it incorporates the complete parameters of the sets. We use (5) and (6) to find the similarity between equal PFSs in Example 3.1 and get

$$\tilde{\Gamma}(\ell_1,\ell_2) = \tilde{\Gamma}_*(\ell_1,\ell_2) = 1,$$

which satisfies $\Gamma(\ell_1, \ell_2) = 1 \Leftrightarrow \ell_1 = \ell_2$. This is the second advantage of the corrected version over Zhang et al.'s methods.

In addition, the corrected approaches produce results that are within $0 \leq \Gamma(\ell_1, \ell_2) \leq 1$. To see this, we consider Example 3.2 with $\omega_j = \{0.2, 0.4, 0.4\}$, and get the following results:

$$\tilde{\Gamma}(\ell_j,\ell) = 0.6857, 0.7427, 0.8251,$$
$$\tilde{\Gamma}_*(\ell_j,\ell) = 0.6371, 0.7304, 0.7956,$$

for $j = 1, 2, 3$. Clearly, these results are better than the results from Zhang et al.'s approaches. The results from Zhang et al.'s methods and the corrected form are displayed in Table 1.

Table 1: Results for Comparison

PFSOs	Example 3.1	Example 3.2
Γ_1 [41]	0.3333	$-0.0626, 0.0142, 0.2675$
Γ_2 [41]	0.3333	$0.0770, 0.1268, 0.2887$
Γ_3 [41]	0.3333	$-0.0550, 0.0142, 0.2844$
Γ_4 [41]	0.3333	$-0.0626, 0.0142, 0.2675$
$\tilde{\Gamma}_*$	1.0000	$0.6371, 0.7304, 0.7956$

The results in Table 1 justify the faults with the methods in [41] and the superiority of the corrected form. While the results of Zhang et al.'s methods (i.e., Example 3.2) show that weak resemblance exist between the PFSs, the new method shows that the PFSs are well related in agreement to mere observation. Now, we characterize the corrected similarity operator theoretically.

Theorem 4.2. *Suppose ℓ_1 and ℓ_2 are PFSs in $A = \{a_1, a_2, \cdots, a_Q\}$, then*

(i) $\tilde{\Gamma}(\ell_1, \ell_2) = \tilde{\Gamma}(\ell_2, \ell_1)$,

(ii) $\tilde{\Gamma}(\ell_1, \ell_2) = \tilde{\Gamma}(\overline{\ell}_1, \overline{\ell}_2)$,

(iii) $0 \leq \tilde{\Gamma}(\ell_1, \ell_2) \leq 1$,

(iv) $\tilde{\Gamma}(\ell_1, \ell_2) = 1 \Leftrightarrow \ell_1 = \ell_2$.

Proof. The prove of (i) follows because

$$\begin{aligned}\tilde{\Gamma}(\ell_1, \ell_2) &= \sum_{j=i}^{Q}\left[2^{1-\frac{1}{3}\omega_j\left(\left|M^2_{\ell_1}(a_j)-M^2_{\ell_2}(a_j)\right|+\left|N^2_{\ell_1}(a_j)-N^2_{\ell_2}(a_j)\right|+\left|H^2_{\ell_1}(a_j)-H^2_{\ell_2}(a_j)\right|\right)}-1\right]\\ &= \sum_{j=i}^{Q}\left[2^{1-\frac{1}{3}\omega_j\left(\left|M^2_{\ell_2}(a_j)-M^2_{\ell_1}(a_j)\right|+\left|N^2_{\ell_2}(a_j)-N^2_{\ell_1}(a_j)\right|+\left|H^2_{\ell_2}(a_j)-H^2_{\ell_1}(a_j)\right|\right)}-1\right]\\ &= \tilde{\Gamma}(\ell_2, \ell_1).\end{aligned}$$

Similarly, (ii) holds.

To prove $0 \leq \tilde{\Gamma}(\ell_1, \ell_2) \leq 1$, it is sufficient to show that $\tilde{\Gamma}(\ell_1, \ell_2) \leq 1$ since $\tilde{\Gamma}(\ell_1, \ell_2) \geq 0$ is straightforward. Assume that $y = \tilde{\Gamma}(\ell_1, \ell_2)$ and $x = \dfrac{1}{3}\sum_{j=1}^{Q}\omega_j\left(\left|M^2_{\ell_1}(a_j) - M^2_{\ell_2}(a_j)\right| + \left|N^2_{\ell_1}(a_j) - N^2_{\ell_2}(a_j)\right| + \left|H^2_{\ell_1}(a_j) - H^2_{\ell_2}(a_j)\right|\right)$, where $x \in [0, 1]$. Then, we have $y = 2^{1-x} - 1$, which is a curve function with values range from 0 to 1. Thus, $0 \leq y \leq 1$ and hence, $0 \leq \tilde{\Gamma}(\ell_1, \ell_2) \leq 1$ as desired, i.e., (iii) holds.

Next, we establish (iv). Suppose $\tilde{\Gamma}(\ell_1, \ell_2) = 1$. Then, we have

$$2^{1-\frac{1}{3}\omega_j\left(\left|M^2_{\ell_1}(a_j)-M^2_{\ell_2}(a_j)\right|+\left|N^2_{\ell_1}(a_j)-N^2_{\ell_2}(a_j)\right|+\left|H^2_{\ell_1}(a_j)-H^2_{\ell_2}(a_j)\right|\right)} - 1 = 1 \Longrightarrow$$

$$2^{1-\frac{1}{3}\omega_j\left(\left|M^2_{\ell_1}(a_j)-M^2_{\ell_2}(a_j)\right|+\left|N^2_{\ell_1}(a_j)-N^2_{\ell_2}(a_j)\right|+\left|H^2_{\ell_1}(a_j)-H^2_{\ell_2}(a_j)\right|\right)} = 2 \Longrightarrow$$

$$1 - \frac{1}{3}\omega_j\left(\left|M^2_{\ell_1}(a_j) - M^2_{\ell_2}(a_j)\right| + \left|N^2_{\ell_1}(a_j) - N^2_{\ell_2}(a_j)\right| + \left|H^2_{\ell_1}(a_j) - H^2_{\ell_2}(a_j)\right|\right) = 1 \Longrightarrow$$

$$\frac{1}{3}\omega_j\left(\left|M^2_{\ell_1}(a_j) - M^2_{\ell_2}(a_j)\right| + \left|N^2_{\ell_1}(a_j) - N^2_{\ell_2}(a_j)\right| + \left|H^2_{\ell_1}(a_j) - H^2_{\ell_2}(a_j)\right|\right) = 0 \Longrightarrow$$

$$\big(\big|M^2_{\ell_1}(a_j) - M^2_{\ell_2}(a_j)\big| + \big|N^2_{\ell_1}(a_j) - N^2_{\ell_2}(a_j)\big| + \big|H^2_{\ell_1}(a_j) - H^2_{\ell_2}(a_j)\big|\big) = 0 \Longrightarrow$$
$$M_{\ell_1}(a_j) = M_{\ell_2}(a_j), N_{\ell_1}(a_j) = N_{\ell_2}(a_j), H_{\ell_1}(a_j) = H_{\ell_2}(a_j).$$

Hence, $\ell_1 = \ell_2$.

Conversely, if $\ell_1 = \ell_2$. Then, $\big|M^2_{\ell_1}(a_j)-M^2_{\ell_2}(a_j)\big| = 0$, $\big|N^2_{\ell_1}(a_j)-N^2_{\ell_2}(a_j)\big| = 0$, and $\big|H^2_{\ell_1}(a_j)-H^2_{\ell_2}(a_j)\big| = 0$. Thus,

$$\frac{1}{3}\omega_j\big(\big|M^2_{\ell_1}(a_j) - M^2_{\ell_2}(a_j)\big| + \big|N^2_{\ell_1}(a_j) - N^2_{\ell_2}(a_j)\big| + \big|H^2_{\ell_1}(a_j) - H^2_{\ell_2}(a_j)\big|\big) = 0,$$

and hence $\tilde{\Gamma}(\ell_1, \ell_2) = 1$.

□

Theorem 4.3. *Suppose ℓ_1 and ℓ_2 are PFSs in $A = \{a_1, a_2, \cdots, a_Q\}$, then*

(i) $\tilde{\Gamma}_*(\ell_1, \ell_2) = \tilde{\Gamma}_*(\ell_2, \ell_1)$,

(ii) $\tilde{\Gamma}_*(\ell_1, \ell_2) = \tilde{\Gamma}_*(\bar{\ell}_1, \bar{\ell}_2)$,

(iii) $0 \leq \tilde{\Gamma}_*(\ell_1, \ell_2) \leq 1$,

(iv) $\tilde{\Gamma}_*(\ell_1, \ell_2) = 1 \Leftrightarrow \ell_1 = \ell_2$.

Proof. Follow from Theorem 4.2. □

Theorem 4.4. *Given that ℓ_1, ℓ_2, and ℓ_3 are PFSs in $A = \{a_1, a_2, \cdots, a_Q\}$ such that $\ell_1 \subseteq \ell_2 \subseteq \ell_3$. Then*

(i) $\tilde{\Gamma}(\ell_1, \ell_3) \leq \tilde{\Gamma}(\ell_1, \ell_2)$,

(ii) $\tilde{\Gamma}(\ell_1, \ell_3) \leq \tilde{\Gamma}(\ell_2, \ell_3)$,

(iii) $\tilde{\Gamma}_*(\ell_1, \ell_3) \leq \tilde{\Gamma}_*(\ell_1, \ell_2)$,

(iv) $\tilde{\Gamma}_*(\ell_1, \ell_3) \leq \tilde{\Gamma}_*(\ell_2, \ell_3)$.

Proof. Because $\ell_1 \subseteq \ell_2 \subseteq \ell_3$, we have $M_{\ell_1}(a_j) \leq M_{\ell_2}(a_j) \leq M_{\ell_3}(a_j)$ and $N_{\ell_1}(a_j) \leq N_{\ell_2}(a_j) \leq N_{\ell_3}(a_j)$ $\forall a_j \in A$. Thus,

$$\begin{aligned}\big|M^2_{\ell_1}(a_j) - M^2_{\ell_3}(a_j)\big| &\geq \big|M^2_{\ell_1}(a_j) - M^2_{\ell_2}(a_j)\big|,\\ \big|N^2_{\ell_1}(a_j) - N^2_{\ell_3}(a_j)\big| &\geq \big|N^2_{\ell_1}(a_j) - N^2_{\ell_2}(a_j)\big|,\\ \big|H^2_{\ell_1}(a_j) - H^2_{\ell_3}(a_j)\big| &\geq \big|H^2_{\ell_1}(a_j) - H^2_{\ell_2}(a_j)\big|,\end{aligned}$$

such that

$$\begin{aligned}\big|M^2_{\ell_1}(a_j) - M^2_{\ell_3}(a_j)\big| &+ \big|N^2_{\ell_1}(a_j) - N^2_{\ell_3}(a_j)\big|\\ +\big|H^2_{\ell_1}(a_j) - H^2_{\ell_3}(a_j)\big| &\geq \big|M^2_{\ell_1}(a_j) - M^2_{\ell_2}(a_j)\big|\\ +\big|N^2_{\ell_1}(a_j) - N^2_{\ell_2}(a_j)\big| &+ \big|H^2_{\ell_1}(a_j) - H^2_{\ell_2}(a_j)\big|.\end{aligned}$$

Clearly, $\tilde{\Gamma}(\ell_1, \ell_3) \leq \tilde{\Gamma}(\ell_1, \ell_2)$ which proves (i). By using the same logic, the proofs of (ii), (iii), and (iv) hold. □

Corollary 4.5. *If ℓ_1, ℓ_2, and ℓ_3 are PFSs in $A = \{a_1, a_2, \cdots, a_Q\}$ and $\ell_1 \subseteq \ell_2 \subseteq \ell_3$. Then $\tilde{\Gamma}(\ell_1, \ell_3) \leq \min\big\{\tilde{\Gamma}(\ell_2, \ell_3), \tilde{\Gamma}(\ell_1, \ell_2)\big\}$ and $\tilde{\Gamma}_*(\ell_1, \ell_3) \leq \min\big\{\tilde{\Gamma}_*(\ell_2, \ell_3), \tilde{\Gamma}_*(\ell_1, \ell_2)\big\}$.*

Proof. From Theorem 4.4, $\tilde{\Gamma}(\ell_1,\ell_3) \leq \tilde{\Gamma}(\ell_1,\ell_2)$ and $\tilde{\Gamma}(\ell_1,\ell_3) \leq \tilde{\Gamma}(\ell_2,\ell_3)$. Hence, $\tilde{\Gamma}(\ell_1,\ell_3) \leq \min\left\{\tilde{\Gamma}(\ell_2,\ell_3), \tilde{\Gamma}(\ell_1,\ell_2)\right\}$. Similarly, $\tilde{\Gamma}_*(\ell_1,\ell_3) \leq \min\left\{\tilde{\Gamma}_*(\ell_2,\ell_3), \tilde{\Gamma}_*(\ell_1,\ell_2)\right\}$. □

Theorem 4.6. *Suppose $\ell_1 \subseteq \ell_2 \subseteq \ell_3$ are PFSs in $A = \{a_1, a_2, \cdots, a_Q\}$, then*

(i) $\tilde{\Gamma}(\ell_1,\ell_2) + \tilde{\Gamma}(\ell_2,\ell_3) \geq \tilde{\Gamma}(\ell_1,\ell_3)$,

(ii) $\tilde{\Gamma}_*(\ell_1,\ell_2) + \tilde{\Gamma}_*(\ell_2,\ell_3) \geq \tilde{\Gamma}_*(\ell_1,\ell_3)$,

(iii) $\tilde{\Gamma}(\ell_1,\ell_2) = \tilde{\Gamma}(\ell_1 \cap \ell_2, \ell_1 \cup \ell_2)$,

(iv) $\tilde{\Gamma}_*(\ell_1,\ell_2) = \tilde{\Gamma}_*(\ell_1 \cap \ell_2, \ell_1 \cup \ell_2)$.

Proof. Suppose $\ell_1 \subseteq \ell_2 \subseteq \ell_3$. Then, $\tilde{\Gamma}(\ell_1,\ell_3) \leq \tilde{\Gamma}(\ell_1,\ell_2)$ and $\tilde{\Gamma}(\ell_1,\ell_3) \leq \tilde{\Gamma}(\ell_2,\ell_3)$ from Theorem 4.4. Thus, $\tilde{\Gamma}(\ell_1,\ell_3) \leq \tilde{\Gamma}(\ell_1,\ell_2) + \tilde{\Gamma}(\ell_2,\ell_3)$, which proves (i). Similarly, (ii) follows from (i).

The proof of (iii) follows by using intersection and union of PFSs in terms of $\tilde{\Gamma}$. Thus,

$$\begin{aligned}\tilde{\Gamma}(\ell_1 \cap \ell_2, \ell_1 \cup \ell_2) &= \sum_{j=i}^{Q}\Big[2 \times 2^{\frac{2}{3}\omega_j\left|\left(\min\{M_{\ell_1}(a_j),M_{\ell_2}(a_j)\}\right)^2-\left(\max\{M_{\ell_1}(a_j),M_{\ell_2}(a_j)\}\right)^2\right|} \\ &\quad \times 2^{-\omega_j\left(\left|\left(\max\{N_{\ell_1}(a_j),N_{\ell_2}(a_j)\}\right)^2-\left(\min\{N_{\ell_1}(a_j),N_{\ell_2}(a_j)\}\right)^2\right|+\left|H^2_{\ell_1\cap\ell_2}(a_j)-H^2_{\ell_1\cup\ell_2}(a_j)\right|\right)} - 1\Big] \\ &= \sum_{j=i}^{Q}\Big[2 \times 2^{\frac{2}{3}\omega_j\left|M^2_{\ell_1}(a_j)-M^2_{\ell_2}(a_j)\right|} \times 2^{-\omega_j\left(\left|N^2_{\ell_1}(a_j)-N^2_{\ell_2}(a_j)\right|+\left|H^2_{\ell_1}(a_j)-H^2_{\ell_2}(a_j)\right|\right)} - 1\Big] \\ &= \tilde{\Gamma}(\ell_1,\ell_2),\end{aligned}$$

which proves (iii). The proof of (iv) is similar to (iii). □

5 Application in Questionnaire Analysis

This section deliberates on the use of the new PFSO in the analysis of questionnaire due to the fuzziness in filling questionnaire. The questionnaire is constructed to measure the extents of awareness and use of virtual library esources (VLR) by undergraduate medical students. Virtual library (VL) is the incorporation of ICT into library services, and this has brought remarkable progress in the academic performance of students in universities [42]. The majority of works done on virtual library made used of questionnaire to decide their aim and objectives. The process of filling questionnaire is characterized with hesitation on the part of the respondents and equally, some of the questions in the questionnaire could be ambiguous. This is the reason why PFS is necessary for questionnaire analysis. This work is governed by the following questions, namely: (i) what is the level of awareness of the VLR in the department by the students? (ii) what are the effects of VL on the medical students' academic wellbeing in the department?

5.1 Data description and presentation

The data for the analysis is drawn from 198 students in the Department of Medicine and Surgery, Benue State University, Makurdi, Nigeria. 198 students out of the 392 students in the department are gotten by using the Yamane's sampling technique [43]. The collected data are presented in Tables 2 and 3, where strongly agree is represented by ℓ_1, agree is ℓ_2, disagree is ℓ_3, strongly disagree is ℓ_4, and the questions are Q_1, Q_2, Q_3, Q_4 and Q_5, respectively.

Table 2: Level of Awareness on the Availability of VL

Questions/Scales	ℓ_1	%	ℓ_2	%	ℓ_3	%	ℓ_4	%
Q_1	95	48	63	31.8	22	11.1	18	9.1
Q_2	68	34.3	51	25.8	49	24.7	30	15.2
Q_3	39	19.7	50	25.3	78	39.4	31	15.7
Q_4	34	17.2	78	39.4	65	32.8	21	10.6
Q_5	105	53	73	36.9	16	8.1	4	2

Table 3: Effects of VL on Academic Performance

Questions/Scales	ℓ_1	%	ℓ_2	%	ℓ_3	%	ℓ_4	%
Q_1	47	23.7	81	40.9	45	22.7	25	12.6
Q_2	51	25.8	75	37.9	46	23.2	26	13.1
Q_3	34	17.2	83	41.9	50	25.3	31	15.7
Q_4	42	21.2	72	36.4	57	28.8	27	13.6
Q_5	52	26.3	62	31.3	56	28.3	28	14.1

Due to the fuzziness in filling the questionnaire, we transform the data in Tables 2 and 3 into PFD as displayed in Tables 4 and 5, by taking the percentages of each of the scales as the MGs while 1−MGs are the NMGs.

Table 4: Data on Level of Awareness of VL

Scales	Q_1	Q_2	Q_3	Q_4	Q_5
ℓ_1	$(0.480, 0.520)$	$(0.343, 0.657)$	$(0.197, 0.803)$	$(0.172, 0.828)$	$(0.53, 0.47)$
ℓ_2	$(0.318, 0.682)$	$(0.258, 0.742)$	$(0.253, 0.747)$	$(0.394, 0.606)$	$(0.369, 0.631)$
ℓ_3	$(0.111, 0.889)$	$(0.247, 0.753)$	$(0.394, 0.606)$	$(0.328, 0.672)$	$(0.081, 0.919)$
ℓ_4	$(0.091, 0.909)$	$(0.152, 0.848)$	$(0.157, 0.843)$	$(0.106, 0.894)$	$(0.02, 0.98)$

Table 5: Data on Effects of Virtual Library

Scales	Q_1	Q_2	Q_3	Q_4	Q_5
ℓ_1	$(0.237, 0.763)$	$(0.258, 0.742)$	$(0.172, 0.828)$	$(0.212, 0.788)$	$(0.263, 0.737)$
ℓ_2	$(0.409, 0.591)$	$(0.379, 0.621)$	$(0.419, 0.581)$	$(0.364, 0.636)$	$(0.313, 0.687)$
ℓ_3	$(0.227, 0.773)$	$(0.232, 0.768)$	$(0.253, 0.747)$	$(0.288, 0.712)$	$(0.283, 0.717)$
ℓ_4	$(0.126, 0.874)$	$(0.131, 0.869)$	$(0.157, 0.843)$	$(0.136, 0.864)$	$(0.141, 0.859)$

Now, we find the similarity between the scales in Tables 4 and 5 using the new similarity operator (5) and get the outcomes in Table 6, which are presented in Figure 1.

Table 6: Results for Analysis

Awareness/Effects	$\tilde{\Gamma}_*(\ell_1, \ell_2)$	$\tilde{\Gamma}_*(\ell_1, \ell_3)$	$\tilde{\Gamma}_*(\ell_1, \ell_4)$	$\tilde{\Gamma}_*(\ell_2, \ell_3)$	$\tilde{\Gamma}_*(\ell_2, \ell_4)$	$\tilde{\Gamma}_*(\ell_3, \ell_4)$
Awareness	0.8410	0.6950	0.6994	0.8128	0.7129	0.8243
Effects	0.8178	0.9408	0.8692	0.8545	0.6990	0.8323

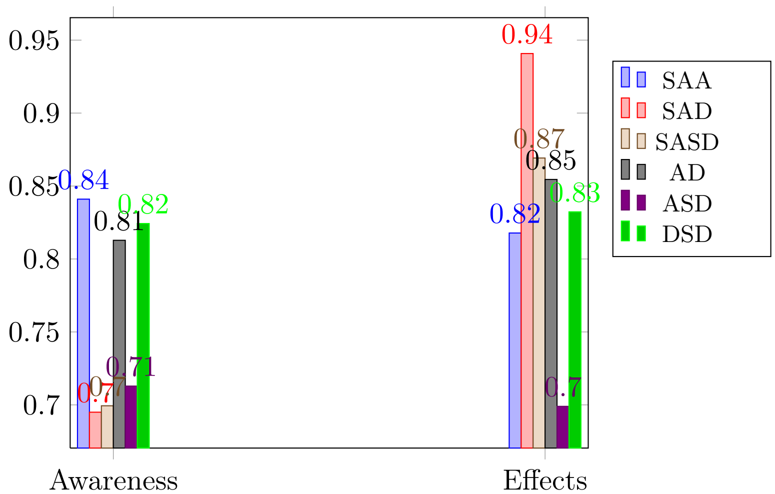

Figure 1: Plot of Results

From these results, for the case of level of awareness of VL, we see that the similarity between SA and A (i.e., SAA) is the greatest, which implies that the undergraduate medical students are aware of the virtual library resources in their department. Similarly, for the case of effects of virtual library, it is observed that the similarity between SA and D (i.e., SAD) is the closest, which implies that the effect of virtual library on the academic performance of the students is not satisfactory.

5.2 Comparison I

To determine the effectiveness of the corrected similarity method, we show its results side by side with the results from the methods in [41]. The comparative results are expressed in Tables 7 and 8.

Table 7: Level of Awareness of VL

Methods	(ℓ_1, ℓ_2)	(ℓ_1, ℓ_3)	(ℓ_1, ℓ_4)	(ℓ_2, ℓ_3)	(ℓ_2, ℓ_4)	(ℓ_3, ℓ_4)
$\tilde{\Gamma}_*$	0.8410	0.6950	0.6994	0.8128	0.7129	0.8243
Γ_1	0.0486	−0.0338	−0.0254	0.0440	−0.0089	0.0576
Γ_2	0.0149	−0.0844	−0.0821	−0.0086	−0.0749	0.0007
Γ_3	0.0149	−0.0844	−0.0821	−0.0086	−0.0749	0.0007
Γ_4	0.0149	−0.0844	−0.0821	−0.0086	−0.0749	0.0007

From Table 7, we see that the similarity between SA and A, and D and SD are very close using the corrected similarity operator. Among the relations, the similarity between scales SA and A is the greatest. This implies that the medical students are aware of the existent of VL on their campus. It is observed that the methods in [41] fail a similarity condition by giving negative results. Therefore, the methods are not appropriate PFSOs, which justifies the effected correction.

Table 8: Effects of VL

Methods	(ℓ_1, ℓ_2)	(ℓ_1, ℓ_3)	(ℓ_1, ℓ_4)	(ℓ_2, ℓ_3)	(ℓ_2, ℓ_4)	(ℓ_3, ℓ_4)
$\tilde{\Gamma}_*$	0.8178	0.9408	0.8692	0.8545	0.6990	0.8323
$\varGamma_1$	0.0392	0.1451	0.0926	0.0637	−0.0250	0.0654
$\varGamma_2$	−0.0046	0.1193	0.0409	0.0270	−0.0823	0.0074
$\varGamma_3$	−0.0046	0.1193	0.0409	0.0270	−0.0823	0.0074
$\varGamma_4$	−0.0046	0.1193	0.0409	0.0270	−0.0823	0.0074

The information in Table 8 shows that the similarity between the scales SA and D is the closest based on the corrected PFSO. The implication of this is that, VL has not effect on the academic wellbeing of the medical students because awareness does not translates into effectiveness if the VLR are not put into use. We observe that the defective methods in [41] produce outcomes that are undefined in the range of the similarity values. Throughout the study, we see that $\varGamma_2$, $\varGamma_3$, and $\varGamma_4$ in [41] yield the same results.

5.3 PFSO based-MCDM Approach of Analyzing Questionnaire

MCDM is a process of choice making in social sciences, medicine, engineering, etc. MCDM determines the best option by assessing more than one criteria for the purpose of selection. Due to the present of imprecision in choice making, MCDM has been studied under PFSs using various information measures. Here, we present the MCDM approach of analyzing questionnaire of VL based on the corrected similarity operator because it has been proven to be effective, consistent and reliable with the most precise results compare to the methods in [41].

5.3.1 Algorithm for the MCDM

The algorithm are as follows:
Step 1. Obtain the Pythagorean fuzzy decision matrix (PFDM) denoted by $\tilde{\ell}_j = \{Q_i(\tilde{\ell}_j)\}_{(m\times n)}$ for $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$, where Q_i are the questions.
Step 2. Formulate the normalized PFDM $\tilde{\ell} = \langle M_{\tilde{\ell}^*_j}(Q_i), N_{\tilde{\ell}^*_j}(Q_i)\rangle_{m\times n}$, where $\langle M_{\tilde{\ell}^*_j}(Q_i), N_{\tilde{\ell}^*_j}(Q_i)\rangle$ are the PFD, and $\tilde{\ell}$ is defined as:

$$\langle M_{\tilde{\ell}^*_j}(Q_i), N_{\tilde{\ell}^*_j}(Q_i)\rangle = \begin{cases} \langle M_{\tilde{\ell}_j}(Q_i), N_{\tilde{\ell}_j}(Q_i)\rangle, & \text{for benefit criterion of } \tilde{\ell} \\ \langle N_{\tilde{\ell}_j}(Q_i), M_{\tilde{\ell}_j}(Q_i)\rangle, & \text{for cost criterion of } \tilde{\ell} \end{cases} \tag{7}$$

Step 3. Compute PIS (positive ideal solution) and NIS (negative ideal solution) given by

$$\begin{aligned} \tilde{\ell}^+ &= \{\tilde{\ell}^+_1, \cdots, \tilde{\ell}^+_n\} \\ \tilde{\ell}^- &= \{\tilde{\ell}^-_1, \cdots, \tilde{\ell}^-_n\} \end{aligned} \tag{8}$$

where

$$\tilde{\ell}^+ = \begin{cases} \langle \max\{M_{\tilde{\ell}_j}(Q_i)\}, \min\{N_{\tilde{\ell}_j}(Q_i)\}\rangle, & \text{if } Q_i \text{ is the BC} \\ \langle \min\{M_{\tilde{\ell}_j}(Q_i)\}, \max\{N_{\tilde{\ell}_j}(Q_i)\}\rangle, & \text{if } Q_i \text{ is the CC,} \end{cases} \tag{9}$$

$$\tilde{\ell}^- = \begin{cases} \langle \min\{M_{\tilde{\ell}_j}(Q_i)\}, \max\{N_{\tilde{\ell}_j}(Q_i)\}\rangle, & \text{if } Q_i \text{ is the BC} \\ \langle \max\{M_{\tilde{\ell}_j}(Q_i)\}, \min\{N_{\tilde{\ell}_j}(Q_i)\}\rangle, & \text{if } Q_i \text{ is the CC,} \end{cases} \tag{10}$$

where BC is benefit criterion and CC is cost criterion.
Step 4. Compute the similarities $\tilde{\Gamma}_*(\tilde{\ell}_j, \tilde{\ell}^-)$ and $\tilde{\Gamma}_*(\tilde{\ell}_j, \tilde{\ell}^+)$.
Step 5. Find the closeness coefficients $\nabla_*(\tilde{\ell}_j)$ by

$$\nabla_*(\tilde{\ell}_j) = \frac{\tilde{\Gamma}_*(\tilde{\ell}_j, \tilde{\ell}^+)}{\tilde{\Gamma}_*(\tilde{\ell}_j, \tilde{\ell}^+) + \tilde{\Gamma}_*(\tilde{\ell}_j, \tilde{\ell}^-)}, \tag{11}$$

for $j = 1, 2, \cdots, n$.
Step 6. Determine the greatest closeness coefficient for the interpretation.

In case either $\tilde{\Gamma}_*(\tilde{\ell}_j, \tilde{\ell}^-)$ or $\tilde{\Gamma}_*(\tilde{\ell}_j, \tilde{\ell}^+)$ is negative (which ought not to happen except the similarity operator is defective), we find $\nabla^+(\tilde{\ell}_j)$ and $\nabla^-(\tilde{\ell}_j)$ thus:

$$\nabla^+(\tilde{\ell}_j) = \frac{\tilde{\Gamma}_*(\tilde{\ell}_j, \tilde{\ell}^+) - \tilde{\Gamma}_{\min}(\tilde{\ell}_j, \tilde{\ell}^+)}{\tilde{\Gamma}_{\max}(\tilde{\ell}_j, \tilde{\ell}^+) - \tilde{\Gamma}_{\min}(\tilde{\ell}_j, \tilde{\ell}^+)}, \tag{12}$$

$$\nabla^-(\tilde{\ell}_j) = \frac{\tilde{\Gamma}_*(\tilde{\ell}_j, \tilde{\ell}^-) - \tilde{\Gamma}_{\min}(\tilde{\ell}_j, \tilde{\ell}^-)_{\min}}{\tilde{\Gamma}_{\max}(\tilde{\ell}_j, \tilde{\ell}^-) - \tilde{\Gamma}_{\min}(\tilde{\ell}_j, \tilde{\ell}^-)} \tag{13}$$

before Step 5. Then (11) becomes:

$$\nabla_*(\tilde{\ell}_j) = \frac{\nabla^+(\tilde{\ell}_j)}{\nabla^+(\tilde{\ell}_j) + \nabla^-(\tilde{\ell}_j)}, \tag{14}$$

for $j = 1, 2, \cdots, n$.

Note that

$$\tilde{\Gamma}_{\max}(\tilde{\ell}_j, \tilde{\ell}^+) = \max_{1 \le j \le n} \{\tilde{\Gamma}_*(\tilde{\ell}_j, \tilde{\ell}^+)\},$$
$$\tilde{\Gamma}_{\min}(\tilde{\ell}_j, \tilde{\ell}^+) = \min_{1 \le j \le n} \{\tilde{\Gamma}_*(\tilde{\ell}_j, \tilde{\ell}^+)\},$$

$$\tilde{\Gamma}_{\max}(\tilde{\ell}_j, \tilde{\ell}^-) = \max_{1 \le j \le n} \{\tilde{\Gamma}_*(\tilde{\ell}_j, \tilde{\ell}^-)\},$$
$$\tilde{\Gamma}_{\min}(\tilde{\ell}_j, \tilde{\ell}^-) = \min_{1 \le j \le n} \{\tilde{\Gamma}_*(\tilde{\ell}_j, \tilde{\ell}^-)\}.$$

The algorithm is captured in Figure 2.

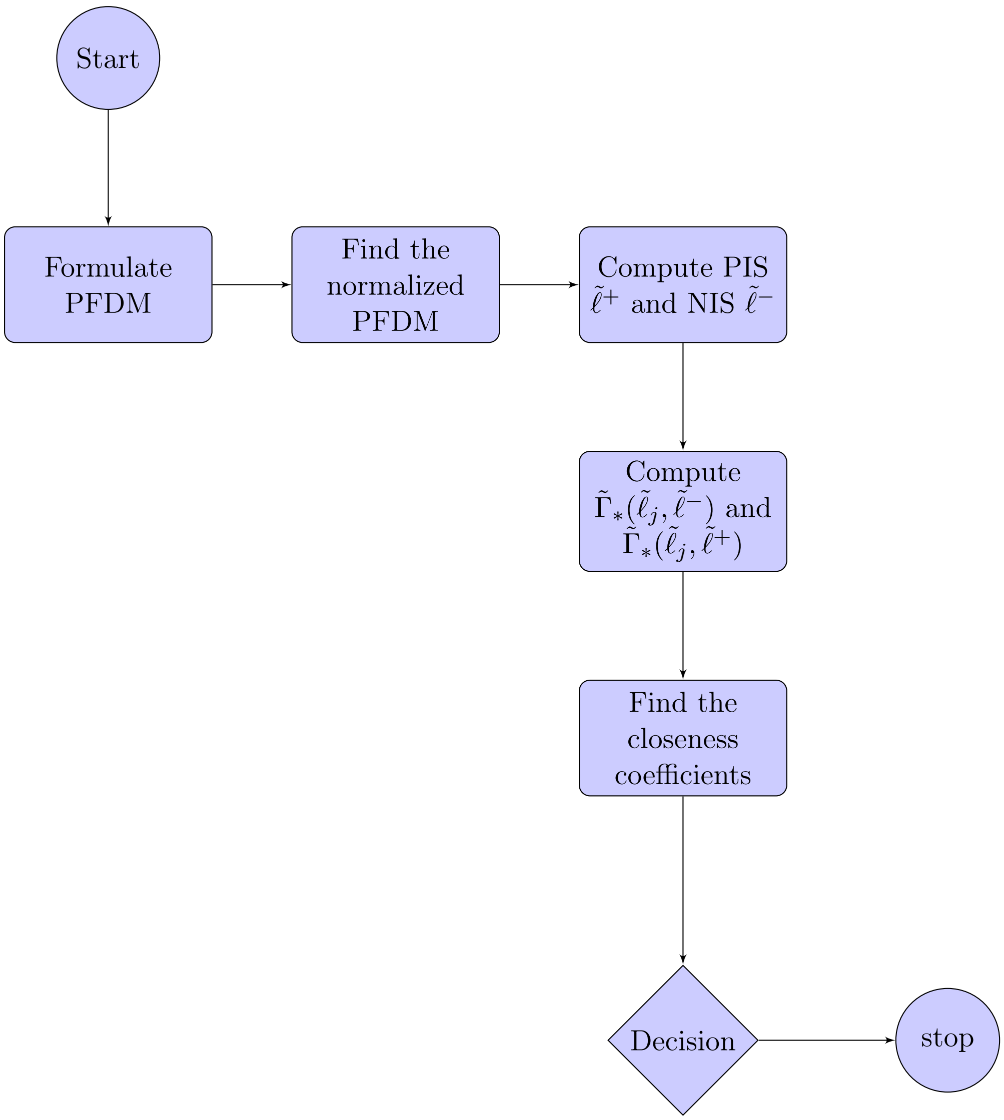

Figure 2: Flowchart for Implementation

5.3.2 Case I

Here, we discuss the questionnaire on the level of awareness of VL as presented in Table 4 via MCDM technique, where Q_5 is cost criterion since the question gives the least MDs. By Step 2, we get Table 9.

Table 9: Normalized PFDM for Level of Awareness

Scales	$\tilde{\ell}_1$	$\tilde{\ell}_2$	$\tilde{\ell}_3$	$\tilde{\ell}_4$
Q_1	$(0.48, 0.52)$	$(0.318, 0.682)$	$(0.111, 0.889)$	$(0.091, 0.909)$
Q_2	$(0.343, 0.657)$	$(0.258, 0.742)$	$(0.247, 0.753)$	$(0.152, 0.848)$
Q_3	$(0.197, 0.803)$	$(0.253, 0.747)$	$(0.394, 0.606)$	$(0.157, 0.843)$
Q_4	$(0.172, 0.828)$	$(0.394, 0.606)$	$(0.328, 0.672)$	$(0.106, 0.894)$
Q_5	$(0.47, 0.53)$	$(0.631, 0.369)$	$(0.919, 0.081)$	$(0.98, 0.02)$

Using Step 3, we obtain the PIS and NIS in Table 10.

Table 10: PIS and NIS for Level of Awareness

Scales	$\tilde{\ell}^-$	$\tilde{\ell}^+$
Q_1	$(0.091, 0.909)$	$(0.48, 0.52)$
Q_2	$(0.152, 0.848)$	$(0.343, 0.657)$
Q_3	$(0.157, 0.843)$	$(0.394, 0.606)$
Q_4	$(0.106, 0.894)$	$(0.394, 0.606)$
Q_5	$(0.98, 0.02)$	$(0.47, 0.53)$

By Step 4, we compute the similarities between $\tilde{\ell}_j$ and $\tilde{\ell}^-$, and $\tilde{\ell}_j$ and $\tilde{\ell}^+$ using (5) to obtain the results in Table 11.

Table 11: Similarities of $(\tilde{\ell}_j, \tilde{\ell}^-)$ and $(\tilde{\ell}_j, \tilde{\ell}^+)$

Scales	$\tilde{\Gamma}_*(\tilde{\ell}_j, \tilde{\ell}^-)$	$\tilde{\Gamma}_*(\tilde{\ell}_j, \tilde{\ell}^+)$
$\tilde{\ell}_1$	0.7088	0.8824
$\tilde{\ell}_2$	0.6719	0.8883
$\tilde{\ell}_3$	0.6883	0.7730
$\tilde{\ell}_4$	0.8302	0.6173

Next, by using (11) in Step 6, we obtain the closeness coefficients in Table 12.

Table 12: Closeness Coefficients for Level of Awareness

Scales	$\nabla_*(\tilde{\ell}_j)$	Ranking
$\tilde{\ell}_1$	0.5546	2^{nd}
$\tilde{\ell}_2$	0.5694	1^{st}
$\tilde{\ell}_3$	0.5290	3^{rd}
$\tilde{\ell}_4$	0.4265	4^{th}

From Table 12, we see that the medical students are aware of the existent of VLR because the scale $\tilde{\ell}_2$ (i.e., A) is ranked first, which tallies with the finding in Table 7.

5.3.3 Case II

Now, we consider the questionnaire on the effects of VLR on the academic wellbeing via MCDM method using the PFDM in Table 5, where Q_3 is taken as the cost criterion. By Step 2, we get Table 13.

Table 13: Normalized PFDM for Effects of VL

Scales	$\tilde{\ell}_1$	$\tilde{\ell}_2$	$\tilde{\ell}_3$	$\tilde{\ell}_4$
Q_1	$(0.237, 0.763)$	$(0.409, 0.591)$	$(0.227, 0.773)$	$(0.126, 0.874)$
Q_2	$(0.258, 0.742)$	$(0.379, 0.621)$	$(0.232, 0.768)$	$(0.131, 0.869)$
Q_3	$(0.828, 0.172)$	$(0.581, 0.419)$	$(0.747, 0.253)$	$(0.843, 0.157)$
Q_4	$(0.212, 0.788)$	$(0.364, 0.636)$	$(0.288, 0.712)$	$(0.136, 0.864)$
Q_5	$(0.263, 0.737)$	$(0.313, 0.687)$	$(0.283, 0.717)$	$(0.141, 0.859)$

Using Step 3, we obtain the PIS and NIS in Table 14.

Table 14: PIS and NIS for Effects of VL

Scales	$\tilde{\ell}^-$	$\tilde{\ell}^+$
Q_1	$(0.126, 0.874)$	$(0.409, 0.591)$
Q_2	$(0.131, 0.869)$	$(0.379, 0.621)$
Q_3	$(0.843, 0.157)$	$(0.581, 0.419)$
Q_4	$(0.136, 0.864)$	$(0.364, 0.636)$
Q_5	$(0.141, 0.859)$	$(0.313, 0.687)$

By Step 4 via (5), we get Table 15.

Table 15: Similarities between $(\tilde{\ell}_j, \tilde{\ell}^-)$ and $(\tilde{\ell}_j, \tilde{\ell}^+)$

Scales	$\tilde{\Gamma}_*(\tilde{\ell}_j, \tilde{\ell}^-)$	$\tilde{\Gamma}_*(\tilde{\ell}_j, \tilde{\ell}^+)$
$\tilde{\ell}_1$	0.7593	0.7908
$\tilde{\ell}_2$	0.6737	0.9703
$\tilde{\ell}_3$	0.7505	0.8269
$\tilde{\ell}_4$	0.8771	0.6737

Next, we find the closeness coefficients for the similarity values using (11) and get Table 16.

Table 16: Closeness Coefficients for Effects of Virtual Library

Scales	$\nabla_*(\tilde{\ell}_j)$	Ranking
$\tilde{\ell}_1$	0.5102	3rd
$\tilde{\ell}_2$	0.5902	1st
$\tilde{\ell}_3$	0.5242	2nd
$\tilde{\ell}_4$	0.4344	4th

The values of the closeness coefficient indicate that $\tilde{\ell}_2 \succeq \tilde{\ell}_3 \succeq \tilde{\ell}_1 \succeq \tilde{\ell}_4$. The interpretation of the ranking is somehow confusing because it oscillates between agree and disagree, which infers that the medical students agree to a minimal effect of VLR on their academic wellbeing possibly due to a very poor use of the VLR, which may be caused by technological barriers, user interface issues, and competing academic commitments.

5.4 Comparison II

Again, we show the effectiveness of the corrected similarity method via MCDM in comparison with the defective methods in [41]. The comparative results are shown in Tables 17 and 18, and Figures 3 and 4.

Table 17: MCDM Comparative Results for Case 1

Methods	$\nabla_*(\tilde{\ell}_1)$	$\nabla_*(\tilde{\ell}_2)$	$\nabla_*(\tilde{\ell}_3)$	$\nabla_*(\tilde{\ell}_4)$	Ordering	Verdict
$\tilde{\Gamma}_*$	0.5546	0.5694	0.5290	0.4265	$\tilde{\ell}_2 \succ \tilde{\ell}_1 \succ \tilde{\ell}_3 \succ \tilde{\ell}_4$	$\tilde{\ell}_2$
Γ_1 [41]	0.8403	1	0.8513	0	$\tilde{\ell}_2 \succ \tilde{\ell}_3 \succ \tilde{\ell}_1 \succ \tilde{\ell}_4$	$\tilde{\ell}_2$
Γ_2 [41]	0.6187	0.7997	1	0	$\tilde{\ell}_3 \succ \tilde{\ell}_2 \succ \tilde{\ell}_1 \succ \tilde{\ell}_4$	$\tilde{\ell}_3$
Γ_3 [41]	0.8491	0.9359	0.4522	0	$\tilde{\ell}_2 \succ \tilde{\ell}_1 \succ \tilde{\ell}_3 \succ \tilde{\ell}_4$	$\tilde{\ell}_2$
Γ_4 [41]	0.8403	1	0.8513	0	$\tilde{\ell}_2 \succ \tilde{\ell}_3 \succ \tilde{\ell}_1 \succ \tilde{\ell}_4$	$\tilde{\ell}_2$

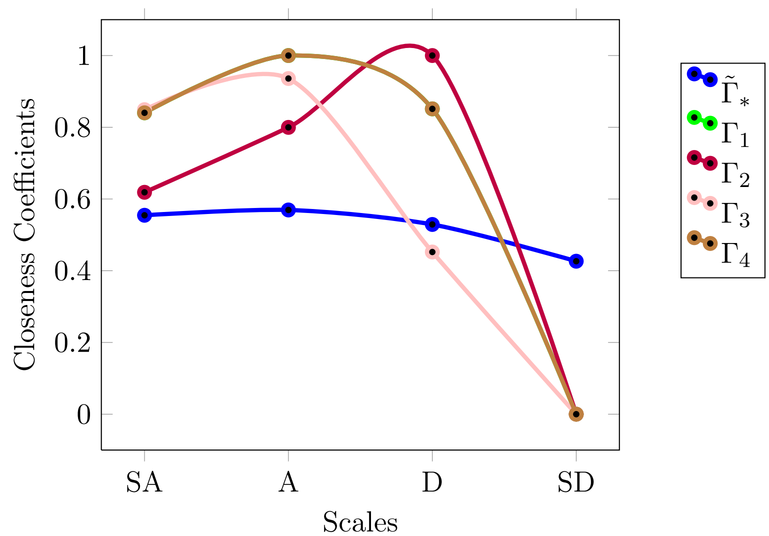

Figure 3: Plot for Comparison

From Table 17, we see that the medical students agree that they are aware of the existent of VLR in their department. While all the methods give the same interpretation, Γ_2 gives different interpretation. The existing methods give zero and one closeness coefficients due to their defectiveness. From Figure 3, it is only the corrected similarity operator that shows consistency.

Table 18: MCDM Comparative Results for Case 2

Methods	$\nabla_*(\tilde{\ell}_1)$	$\nabla_*(\tilde{\ell}_2)$	$\nabla_*(\tilde{\ell}_3)$	$\nabla_*(\tilde{\ell}_4)$	Ordering	Verdict
$\tilde{\Gamma}_*$	0.5102	0.5902	0.5242	0.4344	$\tilde{\ell}_2 \succ \tilde{\ell}_3 \succ \tilde{\ell}_1 \succ \tilde{\ell}_4$	$\tilde{\ell}_2$
Γ_1 [41]	0.4531	1	0.5687	0	$\tilde{\ell}_2 \succ \tilde{\ell}_3 \succ \tilde{\ell}_1 \succ \tilde{\ell}_4$	$\tilde{\ell}_2$
Γ_2 [41]	0.4731	1	0.5318	0	$\tilde{\ell}_2 \succ \tilde{\ell}_3 \succ \tilde{\ell}_1 \succ \tilde{\ell}_4$	$\tilde{\ell}_2$
Γ_3 [41]	0.5881	1	0.7151	0	$\tilde{\ell}_2 \succ \tilde{\ell}_3 \succ \tilde{\ell}_1 \succ \tilde{\ell}_4$	$\tilde{\ell}_2$
Γ_4 [41]	0.4531	1	0.5687	0	$\tilde{\ell}_2 \succ \tilde{\ell}_3 \succ \tilde{\ell}_1 \succ \tilde{\ell}_4$	$\tilde{\ell}_2$

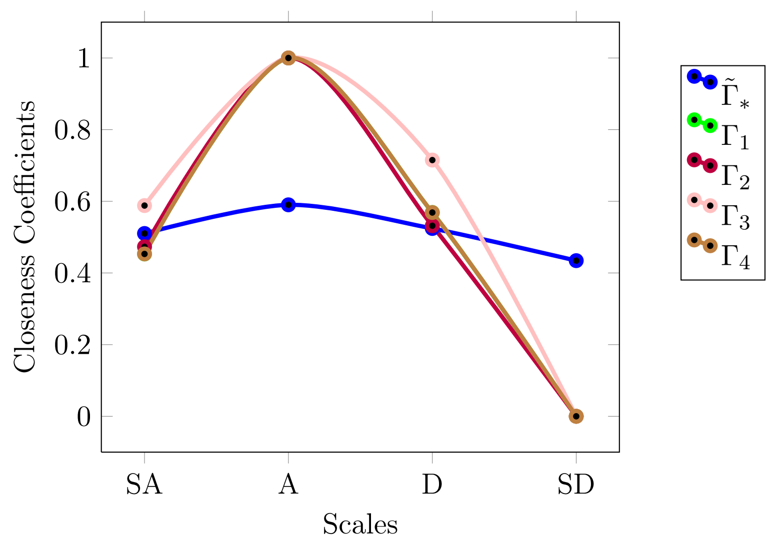

Figure 4: Plot for Comparison

From Table 18, it follows that the medical students agree that the use of VLR has effect on their academic wellbeing. We observe that the closeness coefficients based on the existing PFSOs [41] for $\tilde{\ell}_2$ and $\tilde{\ell}_4$ give zero and one, respectively due to their defectiveness unlike the corrected PFSO. From Figure 4, we see the consistency of the corrected PFSO.

6 Conclusion

In this paper, a new PFSO was developed to ease decision-making in imprecise environments. The new PFSO is the corrected form of the PFSOs in [41], where four PFSOs were constructed which we have demonstrated to be defective. The new PFSO can be used with or without weight vector. Some numerical illustrations were used to showcase the defectiveness of the PFSOs in [41] and to demonstrate the overriding significant of the new PFSO. While the PFSOs in [41] violated the axioms of similarity function, the new PFSO yields reliable and precise results which are consistent with the axioms of similarity function. In addition, some theoretic results of the new PFSO were considered and proved. Furthermore, the new PFSO was used to analyze questionnaire on VL where the collected data were transformed to Pythagorean fuzzy data (PFD). The questionnaire was designed and distributed to 198 undergraduate medical students for the purpose of data collection, after which the data were converted to PFD. It is observed that the corrected version of PFSO could be helpful in decision-making under indeterminate domains since the PFSO is well equipped to control hesitations that may constitute bottleneck for decision-makers. Exploring the potential real-world applications of the new PFSO in different imprecise domains is an interesting research direction for future endeavor. The construction of the modified PFSO limits its application to only Pythagorean fuzzy environment. Thus, the modified PFSO cannot be used to model decision-making problems under picture fuzzy sets [44], q-rung orthopair fuzzy sets [45], Fermatean fuzzy sets [46], etc. because the distinct properties of these sets are not represented in the modified PFSO. However, with some alterations, the modified PFSO could be stretched to the aforementioned domains and use to solve real-world applications.

Conflict of Interest: The author declares no conflict of interest.

Data Availability Statement: The datasets used for this research are contained in the work.

Funding: This article has no Funding resources.

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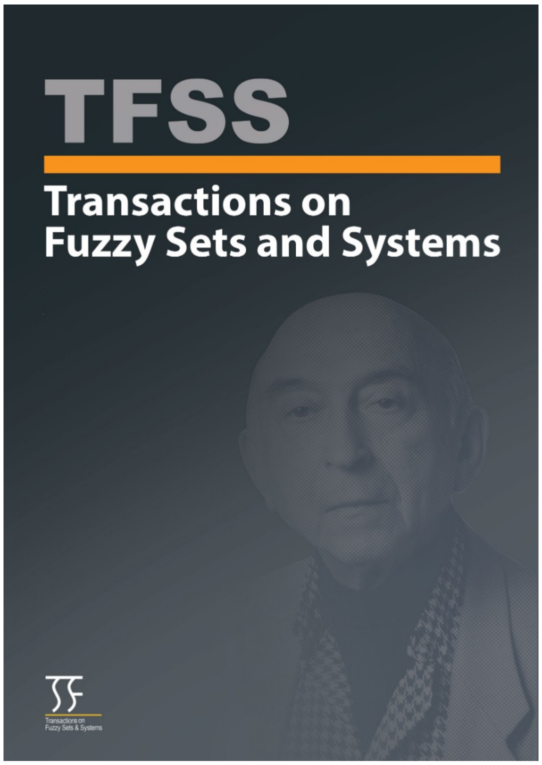
Transactions on Fuzzy Sets and Systems



Transactions on Fuzzy Sets and Systems

ISSN: **2821-0131**

https://sanad.iau.ir/journal/tfss/

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Vol.4, No.1, (2025), 129-158. DOI: https://doi.org/10.71602/tfss.2025.1189142

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Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://sanad.iau.ir/journal/tfss/
Online ISSN: 2821-0131
Vol.4, No.1, (2025), 129-158
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Article Type: Original Research Article

Fuzzy MABAC Deep Learning for Diagnosis of Alzheimers Disease: Analysis of Complex Propositional Linear Diophantine Fuzzy Power Aggregation Insights

Zeeshan Ali

(This paper is dedicated to Professor "John N. Mordeson" on the occasion of his 91st birthday.)

Abstract. Alzheimers disease is an unpredictable and progressive neurodegenerative disorder that initially affects memory thinking and behavior. Some key features of Alzheimers disease are memory loss, cognitive decline, behavioral changes, disorientation, and physical symptoms. In this article, we design the procedure of a multi-attributive border approximation area comparison deep learning algorithm for the diagnosis of Alzheimers Disease. For this, first, we goal to design the model of complex propositional linear Diophantine fuzzy information with their basic operational laws. In addition, we analyze the model of complex propositional linear Diophantine fuzzy power average operator, complex propositional linear Diophantine fuzzy weighted power average operator, complex propositional linear Diophantine fuzzy power geometric operator, complex propositional linear Diophantine fuzzy weighted power geometric operator, and also initiate their major properties. Additionally, the key role of this paper is to arrange relevant from different sources for diagnosing Alzheimers disease under the consideration of the designed technique. Finally, we compare both (proposed and existing) ranking information to address the supremacy and strength of the designed models.

AMS Subject Classification 2020: 03B52; 68T27; 68T37; 94D05; 03E72
Keywords and Phrases: Alzheimers Disease, Complex Propositional linear Diophantine fuzzy sets, MABAC deep learning methods, Power aggregation operators.

