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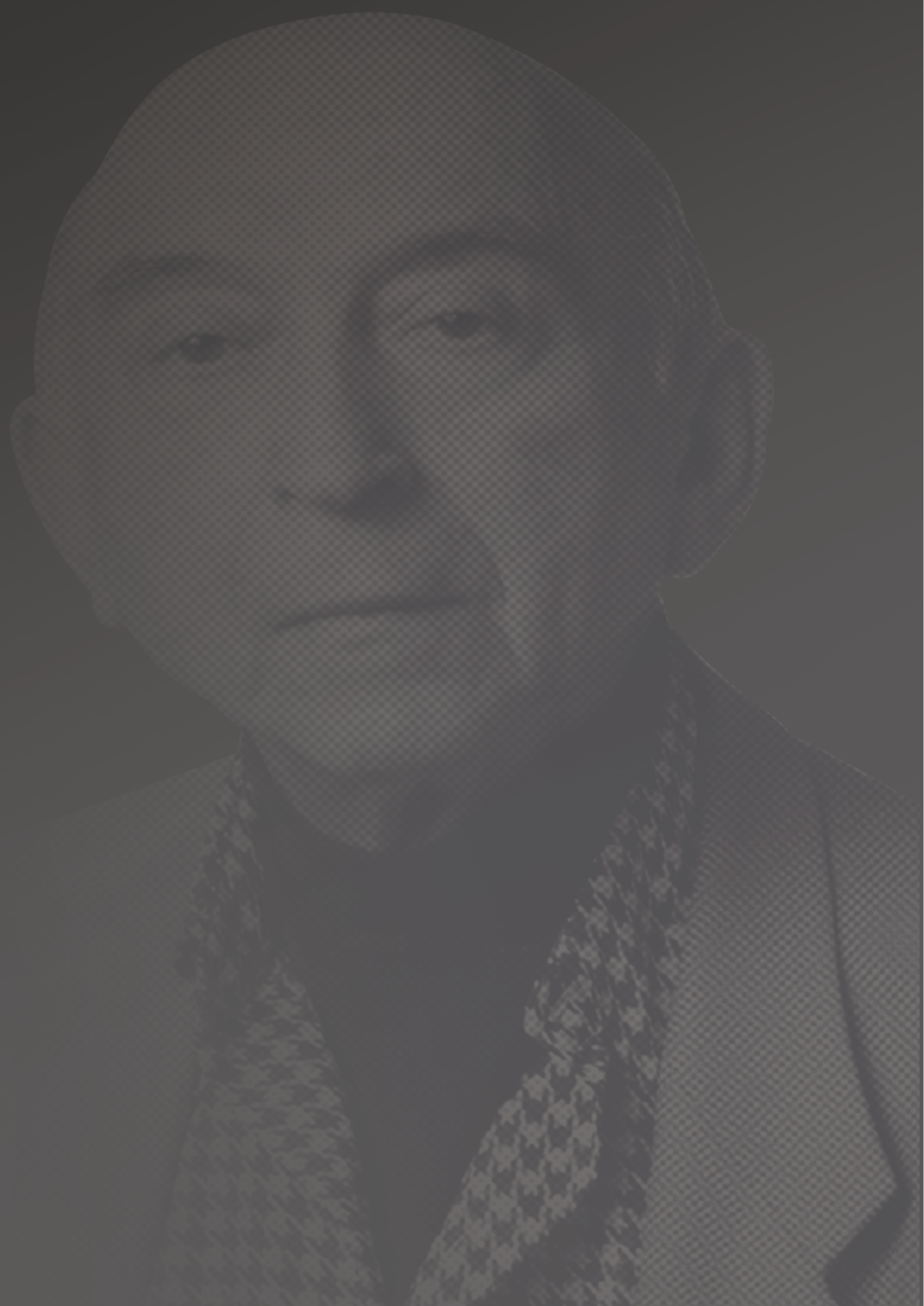