1 Introduction

Diagnosing Alzheimers disease is very ambiguous and uncertain, connected with memory loss and changing behavior because of progressive neurodegenerative disorder [1]. The analysis of Alzheimers disease has been done by different scholars according to consider the information of crisp data [2], but to analyze the best one among the collection of data, we needed a soft and valuable technique that can help us in the evaluation of the procedure of decision-making models [3]. A lot of data has been lost in numerous decision-making procedures because of limited information and due to this, various problems are unsolved [4]. For this, Zadeh [5] prepared the fuzzy sets (FSs). FSs theory developed with just a function, called truth degree, defined from fixed sets to unit intervals. In addition, it is quite complex to deal with genuine life problems in the presence of just FS theory, because truth and falsity, yes and no, supporting and supporting against information are

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Received: 1 November 2024; **Revised**: 5 December 2024; **Accepted**: 10 December 2024; **Available Online**: 26 December 2024; **Published Online**: 7 May 2025.

How to cite: Zeeshan Ali. Fuzzy MABAC Deep Learning for Diagnosis of Alzheimers Disease: Analysis of Complex Propositional Linear Diophantine Fuzzy Power Aggregation Insights. *Transactions on Fuzzy Sets and Systems.* 2025; 4(1): 129-158. DOI: https://doi.org/10.71602/tfss.2025.1189142

the key parts of various real-life scenarios. For this, the model of FSs is not suitable, therefore, Atanassov [6, 7] designed the intuitionistic FSs (IFSs). IFSs designed with two different functions but with the same range, called truth degree and falsity degree with a characteristic that is the sum of both functions belonging to a unit interval.

In genuine life situations, all experts are independent and they are not restricted to following the condition of IFSs, because the provided information of experts will exceed form unit interval. For this, the model of Pythagorean FSs (PFSs) was designed by Yager [8]. PFSs are constructed with truth and falsity degrees with a characteristic that is the sum of the squares of both functions belonging to a unit interval. In addition, Yager [9] designed the q-rung orthopair FSs (q-ROFSs) in 2016. The model of q-ROFSs has also developed with truth and falsity information with a model that is the sum of the q-power of both functions belonging to the unit interval. These techniques are very useful and dominant because of their characteristics and due to this reason, many scholars have utilized them in various fields. Riaz and Hashmi [10] organized the linear Diophantine FSs (LDFSs) with a truth and falsity function $\left(\mathcal{F}^{\omega}_{rp}(\tau), \Finv^{\omega}_{rp}(\tau)\right)$ with parameters $\left(\zeta^{\omega}_{rp}(\tau), \Gamma^{\omega}_{rp}(\tau)\right)$.The prominent characteristics of LDFSs, such as $\zeta^{\omega}_{rp}(\tau) * \mathcal{F}^{\omega}_{rp}(\tau) + \Gamma^{\omega}_{rp}(\tau) * \Finv^{\omega}_{rp}(\tau) \in [0,1]$, where $\zeta^{\omega}_{rp}(\tau) + \Gamma^{\omega}_{rp}(\tau) \in [0,1]$. The model of LDFSs is more powerful and more dominant because of their features, the condition of LDFSs is developed based on linear Diophantine equation $ax + by = c$.

Ramot et al. [11] designed the complex FSs (CFSs), the function in CFSs is developed in the form of complex-valued information, where the real and unreal parts of the truth function are limited to unit interval. In various situations, we will cope with complex problems with the help of two-dimensional information, called the complex-valued truth function. Further, Alkouri and Salleh [12] designed the complex IFSs (CIFSs) with complex-valued functions, the condition of CIFSs is that the sum of both functions (for both functions, real and unreal) belongs to the unit interval. Ullah et al. [13] derived the complex PFSs (CPFSs), the projecting condition of CPFSs is the sum of the square of both functions (for both functions, real and unreal) belonging to the unit interval. In 2019, Liu et al. [14] invented the complex q-ROFSs (Cq-ROFSs), the projecting condition of Cq-ROFSs is the sum of the q-power of both functions (for both functions, real and unreal) belonging to the unit interval. In 2020, Ali and Mahmood [15] evaluated the Maclaurin Symmetric mean operators for Cq-ROFSs. In 2022, Kamaci [16] designed the invented the complex LDFSs (CLDFSs), such as $\tilde{H} = \left\{ \left(\tau, \left(\mathcal{F}^{\omega}_{rp}(\tau), \mathcal{F}^{\omega}_{ip}(\tau) \right), \left(\Finv^{\omega}_{rp}(\tau), \Finv^{\omega}_{ip}(\tau) \right), \left(\zeta^{\omega}_{rp}(\tau), \zeta^{\omega}_{ip}(\tau) \right), \left(\Gamma^{\omega}_{rp}(\tau), \Gamma^{\omega}_{ip}(\tau) \right) \right) : \tau \in \mathbb{X} \right\}$, where the model of complex-valued membership (non-membership) function is defined by: $\left(\mathcal{F}^{\omega}_{rp}, \mathcal{F}^{\omega}_{ip}\right) : \mathbb{X} \to [0,1], \left(\left(\Finv^{\omega}_{rp}, \Finv^{\omega}_{ip}\right) : \mathbb{X} \to [0,1]\right)$ with $\zeta^{\omega}_{rp}(t) * \mathcal{F}^{\omega}_{rp}(t) + \Gamma^{\omega}_{rp}(\tau) * \Finv^{\omega}_{rp}(\tau) \in [0,1], \left(\zeta^{\omega}_{ip}(\tau) * \mathcal{F}^{\omega}_{ip}(\tau) + \Gamma^{\omega}_{ip}(\tau) * \Finv^{\omega}_{ip}(\tau) \in [0,1]\right)$ and $\zeta^{\omega}_{rp}(\tau) + \Gamma^{\omega}_{rp}(\tau) \in [0,1], \left(\zeta^{\omega}_{ip}(\tau) + \Gamma^{\omega}_{ip}(\tau) \in [0,1]\right)$, where, the model of complex-valued parameters is defined by: $\zeta^{\omega}_{rp}, \zeta^{\omega}_{ip}, \Gamma^{\omega}_{rp}, \Gamma^{\omega}_{ip} : \mathbb{X} \to [0,1]$ where $\varepsilon^{\omega}_{rp}(\tau) = 1 - \left(\zeta^{\omega}_{rp}(\tau) * \mathcal{F}^{\omega}_{rp}(\tau) + \Gamma^{\omega}_{rp}(\tau) * \Finv^{\omega}_{rp}(\tau)\right), \varepsilon^{\omega}_{ip}(\tau) = 1 - \left(\zeta^{\omega}_{ip}(\tau) * \mathcal{F}^{\omega}_{ip}(\tau) + \Gamma^{\omega}_{ip}(\tau) * \Finv^{\omega}_{ip}(\tau)\right)$, called the refusal function.

In 1980, Gottwald [17] designed the fuzzy propositional logic, a modified version of the FSs theory. In 1988, Atanassov [18] derived the intuitionistic fuzzy propositional calculus with two variants. In 2020, Wang et al. [19] presented the intuitionistic fuzzy propositional logic with novel plausible reasoning-based decision-making models. In 2024, Kahraman [20] introduced propositional PFSs with analytical hierarchal process extensions. In addition, Pamucar and Cirovic [21] invented the (multi-attributive border approximation area comparison) MABAC technique for classical set theory. Further, Yager [22] evaluated the power averaging (PoA) technique. In 2009, Xu and Yager [23] introduced the power geometric (PoG) technique for classical set theory. Jiang et al. [24] derived the power operators for IFSs. Wei and Lu [25] examined the power operators for PFSs. Garg et al. [26] initiated the power operators for Cq-ROFSs. Liu et al. [27] derived the power Dombi operators for CPFSs. Rani and Garg [28] evaluated the power operators for CIFSs. Ali [29] presented the power interaction operator for CIFSs. Ali et al. [30] described the power operators for complex intuitionistic fuzzy soft sets. Moslem [31] designed the parsimonious spherical fuzzy AHP models. Moslem et al. [32] evaluated the fuzzy analytical hierarchy model. Moslem and Pilla [33] invented the spherical fuzzy group decision-making techniques. Acharya et al. [34] designed the stability analysis for neutrosophic fuzzy information. Singh et al. [35] evaluated the malaria disease model in crisp and fuzzy information. Momena et al. [36] initiated the generalized dual hesitant hexagonal fuzzy decision-making techniques. Acharya et al. [37] constructed the neutrosophic differential equation with decision-making techniques. During the assessment of the existing models, we noticed or missed that the technique of complex propositional linear Diophantine fuzzy sets (CPLDFS) needed to be introduced because the above techniques are special cases of proposed models. In addition, we also noticed that to propose the technique of power operators and MABAC for CPLDFSs. The key and major contributions of the designed techniques are listed below:

1. To design the procedure of a MABAC deep learning algorithm for the diagnosis of Alzheimers Disease.
2. To design the model of complex propositional linear Diophantine fuzzy (CPLDF) information with their basic operational laws.
3. To analyze the model of CPLDF power average (CPLDFPoA) operator, CPLDF weighted power average (CPLDFWPoA) operator, CPLDF power geometric (CPLDFPoG) operator, CPLDF weighted power geometric (CPLDFWPoG) operator, and also initiate their major properties.
4. To arrange relevant from different sources for diagnosing Alzheimers disease under the consideration of the designed technique.
5. To compare both (proposed and existing) ranking information to address the supremacy and strength of the designed models. The graphical interpretation of the designed technique is derived in the form of Figure 1.

abstract of the proposed theory..png abstract of the proposed theory.bb

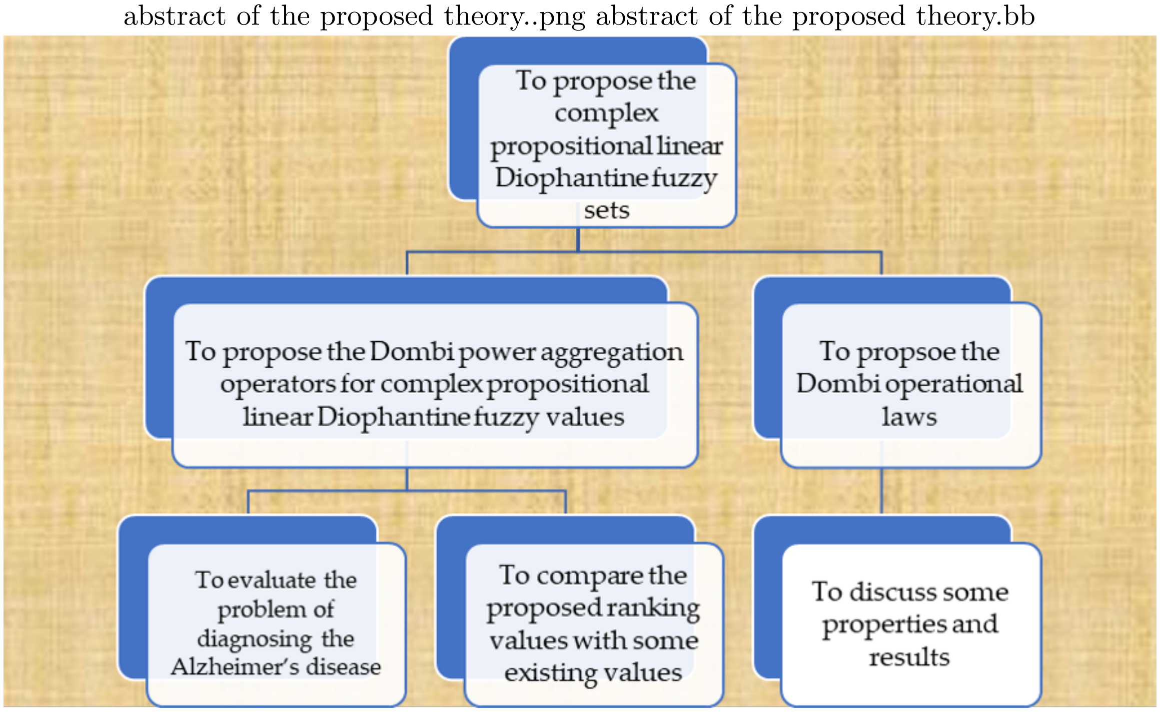

Figure 1: Graphical abstract of the proposed theory

This article is organized in the following ways: In Section 2, we explained the revised techniques of CLDFSs with basic definitions. In addition, we also reviewed the PA operator, and PG operator for the group of any positive integers. In Section 3, we designed the model of CPLDF information with their basic operational laws. In Section 4, we analyzed the model of CPLDFPoA, CPLDFWPoA, CPLDFPoG, and CPLDFWPoG operators, and also initiated their major properties. In Section 5, we designed the procedure of a MABAC deep learning algorithm for the diagnosis of Alzheimers Disease. In Section 6, we arranged relevant from different sources for diagnosing Alzheimers disease under the consideration of the designed technique. In Section 7, we compared both (proposed and existing) ranking information to address the supremacy and strength of the designed models. Some concluding remarks are described in Section 8.

2 Preliminaries

The model of complex linear Diophantine fuzzy information is the reformed version of numerous techniques and very reliable ideas for controlling imprecise and inexact data. This section goals to explain the revised techniques of CLDFSs with basic definitions. In addition, we also reviewed the PA operator, and PG operator for the group of any positive integers.

Definition 2.1. [16] A methodology of CLDFSs for $\mathbb{X}$ (universal set), is designed and deliberated by:

$$\tilde{H}=\left\{\left(\tau,\left(\mathcal{F}_{rp}^{\omega}(\tau),\mathcal{F}_{ip}^{\omega}(\tau)\right),\left(Ⅎ_{rp}^{\omega}(\tau),Ⅎ_{ip}^{\omega}(\tau)\right),\left(\zeta_{rp}^{\omega}(\tau),\zeta_{ip}^{\omega}(\tau)\right),\left(\Gamma_{rp}^{\omega}(\tau),\Gamma_{ip}^{\omega}(\tau)\right)\right):\tau\in\mathbb{X}\right\}$$

Where the model of complex-valued membership (non-membership) function is defined by:

$$\left(\mathcal{F}_{rp}^{\omega},\mathcal{F}_{ip}^{\omega}\right):\mathbb{X}\rightarrow[0,1],\left(\left(Ⅎ_{rp}^{\omega},Ⅎ_{ip}^{\omega}\right):\mathbb{X}\rightarrow[0,1]\right)$$

with $\zeta_{rp}^{\omega}(\tau)*\mathcal{F}_{rp}^{\omega}(\tau)+\Gamma_{rp}^{\omega}(\tau)*Ⅎ_{rp}^{\omega}(\tau)\in[0,1],\left(\zeta_{ip}^{\omega}(\tau)*\mathcal{F}_{ip}^{\omega}(\tau)+\Gamma_{ip}^{\omega}(\tau)*Ⅎ_{ip}^{\omega}(\tau)\in[0,1]\right)$ and $\zeta_{rp}^{\omega}(\tau)+\Gamma_{rp}^{\omega}(\tau)\in[0,1],\left(\zeta_{ip}^{\omega}(\tau)+\Gamma_{ip}^{\omega}(\tau)\in[0,1]\right)$, where, the model of complex-valued parameters is defined by: $\zeta_{rp}^{\omega},\zeta_{ip}^{\omega},\Gamma_{rp}^{\omega},\Gamma_{ip}^{\omega}:\mathbb{X}\rightarrow[0,1]$ where $\varepsilon_{rp}^{\omega}(\tau)=1-\left(\zeta_{rp}^{\omega}(\tau)*\mathcal{F}_{rp}^{\omega}(\tau)+\Gamma_{rp}^{\omega}(\tau)*Ⅎ_{rp}^{\omega}(\tau)\right),\varepsilon_{ip}^{\omega}(\tau)=1-\left(\zeta_{ip}^{\omega}(\tau)*\mathcal{F}_{ip}^{\omega}(\tau)+\Gamma_{ip}^{\omega}(\tau)*Ⅎ_{ip}^{\omega}(\tau)\right)$, called the refusal function. The simple version of CLDFN is mentioned in the following form, such as:

$$\tilde{H}_{\&}=\left(\left(\mathcal{F}_{rp}^{\omega_{\&}},\mathcal{F}_{ip}^{\omega_{\&}}\right),\left(Ⅎ_{rp}^{\omega_{\&}},Ⅎ_{ip}^{\omega_{\&}}\right),\left(\zeta_{rp}^{\omega_{\&}},\zeta_{ip}^{\omega_{\&}}\right),\left(\Gamma_{rp}^{\omega_{\&}},\Gamma_{ip}^{\omega_{\&}}\right)\right),\&=1,2,\cdots,\ni$$

In addition, we goal to describe numerous operational laws for the above existing models, such as algebraic operational laws, briefly discussed below.

Definition 2.2. [16] Let $\tilde{H}_{\&}=\left(\left(\mathcal{F}_{rp}^{\omega_{\&}},\mathcal{F}_{ip}^{\omega_{\&}}\right),\left(Ⅎ_{rp}^{\omega_{\&}},Ⅎ_{ip}^{\omega_{\&}}\right),\left(\zeta_{rp}^{\omega_{\&}},\zeta_{ip}^{\omega_{\&}}\right),\left(\Gamma_{rp}^{\omega_{\&}},\Gamma_{ip}^{\omega_{\&}}\right)\right),\&=1,2$ be two CLDFN. Thus

$$\tilde{H}_1\oplus\tilde{H}_2=\begin{pmatrix}\left(\mathcal{F}_{rp}^{\omega_1}+\mathcal{F}_{rp}^{\omega_2}-\mathcal{F}_{rp}^{\omega_1}\mathcal{F}_{rp}^{\omega_2},\mathcal{F}_{ip}^{\omega_1}+\mathcal{F}_{ip}^{\omega_2}-\mathcal{F}_{ip}^{\omega_1}\mathcal{F}_{ip}^{\omega_2}\right),\left(Ⅎ_{rp}^{\omega_1}Ⅎ_{rp}^{\omega_2},Ⅎ_{ip}^{\omega_1}Ⅎ_{ip}^{\omega_2}\right),\\\left(\zeta_{rp}^{\omega_1}+\zeta_{rp}^{\omega_2}-\zeta_{rp}^{\omega_1}\zeta_{rp}^{\omega_2},\zeta_{ip}^{\omega_1}+\zeta_{ip}^{\omega_2}-\zeta_{ip}^{\omega_1}\zeta_{ip}^{\omega_2}\right),\left(\Gamma_{rp}^{\omega_1}\Gamma_{rp}^{\omega_2},\Gamma_{ip}^{\omega_1}\Gamma_{ip}^{\omega_2}\right)\end{pmatrix}$$

$$\tilde{H}_1\otimes\tilde{H}_2=\begin{pmatrix}\left(\mathcal{F}_{rp}^{\omega_1}\mathcal{F}_{rp}^{\omega_2},\mathcal{F}_{ip}^{\omega_1}\mathcal{F}_{ip}^{\omega_2}\right),\left(Ⅎ_{rp}^{\omega_1}+Ⅎ_{rp}^{\omega_2}-Ⅎ_{rp}^{\omega_1}Ⅎ_{rp}^{\omega_2},Ⅎ_{ip}^{\omega_1}+Ⅎ_{ip}^{\omega_2}-Ⅎ_{ip}^{\omega_1}Ⅎ_{ip}^{\omega_2}\right),\\\left(\zeta_{rp}^{\omega_1}\zeta_{rp}^{\omega_2},\zeta_{ip}^{\omega_1}\zeta_{ip}^{\omega_2}\right),\left(\Gamma_{rp}^{\omega_1}+\Gamma_{rp}^{\omega_2}-\Gamma_{rp}^{\omega_1}\Gamma_{rp}^{\omega_2},\Gamma_{ip}^{\omega_1}+\Gamma_{ip}^{\omega_2}-\Gamma_{ip}^{\omega_1}\Gamma_{ip}^{\omega_2}\right)\end{pmatrix}$$

$$\tilde{\eta}_{\Theta}\tilde{H}_{\&} = \begin{pmatrix} \left(1-\left(1-\mathcal{F}_{rp}^{\omega_{\&}}\right)^{\tilde{\eta}_{\Theta}}, 1-\left(1-\mathcal{F}_{ip}^{\omega_{\&}}\right)^{\tilde{\eta}_{\Theta}}\right), \left(\left(\text{Ⅎ}_{rp}^{\omega_{\&}}\right)^{\tilde{\eta}_{\Theta}}, \left(\text{Ⅎ}_{ip}^{\omega_{\&}}\right)^{\tilde{\eta}_{\Theta}}\right), \\ \left(1-\left(1-\zeta_{rp}^{\omega_{\&}}\right)^{\tilde{\eta}_{\Theta}}, 1-\left(1-\zeta_{ip}^{\omega_{\&}}\right)^{\tilde{\eta}_{\Theta}}\right), \left(\left(\Gamma_{rp}^{\omega_{\&}}\right)^{\tilde{\eta}_{\Theta}}, \left(\Gamma_{ip}^{\omega_{\&}}\right)^{\tilde{\eta}_{\Theta}}\right) \end{pmatrix}$$

$$\left(\tilde{H}_{\&}\right)^{\tilde{\eta}_{\Theta}} = \begin{pmatrix} \left(\left(\mathcal{F}_{rp}^{\omega_{\&}}\right)^{\tilde{\eta}_{\Theta}}, \left(\mathcal{F}_{ip}^{\omega_{\&}}\right)^{\tilde{\eta}_{\Theta}}\right), \left(1-\left(1-\text{Ⅎ}_{rp}^{\omega_{\&}}\right)^{\tilde{\eta}_{\Theta}}, 1-\left(1-\text{Ⅎ}_{ip}^{\omega_{\&}}\right)^{\tilde{\eta}_{\Theta}}\right), \\ \left(\left(\zeta_{rp}^{\omega_{\&}}\right)^{\tilde{\eta}_{\Theta}}, \left(\zeta_{ip}^{\omega_{\&}}\right)^{\tilde{\eta}_{\Theta}}\right), \left(1-\left(1-\Gamma_{rp}^{\omega_{\&}}\right)^{\tilde{\eta}_{\Theta}}, 1-\left(1-\Gamma_{ip}^{\omega_{\&}}\right)^{\tilde{\eta}_{\Theta}}\right) \end{pmatrix}$$

Moreover, we target to revise the information of score value and accuracy value, for evaluating the relationship among any two complex linear Diophantine fuzzy numbers.

Definition 2.3. [16] Let $\tilde{H}_{\&} = \left(\left(\mathcal{F}_{rp}^{\omega_{\&}}, \mathcal{F}_{ip}^{\omega_{\&}}\right), \left(\text{Ⅎ}_{rp}^{\omega_{\&}}, \text{Ⅎ}_{ip}^{\omega_{\&}}\right), \left(\zeta_{rp}^{\omega_{\&}}, \zeta_{ip}^{\omega_{\&}}\right), \left(\Gamma_{rp}^{\omega_{\&}}, \Gamma_{ip}^{\omega_{\&}}\right)\right), \& = 1$ be a CLDFN. Thus

$$SC\left(\tilde{H}_{\&}\right) = \frac{1}{4}\left(\left(\mathcal{F}_{rp}^{\omega_{\&}} + \mathcal{F}_{ip}^{\omega_{\&}}\right) - \left(\text{Ⅎ}_{rp}^{\omega_{\&}} + \text{Ⅎ}_{ip}^{\omega_{\&}}\right) + \left(\zeta_{rp}^{\omega_{\&}} + \zeta_{ip}^{\omega_{\&}}\right) - \left(\Gamma_{rp}^{\omega_{\&}} + \Gamma_{ip}^{\omega_{\&}}\right)\right) \in [-1, 1]$$

$$AC\left(\tilde{H}_{\&}\right) = \frac{1}{4}\left(\left(\mathcal{F}_{rp}^{\omega_{\&}} + \mathcal{F}_{ip}^{\omega_{\&}}\right) + \left(\text{Ⅎ}_{rp}^{\omega_{\&}} + \text{Ⅎ}_{ip}^{\omega_{\&}}\right) + \left(\zeta_{rp}^{\omega_{\&}} + \zeta_{ip}^{\omega_{\&}}\right) + \left(\Gamma_{rp}^{\omega_{\&}} + \Gamma_{ip}^{\omega_{\&}}\right)\right) \in [0, 1]$$

Thus, if $SC\left(\tilde{H}_1\right) > SC\left(\tilde{H}_2\right) \Rightarrow \tilde{H}_1 > \tilde{H}_2$, then if $AC\left(\tilde{H}_1\right) > AC\left(\tilde{H}_2\right) \Rightarrow \tilde{H}_1 > \tilde{H}_2$. Further, we goal to discuss the technique of PoA and PoG techniques.

Definition 2.4. [22, 23] Let $\tilde{H}_{\&}, \& = 1, 2, \cdots, \ni$, be a group of non-negative information. Then

$$PoA\left(\tilde{H}_1, \tilde{H}_2, \cdots, \tilde{H}_{\ni}\right) = \frac{\left(1+\tilde{\eta}\left(\tilde{H}_1\right)\right)}{\sum_{\&=1}^{\ni}\left(1+\tilde{\eta}\left(\tilde{H}_{\&}\right)\right)}\tilde{H}_1 \oplus \frac{\left(1+\tilde{\eta}\left(\tilde{H}_2\right)\right)}{\sum_{\&=1}^{\ni}\left(1+\tilde{\eta}\left(\tilde{H}_{\&}\right)\right)}\tilde{H}_2 \oplus \cdots \oplus \frac{\left(1+\tilde{\eta}\left(\tilde{H}_{\ni}\right)\right)}{\sum_{\&=1}^{\ni}\left(1+\tilde{\eta}\left(\tilde{H}_{\&}\right)\right)}\tilde{H}_{\ni}$$
$$= \sum_{\&=1}^{\ni} \frac{\left(1+\tilde{\eta}\left(\tilde{H}_{\&}\right)\right)}{\sum_{\&=1}^{\ni}\left(1+\tilde{\eta}\left(\tilde{H}_{\&}\right)\right)}\tilde{H}_{\&}$$

signified the PoA operators, and the technique

$$PoG\left(\tilde{H}_1, \tilde{H}_2, \cdots, \tilde{H}_{\ni}\right) = \left(\tilde{H}_1\right)^{\frac{\left(1+\tilde{\eta}\left(\tilde{H}_1\right)\right)}{\sum_{\&=1}^{\ni}\left(1+\tilde{\eta}\left(\tilde{H}_{\&}\right)\right)}} \otimes \left(\tilde{H}_2\right)^{\frac{\left(1+\tilde{\eta}\left(\tilde{H}_2\right)\right)}{\sum_{\&=1}^{\ni}\left(1+\tilde{\eta}\left(\tilde{H}_{\&}\right)\right)}} \otimes \cdots \otimes \left(\tilde{H}_{\ni}\right)^{\frac{\left(1+\tilde{\eta}\left(\tilde{H}_{\ni}\right)\right)}{\sum_{\&=1}^{\ni}\left(1+\tilde{\eta}\left(\tilde{H}_{\&}\right)\right)}}$$
$$= \prod_{\&=1}^{\ni}\left(\tilde{H}_{\&}\right)^{\frac{\left(1+\tilde{\eta}\left(\tilde{H}_{\&}\right)\right)}{\sum_{\&=1}^{\ni}\left(1+\tilde{\eta}\left(\tilde{H}_{\&}\right)\right)}}$$

called PoG operator with $\tilde{\eta}\left(\tilde{H}_{\&}\right) = \sum_{i\neq\&=1}^{\ni} S\left(\tilde{H}_i, \tilde{H}_{\&}\right)$, and $S\left(\tilde{H}_i, \tilde{H}_{\&}\right) = 1 - D\left(\tilde{H}_i, \tilde{H}_{\&}\right)$, thus

1. $S\left(\tilde{H}_i, \tilde{H}_{\&}\right) \in [0, 1]$.
2. $S\left(\tilde{H}_i, \tilde{H}_{\&}\right) = S\left(\tilde{H}_{\&}, \tilde{H}_i\right)$.
3. When $S\left(\tilde{H}_i, \tilde{H}_{\&}\right) \geq S\left(\tilde{H}_k, \tilde{H}_l\right)$, then $D\left(\tilde{H}_i, \tilde{H}_{\&}\right) \leq D\left(\tilde{H}_k, \tilde{H}_l\right)$.

3 CPLDFSs: Complex Propositional Linear Diophantine Fuzzy Sets

This section goals to explain the new techniques of CPLDFSs with basic definitions. Further, we designed some algebraic operational laws for CPLDFSs.

Definition 3.1. A methodology of CPLDFSs for $\mathbb{X}$ (universal set), is designed and deliberated by:

$$\tilde{H}=\left\{\left(\tau,\left(\mathcal{F}_{rp}^{\omega}(\tau),\mathcal{F}_{ip}^{\omega}(\tau)\right),\left(\Finv_{rp}^{\omega}(\tau),\Finv_{ip}^{\omega}(\tau)\right),\left(\zeta_{rp}^{\omega}(\tau),\zeta_{ip}^{\omega}(\tau)\right),\left(\Gamma_{rp}^{\omega}(\tau),\Gamma_{ip}^{\omega}(\tau)\right)\right):\tau\in\mathbb{X}\right\}$$

In addition, we define the truth and parameter function according to their real and imaginary parts, such as

$$\mathcal{F}_{rp}^{\omega}(\tau)=\mathbb{L}_{rp}^{1}\left(\Finv_{rp}^{\omega}(\tau)\right),\left(\mathcal{F}_{ip}^{\omega}(\tau)=\mathbb{L}_{ip}^{1}\left(\Finv_{ip}^{\omega}(\tau)\right)\right)$$

and

$$\zeta_{rp}^{\omega}(\tau)=\mathbb{L}_{rp}^{2}\left(\Gamma_{rp}^{\omega}(\tau)\right),\left(\zeta_{ip}^{\omega}(\tau)=\mathbb{L}_{ip}^{2}\left(\Gamma_{ip}^{\omega}(\tau)\right)\right)$$

Then

$$\mathbb{L}_{rp}^{2}\left(\Gamma_{rp}^{\omega}(\tau)\right)*\mathbb{L}_{rp}^{1}\left(\Finv_{rp}^{\omega}(\tau)\right)+\Gamma_{rp}^{\omega}(\tau)*\Finv_{rp}^{\omega}(\tau)\leq 1$$
$$\Rightarrow\Gamma_{rp}^{\omega}(\tau)*\Finv_{rp}^{\omega}(\tau)\left(1+\mathbb{L}_{rp}^{1}\mathbb{L}_{rp}^{2}\right)\leq 1\Rightarrow\Gamma_{rp}^{\omega}(\tau)*\Finv_{rp}^{\omega}(\tau)\leq\frac{1}{1+\mathbb{L}_{rp}^{1}\mathbb{L}_{rp}^{2}}$$

Similarly, we have imaginary parts, such as

$$\Gamma_{ip}^{\omega}(\tau)*\Finv_{ip}^{\omega}(\tau)\leq\frac{1}{1+\mathbb{L}_{ip}^{1}\mathbb{L}_{ip}^{2}}$$

thus

$$\varepsilon_{rp}^{\omega}(\tau)=1-\left(\zeta_{rp}^{\omega}(\tau)*\mathcal{F}_{rp}^{\omega}(\tau)+\Gamma_{rp}^{\omega}(\tau)*\Finv_{rp}^{\omega}(\tau)\right)=1-\left(\mathbb{L}_{rp}^{2}\left(\Gamma_{rp}^{\omega}(\tau)\right)*\mathbb{L}_{rp}^{1}\left(\Finv_{rp}^{\omega}(\tau)\right)+\Gamma_{rp}^{\omega}(\tau)*\Finv_{rp}^{\omega}(\tau)\right)$$
$$\Rightarrow 1-\left(\Gamma_{rp}^{\omega}(\tau)*\Finv_{rp}^{\omega}(\tau)\left(1+\mathbb{L}_{rp}^{1}\mathbb{L}_{rp}^{2}\right)\right)$$

then

$$1-\varepsilon_{rp}^{\omega}(\tau)=\Gamma_{rp}^{\omega}(\tau)*\Finv_{rp}^{\omega}(\tau)\left(1+\mathbb{L}_{rp}^{1}\mathbb{L}_{rp}^{2}\right)$$

and

$$\Gamma_{rp}^{\omega}(\tau)*\Finv_{rp}^{\omega}(\tau)=\frac{1-\varepsilon_{rp}^{\omega}(\tau)}{\left(1+\mathbb{L}_{rp}^{1}\mathbb{L}_{rp}^{2}\right)}$$

Similarly, we have

$$\Gamma_{ip}^{\omega}(\tau)*\Finv_{ip}^{\omega}(\tau)=\frac{1-\varepsilon_{ip}^{\omega}(\tau)}{\left(1+\mathbb{L}_{ip}^{1}\mathbb{L}_{ip}^{2}\right)}$$

But if we use the condition of IFSs, thus we have

$$\mathbb{L}_{rp}^{1}\left(\Finv_{rp}^{\omega}(\tau)\right)+\Finv_{rp}^{\omega}(\tau)\leq 1$$
$$\Rightarrow\Finv_{rp}^{\omega}(\tau)\left(1+\mathbb{L}_{rp}^{1}\right)\leq 1\Rightarrow\Finv_{rp}^{\omega}(\tau)\leq\frac{1}{1+\mathbb{L}_{rp}^{1}}$$

Similarly, we have imaginary parts, such as

$$\Finv_{ip}^{\omega}(\tau) \leq \frac{1}{1+\mathbb{L}_{ip}^{1}}$$

Thus, we have the condition of refusal function in IFSs, such as

$$\varepsilon_{rp}^{\omega}(\tau) = 1-\left(\mathcal{F}_{rp}^{\omega}(\tau)+\Finv_{rp}^{\omega}(\tau)\right) = 1-\left(\mathbb{L}_{rp}^{1}\left(\Finv_{rp}^{\omega}(\tau)\right)+\Finv_{rp}^{\omega}(\tau)\right) = 1-\left(\Finv_{rp}^{\omega}(\tau)\left(1+\mathbb{L}_{rp}^{1}\right)\right)$$

then

$$1-\varepsilon_{rp}^{\omega}(\tau) = \Finv_{rp}^{\omega}(\tau)\left(1+\mathbb{L}_{rp}^{1}\right)$$

and

$$\Finv_{rp}^{\omega}(\tau) = \frac{1-\varepsilon_{rp}^{\omega}(\tau)}{\left(1+\mathbb{L}_{rp}^{1}\right)}, \left(\zeta_{rp}^{\omega}(\tau) = \frac{1-\varepsilon_{rp}^{\omega}(\tau)}{\left(1+\mathbb{L}_{rp}^{2}\right)}\right)$$

Similarly, we have

$$\Finv_{ip}^{\omega}(\tau) = \frac{1-\varepsilon_{ip}^{\omega}(\tau)}{\left(1+\mathbb{L}_{rp}^{1}\right)}, \left(\zeta_{ip}^{\omega}(\tau) = \frac{1-\varepsilon_{ip}^{\omega}(\tau)}{\left(1+\mathbb{L}_{ip}^{2}\right)}\right)$$

if $\varepsilon_{rp}^{\omega}(\tau) = \varepsilon_{ip}^{\omega}(\tau) = 0$, thus $\Finv_{rp}^{\omega}(\tau) = \frac{1}{\left(1+\mathbb{L}_{rp}^{1}\right)}$ and $\Finv_{ip}^{\omega}(\tau) = \frac{1}{\left(1+\mathbb{L}_{ip}^{1}\right)}$. Then

$$\mathcal{F}_{rp}^{\omega}(\tau) = \mathbb{L}_{rp}^{1}\left(\frac{1}{\left(1+\mathbb{L}_{rp}^{1}\right)}\right), \left(\mathcal{F}_{ip}^{\omega}(\tau) = \mathbb{L}_{ip}^{1}\left(\frac{1}{\left(1+\mathbb{L}_{ip}^{1}\right)}\right)\right)$$

and

$$\zeta_{rp}^{\omega}(\tau) = \mathbb{L}_{rp}^{2}\left(\frac{1}{\left(1+\mathbb{L}_{rp}^{2}\right)}\right), \left(\zeta_{ip}^{\omega}(\tau) = \mathbb{L}_{ip}^{2}\left(\frac{1}{\left(1+\mathbb{L}_{ip}^{2}\right)}\right)\right)$$

Then

$$\tilde{H} = \left\{\left(\begin{array}{c}\tau, \left(\left(\mathbb{L}_{rp}^{1}\left(\frac{1}{1+\mathbb{L}_{rp}^{1}}\right), \mathbb{L}_{ip}^{1}\left(\frac{1}{1+\mathbb{L}_{ip}^{1}}\right)\right), \left(\frac{1}{1+\mathbb{L}_{rp}^{1}}, \frac{1}{1+\mathbb{L}_{rp}^{1}}\right)\right), \\ \left(\left(\mathbb{L}_{rp}^{2}\left(\frac{1}{1+\mathbb{L}_{rp}^{2}}\right), \mathbb{L}_{ip}^{2}\left(\frac{1}{1+\mathbb{L}_{ip}^{2}}\right)\right), \left(\frac{1}{1+\mathbb{L}_{rp}^{2}}, \frac{1}{1+\mathbb{L}_{rp}^{2}}\right)\right)\end{array}\right) : \tau \in \mathbb{X}\right\}$$

Thus, we have the following final shape, such as

$$\tilde{H}_{\&} = \left(\begin{array}{c}\left(\left(\mathbb{L}_{rp}^{1_{\&}}\left(\frac{1-\varepsilon_{rp}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right), \mathbb{L}_{ip}^{1_{\&}}\left(\frac{1-\varepsilon_{ip}{}_{\&}^{\omega}}{1+\mathbb{L}_{ip}^{1_{\&}}}\right)\right), \left(\left(\frac{1-\varepsilon_{rp}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right), \left(\frac{1-\varepsilon_{ip}{}_{\&}^{\omega}}{1+\mathbb{L}_{ip}^{1_{\&}}}\right)\right)\right), \\ \left(\left(\mathbb{L}_{rp}^{2_{\&}}\left(\frac{1-\varepsilon_{rp}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right), \mathbb{L}_{ip}^{2_{\&}}\left(\frac{1-\varepsilon_{ip}{}_{\&}^{\omega}}{1+\mathbb{L}_{ip}^{2_{\&}}}\right)\right), \left(\left(\frac{1-\varepsilon_{rp}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right), \left(\frac{1-\varepsilon_{ip}{}_{\&}^{\omega}}{1+\mathbb{L}_{ip}^{2_{\&}}}\right)\right)\right)\end{array}\right), \& = 1, 2, \cdots, \ni.$$

Definition 3.2. For any $\tilde{H}_{\&} = \begin{pmatrix} \left(\left(\mathbb{L}_{rp}^{1_{\&}}\left(\frac{1-\varepsilon_{rp_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right), \mathbb{L}_{ip}^{1_{\&}}\left(\frac{1-\varepsilon_{ip_{\&}}^{\omega}}{1+\mathbb{L}_{ip}^{1_{\&}}}\right)\right), \left(\left(\frac{1-\varepsilon_{rp_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right), \left(\frac{1-\varepsilon_{ip_{\&}}^{\omega}}{1+\mathbb{L}_{ip}^{1_{\&}}}\right)\right)\right), \\ \left(\left(\mathbb{L}_{rp}^{2_{\&}}\left(\frac{1-\varepsilon_{rp_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right), \mathbb{L}_{ip}^{2_{\&}}\left(\frac{1-\varepsilon_{ip_{\&}}^{\omega}}{1+\mathbb{L}_{ip}^{2_{\&}}}\right)\right), \left(\left(\frac{1-\varepsilon_{rp_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right), \left(\frac{1-\varepsilon_{ip_{\&}}^{\omega}}{1+\mathbb{L}_{ip}^{2_{\&}}}\right)\right)\right) \end{pmatrix}, \& = 1, 2$, we have

$$\tilde{H}_1 \oplus \tilde{H}_2 = \begin{pmatrix} \begin{pmatrix} \mathbb{L}_{rp}^{1_1}\left(\frac{1-\varepsilon_{rp_1}^{\omega}}{1+\mathbb{L}_{rp}^{1_1}}\right) + \mathbb{L}_{rp}^{1_2}\left(\frac{1-\varepsilon_{rp_2}^{\omega}}{1+\mathbb{L}_{rp}^{1_2}}\right) - \mathbb{L}_{rp}^{1_1}\left(\frac{1-\varepsilon_{rp_1}^{\omega}}{1+\mathbb{L}_{rp}^{1_1}}\right)\mathbb{L}_{rp}^{1_2}\left(\frac{1-\varepsilon_{rp}}{1+\mathbb{L}_{rp}^{1_2}}\right), \\ \mathbb{L}_{ip}^{1_1}\left(\frac{1-\varepsilon_{ip_1}^{\omega}}{1+\mathbb{L}_{ip}^{1_1}}\right) + \mathbb{L}_{ip}^{1_2}\left(\frac{1-\varepsilon_{ip_2}^{\omega}}{1+\mathbb{L}_{ip}^{1_2}}\right) - \mathbb{L}_{ip}^{1_1}\left(\frac{1-\varepsilon_{ip_1}^{\omega}}{1+\mathbb{L}_{ip}^{1_1}}\right)\mathbb{L}_{ip}^{1_2}\left(\frac{1-\varepsilon_{ip_2}^{\omega}}{1+\mathbb{L}_{ip}^{1_2}}\right) \end{pmatrix}, \\ \left(\left(\frac{1-\varepsilon_{rp_1}^{\omega}}{1+\mathbb{L}_{rp}^{1_1}}\right)\left(\frac{1-\varepsilon_{rp_2}^{\omega}}{1+\mathbb{L}_{rp}^{1_2}}\right), \left(\frac{1-\varepsilon_{ip_1}^{\omega}}{1+\mathbb{L}_{ip}^{1_1}}\right)\left(\frac{1-\varepsilon_{ip_2}^{\omega}}{1+\mathbb{L}_{ip}^{1_2}}\right)\right), \\ \begin{pmatrix} \mathbb{L}_{rp}^{2_1}\left(\frac{1-\varepsilon_{rp_1}^{\omega}}{1+\mathbb{L}_{rp}^{2}}\right) + \mathbb{L}_{rp}^{2_2}\left(\frac{1-\varepsilon_{rp_2}^{\omega}}{1+\mathbb{L}_{rp}^{2_2}}\right) - \mathbb{L}_{rp}^{2_1}\left(\frac{1-\varepsilon_{rp_1}^{\omega}}{1+\mathbb{L}_{rp}^{2_1}}\right)\mathbb{L}_{rp}^{2_2}\left(\frac{1-\varepsilon_{rp}}{1+\mathbb{L}_{rp}^{2_2}}\right), \\ \mathbb{L}_{ip}^{2_1}\left(\frac{1-\varepsilon_{ip_1}^{\omega}}{1+\mathbb{L}_{ip}^{2_1}}\right) + \mathbb{L}_{ip}^{2_2}\left(\frac{1-\varepsilon_{ip_2}^{\omega}}{1+\mathbb{L}_{ip}^{2_2}}\right) - \mathbb{L}_{ip}^{2_1}\left(\frac{1-\varepsilon_{ip_1}^{\omega}}{1+\mathbb{L}_{ip}^{2_1}}\right)\mathbb{L}_{ip}^{2_2}\left(\frac{1-\varepsilon_{ip_2}^{\omega}}{1+\mathbb{L}_{ip}^{2_2}}\right) \end{pmatrix}, \\ \left(\left(\frac{1-\varepsilon_{rp_1}^{\omega}}{1+\mathbb{L}_{rp}^{2_1}}\right)\left(\frac{1-\varepsilon_{rp_2}^{\omega}}{1+\mathbb{L}_{rp}^{2_2}}\right), \left(\frac{1-\varepsilon_{ip_1}^{\omega}}{1+\mathbb{L}_{ip}^{2_1}}\right)\left(\frac{1-\varepsilon_{ip_2}^{\omega}}{1+\mathbb{L}_{ip}^{2_2}}\right)\right) \end{pmatrix}$$

$$\tilde{H}_1 \otimes \tilde{H}_2 = \begin{pmatrix} \left(\mathbb{L}_{rp}^{1_1}\left(\frac{1-\varepsilon_{rp_1}^{\omega}}{1+\mathbb{L}_{rp}^{1_1}}\right)\mathbb{L}_{rp}^{1_2}\left(\frac{1-\varepsilon_{rp}}{1+\mathbb{L}_{rp}^{1_2}}\right), \mathbb{L}_{ip}^{1_1}\left(\frac{1-\varepsilon_{ip_1}^{\omega}}{1+\mathbb{L}_{ip}^{1_1}}\right)\mathbb{L}_{ip}^{1_2}\left(\frac{1-\varepsilon_{ip_2}^{\omega}}{1+\mathbb{L}_{ip}^{1_2}}\right)\right), \\ \begin{pmatrix} \left(\frac{1-\varepsilon_{rp_1}^{\omega}}{1+\mathbb{L}_{rp}^{1_1}}\right) + \left(\frac{1-\varepsilon_{rp_2}^{\omega}}{1+\mathbb{L}_{rp}^{1_2}}\right) - \left(\frac{1-\varepsilon_{rp_1}^{\omega}}{1+\mathbb{L}_{rp}^{1_1}}\right)\left(\frac{1-\varepsilon_{rp_2}^{\omega}}{1+\mathbb{L}_{rp}^{1_2}}\right), \\ \left(\frac{1-\varepsilon_{ip_1}^{\omega}}{1+\mathbb{L}_{ip}^{1_1}}\right) + \left(\frac{1-\varepsilon_{ip_2}^{\omega}}{1+\mathbb{L}_{ip}^{1_2}}\right) - \left(\frac{1-\varepsilon_{ip_1}^{\omega}}{1+\mathbb{L}_{ip}^{1_1}}\right)\left(\frac{1-\varepsilon_{ip_2}^{\omega}}{1+\mathbb{L}_{ip}^{1_2}}\right) \end{pmatrix}, \\ \left(\mathbb{L}_{rp}^{2_1}\left(\frac{1-\varepsilon_{rp_1}^{\omega}}{1+\mathbb{L}_{rp}^{2_1}}\right)\mathbb{L}_{rp}^{2_2}\left(\frac{1-\varepsilon_{rp}}{1+\mathbb{L}_{rp}^{2_2}}\right), \mathbb{L}_{ip}^{2_1}\left(\frac{1-\varepsilon_{ip_1}^{\omega}}{1+\mathbb{L}_{ip}^{2_1}}\right)\mathbb{L}_{ip}^{2_2}\left(\frac{1-\varepsilon_{ip_2}^{\omega}}{1+\mathbb{L}_{ip}^{2_2}}\right)\right), \\ \begin{pmatrix} \left(\frac{1-\varepsilon_{rp_1}^{\omega}}{1+\mathbb{L}_{rp}^{2_1}}\right) + \left(\frac{1-\varepsilon_{rp_2}^{\omega}}{1+\mathbb{L}_{rp}^{2_2}}\right) - \left(\frac{1-\varepsilon_{rp_1}^{\omega}}{1+\mathbb{L}_{rp}^{2_1}}\right)\left(\frac{1-\varepsilon_{rp_2}^{\omega}}{1+\mathbb{L}_{rp}^{2_2}}\right), \\ \left(\frac{1-\varepsilon_{ip_1}^{\omega}}{1+\mathbb{L}_{ip}^{2_1}}\right) + \left(\frac{1-\varepsilon_{ip_2}^{\omega}}{1+\mathbb{L}_{ip}^{2_2}}\right) - \left(\frac{1-\varepsilon_{ip_1}^{\omega}}{1+\mathbb{L}_{ip}^{2_1}}\right)\left(\frac{1-\varepsilon_{ip_2}^{\omega}}{1+\mathbb{L}_{ip}^{2_2}}\right) \end{pmatrix} \end{pmatrix}$$

$$\tilde{\eta}_{\Theta}\tilde{H}_{\&} = \begin{pmatrix} \left(\left(1-\left(1-\mathbb{L}_{rp}^{1_{\&}}\left(\frac{1-\varepsilon_{rp_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right)\right)^{\tilde{\eta}_{\Theta}}, 1-\left(1-\mathbb{L}_{rp}^{1_{\&}}\left(\frac{1-\varepsilon_{ip_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right)\right)^{\tilde{\eta}_{\Theta}}\right), \left(\left(\left(\frac{1-\varepsilon_{rp_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right)\right)^{\tilde{\eta}_{\Theta}}, \left(\left(\frac{1-\varepsilon_{ip_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right)\right)^{\tilde{\eta}_{\Theta}}\right)\right), \\ \left(\left(1-\left(1-\mathbb{L}_{rp}^{2_{\&}}\left(\frac{1-\varepsilon_{rp_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right)\right)^{\tilde{\eta}_{\Theta}}, 1-\left(1-\mathbb{L}_{rp}^{2_{\&}}\left(\frac{1-\varepsilon_{ip_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right)\right)^{\tilde{\eta}_{\Theta}}\right), \left(\left(\left(\frac{1-\varepsilon_{rp_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right)\right)^{\tilde{\eta}_{\Theta}}, \left(\left(\frac{1-\varepsilon_{ip_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right)\right)^{\tilde{\eta}_{\Theta}}\right)\right) \end{pmatrix}$$

$$\left(\tilde{H}_{\&}\right)^{\tilde{\eta}_{\Theta}} = \begin{pmatrix} \left(\left(\left(\mathbb{L}_{rp}^{1_{\&}}\left(\frac{1-\varepsilon_{rp_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right)\right)^{\tilde{\eta}_{\Theta}}, \left(\mathbb{L}_{rp}^{1_{\&}}\left(\frac{1-\varepsilon_{ip_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right)\right)^{\tilde{\eta}_{\Theta}}\right), \left(1-\left(1-\left(\frac{1-\varepsilon_{rp_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right)\right)^{\tilde{\eta}_{\Theta}}, 1-\left(1-\left(\frac{1-\varepsilon_{ip_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right)\right)^{\tilde{\eta}_{\Theta}}\right)\right), \\ \left(\left(\left(\mathbb{L}_{rp}^{2_{\&}}\left(\frac{1-\varepsilon_{rp_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right)\right)^{\tilde{\eta}_{\Theta}}, \left(\mathbb{L}_{rp}^{2_{\&}}\left(\frac{1-\varepsilon_{ip_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right)\right)^{\tilde{\eta}_{\Theta}}\right), \left(1-\left(1-\left(\frac{1-\varepsilon_{rp_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right)\right)^{\tilde{\eta}_{\Theta}}, 1-\left(1-\left(\frac{1-\varepsilon_{ip_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right)\right)^{\tilde{\eta}_{\Theta}}\right)\right) \end{pmatrix}$$

Definition 3.3. For any $\tilde{H}_{\&} = \begin{pmatrix} \left(\left(\mathbb{L}_{rp}^{1\&}\left(\frac{1-\varepsilon_{rp\&}^{\omega}}{1+\mathbb{L}_{rp}^{1\&}}\right), \mathbb{L}_{ip}^{1\&}\left(\frac{1-\varepsilon_{ip\&}^{\omega}}{1+\mathbb{L}_{ip}^{1\&}}\right)\right), \left(\left(\frac{1-\varepsilon_{rp\&}^{\omega}}{1+\mathbb{L}_{rp}^{1\&}}\right), \left(\frac{1-\varepsilon_{ip\&}^{\omega}}{1+\mathbb{L}_{ip}^{1\&}}\right)\right)\right), \\ \left(\left(\mathbb{L}_{rp}^{2\&}\left(\frac{1-\varepsilon_{rp\&}^{\omega}}{1+\mathbb{L}_{rp}^{2\&}}\right), \mathbb{L}_{ip}^{2\&}\left(\frac{1-\varepsilon_{ip\&}^{\omega}}{1+\mathbb{L}_{ip}^{2\&}}\right)\right), \left(\left(\frac{1-\varepsilon_{rp\&}^{\omega}}{1+\mathbb{L}_{rp}^{2\&}}\right), \left(\frac{1-\varepsilon_{ip\&}^{\omega}}{1+\mathbb{L}_{ip}^{2\&}}\right)\right)\right) \end{pmatrix}, \& = 1,$ we have

$$S\left(\tilde{H}_{\&}\right) = \frac{1}{4}\begin{pmatrix} \left(\left(\mathbb{L}_{rp}^{1\&}\left(\frac{1-\varepsilon_{rp\&}^{\omega}}{1+\mathbb{L}_{rp}^{1\&}}\right) + \mathbb{L}_{ip}^{1\&}\left(\frac{1-\varepsilon_{ip\&}^{\omega}}{1+\mathbb{L}_{ip}^{1\&}}\right)\right) - \left(\left(\frac{1-\varepsilon_{rp\&}^{\omega}}{1+\mathbb{L}_{rp}^{1\&}}\right) + \left(\frac{1-\varepsilon_{ip\&}^{\omega}}{1+\mathbb{L}_{ip}^{1\&}}\right)\right)\right) + \\ \left(\left(\mathbb{L}_{rp}^{2\&}\left(\frac{1-\varepsilon_{rp\&}^{\omega}}{1+\mathbb{L}_{rp}^{2\&}}\right) + \mathbb{L}_{ip}^{2\&}\left(\frac{1-\varepsilon_{ip\&}^{\omega}}{1+\mathbb{L}_{ip}^{2\&}}\right)\right) - \left(\left(\frac{1-\varepsilon_{rp\&}^{\omega}}{1+\mathbb{L}_{rp}^{2\&}}\right) + \left(\frac{1-\varepsilon_{ip\&}^{\omega}}{1+\mathbb{L}_{ip}^{2\&}}\right)\right)\right) \end{pmatrix} \in [-1, 1]$$

$$A\left(\tilde{H}_{\&}\right) = \frac{1}{4}\begin{pmatrix} \left(\left(\mathbb{L}_{rp}^{1\&}\left(\frac{1-\varepsilon_{rp\&}^{\omega}}{1+\mathbb{L}_{rp}^{1\&}}\right) + \mathbb{L}_{ip}^{1\&}\left(\frac{1-\varepsilon_{ip\&}^{\omega}}{1+\mathbb{L}_{ip}^{1\&}}\right)\right) + \left(\left(\frac{1-\varepsilon_{rp\&}^{\omega}}{1+\mathbb{L}_{rp}^{1\&}}\right) + \left(\frac{1-\varepsilon_{ip\&}^{\omega}}{1+\mathbb{L}_{ip}^{1\&}}\right)\right)\right) + \\ \left(\left(\mathbb{L}_{rp}^{2\&}\left(\frac{1-\varepsilon_{rp\&}^{\omega}}{1+\mathbb{L}_{rp}^{2\&}}\right) + \mathbb{L}_{ip}^{2\&}\left(\frac{1-\varepsilon_{ip\&}^{\omega}}{1+\mathbb{L}_{ip}^{2\&}}\right)\right) + \left(\left(\frac{1-\varepsilon_{rp\&}^{\omega}}{1+\mathbb{L}_{rp}^{2\&}}\right) + \left(\frac{1-\varepsilon_{ip\&}^{\omega}}{1+\mathbb{L}_{ip}^{2\&}}\right)\right)\right) \end{pmatrix} \in [-1, 1]$$

Thus, if $SC\left(\tilde{H}_1\right) > SC\left(\tilde{H}_2\right) \Rightarrow \tilde{H}_1 > \tilde{H}_2$, if $SC\left(\tilde{H}_1\right) = SC\left(\tilde{H}_2\right)$, then if $AC\left(\tilde{H}_1\right) > AC\left(\tilde{H}_2\right) \Rightarrow \tilde{H}_1 > \tilde{H}_2$.

4 CPLDF Power Aggregation Insights

This section is famous for the analysis of the power operators for CPLDFSs, called the CPLDFPoA operator, CPLDFWPoA operator, CPLDFPoG operator, CPLDFWPoG operator, and their genuine properties.

Definition 4.1. Let $\tilde{H}_{\&} = \begin{pmatrix} \left(\left(\mathbb{L}_{rp}^{1\&}\left(\frac{1-\varepsilon_{rp\&}^{\omega}}{1+\mathbb{L}_{rp}^{1\&}}\right), \mathbb{L}_{ip}^{1\&}\left(\frac{1-\varepsilon_{ip\&}^{\omega}}{1+\mathbb{L}_{ip}^{1\&}}\right)\right), \left(\left(\frac{1-\varepsilon_{rp\&}^{\omega}}{1+\mathbb{L}_{rp}^{1\&}}\right), \left(\frac{1-\varepsilon_{ip\&}^{\omega}}{1+\mathbb{L}_{ip}^{1\&}}\right)\right)\right), \\ \left(\left(\mathbb{L}_{rp}^{2\&}\left(\frac{1-\varepsilon_{rp\&}^{\omega}}{1+\mathbb{L}_{rp}^{2\&}}\right), \mathbb{L}_{ip}^{2\&}\left(\frac{1-\varepsilon_{ip\&}^{\omega}}{1+\mathbb{L}_{ip}^{2\&}}\right)\right), \left(\left(\frac{1-\varepsilon_{rp\&}^{\omega}}{1+\mathbb{L}_{rp}^{2\&}}\right), \left(\frac{1-\varepsilon_{ip\&}^{\omega}}{1+\mathbb{L}_{ip}^{2\&}}\right)\right)\right) \end{pmatrix}, \& = 1, 2, \cdots, \ni,$ be a group of CPLDF information. Then

$$CPLDFPoA\left(\tilde{H}_1, \tilde{H}_2, \cdots, \tilde{H}_{\ni}\right) = \frac{\left(1+\tilde{\eta}\left(\tilde{H}_1\right)\right)}{\sum_{\&=1}^{\ni}\left(1+\tilde{\eta}\left(\tilde{H}_{\&}\right)\right)}\tilde{H}_1 \oplus \frac{\left(1+\tilde{\eta}\left(\tilde{H}_2\right)\right)}{\sum_{\&=1}^{\ni}\left(1+\tilde{\eta}\left(\tilde{H}_{\&}\right)\right)}\tilde{H}_2 \oplus \cdots$$
$$\oplus \frac{\left(1+\tilde{\eta}\left(\tilde{H}_{\ni}\right)\right)}{\sum_{\&=1}^{\ni}\left(1+\tilde{\eta}\left(\tilde{H}_{\&}\right)\right)}\tilde{H}_{\ni} = \oplus_{\&=1}^{\ni}\frac{\left(1+\tilde{\eta}\left(\tilde{H}_{\&}\right)\right)}{\sum_{\&=1}^{\ni}\left(1+\tilde{\eta}\left(\tilde{H}_{\&}\right)\right)}\tilde{H}_{\&}$$

Signified the CPLDFPoA operators with $\tilde{\eta}\left(\tilde{H}_{\&}\right) = \sum_{i\neq\&=1}^{\ni} S\left(\tilde{H}_i, \tilde{H}_{\&}\right)$, and $S\left(\tilde{H}_i, \tilde{H}_{\&}\right) = 1 - D\left(\tilde{H}_i, \tilde{H}_{\&}\right)$, thus

1. $S\left(\tilde{H}_i, \tilde{H}_{\&}\right) \in [0, 1]$.

2. $S\left(\tilde{H}_i, \tilde{H}_{\&}\right) = S\left(\tilde{H}_{\&}, \tilde{H}_i\right)$.

3. When $S\left(\tilde{H}_i, \tilde{H}_{\&}\right) \geq S\left(\tilde{H}_k, \tilde{H}_l\right)$, then $D\left(\tilde{H}_i, \tilde{H}_{\&}\right) \leq D\left(\tilde{H}_k, \tilde{H}_l\right)$.

Theorem 4.2. *Let* $\tilde{H}_{\&} = \left(\begin{matrix}\left(\left(\left(\mathbb{L}_{rp}^{1\&}\left(\frac{1-\varepsilon_{rp\&}^{\omega}}{1+\mathbb{L}_{rp}^{1\&}}\right), \mathbb{L}_{ip}^{1\&}\left(\frac{1-\varepsilon_{ip\&}^{\omega}}{1+\mathbb{L}_{ip}^{1\&}}\right)\right), \left(\left(\frac{1-\varepsilon_{rp\&}^{\omega}}{1+\mathbb{L}_{rp}^{1\&}}\right), \left(\frac{1-\varepsilon_{ip\&}^{\omega}}{1+\mathbb{L}_{ip}^{1\&}}\right)\right)\right),\\ \left(\left(\mathbb{L}_{rp}^{2\&}\left(\frac{1-\varepsilon_{rp\&}^{\omega}}{1+\mathbb{L}_{rp}^{2\&}}\right), \mathbb{L}_{ip}^{2\&}\left(\frac{1-\varepsilon_{ip\&}^{\omega}}{1+\mathbb{L}_{ip}^{2\&}}\right)\right), \left(\left(\frac{1-\varepsilon_{rp\&}^{\omega}}{1+\mathbb{L}_{rp}^{2\&}}\right), \left(\frac{1-\varepsilon_{ip\&}^{\omega}}{1+\mathbb{L}_{ip}^{2\&}}\right)\right)\right)\end{matrix}\right), \& = 1, 2, \cdots, \ni.,$ *be a group of CPLDF information. Then, using the information in Def. (6), we evaluate the information in Def. (8), such as*

$$CPLDFPoA\left(\tilde{H}_1, \tilde{H}_2, \cdots, \tilde{H}_{\ni}\right)$$

$$= \left(\begin{matrix}\left(\begin{matrix}\left(1-\prod_{\&=1}^{\ni}\left(1-\mathbb{L}_{rp}^{1\&}\left(\frac{1-\varepsilon_{rp\&}^{\omega}}{1+\mathbb{L}_{rp}^{1\&}}\right)\right)^{\frac{(1+\tilde{\eta}(\tilde{H}_{\&}))}{\sum_{\&=1}^{\ni}(1+\tilde{\eta}(\tilde{H}_{\&}))}}, 1-\prod_{\&=1}^{\ni}\left(1-\mathbb{L}_{rp}^{1\&}\left(\frac{1-\varepsilon_{ip\&}^{\omega}}{1+\mathbb{L}_{rp}^{1\&}}\right)\right)^{\frac{(1+\tilde{\eta}(\tilde{H}_{\&}))}{\sum_{\&=1}^{\ni}(1+\tilde{\eta}(\tilde{H}_{\&}))}}\right),\\ \left(\prod_{\&=1}^{\ni}\left(\left(\frac{1-\varepsilon_{rp\&}^{\omega}}{1+\mathbb{L}_{rp}^{1\&}}\right)\right)^{\frac{(1+\tilde{\eta}(\tilde{H}_{\&}))}{\sum_{\&=1}^{\ni}(1+\tilde{\eta}(\tilde{H}_{\&}))}}, \prod_{\&=1}^{\ni}\left(\left(\frac{1-\varepsilon_{ip\&}^{\omega}}{1+\mathbb{L}_{rp}^{1\&}}\right)\right)^{\frac{(1+\tilde{\eta}(\tilde{H}_{\&}))}{\sum_{\&=1}^{\ni}(1+\tilde{\eta}(\tilde{H}_{\&}))}}\right)\end{matrix}\right),\\ \left(\begin{matrix}\left(1-\prod_{\&=1}^{\ni}\left(1-\mathbb{L}_{rp}^{2\&}\left(\frac{1-\varepsilon_{rp\&}^{\omega}}{1+\mathbb{L}_{rp}^{2\&}}\right)\right)^{\frac{(1+\tilde{\eta}(\tilde{H}_{\&}))}{\sum_{\&=1}^{\ni}(1+\tilde{\eta}(\tilde{H}_{\&}))}}, 1-\prod_{\&=1}^{\ni}\left(1-\mathbb{L}_{rp}^{2\&}\left(\frac{1-\varepsilon_{ip\&}^{\omega}}{1+\mathbb{L}_{rp}^{2\&}}\right)\right)^{\frac{(1+\tilde{\eta}(\tilde{H}_{\&}))}{\sum_{\&=1}^{\ni}(1+\tilde{\eta}(\tilde{H}_{\&}))}}\right),\\ \left(\prod_{\&=1}^{\ni}\left(\left(\frac{1-\varepsilon_{rp\&}^{\omega}}{1+\mathbb{L}_{rp}^{2\&}}\right)\right)^{\frac{(1+\tilde{\eta}(\tilde{H}_{\&}))}{\sum_{\&=1}^{\ni}(1+\tilde{\eta}(\tilde{H}_{\&}))}}, \prod_{\&=1}^{\ni}\left(\left(\frac{1-\varepsilon_{ip\&}^{\omega}}{1+\mathbb{L}_{rp}^{2\&}}\right)\right)^{\frac{(1+\tilde{\eta}(\tilde{H}_{\&}))}{\sum_{\&=1}^{\ni}(1+\tilde{\eta}(\tilde{H}_{\&}))}}\right)\end{matrix}\right)\end{matrix}\right)$$

Property 4.3. Let $\tilde{H}_{\&} = \left(\begin{matrix}\left(\left(\left(\mathbb{L}_{rp}^{1\&}\left(\frac{1-\varepsilon_{rp\&}^{\omega}}{1+\mathbb{L}_{rp}^{1\&}}\right), \mathbb{L}_{ip}^{1\&}\left(\frac{1-\varepsilon_{ip\&}^{\omega}}{1+\mathbb{L}_{ip}^{1\&}}\right)\right), \left(\left(\frac{1-\varepsilon_{rp\&}^{\omega}}{1+\mathbb{L}_{rp}^{1\&}}\right), \left(\frac{1-\varepsilon_{ip\&}^{\omega}}{1+\mathbb{L}_{ip}^{1\&}}\right)\right)\right),\\ \left(\left(\mathbb{L}_{rp}^{2\&}\left(\frac{1-\varepsilon_{rp\&}^{\omega}}{1+\mathbb{L}_{rp}^{2\&}}\right), \mathbb{L}_{ip}^{2\&}\left(\frac{1-\varepsilon_{ip\&}^{\omega}}{1+\mathbb{L}_{ip}^{2\&}}\right)\right), \left(\left(\frac{1-\varepsilon_{rp\&}^{\omega}}{1+\mathbb{L}_{rp}^{2\&}}\right), \left(\frac{1-\varepsilon_{ip\&}^{\omega}}{1+\mathbb{L}_{ip}^{2\&}}\right)\right)\right)\end{matrix}\right), \& = 1, 2, \cdots, \ni.,$ be a group of CPLDF information.

1. If $\tilde{H}_{\&} = \tilde{H}, \& = 1, 2, \cdots, \ni$, thus $CPLDFPoA\left(\tilde{H}_1, \tilde{H}_2, \cdots, \tilde{H}_{\ni}\right) = \tilde{H}$, called the idempotency.
2. If $\tilde{H}_{\&} \leq \tilde{H}'_{\&}, \& = 1, 2, \cdots, \ni$, thus $CPLDFPoA\left(\tilde{H}_1, \tilde{H}_2, \cdots, \tilde{H}_{\ni}\right) \leq CPLDFPoA\left(\tilde{H}'_1, \tilde{H}'_2, \cdots, \tilde{H}'_{\ni}\right)$, called the monotonicity.
3. If $\tilde{H}_{-} = min\left(\tilde{H}_1, \tilde{H}_2, \cdots, \tilde{H}_{\ni}\right)$, and $\tilde{H}_{+} = max\left(\tilde{H}_1, \tilde{H}_2, \cdots, \tilde{H}_{\ni}\right), \& = 1, 2, \cdots, \ni$, thus $\tilde{H}_{-} \leq CPLDFPoA\left(\tilde{H}_1, \tilde{H}_2, \cdots, \tilde{H}_{\ni}\right) \leq \tilde{H}_{+}$, called the boundedness.

Definition 4.4. Let $\tilde{H}_{\&} = \left(\begin{matrix}\left(\left(\left(\mathbb{L}_{rp}^{1\&}\left(\frac{1-\varepsilon_{rp\&}^{\omega}}{1+\mathbb{L}_{rp}^{1\&}}\right), \mathbb{L}_{ip}^{1\&}\left(\frac{1-\varepsilon_{ip\&}^{\omega}}{1+\mathbb{L}_{ip}^{1\&}}\right)\right), \left(\left(\frac{1-\varepsilon_{rp\&}^{\omega}}{1+\mathbb{L}_{rp}^{1\&}}\right), \left(\frac{1-\varepsilon_{ip\&}^{\omega}}{1+\mathbb{L}_{ip}^{1\&}}\right)\right)\right),\\ \left(\left(\mathbb{L}_{rp}^{2\&}\left(\frac{1-\varepsilon_{rp\&}^{\omega}}{1+\mathbb{L}_{rp}^{2\&}}\right), \mathbb{L}_{ip}^{2\&}\left(\frac{1-\varepsilon_{ip\&}^{\omega}}{1+\mathbb{L}_{ip}^{2\&}}\right)\right), \left(\left(\frac{1-\varepsilon_{rp\&}^{\omega}}{1+\mathbb{L}_{rp}^{2\&}}\right), \left(\frac{1-\varepsilon_{ip\&}^{\omega}}{1+\mathbb{L}_{ip}^{2\&}}\right)\right)\right)\end{matrix}\right), \& = 1, 2, \cdots, \ni.,$ be a group of CPLDF information. Then

$$CPLDFWPoA\left(\tilde{H}_1, \tilde{H}_2, \cdots, \tilde{H}_{\ni}\right) = \frac{\aleph_1\left(1+\tilde{\eta}\left(\tilde{H}_1\right)\right)}{\sum_{\&=1}^{\ni}\aleph_{\&}\left(1+\tilde{\eta}\left(\tilde{H}_{\&}\right)\right)}\tilde{H}_1 \oplus \frac{\aleph_2\left(1+\tilde{\eta}\left(\tilde{H}_2\right)\right)}{\sum_{\&=1}^{\ni}\aleph_{\&}\left(1+\tilde{\eta}\left(\tilde{H}_{\&}\right)\right)}\tilde{H}_2 \oplus \cdots$$

$$\oplus \frac{\aleph_{\ni}\left(1+\tilde{\eta}\left(\tilde{H}_{\ni}\right)\right)}{\sum_{\&=1}^{\ni}\aleph_{\&}\left(1+\tilde{\eta}\left(\tilde{H}_{\&}\right)\right)}\tilde{H}_{\ni} = \oplus_{\&=1}^{\ni}\frac{\aleph_{\&}\left(1+\tilde{\eta}\left(\tilde{H}_{\&}\right)\right)}{\sum_{\&=1}^{\ni}\aleph_{\&}\left(1+\tilde{\eta}\left(\tilde{H}_{\&}\right)\right)}\tilde{H}_{\&}$$

Signified the CPLDFWPoA operators with $\tilde{\eta}\left(\tilde{H}_{\&}\right)=\sum_{i\neq\&=1}^{\ni} S\left(\tilde{H}_i,\tilde{H}_{\&}\right)$, and $S\left(\tilde{H}_i,\tilde{H}_{\&}\right)=1-D\left(\tilde{H}_i,\tilde{H}_{\&}\right)$, thus

4. $S\left(\tilde{H}_i,\tilde{H}_{\&}\right)\in[0,1]$.

5. $S\left(\tilde{H}_i,\tilde{H}_{\&}\right)=S\left(\tilde{H}_{\&},\tilde{H}_i\right)$.

6. When $S\left(\tilde{H}_i,\tilde{H}_{\&}\right)\geq S\left(\tilde{H}_k,\tilde{H}_l\right)$, then $D\left(\tilde{H}_i,\tilde{H}_{\&}\right)\leq D\left(\tilde{H}_k,\tilde{H}_l\right)$.

Where $\sum_{\&=1}^{\ni}\aleph_{\&}=1$, called weight vector.

Theorem 4.5. *Let* $\tilde{H}_{\&}=\begin{pmatrix}\left(\left(\mathbb{L}_{rp}^{1\&}\left(\frac{1-\varepsilon_{rp}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{1\&}}\right),\mathbb{L}_{ip}^{1\&}\left(\frac{1-\varepsilon_{ip}{}_{\&}^{\omega}}{1+\mathbb{L}_{ip}^{1\&}}\right)\right),\left(\left(\frac{1-\varepsilon_{rp}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{1\&}}\right),\left(\frac{1-\varepsilon_{ip}{}_{\&}^{\omega}}{1+\mathbb{L}_{ip}^{1\&}}\right)\right)\right),\\\left(\left(\mathbb{L}_{rp}^{2\&}\left(\frac{1-\varepsilon_{rp}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{2\&}}\right),\mathbb{L}_{ip}^{2\&}\left(\frac{1-\varepsilon_{ip}{}_{\&}^{\omega}}{1+\mathbb{L}_{ip}^{2\&}}\right)\right),\left(\left(\frac{1-\varepsilon_{rp}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{2\&}}\right),\left(\frac{1-\varepsilon_{ip}{}_{\&}^{\omega}}{1+\mathbb{L}_{ip}^{2\&}}\right)\right)\right)\end{pmatrix},\&=1,2,\cdots,\ni.$
be a group of CPLDF information. Then, using the information in Def. (6), we evaluate the information in Def. (9), such as

$$CPLDFWPoA\left(\tilde{H}_1,\tilde{H}_2,\cdots,\tilde{H}_{\ni}\right)$$

$$=\begin{pmatrix}\begin{pmatrix}\left(1-\prod_{\&=1}^{\ni}\left(1-\mathbb{L}_{rp}^{1\&}\left(\frac{1-\varepsilon_{rp}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{1\&}}\right)\right)^{\frac{\aleph_{\&}(1+\tilde{\eta}(\tilde{H}_{\&}))}{\sum_{\&=1}^{\ni}\aleph_{\&}(1+\tilde{\eta}(\tilde{H}_{\&}))}},1-\prod_{\&=1}^{\ni}\left(1-\mathbb{L}_{rp}^{1\&}\left(\frac{1-\varepsilon_{ip}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{1\&}}\right)\right)^{\frac{\aleph_{\&}(1+\tilde{\eta}(\tilde{H}_{\&}))}{\sum_{\&=1}^{\ni}\aleph_{\&}(1+\tilde{\eta}(\tilde{H}_{\&}))}}\right),\\\left(\prod_{\&=1}^{\ni}\left(\left(\frac{1-\varepsilon_{rp}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{1\&}}\right)\right)^{\frac{\aleph_{\&}(1+\tilde{\eta}(\tilde{H}_{\&}))}{\sum_{\&=1}^{\ni}\aleph_{\&}(1+\tilde{\eta}(\tilde{H}_{\&}))}},\prod_{\&=1}^{\ni}\left(\left(\frac{1-\varepsilon_{ip}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{1\&}}\right)\right)^{\frac{\aleph_{\&}(1+\tilde{\eta}(\tilde{H}_{\&}))}{\sum_{\&=1}^{\ni}\aleph_{\&}(1+\tilde{\eta}(\tilde{H}_{\&}))}}\right)\end{pmatrix},\\\begin{pmatrix}\left(1-\prod_{\&=1}^{\ni}\left(1-\mathbb{L}_{rp}^{2\&}\left(\frac{1-\varepsilon_{rp}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{2\&}}\right)\right)^{\frac{\aleph_{\&}(1+\tilde{\eta}(\tilde{H}_{\&}))}{\sum_{\&=1}^{\ni}\aleph_{\&}(1+\tilde{\eta}(\tilde{H}_{\&}))}},1-\prod_{\&=1}^{\ni}\left(1-\mathbb{L}_{rp}^{2\&}\left(\frac{1-\varepsilon_{ip}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{2\&}}\right)\right)^{\frac{\aleph_{\&}(1+\tilde{\eta}(\tilde{H}_{\&}))}{\sum_{\&=1}^{\ni}\aleph_{\&}(1+\tilde{\eta}(\tilde{H}_{\&}))}}\right),\\\left(\prod_{\&=1}^{\ni}\left(\left(\frac{1-\varepsilon_{rp}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{2\&}}\right)\right)^{\frac{\aleph_{\&}(1+\tilde{\eta}(\tilde{H}_{\&}))}{\sum_{\&=1}^{\ni}\aleph_{\&}(1+\tilde{\eta}(\tilde{H}_{\&}))}},\prod_{\&=1}^{\ni}\left(\left(\frac{1-\varepsilon_{ip}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{2\&}}\right)\right)^{\frac{\aleph_{\&}(1+\tilde{\eta}(\tilde{H}_{\&}))}{\sum_{\&=1}^{\ni}\aleph_{\&}(1+\tilde{\eta}(\tilde{H}_{\&}))}}\right)\end{pmatrix}\end{pmatrix}$$

Property 4.6. Let $\tilde{H}_{\&}=\begin{pmatrix}\left(\left(\mathbb{L}_{rp}^{1\&}\left(\frac{1-\varepsilon_{rp}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{1\&}}\right),\mathbb{L}_{ip}^{1\&}\left(\frac{1-\varepsilon_{ip}{}_{\&}^{\omega}}{1+\mathbb{L}_{ip}^{1\&}}\right)\right),\left(\left(\frac{1-\varepsilon_{rp}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{1\&}}\right),\left(\frac{1-\varepsilon_{ip}{}_{\&}^{\omega}}{1+\mathbb{L}_{ip}^{1\&}}\right)\right)\right),\\\left(\left(\mathbb{L}_{rp}^{2\&}\left(\frac{1-\varepsilon_{rp}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{2\&}}\right),\mathbb{L}_{ip}^{2\&}\left(\frac{1-\varepsilon_{ip}{}_{\&}^{\omega}}{1+\mathbb{L}_{ip}^{2\&}}\right)\right),\left(\left(\frac{1-\varepsilon_{rp}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{2\&}}\right),\left(\frac{1-\varepsilon_{ip}{}_{\&}^{\omega}}{1+\mathbb{L}_{ip}^{2\&}}\right)\right)\right)\end{pmatrix},\&=1,2,\cdots,\ni.$,
be a group of CPLDF information.

1. If $\tilde{H}_{\&}=\tilde{H},\&=1,2,\cdots,\ni$, thus $CPLDFWPoA\left(\tilde{H}_1,\tilde{H}_2,\cdots,\tilde{H}_{\ni}\right)=\tilde{H}$, called the idempotency.

2. If $\tilde{H}_{\&}\leq\tilde{H}'_{\&},\&=1,2,\cdots,\ni$, thus $CPLDFWPoA\left(\tilde{H}_1,\tilde{H}_2,\cdots,\tilde{H}_{\ni}\right)\leq CPLDFWPoA\left(\tilde{H}'_1,\tilde{H}'_2,\cdots,\tilde{H}'_{\ni}\right)$, called the monotonicity.

3. If $\tilde{H}_{-}=min\left(\tilde{H}_1,\tilde{H}_2,\cdots,\tilde{H}_{\ni}\right)$, and $\tilde{H}_{+}=max\left(\tilde{H}_1,\tilde{H}_2,\cdots,\tilde{H}_{\ni}\right),\&=1,2,\cdots,\ni$, thus $\tilde{H}_{-}\leq CPLDFWPoA\left(\tilde{H}_1,\tilde{H}_2,\cdots,\tilde{H}_{\ni}\right)\leq\tilde{H}_{+}$, called the boundedness.

Definition 4.7. Let $\tilde{H}_{\&} = \begin{pmatrix} \left(\left(\mathbb{L}_{rp}^{1_{\&}}\left(\frac{1-\varepsilon_{rp}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right), \mathbb{L}_{ip}^{1_{\&}}\left(\frac{1-\varepsilon_{ip}{}_{\&}^{\omega}}{1+\mathbb{L}_{ip}^{1_{\&}}}\right)\right), \left(\left(\frac{1-\varepsilon_{rp}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right), \left(\frac{1-\varepsilon_{ip}{}_{\&}^{\omega}}{1+\mathbb{L}_{ip}^{1_{\&}}}\right)\right)\right), \\ \left(\left(\mathbb{L}_{rp}^{2_{\&}}\left(\frac{1-\varepsilon_{rp}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right), \mathbb{L}_{ip}^{2_{\&}}\left(\frac{1-\varepsilon_{ip}{}_{\&}^{\omega}}{1+\mathbb{L}_{ip}^{2_{\&}}}\right)\right), \left(\left(\frac{1-\varepsilon_{rp}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right), \left(\frac{1-\varepsilon_{ip}{}_{\&}^{\omega}}{1+\mathbb{L}_{ip}^{2_{\&}}}\right)\right)\right) \end{pmatrix}, \& = 1, 2, \cdots, \ni.,$ be a group of CPLDF information. Then

$$CPLDFPoG\left(\tilde{H}_1, \tilde{H}_2, \cdots, \tilde{H}_{\ni}\right) = \tilde{H}_1^{\frac{(1+\tilde{\eta}(\tilde{H}_1))}{\sum_{\&=1}^{\ni}(1+\tilde{\eta}(\tilde{H}_{\&}))}} \otimes \tilde{H}_2^{\frac{(1+\tilde{\eta}(\tilde{H}_2))}{\sum_{\&=1}^{\ni}(1+\tilde{\eta}(\tilde{H}_{\&}))}} \otimes \cdots \otimes \tilde{H}_{\ni}^{\frac{(1+\tilde{\eta}(\tilde{H}_{\ni}))}{\sum_{\&=1}^{\ni}(1+\tilde{\eta}(\tilde{H}_{\&}))}} = \otimes_{\&=1}^{\ni} \tilde{H}_{\&}^{\frac{(1+\tilde{\eta}(\tilde{H}_{\&}))}{\sum_{\&=1}^{\ni}(1+\tilde{\eta}(\tilde{H}_{\&}))}}$$

Signified the CPLDFPoG operators with $\tilde{\eta}\left(\tilde{H}_{\&}\right) = \sum_{i \neq \&=1}^{\ni} S\left(\tilde{H}_i, \tilde{H}_{\&}\right)$, and $S\left(\tilde{H}_i, \tilde{H}_{\&}\right) = 1 - D\left(\tilde{H}_i, \tilde{H}_{\&}\right)$, thus

1. $S\left(\tilde{H}_i, \tilde{H}_{\&}\right) \in [0, 1]$.
2. $S\left(\tilde{H}_i, \tilde{H}_{\&}\right) = S\left(\tilde{H}_{\&}, \tilde{H}_i\right)$.
3. When $S\left(\tilde{H}_i, \tilde{H}_{\&}\right) \geq S\left(\tilde{H}_k, \tilde{H}_l\right)$, then $D\left(\tilde{H}_i, \tilde{H}_{\&}\right) \leq D\left(\tilde{H}_k, \tilde{H}_l\right)$.

Theorem 4.8. *Let* $\tilde{H}_{\&} = \begin{pmatrix} \left(\left(\mathbb{L}_{rp}^{1_{\&}}\left(\frac{1-\varepsilon_{rp}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right), \mathbb{L}_{ip}^{1_{\&}}\left(\frac{1-\varepsilon_{ip}{}_{\&}^{\omega}}{1+\mathbb{L}_{ip}^{1_{\&}}}\right)\right), \left(\left(\frac{1-\varepsilon_{rp}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right), \left(\frac{1-\varepsilon_{ip}{}_{\&}^{\omega}}{1+\mathbb{L}_{ip}^{1_{\&}}}\right)\right)\right), \\ \left(\left(\mathbb{L}_{rp}^{2_{\&}}\left(\frac{1-\varepsilon_{rp}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right), \mathbb{L}_{ip}^{2_{\&}}\left(\frac{1-\varepsilon_{ip}{}_{\&}^{\omega}}{1+\mathbb{L}_{ip}^{2_{\&}}}\right)\right), \left(\left(\frac{1-\varepsilon_{rp}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right), \left(\frac{1-\varepsilon_{ip}{}_{\&}^{\omega}}{1+\mathbb{L}_{ip}^{2_{\&}}}\right)\right)\right) \end{pmatrix}, \& = 1, 2, \cdots, \ni.,$ *be a group of CPLDF information. Then, using the information in Def. (6), we evaluate the information in Def. (10), such as*

$$CPLDFPoG\left(\tilde{H}_1, \tilde{H}_2, \cdots, \tilde{H}_{\ni}\right)$$
$$= \begin{pmatrix} \begin{pmatrix} \left(\prod_{\&=1}^{\ni}\left(\mathbb{L}_{rp}^{1_{\&}}\left(\frac{1-\varepsilon_{rp}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right)\right)^{\frac{(1+\tilde{\eta}(\tilde{H}_{\&}))}{\sum_{\&=1}^{\ni}(1+\tilde{\eta}(\tilde{H}_{\&}))}}, \prod_{\&=1}^{\ni}\left(\mathbb{L}_{rp}^{1_{\&}}\left(\frac{1-\varepsilon_{ip}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right)\right)^{\frac{(1+\tilde{\eta}(\tilde{H}_{\&}))}{\sum_{\&=1}^{\ni}(1+\tilde{\eta}(\tilde{H}_{\&}))}}\right), \\ \left(1-\prod_{\&=1}^{\ni}\left(1-\left(\frac{1-\varepsilon_{rp}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right)\right)^{\frac{(1+\tilde{\eta}(\tilde{H}_{\&}))}{\sum_{\&=1}^{\ni}(1+\tilde{\eta}(\tilde{H}_{\&}))}}, 1-\prod_{\&=1}^{\ni}\left(1-\left(\frac{1-\varepsilon_{ip}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right)\right)^{\frac{(1+\tilde{\eta}(\tilde{H}_{\&}))}{\sum_{\&=1}^{\ni}(1+\tilde{\eta}(\tilde{H}_{\&}))}}\right) \end{pmatrix}, \\ \begin{pmatrix} \left(\prod_{\&=1}^{\ni}\left(\mathbb{L}_{rp}^{2_{\&}}\left(\frac{1-\varepsilon_{rp}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right)\right)^{\frac{(1+\tilde{\eta}(\tilde{H}_{\&}))}{\sum_{\&=1}^{\ni}(1+\tilde{\eta}(\tilde{H}_{\&}))}}, \prod_{\&=1}^{\ni}\left(\mathbb{L}_{rp}^{2_{\&}}\left(\frac{1-\varepsilon_{ip}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right)\right)^{\frac{(1+\tilde{\eta}(\tilde{H}_{\&}))}{\sum_{\&=1}^{\ni}(1+\tilde{\eta}(\tilde{H}_{\&}))}}\right), \\ \left(1-\prod_{\&=1}^{\ni}\left(1-\left(\frac{1-\varepsilon_{rp}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right)\right)^{\frac{(1+\tilde{\eta}(\tilde{H}_{\&}))}{\sum_{\&=1}^{\ni}(1+\tilde{\eta}(\tilde{H}_{\&}))}}, 1-\prod_{\&=1}^{\ni}\left(1-\left(\frac{1-\varepsilon_{ip}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right)\right)^{\frac{(1+\tilde{\eta}(\tilde{H}_{\&}))}{\sum_{\&=1}^{\ni}(1+\tilde{\eta}(\tilde{H}_{\&}))}}\right) \end{pmatrix} \end{pmatrix}$$

Property 4.9. Let $\tilde{H}_{\&} = \begin{pmatrix} \left(\left(\mathbb{L}_{rp}^{1_{\&}}\left(\frac{1-\varepsilon_{rp}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right), \mathbb{L}_{ip}^{1_{\&}}\left(\frac{1-\varepsilon_{ip}{}_{\&}^{\omega}}{1+\mathbb{L}_{ip}^{1_{\&}}}\right)\right), \left(\left(\frac{1-\varepsilon_{rp}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right), \left(\frac{1-\varepsilon_{ip}{}_{\&}^{\omega}}{1+\mathbb{L}_{ip}^{1_{\&}}}\right)\right)\right), \\ \left(\left(\mathbb{L}_{rp}^{2_{\&}}\left(\frac{1-\varepsilon_{rp}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right), \mathbb{L}_{ip}^{2_{\&}}\left(\frac{1-\varepsilon_{ip}{}_{\&}^{\omega}}{1+\mathbb{L}_{ip}^{2_{\&}}}\right)\right), \left(\left(\frac{1-\varepsilon_{rp}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right), \left(\frac{1-\varepsilon_{ip}{}_{\&}^{\omega}}{1+\mathbb{L}_{ip}^{2_{\&}}}\right)\right)\right) \end{pmatrix}, \& = 1, 2, \cdots, \ni.,$ be a group of CPLDF information.

1. If $\tilde{H}_{\&} = \tilde{H}, \& = 1, 2, \cdots, \ni$, thus $CPLDFPoG\left(\tilde{H}_1, \tilde{H}_2, \cdots, \tilde{H}_{\ni}\right) = \tilde{H}$, called the idempotency.
2. If $\tilde{H}_{\&} \leq \tilde{H}'_{\&}, \& = 1, 2, \cdots, \ni$, thus $CPLDFPoG\left(\tilde{H}_1, \tilde{H}_2, \cdots, \tilde{H}_{\ni}\right) \leq CPLDFPoG\left(\tilde{H}'_1, \tilde{H}'_2, \cdots, \tilde{H}'_{\ni}\right)$, called the monotonicity.

3. If $\tilde{H}_{-} = min\left(\tilde{H}_1, \tilde{H}_2, \cdots, \tilde{H}_\ni\right)$, and $\tilde{H}_{+} = max\left(\tilde{H}_1, \tilde{H}_2, \cdots, \tilde{H}_\ni\right), \& = 1, 2, \cdots, \ni$, thus $\tilde{H}_{-} \leq CPLDFPoG\left(\tilde{H}_1, \tilde{H}_2, \cdots, \tilde{H}_\ni\right) \leq \tilde{H}_{+}$, called the boundedness.

Definition 4.10. Let $\tilde{H}_\& = \begin{pmatrix} \left(\left(\mathbb{L}_{rp}^{1_\&}\left(\frac{1-\varepsilon_{rp}{}_\&^\omega}{1+\mathbb{L}_{rp}^{1_\&}}\right), \mathbb{L}_{ip}^{1_\&}\left(\frac{1-\varepsilon_{ip}{}_\&^\omega}{1+\mathbb{L}_{ip}^{1_\&}}\right)\right), \left(\left(\frac{1-\varepsilon_{rp}{}_\&^\omega}{1+\mathbb{L}_{rp}^{1_\&}}\right), \left(\frac{1-\varepsilon_{ip}{}_\&^\omega}{1+\mathbb{L}_{ip}^{1_\&}}\right)\right)\right), \\ \left(\left(\mathbb{L}_{rp}^{2_\&}\left(\frac{1-\varepsilon_{rp}{}_\&^\omega}{1+\mathbb{L}_{rp}^{2_\&}}\right), \mathbb{L}_{ip}^{2_\&}\left(\frac{1-\varepsilon_{ip}{}_\&^\omega}{1+\mathbb{L}_{ip}^{2_\&}}\right)\right), \left(\left(\frac{1-\varepsilon_{rp}{}_\&^\omega}{1+\mathbb{L}_{rp}^{2_\&}}\right), \left(\frac{1-\varepsilon_{ip}{}_\&^\omega}{1+\mathbb{L}_{ip}^{2_\&}}\right)\right)\right) \end{pmatrix}, \& = 1, 2, \cdots, \ni.$, be a group of CPLDF information. Then

$$CPLDFWPoG\left(\tilde{H}_1, \tilde{H}_2, \cdots, \tilde{H}_\ni\right) = \tilde{H}_1^{\frac{\aleph_1(1+\tilde{\eta}(\tilde{H}_1))}{\sum_{\&=1}^{\ni}\aleph_\&(1+\tilde{\eta}(\tilde{H}_\&))}} \otimes \tilde{H}_2^{\frac{\aleph_2(1+\tilde{\eta}(\tilde{H}_2))}{\sum_{\&=1}^{\ni}\aleph_\&(1+\tilde{\eta}(\tilde{H}_\&))}} \otimes \cdots \otimes \tilde{H}_\ni^{\frac{\aleph_\ni(1+\tilde{\eta}(\tilde{H}_\ni))}{\sum_{\&=1}^{\ni}\aleph_\&(1+\tilde{\eta}(\tilde{H}_\&))}}$$
$$= \otimes_{\&=1}^{\ni}\tilde{H}_\&^{\frac{\aleph_\&(1+\tilde{\eta}(\tilde{H}_\&))}{\sum_{\&=1}^{\ni}\aleph_\&(1+\tilde{\eta}(\tilde{H}_\&))}}$$

Signified the CPLDFWPoG operators with $\tilde{\eta}\left(\tilde{H}_\&\right) = \sum_{i\neq\&=1}^{\ni} S\left(\tilde{H}_i, \tilde{H}_\&\right)$, and $S\left(\tilde{H}_i, \tilde{H}_\&\right) = 1 - D\left(\tilde{H}_i, \tilde{H}_\&\right)$, thus

4. $S\left(\tilde{H}_i, \tilde{H}_\&\right) \in [0, 1]$.
5. $S\left(\tilde{H}_i, \tilde{H}_\&\right) = S\left(\tilde{H}_\&, \tilde{H}_i\right)$.
6. When $S\left(\tilde{H}_i, \tilde{H}_\&\right) \geq S\left(\tilde{H}_k, \tilde{H}_l\right)$, then $D\left(\tilde{H}_i, \tilde{H}_\&\right) \leq D\left(\tilde{H}_k, \tilde{H}_l\right)$.

Where $\sum_{\&=1}^{\ni}\aleph_\& = 1$, called weight vector.

Theorem 4.11. *Let* $\tilde{H}_\& = \begin{pmatrix} \left(\left(\mathbb{L}_{rp}^{1_\&}\left(\frac{1-\varepsilon_{rp}{}_\&^\omega}{1+\mathbb{L}_{rp}^{1_\&}}\right), \mathbb{L}_{ip}^{1_\&}\left(\frac{1-\varepsilon_{ip}{}_\&^\omega}{1+\mathbb{L}_{ip}^{1_\&}}\right)\right), \left(\left(\frac{1-\varepsilon_{rp}{}_\&^\omega}{1+\mathbb{L}_{rp}^{1_\&}}\right), \left(\frac{1-\varepsilon_{ip}{}_\&^\omega}{1+\mathbb{L}_{ip}^{1_\&}}\right)\right)\right), \\ \left(\left(\mathbb{L}_{rp}^{2_\&}\left(\frac{1-\varepsilon_{rp}{}_\&^\omega}{1+\mathbb{L}_{rp}^{2_\&}}\right), \mathbb{L}_{ip}^{2_\&}\left(\frac{1-\varepsilon_{ip}{}_\&^\omega}{1+\mathbb{L}_{ip}^{2_\&}}\right)\right), \left(\left(\frac{1-\varepsilon_{rp}{}_\&^\omega}{1+\mathbb{L}_{rp}^{2_\&}}\right), \left(\frac{1-\varepsilon_{ip}{}_\&^\omega}{1+\mathbb{L}_{ip}^{2_\&}}\right)\right)\right) \end{pmatrix}, \& = 1, 2, \cdots, \ni.$, *be a group of CPLDF information. Then, using the information in Def. (6), we evaluate the information in Def. (11), such as*

$$CPLDFWPoG\left(\tilde{H}_1, \tilde{H}_2, \cdots, \tilde{H}_\ni\right)$$
$$= \begin{pmatrix} \begin{pmatrix} \left(\prod_{\&=1}^{\ni}\left(\mathbb{L}_{rp}^{1_\&}\left(\frac{1-\varepsilon_{rp}{}_\&^\omega}{1+\mathbb{L}_{rp}^{1_\&}}\right)\right)^{\frac{\aleph_\&(1+\tilde{\eta}(\tilde{H}_\&))}{\sum_{\&=1}^{\ni}\aleph_\&(1+\tilde{\eta}(\tilde{H}_\&))}}, \prod_{\&=1}^{\ni}\left(\mathbb{L}_{rp}^{1_\&}\left(\frac{1-\varepsilon_{ip}{}_\&^\omega}{1+\mathbb{L}_{rp}^{1_\&}}\right)\right)^{\frac{\aleph_\&(1+\tilde{\eta}(\tilde{H}_\&))}{\sum_{\&=1}^{\ni}\aleph_\&(1+\tilde{\eta}(\tilde{H}_\&))}}\right), \\ \left(1-\prod_{\&=1}^{\ni}\left(1-\left(\frac{1-\varepsilon_{rp}{}_\&^\omega}{1+\mathbb{L}_{rp}^{1_\&}}\right)\right)^{\frac{\aleph_\&(1+\tilde{\eta}(\tilde{H}_\&))}{\sum_{\&=1}^{\ni}\aleph_\&(1+\tilde{\eta}(\tilde{H}_\&))}}, 1-\prod_{\&=1}^{\ni}\left(1-\left(\frac{1-\varepsilon_{ip}{}_\&^\omega}{1+\mathbb{L}_{rp}^{1_\&}}\right)\right)^{\frac{\aleph_\&(1+\tilde{\eta}(\tilde{H}_\&))}{\sum_{\&=1}^{\ni}\aleph_\&(1+\tilde{\eta}(\tilde{H}_\&))}}\right) \end{pmatrix}, \\ \begin{pmatrix} \left(\prod_{\&=1}^{\ni}\left(\mathbb{L}_{rp}^{2_\&}\left(\frac{1-\varepsilon_{rp}{}_\&^\omega}{1+\mathbb{L}_{rp}^{2_\&}}\right)\right)^{\frac{\aleph_\&(1+\tilde{\eta}(\tilde{H}_\&))}{\sum_{\&=1}^{\ni}\aleph_\&(1+\tilde{\eta}(\tilde{H}_\&))}}, \prod_{\&=1}^{\ni}\left(\mathbb{L}_{rp}^{2_\&}\left(\frac{1-\varepsilon_{ip}{}_\&^\omega}{1+\mathbb{L}_{rp}^{2_\&}}\right)\right)^{\frac{\aleph_\&(1+\tilde{\eta}(\tilde{H}_\&))}{\sum_{\&=1}^{\ni}\aleph_\&(1+\tilde{\eta}(\tilde{H}_\&))}}\right), \\ \left(1-\prod_{\&=1}^{\ni}\left(1-\left(\frac{1-\varepsilon_{rp}{}_\&^\omega}{1+\mathbb{L}_{rp}^{2_\&}}\right)\right)^{\frac{\aleph_\&(1+\tilde{\eta}(\tilde{H}_\&))}{\sum_{\&=1}^{\ni}\aleph_\&(1+\tilde{\eta}(\tilde{H}_\&))}}, 1-\prod_{\&=1}^{\ni}\left(1-\left(\frac{1-\varepsilon_{ip}{}_\&^\omega}{1+\mathbb{L}_{rp}^{2_\&}}\right)\right)^{\frac{(1+\tilde{\eta}(\tilde{H}_\&))}{\sum_{\&=1}^{\ni}(1+\tilde{\eta}(\tilde{H}_\&))}}\right) \end{pmatrix} \end{pmatrix}$$

Property 4.12. Let $\tilde{H}_{\&}=\left(\begin{array}{c}\left(\left(\mathbb{L}_{rp}^{1_{\&}}\left(\frac{1-\varepsilon_{rp}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right),\mathbb{L}_{ip}^{1_{\&}}\left(\frac{1-\varepsilon_{ip}{}_{\&}^{\omega}}{1+\mathbb{L}_{ip}^{1_{\&}}}\right)\right),\left(\left(\frac{1-\varepsilon_{rp}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right),\left(\frac{1-\varepsilon_{ip}{}_{\&}^{\omega}}{1+\mathbb{L}_{ip}^{1_{\&}}}\right)\right)\right),\\ \left(\left(\mathbb{L}_{rp}^{2_{\&}}\left(\frac{1-\varepsilon_{rp}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right),\mathbb{L}_{ip}^{2_{\&}}\left(\frac{1-\varepsilon_{ip}{}_{\&}^{\omega}}{1+\mathbb{L}_{ip}^{2_{\&}}}\right)\right),\left(\left(\frac{1-\varepsilon_{rp}{}_{\&}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right),\left(\frac{1-\varepsilon_{ip}{}_{\&}^{\omega}}{1+\mathbb{L}_{ip}^{2_{\&}}}\right)\right)\right)\end{array}\right), \&=1,2,\cdots,\ni.$, be a group of CPLDF information.

1. If $\tilde{H}_{\&}=\tilde{H}, \&=1,2,\cdots,\ni$, thus $CPLDFWPoG\left(\tilde{H}_1,\tilde{H}_2,\cdots,\tilde{H}_{\ni}\right)=\tilde{H}$, called the idempotency.

2. If $\tilde{H}_{\&}\leq\tilde{H}'_{\&}, \&=1,2,\cdots,\ni$, thus $CPLDFWPoG\left(\tilde{H}_1,\tilde{H}_2,\cdots,\tilde{H}_{\ni}\right)\leq CPLDFWPoG\left(\tilde{H}'_1,\tilde{H}'_2,\cdots,\tilde{H}'_{\ni}\right)$, called the monotonicity.

3. If $\tilde{H}_{-}=min\left(\tilde{H}_1,\tilde{H}_2,\cdots,\tilde{H}_{\ni}\right)$, and $\tilde{H}_{+}=max\left(\tilde{H}_1,\tilde{H}_2,\cdots,\tilde{H}_{\ni}\right), \&=1,2,\cdots,\ni$, thus $\tilde{H}_{-}\leq CPLDFWPoG\left(\tilde{H}_1,\tilde{H}_2,\cdots,\tilde{H}_{\ni}\right)\leq\tilde{H}_{+}$, called the boundedness.

5 CPLDF MABAC Techniques

In this section, we analyze the MABAC technique for designed operators, called CPLDFPoA operator and CPLDFPoG operator to deliberate the consistency of the suggested theory. The graphical interpretation of the proposed application is given in the form of Figure 2.

For this, we have a group of alternatives $\tilde{H}_1,\tilde{H}_2,...,\tilde{H}_{\ni}$ with $A_1,A_2,...,A_n$, called attributes for each alternative with the same order of weighted information, such as $\aleph_{\&}\in[0,1]$ with $\sum_{\&=1}^{\ni}\aleph_{\&}=1$, thus, we design a matrix by putting their information in the form of CPLDFSs, such as

$$\tilde{H}=\left\{\left(\tau,\left(\mathcal{F}_{rp}^{\omega}(\tau),\mathcal{F}_{ip}^{\omega}(\tau)\right),\left(\text{Ⅎ}_{rp}^{\omega}(\tau),\text{Ⅎ}_{ip}^{\omega}(\tau)\right),\left(\zeta_{rp}^{\omega}(\tau),\zeta_{ip}^{\omega}(\tau)\right),\left(\Gamma_{rp}^{\omega}(\tau),\Gamma_{ip}^{\omega}(\tau)\right)\right):\tau\in\mathbb{X}\right\}$$

In addition, we define the truth and parameter function according to their real and imaginary parts, such as

$$\mathcal{F}_{rp}^{\omega}(\tau)=\mathbb{L}_{rp}^{1}\left(\text{Ⅎ}_{rp}^{\omega}(\tau)\right),\left(\mathcal{F}_{ip}^{\omega}(\tau)=\mathbb{L}_{ip}^{1}\left(\text{Ⅎ}_{ip}^{\omega}(\tau)\right)\right)$$

and

$$\zeta_{rp}^{\omega}(\tau)=\mathbb{L}_{rp}^{2}\left(\Gamma_{rp}^{\omega}(\tau)\right),\left(\zeta_{ip}^{\omega}(\tau)=\mathbb{L}_{ip}^{2}\left(\Gamma_{ip}^{\omega}(\tau)\right)\right)$$

Then

$$\mathbb{L}_{rp}^{2}\left(\Gamma_{rp}^{\omega}(\tau)\right)*\mathbb{L}_{rp}^{1}\left(\text{Ⅎ}_{rp}^{\omega}(\tau)\right)+\Gamma_{rp}^{\omega}(\tau)*\text{Ⅎ}_{rp}^{\omega}(\tau)\leq 1$$
$$\Rightarrow\Gamma_{rp}^{\omega}(\tau)*\text{Ⅎ}_{rp}^{\omega}(\tau)\left(1+\mathbb{L}_{rp}^{1}\mathbb{L}_{rp}^{2}\right)\leq 1\Rightarrow\Gamma_{rp}^{\omega}(\tau)*\text{Ⅎ}_{rp}^{\omega}(\tau)\leq\frac{1}{1+\mathbb{L}_{rp}^{1}\mathbb{L}_{rp}^{2}}$$

form of the proposed technique..png form of the proposed technique.bb

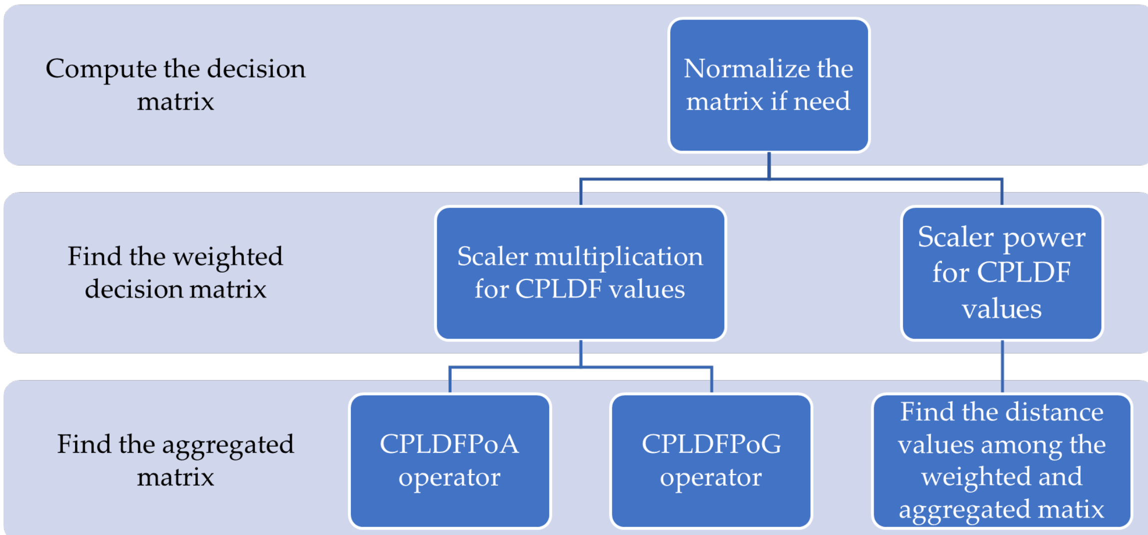

Figure 2: Graphical form of the proposed technique.

Thus, we have the following final shape, such as

$$\tilde{H}_{\&} = \left(\begin{array}{c} \left(\left(\mathbb{L}_{rp}^{1\&} \left(\frac{1-\varepsilon_{rp_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{1\&}} \right), \mathbb{L}_{ip}^{1\&} \left(\frac{1-\varepsilon_{ip_{\&}}^{\omega}}{1+\mathbb{L}_{ip}^{1\&}} \right) \right), \left(\left(\frac{1-\varepsilon_{rp_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{1\&}} \right), \left(\frac{1-\varepsilon_{ip_{\&}}^{\omega}}{1+\mathbb{L}_{ip}^{1\&}} \right) \right) \right), \\ \left(\left(\mathbb{L}_{rp}^{2\&} \left(\frac{1-\varepsilon_{rp_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{2\&}} \right), \mathbb{L}_{ip}^{2\&} \left(\frac{1-\varepsilon_{ip_{\&}}^{\omega}}{1+\mathbb{L}_{ip}^{2\&}} \right) \right), \left(\left(\frac{1-\varepsilon_{rp_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{2\&}} \right), \left(\frac{1-\varepsilon_{ip_{\&}}^{\omega}}{1+\mathbb{L}_{ip}^{2\&}} \right) \right) \right) \end{array} \right), \& = 1, 2, \cdots, \ni$$

After constructing the decision matrix, we goal to design the procedure of the decision-making model for evaluating numerous genuine life problems. Therefore, we will follow the following technique for evaluating any type of problem, such as

Step 1: Construction of matrix: We focus on designing a matrix, where the value of the matrix must be the form of CPLDFNs, such as

$$DM = \left[\tilde{H}_{i \times \&} \right]_{n \times \ni} = \begin{bmatrix} \tilde{H}_{11} & \tilde{H}_{12} & \cdots & \tilde{H}_{1\ni} \\ \tilde{H}_{21} & \tilde{H}_{22} & \cdots & \tilde{H}_{2\ni} \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{H}_{n1} & \tilde{H}_{n2} & \cdots & \tilde{H}_{n\ni} \end{bmatrix}$$

After the construction of the complex propositional linear Diophantine fuzzy matrix, we goal to normalize the data.