Transactions on
Fuzzy Sets & Systems

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Fahim Uddin; Muhammad Saeed; Khaleel Ahmad; Umar Ishtiaq; Salvatore Sessa

Fixed Point Theorems in Orthogonal Intuitionistic Fuzzy b-metric Spaces with an Application to Fredholm Integral Equation **1**

Zahra Behdani; Majid Darehmiraki

Fuzzy Logistic Regression Analysis Using the Least Squares Method **23**

Muhammed Raji; Arvind Kumar Rajpoot; Laxmi Rathour; Lakshmi Narayan Mishra; Vishnu Narayan Mishra

Nonlinear Contraction Mappings in b-metric Space and Related Fixed Point Results with Application **37**

Musa Adeku Ibrahim; Muhammed Raji; Kamilu Rauf

Fuzzy Metric Spaces and Corresponding Fixed Point Theorems for Fuzzy Type Contraction **51**

Mohd Hasan

Novel Generalisation of Some Fixed Point Results Using a New Type of Simulation Function **66**

Muhammed Raji; Laxmi Rathour; Lakshmi Narayan Mishra; Vishnu Narayan Mishra

Fuzzy Cone Metric Spaces and Fixed Point Theorems for Fuzzy Type Contraction **82**

Reza Chaharpashlou; Ehsan Lotfali Ghasab

Some Result on Fuzzy Integration **100**

Mijanur Rahaman Seikh; Arnab Mukherjee

Fermatean Fuzzy CRADIS Approach Based on Triangular Divergence for Selecting Online Shopping Platform **108**

Xiaobin Guo; Xiangyang Fan; Hangru Lin

Approximate Solution of Complex LR Fuzzy Linear Matrix Equation **128**

Alfredo Cuzzocrea; Enzo Mumolo; Islam Belmerabet; Abderraouf Hafsaoui

A Stochastic-Process Methodology for Detecting Anomalies at Runtime in Embedded Systems **142**

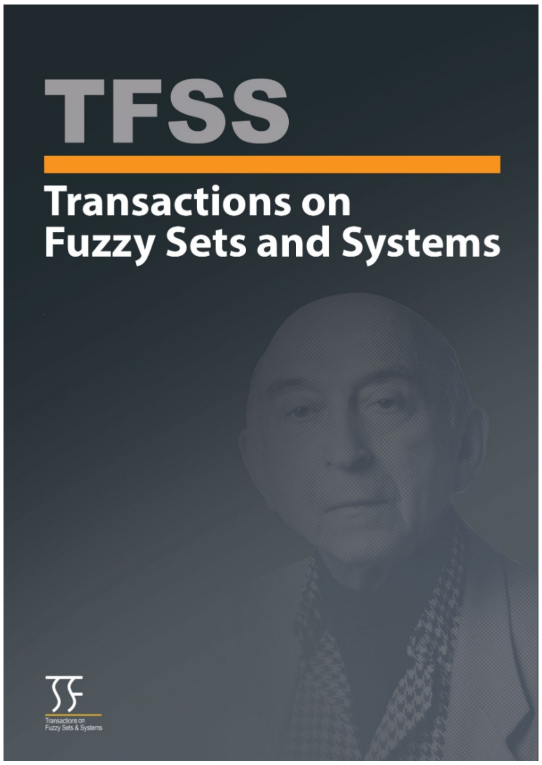
Zohreh Nazari; Elham Zangiabadi

Logical Entropy of Partitions for Interval-Valued Intuitionistic Fuzzy Sets **172**

Xinyi Duan; Hangru Lin; Xiaobin Guo

A Method for Finding LR Fuzzy Eigenvectors of Real Symmetric Matrix **186**

Transactions on Fuzzy Sets and Systems



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Fixed Point Theorems in Orthogonal Intuitionistic Fuzzy b-metric Spaces with an Application to Fredholm Integral Equation

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Fixed Point Theorems in Orthogonal Intuitionistic Fuzzy b-metric Spaces with an Application to Fredholm Integral Equation

Fahim Uddin, **Muhammad Saeed**, **Khaleel Ahmed**, **Umar Ishtiaq**, **Salvatore Sessa***

Abstract. In this manuscript, the concept of an orthogonal intuitionistic fuzzy b-metric space is initiated as a generalization of an intuitionistic fuzzy b-metric space. We presented some fixed point results in this setting. For the validity of the obtained results, some non-trivial examples are given. In the last part, we established an application on the existence of a unique solution of a Fredholm-type integral equation.

AMS Subject Classification 2020: 47H10; 54H25
Keywords and Phrases: Orthogonal set, Intuitionistic fuzzy metric space, Unique solution, Integral equation.

1 Introduction

A publication showing there are solutions to differential equations established fixed-point theory in the second quarter of the eighteenth century (Joseph Liouville, 1837). This approach was further improved as a sequential approximation technique (Charles Emile Picard, 1890), and in the setting of complete normed space, it was generalized as a fixed-point theorem (Stefan Banach, 1922). It presents the a priori and a posteriori approximations for the convergence rate as well as a general way to actually determine the fixed point. Additionally, it ensures that a fixed point exists and is distinct. This information is helpful for studying metric spaces. Stefan Banach is acknowledged for developing fixed-point theory after that. Fixed-point theorems allow us to guarantee that the main problem has been resolved, as has the existence of a fixed point for a given function. In a large variety of scientific problems that are derive from many different branches of mathematics, the existence of a solution is equivalent to the existence of a fixed point for a suitable mapping.

In 1989, Bakhtin [1] established the notion of quasi-metric spaces and established some results for contraction mappings. In 1993, Czerwik [2] established the concept of b-metric spaces and discussed several fixed-point results. Eshaghi et al. [3] introduced the notion of orthogonal metric spaces and derived well-known Banach fixed point theorem. Uddin et al. [4] established orthogonal m-metric spaces and solve the integral equation. Eshaghi and Habibia [5] derived several fixed point results in the context of generalized orthogonal metric space. Senapati et al. [6] established some new fixed point theorems in the context of orthogonal metric spaces. In 1965, Zadeh [7] established the notion of fuzzy sets (FSs) to deal with those problems that do have not any clear boundaries.

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In 1960, Schweizer [8] introduced the notion of continuous t-norm and worked on statistical metric spaces. In 1975, the combination of metric spaces and FSs, named fuzzy metric spaces (FMSs), have been introduced by Kramosil and Michlek [9]. In 1994, George and Veeramani [10] modified the notion of FMSs and gave an interesting analysis of FMSs in 1997 in a research paper [11]. Deng [12] established the notion of fuzzy pseudo-metric spaces and proved neumours results in the existence and uniqueness of a solution. Shukla and Abbas [13] established the notion of fuzzy metric-like spaces as a generalization of FMSs. Hezarjaribi [14] established the notion of orthogonal FMSs as a generalization of FMSs. Ndban [15] established the concept of fuzzy b-metric spaces (FBMSs) and Jeved et al. [16] introduced fuzzy b-metric like spaces as a generalization of FBMSs. The authors [17, 18, 19, 20] derived several fixed points results under some circumstances in the context of FBMSs. In 2004, Park [21] introduced the notion of intuitionistic fuzzy metric spaces (IFMSs), in which he combined the notions of continuous t-norm, continuous t-conorm, FSs and metric space.

Rafi and Noorani [22], Sintunavarat and Kumam [23], Alaca et al. [24] and Mohamad [25] derived some fixed point results for contraction mappings in the context of IFMSs. Konwar [26] introduced the notion of intuitionistic fuzzy b-metric spaces (IFBMSs) as a generalization of IFMSs and derived fixed point results. Baleanu and Rezapour [27] and Sudsutad and Tariboon [28] worked on fractional differential equations. In this manuscript, we aim to toss the notion of orthogonal Intuitionistic fuzzy b-metric spaces (OIFBMSs) as a generalization of IFBMSs. We provide some related fixed point theorems, including non-trivial examples and an application. Some of the following notions are used throughout this paper, as CTN for a continuous t-norm, CTCN for a continuous t-conorm and FP for fixed point.