Step 2: Unvarying the matrix: We goal to normalize the data, if Cost types of data occurrences, such as

$$\tilde{H}=\left[\begin{array}{l}\left(\begin{array}{l}\left(\left(\mathbb{L}_{rp}^{1_{\&}}\left(\frac{1-\varepsilon_{rp_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right),\mathbb{L}_{ip}^{1_{\&}}\left(\frac{1-\varepsilon_{ip_{\&}}^{\omega}}{1+\mathbb{L}_{ip}^{1_{\&}}}\right)\right),\left(\left(\frac{1-\varepsilon_{rp_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right),\left(\frac{1-\varepsilon_{ip_{\&}}^{\omega}}{1+\mathbb{L}_{ip}^{1_{\&}}}\right)\right)\right),\\ \left(\left(\mathbb{L}_{rp}^{2_{\&}}\left(\frac{1-\varepsilon_{rp_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right),\mathbb{L}_{ip}^{2_{\&}}\left(\frac{1-\varepsilon_{ip_{\&}}^{\omega}}{1+\mathbb{L}_{ip}^{2_{\&}}}\right)\right),\left(\left(\frac{1-\varepsilon_{rp_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right),\left(\frac{1-\varepsilon_{ip_{\&}}^{\omega}}{1+\mathbb{L}_{ip}^{2_{\&}}}\right)\right)\right)\end{array}\right) benefit\\ \left(\begin{array}{l}\left(\left(\left(\frac{1-\varepsilon_{rp_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right),\left(\frac{1-\varepsilon_{ip_{\&}}^{\omega}}{1+\mathbb{L}_{ip}^{1_{\&}}}\right)\right),\left(\mathbb{L}_{rp}^{1_{\&}}\left(\frac{1-\varepsilon_{rp_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right),\mathbb{L}_{ip}^{1_{\&}}\left(\frac{1-\varepsilon_{ip_{\&}}^{\omega}}{1+\mathbb{L}_{ip}^{1_{\&}}}\right)\right)\right),\\ \left(\left(\left(\frac{1-\varepsilon_{rp_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right),\left(\frac{1-\varepsilon_{ip_{\&}}^{\omega}}{1+\mathbb{L}_{ip}^{2_{\&}}}\right)\right),\left(\mathbb{L}_{rp}^{2_{\&}}\left(\frac{1-\varepsilon_{rp_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right),\mathbb{L}_{ip}^{2_{\&}}\left(\frac{1-\varepsilon_{ip_{\&}}^{\omega}}{1+\mathbb{L}_{ip}^{2_{\&}}}\right)\right)\right)\end{array}\right) cost\end{array}\right.$$

In another case, we goal to go to the next step.

Step 3: Weighted matrix construction: We goal to develop the weighted matrix, such as

$$\begin{array}{l}\tilde{\eta}_{\Theta}\tilde{H}_{\&}\\ =\left(\begin{array}{l}\left(\left(1-\left(1-\mathbb{L}_{rp}^{1_{\&}}\left(\frac{1-\varepsilon_{rp_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right)\right)^{\tilde{\eta}_{\Theta}},1-\left(1-\mathbb{L}_{rp}^{1_{\&}}\left(\frac{1-\varepsilon_{ip_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right)\right)^{\tilde{\eta}_{\Theta}}\right),\left(\left(\left(\frac{1-\varepsilon_{rp_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right)\right)^{\tilde{\eta}_{\Theta}},\left(\left(\frac{1-\varepsilon_{ip_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right)\right)^{\tilde{\eta}_{\Theta}}\right)\right),\\ \left(\left(1-\left(1-\mathbb{L}_{rp}^{2_{\&}}\left(\frac{1-\varepsilon_{rp_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right)\right)^{\tilde{\eta}_{\Theta}},1-\left(1-\mathbb{L}_{rp}^{2_{\&}}\left(\frac{1-\varepsilon_{ip_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right)\right)^{\tilde{\eta}_{\Theta}}\right),\left(\left(\left(\frac{1-\varepsilon_{rp_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right)\right)^{\tilde{\eta}_{\Theta}},\left(\left(\frac{1-\varepsilon_{ip_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right)\right)^{\tilde{\eta}_{\Theta}}\right)\right)\end{array}\right)\end{array}$$

$$\begin{array}{l}\left(\tilde{H}_{\&}\right)^{\tilde{\eta}_{\Theta}}\\ =\left(\begin{array}{l}\left(\left(\left(\mathbb{L}_{rp}^{1_{\&}}\left(\frac{1-\varepsilon_{rp_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right)\right)^{\tilde{\eta}_{\Theta}},\left(\mathbb{L}_{rp}^{1_{\&}}\left(\frac{1-\varepsilon_{ip_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right)\right)^{\tilde{\eta}_{\Theta}}\right),\left(1-\left(1-\left(\frac{1-\varepsilon_{rp_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right)\right)^{\tilde{\eta}_{\Theta}},1-\left(1-\left(\frac{1-\varepsilon_{ip_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right)\right)^{\tilde{\eta}_{\Theta}}\right)\right),\\ \left(\left(\left(\mathbb{L}_{rp}^{2_{\&}}\left(\frac{1-\varepsilon_{rp_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right)\right)^{\tilde{\eta}_{\Theta}},\left(\mathbb{L}_{rp}^{2_{\&}}\left(\frac{1-\varepsilon_{ip_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right)\right)^{\tilde{\eta}_{\Theta}}\right),\left(1-\left(1-\left(\frac{1-\varepsilon_{rp_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right)\right)^{\tilde{\eta}_{\Theta}},1-\left(1-\left(\frac{1-\varepsilon_{ip_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right)\right)^{\tilde{\eta}_{\Theta}}\right)\right)\end{array}\right)\end{array}$$

After evaluating the weighted decision matrix, we goal to address the aggregated matrix.

Step 4: Aggregation matrix construction: We goal to construct the aggregated values matrix by using the CPLDFPoA operator and CPLDFPoG operator, such as

$$\begin{array}{l}CPLDFPoA\left(\tilde{H}_{1},\tilde{H}_{2},\cdots,\tilde{H}_{\ni}\right)\\ =\left(\begin{array}{l}\left(\begin{array}{c}\left(1-\prod_{\&=1}^{\ni}\left(1-\mathbb{L}_{rp}^{1_{\&}}\left(\frac{1-\varepsilon_{rp_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right)\right)^{\frac{(1+\tilde{\eta}(\tilde{H}_{\&}))}{\sum_{\&=1}^{\ni}(1+\tilde{\eta}(\tilde{H}_{\&}))}},1-\prod_{\&=1}^{\ni}\left(1-\mathbb{L}_{rp}^{1_{\&}}\left(\frac{1-\varepsilon_{ip_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right)\right)^{\frac{(1+\tilde{\eta}(\tilde{H}_{\&}))}{\sum_{\&=1}^{\ni}(1+\tilde{\eta}(\tilde{H}_{\&}))}}\right),\\ \left(\prod_{\&=1}^{\ni}\left(\left(\frac{1-\varepsilon_{rp_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right)\right)^{\frac{(1+\tilde{\eta}(\tilde{H}_{\&}))}{\sum_{\&=1}^{\ni}(1+\tilde{\eta}(\tilde{H}_{\&}))}},\prod_{\&=1}^{\ni}\left(\left(\frac{1-\varepsilon_{ip_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{1_{\&}}}\right)\right)^{\frac{(1+\tilde{\eta}(\tilde{H}_{\&}))}{\sum_{\&=1}^{\ni}(1+\tilde{\eta}(\tilde{H}_{\&}))}}\right)\end{array}\right),\\ \left(\begin{array}{c}\left(1-\prod_{\&=1}^{\ni}\left(1-\mathbb{L}_{rp}^{2_{\&}}\left(\frac{1-\varepsilon_{rp_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right)\right)^{\frac{(1+\tilde{\eta}(\tilde{H}_{\&}))}{\sum_{\&=1}^{\ni}(1+\tilde{\eta}(\tilde{H}_{\&}))}},1-\prod_{\&=1}^{\ni}\left(1-\mathbb{L}_{rp}^{2_{\&}}\left(\frac{1-\varepsilon_{ip_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right)\right)^{\frac{(1+\tilde{\eta}(\tilde{H}_{\&}))}{\sum_{\&=1}^{\ni}(1+\tilde{\eta}(\tilde{H}_{\&}))}}\right),\\ \left(\prod_{\&=1}^{\ni}\left(\left(\frac{1-\varepsilon_{rp_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right)\right)^{\frac{(1+\tilde{\eta}(\tilde{H}_{\&}))}{\sum_{\&=1}^{\ni}(1+\tilde{\eta}(\tilde{H}_{\&}))}},\prod_{\&=1}^{\ni}\left(\left(\frac{1-\varepsilon_{ip_{\&}}^{\omega}}{1+\mathbb{L}_{rp}^{2_{\&}}}\right)\right)^{\frac{(1+\tilde{\eta}(\tilde{H}_{\&}))}{\sum_{\&=1}^{\ni}(1+\tilde{\eta}(\tilde{H}_{\&}))}}\right)\end{array}\right)\end{array}\right)\end{array}$$

and

$$CPLDFPoG\left(\tilde{H}_1,\tilde{H}_2,\cdots,\tilde{H}_\ni\right)$$
$$=\left(\begin{array}{c}\left(\begin{array}{c}\left(\prod_{\&=1}^{\ni}\left(\mathbb{L}_{rp}^{1_\&}\left(\frac{1-\varepsilon rp_\&^\omega}{1+\mathbb{L}_{rp}^{1_\&}}\right)\right)^{\frac{(1+\tilde{\eta}(\tilde{H}_\&))}{\sum_{\&=1}^{\ni}(1+\tilde{\eta}(\tilde{H}_\&))}},\prod_{\&=1}^{\ni}\left(\mathbb{L}_{rp}^{1_\&}\left(\frac{1-\varepsilon ip_\&^\omega}{1+\mathbb{L}_{rp}^{1_\&}}\right)\right)^{\frac{(1+\tilde{\eta}(\tilde{H}_\&))}{\sum_{\&=1}^{\ni}(1+\tilde{\eta}(\tilde{H}_\&))}}\right),\\ \left(1-\prod_{\&=1}^{\ni}\left(1-\left(\frac{1-\varepsilon rp_\&^\omega}{1+\mathbb{L}_{rp}^{1_\&}}\right)\right)^{\frac{(1+\tilde{\eta}(\tilde{H}_\&))}{\sum_{\&=1}^{\ni}(1+\tilde{\eta}(\tilde{H}_\&))}},1-\prod_{\&=1}^{\ni}\left(1-\left(\frac{1-\varepsilon ip_\&^\omega}{1+\mathbb{L}_{rp}^{1_\&}}\right)\right)^{\frac{(1+\tilde{\eta}(\tilde{H}_\&))}{\sum_{\&=1}^{\ni}(1+\tilde{\eta}(\tilde{H}_\&))}}\right)\end{array}\right),\\ \left(\begin{array}{c}\left(\prod_{\&=1}^{\ni}\left(\mathbb{L}_{rp}^{2_\&}\left(\frac{1-\varepsilon rp_\&^\omega}{1+\mathbb{L}_{rp}^{2_\&}}\right)\right)^{\frac{(1+\tilde{\eta}(\tilde{H}_\&))}{\sum_{\&=1}^{\ni}(1+\tilde{\eta}(\tilde{H}_\&))}},\prod_{\&=1}^{\ni}\left(\mathbb{L}_{rp}^{2_\&}\left(\frac{1-\varepsilon ip_\&^\omega}{1+\mathbb{L}_{rp}^{2_\&}}\right)\right)^{\frac{(1+\tilde{\eta}(\tilde{H}_\&))}{\sum_{\&=1}^{\ni}(1+\tilde{\eta}(\tilde{H}_\&))}}\right),\\ \left(1-\prod_{\&=1}^{\ni}\left(1-\left(\frac{1-\varepsilon rp_\&^\omega}{1+\mathbb{L}_{rp}^{2_\&}}\right)\right)^{\frac{(1+\tilde{\eta}(\tilde{H}_\&))}{\sum_{\&=1}^{\ni}(1+\tilde{\eta}(\tilde{H}_\&))}},1-\prod_{\&=1}^{\ni}\left(1-\left(\frac{1-\varepsilon ip_\&^\omega}{1+\mathbb{L}_{rp}^{2_\&}}\right)\right)^{\frac{(1+\tilde{\eta}(\tilde{H}_\&))}{\sum_{\&=1}^{\ni}(1+\tilde{\eta}(\tilde{H}_\&))}}\right)\end{array}\right)\end{array}\right)$$

To assess the values of the aggregated matrix, we will find the distance values among the information of weighted value and aggregated values.

Step 5: Distance matrix construction: We goal to design the values by distance function, such as

$$\tilde{H}_{\&k}=\begin{cases}D\left(\tilde{H}_\&,\tilde{H}_k\right) & if\tilde{H}_\&>\tilde{H}_k\\ 0 & if\tilde{H}_\&=\tilde{H}_k\\ -D\left(\tilde{H}_\&,\tilde{H}_k\right) & if\tilde{H}_\&<\tilde{H}_k\end{cases}$$

where

$$\begin{aligned}D\left(\tilde{H}_\&,\tilde{H}_k\right)=&\frac{1}{8}\Bigg(\left|\mathbb{L}_{rp}^{1_\&}\left(\frac{1-\varepsilon rp_\&^\omega}{1+\mathbb{L}_{rp}^{1_\&}}\right)-\mathbb{L}_{rp}^{1_k}\left(\frac{1-\varepsilon rp_k^\omega}{1+\mathbb{L}_{rp}^{1_k}}\right)\right|+\left|\mathbb{L}_{ip}^{1_\&}\left(\frac{1-\varepsilon ip_\&^\omega}{1+\mathbb{L}_{ip}^{1_\&}}\right)-\mathbb{L}_{ip}^{1_k}\left(\frac{1-\varepsilon ip_k^\omega}{1+\mathbb{L}_{ip}^{1_k}}\right)\right|\\ &+\left|\left(\frac{1-\varepsilon rp_\&^\omega}{1+\mathbb{L}_{rp}^{1_\&}}\right)-\left(\frac{1-\varepsilon rp_k^\omega}{1+\mathbb{L}_{rp}^{1_k}}\right)\right|+\left|\left(\frac{1-\varepsilon ip_\&^\omega}{1+\mathbb{L}_{ip}^{1_\&}}\right)-\left(\frac{1-\varepsilon ip_k^\omega}{1+\mathbb{L}_{ip}^{1_k}}\right)\right|\\ &+\left|\mathbb{L}_{rp}^{2_\&}\left(\frac{1-\varepsilon rp_\&^\omega}{1+\mathbb{L}_{rp}^{2_\&}}\right)-\mathbb{L}_{rp}^{2_k}\left(\frac{1-\varepsilon rp_k^\omega}{1+\mathbb{L}_{rp}^{2_k}}\right)\right|+\left|\mathbb{L}_{ip}^{2_\&}\left(\frac{1-\varepsilon ip_\&^\omega}{1+\mathbb{L}_{ip}^{2_\&}}\right)-\mathbb{L}_{ip}^{2_k}\left(\frac{1-\varepsilon ip_k^\omega}{1+\mathbb{L}_{ip}^{2_k}}\right)\right|\\ &+\left|\left(\frac{1-\varepsilon rp_\&^\omega}{1+\mathbb{L}_{rp}^{2_\&}}\right)-\left(\frac{1-\varepsilon rp_k^\omega}{1+\mathbb{L}_{rp}^{2_k}}\right)\right|+\left|\left(\frac{1-\varepsilon ip_\&^\omega}{1+\mathbb{L}_{ip}^{2_\&}}\right)-\left(\frac{1-\varepsilon ip_k^\omega}{1+\mathbb{L}_{ip}^{2_k}}\right)\right|\Bigg)\end{aligned}$$

Step 6: Appraisal matrix: We goal to address the appraisal information, such as

$$S_\&=\frac{1}{\ni}\sum_{k=1}^{\ni}D\left(\tilde{H}_\&,\tilde{H}_k\right)$$

Step 8: Ranking matrix: Calculate the ranking data according to the appraisal function for addressing the best one amid the group of a finite number of values.

6 CPLDF MABAC Deep Learning for Diagnosis of Alzheimers Disease

In this section, we goal to address the problem of the CPLDF MABAC deep learning model for diagnosis of Alzheimers disease for initiated techniques. Alzheimers disease is an unpredictable and progressive neurodegenerative disorder that initially affects memory thinking and behavior. The analysis of Alzheimers disease has been done by different scholars according to the information of crisp data, but to analyze the best one among the collection of data, we needed a soft and valuable technique that can help us in the evaluation of the procedure of decision-making models. Some key features of Alzheimers disease are memory loss, cognitive decline, behavioral changes, disorientation, and physical symptoms. In this article, we design the procedure of a multi-attributive border approximation area comparison deep learning algorithm for the diagnosis of Alzheimers Disease. Application point of view, we target data collection for diagnosing Alzheimers disease involves collecting a brief set of data from different sources, thus with the help of the above model, we aim to select the major means the best and worst ones among the collecting five, such as

1. Cognitive Assessments.
2. Neuroimaging Results.
3. Genetic Information.
4. Biomarkers.
5. Clinical History and Physical Examination.

Once selected, this information can be integrated into machine learning or deep learning models to analyze patterns and support diagnostic decision-making. Further, we have some attributes for the above alternatives, such as

1. Memory Loss.
2. Cognitive Decline.
3. Behavioral Changes.
4. Disorientation.
5. Physical symptom.

Therefore, to evaluate the above problems, we have a group of alternatives $\tilde{H}_1, \tilde{H}_2, \cdots, \tilde{H}_\ni$ with $A_1, A_2, \cdots, A_n$, called attributes for each alternative with the same order of weighted information, such as $\aleph_{\&} =\in [0,1]$ with $\sum_{\&=1}^{\ni} \aleph_{\&} = 1$, thus, we design a matrix by putting their information in the form of CPLDFSs, such as

$$\tilde{H} = \left\{ \left(\tau, \left(\mathcal{F}_{rp}^{\omega}(\tau), \mathcal{F}_{ip}^{\omega}(\tau) \right), \left(Ⅎ_{rp}^{\omega}(\tau), Ⅎ_{ip}^{\omega}(\tau) \right), \left(\zeta_{rp}^{\omega}(\tau), \zeta_{ip}^{\omega}(\tau) \right), \left(\Gamma_{rp}^{\omega}(\tau), \Gamma_{ip}^{\omega}(\tau) \right) \right) : \tau \in \mathbb{X} \right\}$$

In addition, we define the truth and parameter function according to their real and imaginary parts, such as

$$\mathcal{F}_{rp}^{\omega}(\tau) = \mathbb{L}_{rp}^{1} \left(Ⅎ_{rp}^{\omega}(\tau) \right), \left(\mathcal{F}_{ip}^{\omega}(\tau) = \mathbb{L}_{ip}^{1} \left(Ⅎ_{ip}^{\omega}(\tau) \right) \right)$$

and

$$\zeta_{rp}^{\omega}(\tau) = \mathbb{L}_{rp}^{2} \left(\Gamma_{rp}^{\omega}(\tau) \right), \left(\zeta_{ip}^{\omega}(\tau) = \mathbb{L}_{ip}^{2} \left(\Gamma_{ip}^{\omega}(\tau) \right) \right)$$

Then

$$\mathbb{L}_{rp}^{2} \left(\Gamma_{rp}^{\omega}(\tau) \right) * \mathbb{L}_{rp}^{1} \left(Ⅎ_{rp}^{\omega}(\tau) \right) + \Gamma_{rp}^{\omega}(\tau) * Ⅎ_{rp}^{\omega}(\tau) \leq 1$$
$$\Rightarrow \Gamma_{rp}^{\omega}(\tau) * Ⅎ_{rp}^{\omega}(\tau) \left(1 + \mathbb{L}_{rp}^{1} \mathbb{L}_{rp}^{2} \right) \leq 1 \Rightarrow \Gamma_{rp}^{\omega}(\tau) * Ⅎ_{rp}^{\omega}(\tau) \leq \frac{1}{1 + \mathbb{L}_{rp}^{1} \mathbb{L}_{rp}^{2}}$$

Similarly, we have imaginary parts, such as

$$\Gamma_{ip}^{\omega}(\tau) * \Finv_{ip}^{\omega}(\tau) \leq \frac{1}{1+\mathbb{L}_{ip}^{1}\mathbb{L}_{ip}^{2}}$$

thus

$$\begin{aligned}\varepsilon_{rp}^{\omega}(\tau) &= 1-\left(\zeta_{rp}^{\omega}(\tau) * \mathcal{F}_{rp}^{\omega}(\tau) * +\Gamma_{rp}^{\omega}(\tau) * \Finv_{rp}^{\omega}(\tau)\right) = 1-\left(\mathbb{L}_{rp}^{2}\left(\Gamma_{rp}^{\omega}(\tau)\right) * \mathbb{L}_{rp}^{1}\left(\Finv_{rp}^{\omega}(\tau)\right)+\Gamma_{rp}^{\omega}(\tau) * \Finv_{rp}^{\omega}(\tau)\right)\\ &= 1-\left(\Gamma_{rp}^{\omega}(\tau) * \Finv_{rp}^{\omega}(\tau)\left(1+\mathbb{L}_{rp}^{1}\mathbb{L}_{rp}^{2}\right)\right)\end{aligned}$$

then

$$1-\varepsilon_{rp}^{\omega}(\tau) = \Gamma_{rp}^{\omega}(\tau) * \Finv_{rp}^{\omega}(\tau)\left(1+\mathbb{L}_{rp}^{1}\mathbb{L}_{rp}^{2}\right)$$

and

$$\Gamma_{rp}^{\omega}(\tau) * \Finv_{rp}^{\omega}(\tau) = \frac{1-\varepsilon_{rp}^{\omega}(\tau)}{\left(1+\mathbb{L}_{rp}^{1}\mathbb{L}_{rp}^{2}\right)}$$

Similarly, we have

$$\Gamma_{ip}^{\omega}(\tau) * \Finv_{ip}^{\omega}(\tau) = \frac{1-\varepsilon_{ip}^{\omega}(\tau)}{\left(1+\mathbb{L}_{ip}^{1}\mathbb{L}_{ip}^{2}\right)}$$

But if we use the condition of IFSs, thus we have

$$\begin{gathered}\mathbb{L}_{rp}^{1}\left(\Finv_{rp}^{\omega}(\tau)\right)+\Finv_{rp}^{\omega}(\tau) \leq 1\\ \Rightarrow \Finv_{rp}^{\omega}(\tau)\left(1+\mathbb{L}_{rp}^{1}\right) \leq 1 \Rightarrow \Finv_{rp}^{\omega}(\tau) \leq \frac{1}{\left(1+\mathbb{L}_{rp}^{1}\right)}\end{gathered}$$

Similarly, we have imaginary parts, such as

$$\Finv_{ip}^{\omega}(\tau) \leq \frac{1}{\left(1+\mathbb{L}_{ip}^{1}\right)}$$

Thus, we have the condition of refusal function in IFSs, such as

$$\varepsilon_{rp}^{\omega}(\tau) = 1-\left(\mathcal{F}_{rp}^{\omega}(\tau)+\Finv_{rp}^{\omega}(\tau)\right) = 1-\left(\mathbb{L}_{rp}^{1}\left(\Finv_{rp}^{\omega}(\tau)\right)+\Finv_{rp}^{\omega}(\tau)\right) = 1-\left(\Finv_{rp}^{\omega}(\tau)\left(1+\mathbb{L}_{rp}^{1}\right)\right)$$

then

$$1-\varepsilon_{rp}^{\omega}(\tau) = \Finv_{rp}^{\omega}(\tau)\left(1+\mathbb{L}_{rp}^{1}\right)$$

and

$$\Finv_{rp}^{\omega}(\tau) = \frac{1-\varepsilon_{rp}^{\omega}(\tau)}{\left(1+\mathbb{L}_{rp}^{1}\right)}, \left(\zeta_{rp}^{\omega}(\tau) = \frac{1-\varepsilon_{rp}^{\omega}(\tau)}{\left(1+\mathbb{L}_{rp}^{2}\right)}\right)$$

Similarly, we have

$$\Finv_{ip}^{\omega}(\tau) = \frac{1-\varepsilon_{ip}^{\omega}(\tau)}{\left(1+\mathbb{L}_{rp}^{1}\right)}, \left(\zeta_{ip}^{\omega}(\tau) = \frac{1-\varepsilon_{ip}^{\omega}(\tau)}{\left(1+\mathbb{L}_{ip}^{2}\right)}\right)$$

If $\varepsilon^{\omega}_{rp}(\tau)=\varepsilon^{\omega}_{ip}(\tau)=0$, thus $\Finv^{\omega}_{rp}(\tau)=\frac{1}{(1+\mathbb{L}^{1}_{rp})}$ and $\Finv^{\omega}_{ip}(\tau)=\frac{1}{(1+\mathbb{L}^{1}_{ip})}$. Then

$$\mathcal{F}^{\omega}_{rp}(\tau)=\mathbb{L}^{1}_{rp}\left(\frac{1}{\left(1+\mathbb{L}^{1}_{rp}\right)}\right),\left(\mathcal{F}^{\omega}_{ip}(\tau)=\mathbb{L}^{1}_{ip}\left(\frac{1}{\left(1+\mathbb{L}^{1}_{ip}\right)}\right)\right)$$

and

$$\zeta^{\omega}_{rp}(\tau)=\mathbb{L}^{2}_{rp}\left(\frac{1}{\left(1+\mathbb{L}^{2}_{rp}\right)}\right),\left(\zeta^{\omega}_{ip}(\tau)=\mathbb{L}^{2}_{ip}\left(\frac{1}{\left(1+\mathbb{L}^{2}_{ip}\right)}\right)\right)$$

Then

$$\tilde{H}=\left\{\left(\begin{array}{l}\tau,\left(\left(\mathbb{L}^{1}_{rp}\left(\frac{1}{1+\mathbb{L}^{1}_{rp}}\right),\mathbb{L}^{1}_{ip}\left(\frac{1}{1+\mathbb{L}^{1}_{ip}}\right)\right)\right),\left(\frac{1}{1+\mathbb{L}^{1}_{rp}},\frac{1}{1+\mathbb{L}^{1}_{rp}}\right)\right),\\ \quad\left(\left(\mathbb{L}^{2}_{rp}\left(\frac{1}{1+\mathbb{L}^{2}_{rp}}\right),\mathbb{L}^{2}_{ip}\left(\frac{1}{1+\mathbb{L}^{2}_{ip}}\right)\right)\right),\left(\frac{1}{1+\mathbb{L}^{2}_{rp}},\frac{1}{1+\mathbb{L}^{2}_{rp}}\right)\right)\end{array}\right):\tau\in\mathbb{X}\right\}$$

Thus, we have the following final shape, such as

$$\tilde{H}_{\&}=\left(\begin{array}{l}\left(\left(\left(\mathbb{L}^{1_{\&}}_{rp}\left(\frac{1-\varepsilon_{rp}{}^{\omega}_{\&}}{1+\mathbb{L}^{1_{\&}}_{rp}}\right),\mathbb{L}^{1_{\&}}_{ip}\left(\frac{1-\varepsilon_{ip}{}^{\omega}_{\&}}{1+\mathbb{L}^{1_{\&}}_{ip}}\right)\right)\right),\left(\left(\frac{1-\varepsilon_{rp}{}^{\omega}_{\&}}{1+\mathbb{L}^{1_{\&}}_{rp}}\right),\left(\frac{1-\varepsilon_{ip}{}^{\omega}_{\&}}{1+\mathbb{L}^{1_{\&}}_{rp}}\right)\right)\right)\right),\\ \left(\left(\mathbb{L}^{2_{\&}}_{rp}\left(\frac{1-\varepsilon_{rp}{}^{\omega}_{\&}}{1+\mathbb{L}^{2_{\&}}_{rp}}\right),\mathbb{L}^{2_{\&}}_{ip}\left(\frac{1-\varepsilon_{ip}{}^{\omega}_{\&}}{1+\mathbb{L}^{2_{\&}}_{ip}}\right)\right)\right),\left(\left(\frac{1-\varepsilon_{rp}{}^{\omega}_{\&}}{1+\mathbb{L}^{2_{\&}}_{rp}}\right),\left(\frac{1-\varepsilon_{ip}{}^{\omega}_{\&}}{1+\mathbb{L}^{2_{\&}}_{rp}}\right)\right)\right)\end{array}\right),\&=1,2,\cdots,\backepsilon.$$

Therefore, we will follow the following technique for evaluating any type of problem, such as
Step 1: Construction of matrix: We focus on designing a matrix, where the value of the matrix must be the form of CPLDFNs, see Table 1.

Table 1: CPLDF information decision matrix.

	***A*₁**	***A*₂**	***A*₃**	***A*₄**	***A*₅**
$\tilde{H}_1$	$((4,3),(4,2))$	$((5,1),(5,3))$	$((6,2),(6,3))$	$((7,4),(7,4))$	$((1,3),(8,5))$
$\tilde{H}_2$	$((1,3),(2,6))$	$((2,2),(3,5))$	$((3,4),(4,4))$	$((4,4),(5,3))$	$((1,1),(8,5))$
$\tilde{H}_3$	$((3,3),(1,5))$	$((4,2),(4,4))$	$((5,3),(3,3))$	$((7,4),(5,2))$	$((1,3),(6,1))$
$\tilde{H}_4$	$((6,4),(1,2))$	$((5,1),(2,1))$	$((4,2),(3,2))$	$((3,3),(1,2))$	$((1,3),(2,1))$
$\tilde{H}_5$	$((1,5),(2,5))$	$((2,4),(3,4))$	$((3,3),(4,3))$	$((1,2),(1,2))$	$((1,3),(2,1))$

Step 2: Unvarying the matrix: We goal to normalize the data, if Cost types of data occurrences, such as

$$\tilde{H}=\left[\begin{array}{l}\left(\begin{array}{l}\left(\left(\left(\mathbb{L}^{1_{\&}}_{rp}\left(\frac{1-\varepsilon_{rp}{}^{\omega}_{\&}}{1+\mathbb{L}^{1_{\&}}_{rp}}\right),\mathbb{L}^{1_{\&}}_{ip}\left(\frac{1-\varepsilon_{ip}{}^{\omega}_{\&}}{1+\mathbb{L}^{1_{\&}}_{ip}}\right)\right)\right),\left(\left(\frac{1-\varepsilon_{rp}{}^{\omega}_{\&}}{1+\mathbb{L}^{1_{\&}}_{rp}}\right),\left(\frac{1-\varepsilon_{ip}{}^{\omega}_{\&}}{1+\mathbb{L}^{1_{\&}}_{ip}}\right)\right)\right)\right),\\ \left(\left(\mathbb{L}^{2_{\&}}_{rp}\left(\frac{1-\varepsilon_{rp}{}^{\omega}_{\&}}{1+\mathbb{L}^{2_{\&}}_{rp}}\right),\mathbb{L}^{2_{\&}}_{ip}\left(\frac{1-\varepsilon_{ip}{}^{\omega}_{\&}}{1+\mathbb{L}^{2_{\&}}_{ip}}\right)\right)\right),\left(\left(\frac{1-\varepsilon_{rp}{}^{\omega}_{\&}}{1+\mathbb{L}^{2_{\&}}_{rp}}\right),\left(\frac{1-\varepsilon_{ip}{}^{\omega}_{\&}}{1+\mathbb{L}^{2_{\&}}_{ip}}\right)\right)\right)\end{array}\right) benefit\\ \left(\begin{array}{l}\left(\left(\left(\left(\frac{1-\varepsilon_{rp}{}^{\omega}_{\&}}{1+\mathbb{L}^{1_{\&}}_{rp}}\right),\left(\frac{1-\varepsilon_{ip}{}^{\omega}_{\&}}{1+\mathbb{L}^{1_{\&}}_{ip}}\right)\right)\right),\left(\mathbb{L}^{1_{\&}}_{rp}\left(\frac{1-\varepsilon_{rp}{}^{\omega}_{\&}}{1+\mathbb{L}^{1_{\&}}_{rp}}\right),\mathbb{L}^{1_{\&}}_{ip}\left(\frac{1-\varepsilon_{ip}{}^{\omega}_{\&}}{1+\mathbb{L}^{1_{\&}}_{ip}}\right)\right)\right)\right),\\ \left(\left(\left(\left(\frac{1-\varepsilon_{rp}{}^{\omega}_{\&}}{1+\mathbb{L}^{2_{\&}}_{rp}}\right),\left(\frac{1-\varepsilon_{ip}{}^{\omega}_{\&}}{1+\mathbb{L}^{2_{\&}}_{ip}}\right)\right)\right),\left(\mathbb{L}^{2_{\&}}_{rp}\left(\frac{1-\varepsilon_{rp}{}^{\omega}_{\&}}{1+\mathbb{L}^{2_{\&}}_{rp}}\right),\mathbb{L}^{2_{\&}}_{ip}\left(\frac{1-\varepsilon_{ip}{}^{\omega}_{\&}}{1+\mathbb{L}^{2_{\&}}_{ip}}\right)\right)\right)\right)\end{array}\right) cost\end{array}\right.$$

In another case, we goal to go to the next step. So here we have benefit types of data in Table 1, so we will go to the next step.

Step 3: Weighted matrix construction: We goal to develop the weighted matrix, where $\tilde{\eta}_{\Theta} = 2$, see Table 2.

Table 2: CPLDF weighted information matrix.

	A_1	A_2	A_3	A_4	A_5
$\tilde{H}_1$	((0.96,0.9375), (0.36,0.4375), (0.64,0.4444), (0.04.0.111))	((0.9722,0.75), (0.3056,0.75), (0.6944,0.5625), (0.0278,0.0625))	((0.9796,0.8889), (0.2653,0.5556), (0.7347,0.5625), (0.0204,0.0625))	((0.9844,0.96), (0.2344,0.36), (0.7656,0.64), (0.0156,0.04))	((0.75,0.9375), (0.75,0.4375), (0.7901,0.6944), (0.0123,0.0278))
$\tilde{H}_2$	((0.75,0.9375), (0.75,0.4375), (0.4444,0.7347), (0.1111,0.0204))	((0.8889,0.8889), (0.5556,0.5556), (0.5625,0.6944), (0.0625,0.0278))	((0.9375,0.96), (0.4375,0.36), (0.64,0.64), (0.04,0.04))	((0.96,0.96), (0.36,0.36), (0.6944,0.5625), (0.0278,0.0625))	((0.75,0.75), (0.75,0.75), (0.7901,0.6944), (0.0123,0.0278))
$\tilde{H}_3$	((0.9375,0.9375), (0.4375,0.4375), (0.25,0.6944), (0.25,0.0278))	((0.96,0.8889), (0.36,0.5556), (0.64,0.64), (0.04,0.04))	((0.9722,0.9375), (0.3056,0.4375), (0.5625,0.5625), (0.0625,0.0625))	((0.9844,0.96), (0.2344,0.36), (0.6944,0.4444), (0.0278,0.1111))	((0.75,0.9375), (0.75,0.4375), (0.7347,0.25), (0.0204,0.25))
$\tilde{H}_4$	((0.9796,0.96), (0.2653,0.36), (0.25,0.4444), (0.25,0.1111))	((0.9722,0.75), (0.3056,0.75), (0.4444,0.25), (0.111,0.25))	((0.96,0.8889), (0.36,0.5556), (0.5625,0.4444), (0.0625,0.1111))	((0.9375,0.9375), (0.4375,0.4375), (0.25,0.4444), (0.25,0.1111))	((0.75,0.9375), (0.75,0.4375), (0.4444,0.25), (0.1111,0.25))
$\tilde{H}_5$	((0.75,0.9722), (0.75,0.3056), (0.4444,0.6944), (0.111,0.0278))	((0.8889,0.96), (0.5556,0.36), (0.5625,0.64), (0.0625,0.04))	((0.9375,0.9375), (0.4375,0.4375), (0.64,0.5625), (0.04,0.0625))	((0.75,0.8889), (0.75,0.5556), (0.25,0.4444), (0.25,0.1111))	((0.75,0.9375), (0.75,0.4375), (0.4444,0.25), (0.1111,0.25))

Step 4: Aggregation matrix construction: We goal to construct the aggregated values matrix by using the CPLDFPoA operator and CPLDFPoG operator, see Table 3.

Table 3: CPLDF aggregated information matrix.

	CPLDFPoA	***CPLDFPoG***	**Weighted vector obtained with the help of power operators**
$\tilde{H}_1$	((0.9618,0.9154), (0.4214,0.5317), (0.7226,0.5735), (0.0214,0.0548))	((0.9261,0.8913), (0.3458,0.4916), (0.7296,0.5883), (0.0233,0.0615))	0.2011,0.199,0.2044,0.2024,0.1931
$\tilde{H}_2$	((0.8891,0.9231), (0.6002,0.5177), (0.6139,0.6623), (0.0397,0.0331))	((0.8534,0.8965), (0.5462,0.4716), (0.6442,0.67), (0.0515,0.0358))	0.1991,0.2036,0.2932,0.1995,0.1946
$\tilde{H}_3$	((0.9525,0.9358), (0.4499,0.4497), (0.5409,0.4916), (0.0513,0.0712))	((0.9186,0.9618), (0.3813,0.4416), (0.605,0.5442), (0.084,0.1005))	0.1962,0.204,0.2065,0.2026,0.1907
$\tilde{H}_4$	((0.9491,0.9156), (0.4579,0.531), (0.3692,0.3546), (0.1374,0.1528))	((0.9164,0.8918), (0.3936,0.491), (0.4022,0.375), (0.1612,0.1685))	0.2014,0.1973,0.2024,0.2037,0.1953
$\tilde{H}_5$	((0.8389,0.9457), (0.67,0.4247), (0.4475,0.4893), (0.0947,0.0716))	((0.8114,0.9391), (0.6341,0.4099), (0.4848,0.5436), (0.1176,0.1018))	0.2027,0.2022,0.1988,0.1968,0.1996

Step 5: Distance matrix construction: We goal to design the values by distance function, see Table 4.

Table 4: CPLDF distance values.

	CPLDFPoA	***CPLDFPoG***
$\tilde{H}_1$	0.0583,0.0704,0.032,0.0695,0.1102	0.056,0.0691,0.0294,0.0609,0.115
$\tilde{H}_2$	0.0887,0.0286,0.0577,0.0909,0.1169	0.0902,0.0327,0.0522,0.0845,0.1187
$\tilde{H}_3$	0.0972,0.0675,0.0363,0.0782,0.1452	0.1036,0.0569,0.0324,0.0723,0.148
$\tilde{H}_4$	0.1002,0.1079,0.07,0.0638,0.1137	0.0948,0.1054,0.0663,0.0665,0.1151
$\tilde{H}_5$	0.0729,0.0716,0.0852,0.0992,0.0784	0.0733,0.0647,0.0804,0.1061,0.0869

Step 6: Appraisal matrix: We goal to address the appraisal information, see Table 5.

Table 5: CPLDF ranking values.

	CPLDFPoA	***CPLDFPoG***
$\tilde{H}_1$	0.0681	0.0654
$\tilde{H}_2$	0.0765	0.0757
$\tilde{H}_3$	0.0849	0.0826
$\tilde{H}_4$	0.0911	0.0896
$\tilde{H}_5$	0.0815	0.0823

Step 8: Ranking matrix: Calculate the ranking data according to the appraisal function for addressing the best one amid the group of a finite number of values, see Table 6.

Table 6: CPLDF ranking values.

Methods	**Ranking values**	**Best idea**
CPLDFPoA operator	$\tilde{H}_4 > \tilde{H}_5 > \tilde{H}_3 > \tilde{H}_2 > \tilde{H}_1$	$\tilde{H}_4$
CPLDFPoG operator	$\tilde{H}_4 > \tilde{H}_5 > \tilde{H}_3 > \tilde{H}_2 > \tilde{H}_1$	$\tilde{H}_4$

According to the data in Table 6, the most preferable decision is $\tilde{H}_4$, called the Biomarkers for the MABAC model based on both operators. The simple representation of the data in Table 5 is available in the form of Figure 3.

form of data in Table 5..png form of data in Table 5.bb

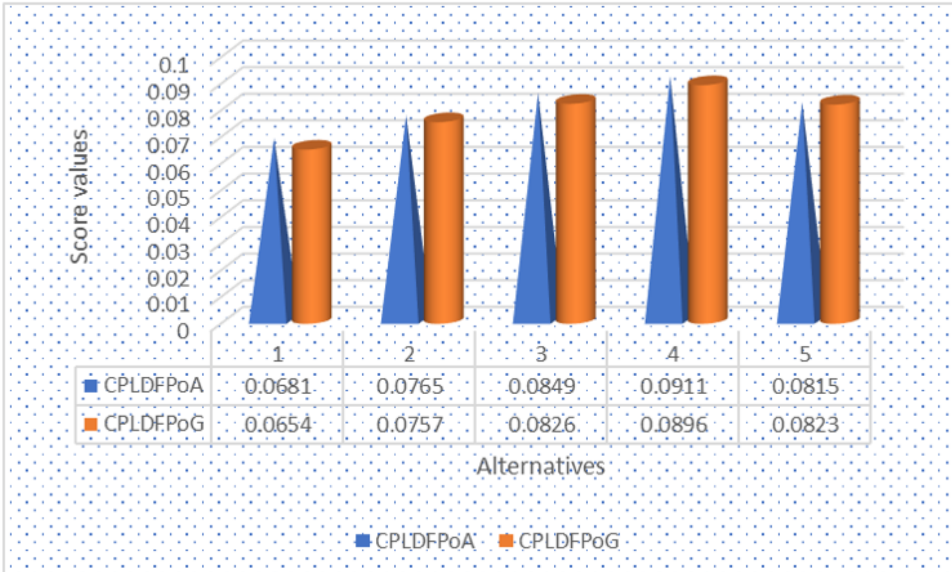

Figure 3: Graphical form of data in Table 5.

In addition, we will consider the data in Table 1 and, will evaluate it with the help of operators without the MABAC technique. Thus, the aggregated values matrix by using the CPLDFPoA operator and CPLDFPoG operator, see Table 7

Table 7: CPLDF aggregated matrix.

	CPLDFPoA	***CPLDFPoG***
$\tilde{H}_1$	$\begin{pmatrix}(0.8046,0.7091),\\(0.2393,0.3157),\\(0.8501,0.7573),\\(0.1462,0.2341)\end{pmatrix}$	$\begin{pmatrix}(0.7607,0.6843),\\(0.1954,0.2909),\\(0.8538,0.7659),\\(0.1499,0.2427)\end{pmatrix}$
$\tilde{H}_2$	$\begin{pmatrix}(0.6669,0.7227),\\(0.3677,0.3055),\\(0.7835,0.8138),\\(0.1993,0.1819)\end{pmatrix}$	$\begin{pmatrix}(0.6323,0.6945),\\(0.3331,0.2773),\\(0.8007,0.8181),\\(0.2165,0.1862)\end{pmatrix}$
$\tilde{H}_3$	$\begin{pmatrix}(0.7821,0.7466),\\(0.2583,0.2582),\\(0.7355,0.7011),\\(0.2266,0.2669)\end{pmatrix}$	$\begin{pmatrix}(0.7417,0.7418),\\(0.2179,0.2534),\\(0.7734,0.7331),\\(0.2645,0.2989)\end{pmatrix}$
$\tilde{H}_4$	$\begin{pmatrix}(0.7744,0.7095),\\(0.2637,0.3151),\\(0.6077,0.5955),\\(0.3706,0.3909)\end{pmatrix}$	$\begin{pmatrix}(0.7363,0.6849),\\(0.2256,0.2905),\\(0.6294,0.6091),\\(0.3923,0.4045)\end{pmatrix}$
$\tilde{H}_5$	$\begin{pmatrix}(0.5986,0.7671),\\(0.4256,0.2415),\\(0.669,0.6995),\\(0.3077,0.2675)\end{pmatrix}$	$\begin{pmatrix}(0.5744,0.7585),\\(0.4014,0.2329),\\(0.6923,0.7325),\\(0.331,0.3005)\end{pmatrix}$

Score value matrix: We goal to address the Score information, see Table 8.

Table 8: CPLDF ranking values.

	CPLDFPoA	***CPLDFPoG***
$\tilde{H}_1$	0.5464	0.5464
$\tilde{H}_2$	0.4832	0.4832
$\tilde{H}_3$	0.4888	0.4888
$\tilde{H}_4$	0.3367	0.3367
$\tilde{H}_5$	0.373	0.373

Ranking matrix: Calculate the ranking data according to the Score function for addressing the best one amid the group of a finite number of values, see Table 9.

Table 9: CPLDF ranking values.

Methods	**Ranking values**	**Best idea**
CPLDFPoA operator	$\tilde{H}_3 > \tilde{H}_2 > \tilde{H}_1 > \tilde{H}_5 > \tilde{H}_4$	$\tilde{H}_3$
CPLDFPoG operator	$\tilde{H}_3 > \tilde{H}_2 > \tilde{H}_1 > \tilde{H}_5 > \tilde{H}_4$	$\tilde{H}_3$

According to the data in Table 9, the most preferable decision is $\tilde{H}_3$, called the Genetic Information for both operators. The sensitivity of the proposed information for different values of parameters $\tilde{\eta}_\Theta$ is described in Table 10.

Table 10: Representation of the sensitive analysis.

$\tilde{\eta}_\Theta$	**Ranking values**	**Best idea**
2	$\tilde{H}_3 > \tilde{H}_2 > \tilde{H}_1 > \tilde{H}_5 > \tilde{H}_4$	$\tilde{H}_3$
4	$\tilde{H}_3 > \tilde{H}_2 > \tilde{H}_1 > \tilde{H}_5 > \tilde{H}_4$	$\tilde{H}_3$
6	$\tilde{H}_3 > \tilde{H}_2 > \tilde{H}_1 > \tilde{H}_5 > \tilde{H}_4$	$\tilde{H}_3$
8	$\tilde{H}_3 > \tilde{H}_2 > \tilde{H}_1 > \tilde{H}_5 > \tilde{H}_4$	$\tilde{H}_3$
10	$\tilde{H}_3 > \tilde{H}_2 > \tilde{H}_1 > \tilde{H}_5 > \tilde{H}_4$	$\tilde{H}_3$
12	$\tilde{H}_3 > \tilde{H}_2 > \tilde{H}_1 > \tilde{H}_5 > \tilde{H}_4$	$\tilde{H}_3$

According to the data in Table 10, the most preferable decision is $\tilde{H}_3$, called the Genetic Information for both operators for different values of parameters, anyhow, the proposed model is stable for all possible values of parameters, and the best value is $\tilde{H}_3$ for all values of the parameter. The simple representation of the data in Table 8 is available in the form of Figure 4.

form of data in Table 8..png form of data in Table 8.bb

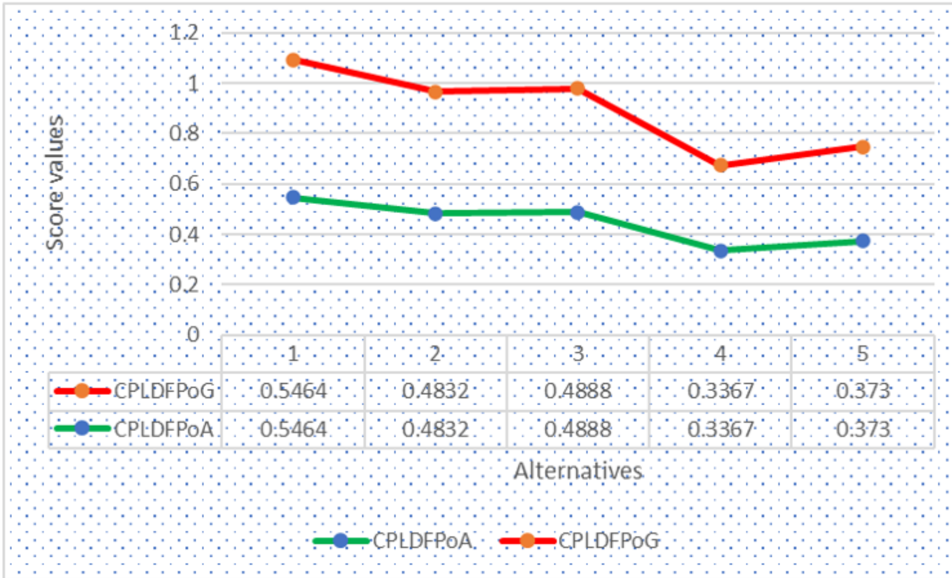

Figure 4: Graphical form of data in Table 8.

Additionally, we will compare the proposed ranking data with the ranking information of various existing techniques to discuss the efficiency of the invented theory.

7 Comparative Analysis

In this section, we scrutinize and deliberate the supremacy and validity of the designed technique and models by comparing their ranking values with the ranking values of various models. For this, we goal to collect various necessary techniques based on fuzzy models and their extensions, then we will evaluate the data in Table 1 with the help of considered information, such as Pamucar and Cirovic [21] invented the (multi-attributive border approximation area comparison) MABAC technique for classical set theory. Further, Yager [22] evaluated the power averaging (PoA) technique. In 2009, Xu and Yager [23] introduced the power geometric (PoG) technique for classical set theory. Jiang et al. [24] derived the power operators for IFSs. Wei and Lu [25] examined the power operators for PFSs. Garg et al. [26] initiated the power operators for Cq-ROFSs. Liu et al. [27] derived the power Dombi operators for CPFSs. Rani and Garg [28] evaluated the power operators for CIFSs. Ali [29] presented the power interaction operator for CIFSs. Ali et al. [30] described the power operators for complex intuitionistic fuzzy soft sets. Thus, the final ranking values are illustrated in Table 11.

Table 11: CPLDF comparative model.

Methods	**Score values**	**Ranking values**
Pamucar and Cirovic [21]	0.0,0.0,0.0,0.0,0.0	No
Yager [22]	0.0,0.0,0.0,0.0,0.0	No
Xu and Yager [23]	0.0,0.0,0.0,0.0,0.0	No
Jiang et al. [24]	0.0,0.0,0.0,0.0,0.0	No
Wei and Lu [25]	0.0,0.0,0.0,0.0,0.0	No
Garg et al. [26]	0.0,0.0,0.0,0.0,0.0	No
Liu et al. [27]	0.0,0.0,0.0,0.0,0.0	No
Rani and Garg [28]	0.0,0.0,0.0,0.0,0.0	No
Ali [29]	0.0,0.0,0.0,0.0,0.0	No
Ali et al. [30]	0.0,0.0,0.0,0.0,0.0	No
CPLDFPoA-MABAC	0.0681,0.0765,0.0849,0.0911,0.0815	$\tilde{H}_4 > \tilde{H}_3 > \tilde{H}_5 > \tilde{H}_2 > \tilde{H}_1$
CPLDFPoG-MABAC	0.0654,0.0757,0.0826,0.0896,0.0823	$\tilde{H}_4 > \tilde{H}_3 > \tilde{H}_5 > \tilde{H}_2 > \tilde{H}_1$
CPLDFPoA	0.5464,0.4832,0.4888,0.3367,0.373	$\tilde{H}_3 > \tilde{H}_2 > \tilde{H}_1 > \tilde{H}_5 > \tilde{H}_4$
CPLDFPoG	0.5464,0.4832,0.4888,0.3367,0.373	$\tilde{H}_3 > \tilde{H}_2 > \tilde{H}_1 > \tilde{H}_5 > \tilde{H}_4$

According to the data in Table 6, the most preferable decision is $\tilde{H}_4$, called the Biomarkers for the MABAC model based on both operators. But, according to the data in Table 11, the most preferable decision is $\tilde{H}_3$, called the Genetic Information for both operators. In addition, the limitation of the existing models is described in Table 12.

Table 12: CPLDF theoretical comparison.

Methods	Truth value	Falsity value	Crisp function	Parameters for both function	Aggregation operators	Techniques/methods	Strong condition/not failed	Periodic function
Pamucar and Cirovic [21]	no	no	yes	no	Yes	yes	no	no
Yager [22]	no	no	yes	no	Yes	no	no	no
Xu and Yager [23]	yes	yes	yes	no	Yes	no	no	no
Jiang et al. [24]	yes	yes	yes	no	Yes	no	no	no
Wei and Lu [25]	yes	yes	yes	no	yes	no	no	no
Garg et al. [26]	yes	yes	yes	no	yes	no	no	yes
Liu et al. [27]	yes	yes	yes	no	yes	no	no	yes
Rani and Garg [28]	yes	yes	yes	no	yes	no	Yes	yes
Ali [29]	yes	yes	yes	no	yes	no	no	yes
Ali et al. [30]	yes	yes	yes	no	yes	no	no	yes
Proposed models	yes	yes	yes	Yes	yes	yes	Yes	yes

Finally, from the information in Table 12, we analyze that the existing techniques and models contain various limitations because of their features. Every point of view, we have discussed in Table 12, and from the data in Table 12, and Table 11, we concluded that the existing models are the special cases of the proposed theory. Hence, the designed techniques are more powerful and more reliable compared to existing models.

8 Conclusion

The complex propositional linear Diophantine fuzzy technique is a very powerful model for handling vague and uncertain data. The technique of complex propositional linear Diophantine fuzzy sets is the combination of numerous valuable ideas, where the key and major contributions of the designed techniques are followed, such as designing the procedure of a MABAC deep learning algorithm for the diagnosis of Alzheimers Disease. Further, we design the model of CPLDF information with their basic operational laws. In addition, we analyze the model of the CPLDFPoA operator, CPLDFWPoA operator, CPLDFPoG operator, and CPLDFWPoG operator, and also initiate their major properties. Moreover, we arrange relevant from different sources for diagnosing Alzheimers disease under the consideration of the designed technique. Lastly, we compare both (proposed and existing) ranking information to address the supremacy and strength of the designed models.

In the future, we will begin the model of complex propositional (p, q) Diophantine fuzzy sets with some new extensions. In addition, we will evaluate the model of operator, measures, and methods for designed models and discuss their application in decision-making, artificial intelligence, and data mining to improve the worth of fuzzy set theory.

Conflict of interest: About the publication of this manuscript, the authors declare that they have no conflict of interest.

Funding: No external funding has been received for this submission.

Ethical approval: The authors state that this is their original work and it is neither submitted nor under consideration in any other journal simultaneously.

Human and animal participants: This article does not contain any studies with human participants or animals performed by any of the authors.

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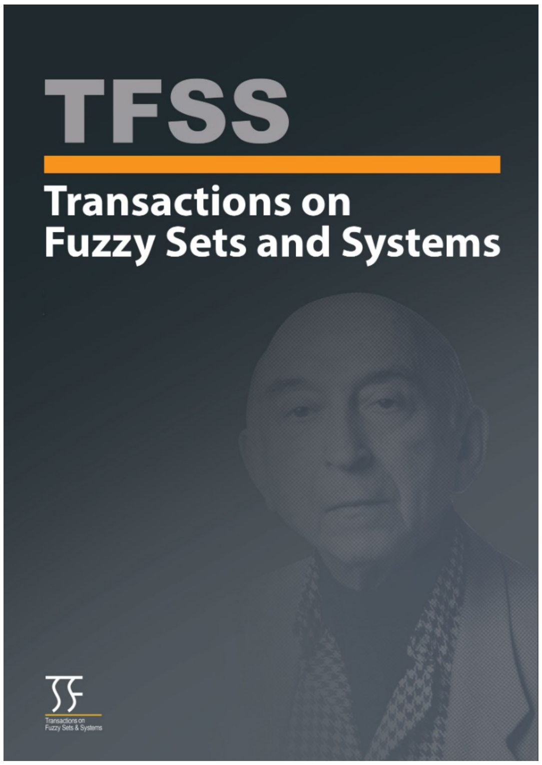

Transactions on Fuzzy Sets and Systems

ISSN: **2821-0131**

https://sanad.iau.ir/journal/tfss/

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Vol.4, No.1, (2025), 159-180. DOI: https://doi.org/10.71602/tfss.2025.1127340

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Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://sanad.iau.ir/journal/tfss/
Online ISSN: 2821-0131
Vol.4, No.1, (2025), 159-180
DOI: https://doi.org/10.71602/tfss.2025.1127340

Article Type: Original Research Article

A Novel Generalization of Hesitant Fuzzy Model with Application in Sustainable Supply Chain Optimization

Wajid Ali*, Tanzeela Shaheen, Iftikhar Ul Haq, Mohammad Mehedi Hassan

(This paper is dedicated to Professor "John N. Mordeson" on the occasion of his 91st birthday.)

Abstract. The n,m-rung orthopair fuzzy set theory is a robust model for managing uncertainty, particularly in multi-attribute decision-making. Meanwhile, the hesitant fuzzy model is a well-established tool in decision-making processes. Recognizing the similarities between these models, we propose a new framework called "c,d-rung orthopair hesitant fuzzy sets," which integrates both approaches. We examine key operations such as union, intersection, complement, subset, and equality, and introduce aggregation operators like the c,d-RHFPA, c,d-RHFWA, c,d-RHFPG, and c,d-RHFWPG operators. Additionally, an algorithm for multi-attribute decision-making is developed, which is applied to determine optimal business strategies for sustainable supply chain management. A comparative analysis with existing methods demonstrates the model's effectiveness, offering insights into its strengths and limitations. This paper introduces a novel approach to decision-making, outlining its real-world application and future research directions.

AMS Subject Classification 2020: 03E72; 94D05
Keywords and Phrases: Hesitant fuzzy sets, c,d-rung orthopair fuzzy sets, Decision making, MADM, Sustainable supply chain.

1 Introduction

1.1 Sustainable supply chain (SSC)

A sustainable supply chain, often referred to as an environmentally friendly or eco-conscious supply chain, is a business strategy that significantly focuses on incorporating environmentally and socially responsible practices throughout all phases of the supply chain process. This model has gained prominence in recent years due to growing concerns about climate change, natural resource depletion, social responsibility, and the need to mitigate the environmental and social impacts of business operations. Various re-searchers have explored this field from different perspectives. In 2015, Eskandarpour et al. [1] developed a supply chain network, and [2] outlined multi-objective optimization for sustainable supply chain. Linton et al. [3], in 2007, introduced the standard model for a sustainable supply chain, as well as decision models for its design and management in [4]. Resat et al. [5] developed an innovative model for multi-objective optimization approaches to sustainable supply chain management. Zhao et al. [6] applied a supply chain optimization model to continuous process industries with sustainability considerations. Eskandari [7] formulated and optimized a sustainable supply chain network for a blood platelet bank under conditions of uncertainty. Zhang et al. [8] introduced a novel

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Received: 26 July 2024; **Revised**: 22 September 2024; **Accepted**: 25 September 2024; **Available Online**: 16 January 2025; **Published Online**: 7 May 2025.

How to cite: Ali W, Shaheen T, Haq IUl, Hassan MM. A novel generalization of hesitant fuzzy model with application in sustainable supply chain optimization. *Transactions on Fuzzy Sets and Systems.* 2025; 4(1): 159-180. DOI: https://doi.org/10.71602/tfss.2025.1127340

multi-objective optimization model for sustainable supply chain network design, considering multiple distribution channels. Yu et al. [9] proposed two distinct frameworks for managing supply chain uncertainty by integrating a fuzzy structure with supply chain network optimization. Kazancoglu et al. [10] focused on leveraging emerging technologies to enhance the sustainability and resilience of supply chains in a fuzzy environment, particularly in the context of the COVID-19 pandemic. Goodarzian et al. [11] defined a new bi-objective green medicine supply chain network design under a fuzzy environment.

Liu et al. [12] demonstrated the application of a supply chain system in agriculture with a Crop Harvest Time Prediction Model for Better Sustainability, Integrating Feature Selection and Artificial Intelligence Methods. Yadav et al. [13] developed a sustainable supply chain model for multi-stage manufacturing with partial backlogging under a fuzzy environment, considering the effect of learning in the screening process. In 2020, Poujavad [14] designed a hybrid model for analyzing the risks of green supply chains in a fuzzy environment. Alsaed et al. [15] established a sustainable green A Novel Generalization of Hesitant Fuzzy Model with Application in Sustainable Supply Chain Optimization [16, 17] worked on a green Supply Chain Member Selection Method Considering Green Innovation Capability in a Hesitant Fuzzy Environment. Mistarihi et al. [18] developed a Strategic Framework for Disruption Management under a Fuzzy Environment. Liu [19] utilized q-rung interval-valued orthopair fuzzy data in a large-scale green supplier selection approach. Chang et al. [20] introduced a fuzzy optimization model for decision-making in supply chain management. Rehman et al. [21] constructed the application of a supply chain model in enhancing healthcare supply chain resilience by fuzzy decision-making.

1.2 Fuzzy Sets and their Generalizations

In 1965, Zadeh introduced the concept of Fuzzy sets [22] as a tool to address uncertainty. Fuzzy sets are ordered pairs where elements from a universal set are assigned member-ship values ranging from 0 to 1. Dubois et al. [23] authored a book on the fundamentals of fuzzy sets, discussing applications in detail. Attansove et al. [24] designed an extension of fuzzy sets called intuitionistic fuzzy sets, and Fermatean fuzzy sets were introduced by Senapati et al. [25]. Picture fuzzy sets and generalized orthopair fuzzy sets were introduced in [26, 27]. Torra [28] extended the FS model into a Hesitant fuzzy structure and discussed the generalized membership grade. Numerous researchers have contributed to the field of fuzzy sets and its generalizations [29-35]. Recently, Shahzadi et al. [36] introduced the latest extension of q-rung orthopair fuzzy sets, known as p,q-rung orthopair fuzzy sets, applied in multi-criteria decision-making. Ibrahim et al. [37] defined a topological approach for n, m-Rung orthopair fuzzy sets with applications to the diagnosis of learning disabilities. Continuously, Ibrahim et al. [38] combined two fuzzy frame-works—bipolar fuzzy sets and n,m-rung orthopair fuzzy sets—and defined an approach for multi-attribute group decision-making based on bipolar n, m-rung orthopair fuzzy sets. Furthermore, Ibrahim et al. [39] worked on an innovative method for group decision-making using n, m-rung orthopair fuzzy soft expert set knowledge. Mahmood at al [40] combined intuitionistic fuzzy sets and hesitant fuzzy sets and called intuitionistic hesitant fuzzy sets with their application in decision-making. Qahtan et al. [41] used Pythagorean hesitant fuzzy sets for supply chain systems and multiple-attribute decision-making in [42]. Krisci et al. [43] developed Fermatean hesitant fuzzy sets with medical decision-making applications. Liu et al. [44] constructed q-rung hesitant fuzzy sets and their application in multi-criteria decision-making. Sarwar et al. [45] established a decision-making model for failure modes and effects analysis based on rough fuzzy integrated clouds. Punnam et al. [46] explored a Linear Diophantine Fuzzy Soft Set-Based Decision-Making Approach Using a Revised Max-Min Average Composition Method. Recently, some novel extensions and generalizations of fuzzy models have been developed with their applications [47-49]. Aggregation operators play
a crucial role in information calculation, leading to the development of several aggregation operators in literature. Yager [50] introduced power average operators in 2001, and Xu et al. [51] developed pow-er geometric operators and their application in group decision-making. Yager and Ronald [52] designed generalized OWA aggregation operators. Dhankhar et al. [53] discussed multi-attribute decision-making based on the q-rung orthopair fuzzy Yager power weighted geometric aggregation operator of q-rung orthopair fuzzy values. Ali et al. [54] developed an Innovative Decision Model Utilizing Intuitionistic Hesitant Fuzzy Aczel-Alsina Aggregation Operators and Its Application. Haq et al. [55] designed a novel Fermatean Fuzzy Aczel–Alsina Model for Investment Strategy Selection.

1.3 Motivation and Contribution

In our comprehensive literature review, we have identified key areas of interest and gaps in current research. This review underscores the significance of fuzzy sets and their extensions, as well as highlighting the burgeoning field of 'Sustainable Supply Chain with Multi-Objective Decision-Making' and its applications. Addressing a notable gap in existing literature, our research primarily focuses on bridging the disconnect between the advanced "n,m-rung orthopair fuzzy model" and its application in sustainable supply chain systems. We have recognized the absence of methodologies based on "n,m-rung orthopair hesitant fuzzy sets" that facilitate multiple-attribute decision-making within sustainable supply chain contexts. In this paper, we have made several groundbreaking contributions to address these existing gaps as follows:

i. We successfully designed and developed an innovative combination of n,m-rung orthopair fuzzy sets and hesitant fuzzy sets, which we have named c,d-rung orthopair hesitant fuzzy sets. This pioneering work utilizes the synergies between the n,m and c,d models to propel the field forward.

ii. We have introduced and validated a comprehensive series of theorems and properties specific to our proposed model. This effort has significantly strengthened the theoretical underpinnings of our research.

iii. We have completed the development of an extensive series of aggregation operators for the c,d-rung orthopair hesitant fuzzy sets. This series includes the c,d-rung orthopair hesitant fuzzy power averaging (c,d-RHFPA) operator, the c,d-rung orthopair hesitant fuzzy power weighted averaging (c,d-RHFPWA) operator, the c,d-rung orthopair hesitant fuzzy power geometric (c,d-RHFPG) operator, and the c,d-rung orthopair hesitant fuzzy power weighted geometric (c,d-RHFWPG) operator.

iv. We have established a detailed algorithm for multiple criteria decision-making using c,d-RHF information. This robust framework is tailored for navigating com-plex decision processes efficiently.

v. Our research has successfully applied the developed multiple criteria decision-making model to identify optimal strategies for maintaining sustainable supply chain systems under c,d-rung orthopair hesitant fuzzy information.

vi. We conducted a thorough comparison of our model with existing techniques, demonstrating its consistency and superiority in the field.

vii. Finally, we have clearly articulated the benefits and advantages of our proposed model, emphasizing its significant impact and practical applications in the realm of sustainable supply chain management.

The article is structured as follows: Section 2 introduces the fundamental concepts relevant to our proposed approach. Section 3 develops the novel concept of "c,d-rung orthopair hesitant fuzzy Sets," including their operations and properties. Section 4 details the creation of aggregation operators for c,d-rung orthopair hesitant fuzzy sets, along with essential results and proofs. Section 5 elucidates the MCDM algorithm using c,d-rung orthopair hesitant fuzzy power averaging and geometric operators. Section 6 applies these concepts to a sustainable supply chain (SSC) model, providing a comprehensive exploration of MCDM. Section 7 presents a comparative analysis with existing techniques, highlighting the strengths and limitations of our approach. Finally, Section 8 concludes the paper and outlines future research directions. Figure 1 illustrates the manuscript's workflow.

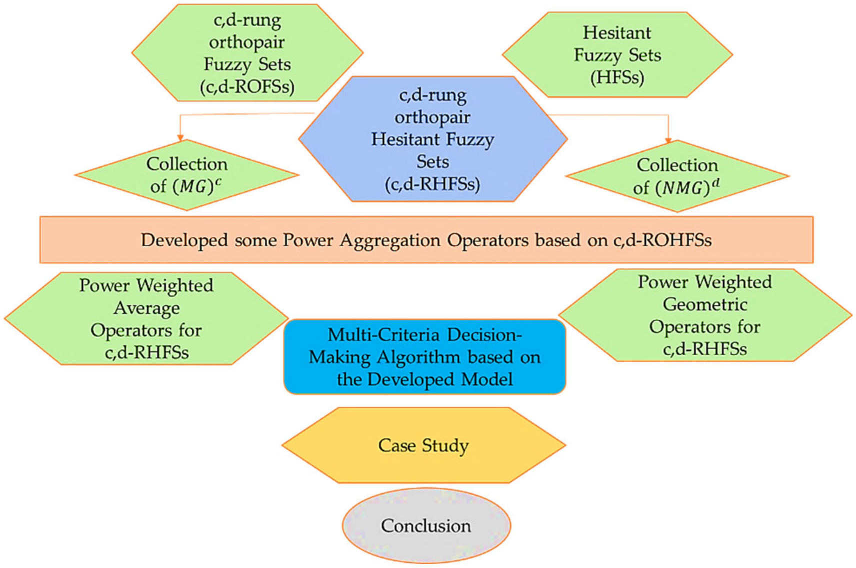

Figure 1: Flow chart of the sequence of research

2 Preliminaries

In this section, we recall the basic concepts such as intuitionistic hesitant fuzzy sets, Pythagorean hesitant fuzzy sets, Q-ROHFSs and c,d-rung rung orthopair fuzzy sets. Table 1 shows the symbols, and their descriptions used in the article.

Definition 2.1. let ∂ be a universal set. Then, $L = \{u, \pi_L(u), \psi_L(u): u \in \partial\}$ is called.

1. an intuitionistic hesitant fuzzy set (IHFS) [40] if $0 \leq \max(\pi_L(u)) + \max(\psi_L(u)) \leq 1$ where $\pi_L(u)$ and $\psi_L(u)$ is a collection of distinct elements from [0,1]

2. a Pythagorean hesitant fuzzy set (PHFS) [42] if $0 \leq \max(\pi_L(u))^2 + \max(\psi_L(u))^2 \leq 1$ where $\pi_L(u)$ and $\psi_L(u)$ is a collection of distinct elements from [0,1]

3. a Fermatean hesitant fuzzy set (FHFS) [43] if $0 \leq \max(\pi_L(u))^3 + \max(\psi_L(u))^3 \leq 1$ where $\pi_L(u)$ and $\psi_L(u)$ is a collection of distinct elements from [0,1].

4. a Q-ROFS [44] if $0 \leq \max(\pi_L(u))^q + \max(\psi_L(u))^q \leq 1$, for $q \geq 1$. where $\pi_L(u)$ and $\psi_L(u)$ is a collection of distinct elements from [0,1].

Where $\pi_L(u), \psi_L(u): \partial \to [0,1]$ are MG and NMG, respectively.

Definition 2.2. [38] let ∂ be a universal set. Then, $L = \{u, \pi_L(u), \psi_L(u): u \in \partial\}$ is called a c,d-rung orthopair fuzzy set if $0 \leq (\pi_L(u))^c + (\psi_L(u))^d \leq 1$ such that $c, d \in N$. The degree of indeterminacy for $u \in \partial$ to L is given as,

$$\gamma_L(u) = \sqrt[c+d]{1 - \left[(\pi_L(u))^c + (\psi_L(u))^d\right]}, \quad \gamma_L(u) \in [0,1]$$

Definition 2.3. For a c,d-rung orthopair fuzzy set $L = (\pi_L(u), \psi_L(u))$, The score (SF) and accuracy functions (AF) are defined as

$$S(L) = (\pi_L(u))^c - (\psi_L(u))^d, \quad A(L) = (\pi_L(u))^c + (\psi_L(u))^d$$

Wherever, $S(L) \in [-1,1]$ and $A(L) \in [0,1]$.

3 An idea of c,d-rung orthopair hesitant fuzzy sets (c,d RHFSs)

The concept of the novel model called c,d-RHFSs with their basic properties and operators are briefly discussed in this section.

Table 1: Symbols and their descriptions

Symbols	Descriptions	Symbols	Descriptions
Q-ROFSS	q-rung orthopair fuzzy sets	c,d RHFSs	c,d-rung hesitant fuzzy sets
MG	Membership grade	c,d RHFPA	c,d-rung hesitant fuzzy power average
NMG	Non-membership grade	c,d RHFPG	c,d-rung hesitant fuzzy power geometric
SSC	Sustainable supply chain	MCDM	Multi criteria decision making
SF	Score function	AF	Accuracy function

Definition 3.1. let ∂ be a universal set. A c,d rung orthopair hesitant fuzzy sets(c,d-RHFSs) L in ∂ is stated as, $L = \{u, \pi_L(u), \psi_L(u) : u \in \partial\}$ where $\pi_L(u)$ and $\psi_L(u)$ is a set of elements from the [0,1] and $0 \leq max(\pi_L(u))^c + max(\psi_L(u))^d \leq 1$ such that $c, d \in N$. The degree of indeterminacy for $u \in \partial$ to L is given as,

$$\gamma_L(u) = \bigcup_{\substack{e \in \pi_L(u) \\ f \in \psi_L(u)}} \sqrt[c+d]{1 - \left[\left(e_L(u)\right)^c + \left(f_L(u)\right)^d\right]}$$

and $\gamma_L(u) \in [0,1]$

Throughout the paper, for our easiness a c,d-RHFS is represented as $L = (\pi_L, \psi_L)$.

Remark 3.2. If c=d for a c,d-RHFS $L = (\pi_L, \psi_L)$, then we call $L = (\pi_L, \psi_L)$ is a Q-RHFS where q=c=d.

Definition 3.3. let $L = (\pi_L, \psi_L)$, $L_1 = \left(\pi_{L_1}, \psi_{L_1}\right)$ and $L_2 = \left(\pi_{L_2}, \psi_{L_2}\right)$ be three c,d-rung orthopair hesitant fuzzy sets then,

1. $L_1 \wedge L_2$

$$= \bigcup_{\substack{e_1 \in \pi_{L_1} \\ e_2 \in \pi_{L_2} \\ f_1 \in \psi_{L_1} \\ f_2 \in \psi_{L_2}}} (\min\{e_1, e_2\}, \max\{f_1, f_2\})$$

2. $L_1 \vee L_2$

$$= \bigcup_{\substack{e_1 \in \pi_{L_1} \\ e_2 \in \pi_{L_2} \\ f_1 \in \psi_{L_1} \\ f_2 \in \psi_{L_2}}} (\max\{e_1, e_2\}, \min\{f_1, f_2\})$$

3. L'

$$= \bigcup_{\substack{e \in \pi_L \\ f \in \psi_L}} \left(f^{\frac{d}{c}}, e^{\frac{c}{d}}\right)$$

Theorem 3.4. If $L = (\pi_L, \psi_L)$ is a c,d-rung orthopair hesitant fuzzy sets then L' is also (c,d-RHFSs) and $(L')' = L$.

Proof. Let $0 \leq \pi_L^c + \psi_L^d \leq 1$, then.

$0 \leq \left(f^{\frac{d}{c}}\right)^c + \left(e^{\frac{c}{d}}\right)^d = (e)^c + (\boldsymbol{f})^d \leq 1$ where $e \in \pi_L$, $f \in \psi_L$. Thus, L' is also (c,d-RHFS) and it is obvious

$$(L')' = \bigcup_{\substack{e \in \pi_L \\ f \in \psi_L}} \left(\left(f^{\frac{d}{c}}, e^{\frac{c}{d}}\right)\right)' = \bigcup_{\substack{e \in \pi_L \\ f \in \psi_L}} \left(\left(e^{\frac{c}{d}}\right)^{\frac{d}{c}}, \left(f^{\frac{d}{c}}\right)^{\frac{c}{d}}\right)$$

Which is again L. □

Remark 3.5. If $L_1 = (\pi_{L_1}, \psi_{L_1})$ and $L_2 = (\pi_{L_2}, \psi_{L_2})$are two c,d-rung orthopair hesitant fuzzy sets then $L_1 \wedge L_2$ and $L_1 \vee L_2$ are also (c,d-RHFSs).

Theorem 3.6. let $L_1 = (\pi_{L_1}, \psi_{L_1})$ and $L_2 = (\pi_{L_2}, \psi_{L_2})$ be two c,d-rung orthopair hesitant fuzzy sets then,

1. $L_1 \wedge L_2 = L_2 \wedge L_1$
2. $L_1 \vee L_2 = L_2 \vee L_1$
3. $(L_1 \wedge L_2) \vee L_2 = L_2$
4. $(L_1 \vee L_2) \wedge L_2 = L_2$

Proof. From Definition 3.3, we have:

1. $L_1 \wedge L_2$

$$= \bigcup_{\substack{e_1 \in \pi_{L_1} \\ e_2 \in \pi_{L_2} \\ f_1 \in \psi_{L_1} \\ f_2 \in \psi_{L_2}}} (\min\{e_1, e_2\}, \max\{f_1, f_2\}) = \bigcup_{\substack{e_1 \in \pi_{L_1} \\ e_2 \in \pi_{L_2} \\ f_1 \in \psi_{L_1} \\ f_2 \in \psi_{L_2}}} (\min\{e_2, e_1\}, \max\{f_2, f_1\}) = L_2 \wedge L_1$$

2. $L_1 \vee L_2$

$$= \bigcup_{\substack{e_1 \in \pi_{L_1} \\ e_2 \in \pi_{L_2} \\ f_1 \in \psi_{L_1} \\ f_2 \in \psi_{L_2}}} (\max\{e_1, e_2\}, \min\{f_1, f_2\}) = \bigcup_{\substack{e_1 \in \pi_{L_1} \\ e_2 \in \pi_{L_2} \\ f_1 \in \psi_{L_1} \\ f_2 \in \psi_{L_2}}} (\max\{e_2, e_1\}, \min\{f_2, f_1\}) = L_2 \vee L_1$$

3. $(L_1 \wedge L_2) \vee L_2$

$$= \bigcup_{\substack{e_1 \in \pi_{L_1} \\ e_2 \in \pi_{L_2} \\ f_1 \in \psi_{L_1} \\ f_2 \in \psi_{L_2}}} (\min\{e_1, e_2\}, \max\{f_1, f_2\}) \vee \left(\bigcup_{\substack{e_2 \in \pi_{L_2} \\ f_2 \in \psi_{L_2}}} (e_2, f_2) \right) = \bigcup_{\substack{e_1 \in \pi_{L_1} \\ e_2 \in \pi_{L_2} \\ f_1 \in \psi_{L_1} \\ f_2 \in \psi_{L_2}}} \left(\max(\min\{e_2, e_1\}, e_2), \min\left(\max\{f_2, f_1\}, f_2\right)\right) = L_2$$

4. $(L_1 \vee L_2) \wedge L_2$

$$= \bigcup_{\substack{e_1 \in \pi_{L_1} \\ e_2 \in \pi_{L_2} \\ f_1 \in \psi_{L_1} \\ f_2 \in \psi_{L_2}}} (\max\{e_1, e_2\}, \min\{f_1, f_2\}) \wedge \left(\bigcup_{\substack{e_2 \in \pi_{L_2} \\ f_2 \in \psi_{L_2}}} (e_2, f_2) \right) = \bigcup_{\substack{e_1 \in \pi_{L_1} \\ e_2 \in \pi_{L_2} \\ f_1 \in \psi_{L_1} \\ f_2 \in \psi_{L_2}}} (\min(\max\{e_2, e_1\}, e_2), \max(\min\{f_2, f_1\}, f_2)) = L_2$$

□

Theorem 3.7. let $L_1 = (\pi_{L_1}, \psi_{L_1})$ and $L_2 = (\pi_{L_2}, \psi_{L_2})$ be two c,d-rung orthopair hesitant fuzzy sets then,

1. $(L_1 \wedge L_2)' = L_1' \vee L_2'$
2. $(L_1 \vee L_2)' = L_1' \wedge L_2'$

Proof. For the c,d-RHFSs L_1 and L_2, we have:

1. $(L_1 \wedge L_2)'$

$$= \bigcup_{\substack{e_1 \in \pi_{L_1} \\ e_2 \in \pi_{L_2} \\ f_1 \in \psi_{L_1} \\ f_2 \in \psi_{L_2}}} (\min\{e_1, e_2\}, \max\{f_1, f_2\})' = \bigcup_{\substack{e_1 \in \pi_{L_1} \\ e_2 \in \pi_{L_2} \\ f_1 \in \psi_{L_1} \\ f_2 \in \psi_{L_2}}} \left(\max\left\{e_1^{\frac{d}{c}}, e_2^{\frac{d}{c}}\right\}, \min\left\{f_1^{\frac{c}{d}}, f_2^{\frac{c}{d}}\right\}\right) = \bigcup_{\substack{e_1 \in \pi_{L_1} \\ f_1 \in \psi_{L_1}}} \left(e_1^{\frac{d}{c}}, f_1^{\frac{c}{d}}\right) \vee \bigcup_{\substack{e_2 \in \pi_{L_2} \\ f_2 \in \psi_{L_2}}} \left(e_2^{\frac{d}{c}}, f_2^{\frac{c}{d}}\right)$$

$$= L_1' \vee L_2'$$

2. Similar to (1). □

Definition 3.8. let $L = (\pi_L, \psi_L)$, $L_1 = (\pi_{L_1}, \psi_{L_1})$ and $L_2 = (\pi_{L_2}, \psi_{L_2})$ be three c,d-rung orthopair hesitant fuzzy sets, and Δ is a positive real number ($\Delta > 0$),then

1. $L_1 \oplus L_2$

$$= \bigcup_{\substack{e_1 \in \pi_{L_1} \\ e_2 \in \pi_{L_2} \\ f_1 \in \psi_{L_1} \\ f_2 \in \psi_{L_2}}} \left(\sqrt[c]{e_1^c + e_2^c - e_1^c e_2^c}, f_1 f_2\right)$$

2. $L_1 \otimes L_2$

$$= \bigcup_{\substack{e_1 \in \pi_{L_1} \\ e_2 \in \pi_{L_2} \\ f_1 \in \psi_{L_1} \\ f_2 \in \psi_{L_2}}} \left(e_1 e_2, \sqrt[d]{f_1^d + f_2^d - f_1^d f_2^d}\right)$$

3. ΔL

$$= \bigcup_{\substack{e \in \pi_L \\ f \in \psi_L}} \left(\sqrt[c]{1 - (1 - e^c)^{\Delta}}, (f)^{\Delta}\right)$$

4. L^{Δ}

$$= \bigcup_{\substack{e \in \pi_L \\ f \in \psi_L}} \left(e^{\Delta}, \sqrt[d]{1 - (1 - f^d)^{\Delta}}\right)$$

Theorem 3.9. let $L_1 = (\pi_{L_1}, \psi_{L_1})$, $L_2 = (\pi_{L_2}, \psi_{L_2})$ and $L_3 = (\pi_{L_3}, \psi_{L_3})$ be three c,d-RHFSs, and Δ is a positive real number ($\Delta > 0$),then

1. $L_1 \oplus L_2 = L_2 \oplus L_1$
2. $L_1 \otimes L_2 = L_2 \otimes L_1$
3. $L_1 \oplus L_2 \oplus L_3 = L_1 \oplus L_3 \oplus L_2$
4. $L_1 \otimes L_2 \otimes L_3 = L_1 \otimes L_3 \otimes L_2$

Proof. From Definition 3.8 we have:

1. $L_1 \oplus L_2$

$$= \bigcup_{\substack{e_1 \in \pi_{L_1} \\ e_2 \in \pi_{L_2} \\ f_1 \in \psi_{L_1} \\ f_2 \in \psi_{L_2}}} \left(\sqrt[c]{e_1^c + e_2^c - e_1^c e_2^c}, f_1 f_2 \right) = \bigcup_{\substack{e_1 \in \pi_{L_1} \\ e_2 \in \pi_{L_2} \\ f_1 \in \psi_{L_1} \\ f_2 \in \psi_{L_2}}} \left(\sqrt[c]{e_2^c + e_1^c - e_2^c e_1^c}, f_2 f_1 \right) = L_2 \oplus L_1$$

2. $L_1 \otimes L_2$

$$= \bigcup_{\substack{e_1 \in \pi_{L_1} \\ e_2 \in \pi_{L_2} \\ f_1 \in \psi_{L_1} \\ f_2 \in \psi_{L_2}}} \left(e_1 e_2, \sqrt[d]{f_1^d + f_2^d - f_1^d f_2^d} \right) = \bigcup_{\substack{e_1 \in \pi_{L_1} \\ e_2 \in \pi_{L_2} \\ f_1 \in \psi_{L_1} \\ f_2 \in \psi_{L_2}}} \left(e_2 e_1, \sqrt[d]{f_2^d + f_1^d - f_2^d f_1^d} \right) = L_2 \otimes L_1$$

3. It is similar to 1.

4. It is similar to 2. □

Theorem 3.10. let $L = (\pi_L, \psi_L)$, $L_1 = (\pi_{L_1}, \psi_{L_1})$ and $L_2 = (\pi_{L_2}, \psi_{L_2})$ be three c,d-rung orthopair hesitant fuzzy sets, and Δ is a positive real number $(\Delta > 0)$,then

1. $(L_1 \oplus L_2)' = L_1' \otimes L_2'$
2. $(L_1 \otimes L_2)' = L_1' \oplus L_2'$
3. $(L')^{\Delta} = (\Delta L)'$
4. $\Delta(L)' = (L^{\Delta})'$

Proof. For the c,d-RHFSs L, L_1 and L_3, we have

1. $(L_1 \oplus L_2)'$

$$= \bigcup_{\substack{e_1 \in \pi_{L_1} \\ e_2 \in \pi_{L_2} \\ f_1 \in \psi_{L_1} \\ f_2 \in \psi_{L_2}}} \left(\sqrt[c]{e_1^c + e_2^c - e_1^c e_2^c}, f_1 f_2 \right)' = \bigcup_{\substack{e_1 \in \pi_{L_1} \\ e_2 \in \pi_{L_2} \\ f_1 \in \psi_{L_1} \\ f_2 \in \psi_{L_2}}} \left((f_1 f_2)^{\frac{d}{c}}, \left(\sqrt[c]{e_1^c + e_2^c - e_1^c e_2^c} \right)^{\frac{c}{d}} \right)$$

$$= \bigcup_{\substack{e_1 \in \pi_{L_1} \\ f_1 \in \psi_{L_1}}} \left(f_1^{\frac{d}{c}}, e_1^{\frac{c}{d}} \right) \otimes \bigcup_{\substack{e_2 \in \pi_{L_2} \\ f_2 \in \psi_{L_2}}} \left(f_2^{\frac{d}{c}}, e_2^{\frac{c}{d}} \right) = L_1' \otimes L_2'$$

2. It is similar to 1.

3. $(L')^{\Delta}$

$$= \bigcup_{\substack{e\in\pi_L\\ f\in\psi_L}} \left(f^{\frac{d}{c}}, e^{\frac{c}{d}}\right)^{\Delta} = \bigcup_{\substack{e\in\pi_L\\ f\in\psi_L}} \left(\left(f^{\frac{d}{c}}\right)^{\Delta}, \left(1-\left(1-\left(e^{\frac{c}{d}}\right)^{d}\right)^{\Delta}\right)^{\frac{1}{d}}\right) = \bigcup_{\substack{e\in\pi_L\\ f\in\psi_L}} \left((f^{\Delta})^{\frac{d}{c}}, \left(1-(1-(e^{c})^{\Delta})^{\frac{1}{d}}\right)^{\frac{c}{d}}\right)$$

$$= \bigcup_{\substack{e\in\pi_L\\ f\in\psi_L}} \left((1-(1-e^{c})^{\Delta})^{\frac{1}{c}} \ \ , f^{\Delta}\right) = (\Delta L)'$$

4. $\Delta(L)'$

$$= \Delta \bigcup_{\substack{e\in\pi_L\\ f\in\psi_L}} \left(f^{\frac{d}{c}}, e^{\frac{c}{d}}\right) = \bigcup_{\substack{e\in\pi_L\\ f\in\psi_L}} \left(\left((1-\left(1-\left(f^{\frac{d}{c}}\right)^{c}\right)^{\Delta}\right)^{\frac{1}{c}}, \left((e)^{\frac{c}{d}}\right)^{\Delta}\right)$$

$$= \bigcup_{\substack{e\in\pi_L\\ f\in\psi_L}} \left(e^{\Delta}, \left((1-\left(1-(f^{d})\right)\right)^{\Delta}\right)^{\frac{1}{d}}\right) = (L^{\Delta})' \qquad \square$$

4 c,d-rung orthopair hesitant fuzzy aggregation operators

Here, a series of average and geometric operators for c,d-RHFSs is briefly discussed. Moreover, their basic properties are also explained.

Definition 4.1. let $L_i = \left(\pi_{L_i}, \psi_{L_i}\right), (i = 1,2, \dots, k)$ be a set of c,d-RHFNs and $\tau = (\tau_i)^T$ be weight vector of L_i with $\tau_i > 0$ such that $\sum_{i=1}^{k} \tau_i = 1$ then, the

1. c,d-rung orthopair hesitant fuzzy weighted averaging (c,d-RHFWA) operator is a mapping $c, d - RHFWA: L^k \to L$ such that $c, d - RHFWA(L_1, L_2, \dots, L_k) = \bigoplus_{i=1}^{k} \tau_i L_i = \tau_1 L_1 \oplus \tau_2 L_2 \dots . \oplus \tau_k L_k$
2. c,d-rung orthopair hesitant fuzzy weighted geometric (c,d-RHFWG) operator is a mapping $c, d - RHFWG: L^k \to L$ such that $c, d - RHFWA(L_1, L_2, \dots, L_k) = \bigotimes_{i=1}^{k} L_i^{\tau_i} = L_1^{\tau_1} \otimes L_2^{\tau_2} \dots . \otimes L_k^{\tau_k}$.

Theorem 4.2. let $L_i = \left(\pi_{L_i}, \psi_{L_i}\right), (i = 1,2, \dots, k)$ be a set of c,d-RHFNs and $\tau = (\tau_i)^T$ be weight vector of L_i with $\tau_i > 0$ such that $\sum_{i=1}^{k} \tau_i = 1$ then,

1. The aggregation value of c,d-RHFNs $L_i(i = 1,2, \dots, k)$ by using $c, d - RHFWA$ operator is also c,d-RHFN. And

$$c, d - RHFWA(L_1, L_2, \dots, L_k) = \bigcup_{\substack{e_{L_i}\in\pi_{L_i}\\ f_{L_i}\in\psi_{L_i}}} \langle \left(1-\prod_{i=1}^{k}\left(1-\left(e_{L_i}\right)^{c}\right)^{\tau_i}\right)^{\frac{1}{c}}, \prod_{i=1}^{k}\left(f_{L_i}\right)^{\tau_i} \rangle$$

2. The aggregation value of c,d-RHFNs $L_i(i = 1,2, \dots, k)$ by using $c, d - RHFWG$ operator is also c,d-RHFN. And

$$c, d - RHFWG(L_1, L_2, \dots, L_k) = \bigcup_{\substack{e_{L_i}\in\pi_{L_i}\\ f_{L_i}\in\psi_{L_i}}} \langle \prod_{i=1}^{k}\left(e_{L_i}\right)^{\tau_i}, \left(1-\prod_{i=1}^{k}\left(1-\left(f_{L_i}\right)^{c}\right)^{\tau_i}\right)^{\frac{1}{c}} \rangle$$

Proof.

1. We can provide proof of the abovementioned results by using mathematical induction. Therefore, we follow as,

(i). For $i = 2$ since

$$\tau_1 L_1 = \bigcup_{\substack{e_{L_1}\in\pi_{L_1}\\ f_{L_1}\in\psi_{L_1}}} \langle \left(1-\left(1-\left(e_{L_1}\right)^{c}\right)^{\tau_1}\right)^{\frac{1}{c}}, \left(f_{L_1}\right)^{\tau_1} \rangle$$

and

$$\tau_2 L_2 = \bigcup_{\substack{e_{L_2}\in\pi_{L_2}\\ f_{L_2}\in\psi_{L_2}}} \langle\left(1-\left(1-\left(e_{L_2}\right)^c\right)^{\tau_2}\right)^{\frac{1}{c}}, \left(f_{L_2}\right)^{\tau_2}\rangle$$

then $c,d-RHFWA(L_1,L_2)=\tau_1L_1\oplus\tau_2L_2$=

$$\bigcup_{\substack{e_{L_1}\in\pi_{L_1}\\ f_{L_1}\in\psi_{L_1}}} \langle\left(1-\left(1-\left(e_{L_1}\right)^c\right)^{\tau_1}\right)^{\frac{1}{c}}, \left(f_{L_1}\right)^{\tau_1}\rangle \oplus \bigcup_{\substack{e_{L_2}\in\pi_{L_2}\\ f_{L_2}\in\psi_{L_2}}} \langle\left(1-\left(1-\left(e_{L_2}\right)^c\right)^{\tau_2}\right)^{\frac{1}{c}}, \left(f_{L_2}\right)^{\tau_2}\rangle$$

$$= \bigcup_{\substack{e_{L_1}\in\pi_{L_1}\\ e_{L_2}\in\pi_{L_2}\\ f_{L_1}\in\psi_{L_1}\\ f_{L_2}\in\psi_{L_2}}} \langle\left(1-\left(1-\left(e_{L_1}\right)^c\right)^{\tau_1}+1-\left(1-\left(e_{L_2}\right)^c\right)^{\tau_2}\right)^{\frac{1}{c}}, \left(f_{L_1}\right)^{\tau_1}\left(f_{L_2}\right)^{\tau_2}\rangle$$

$$= \bigcup_{\substack{e_{L_1}\in\pi_{L_1}\\ e_{L_2}\in\pi_{L_2}\\ f_{L_1}\in\psi_{L_1}\\ f_{L_2}\in\psi_{L_2}}} \langle\left(1-\left(1-\left(e_{L_1}\right)^c\right)^{\tau_1}\left(1-\left(e_{L_2}\right)^c\right)^{\tau_2}\right)^{\frac{1}{c}}, \left(f_{L_1}\right)^{\tau_1}\left(f_{L_2}\right)^{\tau_2}\rangle$$

$$= \bigcup_{\substack{e_{L_1}\in\pi_{L_1}\\ e_{L_2}\in\pi_{L_2}\\ f_{L_1}\in\psi_{L_1}\\ f_{L_2}\in\psi_{L_2}}} \langle\left(1-\left(1-\left(e_{L_1}\right)^c\right)^{\tau_1}\left(1-\left(e_{L_2}\right)^c\right)^{\tau_2}\right)^{\frac{1}{c}}, \left(f_{L_1}\right)^{\tau_1}\left(f_{L_2}\right)^{\tau_2}\rangle$$

$$= \bigcup_{\substack{e_{L_1}\in\pi_{L_1}\\ e_{L_2}\in\pi_{L_2}\\ f_{L_1}\in\psi_{L_1}\\ f_{L_2}\in\psi_{L_2}}} \langle\left(1-\prod_{i=1}^{2}\left(1-\left(e_{L_i}\right)^c\right)^{\tau_i}\right)^{\frac{1}{c}}, \prod_{i=1}^{2}\left(f_{L_i}\right)^{\tau_i}\rangle$$

(ii). Suppose that this result is satisfied for $i=r$ which is,

$$c,d-RHFWA(L_1,L_2,\dots,L_r)=\tau_1L_1\oplus\tau_2L_2\dots.\oplus\tau_rL_r=\bigcup_{\substack{e_{L_i}\in\pi_{L_i}\\ f_{L_i}\in\psi_{L_i}}} \langle\left(1-\prod_{i=1}^{r}\left(1-\left(e_{L_i}\right)^c\right)^{\tau_i}\right)^{\frac{1}{c}}, \prod_{i=1}^{r}\left(f_{L_i}\right)^{\tau_i}\rangle$$

Now, we will prove that the result is true for $i=r+1$ by using (i) and (ii) we have.

$$c,d-RHFWA(L_1,L_2,\dots,L_{r+1})=\tau_1L_1\oplus\tau_2L_2\dots.\oplus\tau_{r+1}L_{r+1}=$$

$$\bigcup_{\substack{e_{L_i}\in\pi_{L_i}\\ f_{L_i}\in\psi_{L_i}}} \langle\left(1-\prod_{i=1}^{r}\left(1-\left(e_{L_i}\right)^c\right)^{\tau_i}\right)^{\frac{1}{c}}, \prod_{i=1}^{r}\left(f_{L_i}\right)^{\tau_i}\rangle \oplus \bigcup_{\substack{e_{L_{r+1}}\in\pi_{L_{r+1}}\\ f_{L_{r+1}}\in\psi_{L_{r+1}}}} \langle\left(1-\left(1-\left(e_{L_{r+1}}\right)^c\right)^{\tau_{r+1}}\right)^{\frac{1}{c}}, \left(f_{L_{r+1}}\right)^{\tau_{r+1}}\rangle$$

$$= \bigcup_{\substack{e_{L_i}\in\pi_{L_i}\\ f_{L_i}\in\psi_{L_i}}} \langle\left(1-\prod_{i=1}^{r}\left(1-\left(e_{L_i}\right)^c\right)^{\tau_i}+1-\left(1-\left(e_{L_{r+1}}\right)^c\right)^{\tau_{r+1}}-\left(1-\prod_{i=1}^{r}\left(1-\left(e_{L_i}\right)^c\right)^{\tau_i}\right)(1$$

$$-\left(1-\left(e_{L_{r+1}}\right)^c\right)^{\tau_{r+1}}\Bigg)^{\frac{1}{c}}, \prod_{i=1}^{r}\left(f_{L_i}\right)^{\tau_i}\left(f_{L_{r+1}}\right)^{\tau_{r+1}}\rangle$$

$$= \bigcup_{\substack{e_{L_i}\in\pi_{L_i}\\ f_{L_i}\in\psi_{L_i}}} \langle\left(1-\prod_{i=1}^{r}\left(1-\left(e_{L_i}\right)^c\right)^{\tau_i}\left(1-\left(e_{L_{r+1}}\right)^c\right)^{\tau_{r+1}}\right)^{\frac{1}{c}}, \prod_{i=1}^{r}\left(f_{L_i}\right)^{\tau_i}\left(f_{L_{r+1}}\right)^{\tau_{r+1}}\rangle$$

$$= \bigcup_{\substack{e_{L_i}\in\pi_{L_i}\\ f_{L_i}\in\psi_{L_i}}} \langle \left(1-\prod_{i=1}^{r+1}\left(1-\left(e_{L_i}\right)^c\right)^{\tau_i}\right)^{\frac{1}{c}}, \prod_{i=1}^{r+1}\left(f_{L_i}\right)^{\tau_i} \rangle$$

The theorem is meeting for $i = r+1$. Thus, theorem is fulfilled for whole i.

2. The proof is same as part 1. □

Theorem 4.3. (Idempotence) let $L_i = \left(\pi_{L_i}, \psi_{L_i}\right), (i = 1,2,\dots,k)$ be a set of c,d-rung orthopair hesitant fuzzy numbers and $\tau = (\tau_i)^T$ be weight vector of L_i with $\tau_i > 0$ such that $\sum_{i=1}^{k}\tau_i = 1$.If all of $L_i = \left(\pi_{L_i}, \psi_{L_i}\right), (i = 1,2,\dots,k)$ are identical to $L = (\pi_L, \psi_L)$ then

1. $c,d-RHFWA(L_1, L_2, \dots, L_k) = L$

2. $c,d-RHFWG(L_1, L_2, \dots, L_k) = L$

Proof.

1. Since $L_i = L = (\pi_L, \psi_L)(i = 1,2,\dots.,k)$ then$c,d-RHFWG(L_1, L_2, \dots, L_k)$

$$= \bigcup_{\substack{e_{L_i}\in\pi_{L_i}\\ f_{L_i}\in\psi_{L_i}}} \langle \left(1-\prod_{i=1}^{k}\left(1-\left(e_{L_i}\right)^c\right)^{\tau_i}\right)^{\frac{1}{c}}, \prod_{i=1}^{k}\left(f_{L_i}\right)^{\tau_i} \rangle$$

$$= \bigcup_{\substack{e_L\in\pi_L\\ f_L\in\psi_L}} \langle \left(1-\prod_{i=1}^{k}(1-(e_L)^c)^{\tau_i}\right)^{\frac{1}{c}}, \prod_{i=1}^{k}(f_L)^{\tau_i} \rangle$$

$$= \bigcup_{\substack{e_L\in\pi_L\\ f_L\in\psi_L}} \langle \left(1-(1-(e_L)^c)^{\sum_{i=1}^{k}\tau_i}\right)^{\frac{1}{c}}, \prod_{i=1}^{k}(f_L)^{\sum_{i=1}^{k}\tau_i} \rangle = \bigcup_{\substack{e_L\in\pi_L\\ f_L\in\psi_L}} \langle \left(1-(1-(e_L)^c)\right)^{\frac{1}{c}}, \prod_{i=1}^{k}(f_L) \rangle = L$$

2. The proof is same as 1. □

Theorem 4.4. (Boundedness) let $L_i = \left(\pi_{L_i}, \psi_{L_i}\right), (i = 1,2,\dots,k)$ be a set of c,d-rung orthopair hesitant fuzzy numbers and $\tau = (\tau_i)^T$ be weight vector of L_i with $\tau_i > 0$ such that $\sum_{i=1}^{k}\tau_i = 1$. Suppose that $e_L^- = \min_{1\le i\le k}\{e_i\}_{e_i\in\pi_{L_i}}$ and $e_L^+ = \max_{1\le i\le k}\{e_i\}_{e_i\in\pi_{L_i}}$ $f_L^+ = \max_{1\le i\le k}\{f_i\}_{f_i\in\psi_{L_i}}$, and $f_L^- = \min_{1\le i\le k}\{f_i\}_{f_i\in\psi_{L_i}}$. Then,

1. $(e_L^-, f_L^-) \le c,d-RHFWA(L_1, L_2, \dots, L_k) \le (e_L^+, f_L^+)$
2. $(e_L^-, f_L^-) \le c,d-RHFWG(L_1, L_2, \dots, L_k) \le (e_L^+, f_L^+)$

Proof.

1. For any $L_i = \left(\pi_{L_i}, \psi_{L_i}\right), (i = 1,2,\dots.,k)$ we can get $e_L^- \le e_i \le e_L^+$ and $f_L^- \le f_i \le f_L^+$..Then we have

$$e_L^- = \left(1-(1-(e_L^-)^c)^{\sum_{i=1}^{k}\tau_i}\right)^{\frac{1}{c}} = \left(1-\prod_{i=1}^{k}(1-(e_L^-)^c)^{\tau_i}\right)^{\frac{1}{c}} \le \left(1-\prod_{i=1}^{k}\left(1-\left(e_{L_i}\right)^c\right)^{\tau_i}\right)^{\frac{1}{c}}$$

$$\le \left(1-\prod_{i=1}^{k}(1-(e_L^+)^c)^{\tau_i}\right)^{\frac{1}{c}} = \left(1-(1-(e_L^+)^c)^{\sum_{i=1}^{k}\tau_i}\right)^{\frac{1}{c}} = e_L^+$$

and

$$f_L^- = \left(1-(1-(f_L^-)^c)^{\sum_{i=1}^{k}\tau_i}\right)^{\frac{1}{c}} = \left(1-\prod_{i=1}^{k}(1-(f_L^-)^c)^{\tau_i}\right)^{\frac{1}{c}} \le \left(1-\prod_{i=1}^{k}\left(1-\left(f_{L_i}\right)^c\right)^{\tau_i}\right)^{\frac{1}{c}}$$

$$\le \left(1-\prod_{i=1}^{k}(1-(f_L^+)^c)^{\tau_i}\right)^{\frac{1}{c}} = \left(1-(1-(f_L^+)^c)^{\sum_{i=1}^{k}\tau_i}\right)^{\frac{1}{c}} = f_L^+$$

Therefore,

$$(e_L^-, f_L^-) \le c,d-RHFWA(L_1, L_2, \dots, L_k) \le (e_L^+, f_L^+)$$

2. The proof is same as part 1. □

Theorem 4.5. (Monotonicity) let $L_i = (\pi_{L_i}, \psi_{L_i})$and $M_i = (\pi_{M_i}, \psi_{M_i})(i = 1,2, \dots, k)$ be two sets of c,d-rung orthopair hesitant fuzzy numbers. If $L_i \subseteq M_i, \forall i$,then

1. $c, d - RHFWA(L_1, L_2, \dots, L_k) \leq c, d - RHFWA(M_1, M_2, \dots, M_k)$
2. $c, d - RHFWG(L_1, L_2, \dots, L_k) \leq c, d - RHFWG(M_1, M_2, \dots, M_k)$

Proof.

1. Since for all i, we have $e_{L_i} \leq e_{M_i}, f_{L_i} \geq f_{M_i}$then

$$\bigcup_{\substack{e_{L_i} \in \pi_{L_i} \\ f_{L_i} \in \psi_{L_i}}} \langle \left(1 - \prod_{i=1}^{k} \left(1 - (e_{L_i})^c\right)^{\tau_i}\right)^{\frac{1}{c}} \rangle \leq \bigcup_{\substack{e_{M_i} \in \pi_{M_i} \\ f_{M_i} \in \psi_{M_i}}} \langle \left(1 - \prod_{i=1}^{k} \left(1 - (e_{M_i})^c\right)^{\tau_i}\right)^{\frac{1}{c}} \rangle, \prod_{i=1}^{k} (f_{L_i})^{\tau_i} \leq \prod_{i=1}^{k} (f_{M_i})^{\tau_i}$$

Therefore,

$$c, d - RHFWA(L_1, L_2, \dots, L_k) = \bigcup_{\substack{e_{L_i} \in \pi_{L_i} \\ f_{L_i} \in \psi_{L_i}}} \langle \left(1 - \prod_{i=1}^{k} \left(1 - (e_{L_i})^c\right)^{\tau_i}\right)^{\frac{1}{c}}, \prod_{i=1}^{k} (f_{L_i})^{\tau_i} \rangle$$

$$\leq \bigcup_{\substack{e_{M_i} \in \pi_{M_i} \\ f_{m_i} \in \psi_{M_i}}} \langle \left(1 - \prod_{i=1}^{k} \left(1 - (e_{M_i})^c\right)^{\tau_i}\right)^{\frac{1}{c}}, \prod_{i=1}^{k} (f_{M_i})^{\tau_i} \rangle = \leq c, d - RHFWA(M_1, M_2, \dots, M_k)$$

2. The proof is similar to (1). □

The important function for ranking two c,d-rung hesitant fuzzy sets is known as score function and accuracy function. Here, we will introduce these functions.

Definition 4.6. Let $L = (\pi_L, \psi_L)$ a c,d-rung orthopair hesitant fuzzy numbers. Then

1. SF of L is defined as follows:

$$H(L) = \frac{S(\pi_L) - S(\psi_L)}{2}$$

2. AF of L is defined as follows:

$$A(L) = \frac{S(\pi_L) + S(\psi_L)}{2}$$

Where

$$S(\pi_L) = \frac{\sum_{i=1}^{k} e_{L\,i}^{c}}{k}$$

and

$$S(\psi_L) = \frac{\sum_{i=1}^{k} f_{L\,i}^{c}}{k}$$

Remark 4.7. Let $L = (\pi_L, \psi_L)$ a c,d-rung orthopair hesitant fuzzy number. Then it is suggested that,

1. SF $H(L) \in [-1,1]$
2. AF $A(L) \in [0,1]$

Note 4.8. Let $L_1 = (\pi_{L_1}, \psi_{L_1})$and $L_2 = (\pi_{L_2}, \psi_{L_2})$ be two c,d-rung orthopair hesitant fuzzy numbers. Then comparison techniques supposed as,

1. If $H(L_1) < H(L_2)$, then $L_1 \prec L_2$,
2. If $H(L_1) > H(L_2)$, then $L_1 \succ L_2$,
3. If $H(L_1) = H(L_2)$, then

(a) If $A(L_1) < H(L_2)$, then $L_1 \prec L_2$,
(b) If $A(L_1) > A(L_2)$, then $L_1 \succ L_2$,
(c) If $A(L_1) = A(L_2)$, then $L_1 \approx L_2$.