2 preliminaries

In this section, we will discuss some important definitions that support our main result.

Definition 2.1. [1] Suppose $\Xi \neq \phi$. Given a five tuple $(\Xi, G, H, *, \Delta)$ where $*$ is a CTN, Δ is a CTCN, $\theta \geq 1$ and G, H are FSs on $\Xi \times \Xi \times (0, \infty)$. If $(\Xi, G, H, *, \Delta)$ meets the below conditions for all $\mathrm{w}, \mathrm{k} \in \Xi$ and $\pi, \tau > 0$:

(B1) $G(\mathrm{w}, \mathrm{k}, \tau) + H(\mathrm{w}, \mathrm{k}, \tau) \leq 1$;

(B2) $G(\mathrm{w}, \mathrm{k}, \tau) > 0$;

(B3) $G(\mathrm{w}, \mathrm{k}, \tau) = 1 \Leftrightarrow \mathrm{w} = \mathrm{k}$;

(B4) $G(\mathrm{w}, \mathrm{k}, \tau) = G(\mathrm{k}, \mathrm{w}, \tau)$;

(B5) $G(\mathrm{w}, e, \theta(\tau + \pi)) \geq G(\mathrm{w}, \mathrm{k}, \tau) * G(\mathrm{k}, e, \Pi)$;

(B6) $G(\mathrm{w}, \mathrm{k}, \cdot)$ is a non decreasing function of R^+ and $lim_{\tau \to \infty} G(\mathrm{w}, \mathrm{k}, \tau) = 1$;

(B7) $H(\mathrm{w}, \mathrm{k}, \tau) > 0$;

(B8) $H(\mathrm{w}, \mathrm{k}, \tau) = 0 \Leftrightarrow \mathrm{w} = \mathrm{k}$;

(B9) $H(\mathrm{w}, \mathrm{k}, \tau) = H(\mathrm{k}, \mathrm{w}, \tau)$;

(B10) $H(\mathrm{w}, e, \theta(\tau + \pi)) \leq H(\mathrm{w}, \mathrm{k}, \tau) \Delta H(\mathrm{k}, e, \Pi)$;

(B11) $H(\mathrm{w}, \mathrm{k}, \cdot)$ is a non increasing function of R^+ and $lim_{\tau \to \infty} H(\mathrm{w}, \mathrm{k}, \tau) = 1$;

Then $(\Xi, G, H, *, \Delta)$ is an IFBMS.

Definition 2.2. Assume $\Xi \neq \phi$. Let $\perp \in \Xi \times \Xi$ be a binary relation. Suppose there exists $\mathrm{w}_0 \in \Xi$ such that $\mathrm{w}_0 \perp \mathrm{w}$ or $\mathrm{w} \perp \mathrm{w}_0$ for all $\mathrm{w} \in \Xi$. Thus, Ξ is known as orthogonal set (OS) and denoted by $(\Xi, \perp)$

Definition 2.3. Assume that $(\Xi, \perp)$ is an OS. A sequence $\{\mathrm{w}_n\}$ for $n \in \mathbf{N}$ is known to be an O-sequence if $(\forall n, \mathrm{w}_n \perp \mathrm{w}_{n+1})$ or $(\forall n, \mathrm{w}_{n+1} \perp \mathrm{w}_n)$

3 Orthogonal Intuitionistic Fuzzy b-metric Spaces

Now, we establish the notion of OIFBMSs and derive several FP results with non-trivial examples.