5 Decision making on c,d-rung orthopair hesitant fuzzy sets

This section includes the establishment of a model to use the proposed operators for MCDM under c,d-RHFNs. For a MCDM problem, assume that $L = \{L_1, L_2, \dots, L_m\}$ is a finite set of alternatives and $Q = \{Q_1, Q_2, \dots, Q_k\}$ is a set of criteria. Let $B = [L_{ij}] = [\pi_{L_{ij}}, \psi_{L_{ij}}]_{m\times k}$ be a decision matrix be provided by decision makers. A set of weight vector $\tau = (\tau_1, \tau_2, \dots, \tau_k)^T$ with $\tau_i > 0$ such that $\sum_{i=1}^{k} \tau_i = 1$ then, the model (Algorithm 1) of managing the MCDM troubles as follows:

Algorithm 1

1. We will establish a decision matrix based on c,d-rung orthopair hesitant fuzzy numbers $\boldsymbol{B} = [\boldsymbol{L_{ij}}]$ for MCDM.

2. Create a normalized c,d-rung orthopair hesitant fuzzy numbers decision matrix $\boldsymbol{B} = [\boldsymbol{L_{ij}}]$ from c,d-rung orthopair hesitant fuzzy numbers

3. Calculate the alternatives values $\boldsymbol{L_{ij}}$ by using the set of weight vector $\boldsymbol{\tau} = (\boldsymbol{\tau_1}, \boldsymbol{\tau_2}, \dots, \boldsymbol{\tau_k})^T$ and the averaging and geometric aggregation operaotrs discussed in Section 4.

$$\boldsymbol{L_j = c, d - RHFWA(L_{j1}, L_{j2}, \dots, L_{jk})} = \bigcup_{\substack{e_{L_i} \in \pi_{L_i} \\ f_{L_i} \in \psi_{L_i}}} \left\langle \left(1 - \prod_{i=1}^{k} \left(1 - (e_{L_i})^c\right)^{\tau_i} \right)^{\frac{1}{c}}, \prod_{i=1}^{k} (f_{L_i})^{\tau_i} \right\rangle$$

or

$$\boldsymbol{L_j = c, d - RHFWG(L_{j1}, L_{j2}, \dots, L_{jk})} = \bigcup_{\substack{e_{L_i} \in \pi_{L_i} \\ f_{L_i} \in \psi_{L_i}}} \left\langle \prod_{i=1}^{k} (e_{L_i})^{\tau_i}, \left(1 - \prod_{i=1}^{k} \left(1 - (f_{L_i})^c\right)^{\tau_i} \right)^{\frac{1}{c}} \right\rangle$$

For all $\boldsymbol{j = 1, 2, \dots, m}$.

4.Calculate the score results for all c,d-rung orthopair hesitant fuzzy numbers of $\boldsymbol{L_j}$ obtained from Step 3.

5. The best option can be found by obtaining from the comparing techniques using score values and accuracy values.

6 Case Study via c,d-rung orthopair hesitant aggregation operators

In a rapidly evolving global marketplace, a multinational manufacturing company, NB sons Ltd, is committed to enhancing its sustainability practices throughout its supply chain. The company recognizes that achieving sustainability goals requires making strategic decisions that balance economic, environmental, and social factors. NB Sons Ltd has adopted a c,d-rung orthopair hesitant Information System to evaluate potential solutions for optimizing its supply chain sustainability.

Background:

NB Sons Ltd operates in the electronics industry, producing consumer devices. Their supply chain consists of numerous suppliers, transportation networks, and manufacturing facilities distributed across various countries. They aim to reduce their environmental footprint, improve working conditions, and maintain cost-effectiveness. Four alternative strategies have been identified for supply chain optimization, each with four attributes:

Alternatives and Criteria for MCDM:

A set of alternatives $L = \{L_1, L_2, L_3, L_4\}$ and $Q = \{Q_1, Q_2, Q_3, Q_4\}$ are described in our scenario is, $\boldsymbol{L_1}$ **(Local Sourcing):** Emphasizing local suppliers and shortening transportation distances. $\boldsymbol{Q_1}$ Cost Efficiency: Lower transportation costs. $\boldsymbol{Q_2}$ Environmental Impact**:** Reduced carbon emissions due to shorter distances. $\boldsymbol{Q_3}$ Social Responsibility**:** Support for local economies and labor conditions. $\boldsymbol{Q_4}$ Product Quality: Product Quality should be attractive. $\boldsymbol{L_2}$ **(Global Sourcing):** Seeking suppliers from low-cost regions for cost savings.$\boldsymbol{Q_1}$ Cost Efficiency: Lower procurement costs. $\boldsymbol{Q_2}$ Environmental Impact: Increased transportation-related emissions. $\boldsymbol{Q_3}$Social Responsibility**:** Ethical concerns related to labor practices abroad. $\boldsymbol{Q_4}$ Product Quality: Product Quality should be attractive. $\boldsymbol{L_3}$ **(Green Logistics):** Investing in eco-friendly transportation and warehousing.$\boldsymbol{Q_1}$ Cost Efficiency: Higher initial investment but potential long-term savings. $\boldsymbol{Q_2}$Environmental Impact: Reduced emissions from sustainable logistics. $\boldsymbol{Q_3}$Social Responsibility**:** Improved supply chain sustainability practices. $\boldsymbol{Q_4}$ Product Quality: Product Quality should be attractive.

L_4 **Supplier Collaboration:** Partnering closely with suppliers for sustainable practices. Q_1Cost Efficiency: Potential for cost savings through collaborative efforts. Q_2Environmental Impact: Reduction in the overall supply chain's carbon footprint. Q_3Social Responsibility: Enhanced labor conditions and ethical sourcing. Q_4 Product Quality: Product Quality should be attractive.

Objective: NB Sons Ltd aims to select the supply chain strategy that best aligns with its sustainability objectives. The decision-making process involves evaluating the four alternatives based on the three attributes: Cost Efficiency, Environmental Impact, and Social Responsibility. However, the decision-makers recognize that they have bipolar hesitant information, meaning they may have conflicting feelings or uncertainties regarding each attribute's importance and performance for each alternative.

The MCDM matrix is given in the form of Table 2 based on the c,d-rung orthopair hesitant information.

Table 2: c,d-RHF information

Alternatives	Q1	Q2	Q3	Q4
L_1	$\{0.7,0.3\},\{0.6,0.5\}$	$\{0.5,0.4\},\{0.8,0.4\}$	$\{0.6,0.5\},\{0.7,0.4\}$	$\{0.2,0.1\},\{0.6,0.2\}$
L_2	$\{0.5,0.2\},\{0.7,0.6\}$	$\{0.3,0.1\},\{0.6,0.4\}$	$\{0.6,0.3\},\{0.7,0.1\}$	$\{0.3,0.2\},\{0.8,0.3\}$
L_3	$\{0.6,0.3\},\{0.6,0.4\}$	$\{0.7,0.1\},\{0.7,0.2\}$	$\{0.5,0.2\},\{0.8,0.3\}$	$\{0.4,0.4\},\{0.9,0.4\}$
L_4	$\{0.8,0.5\},\{0.6,0.3\}$	$\{0.5,0.4\},\{0.8,0.5\}$	$\{0.4,0.1\},\{0.7,0.4\}$	$\{0.6,0.2\},\{0.5,0.3\}$

To aggregate the information given in Table 2, proposed aggregation operators are used where $c = 3\ and\ d = 1$ and weights for each attribute is taken as $\tau_1 = 0.1, \tau_1 = 0.2, \tau_1 = 0.2$ and $\tau_1 = 0.5$

$$c,d-RHFWA\left(L_{j1},L_{j2},\ldots,L_{j4}\right)=\bigcup_{\substack{e_{L_i}\in\pi_{L_i}\\ f_{L_i}\in\psi_{L_i}}}\langle\left(1-\prod_{i=1}^{4}\left(1-\left(e_{L_i}\right)^c\right)^{\tau_i}\right)^{\frac{1}{c}},\prod_{i=1}^{4}\left(f_{L_i}\right)^{\tau_i}\rangle$$

or

$$L_j=c,d-RHFWG\left(L_{j1},L_{j2},\ldots,L_{j4}\right)=\bigcup_{\substack{e_{L_i}\in\pi_{L_i}\\ f_{L_i}\in\psi_{L_i}}}\langle\prod_{i=1}^{4}\left(e_{L_i}\right)^{\tau_i},\left(1-\prod_{i=1}^{4}\left(1-\left(f_{L_i}\right)^c\right)^{\tau_i}\right)^{\frac{1}{c}}\rangle$$

After applying these aggregation operators, we obtain the calculated values shown in Table 3

Table 3: Aggregated results of c,d-RHF information

Alternatives	c,d-RFWA	c,d-RHFWG
L_1	$\{0.0432,0.0106\},\{0.6454,0.2957\}$	$\{0.3444,0.1930\},\{0.2211,0.0610\}$
L_2	$\{0.0225,0.0028\},\{0.7256,0.3270\}$	$\{0.3561,0.1813\},\{0.2758,0.0791\}$
L_3	$\{0.0546,0.0129\},\{0.7799,0.3383\}$	$\{0.4961,0.2670\},\{0.3427,0.0651\}$
L_4	$\{0.0860,0.0143\},\{0.5891,0.3419\}$	$\{0.5874,0.2574\},\{0.1981,0.0657\}$

By applying the score function for c,d-RHF information, we have results in Table 4

Table 4: Score values on different c,d parameters

Alternatives	Score (3,2-RFWA)	Score (3,2-RHFWG)	Score(1,1-RFWA)	Score(1,1-RFWG)
L_1	-0.1259	-0.0011	-0.1771	0.0101
L_2	-0.1583	-0.0078	-0.2154	-0.00542
L_3	-0.1806	0.0048	-0.2098	0.0434
L_4	-0.1158	0.0443	-0.1556	0.0893

Ranking results based on the score values presented in Table 5 are displayed in Table 5

Table 5: Ranking of alternatives derived from score values

Alternatives	Ranking	Best
3,2-RHFWA	$L4 > \mathrm{L1} > \mathrm{L3} > \mathrm{L2}$	$L4$
3,2-RHFWG	$L4 > \mathrm{L3} > \mathrm{L1} > \mathrm{L2}$	$L4$
1,1-RHFWA	$L4 > \mathrm{L1} > \mathrm{L3} > \mathrm{L2}$	$L4$
1,1-RHFWG	$L4 > \mathrm{L3} > \mathrm{L1} > \mathrm{L2}$	$L4$

This ranking shows that the alternative $\boldsymbol{L_4}$ **Supplier Collaboration:** Partnering closely with suppliers for sustainable practices is the best strategy identified for supply chain optimization.

7 Comparative analysis

In this section, we will compare the established approach with existing techniques and analyze the difference between these models. The comparison outcomes are presented in Table 6.

Table 6: Comparative analysis

Approaches	Alternatives	Ranking	Best
Proposed	2,3-RHFWA	$L4 > \mathrm{L2} > \mathrm{L1} > \mathrm{L3} > \mathrm{L5}$	$L4$
	3,2-RHFWA	$L4 > \mathrm{L2} > \mathrm{L1} > \mathrm{L3} > \mathrm{L5}$	$L4$
Ibrahim et al [38]	2,3-RFWA	$L4 > \mathrm{L2} > \mathrm{L1} > \mathrm{L3} > \mathrm{L5}$	$L4$
	3,2-RFWA	$L4 > \mathrm{L2} > \mathrm{L1} > \mathrm{L3} > \mathrm{L5}$	$L4$
Mahmood et al [40]	1,1-RFWA(IFWA)	$L4 > \mathrm{L2} > \mathrm{L1} > \mathrm{L3} > \mathrm{L5}$	$L4$
Khan et al [42]	2,2-RFWA(PFWA)	$L4 > \mathrm{L2} > \mathrm{L1} > \mathrm{L3} > \mathrm{L5}$	$L4$
Krisci [43]	3,3-RFWA(FFWA)	$L4 > \mathrm{L2} > \mathrm{L1} > \mathrm{L3} > \mathrm{L5}$	$L4$

It is evident that when we modified the c,d parameters of the proposed c,d-RHFS model, the results consistently aligned with those of existing approaches. The proposed model demonstrated compatibility with the outcomes achieved using n,m-rung orthopair fuzzy sets as presented by Ibrahim [38]. Similarly, our method exhibited promising results when compared to Intuitionistic hesitant fuzzy sets [40], Pythagorean hesitant fuzzy sets [42], and Fermatean hesitant fuzzy sets [43]. Through this comparison, we have discovered some significant characteristics of our suggested approach. In the subsequent section, we will examine in a detailed discussion the advantages of this methodology.

7.1 Benefits and limitations of the proposed technique

The benefits of the proposed approach are discussed as follows,

i. The proposed approach is a more generalized structure, Figure 2 shows this generic structure.
ii. By taking singleton elements in MG and NMG in c,d-RHFS then this is converted into n,m-rung orthopair fuzzy sets [38].
iii. By taking c=d, c,d-RHFS is converted into Q-RHFS [44].
iv. By taking c=d=3, c,d-RHFS is converted into Fermatean hesitant fuzzy set[43].
v. By taking c=d=2, c,d-RHFS is converted into Pythagorean hesitant fuzzy set[42].
vi. By taking c=d=1, c,d-RHFS is converted into intuitionistic hesitant fuzzy set[40].
vii. The proposed approach facilitates the selection process within a multi-attribute decision-making model.
viii. This approach can be expanded to accommodate other decision-making processes such as MULTIMORA, TOPSIS, and VIKOR models.

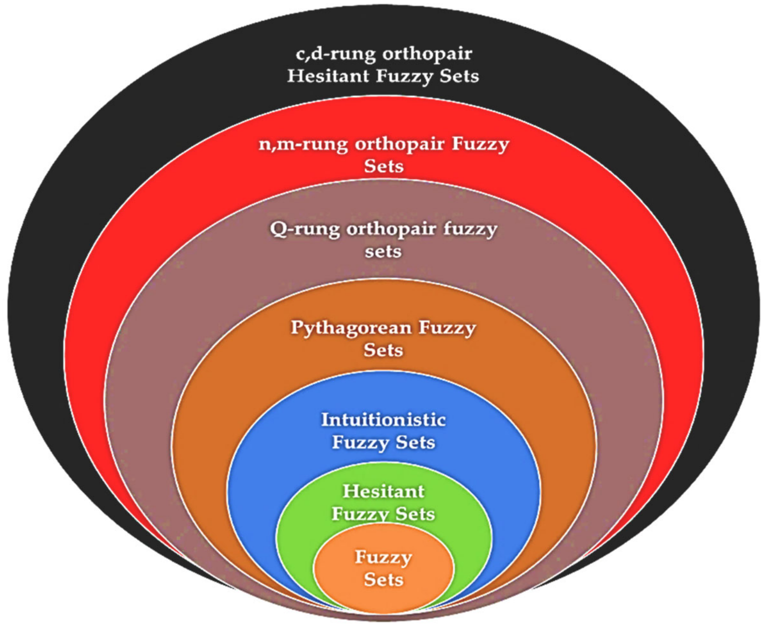

Figure 2: Generalizations of Fuzzy Sets

8 Conclusion

In this research article, we have explored the dimensions of Q-RHFSs, which encapsulate membership and non-membership grades within the [0,1] interval for each element in a given universe. The evolution of Q-RHFSs has led to the development of n,m-rung orthopair fuzzy sets, delineating a more expansive version of the original concept. Our research progresses this idea by amalgamating it with a hesitant fuzzy model, thereby forging the innovative concept of the c,d-rung orthopair hesitant fuzzy model. This innovative model is particularly suited for effectively managing scenarios laden with uncertainty.

Our study rigorously confirms that the proposed c,d-rung orthopair hesitant fuzzy model aligns seamlessly with the core principles and operational mechanisms fundamental to fuzzy set theory. We have innovatively designed a series of power averaging and geometric aggregation operators, offering an exhaustive elucidation of their roles in the calculation of fuzzy information. Further, we have applied this model to tackle a critical global challenge: the development of sustainable supply chain systems. This application focuses on the strategic selection process for corporations, considering a multitude of attributes. To facilitate this complex decision-making process, we have devised a tailored multiple-attribute decision-making model, which is attuned to the nuances of the c,d-rung orthopair hesitant fuzzy information.

Our contribution to the academic field is twofold. Firstly, we conduct a comprehensive comparative analysis with existing models, thereby underscoring the distinctive advantages of our innovative approach. Secondly, the integration of hesitant fuzzy modeling into the c,d-orthopair fuzzy sets framework markedly improves our capacity to make well-informed decisions in environments characterized by significant uncertainty.

Looking forward, we are poised to implement our methodology within the dynamic spheres of machine learning and artificial intelligence. We will further refine and elaborate on existing models by incorporating a variety of aggregation operators and undertaking comparative analyses. Additionally, we anticipate expanding our model to include the methodologies discussed in references [56-58]. This expansion will enable us to thoroughly assess the applicability and efficacy of our methods across an array of techniques and domains, thereby enriching the landscape of fuzzy set theory and its applications.

Acknowledgements: We would like to thank the reviewers for their thoughtful comments and efforts towards improving our manuscript.

Conflict of Interest: The authors declare no conflict of interest.

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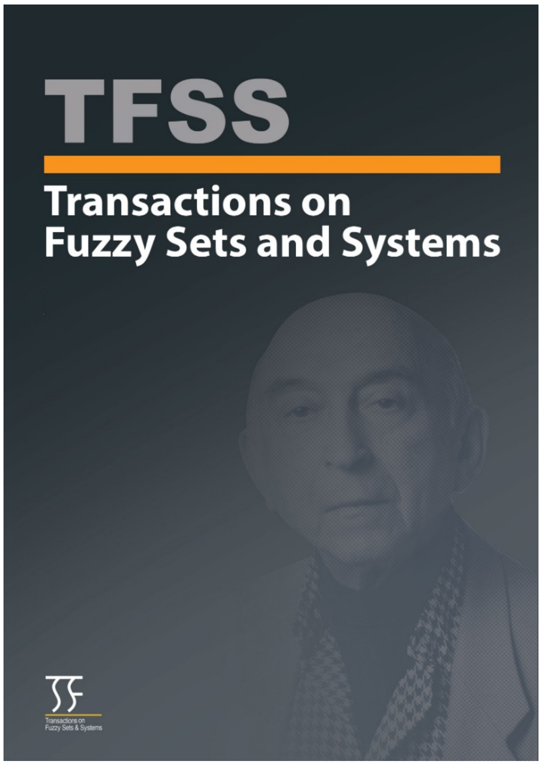

Transactions on Fuzzy Sets and Systems

ISSN: **2821-0131**

https://sanad.iau.ir/journal/tfss/

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Vol.4, No.1, (2025), 181-194. DOI: https://doi.org/10.71602/tfss.2025.1186791

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Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://sanad.iau.ir/journal/tfss/
Online ISSN: 2821-0131
Vol.4, No.1, (2025), 181-194
DOI: https://doi.org/10.71602/tfss.2025.1186791

Article Type: Original Research Article

A Note on the Fuzzy Leonardo Numbers

Elen Viviani Pereira Spreafico*, **Eudes Antonio Costa**, **Paula Maria Machado Cruz Catarino**

(This paper is dedicated to Professor "John N. Mordeson" on the occasion of his 91st birthday.)

Abstract. In this work, we define a new sequence denominated by fuzzy Leonardo numbers. Some algebraic properties of this new sequence are studied and several identities are established. Moreover, the relations between the fuzzy Fibonacci and fuzzy Lucas numbers are explored, and several results are given. In addition, some sums involving fuzzy Leonardo numbers are provided.

AMS Subject Classification 2020: 03E72; 11B39
Keywords and Phrases: Triangular fuzzy numbers, Fuzzy Fibonacci numbers, Fuzzy Lucas numbers, Leonardo numbers, Algebraic properties, Identities, Sum identities.

1 Introduction

Recently, several researchers have worked enthusiastically with numerical sequences. Their studies cover a wide range of fascinating aspects, including exploring unique properties, revealing previously known identities, and even unlocking the secrets behind generating functions and matrices. One of these interesting sequences is the Fibonacci sequence of numbers. The sequence of Fibonacci $\{F_n\}_{n\geq 0}$ is defined by a recurrence relation of order two, given by

$$F_n = F_{n-1} + F_{n-2}, \ (n \geq 2), \tag{1}$$

with initial conditions $F_0 = 0$ and $F_1 = 1$. Other classical sequence is the sequence of Lucas numbers $\{L_n\}_{n\geq 0}$, defined by the same recurrence relation of Fibonacci sequence,

$$L_n = L_{n-1} + L_{n-2}, \ (n \geq 2), \tag{2}$$

but with different initial conditions, $L_0 = 2$ and $L_1 = 1$. The Fibonacci sequence has motivated the study of many other numerical sequences. We can find not only properties of the sequences of Fibonacci but also the correlated sequences such as Lucas, Pell, and Pell-Lucas and their applications in the following works [1], [2] and [3].

One of these correlated sequences is the sequence of Leonardo, introduced by Catarino and Borges in [4], and defined by the recurrence relation

$$Le_n = Le_{n-1} + Le_{n-2} + 1, \ (n \geq 2), \tag{3}$$

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Received: 11 October 2024; **Revised**: 12 December 2024; **Accepted**: 15 December 2024; **Available Online**: 22 January 2025; **Published Online**: 7 May 2025.

How to cite: Spreafico EVP, Costa EA, Catarino PMMC. A Note on the Fuzzy Leonardo Numbers. *Transactions on Fuzzy Sets and Systems.* 2025; 4(1): 181-194. DOI: https://doi.org/10.71602/tfss.2025.1186791

with initial conditions $Le_0 = Le_1 = 1$. This recurrence relation can have the equivalent form

$$Le_{n+1} = 2Le_n - Le_{n-2},\ (n \geq 2).$$

The relation between Leonardo and Fibonacci numbers is given by

$$Le_n = 2F_{n+1} - 1, \tag{4}$$

according to Proposition 2.2 in [4].

The Leonardo sequence has given rise to many related research studies, which are, for example, those of Alp and Koçer in [5], Alves and Vieira in [6], Catarino and Borges in [7], Gokbas in [8], Kara and Yilmaz in [9], Kuhapatanakul and Chobsorn in [10], and Tan and Leung in [11], among others.

On the other hand, since fuzzy set theory has a lot of applications in real life, the interest in workings and researching has increased in recent years [12, 13, 14]. To face the challenges of ambiguity in various areas, Zadeh, in the article [15], introduced the fuzzy set theory. The fuzzy set theory is based on the fuzzy membership function. Given a set A, the membership function denoted by μ_A is a function that associates an element of a set A to an element in the interval $[0, 1]$. A fuzzy set A is described by its membership function μ_A, and by the fuzzy membership function, we can determine the membership grade of an element concerning a set (see more details in [16, 17, 18, 19]). Following Duman, in [20], there are many fuzzy membership function types, which most commonly used are the triangular, trapezoidal, Gaussian, and generalized Bell. Fuzzy operations on fuzzy sets are defined as crisp operations performed on crisp sets. Operations on fuzzy sets are done using fuzzy membership functions. Operations such as addition, subtraction, multiplication, and division are defined in a fuzzy set, [16, 21]. When fuzzy set operations are applied to a set, the result is a fuzzy set. But these sets need to be converted to a real number, that is, an inference must be made. This process is called defuzzyfication, which means inversion of fuzzyfication [22].

Recently, a bridge between fuzzy sets and number theory was built when fuzzy Fibonacci and Lucas number sequences were defined using the triangular membership function by [23], and also several identities were provided. In addition, other properties are investigated in [20].

We aim to introduce the fuzzy Leonardo numbers using the triangular membership function and give some new properties of this new sequence. The article is organized as follows. In Section 2, we present the triangular fuzzy numbers with their operations. Also, the definitions of fuzzy Fibonacci numbers and fuzzy Lucas numbers are given as identities related to these sequences, which will be useful for the next sections. Section 3 introduces the fuzzy Leonardo numbers and establishes some properties and identities of this new set of numbers. In Section 4, some sums involving fuzzy Leonardo numbers are provided. Finally, some conclusions are stated.

2 Preliminaries concepts

In this section, we will present the definition of triangular fuzzy numbers, such as their arithmetic operations of the α-cut, $\alpha \in [0, 1]$. In addition, the definitions of fuzzy Fibonacci and fuzzy Lucas numbers are given, and some properties of these numbers are presented.

First, consider the definition of the triangular fuzzy number given by Irmak and Demirtas in [23]. A triangular fuzzy number, denoted by $\tilde{A} = (a_1, a_2, a_3)$ is represented by three points, two of which are left and right of the interval, such that a_1, a_2, a_3 are real numbers. The triangular membership function with $\tilde{A} = (a_1, a_2, a_3)$ is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \leq a_1 \\ \frac{x-a_1}{a_2-a_1}, & a_1 < x \leq a_2 \\ \frac{a_3-x}{a_3-a_2}, & a_2 < x \leq a_3 \\ 0, & x \geq a_3 \end{cases}.$$

A triangular fuzzy number can be represented by α-cut operation, which denotes A^α. To convert a triangular fuzzy number to $\alpha-$cut interval, we follow that

$$A^\alpha = [a_1^\alpha, a_3^\alpha] = [a_1 + \alpha(a_2 - a_1), a_3 - \alpha(a_3 - a_2)], \tag{5}$$

where $\alpha \in [0,1]$ and a_j for $j = 1, 2, 3$ are real numbers.

Let $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1, b_2, b_3)$ be the triangular fuzzy numbers and $A^\alpha = [a_1^\alpha, a_3^\alpha]$ and $B^\alpha = [b_1^\alpha, b_3^\alpha]$ be the $\alpha-$cut obtained from these numbers. The arithmetic operations of the $\alpha-$cut are given in [23] as follows

$$\begin{aligned} A^\alpha + B^\alpha &= [a_1^\alpha + b_1^\alpha, a_3^\alpha + b_3^\alpha], & (6)\\ A^\alpha - B^\alpha &= [a_1^\alpha - b_1^\alpha, a_3^\alpha - b_3^\alpha], \\ A^\alpha B^\alpha &= [min\{a_1^\alpha b_1^\alpha, a_3^\alpha b_1^\alpha, a_1^\alpha b_3^\alpha, a_3^\alpha b_3^\alpha\}, max\{a_1^\alpha b_1^\alpha, a_3^\alpha b_1^\alpha, a_1^\alpha b_3^\alpha, a_3^\alpha b_3^\alpha\}], & (7)\\ A^\alpha / B^\alpha &= [min\{a_1^\alpha/b_1^\alpha, a_3^\alpha/b_1^\alpha, a_1^\alpha/b_3^\alpha, a_3^\alpha/b_3^\alpha\}, max\{a_1^\alpha/b_1^\alpha, a_3^\alpha/b_1^\alpha, a_1^\alpha/b_3^\alpha, a_3^\alpha/b_3^\alpha\}], \\ kA^\alpha &= [min\{ka_1^\alpha, ka_3^\alpha\}, max\{ka_1^\alpha, ka_3^\alpha\}], \end{aligned}$$

with k real number. Note that, if a_1, b_1, a_2, b_2, a_3 and b_3 are positive real numbers with $a_1 \leq a_2 \leq a_3$, and $b_1 \leq b_2 \leq b_3$, then $A^\alpha B^\alpha = [a_1^\alpha b_1^\alpha, a_3^\alpha b_3^\alpha]$. Moreover, if k is a positive real number then we have $kA^\alpha = [min\{ka_1^\alpha, ka_3^\alpha\}, max\{ka_1^\alpha, ka_3^\alpha\}] = [ka_1^\alpha, ka_3^\alpha]$, (see more details in [16, 21]).

In [23], the author introduced the fuzzy Fibonacci numbers and the fuzzy Lucas numbers, which will be very useful for this article. Let $\{F_n\}_{n\geq 0}$ be the Fibonacci sequence (1). The triangular fuzzy number of Fibonacci is given by $\tilde{F_n} = (F_{n-1}, F_n, F_{n+1})$. Then, we have the following definition.

Definition 2.1. *Let $\{F_n\}_{n\geq 0}$ be the Fibonacci sequence (1). The fuzzy Fibonacci numbers are defined by the expression*

$$F_n^\alpha = [F_{n-1}^\alpha, F_{n+1}^\alpha] = [F_{n-1} + \alpha F_{n-2}, F_{n+1} - \alpha F_{n-1}], \tag{8}$$

for $n \geq 2$, where $\alpha \in [0,1]$ and initial conditions are $F_0^\alpha = [1-\alpha, 1+\alpha]$ and $F_1^\alpha = [\alpha, 1]$.

Similarly, the definition of fuzzy Lucas numbers, as proposed by Irmak and Demirtas in [23], is as follows:

Definition 2.2. *Let $\{L_n\}_{n\geq 0}$ be the Lucas sequence (2). The fuzzy Lucas numbers are defined by the expression*

$$L_n^\alpha = [L_{n-1}^\alpha, L_{n+1}^\alpha] = [L_{n-1} + \alpha L_{n-2}, L_{n+1} - \alpha L_{n-1}], \tag{9}$$

for $n \geq 2$, where $\alpha \in [0,1]$ and initial conditions are $L_0^\alpha = [-1-3\alpha, 1+\alpha]$, and $L_1^\alpha = [2-\alpha, 3-2\alpha]$.

Motivated by the previous definitions, we will introduce the fuzzy Leonardo numbers and study some properties of this new fuzzy sequence of numbers in the next section. Moreover, this article will explore the connection between the fuzzy Leonardo numbers, the fuzzy Fibonacci numbers, and the fuzzy Lucas numbers by considering the following identities for non-negative integers n,

[20, Theorem 3.1] $$F_{n+4}^\alpha + F_n^\alpha = 3F_{n+2}^\alpha\ , \tag{10}$$

[20, Theorem 3.2] $$F_{n+10}^\alpha = 11F_{n+5}^\alpha + F_n^\alpha\ , \tag{11}$$

[20, Theorem 3.3] $$F_{n+2}^\alpha - F_{n+1}^\alpha = (-F_n, F_n, 2F_{n+1})\ , \tag{12}$$

[23, Theorem 3.1] $$F_{n+2}^\alpha - F_{n-2}^\alpha = L_n^\alpha\ , \tag{13}$$

[23, Theorem 3.2(a)] $$2F_{n+2}^\alpha - 3F_n^\alpha = L_n^\alpha\ , \tag{14}$$

[23, Theorem 3.2(d)] $$2F_{n+1}^\alpha - F_n^\alpha = L_n^\alpha\ , \tag{15}$$

[23, Theorem 3.2(g)] $$F_{n+1}^\alpha + F_{n-1}^\alpha = L_n^\alpha \tag{16}$$

[23, Theorem 3.2(b), (c) and (e)] $$5F_n^\alpha = 2L_{n+2}^\alpha - 3L_n^\alpha = L_{n+1}^\alpha + L_{n-1}^\alpha = 2L_{n+1}^\alpha - L_n^\alpha\ . \tag{17}$$

For the reason to establish identities involving the fuzzy Leonardo numbers, in this article, we will consider the following classical identities for Leonardo numbers $\{Le_n\}_{n\geq 0}$ established in Proposition 2.3 [4],

$$Le_n = 2\left(\frac{L_n + L_{n+2}}{5}\right) - 1, \tag{18}$$

$$Le_{n+3} = \left(\frac{L_{n+1} + L_{n+7}}{5}\right) - 1, \tag{19}$$

$$Le_n = L_{n+2} - F_{n+2} - 1, \tag{20}$$

for all non-negative integer n, where F_n is the n-th Fibonacci number given by (1) and L_n is the n-th Lucas number given by (2).

3 The fuzzy Leonardo numbers, properties and identities

In this section, we will introduce the fuzzy Leonardo numbers and provide some properties of this new sequence. Moreover, some identities are established.

3.1 The fuzzy Leonardo numbers and properties

Let $\{Le_n\}_{n\geq 0}$ be the Leonardo sequence of numbers defined by Equation (3) and the triangular fuzzy number of Leonardo given by $\widetilde{Le}_n = (Le_{n-1}, Le_n, Le_{n+1})$. Then, it is natural to consider the $\alpha-$cut of the triangular fuzzy numbers given in the following definition.

Definition 3.1. *Let $\{Le_n\}_{n\geq 0}$ be Leonardo sequence given by (3). The fuzzy Leonardo numbers are defined by the following expression*

$$Le_n^\alpha = [Le_{n-1}^\alpha, Le_{n+1}^\alpha] = [Le_{n-1} + \alpha(Le_{n-2} + 1), Le_{n+1} - \alpha(Le_{n-1} + 1)], \tag{21}$$

for $n \geq 2$, where $\alpha \in [0,1]$ and initial conditions are $Le_0^\alpha = [1-\alpha, 1+\alpha]$, and $Le_1^\alpha = [\alpha, 1]$.

By Definition 3.1, the elements of the sequence $\{Le_n^\alpha\}_{n\geq 0}$ are the $\alpha-$cut obtained from the triangular fuzzy number of Leonardo $\widetilde{Le}_n$, and can be operated by using the $\alpha-$cut operations.

Observe that by considering the triangular fuzzy number $\tilde{1} = (1,1,1)$, and by applying the $\alpha-$cut, we obtain $I^\alpha = [1^\alpha, 1^\alpha] = [1 + \alpha(1-1), 1 - \alpha(1-1)] = [1,1]$.

Then, by using the rule of summation (6), we describe a recurrence relation for the fuzzy Leonardo numbers in the next proposition.

Proposition 3.2. *Consider $\alpha \in [0,1]$. Let $\{Le_n^\alpha\}_{n\geq 0}$ be the sequence of fuzzy Leonardo numbers. Then, it is verified*

$$Le_n^\alpha = Le_{n-1}^\alpha + Le_{n-2}^\alpha + I^\alpha, \tag{22}$$

where $I^\alpha = [1,1]$.

Proof. By considering the sum operation and Expression (21), we have

$$\begin{aligned}
Le_{n-1}^\alpha + Le_{n-2}^\alpha + I^\alpha &= [Le_{n-2}^\alpha, Le_n^\alpha] + [Le_{n-3}^\alpha, Le_{n-1}^\alpha] + [1^\alpha, 1^\alpha] \\
&= [Le_{n-2}^\alpha + Le_{n-3}^\alpha + 1^\alpha, Le_n^\alpha + Le_{n-1}^\alpha + 1^\alpha] \\
&= [Le_{n-2} + \alpha(Le_{n-3} + 1) + Le_{n-3} + \alpha(Le_{n-4} + 1) + 1, \\
&\quad Le_n - \alpha(Le_{n-2} + 1) + Le_{n-1} - \alpha(Le_{n-3} + 1) + 1] \\
&= [Le_{n-1} + \alpha(Le_{n-2} + 1), Le_{n+1} - \alpha(Le_{n-1} + 1)] \\
&= Le_n^\alpha,
\end{aligned}$$

which verifies the result. □

In addition, observe that the Leonardo sequence $\{Le_n\}_{n\geq 0}$ is an increasing sequence of positive integers, then it is verified the scalar operation $kA^\alpha = [ka_1^\alpha, ka_3^\alpha]$, for k positive real number. Moreover, since it is verified the recurrence relation for the Leonardo numbers, $Le_{n+1} = 2Le_n - Le_{n-2}$, $(n \geq 2)$, with the same proceedings done in the proof of Proposition 22 and the scalar product, we can obtain a new equation for the fuzzy Leonardo numbers given by

$$Le_n^\alpha = 2Le_n^\alpha - Le_{n-2}^\alpha.$$

3.2 Some Identities

This subsection will provide some new identities for the fuzzy Leonardo numbers. In addition, we will establish new identities involving the fuzzy Leonardo numbers, the fuzzy Fibonacci numbers, and the fuzzy Lucas numbers.

Recall the relation between the Leonardo and Fibonacci numbers given by (4), namely, $Le_n = 2F_{n+1} - 1$. Therefore, by using the scalar product, Definition 3.1, and Equation (4), we establish the following result.

Proposition 3.3. *Consider $\alpha \in [0,1]$. Let $\{Le_n^\alpha\}_{n\geq 0}$ be the sequence of fuzzy Leonardo numbers and $\{F_n^\alpha\}_{n\geq 0}$ be the sequence of fuzzy Fibonacci numbers. Then, it is verified*

$$Le_n^\alpha = 2F_{n+1}^\alpha - I^\alpha. \tag{23}$$

Proof. Equation (21) shows us that $Le_n^\alpha = [Le_{n-1} + \alpha(Le_{n-2}+1), Le_{n+1} - \alpha(Le_{n-1}+1)]$. Since it is verified $Le_n = 2F_{n+1} - 1$, then

$$\begin{aligned} Le_n^\alpha &= [Le_{n-1} + \alpha(Le_{n-2}+1), Le_{n+1} - \alpha(Le_{n-1}+1)] \\ &= [2F_n + 2\alpha(F_{n-1}) - 1, 2F_{n+2} - 2\alpha(F_n) - 1] \\ &= 2[F_n + \alpha F_{n-1}, F_{n+2} - \alpha F_n] - [1^\alpha, 1^\alpha] \\ &= 2F_{n+1}^\alpha - I^\alpha, \end{aligned}$$

by Equation (8). □

Similarly, recall the identities of the sequence of Leonardo numbers stated in Proposition 2.3 [4], and the operations in α-cut. Then, the next proposition is stated.

Proposition 3.4. *For all non-negative integers n, the following identities hold:*

$$Le_n^\alpha = \frac{2}{5}\left(L_n^\alpha + L_{n+2}^\alpha\right) - I^\alpha, \tag{24}$$

$$Le_{n+3}^\alpha = \frac{1}{5}\left(L_{n+1}^\alpha + L_{n+7}^\alpha\right) - I^\alpha, \tag{25}$$

$$Le_n^\alpha = L_{n+2}^\alpha - F_{n+2}^\alpha - I^\alpha, \tag{26}$$

where $I^\alpha = [1,1]$, Le_n^α is the n-th fuzzy Leonardo numbers, F_n^α is the n-th fuzzy Fibonacci number given by (8), and L_n^α is the n-th fuzzy Lucas number given by (9).

Proof. By combining Definition 3.1 and Identity (18) we obtain

$$\begin{aligned}
& Le_n^\alpha \\
= & \left[Le_{n-1}+\alpha(Le_{n-2}+1), Le_{n+1}-\alpha(Le_{n-1}+1)\right] \\
= & 2\left[\left(\frac{L_{n-1}+L_{n+1}}{5}\right)-1+\alpha\left(\frac{L_{n-2}+L_n}{5}\right),\left(\frac{L_{n+1}+L_{n+3}}{5}\right)-1-\alpha\left(\frac{L_{n-1}+L_{n+1}}{5}\right)\right] \\
= & 2\left[\left(\frac{L_{n-1}+L_{n+1}}{5}\right)+\alpha\left(\frac{L_{n-2}+L_n}{5}\right),\left(\frac{L_{n+1}+L_{n+3}}{5}\right)-\alpha\left(\frac{L_{n-1}+L_{n+1}}{5}\right)\right]-1^\alpha \\
= & \frac{2}{5}\left[(L_{n-1}+L_{n+1})+\alpha\left(L_{n-2}+L_n\right),(L_{n+1}+L_{n+3})-\alpha\left(L_{n-1}+L_{n+1}\right)\right]-1^\alpha \\
= & \frac{2}{5}\left(L_n^\alpha+L_{n+2}^\alpha\right)-I^\alpha
\end{aligned}$$

Similarly, by using Definition 3.1 and Identity (19), we obtain Equation (25), as well as, by using Definition 3.1 and Identity (20) we obtain (26) □

Next, we will provide an identity related to the product of fuzzy Leonardo numbers. To do this, we need to observe the product rule (7) and the fact of the Leonardo sequence $\{Le_n\}_{n\geq 0}$ is an increasing sequence of positive integers. Then we have $Le_m^\alpha Le_k^\alpha = [Le_{m-1}^\alpha Le_{k-1}^\alpha, Le_{m+1}^\alpha Le_{k-1}^\alpha]$.

Theorem 3.5. *Consider m and n non-negative integers and let Le_n^α be the n-th fuzzy Leonardo numbers. Then*

$$\begin{aligned}
Le_m^\alpha Le_{n-m+1}^\alpha + Le_{m+1}^\alpha Le_{n-m}^\alpha = & [8(-1)^{n-m}(Le_{2m-n-1}+1)-Le_{m-1}-Le_{n-m}+Le_m+Le_{n-m-1} \\
& +\alpha(8(-1)^{n-m}(Le_{2m-n-2}+1)+8(-1)^{m-1}(Le_{n-2m-2}+1)+Le_{n-m}+2Le_{n-1}) \\
& +\alpha^2(8(-1)^{n-m-1}(Le_{2m-n-1}+1)+Le_{m-1}+Le_{n-m-2}), \\
& 8(-1)^{n-m+2}(Le_{2m-n}+1)-Le_{m+1}-Le_{n-m+2}+Le_{m+2}+Le_{n-m+1} \\
& -\alpha(12(-1)^{n-m+3}(Le_{2m-n-2}+1)-Le_{n-m+2} \\
& +12(-1)^{n-m+2}(Le_{2m-n}+1)+Le_{m+3}-1) \\
& +\alpha^2(8(-1)^{n-m-1}(Le_{2m-n-1}+1)+2Le_{n-m-2}+Le_{m-1}-Le_m+1)].
\end{aligned}$$

Proof. Using the Definition 21 we obtain

$$\begin{aligned}
Le_m^\alpha Le_{n-m+1}^\alpha + Le_{m+1}^\alpha Le_{n-m}^\alpha = & [Le_{m-1}+\alpha(Le_{m-2}+1), Le_{m+1}-\alpha(Le_{m-1}+1)] \\
& \times[Le_{n-m}+\alpha(Le_{n-m-1}+1), Le_{n-m+2}-\alpha(Le_{n-m}+1)] \\
& +[Le_m+\alpha(Le_{m-1}+1), Le_{m+2}-\alpha(Le_m+1)] \\
& \times[Le_{n-m-1}+\alpha(Le_{n-m-2}+1), Le_{n-m+1}-\alpha(Le_{n-m-1}+1)] \\
= & [(Le_{m-1}+\alpha(Le_{m-2}+1))(Le_{n-m}+\alpha(Le_{n-m-1}+1)), \\
& (Le_{m+1}-\alpha(Le_{m-1}+1))(Le_{n-m+2}-\alpha(Le_{n-m}+1))] \\
& +[(Le_m+\alpha(Le_{m-1}+1))(Le_{n-m-1}+\alpha(Le_{n-m-2}+1)), \\
& (Le_{m+2}-\alpha(Le_m+1))(Le_{n-m+1}-\alpha(Le_{n-m-1}+1))] \\
= & [(Le_{m-1}+\alpha(Le_{m-2}+1))(Le_{n-m}+\alpha(Le_{n-m-1}+1)) \\
& +(Le_m+\alpha(Le_{m-1}+1))(Le_{n-m-1}+\alpha(Le_{n-m-2}+1)), \\
& (Le_{m+1}-\alpha(Le_{m-1}+1))(Le_{n-m+2}-\alpha(Le_{n-m}+1)) \\
& +(Le_{m+2}-\alpha(Le_m+1))(Le_{n-m+1}-\alpha(Le_{n-m-1}+1))].
\end{aligned}$$

Denote $A_n = Le_{n-1}+\alpha Le_{n-2}$ and $B_n = Le_{n+1}-\alpha Le_{n-1}$, then we have

$$\begin{aligned}&(Le_{m-1}+\alpha(Le_{m-2}+1))(Le_{n-m}+\alpha(Le_{n-m-1}+1)\\&=(Le_{m-1}+\alpha Le_{m-2})(Le_{n-m}+\alpha Le_{n-m-1})+\alpha(Le_{m-1}+\alpha Le_{m-2}+Le_{n-m}+\alpha Le_{n-m-1})\\&=A_mA_{n-m+1}+\alpha(Le_{m-1}+Le_{n-m})+\alpha^2(Le_{m-2}+Le_{n-m-1}),\end{aligned}\tag{27}$$

$$\begin{aligned}&(Le_{m+1}-\alpha(Le_{m-1}+1))(Le_{n-m+2}-\alpha(Le_{n-m}+1))\\&=(Le_{m+1}-\alpha Le_{m-1})(Le_{n-m+2}-\alpha Le_{n-m})-\alpha(Le_{m+1}-\alpha Le_{m-1}+Le_{n-m+2}-\alpha Le_{n-m})\\&=B_mB_{n-m+1}-\alpha(Le_{m+1}+Le_{n-m+2})+\alpha^2(Le_{m-1}+Le_{n-m}),\end{aligned}\tag{28}$$

$$\begin{aligned}&(Le_{m}+\alpha(Le_{m-1}+1))(Le_{n-m-1}+\alpha(Le_{n-m-2}+1))\\&=(Le_{m}+\alpha Le_{m-1})(Le_{n-m-1}+\alpha Le_{n-m-2})+\alpha(Le_{m}+\alpha Le_{m-1}+Le_{n-m-1}+\alpha Le_{n-m-2})\\&=A_{m+1}A_{n-m}+\alpha(Le_{m}+Le_{n-m-1})+\alpha^2(Le_{m-1}+Le_{n-m-2}),\end{aligned}\tag{29}$$

and

$$\begin{aligned}&(Le_{m+2}-\alpha(Le_{m}+1))(Le_{n-m+1}-\alpha(Le_{n-m-1}+1))\\&=(Le_{m+2}-\alpha Le_{m})(Le_{n-m+1}-\alpha Le_{n-m-1})+\alpha(Le_{m+2}-\alpha Le_{m}+Le_{n-m+1}-\alpha Le_{n-m-1})\\&=B_{m+1}B_{n-m}-\alpha(Le_{m+2}+Le_{n-m+1})+\alpha^2(Le_{m}+Le_{n-m-1}).\end{aligned}\tag{30}$$

Now, since

$$\begin{aligned}A_mA_{n-m+1}&=Le_{m-1}Le_{n-m}+\alpha(Le_{n-m}Le_{m-2}+Le_{m-1}Le_{n-m-1})+\alpha^2Le_{m-2}Le_{n-m-1},\\A_{m+1}A_{n-m}&=Le_{m}Le_{n-m-1}+\alpha(Le_{n-m-1}Le_{m-1}+Le_{m}Le_{n-m-2})+\alpha^2Le_{m-1}Le_{n-m-2},\\B_mB_{n-m+1}&=Le_{m+1}Le_{n-m+2}-\alpha(Le_{m-1}Le_{n-m+2}+Le_{m+1}Le_{n-m})+\alpha^2Le_{m-1}Le_{n-m},\\B_{m+1}B_{n-m}&=Le_{m+2}Le_{n-m+1}-\alpha(Le_{m}Le_{n-m+1}+Le_{m+2}Le_{n-m-1})+\alpha^2Le_{m}Le_{n-m-1},\end{aligned}$$

then, by summing Equations (27) and (29), we obtain the first component given by

$$\begin{aligned}&A_mA_{n-m+1}+\alpha(Le_{m-1}+Le_{n-m})+\alpha^2(Le_{m-2}+Le_{n-m-1})\\&+A_{m+1}A_{n-m}+\alpha(Le_{m}+Le_{n-m-1})+\alpha^2(Le_{m-1}+Le_{n-m-2})\\&=Le_{m-1}Le_{n-m}+Le_{m}Le_{n-m-1}\\&+\alpha(Le_{n-m}Le_{m-2}+Le_{m-1}Le_{n-m-1}+Le_{n-m-1}Le_{m-1}+Le_{m}Le_{n-m-2}+Le_{m-1}+Le_{n-m}+Le_{m}+Le_{n-m-1})\\&+\alpha^2(Le_{m-2}Le_{n-m-1}+Le_{m-1}Le_{n-m-2}+Le_{m-2}+Le_{n-m-1}).\end{aligned}\tag{31}$$

Similarly, by summing Equations (27) and (29), we obtain the second component given by

$$\begin{aligned}&B_mB_{n-m+1}-\alpha(Le_{m+1}+Le_{n-m+2})+\alpha^2(Le_{m-1}+Le_{n-m})\\&+B_{m+1}B_{n-m}-\alpha(Le_{m+2}+Le_{n-m+1})+\alpha^2(Le_{m}+Le_{n-m-1})\\&=Le_{m+1}Le_{n-m+2}+Le_{m+2}Le_{n-m+1}\\&-\alpha(Le_{m-1}Le_{n-m+2}+Le_{m+1}Le_{n-m}+Le_{m}Le_{n-m+1}+Le_{m+2}Le_{n-m-1}+Le_{m+1}+Le_{n-m+2})\\&+\alpha^2(Le_{m-1}Le_{n-m}+Le_{m}Le_{n-m-1}+Le_{m-1}+Le_{n-m}).\end{aligned}\tag{32}$$

Theorems 2.1 and 2.14 in [5] established

$$Le_{-n} = (-1)^n (Le_{n-2} + 1) - 1,$$
$$Le_k Le_m + Le_s Le_t = 4(-1)^m (Le_{k-s-1} + 1)(Le_{k-t-1} + 1) - Le_k - Le_m + Le_s + Le_t,$$

for positive integers n, k, m, s and t with $k + m = s + t$, then holds:

$$\begin{aligned} Le_{m-1}Le_{n-m} + Le_m Le_{n-m-1} &= 4(-1)^{n-m}(Le_{-2} + 1)(Le_{2m-n-1} + 1) - Le_{m-1} - Le_{n-m} + Le_m + Le_{n-m-1} \\ &= 8(-1)^{n-m}(Le_{2m-n-1} + 1) - Le_{m-1} - Le_{n-m} + Le_m + Le_{n-m-1}, \end{aligned}$$

$$Le_{m-2}Le_{n-m} + Le_{m-1}Le_{n-m-1} = 8(-1)^{n-m}(Le_{2m-n-2} + 1) - Le_{m-2} - Le_{n-m} + Le_{m-1} + Le_{n-m-1},$$

$$Le_{n-m-1}Le_{m-1} + Le_m Le_{n-m-2} = 8(-1)^{m-1}(Le_{n-2m-2} + 1) - Le_{n-m-1} - Le_{m-1} + Le_m + Le_{n-m-2},$$

and

$$Le_{m-1}Le_{n-m} + Le_m Le_{n-m-1} = 8(-1)^{n-m}(Le_{2m-n-1} + 1) - Le_{m-1} - Le_{n-m} + Le_m + Le_{n-m-1}.$$

Therefore, we can rewrite Equation (31) in the form

$$\begin{gathered} A_m A_{n-m+1} + \alpha(Le_{m-1} + Le_{n-m}) + \alpha^2(Le_{m-2} + Le_{n-m-1}) \\ +A_{m+1}A_{n-m} + \alpha(Le_m + Le_{n-m-1}) + \alpha^2(Le_{m-1} + Le_{n-m-2}) \\ = 8(-1)^{n-m}(Le_{2m-n-1} + 1) - Le_{m-1} - Le_{n-m} + Le_m + Le_{n-m-1} \\ +\alpha(8(-1)^{n-m}(Le_{2m-n-2} + 1) + 8(-1)^{m-1}(Le_{n-2m-2} + 1) + Le_{n-m} + 2Le_{n-1}) \\ +\alpha^2(8(-1)^{n-m-1}(Le_{2m-n-1} + 1) + Le_{m-1} + Le_{n-m-2}). \end{gathered} \tag{33}$$

Similarly, we have,

$$Le_{m+1}Le_{n-m+2} + Le_{m+2}Le_{n-m+1} = 8(-1)^{n-m+2}(Le_{2m-n} + 1) - Le_{m+1} - Le_{n-m+2} + Le_{m+2} + Le_{n-m+1},$$

$$Le_{m-1}Le_{n-m+2} + Le_{m+1}Le_{n-m} = 12(-1)^{n-m+3}(Le_{2m-n-2} + 1) - Le_{m-1} - Le_{n-m+2} + Le_{m+1} + Le_{n-m},$$

$$Le_m Le_{n-m+1} + Le_{m+2}Le_{n-m-1} = 12(-1)^{n-m+2}(Le_{2m-n} + 1) - Le_m - Le_{n-m+1} + Le_{m+2} + Le_{n-m-1},$$

and

$$Le_{m-2}Le_{n-m-1} + Le_{m-1}Le_{n-m-2} = 8(-1)^{n-m-1}(Le_{2m-n-1} + 1) - Le_{m-2} - Le_{n-m-1} + Le_{m-1} + Le_{n-m-2}.$$

Therefore

$$\begin{gathered}B_mB_{n-m+1}-\alpha(Le_{m+1}+Le_{n-m+2})+\alpha^2(Le_{m-1}+Le_{n-m})\\+B_{m+1}B_{n-m}-\alpha(Le_{m+2}+Le_{n-m+1})+\alpha^2(Le_m+Le_{n-m-1})\\=8(-1)^{n-m+2}(Le_{2m-n}+1)-Le_{m+1}-Le_{n-m+2}+Le_{m+2}+Le_{n-m+1}\\-\alpha(12(-1)^{n-m+3}(Le_{2m-n-2}+1)-Le_{n-m+2}+12(-1)^{n-m+2}(Le_{2m-n}+1)+Le_{m+3}-1)\\+\alpha^2(8(-1)^{n-m-1}(Le_{2m-n-1}+1)+2Le_{n-m-2}+Le_{m-1}-Le_m+1).\end{gathered}$$

which proves the theorem. □

Next, we will provide identities involving the fuzzy Leonardo numbers and the fuzzy Fibonacci numbers.