Definition 3.1. $(\Xi, G, H, *, \Delta)$ is known to be an OIFBMS if Ξ is a (non empty) OS, $*$ is a CTN, Δ is a CTCN, and G, H are FSs on $\Xi \times \Xi \times (0, \infty)$ verifying the below conditions for a given real number $\theta \geq 1$:

$(B_\perp 1)$ $G(\mathrm{w}, \mathrm{k}, \tau) + H(\mathrm{w}, \mathrm{k}, \tau) \leq 1$ for all $\mathrm{w}, \mathrm{k} \in \Xi$, $\tau > 0$ such that $\mathrm{w} \perp \mathrm{k}$ and $\mathrm{k} \perp \mathrm{w}$;

$(B_\perp 2)$ $G(\mathrm{w}, \mathrm{k}, \tau) > 0$ for all $\mathrm{w}, \mathrm{k} \in \Xi$, $\tau > 0$ such that $\mathrm{w} \perp \mathrm{k}$ and $\mathrm{k} \perp \mathrm{w}$;

$(B_\perp 3)$ $G(\mathrm{w}, \mathrm{k}, \tau) = 1 \Leftrightarrow \mathrm{w} = \mathrm{k}$; for all $\mathrm{w}, \mathrm{k} \in \Xi$, $\tau > 0$ such that $\mathrm{w} \perp \mathrm{k}$ and $\mathrm{k} \perp \mathrm{w}$;

$(B_\perp 4)$ $G(\mathrm{w}, \mathrm{k}, \tau) = G(\mathrm{k}, \mathrm{w}, \tau)$ for all $\mathrm{w}, \mathrm{k} \in \Xi$, $\tau > 0$ such that $\mathrm{w} \perp \mathrm{k}$ and $\mathrm{k} \perp \mathrm{w}$;

$(B_\perp 5)$ $G(\mathrm{w}, e, \theta(\tau + \pi)) \geq G(\mathrm{w}, \mathrm{k}, \tau) * G(\mathrm{k}, e, \Pi)$ for all $\mathrm{w}, \mathrm{k} \in \Xi$, $\tau > 0$ such that $\mathrm{w} \perp \mathrm{k}$ and $\mathrm{k} \perp \mathrm{w}$;

$(B_\perp 6)$ $G(\mathrm{w}, \mathrm{k}, \cdot)$ is a non decreasing function of R^+ and $lim_{\tau \to \infty} G(\mathrm{w}, \mathrm{k}, \tau) = 1$ for all $\mathrm{w}, \mathrm{k} \in \Xi$, $\tau > 0$ such that $\mathrm{w} \perp \mathrm{k}$ and $\mathrm{k} \perp \mathrm{w}$;

$(B_\perp 7)$ $H(\mathrm{w}, \mathrm{k}, \tau) > 0$for all $\mathrm{w}, \mathrm{k} \in \Xi$, $\tau > 0$ such that $\mathrm{w} \perp \mathrm{k}$ and $\mathrm{k} \perp \mathrm{w}$;

$(B_\perp 8)$ $H(\mathrm{w}, \mathrm{k}, \tau) = 0 \Leftrightarrow \mathrm{w} = \mathrm{k}$ for all $\mathrm{w}, \mathrm{k} \in \Xi$, $\tau > 0$ such that $\mathrm{w} \perp \mathrm{k}$ and $\mathrm{k} \perp \mathrm{w}$;

$(B_\perp 9)$ $H(\mathrm{w}, \mathrm{k}, \tau) = H(\mathrm{k}, \mathrm{w}, \tau)$ for all $\mathrm{w}, \mathrm{k} \in \Xi$, $\tau > 0$ such that $\mathrm{w} \perp \mathrm{k}$ and $\mathrm{k} \perp \mathrm{w}$;