Proposition 3.6. *Consider $\alpha\in[0,1]$. Let $\{Le^\alpha_n\}_{n\geq 0}$ be the sequence of fuzzy Leonardo numbers and $\{F^\alpha_n\}_{n\geq 0}$ be the sequence of fuzzy Fibonacci numbers. Then, the following identities hold:*

$$Le^\alpha_{n+9}-Le^\alpha_{n-1}=22F^\alpha_{n+5},\ n\geq 1; \tag{34}$$

$$Le^\alpha_{n+3}+Le^\alpha_{n-1}+2I^\alpha=6F^\alpha_{n+2},\ n\geq 1; \tag{35}$$

$$Le^\alpha_{n+1}-Le^\alpha_{n}=(-2F_n,2F_n,4F_{n+1}),\ n\geq 0; \tag{36}$$

$$Le^\alpha_{n+1}-Le^\alpha_{n-3}=2L^\alpha_n,\ n\geq 3; \tag{37}$$

$$Le^\alpha_{n+1}=3Le^\alpha_{n-1}+2I^\alpha+2L^\alpha_n,\ n\geq 1; \tag{38}$$

$$2Le^\alpha_{n}-Le^\alpha_{n-1}+I^\alpha=2L^\alpha_n,\ n\geq 1; \tag{39}$$

$$Le^\alpha_{n}+Le^\alpha_{n-2}+2I^\alpha=2L^\alpha_n,\ n\geq 2, \tag{40}$$

$$5Le^\alpha_{n-1}+5I^\alpha=2(L^\alpha_{n+1}+L^\alpha_{n-1})=2(2L^\alpha_{n+1}-L^\alpha_n),\ n\geq 1, \tag{41}$$

where F^α_n in the n-the fuzzy Fibonacci number, and where L^α_n in the n-the fuzzy Lucas number.

Proof. First, by combining Equations (23) and (11), we have

$$\begin{aligned}Le^\alpha_{n+9} &= 2F^\alpha_{10}-I^\alpha\\&= 2(11F^\alpha_{n+5}+F^\alpha_n)-I^\alpha\\&= 22F^\alpha_{n+5}+2F^\alpha_n-I^\alpha\\&= 22F^\alpha_{n+5}+Le^\alpha_{n-1},\end{aligned}$$

which proves Equation (34). Similarly, by combining Equations (23) and (10), we obtain Equation (35). By combining Equations (23) and (12) we get Equation (36). Finally, for to prove Equations (37), (38), (39), (40), and (41), we combine Equations (23) and (13), Equations (23) and (14), Equations (23) and (15), Equations (23) and (16), and using Equations (23) and (17), respectively. □

4 Some sums involving fuzzy Leonardo numbers

In this section, we will provide some identities involving the sums of fuzzy Leonardo numbers. First, recall $I^\alpha=[1^\alpha,1^\alpha]=[1,1]$. By definition of the fuzzy number, we obtain $A^\alpha I^\alpha=I^\alpha A^\alpha=A^\alpha$ for all A^α. Therefore, we have the following lemma.

Lemma 4.1. *Consider the fuzzy number* $I^\alpha = [1,1]$. *Then*

$$\sum_{j=1}^{n} I^\alpha = \left[\frac{n(n+1)}{2}\right]^\alpha .$$

Proof. Note that, by summation rule (6),

$$\begin{aligned}\sum_{j=1}^{n} I^\alpha &= \sum_{j=1}^{n}[1^\alpha, 1^\alpha] = \sum_{j=1}^{n}[1,1] = \left[\sum_{j=1}^{n}1, \sum_{j=1}^{n}1\right] \\ &= \left[\frac{n(n+1)}{2}, \frac{n(n+1)}{2}\right] = \left[\frac{n(n+1)}{2}\right]^\alpha ,\end{aligned}$$

as required. □

Theorem 4.2. *Let* $\{Le_j^\alpha\}_{j\geq 0}$ *be the sequence of fuzzy Leonardo numbers. Then the sum of the* n *first terms of the sequence consisting of these fuzzy numbers is given by*

$$\sum_{j=0}^{n} Le_j^\alpha = 2(F_{n+1}^\alpha - F_1^\alpha) - \left[\frac{(n-1)n}{2}\right]^\alpha + [1-\alpha, 1+\alpha].$$

Proof. Combining Theorem 3.5 in [5], Lemma 4.1 and Proposition 3.3, we get

$$\begin{aligned}\sum_{j=0}^{n} Le_j^\alpha &= \sum_{j=1}^{n}\left(2F_{j-1}^\alpha - I^\alpha\right) + Le_0^\alpha \\ &= \left(\sum_{j=0}^{n-1} 2F_j^\alpha - \sum_{j=1}^{n} I^\alpha\right) + Le_0^\alpha \\ &= 2\sum_{j=0}^{n-1} F_j^\alpha - \sum_{j=1}^{n-1} I^\alpha - F_{-1}^\alpha \\ &= 2(F_{n+1}^\alpha - F_1^\alpha) - \left[\frac{(n-1)n}{2}\right]^\alpha + [1-\alpha, 1+\alpha],\end{aligned}$$

as required. □

Proposition 4.3. *Let* $\{Le_j^\alpha\}_{j\geq 0}$ *be the sequence of fuzzy Leonardo numbers. Then the sum of* n *first even terms of the sequence is:*

$$\sum_{j=0}^{n} Le_{2j}^\alpha = 2(F_{2n}^\alpha - F_1^\alpha) - \left[\frac{(n-1)n}{2}\right]^\alpha + [1-\alpha, 1+\alpha].$$

Proof. Note that

$$\begin{aligned}\sum_{j=0}^{n} Le_{2j}^\alpha &= \sum_{j=1}^{n}\left(2F_{2j-1}^\alpha - I^\alpha\right) + Le_0^\alpha \\ &= \left(\sum_{j=0}^{n-1} 2F_{2j-1}^\alpha - \sum_{j=1}^{n} I^\alpha\right) + Le_0^\alpha \\ &= 2\sum_{j=0}^{n-1} F_{2j-1}^\alpha - \sum_{j=1}^{n-1} I^\alpha + [1-\alpha, 1+\alpha] .\end{aligned}$$

According Theorem 3.5 in [5], and Lemma 4.1, we have that

$$\sum_{j=0}^{n} Le_{2j}^{\alpha} \quad = \quad 2(F_{2n}^{\alpha} - F_{1}^{\alpha}) - \left[\frac{(n-1)n}{2}\right]^{\alpha} + [1-\alpha, 1+\alpha],$$

as required. □

Proposition 4.4. *Let $\{Le_j^{\alpha}\}_{j\geq 0}$ be the sequence of fuzzy Leonardo numbers. Then the sum of n first odd terms of the sequence is:*

$$\sum_{j=0}^{n} Le_{2j+1}^{\alpha} = 2(F_{2n+1}^{\alpha} - F_{1}^{\alpha}) - \left[\frac{(n-1)n}{2}\right]^{\alpha} + [1-\alpha, 1+\alpha].$$

Proof. Observe that

$$\begin{aligned} \sum_{j=0}^{n} Le_{2j+1}^{\alpha} &= \sum_{j=1}^{n} \left(2F_{2j}^{\alpha} - I^{\alpha}\right) + Le_0^{\alpha} \\ &= \left(\sum_{j=0}^{n-1} 2F_{2j}^{\alpha} - \sum_{j=1}^{n} I^{\alpha}\right) + Le_0^{\alpha} \\ &= 2\sum_{j=0}^{n} F_{2j}^{\alpha} - \sum_{j=1}^{n-1} I^{\alpha} + [1-\alpha, 1+\alpha] \; . \end{aligned}$$

Therefore, by Theorem 3.5 in [5] we obtain

$$\sum_{j=0}^{n} Le_{2j+1}^{\alpha} \quad = \quad 2(F_{2n+1}^{\alpha} - F_{1}^{\alpha}) - \left[\frac{(n-1)n}{2}\right]^{\alpha} + [1-\alpha, 1+\alpha],$$

as desired. □

A direct and immediate consequence of Proposition 4.3 and Proposition 4.4 is the result we now present, which arises naturally from the established relationships and further reinforces the conclusions derived from the propositions.

Proposition 4.5. *Let $\{Le_n^{\alpha}\}_{jn\geq 0}$ be the sequence of fuzzy Leonardo numbers. For all non-negative integers n, we have the following formulas:*

$$\sum_{j=0}^{n} (-1)^k Le_k^{\alpha} = 2F_{2n}^{\alpha} - 2F_{2n+1}^{\alpha};$$

if the last term is negative and

$$\sum_{j=0}^{n} (-1)^k Le_k^{\alpha} = 2F_{2n+2}^{\alpha} - 2F_{2n+1}^{\alpha} + [2n+1]^{\alpha};$$

if the last term is positive.

Proof. First, consider that the last term is negative, then

$$\begin{aligned}
&\sum_{k=0}^{2n+1}(-1)^k Le_k^\alpha \\
=\ & Le_0^\alpha - Le_1^\alpha + Le_2^\alpha - Le_3^\alpha + \cdots + Le_{2n}^\alpha - Le_{2n+1}^\alpha \\
=\ & (Le_0^\alpha + Le_2^\alpha + \cdots + Le_{2n}^\alpha) - (Le_1^\alpha + Le_3^\alpha + \cdots + Le_{2n+1}^\alpha) \\
=\ & \sum_{k=0}^{n} Le_{2k}^\alpha - \sum_{k=0}^{n} Le_{(2k+1)}^\alpha \\
=\ & \left(2(F_{2n}^\alpha - F_1^\alpha) - \left[\frac{(n-1)n}{2}\right]^\alpha + [1-\alpha, 1+\alpha]\right) \\
& - \left(2(F_{2n+1}^\alpha - F_1^\alpha) - \left[\frac{(n-1)n}{2}\right]^\alpha + [1-\alpha, 1+\alpha]\right) \\
=\ & 2F_{2n}^\alpha - 2F_{2n+1}^\alpha.
\end{aligned}$$

In which case that last term is positive, then

$$\begin{aligned}
&\sum_{k=0}^{2(n+1)}(-1)^k Le_k^\alpha \\
=\ & Le_0^\alpha - Le_1^\alpha + Le_2^\alpha - Le_3^\alpha + \cdots + Le_{2n}^\alpha - Le_{2n+1}^\alpha + Le_{2n+2}^\alpha \\
=\ & \sum_{k=0}^{n+1} Le_{2k}^\alpha - \sum_{k=0}^{n} Le_{2k+1}^\alpha \\
=\ & \left(2(F_{2n+2}^\alpha - F_1^\alpha) - \left[\frac{(n+1)(n+2)}{2}\right]^\alpha + [1-\alpha, 1+\alpha]\right) \\
& - \left(2(F_{2n+1}^\alpha - F_1^\alpha) - \left[\frac{(n-1)n}{2}\right]^\alpha + [1-\alpha, 1+\alpha]\right) \\
=\ & 2F_{2n+2}^\alpha - 2F_{2n+1}^\alpha + [2n+1]^\alpha,
\end{aligned}$$

which verifies the result. □

5 Conclusion

In this study, we introduced a new sequence of fuzzy numbers, namely, the sequence of fuzzy Leonardo numbers. We established the recurrence relation to this new sequence, some properties, as well as some identities. In addition, we explored the relation between fuzzy Leonardo, fuzzy Fibonacci, and fuzzy Lucas numbers, and some identities were given. Moreover, we provided some sums identities for the fuzzy Leonardo numbers.

It seems to us that all results given here are new in the literature.

Number sequences, especially recurring ones, establish patterns in the real world and are therefore used as discrete growth models. Discrete models are easy to solve and, in some cases, can describe solutions with predictions that are as good as continuous models. On the other hand, in some real problems, we have a certain degree of uncertainty about the solution, and that is why we use a fuzzy number to give us flexibility in finding the best solution for that problem. The construction presented in this article, a priori, is simply the immersion of a recurring integer sequence over the fuzzy number structure. However, generally, the combination of both theories can be the premise for establishing discrete growth models that combine the flexibility of fuzzy logic with the structural properties of the discrete models, and then the models can be discussed closer to the real world.

Acknowledgements: "The first author expresses their sincere thanks to the Brazilian National Council for Scientific and Technological Development- CNPq- Brazil and the Federal University of Mato Grosso do Sul UFMS/MEC Brazil for their valuable support. The second author was partially supported by PROPESQ-UFT. The last author is member of the Research Centre CMAT-UTAD (Polo of Research Centre CMAT - Centre of Mathematics of University of Minho) and she thanks the Portuguese Funds through FCT – Fundação para a Ciência e a Tecnologia, within the Projects UIDB/ 00013/2020 and UIDP/00013/2020."

Conflict of Interest: The authors declare no conflict of interest.

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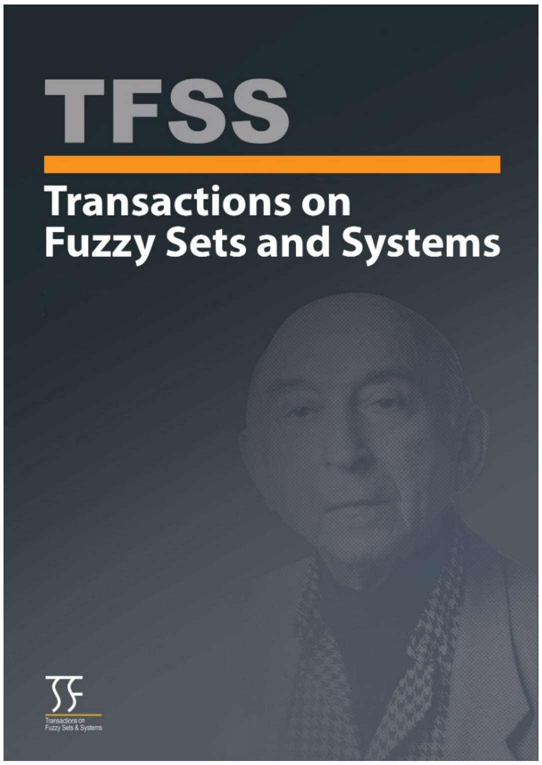

Transactions on Fuzzy Sets and Systems

ISSN: **2821-0131**

https://sanad.iau.ir/journal/tfss/

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Vol.4, No.1, (2025), 195-212. DOI: https://doi.org/10.71602/tfss.2025.1186803

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Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://sanad.iau.ir/journal/tfss/
Online ISSN: 2821-0131
Vol.4, No.1, (2025), 195-212
DOI: https://doi.org/10.71602/tfss.2025.1186803

Article Type: Original Research Article

A Hybrid Method for Numerical Solution of Fuzzy Mixed Delay Volterra-Fredholm Integral Equations System

Bahman Ghazanfari

(This paper is dedicated to Professor "John N. Mordeson" on the occasion of his 91st birthday.)

Abstract. A hybrid method for the numerical solution of the system of delayed linear fuzzy mixed Volterra-Fredholm integral equations (FMDVFIES) is introduced. Using the hybrid of Bernstein polynomials and block-pulse functions (HBBFs), an approximate solution for the equations system is provided. Firstly, the HBBFs and their operational matrices are introduced, and some of their characteristics are described. Then by applying the operational matrices on FMDVFIES convert it to the algebraic equations system. The numerical solution is obtained by solving this algebraic system. Then the convergence is investigated and some numerical examples are presented to show the effectiveness of the method.

AMS Subject Classification 2020: 03E72; 45F99; 65R10
Keywords and Phrases: Fuzzy integral equation, Block pulse function, Bernstein polynomials.

1 Introduction

Some decisions are uncertain or imprecise because they are based on imprecise data. And the dynamics governing these data cannot be stated definitively. This area of imprecise logic was first described by Zadeh in [1]. For the role of fuzzy concepts in real life, I refer the reader to the text of Lotfizadeh's letter in [2]: "The first significant real life applications of fuzzy set theory and fuzzy logic began to appear in the late seventies and early eighties. Among such applications were fuzzy logic controlled cement kilns and production of steel. The first consumer product was Matsushitas shower head, 1986. Soon, many others followed, among them home appliances, photo-graphic equipment, and automobile transmissions. A major real life application was Sendais fuzzy logic control system which began to operate in 1987 and was and is a striking success. In the realm of medical instrumentation, a notable real life application is Omrons fuzzy logic based and widely used blood pressure meter."

This concept of fuzzy quickly spread in most fields of science and engineering. Especially the role of fuzzy mathematics in this expansion has been very significant. It can be claimed that it is used in all branches of classical mathematics. Including mathematical analysis, which has a wide expansion in all its concepts such as derivatives [3, 4] differential equations, [5, 6, 7, 8], the concept of fuzzy integral [9, 10]. Differential and integral equations [11, 12], and various exact and approximate methods for solving them have been

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Received: 13 October 2024; **Revised**: 10 January 2025; **Accepted**: 16 January 2025; **Available Online**: 24 January 2025; **Published Online**: 7 May 2025.

How to cite: Ghazanfari B., A hybrid method for numerical solution of fuzzy mixed delay Volterra-Fredholm integral equations system. *Transactions on Fuzzy Sets and Systems.* 2025; 4(1): 195-212. DOI: https://doi.org/10.71602/tfss.2025.1186803

(This article is the extended version of the article that was presented at The 22nd Iran Fuzzy Systems Conference.)

presented. Abbasbandy et al. [13] applied Rung-Kutta method for fuzzy differential equations, Araghi et al. [14] introduced the Lagrange interpolation based on the extension principle for fuzzy Fredholm integral equations, Ezzati et al. [15] presented numerical solution of two-dimensional fuzzy Fredholm integral equation of the second kind using fuzzy bivariate Bernstein polynomials, Shafiee et al. [16] applied predictor corrector method for nonlinear fuzzy Volterra integral equations, and Amin et al. [17] used Haar wavelet for solution of delay Volterra-Fredholm integral equations.

Many researchers have demonstrated the efficiency and error reduction of the combined Bernstein and Block Pulse methods for various fuzzy and non-fuzzy problems, such as [18] and [19] for fuzzy Fredholm integral equations, and [20] for fractional differential equations, and [21] for a system of linear Fredholm integral equations.

Delayed integral equations are a very important area in mathematics, where many phenomena in physics, biology and economics are modeled by such equations. Therefore, finding an exact or approximate solution for them is very important. Considering that many parameters in these models can have an uncertain nature. Therefore, their solution can be considered based on fuzzy concepts.

A numerical approximation method is proposed using the combination of Bernstein and block-pulse functions (HBBF) to FMDVFIES.

$$\mathbf{y}(t) = \mathbf{f}(t) \oplus \sum_{j=1}^{\sigma}{}^{*} \mathbf{A}_j \odot \mathbf{y}(t-\tau_j) \oplus \int_0^t \int_0^1 \mathbf{k}(s,t) \odot \mathbf{y}(t) dt ds, \ \ \tau_j, t \in [0,1], \tag{1}$$

where $0 \leq \tau_j \leq 1$, $\mathbf{A}_j \in M_{p\times p}$, the set of real $p \times p$ matrices, for $j = 1, ..., \sigma$, and $\mathbf{y}(t) = \mathbf{y}_0(t)$, $t \leq 0$, and

$$\mathbf{y}(t) = [y_1(t), y_2(t), ..., y_p(t)]^T$$

is unknown function for every $t \in (0, 1]$. While $\mathbf{f}(t)$ and $\mathbf{k}(s,t)$ are known vector and matrix functions respectively,

$$\mathbf{f}(t) = [f_1(t), f_2(t), ..., f_p(t)]^T,$$

and

$$\mathbf{k}(s,t) = [k_{ij}(s,t)], \quad i, j = 1, 2, ..., p.$$

The main outlines of the hybrid method to FMDVFIES can be expressed as follows:

- The non-zero coefficients of Bernstein polynomials are natural numbers. Therefore, there is no coefficient error in the computations, a property that some polynomials, such as the Legendre and Bernoulli polynomials, do not have it.
- Presenting the transformation matrix of Bernstein polynomials to block pulse functions.
- Determined operational matrices.
- By substituting these matrices into the fuzzy integral equations system with time delay, we arrive at a system of algebraic equations.
- By solving this system of linear equations, we obtain a numerical solution to the problem.

The structure of the article is as follows: In Section 2, some basic results from Bernstein polynomial, hybrid functions and an overview of fuzzy concepts are given. The main idea are presented in Section 3. In Section 4, uniqueness of the solution and convergence analysis are investigated. The proposed method is tested through two numerical examples in Section 5. The conclusions are given in the last section.

2 Preliminaries

2.1 Bernstein polynomials

The M order of Bernstein polynomials on $[0, 1]$ are defined as [22]:

$$\mathcal{B}_{m,M}(t) = \binom{M}{m} t^m (1-t)^{M-m}, \quad m = 0, 1, ..., M. \tag{2}$$

Hybrid functions $\psi_{nm}(t)$, for $n = 1, 2, ..., N$ and $m = 0, 1, 2, ..., M-1$ on $[0, 1]$ are defined as

$$\psi_{nm}(t) = \begin{cases} \mathcal{B}_{m,M-1}(Nt - n + 1), & \frac{n-1}{N} \leq t < \frac{n}{N} \\ 0, & otherwise \end{cases} \tag{3}$$

where n and M are the number of BPFs and the order of Bernstein polynomials respectively.
A function $f \in L^2[0, 1]$ can be expanded in terms of HBBFs as follows:

$$f(t) \simeq \sum_{n=1}^{N} \sum_{m=0}^{M-1} c_{nm} \psi_{nm}(t) = C^T \Psi(t), \tag{4}$$

where

$$C = [c_{10}, \ \dots \ , c_{1M-1}, c_{20}, \ \dots \ , c_{2M-1}, c_{N0}, \ \dots \ , c_{NM-1},]^T$$

$$\Psi = [\psi_{10}, \ \dots \ , \psi_{1M-1}, \psi_{20}, \ \dots \ , \psi_{2M-1}, \psi_{N0}, \ \dots \ , \psi_{NM-1}]^T,$$

and $c_{nm} = \frac{\langle f(t), \psi_{nm}(t) \rangle}{\langle \psi_{nm}(t), \psi_{nm}(t) \rangle}$ where $\langle \cdot, \cdot \rangle$ denote the inner product on $L^2[0, 1]$.

2.2 An overview of fuzzy concepts

A pair $y = (\underline{y}(r), \overline{y}(r))$ for $r \in [0, 1]$ is called a parametric form of y if
1. $\underline{y}(r)$ is a bounded left continuous monotonic increasing function on $[0, 1]$,
2. $\overline{y}(r)$ is a bounded left continuous monotonic decreasing function on $[0, 1]$,
3. $\forall r \in [0, 1], \ \underline{y}(r) \leq \overline{y}(r)$.
A number $a \in \mathbb{R}$ can be represented as $\underline{y}(r) = \overline{y}(r) = a, \ \forall r \in [0, 1]$.
Suppose E^1 be the set of all upper semi-continuous normal convex fuzzy numbers with bounded $r-$level intervals. It means that if $v \in E^1$ then the $r-$level set

$$[v]_r = \{s | v(s) \geq r\}, \quad 0 < r \leq 1,$$

is a closed bounded interval which is denoted by $[v]_r = [v_1(r), v_2(r)]$.

Lemma 2.1. Let $v, w \in E^1$ and s be scalar. Then for $r \in (0, 1]$

$$[v + w]_r = [v_1(r) + w_1(r), v_2(r) + w_2(r)],$$

$$[v - w]_r = [v_1(r) - w_2(r), v_2(r) - w_1(r)],$$

$$[v \cdot w]_r = [\min\{v_1(r) \cdot w_1(r), v_1(r) \cdot w_2(r), v_2(r) \cdot w_1(r), v_2(r) \cdot w_2(r)\},$$
$$\max\{v_1(r) \cdot w_1(r), v_1(r) \cdot w_2(r), v_2(r) \cdot w_1(r), v_2(r) \cdot w_2(r)\}],$$

$$[sv]_r = s[v]_r.$$

So, the set of all fuzzy numbers E^1 with addition and multiplication which is a convex cone and can be embedded into the Banach space $B = \overline{C}[0, 1] \times \overline{C}[0, 1], \ (B, \| \, . \, \|)$ where

$$\| \, (u, v) \, \| = \sup\{ \max_{0 \leq r \leq 1} |u(r)|, \quad \max_{0 \leq r \leq 1} |v(r)| \}. \tag{5}$$

The distance between u and v can be denoted as:

$$D(u,v) = \sup_{0\le r\le 1}\{\max[|\underline{u}(r)-\underline{v}(r)|, |\overline{u}(r)-\overline{v}(r)|]\}, \tag{6}$$

If $\widetilde{f}(t)$ is continuous in the metric D, then its definite integral exists [23], and

$$\underline{\int_a^b \widetilde{f}(t;r)dt} = \int_a^b \underline{f}(t;r)dt, \quad \overline{\int_a^b \widetilde{f}(t;r)dt} = \int_a^b \overline{f}(t;r)dt.$$

2.3 Block Pulse Functions and transformation matrix

The block-pulse functions (BPFs) and some well-known properties are introduced.

$$\mathfrak{b}_i(t) = \begin{cases} 1, & \frac{(i-1)T}{n} \le t < \frac{iT}{n} \\ 0, & otherwise \end{cases} \tag{7}$$

for $i = 1, 2, \cdots n$ are defined as a set of BPFs that have the following properties:

$$\mathfrak{b}_i(t)\mathfrak{b}_j(t) = \begin{cases} \mathfrak{b}_i(t), & i = j, \\ 0, & i \ne j, \end{cases} \tag{8}$$

$$\int_0^T \mathfrak{b}_i(t)\mathfrak{b}_j(t)dt = \begin{cases} \frac{T}{n}, & i = j, \\ 0, & i \ne j. \end{cases} \tag{9}$$

The set of BPFs is complete.

The BPFs expansion:

The expansion of $f \in L[0,T)$, with respect to BPFs $\mathfrak{B}(t) = (\mathfrak{b}_1(t), \mathfrak{b}_2(t), ..., \mathfrak{b}_n(t))^T$ is defined as [24]:

$$f(t) \simeq (f_1, f_2, ..., f_n)\mathfrak{B}(t) = F^T\mathfrak{B}(t) = \mathfrak{B}^T(t)F,$$

where $F = (f_1, f_2, ..., f_n)^T$ is given by $F = \frac{1}{h}\int_0^T f(t)\mathfrak{B}(t)dt$ and f_i is the block pulse coefficient with respect to $\mathfrak{b}_i(t)$ for $i = 1, 2, \cdots, n$.
Now, assume that $K(s,\tau)$ belongs to $L^2([0,T]\times[0,T])$ we can write

$$K(s,\tau) \simeq \mathfrak{B}^T(s)K\mathfrak{B}(\tau), \ with \ K = \frac{1}{h^2}\int_0^T\int_0^T \mathfrak{B}^T(s)K(s,\tau)\mathfrak{B}(\tau)d\tau ds,$$

and $h = \frac{T}{n}$.
And also, from [24], can be found that

$$\int_0^T \mathfrak{B}(t)\mathfrak{B}^T(t)dt = hI, \tag{10}$$

and $\int_0^t \mathfrak{B}(t)dt \simeq \mathcal{P}\mathfrak{B}(t)$ where

$$\mathcal{P} = \frac{h}{2}\begin{bmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 2 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Therefore,

$$\int_0^T f(t)dt \simeq \int_0^T F^T \mathfrak{B}(t)dt \simeq F^T \mathcal{P}\mathfrak{B}(t). \tag{11}$$

For time delay $\tau = qh$ with a non-negative integer q, we have

$$\mathfrak{B}(t-\tau) = H^q \mathfrak{B}(t), \tag{12}$$

where

$$\begin{array}{c} \text{(q+1)th-column} \\ \downarrow \end{array}$$

$$H^q = \begin{bmatrix} 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & ... & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 1 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

2.4 Transformation matrix

The HBBFs can be expanded into $NM-$terms of BPFs [25] as

$$\Psi_{NM\times 1}(t) = \Phi_{NM\times NM}\mathfrak{B}_{NM\times 1}(t) \tag{13}$$

where

$$\Phi = \begin{bmatrix} A & O & O & \cdots & O \\ 0 & A & O & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & O & A \end{bmatrix}$$

and $A = (a_{m+1,i})_{M\times M}$ whit

$$\begin{aligned} a_{m+1,i} = M \sum_{k=0}^{M-1-m} (-1)^k \begin{pmatrix} M-1 \\ m \end{pmatrix} \begin{pmatrix} M-1-m \\ k \end{pmatrix} \frac{1}{k+m+1} \\ \left[\left(\frac{i+1}{M}\right)^{k+m+1} - \left(\frac{i}{M}\right)^{k+m+1} \right], \\ i = 0, 1, \cdots, NM-1. \end{aligned} \tag{14}$$

For example, with $N = 2$ and $M = 3$,

$$\Psi(t) = [\psi_{10}(t), \psi_{11}(t), \psi_{12}(t), \psi_{20}(t), \psi_{21}(t), \psi_{22}(t)]^T$$

and

$$\mathfrak{B} = [\mathfrak{b}_1(t), \mathfrak{b}_2(t), ..., \mathfrak{b}_6(t)]^T .$$

Such that

$$\left.\begin{array}{l}\psi_{10}(t)=4t^2-4t+1\\ \psi_{11}(t)=-8t^2+4t\\ \psi_{12}(t)=4t^2\end{array}\right\},\quad when\ 0\leq t<\frac{1}{2}, \tag{15}$$

$$\left.\begin{array}{l}\psi_{20}(t)=4t^2-8t+4\\ \psi_{21}(t)=-8t^2+12t-4\\ \psi_{22}(t)=4t^2-4t+1\end{array}\right\},\quad when\ \frac{1}{2}\leq t<1, \tag{16}$$

and

$$\Phi_{6\times6}=\begin{bmatrix}A & O\\ 0 & A\end{bmatrix},$$

with

$$A=\begin{pmatrix}\frac{19}{27} & \frac{7}{27} & \frac{1}{27}\\ \frac{7}{27} & \frac{13}{27} & \frac{7}{27}\\ \frac{1}{27} & \frac{7}{27} & \frac{19}{27}\end{pmatrix}.$$

For more details, see [25].

3 The Main idea

Consider the parametric form of LFMDVFIES as follows:

$$\widetilde{\mathbf{y}}(t)=\widetilde{\mathbf{f}}(t)\oplus\sum_{j=1}^{\sigma}{}^{*}\mathbf{A}_j\odot\widetilde{\mathbf{y}}(t-\tau_j)\oplus\int_0^t\int_0^1\mathbf{k}(s,t)\odot\widetilde{\mathbf{y}}(t)dtds,\ \tau_j,t\in[0,1], \tag{17}$$

where $\widetilde{\mathbf{y}}(t)$ and $\widetilde{\mathbf{f}}(t)$ are parametric form of $\mathbf{y}(t)$ and $\mathbf{f}(t)$ in Eq. (1),

$$\widetilde{y_i}(t)=(\underline{y_i}(t,r),\overline{y_i}(t,r),\ \widetilde{y_i}(t-\tau_j)=(\underline{y_i}(t-\tau_j,r),\overline{y_i}(t-\tau_j,r)),$$
$$\widetilde{f_i}(t)=(\underline{f_i}(t,r),\overline{f_i}(t,r)),\ i=1,2,...,p,\ j=1,2,...,\sigma.$$

The expansion of functions $\widetilde{y_i}(t),\widetilde{f_i}(t)$ and $k_{ij}(s,\tau)$ can be written as follows:

$$\widetilde{y_i}(t)\simeq Y_i^T\odot\Psi(t),\quad \widetilde{f_i}(t)\simeq F_i^T\odot\Psi(t),\quad k_{ij}(s,\tau)\simeq\Psi^T(s)K_{i,j}\Psi(\tau), \tag{18}$$

where Y_i,F_i are vectors of $NM\times1$ and K_{ij} is a matrix of $NM\times NM$.
$h=1/(NM),\tau_j=q_jh,\ j=1,2,...,\sigma.$

$$\begin{aligned}y_i(t-\tau_j)&\simeq Y_i^T\odot\Psi(t-\tau_j)=Y_i^T\odot\Phi\mathfrak{B}(t-\tau_j)\\ &=Y_i^T\odot\Phi H^{q_j}\mathfrak{B}(t)=Y_i^T\odot\Phi H^{q_j}\Phi^{-1}\Psi(t),\end{aligned}$$

$HB^{q_j}=\Phi H^{q_j}\Phi^{-1}$, and also, we have

$$\begin{aligned}&y_i(t-\tau_j)\simeq Y_i^THB^{q_j}\odot\Psi(t),\ if\ t-\tau_j>0,\\ &y_i(t-\tau_j)=y_{0i}(t),\ if\ t-\tau_j\leq0,\end{aligned} \tag{19}$$

where $y_{0i}(t)$ denotes the $i-$th element of the $\mathbf{y}_0(t)$. The integration of vector $\Psi(t)$ can be approximated as

$$(I\Psi)(t)\simeq P\Psi(t), \tag{20}$$

where the $NM \times NM$ matrix P is called the HBBfs operational matrix of integration.

$$(I\Psi)(t) \simeq (I\Phi\mathfrak{B})(t) = \Phi(I\mathfrak{B})(t) \simeq \Phi\mathcal{P}\mathfrak{B}(t) = \Phi\mathcal{P}\Phi^{-1}\Psi(t), \tag{21}$$

so

$$P = \Phi\mathcal{P}\Phi^{-1}.$$

And also

$$\int_0^T \Psi(t)\Psi^T dt = \int_0^T \Phi\mathfrak{B}(t)\mathfrak{B}^T(t)\Phi^T dt = h\Phi\Phi^T = D_h,$$

hence

$$D_h = \frac{T}{NM}\begin{bmatrix} AA^T & O & O & \cdots & O \\ 0 & AA^T & O & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & O & AA^T \end{bmatrix}.$$

By substituting Eqs (18)–(21) into (17), we have

$$\begin{aligned} Y^T \odot \Psi = F^T \odot \Psi \oplus \sum_{j=1}^{\sigma} A^{(j)} \odot Y^T \odot HB^{q_j} \odot \Psi \\ \oplus \int_0^t \int_0^1 \Psi^T K \Psi \odot \psi^T Y dtds. \end{aligned} \tag{22}$$

And hence

$$Y^T = F^T \oplus \sum_{j=1}^{\sigma} Y^T \odot D_{A^{(j)}} \odot HB^{q_j} \oplus Y^T \odot D_h^T K^T P,$$

where

$$D_{A^{(j)}} = \begin{bmatrix} A^{(j)} & O & O & \cdots & O \\ 0 & A^{(j)} & O & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & O & A^{(j)} \end{bmatrix},$$

then

$$Y^T \odot \left(I \ominus \sum_{j=1}^{\sigma} D_{A^{(j)}} \odot HB^{q_j} \ominus D_h^T K^T P \right) = F^T.$$

The approximate solution of Eq.(17) can be found by solving these linear equations and taking $\widetilde{y}(t) = Y^T \odot \Psi(t)$.

4 Uniqueness of the solution and Convergence analysis

Consider the equation

$$\widetilde{\mathbf{y}}(t) = \widetilde{\mathbf{f}}(t) \oplus \widetilde{\mathbf{y}}(t-\tau) \oplus \int_0^t \int_0^1 \mathbf{k}(s,t) \odot \widetilde{\mathbf{y}}(t)dtds,\ \ \tau, t \in [0,1], \tag{23}$$

or of even more general form:

$$\widetilde{\mathbf{y}}(t) = \widetilde{\mathbf{f}}(t) \oplus \sum_{j=1}^{\sigma\ *} \mathbf{A}_j \odot \widetilde{\mathbf{y}}(t-\tau_j) \oplus \int_0^t \int_0^1 \mathbf{k}(s,t) \odot \widetilde{\mathbf{y}}(t)dtds,\ \ \tau_j, t \in [0,1]. \tag{24}$$

4.1 Uniqueness

For Eqs.(23) and (24), the delays $t-\tau$ and $t-\tau_j$ are bounded and $\widetilde{\mathbf{y}}(t)=\widetilde{\mathbf{y_0}}(t),\ t\leq 0$. Then $\widetilde{\mathbf{y}}(t-\tau)$ is a known function of t for $0\leq t\leq\tau$. We then show that Eq.(23) has a unique solution $\widetilde{\mathbf{y}}(t)$ for $-\tau\leq t\leq\tau$ and then can be compute the solution for $-\tau\leq t\leq 2\tau$ and so on. By continuing this process, the existence and uniqueness of the solution for all $-\tau\leq t\leq 1$ is obtained. For any $0\leq\tau\leq 1$, consider $C_0=C([-\tau,0],E^p)$. Suppose that $\widetilde{\mathbf{y}}\in C(J_0,E^p)$ where $J_0=[-\tau,\tau]$ and also $\widetilde{\mathbf{f}}\in C([0,1],E^p)$, and

$$\mathbf{k}=[k_{ij}]_{p\times p},k_{ij}\in C([0,1]\times[0,1],\mathbb{R}),$$

such that

$$\max_{1\leq i,j\leq p}\max_{0\leq s,t\leq 1}\mid k_{ij}(s,t)\mid=K.$$

Theorem 4.1. Suppose that $\widetilde{\mathbf{y}}(t)=\widetilde{\mathbf{y_0}}(t),\ t\leq 0$, and $\widetilde{\mathbf{y_0}}(t)\in C_0$. If

$$\parallel\mathbf{k}\parallel_\infty=\max_{1\leq i\leq p}\sum_{j=1}^{p}\mid k_{ij}(s,t)\mid=pK<1,\ 0\leq s,t\leq 1$$

then Eq.(23) has a unique solution $\widetilde{\mathbf{y}}(t)$ on J_0.

Proof. Define the metric on $C(J_0,E^p)$ by

$$\mathbf{D}(\mathbf{u}(t),\mathbf{v}(t))=\begin{pmatrix}D(u_1(t),v_1(t))\\D(u_2(t),v_2(t))\\\vdots\\D(u_p(t),v_p(t))\end{pmatrix}.$$

We define the operator T on $C(J_0,E^p)$ by

$$\begin{aligned}T\widetilde{\mathbf{u}}(t)&=\widetilde{\mathbf{y_0}}(t),\qquad -\tau\leq t\leq 0,\\T\widetilde{\mathbf{u}}(t)&=\widetilde{\mathbf{f}}(t)\oplus\widetilde{\mathbf{y_0}}(t)\oplus\int_0^t\int_0^1\mathbf{k}(s,z)\odot\widetilde{\mathbf{u}}(z)dzds,t\in[0,\tau],\ \tau>0.\end{aligned}$$

We find that $\mathbf{D}(T\widetilde{\mathbf{u}}(t),T\widetilde{\mathbf{v}}(t))=0,\ -\tau\leq t\leq 0$, and for $0\leq t\leq\tau$,

$$\begin{aligned}\mathbf{D}(T\widetilde{\mathbf{u}}(t),T\widetilde{\mathbf{v}}(t))&=\\&\mathbf{D}\Big(\widetilde{\mathbf{f}}(t)\oplus\widetilde{\mathbf{y_0}}(t)\oplus\int_0^t\int_0^1\mathbf{k}(s,z)\odot\widetilde{\mathbf{u}}(z)dzds,\\&\widetilde{\mathbf{f}}(t)\oplus\widetilde{\mathbf{y_0}}(t)\oplus\int_0^t\int_0^1\mathbf{k}(s,z)\odot\widetilde{\mathbf{v}}(z)dzds,\Big)\\&\leq\int_0^t\int_0^1 pK\mathbf{D}(\widetilde{\mathbf{u}}(z),\widetilde{\mathbf{v}}(z))dzds\end{aligned}$$

$$\parallel\mathbf{D}(T\widetilde{\mathbf{u}}(t),T\widetilde{\mathbf{v}}(t))\parallel_\infty\leq pK\int_0^t\int_0^1\parallel\mathbf{D}(\widetilde{\mathbf{u}}(z),\widetilde{\mathbf{v}}(z))\parallel_\infty dzds$$

Hence we have

$$\parallel\mathbf{D}(T\widetilde{\mathbf{u}}(t),T\widetilde{\mathbf{v}}(t))\parallel_\infty<\parallel\mathbf{D}(\widetilde{\mathbf{u}}(z),\widetilde{\mathbf{v}}(z))\parallel_\infty.$$

So, the operator T is a contraction on $C(J_0,E^p)$. Therefore T has a unique fixed point $\widetilde{\mathbf{y}}\in C(J_0,E^p)$, and consequently this $\widetilde{\mathbf{y}}=\widetilde{\mathbf{y}}(t)$ is the unique solution of Eq.(23) on J_0. □

Theorem 4.2. Suppose that $\widetilde{y}_i(t)_{M,N}$ and $\widetilde{y}_i(t)$ are the approximate solution by HBBFs and exact solution of $i-th$ component of $\widetilde{\mathbf{y}}(t)$ in Eq. (17) respectively. If $k_{ij}(s,t)$ for all $s,t\in[0,1]$ is continuous and bounded, then $\widetilde{y}_i(t)_{M,N}\longrightarrow\widetilde{y}_i(t)$ as $M,N\longrightarrow\infty$, for any $i=1,2,...,p$.

Proof.

$$D(\widetilde{y}_i(t),\widetilde{y}_i(t)_{M,N})\leq\sum_{j=1}^{\sigma}{}^{*}\sum_{l=1}^{p}{}^{*}D(a_{il}^{(j)}\odot\widetilde{y}_l(t-\tau_j),a_{il}^{(j)}\odot\widetilde{y}_l(t-\tau_j)_{M,N})+$$
$$\sum_{l=1}^{p}{}^{*}D\left(\int_0^t\int_0^1 k_{il}(s,\tau)\widetilde{y}_l(\tau)d\tau ds,\int_0^t\int_0^1 k_{il}(s,\tau)\widetilde{y}_l(\tau)_{M,N}d\tau ds\right)$$
$$=\sum_{j=1}^{\sigma}{}^{*}\sum_{l=1}^{p}{}^{*}\mid a_{il}^{(j)}\mid D(\widetilde{y}_l(t-\tau_i),\widetilde{y}_l(t-\tau_i)_{M,N})+$$
$$\sum_{l=1}^{p}{}^{*}D\left(\int_0^t\int_0^1 k_{il}(s,\tau)\widetilde{y}_l(\tau)d\tau ds,\int_0^t\int_0^1 k_{il}(s,\tau)\sum_{n=1}^{N}\sum_{m=0}^{M-1}c_{l,nm}\psi_{n,m}(\tau)d\tau ds\right)$$
$$\leq\sum_{j=1}^{\sigma}{}^{*}\sum_{l=1}^{p}{}^{*}\mid a_{il}^{(j)}\mid D(\widetilde{y}_l(t-\tau_i),\widetilde{y}_l(t-\tau_i)_{M,N})+$$
$$K\sum_{l=1}^{p}{}^{*}D\left(\int_0^t\int_0^1\widetilde{y}_l(\tau)d\tau ds,\int_0^t\int_0^1\sum_{n=1}^{N}\sum_{m=0}^{M-1}c_{l,nm}\psi_{n,m}(\tau)d\tau ds\right),$$

where

$$\max_{1\leq i,j\leq p}\max_{0\leq s,t\leq 1}\mid k_{ij}(s,t)\mid=K.$$

$$\lim_{M,N\to\infty}D\left(\widetilde{y}_i(t),\widetilde{y}_i(t)_{M,N}\right)$$
$$\leq\lim_{M,N\to\infty}\sum_{j=1}^{\sigma}{}^{*}\sum_{l=1}^{p}{}^{*}\mid a_{il}^{(j)}\mid D(\widetilde{y}_l(t-\tau_j),\widetilde{y}_l(t-\tau_j)_{M,N})+$$
$$K\lim_{M,N\to\infty}\sum_{l=1}^{p}{}^{*}D\left(\int_0^t\int_0^1\widetilde{y}_l(\tau)d\tau ds,\int_0^t\int_0^1\sum_{n=1}^{N}\sum_{m=0}^{M-1}c_{l,nm}\psi_{n,m}(\tau)d\tau ds\right)$$
$$\leq\lim_{M,N\to\infty}\sum_{j=1}^{\sigma}{}^{*}\sum_{l=1}^{p}{}^{*}\mid a_{il}^{(j)}\mid D(\widetilde{y}_l(t-\tau_j),\widetilde{y}_l(t-\tau_j)_{M,N})+$$
$$K\sum_{l=1}^{p}{}^{*}D\left(\int_0^t\int_0^1\widetilde{y}_l(\tau)d\tau ds,\int_0^t\int_0^1\lim_{M,N\to\infty}\sum_{n=1}^{N}\sum_{m=0}^{M-1}c_{l,nm}\psi_{n,m}(\tau)d\tau ds\right)$$
$$\leq\lim_{M,N\to\infty}\sum_{j=1}^{\sigma}{}^{*}\sum_{l=1}^{p}{}^{*}\mid a_{il}^{(j)}\mid D(\widetilde{y}_l(t-\tau_j),\widetilde{y}_l(t-\tau_j)_{M,N})+$$
$$K\int_0^t\int_0^1\sum_{l=1}^{p}{}^{*}D\left(\widetilde{y}_l(\tau),\lim_{M,N\to\infty}\sum_{n=1}^{N}\sum_{m=0}^{M-1}c_{l,nm}\psi_{n,m}(\tau)\right)d\tau ds.$$

Since

$$\widetilde{y}_l(t) = \lim_{M,N\to\infty} \sum_{n=1}^{N} \sum_{m=0}^{M-1} c_{l,nm}\psi_{n,m}(t),$$

and for any $t - \tau_j > 0$

$$\lim_{M,N\to\infty} \widetilde{y}_l(t-\tau_j)_{M,N} = \lim_{M,N\to\infty} \sum_{n=1}^{N} \sum_{m=0}^{M-1} c_{l,nm}\psi_{n,m}(t-\tau_j) = \widetilde{y}_l(t-\tau_j),$$

and for $t - \tau_j \leq 0$

$$\widetilde{y}_l(t-\tau_j)_{M,N} = \phi_{0l}(t) = \widetilde{y}_l(t-\tau_j).$$

Hence,

$$\lim_{M,N\to\infty} \sum_{l=1}^{p}{}^{*} \mid a_{il}^{(j)} \mid \widetilde{y}_l(t-\tau_j) = \lim_{M,N\to\infty} \sum_{l=1}^{p}{}^{*} \mid a_{il}^{(j)} \mid \odot \widetilde{y}_l(t-\tau_j)_{M,N}.$$

Therefore, for every $i = 1, 2, ..., p$,

$$\lim_{M,N\to\infty} D\left(\widetilde{y}_i(t), \lim_{M,N\to\infty} \sum_{n=1}^{N} \sum_{m=0}^{M-1} c_{i,nm}\psi_{n,m}(t)\right) = 0.$$

□

5 Examples

Example 5.1. Consider the following FMDVFIES:

$$\begin{aligned} \widetilde{y_1}(t) = \widetilde{f_1}(t) \oplus (1/2)\widetilde{y_1}(t-1/3) \oplus \int_0^t \int_0^1 k_{11} \odot \widetilde{y_1}(t)dtds \\ \oplus \int_0^t \int_0^1 k_{12} \odot \widetilde{y_2}(t)dtds,\ t \in [0,1], \end{aligned}$$

$$\begin{aligned} \widetilde{y_2}(t) = \widetilde{f_2}(t) \oplus \widetilde{y_1}(t-2/3) \oplus 2\widetilde{y_2}(t-1) \oplus \int_0^t \int_0^1 k_{21} \odot \widetilde{y_1}(t)dtds \\ \oplus \int_0^t \int_0^1 k_{22} \odot \widetilde{y_2}(t)dtds, \end{aligned}$$

where $\widetilde{y_1}(t) = \widetilde{y_2}(t) = \widetilde{0}, \quad t \leq 0$, and

$$\begin{aligned} \underline{f_1}(t,r) = (\frac{r+1}{8}) \big(e^{-t} - (2-5/e)(t-t^2/2) - (1/4)(t-t^3/3) \\ -(1/2)e^{-(t-1/3)}H(t-1/3)\big), \end{aligned}$$

$$\begin{aligned} \overline{f_1}(t,r) = (\frac{3-r}{8}) \big(e^{-t} - (2-5/e)(t-t^2/2) - (1/4)(t-t^3/3) \\ -(1/2)e^{-(t-1/3)}H(t-1/3)\big), \end{aligned}$$

$$\underline{f_2}(t,r) = (\frac{r+1}{8})\left(t-(4/e-1)t^2/2-(1/5)(t-t^2/2)\right.$$
$$\left.-e^{-(t-2/3)}H(t-2/3)-(t-1)H(t-1)\right),$$

$$\overline{f_2}(t,r) = (\frac{3-r}{8})\left(t-(4/e-1)t^2/2-(1/5)(t-t^2/2)\right.$$
$$\left.-e^{-(t-2/3)}H(t-2/3)-(t-1)H(t-1)\right),$$

H is the Heaviside function as:

$$H(t) = \begin{cases} 1, & t \geq 0, \\ 0, & otherwise, \end{cases}$$

and

$$k_{11} = t^2(1-s),\ k_{12} = t^2(1-s^2),$$
$$k_{21} = (1-t^2)s,\ k_{22} = t^3(1-s).$$

The exact solutions are as follows:

$$\underline{y_1}(t,r) = \frac{(1+r)e^{-t}}{8},\ \overline{y_1}(t,r) = \frac{(3-r)e^{-t}}{8},$$
$$\underline{y_2}(t,r) = \frac{(1+r)t}{8},\ \overline{y_2}(t,r) = \frac{(3-r)t}{8}.$$

The errors are shown in Figs. 1 and 2 and the numerical values and error values are given in Tables 1, 2, 3 and 4.

Example 5.2. Consider the following FMDVFIES:

$$\widetilde{y_1}(t) = \widetilde{f_1}(t) \oplus \widetilde{y_1}(t-1) \oplus \int_0^t \int_0^1 k_{11} \odot \widetilde{y_1}(t)dtds$$
$$\oplus \int_0^t \int_0^1 k_{12} \odot \widetilde{y_2}(t)dtds,$$

$$\widetilde{y_2}(t) = \widetilde{f_2}(t) \oplus (1/2)\widetilde{y_2}(t-5/6) \oplus \int_0^t \int_0^1 k_{21} \odot \widetilde{y_1}(t)dtds$$
$$\oplus \int_0^t \int_0^1 k_{22} \odot \widetilde{y_2}(t)dtds, \quad t \in [0,1],$$

where

$$\underline{f_1}(t,r) = (r/4)\left(e^{-t}-(6-16/e)(t-t^3/3)-(1/4)(t-t^4/4)\right.$$
$$\left.-e^{-(t-1)}H(t-1)\right),$$

$$\overline{f_1}(t,r) = (1/2-r/4)\left(e^{-t}-(6-16/e)(t-t^3/3)-(1/4)(t-t^4/4)\right.$$
$$\left.-e^{-(t-1)}H(t-1)\right),$$

$$\underline{f_2}(t,r) = (r/4)\left(t - (1/4)(t + t^2/2) - (1/2)(t - 5/6)H(t - 5/6)\right) + (1/2 - r/4)t^3/e,$$

$$\overline{f_2}(t,r) = (1/2 - r/4)\left(t - (1/4)(t + t^2/2) - (1/2)(t - 5/6)H(t - 5/6)\right) + (r/4)t^3/e.$$

H is the Heaviside function, and

$$k_{11} = t^3(1 - s^2),\ k_{12} = t^2(1 - s^3),$$

$$k_{21} = (1 + t^2)s^2,\ k_{22} = t^2(1 + s).$$

The exact solutions are as follows:

$$\underline{y_1}(t,r) = \frac{re^{-t}}{4},\ \overline{y_1}(t,r) = \frac{(2-r)e^{-t}}{4}.$$

$$\underline{y_2}(t,r) = \frac{rt}{4},\ \overline{y_2}(t,r) = \frac{(2-r)t}{4}.$$

The errors are shown in Figs. 3 and 4 and the numerical values and error values are given in Tables 5, 6, 7 and 8.

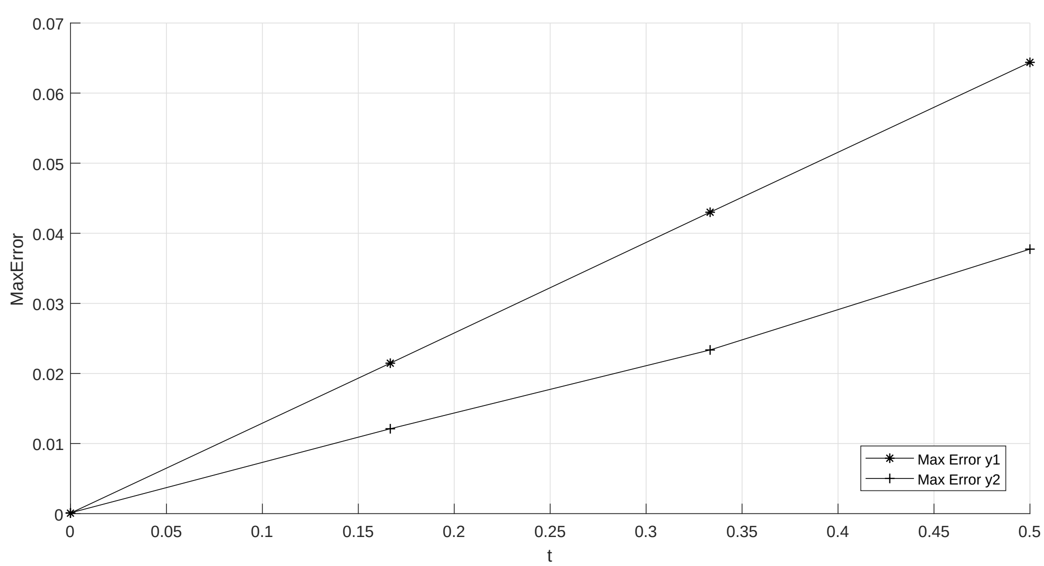

Figure 1: The errors for $M = 3, N = 2$ in Example 5.1.

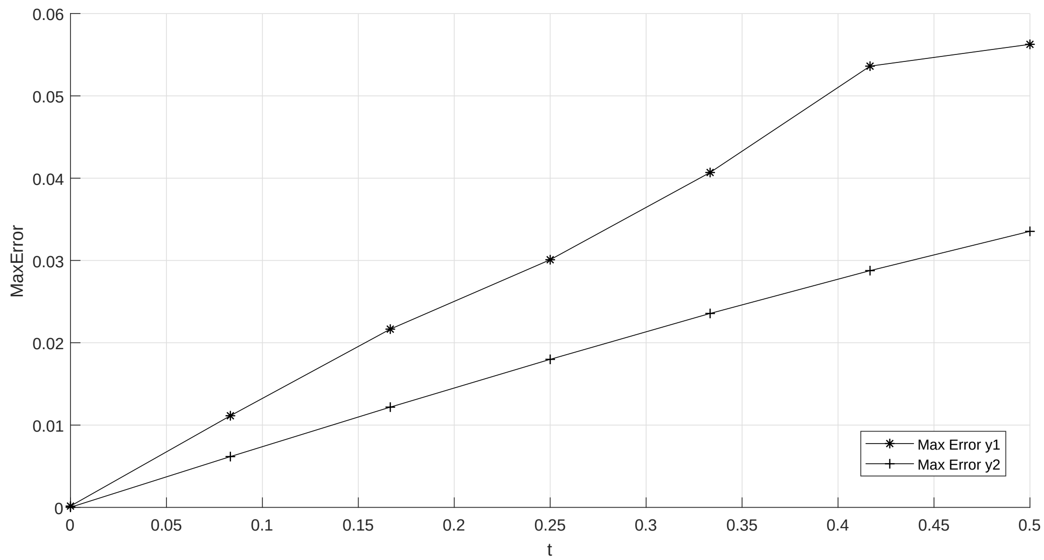

Figure 2: The errors for $M = 3, N = 4$ in Example 5.1.

Table 1: Aproximate and exact solutions and errors for $M = 3$ and $N = 2$ in Example 5.1.

$t = 0.333$ r	App. $\underline{y_1}(t,r)$	Exact $\underline{y_1}(t,r)$	Error	App. $\overline{y_1}(t,r)$	Exact $\overline{y_1}(t,r)$	Error	Max Error
0.00000	0.01575	0.02986	0.01410	0.04656	0.08957	0.04300	0.04300
0.16667	0.01832	0.03483	0.01651	0.04400	0.08459	0.04060	0.04060
0.33333	0.02089	0.03981	0.01892	0.04143	0.07961	0.03819	0.03819
0.50000	0.02346	0.04478	0.02133	0.03886	0.07464	0.03578	0.03578
0.66667	0.02602	0.04976	0.02374	0.03629	0.06966	0.03337	0.03337
0.83333	0.02859	0.05474	0.02614	0.03373	0.06469	0.03096	0.03096
1.00000	0.03116	0.05971	0.02855	0.03116	0.05971	0.02855	0.02855

Table 2: Aproximate and exact solutions and errors for $M = 3$ and $N = 2$ in Example 5.1.

$t = 0.333$ r	App. $\underline{y_2}(t,r)$	Exact $\underline{y_2}(t,r)$	Error	App. $\overline{y_2}(t,r)$	Exact $\overline{y_2}(t,r)$	Error	Max Error
0.00000	0.03373	0.04167	0.00793	0.10164	0.12500	0.02336	0.02336
0.16667	0.03939	0.04861	0.00922	0.09599	0.11806	0.02207	0.02207
0.33333	0.04505	0.05556	0.01050	0.09033	0.11111	0.02078	0.02078
0.50000	0.05071	0.06250	0.01179	0.08467	0.10417	0.01950	0.01950
0.66667	0.05637	0.06944	0.01307	0.07901	0.09722	0.01821	0.01821
0.83333	0.06203	0.07639	0.01436	0.07335	0.09028	0.01693	0.01693
1.00000	0.06769	0.08333	0.01564	0.06769	0.08333	0.01564	0.01564

Table 3: Aproximate and exact solutions and errors for $M = 3$ and $N = 4$ in Example 5.1.

$t = 0.333$ r	App. $\underline{y_1}(t,r)$	Exact $\underline{y_1}(t,r)$	Error	App. $\overline{y_1}(t,r)$	Exact $\overline{y_1}(t,r)$	Error	Max Error
0.00000	0.01411	0.02986	0.01574	0.04888	0.08957	0.04069	0.04069
0.16667	0.01701	0.03483	0.01782	0.04598	0.08459	0.03861	0.03861
0.33333	0.01991	0.03981	0.01990	0.04308	0.07961	0.03653	0.03653
0.50000	0.02281	0.04478	0.02198	0.04019	0.07464	0.03445	0.03445
0.66667	0.02570	0.04976	0.02406	0.03729	0.06966	0.03237	0.03237
0.83333	0.02860	0.05474	0.02614	0.03439	0.06469	0.03029	0.03029
1.00000	0.03150	0.05971	0.02822	0.03150	0.05971	0.02822	0.02822

Table 4: Aproximate and exact solutions and errors for $M = 3$ and $N = 4$ in Example 5.1.

$t = 0.333$ r	App. $\underline{y_2}(t,r)$	Exact $\underline{y_2}(t,r)$	Error	App. $\overline{y_2}(t,r)$	Exact $\overline{y_2}(t,r)$	Error	Max Error
0.00000	0.03381	0.04167	0.00785	0.10144	0.12500	0.02356	0.02356
0.16667	0.03945	0.04861	0.00916	0.09580	0.11806	0.02225	0.02225
0.33333	0.04508	0.05556	0.01047	0.09017	0.11111	0.02094	0.02094
0.50000	0.05072	0.06250	0.01178	0.08453	0.10417	0.01964	0.01964
0.66667	0.05635	0.06944	0.01309	0.07890	0.09722	0.01833	0.01833
0.83333	0.06199	0.07639	0.01440	0.07326	0.09028	0.01702	0.01702
1.00000	0.06762	0.08333	0.01571	0.06762	0.08333	0.01571	0.01571

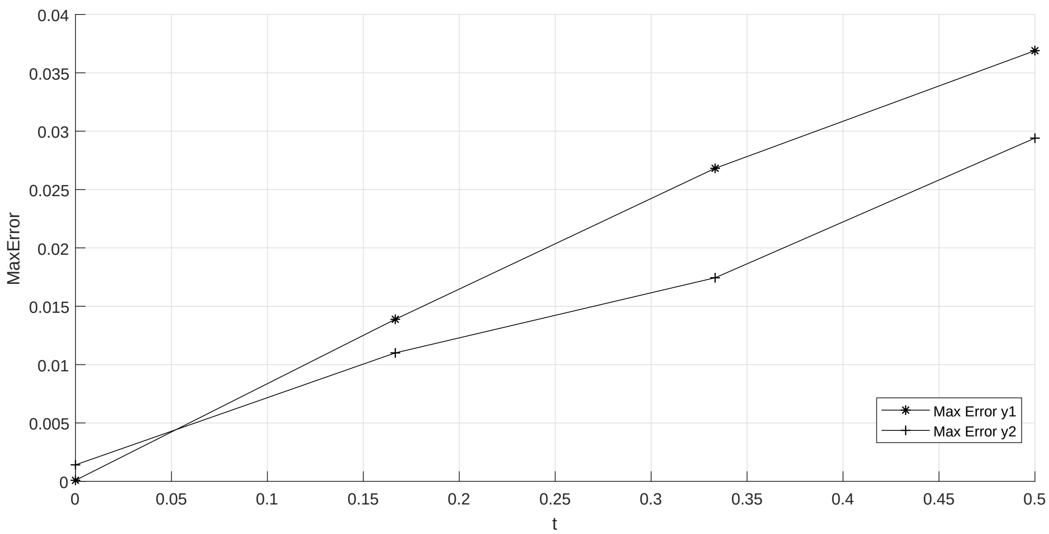

Figure 3: The errors for $M = 3, N = 2$ in Example 5.2.

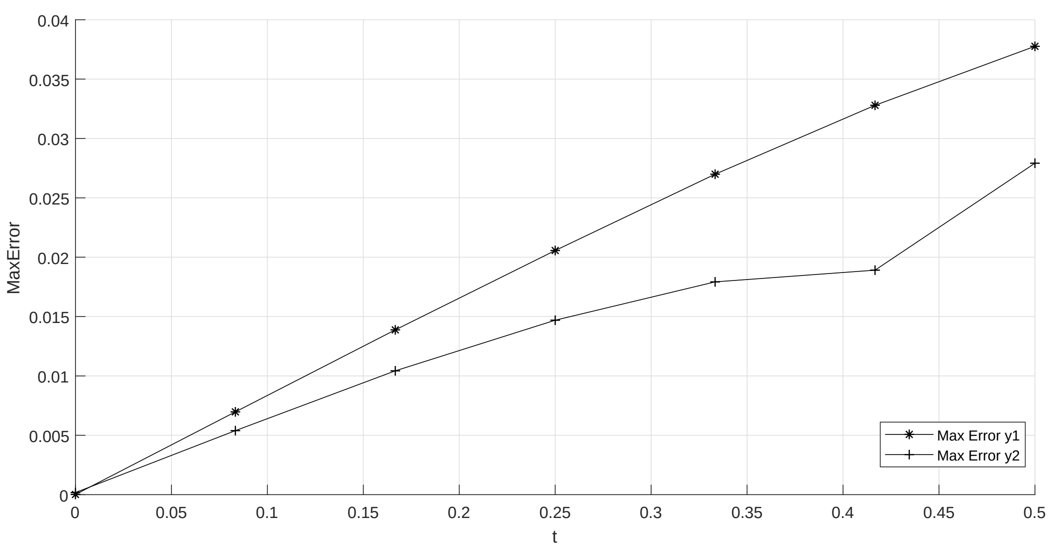

Figure 4: The errors for $M = 3, N = 4$ in Example 5.2.

Table 5: Aproximate and exact solutions and errors for $M = 3$ and $N = 2$ in Example 5.2.

$t = 0.333$ r	App. $\underline{y_1}(t,r)$	Exact $\underline{y_1}(t,r)$	Error	App. $\overline{y_1}(t,r)$	Exact $\overline{y_1}(t,r)$	Error	Max Error
0.00000	0.00000	0.00000	0.00000	0.33627	0.35827	0.02199	0.02199
0.16667	0.02539	0.02986	0.00447	0.30825	0.32841	0.02016	0.02016
0.33333	0.05077	0.05971	0.00894	0.28023	0.29855	0.01833	0.01833
0.50000	0.07616	0.08957	0.01341	0.25220	0.26870	0.01650	0.01650
0.66667	0.10154	0.11942	0.01788	0.22418	0.23884	0.01466	0.01788
0.83333	0.12693	0.14928	0.02235	0.19616	0.20899	0.01283	0.02235
1.00000	0.15231	0.17913	0.02682	0.16814	0.17913	0.01100	0.02682

Table 6: Aproximate and exact solutions and errors for $M = 3$ and $N = 2$ in Example 5.2.

$t = 0.333$ r	App. $\underline{y_2}(t,r)$	Exact $\underline{y_2}(t,r)$	Error	App. $\overline{y_2}(t,r)$	Exact $\overline{y_2}(t,r)$	Error	Max Error
0.00000	0.00747	0.00000	0.00747	0.16343	0.16667	0.00324	0.00747
0.16667	0.01721	0.01389	0.00332	0.15048	0.15278	0.00230	0.00332
0.33333	0.02694	0.02778	0.00083	0.13753	0.13889	0.00136	0.00136
0.50000	0.03668	0.04167	0.00499	0.12458	0.12500	0.00042	0.00499
0.66667	0.04642	0.05556	0.00914	0.11163	0.11111	0.00052	0.00914
0.83333	0.05616	0.06944	0.01329	0.09868	0.09722	0.00146	0.01329
1.00000	0.06589	0.08333	0.01744	0.08573	0.08333	0.00240	0.01744

Table 7: Aproximate and exact solutions and errors for $M = 3$ and $N = 4$ in Example 5.2.

$t = 0.333$ r	App. $\underline{y_1}(t,r)$	Exact $\underline{y_1}(t,r)$	Error	App. $\overline{y_1}(t,r)$	Exact $\overline{y_1}(t,r)$	Error	Max Error
0.00000	0.00000	0.00000	0.00000	0.33595	0.35827	0.02231	0.02231
0.16667	0.02536	0.02986	0.00450	0.30796	0.32841	0.02045	0.02045
0.33333	0.05071	0.05971	0.00900	0.27996	0.29855	0.01859	0.01859
0.50000	0.07607	0.08957	0.01349	0.25197	0.26870	0.01673	0.01673
0.66667	0.10143	0.11942	0.01799	0.22397	0.23884	0.01487	0.01799
0.83333	0.12679	0.14928	0.02249	0.19597	0.20899	0.01301	0.02249
1.00000	0.15214	0.17913	0.02699	0.16798	0.17913	0.01116	0.02699

Table 8: Aproximate and exact solutions and errors for $M = 3$ and $N = 4$ in Example 5.2.

$t = 0.333$ r	App. $\underline{y_2}(t,r)$	Exact $\underline{y_2}(t,r)$	Error	App. $\overline{y_2}(t,r)$	Exact $\overline{y_2}(t,r)$	Error	Max Error
0.00000	0.00817	0.00000	0.00817	0.16091	0.16667	0.00576	0.00817
0.16667	0.01771	0.01389	0.00382	0.14818	0.15278	0.00460	0.00460
0.33333	0.02725	0.02778	0.00052	0.13545	0.13889	0.00344	0.00344
0.50000	0.03679	0.04167	0.00487	0.12272	0.12500	0.00228	0.00487
0.66667	0.04633	0.05556	0.00922	0.11000	0.11111	0.00111	0.00922
0.83333	0.05587	0.06944	0.01357	0.09727	0.09722	0.00005	0.01357
1.00000	0.06541	0.08333	0.01792	0.08454	0.08333	0.00121	0.01792

6 Conclusion

First, the properties of Bernstein polynomials and their combination with block pulse functions are presented. Then, the important transformation matrix is introduced, which is one of the advantages of this work, because it can be generalized to other polynomials and its combination with block pulse functions, where the operational matrices are more easily calculated. We applied the method of combining functions to mixed fuzzy integral equations system with time delay. Then, by using the transformation matrix, we determined other operational, delay, and Fredholm and Volterra integrals matrices. By substituting these matrices into the fuzzy integral equations system with time delay, we arrive at a system of algebraic equations. By solving this system of linear equations, we obtain a solution to the problem. Then, we proved the uniqueness and convergence of the method. And some numerical examples are presented to show the effectiveness of the method. The results showed that the hybrid methods are very useful for these types of systems. For future research, this method can be used for such equations with nonlinear or nonlinear delay functions. And also, it can be applied to other polynomials and its combination with block pulse functions.

Acknowledgements: "I would like to express my gratitude to the editor and reviewers for their valuable and helpful comments in improving my manuscript."

Conflict of interest: The author declares that they have no conflict of interest.

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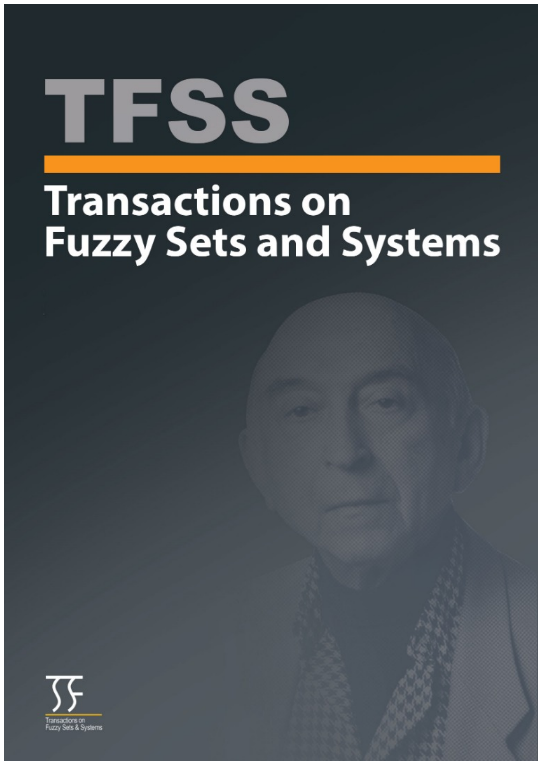

Transactions on Fuzzy Sets and Systems

ISSN: **2821-0131**

https://sanad.iau.ir/journal/tfss/

Comparison Between Fuzzy Number Sequences via Interactive Arithmetic J_0 and Standard Arithmetic

Vol.4, No.1, (2025), 213-224. DOI: https://doi.org/10.71602/tfss.2025.1189025

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Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://sanad.iau.ir/journal/tfss/
Online ISSN: 2821-0131
Vol.4, No.1, (2025), 213-224
DOI: https://doi.org/10.71602/tfss.2025.1189025

Article Type: Original Research Article

Comparison Between Fuzzy Number Sequences via Interactive Arithmetic J_0 and Standard Arithmetic

Vincius Francisco Wasques*, **Nilmara de Jesus Biscaia Pinto**

(This paper is dedicated to Professor "John N. Mordeson" on the occasion of his 91st birthday.)

Abstract. The focus of this work is to study sequences of interactive fuzzy numbers. The interactivity relation is associated with the concept of joint possibility distribution. In this case, the type of interactivity studied is linked to a family of joint possibility distributions (J_γ), in which the parameter γ intrinsically models levels of interactivity between the fuzzy numbers involved. Each element of the sequence of interactive fuzzy numbers is obtained through a discrete equation, and the arithmetic operations present in the equation are extended to this type of fuzzy number. Some simulations are performed to illustrate the behavior of the sequences, called interactive, and to compare them with the sequences obtained by other fuzzy arithmetic operations.

AMS Subject Classification 2020: MSC 39A26; MSC 03B52; MSC 94D05
Keywords and Phrases: Interactive fuzzy numbers, Fuzzy number sequence, Fuzzy discrete equations, Sup-J extension principle.

1 Introduction

Real number sequences have been a subject of study within Mathematics, particularly in the field of Analysis. In this context, the domain of the function that generates the sequence is the set of natural numbers ($\mathbb{N}$) and the image is defined as the set of real numbers ($\mathbb{R}$). In addition, one can define a sequence of real numbers by writing the current value in terms of its predecessors. This type of sequence is also known as a recursive sequence, which must be started from one or more initial conditions, as occurs, for instance, in the Fibonacci and plant growth sequences [1].

There are several well-known real sequences, such as the Lucas sequence, in which the real sequence is the same as in the Fibonacci sequence, but the initial values differ. Also, there is the arithmetic sequence (each term is the sum of the previous term and a constant difference), geometric sequence (each term is the product of the previous term and a constant ratio), triangular number sequence (each term can be arranged in an equilateral triangle), and many others.

This work focuses on the study of an extension of recursive sequences, in the following sense: the domain of the function that generates the sequence remains the set of natural numbers, but its values lie in the set of fuzzy numbers ($\mathbb{R}_\mathcal{F}$). Such sequences are known as fuzzy sequences, and the motivation for working with this approach is based on the uncertainty in determining an exact value for the initial conditions of a recursive sequence, as seen in population dynamics [2].

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Received: 31 October 2024; **Revised**: 20 January 2025; **Accepted**: 28 January 2025; **Available Online**: 11 February 2025; **Published Online**: 7 May 2025.

How to cite: Wasques VF, Pinto NJB. Comparison between fuzzy number sequences via interactive arithmetic J_0 and standard arithmetic. *Transactions on Fuzzy Sets and Systems.* 2025; 4(1): 213-224. DOI: https://doi.org/10.71602/tfss.2025.1189025

In this particular work, the fuzzy sequences, considered here, are given in the form of

$$X_n = f(X_0, X_1, \ldots, X_{n-1}),$$

where $f : \mathbb{R}^n_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ is a linear fuzzy function. An example of this type of fuzzy sequence is given by $X_n = 3X_{n-1} + 2X_{n-2}$. An example of a fuzzy sequence that is not of this type is given by $X_n = X_{n-1} * X_{n-2}$.

For this purpose, the initial conditions of the recursive sequence must be given by fuzzy numbers, consequently, the operations involved in obtaining the n-th value of the sequence, in terms of the previous $n-1$ values, must be appropriated for fuzzy numbers. In the literature, there are various arithmetics for fuzzy numbers. This work will explore only two: the standard arithmetic and the interactive arithmetic.

The standard arithmetic is considered because it is the most common arithmetic operation used in the literature. Moreover, several properties of this arithmetic are well known. For example, it is always possible to compute the standard sum between fuzzy numbers; it is a commutative and associative operation, but it does not satisfy the opposite element; it always produces a fuzzy number with a bigger width than each width of the operands; and so on [2].

On the other hand, the choice of interactive arithmetic arises from the fact that the n-th term of the sequence depends on its predecessors. This dependence is intrinsically modeled by the concept of interactivity [3]. Interactivity is a fuzzy relation that emerges from a joint possibility distribution between fuzzy numbers. This relation is similar, but not equivalent, to the concept of dependence for random variables.

In the context of interactivity, there are several arithmetic operations proposed in the literature, all of which incorporate this relation. Carlsson and Fuller [4] proposed an addition (subtraction) for fuzzy numbers that assumes a linear correlation between the fuzzy numbers. Barros and Santo Pedro [5] explored these operations by proposing a fuzzy derivative. Wasques et al. [6] showed that Hukuara difference and its generalizations incorporate the relation of interactivity, which means that several papers in the literature use the relation of interactivity implicitly or explicitly since these fuzzy differences are widely considered in the fuzzy set theory.

This work addresses fuzzy number sequences that incorporate the interactivity relation, illustrating their advantages over using usual arithmetic for fuzzy numbers. The paper is organized as follows. Section 2 provides the mathematical background for the fuzzy sets theory and the construction of the interactive sum J_0. Section 3 explores fuzzy number sequences with different types of arithmetic operations. Section 4 presents the conclusion of the paper.

2 Mathematical Background

A fuzzy subset A of a universe X is characterized by a membership function $\mu_A : X \to [0,1]$, where $\mu_A(x)$, or simply $A(x)$, indicates the degree to which $x \in X$ belongs to A. Every classical subset A of X is, in particular, a fuzzy set, as it can be described by the characteristic function $\chi_A : X \to \{0,1\}$, which is a particular case of a membership function. One way to handle fuzzy sets computationally is through α-cuts, defined by $[A]^\alpha = \{x \in X : A(x) \geq \alpha\}$ if $0 < \alpha \leq 1$ and $[A]^\alpha = \overline{\{x \in X : A(x) > 0\}}$ if $\alpha = 0$, where $\overline{Y}$ represents the closure of the set $Y \subseteq X$.

The set of fuzzy numbers, denoted by $\mathbb{R}_{\mathcal{F}}$, is formed by fuzzy subsets of $\mathbb{R}$ whose α-cuts are non-empty, bounded, closed, and nested intervals for all $\alpha \in [0,1]$. These α-cuts are denoted by $[A]^\alpha = [a^-_\alpha, a^+_\alpha]$, $\forall \alpha \in [0,1]$ [2]. The set of fuzzy numbers with continuous endpoints $a^-_{(\cdot)}, a^+_{(\cdot)} : [0,1] \to \mathbb{R}$ is denoted by $\mathbb{R}_{\mathcal{F}_C}$. An example of this type of fuzzy number is the triangular fuzzy number, denoted by triple $(a;b;c)$, with $a \leq b \leq c$, and characterized by the α-cuts $[a + \alpha(b-a), c + \alpha(b-c)]$. The width of a fuzzy number A is defined by $width(A) = |a^+_0 - a^-_0|$ [2].

Let A and B be fuzzy numbers. The Pompeiu-Hausdorff distance $D_\infty : \mathbb{R}_\mathcal{F} \times \mathbb{R}_\mathcal{F} \to [0, +\infty)$ is given by

$$D_\infty(A, B) = \sup_{\alpha \in [0,1]} (\max\{|a_\alpha^- - b_\alpha^-|, |a_\alpha^+ - b_\alpha^+|\}), \ \forall A, B \in \mathbb{R}_\mathcal{F}.$$

A sequence of fuzzy numbers is defined by a function $F : \mathbb{N} \to \mathbb{R}_\mathcal{F}$. This sequence is denoted by X_n, where X_n represents the value $F(n)$ and X_n is referred to as the n-th term of the sequence, that is, $F(n) = X_n$, for all $n \in \mathbb{N}$. A sequence X_n converges to X_p if for every $\epsilon > 0$, there exists n_0 such that $D_\infty(X_n, X_p) < \epsilon$, for all $n > n_0$.

A fuzzy relation $J \in \mathcal{F}(\mathbb{R}^2)$ is said to be a joint possibility distribution between the fuzzy numbers $A_1, A_2 \in \mathbb{R}_\mathcal{F}$ if

$$A_i(y) = \sup_{\{(x_1,x_2)\in\mathbb{R}^2 \,:\, x_i = y\}} J(x_1, x_2),$$

for all $y \in \mathbb{R}$ and $\forall i = 1, 2$.

This means that A_1 and A_2 can be obtained by the projection of J in x and y direction, respectively. The fuzzy numbers A_1 and A_2 are also called be the marginals of J.

Let $A_1, A_2 \in \mathbb{R}_\mathcal{F}$ and let J be a joint possibility distribution between them. The fuzzy numbers A_1 and A_2 are said to be non-interactive if

$$J(x_1, x_2) = J_{\min}(x_1, x_2) = \min\{A_1(x_1), A_2(x_2)\}, \quad \forall (x_1, x_2) \in \mathbb{R}^2.$$

Otherwise, that is, if $J \neq J_{\min}$, then A_1 and A_2 are said to be interactive fuzzy numbers.

The above definition states that the concept of interactivity between fuzzy numbers arises from the notion of joint possibility distribution. This idea is similar (but not equivalent) to the definition of dependence in the case of random variables, that is, the relation of dependence is similar to interactivity and independence is similar to non-interactivity.

There are different types of interactivity associated with various joint possibility distributions, such as interactivity via J_L [4, 5, 7]. This joint possibility distribution establishes a linear correlation between the membership functions of the involved fuzzy numbers, which restricts the applicability of J_L [8, 9]. For example, the joint possibility distribution J_L can not be applied for the pair of fuzzy numbers A_1 and A_2, where A_1 is a triangular symmetric fuzzy number (for example $A_1 = (1; 2; 3)$) and A_2 is a triangular non-symmetric fuzzy number (for example $A_1 = (1; 2; 4)$).

The following joint possibility distribution does not have such restrictions. Specifically, it can be applied to any pair of fuzzy numbers in $\mathbb{R}_{\mathcal{F}C}$. Given $A_1, A_2 \in \mathbb{R}_{\mathcal{F}C}$, for each $z \in \mathbb{R}$ and $\alpha \in [0, 1]$, consider the functions [10]:

$$g_1(z, \alpha) = \min_{w \in [A_2]^\alpha} |w + z|, \quad \text{and} \quad g_2(z, \alpha) = \max_{w \in [A_1]^\alpha} |w + z|. \tag{1}$$

Also consider the sets R_α^i and $L^i(z, \alpha)$ defined as follows:

$$R_\alpha^i = \begin{cases} \{a_{i_\alpha}^-, a_{i_\alpha}^+\} & \text{if } \alpha \in [0, 1) \\ [A_i]^1 & \text{if } \alpha = 1 \end{cases},$$

and $L^i(z, \alpha) = [A_{3-i}]^\alpha \cap [-g_i(z, \alpha) - z, g_i(z, \alpha) - z]$, with $i = \{1, 2\}$.

The joint possibility distribution J_0 is defined by the following membership function [10]

$$J_0(x_1, x_2) = \begin{cases} \min\{A_1(x_1), A_2(x_2)\}, & \text{if } (x_1, x_2) \in P \\ \quad 0 \quad , & \text{otherwise} \end{cases}, \tag{2}$$

where

$$P = \bigcup_{i=1}^{2} \bigcup_{\alpha \in [0,1]} P^i(\alpha) \quad \text{with} \quad P^i(\alpha) = \{(x_1, x_2) : x_i \in R^i_\alpha \text{ e } x_{3-i} \in L^i(x_i, \alpha)\}.$$

The following definition is a generalization of Zadeh's extension principle [11], which aims to extend real functions to fuzzy functions. Let $J \in \mathcal{F}(\mathbb{R}^n)$ be a joint possibility distribution of $A_1, ..., A_n \in \mathbb{R}_{\mathcal{F}}$ and $f : \mathbb{R}^n \to \mathbb{R}$. The sup-$J$ extension of the function f applied to $(A_1, ..., A_n)$ is defined by

$$f_J(A_1, ..., A_n)(y) = \sup_{(x_1,...,x_n) \in f^{-1}(y)} J(x_1, ..., x_n),$$

where $f^{-1}(y) = \{(x_1, ..., x_n) \in \mathbb{R}^n : f(x_1, ..., x_n) = y\}$.

Through the sup-J extension principle, the arithmetic between interactive fuzzy numbers is obtained. For example, the interactive sum and difference between A_1 and A_2 is defined as follows:

$$(A_1 +_J A_2)(y) = \sup_{x_1 + x_2 = y} J(x_1, x_2) \quad \text{and} \quad (A_1 -_J A_2)(y) = \sup_{x_1 - x_2 = y} J(x_1, x_2),$$

where J is an arbitrary joint possibility distribution.

Definition 2.1. [6] Let $A, B \in \mathbb{R}_{\mathcal{F}_C}$. The interactive fuzzy sum defined by

$$(A_1 +_0 A_2)(y) = \sup_{x_1 + x_2 = y} J_0(x_1, x_2) \tag{3}$$

is called the J_0-sum.

The J_0-sum for triangular fuzzy numbers can be easily computed according to the following theorem.

Theorem 2.2. *[12] Let $A = (a; b; c)$ and $B = (d; e; f)$ be triangular fuzzy numbers. Let J_0 be the joint possibility distribution between A and B, given by (2). Thus*

$$A +_0 B = \begin{cases} ((a+f) \wedge (b+e); b+e; (b+e) \vee (c+d)), & \text{if } width(A) \geq width(B) \\ ((c+d) \wedge (b+e); b+e; (b+e) \vee (a+f)), & \text{if } width(A) \leq width(B) \end{cases}. \tag{4}$$

For example, the J_0-sum between $A = (1; 2; 3)$ and $B = (0; 2; 4)$ is equal to

$$A +_0 B = (\min\{3+0, 2+2\}; 2+2; \max\{1+4, 2+2\}) = (3; 4; 5).$$

On the other hand, the usual sum is given by

$$A + B = (1+0; 2+2; 3+4) = (1; 4; 7),$$

which has a bigger width than $(3; 4; 5)$.

Also, the subtraction operator can be defined in a similar way.

Definition 2.3. Let $A, B \in \mathbb{R}_{\mathcal{F}}$. The usual fuzzy difference is defined by

$$(A_1 - A_2)(y) = \sup_{x_1 - x_2 = y} \min\{A_1(x_1), A(x_2)\}. \tag{5}$$

Definition 2.4. [6] Let $A, B \in \mathbb{R}_{\mathcal{F}_C}$. The interactive fuzzy difference defined by

$$(A_1 -_I A_2)(y) = \sup_{x_1 - x_2 = y} J_0(x_1, x_2) \tag{6}$$

is called the I-difference.

For example, the I-difference between $A = (1; 3; 4)$ and $B = (1; 2; 3)$ is equal to

$$A -_I B = A +_0 (-B) = (\min\{1 + (-1), 3 + (-2)\}; 3 + (-2); \max\{4 + (-3), 3 + (-2)\}) = (0; 1; 1).$$

On the other hand, the usual sum is given by

$$A - B = A + (-B) = (1 - 3; 3 - 2; 4 - 1) = (-2; 1; 3),$$

which has bigger width than $(0; 1; 1)$. Also, note that

$$A -_I A = A +_0 (-A) = (a; b; c) +_0 (-c; -b; -a) = (0; 0; 0),$$

for all triangular fuzzy numbers A. Indeed, this result holds for any fuzzy number, that is, $A -_I A = 0$ for all $A \in \mathbb{R}_{\mathcal{F}_\mathcal{C}}$ [6].

The next section discusses sequences that are obtained through a discrete equation, where the arithmetic operations involved in the equation are given by interactive arithmetic operations.

3 Fuzzy Number Sequence

The sequences that will be considered here are obtained recurrently, that is, each term $x_n \in \mathbb{R}$ of the sequence is given as a function of the previous terms $x_1, \ldots, x_{n-1}$ from one or more initial conditions. For example, the sequence defined by the Equation (7)

$$x_n = x_{n-1} - r x_{n-2}, \tag{7}$$

where $r \in \mathbb{R}$, with initial conditions x_1 and x_2.

Taking the value of $r = 0.25$ and initial conditions $x_1 = x_2 = 1$, this sequence assumes the following values $\{1; 1; 0.75; 0.5; 0.3125; 0.1875; \ldots\}$, converging to 0.

Considering that the initial conditions are uncertain and given by fuzzy numbers, the sequence given in (7) is extended by the following fuzzy numbers sequence

$$X_n = X_{n-1} \ominus r X_{n-2}, \tag{8}$$

where $r \in \mathbb{R}$, with X_1 and X_2 being fuzzy numbers, and the operation $\ominus$ is a difference between fuzzy numbers.

Two cases will be analyzed here. The first one is when the fuzzy initial conditions are non-interactive, in this case, the usual difference must be considered. In the second case the fuzzy initial conditions are interactive, and thus, an interactive difference must be taken into account.

3.1 Usual Arithmetic Sequence

For the usual difference, we have the following sequence

$$X_n = X_{n-1} - r X_{n-2}. \tag{9}$$

Taking the initial conditions $X_1 = X_2 = (0; 1; 2)$ and $r = 0.25$, we obtain the following sequence of fuzzy numbers represented in Figure 1. Figure 2 shows the 16-th term X_{16} computed in this sequence.

Each element of the sequence X_n given in (9) can be found in Table 1. Note that the width of X_n, that is, the size of the 0-cut of X_n, is increasing with n. This implies that the uncertainty about the elements increases as n increases, this behavior is connected to the usual arithmetic.

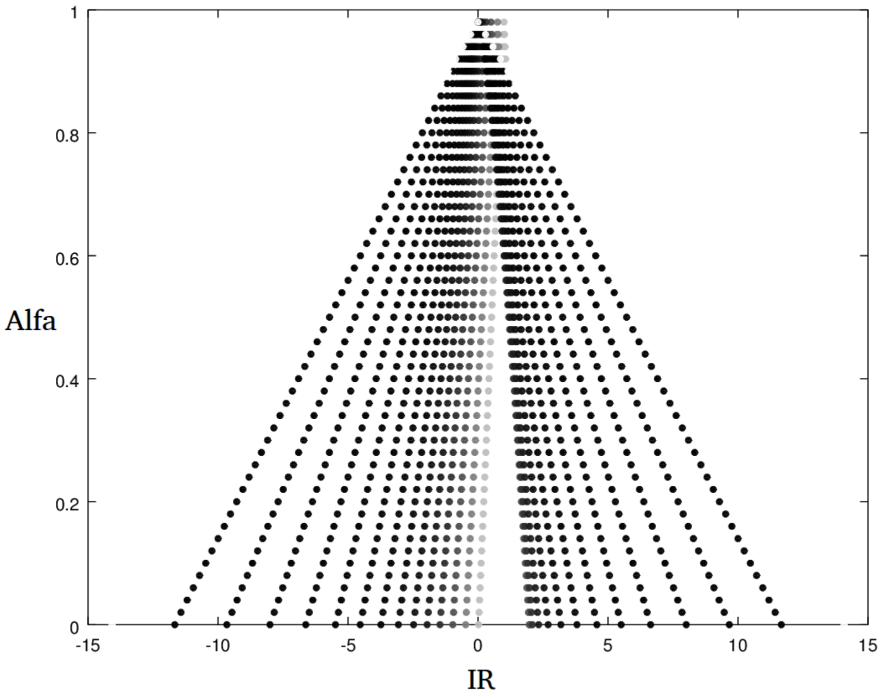

Figure 1: Fuzzy number sequence given by Equation (9) for $n = 16$. Each element of the sequence X_n is represented in shades of gray, with X_1 described by the lightest shade, and X_{16} by the darkest shade.

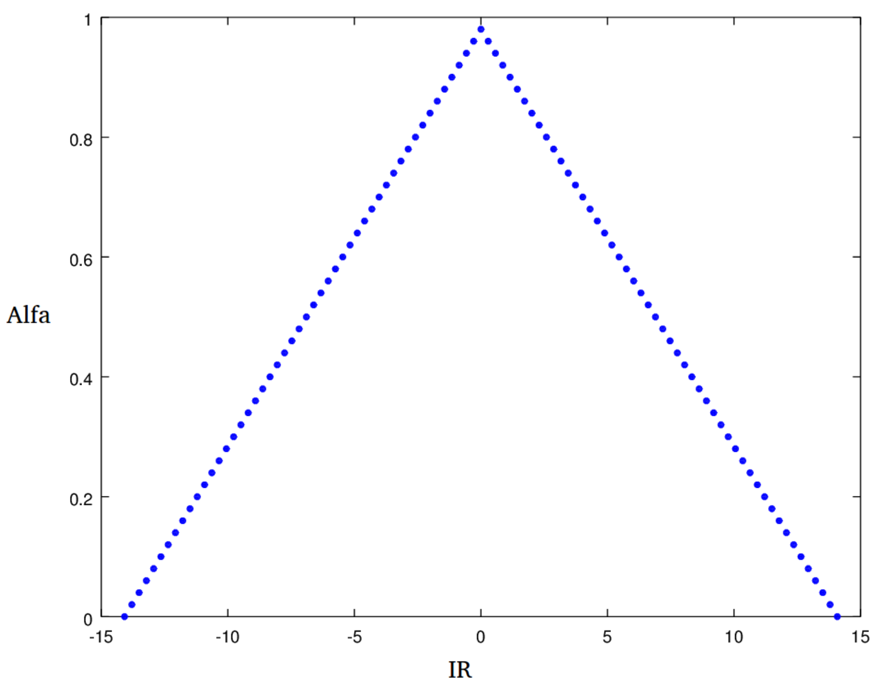

Figure 2: $X_{16} = (-14.080791; 0.00048828; 14.0818)$.

Note that, if the initial conditions are given by triangular fuzzy numbers, then the $(n-1)$-ary term of

Table 1: Sequence of fuzzy numbers obtained from Equation (8) from the initial conditions $X_1 = X_2 = (0; 1; 2)$, $r = 0.25$ and $n = 16$.

Usual Arithmetic	
n	X_n
1	$(0; 1; 2)$
2	$(0; 1; 2)$
3	$(-0.475; 0.75; 1.975)$
4	$(-0.97; 0.5; 1.97)$
5	$(-1.46375; 3.125; 2.0888)$
6	$(-1.95625; 0.1875; 2.3313)$
7	$(-2.478438; 0.10938; 2.6972)$
8	$(-3.06125; 0.0625; 3.1863)$
9	$(-3.735547; 0.035156; 3.8059)$
10	$(-4.532109; 0.019531; 4.5712)$
11	$(-5.483574; 0.010742; 5.5051)$
12	$(-6.626367; 0.0058594; 6.6381)$
13	$(-8.002632; 0.0031738; 8.009)$
14	$(-9.662153; 0.001709; 9.6656)$
15	$(-11.664398; 0.0091553; 11.6662)$
16	$(-14.080791; 0.00048828; 14.0818)$

Interactive Arithmetic	
n	X_n
1	$(0; 1; 2)$
2	$(0; 1; 2)$
3	$(-0.15; 0.75; 1.485)$
4	$(-0.01; 0.5; 0.99)$
5	$(-0.00625; 0.3125; 0.61875)$
6	$(-0.00375; 0.1875; 0.37125)$
7	$(-0.0021875; 0.10938; 0.21656)$
8	$(-0.00125; 0.0625; 0.12375)$
9	$(-0.00070312; 0.035156; 0.069609)$
10	$(-0.00039062; 0.019531; 0.038672)$
11	$(-0.00021484; 0.010742; 0.02127)$
12	$(-0.00011719; 0.0058594; 0.011602)$
13	$(-0.00063477; 0.0031738; 0.0062842)$
14	$(-0.00003418; 0.0017090; 0.0033838)$
15	$(-0.000018311; 0.00091553; 0.0018127)$
16	$(-0.000009765; 0.00048828; 0.0009668)$

this sequence is given by

$$X_{n-1} = (a_{n-2} - rc_{n-3}; b_{n-2} - rb_{n-3}; c_{n-2} - ra_{n-3}),$$

whose α-cuts are given by

$$\begin{aligned} [X_{n-1}]^\alpha &= [a_{n-2} - rc_{n-3} + \alpha(b_{n-2} - rb_{n-3} - (a_{n-2} - rc_{n-3})), \\ &\quad c_{n-2} - ra_{n-3} + \alpha(b_{n-2} - rb_{n-3} - (c_{n-2} - ra_{n-3}))] \\ &= [(a_{n-2} - rc_{n-3})(1-\alpha) + \alpha(b_{n-2} - rb_{n-3}), \\ &\quad (c_{n-2} - ra_{n-3})(1-\alpha) + \alpha(b_{n-2} - rb_{n-3})] \end{aligned}$$

and the n-ary term of this sequence is given by

$$X_n = (a_{n-1} - rc_{n-2}; b_{n-1} - rb_{n-2}; c_{n-1} - ra_{n-2}),$$

whose α-cuts are given by

$$\begin{aligned} [X_n]^\alpha &= [(a_{n-1} - rc_{n-2})(1-\alpha) + \alpha(b_{n-1} - rb_{n-2}), \\ &\quad (c_{n-1} - ra_{n-2})(1-\alpha) + \alpha(b_{n-1} - rb_{n-2})]. \end{aligned}$$

For all $r > 0$, it follows that

$$\begin{aligned} D_\infty(X_n, X_{n-1}) &= \sup_{\alpha\in[0,1]} (\max\{|a^-_\alpha - b^-_\alpha|, |a^+_\alpha - b^+_\alpha|\}) \\ &= \max\{|a_{n-1} - rc_{n-2} - (a_{n-2} - rc_{n-3})|, |c_{n-1} - ra_{n-2} - (c_{n-2} - ra_{n-3})|\} \end{aligned}$$

or

$$\begin{aligned} D_\infty(X_n, X_{n-1}) &= \max\{|b_{n-1} - rb_{n-2} - (b_{n-2} - rb_{n-3})|, |b_{n-1} - rb_{n-2} - (b_{n-2} - rb_{n-3})|\} \\ &= |b_{n-1} - rb_{n-2} - (b_{n-2} - rb_{n-3})|. \end{aligned}$$

Since $width(X_{n-1}) \leq width(X_n)$, it follows that for $r > 1$ the above sequences do not converge. This comment gives raise to the following proposition.

Proposition 3.1. *Let be the fuzzy sequence given by*

$$X_n = X_{n-1} - rX_{n-2},$$

where the subtraction operation $-$ is given by the usual difference for fuzzy numbers. Thus, the fuzzy sequence X_n diverges, for $r > 1$.

3.2 Sequence via Interactive Arithmetic

For interactive arithmetic, several differences can be used, for example, gH-difference [13], L-difference [5] and I-difference [6]. In the simulations performed here, only the I-difference will be considered, since it exists for any pair of fuzzy numbers, in contrast to the gH-difference (which can not be computed for any triangular fuzzy numbers) and L-difference (which can not be computed for triangular fuzzy numbers with different shapes). For the I-difference, the following fuzzy sequence

$$X_n = X_{n-1} -_I rX_{n-2}, \tag{10}$$

is illustrated in Figure 3.

Figure 4 depicts the 16-th term X_{16} computed from the sequence (10). It is possible to observe that the output produced by this sequence is indeed a fuzzy number. Moreover, the operation $-_I$ preserves the shape of the triangular fuzzy number.

As in usual arithmetic, the elements of $[X_n]^1$ are the same as in the classical sequence. Now, due to the interactive arithmetic obtained by the set J_0, the width of each $X_n \in \mathbb{R}_{\mathcal{F}_C}$ is decreasing with n. Therefore, the uncertainty about such elements decreases over time.

The right tabular of Table 1 illustrates the values of each element of the sequence (10). Analyzing the table, it is possible to quantitatively compare each X_n given by (9) and (10). It can be observed that the width of the fuzzy numbers produced by the sequence (10) is smaller or equal than the width of the fuzzy numbers produced by the sequence (9), for all $n \in \mathbb{N}$. Consequently, the uncertainty about the fuzzy sequence given in (8) is smaller using the I-difference than the usual difference.

Moreover, if the initial conditions are given by triangular fuzzy numbers, then the $(n-1)$-ary term of this sequence is given by

$$X_{n-1} = (\min\{a_{n-2} - ra_{n-3}, b_{n-2} - rb_{n-3}\}; b_{n-2} - rb_{n-3}; \max\{b_{n-2} - rb_{n-3}, c_{n-2} - rc_{n-3}\}),$$

if $width(X_{n-2}) \geq width(rX_{n-3})$ or

$$X_{n-1} = (\min\{c_{n-2} - rc_{n-3}, b_{n-2} - rb_{n-3}\}; b_{n-2} - rb_{n-3}; \max\{b_{n-2} - rb_{n-3}, a_{n-2} - ra_{n-3}\}),$$

if $width(X_{n-2}) \leq width(rX_{n-3})$ and the n-ary term of this sequence is given by

$$X_n = (\min\{a_{n-1} - ra_{n-2}, b_{n-1} - rb_{n-2}\}; b_{n-1} - rb_{n-2}; \max\{b_{n-1} - rb_{n-2}, c_{n-1} - rc_{n-2}\}),$$

if $width(X_{n-1}) \leq width(rX_{n-2})$ or

$$X_n = (\min\{c_{n-1} - rc_{n-2}, b_{n-1} - rb_{n-2}\}; b_{n-1} - rb_{n-2}; \max\{b_{n-1} - rb_{n-2}, a_{n-1} - ra_{n-2}\}).$$

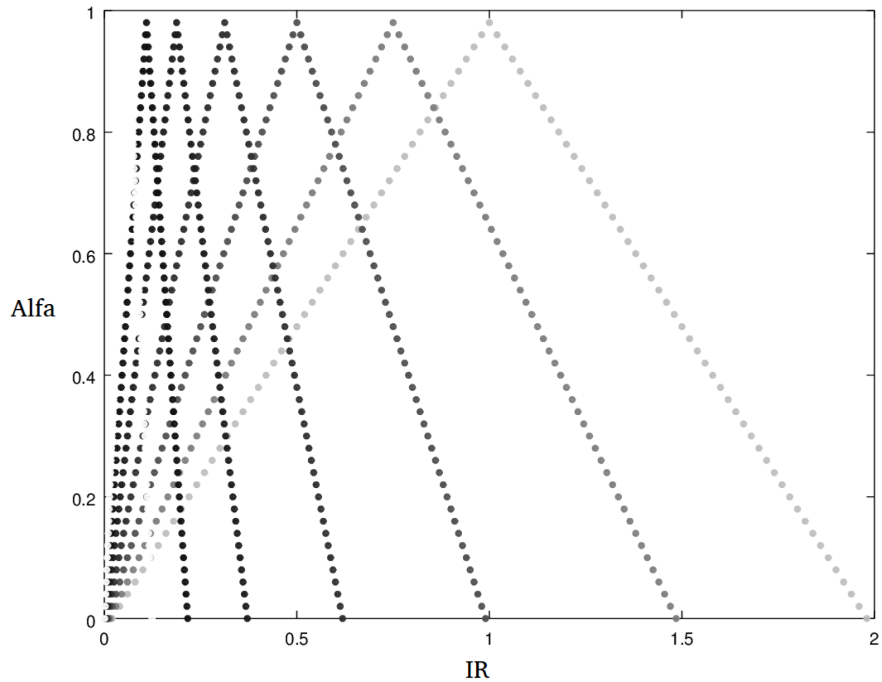

Figure 3: Fuzzy number sequence given by Equation (10) for $n = 16$. Each element of the sequence X_n is represented in shades of gray, with X_1 described by the lightest shade, and X_{16} by the darkest shade.

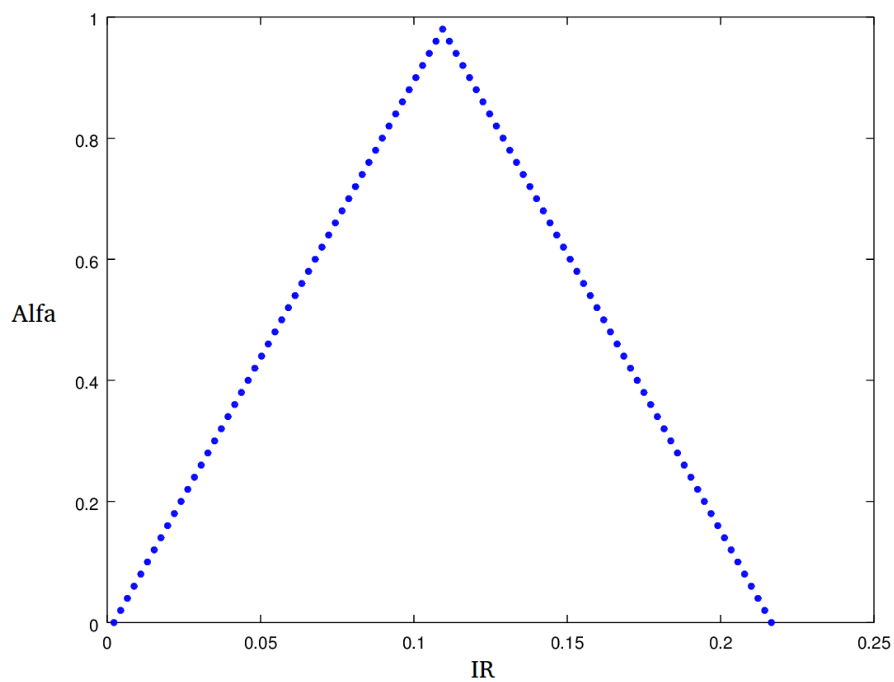

Figure 4: $X_{16} = (0.00009; 0.00488; 0.00966)$

If $0 < r < 1$, then $D_\infty(X_n, X_{n-1})$ is a lower bounded and decreasing function with respect to n, since $width(X_n) \leq width(X_{n-1})$, where $a_{n-2} - ra_{n-3}$, $b_{n-1} - rb_{n-2}$ and $c_{n-1} - rc_{n-2}$ are decreasing sequences.

Such reasoning is summarized in the following proposition.

Proposition 3.2. *Let be the fuzzy sequence given by*

$$X_n = X_{n-1} -_I rX_{n-2},$$

where the subtraction operation $-_I$ *is given by the interactive difference* (2.4)*. Thus, the fuzzy sequence* X_n *converges, for any* $0 < r < 1$.

Similar results would be obtained using the gH-difference and the L-difference, if it were possible to calculate $X_{n-1} \ominus rX_{n-2}$ for each n. This comment is attributed to the fact that every arithmetic operation coming from a joint possibility distribution $J \neq J_{\min}$ produces fuzzy numbers with a smaller width than the usual arithmetic [14].

4 Conclusion

This work studied sequences of fuzzy numbers that assume values in $\mathbb{R}_{\mathcal{F}_C}$. Each element of this sequence is obtained by recurrence according to the equation (8), with fuzzy initial conditions.

Through some simulations, using the I-difference, it was noticed that the interactive arithmetic produces a sequence of elements with a smaller width than the width of the elements obtained by the usual arithmetic. This result is valid for all interactive arithmetic. It is worth mentioning that other interactive arithmetic could have been used, such as the differences gH and L, however, it is not always possible to compute them. The I-difference, on the other hand, does not have such restrictions.

From the point of view of applications, a smaller width implies less uncertainty about the elements of the sequence $\{X_n\}$. The sequence provided by usual arithmetic has an increasing width, and therefore, it propagates uncertainty over its elements. On the other hand, using the I-difference, the width of the sequence decreases, which is better for controlling uncertainty over time. This makes interactive arithmetic more suitable for modeling than the usual one.

It is worth noting that in several applications the usual sum and the gH-difference are used in the same equation. This is not consistent with joint possibility distributions, since the gH-difference is an interactive arithmetic operation [6], and the usual sum is not.

Acknowledgements: The first author would like to acknowledge the financial support of the So Paulo Research Foundation (FAPESP) under grant 2023/03927-0. The second author would like to thank the Brazilian Federal Agency for Support and Evaluation of Graduate Education (CAPES) under grant n 88887.691901/2022-00.

Conflict of Interest: The authors declare no conflict of interest.

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