$(B_\perp 10)$ $H(\mathrm{w}, e, \theta(\tau + \pi)) \leq H(\mathrm{w}, \mathrm{k}, \tau) \Delta H(\mathrm{k}, e, \Pi)$ for all $\mathrm{w}, \mathrm{k} \in \Xi$, $\tau > 0$ such that $\mathrm{w} \perp \mathrm{k}$ and $\mathrm{k} \perp \mathrm{w}$;

$(B_\perp 11)$ $H(\mathrm{w}, \mathrm{k}, \cdot)$ is a non increasing function of R^+ and $\lim_{\tau \to \infty} H(\mathrm{w}, \mathrm{k}, \tau) = 1$ for all $\mathrm{w}, \mathrm{k} \in \Xi$, $\tau > 0$ such that $\mathrm{w} \perp \mathrm{k}$ and $\mathrm{k} \perp \mathrm{w}$;

Then $(\Xi, G, H, *, \Delta)$ is an IFBMS.

Example 3.2. Let $\Xi = R$ and define $\sigma * \theta = \sigma\theta$, $\sigma \Delta \theta = \min\{\sigma, \theta\}$ and $\perp$ by $\mathrm{w} \perp \mathrm{k}$ iff $\mathrm{w} + \mathrm{k} \geq 0$. Let

$$G(\mathrm{w}, \mathrm{k}, \tau) = \begin{cases} 1 \text{ if } \mathrm{w} = \mathrm{k}, \\ \frac{\tau}{\tau + \max\{\mathrm{w}, \mathrm{k}\}^\alpha} \text{ otherwise.} \end{cases} \tag{1}$$

and

$$H(\mathrm{w}, \mathrm{k}, \tau) = \begin{cases} 0 \text{ if } \mathrm{w} = \mathrm{k}, \\ \frac{\max\{\mathrm{w}, \mathrm{k}\}^\alpha}{\tau + \max\{\mathrm{w}, \mathrm{k}\}^\alpha} \text{ otherwise.} \end{cases} \tag{2}$$

for all $\mathrm{w}, \mathrm{k} \in \Xi, \tau > 0$ with α belong to odd natural numbers.
Proof. $(B_\perp 1) - (B_\perp 3)$, $(B_\perp 5) - (B_\perp 9)$ and $(B_\perp 11)$ are obvious. Here, we prove $(B_\perp 4)$ and $(B_\perp 10)$. $(B_\perp 4)$: for a random number $\theta \geq 1$, one writes

$$\max\{\mathrm{w}, e\}^\alpha \leq \theta[\max\{\mathrm{w}, \mathrm{k}\}^\alpha + \max\{\mathrm{k}, e\}^\alpha]$$

Thus,

$$\tau\pi \max\{\mathrm{w}, e\}^\alpha \leq \theta(\tau+\pi)\pi \max\{\mathrm{w}, \mathrm{k}\}^\alpha + \theta(\tau+\pi)\tau \max\{\mathrm{k}, e\}^\alpha.$$

Consequently,

$$\tau\pi \max\{\mathrm{w}, e\}^\alpha \leq \theta(\tau+\pi)\pi \max\{\mathrm{w}, \mathrm{k}\}^\alpha + \theta(\tau+\pi)\tau \max\{\mathrm{k}, e\}^\alpha + \theta(\tau+\pi) \max\{\mathrm{k}, e\}^\alpha.$$

Thus,

$$\tau\pi \max\{\mathrm{w}, e\}^\alpha \leq \theta(\tau+\pi)[\pi \max\{\mathrm{w}, \mathrm{k}\}^\alpha + \tau \max\{\mathrm{k}, e\}^\alpha + \max\{\mathrm{w}, \mathrm{k}\}^\alpha \max\{\mathrm{k}, e\}^\alpha].$$

one write

$$\theta(\tau+\pi)\tau\pi + \tau\pi \max\{\mathrm{w}, e\}^\alpha \leq \theta(\tau+\pi)\tau\pi + \theta(\tau+\pi)[\pi \max\{\mathrm{w}, \mathrm{k}\}^\alpha + \tau \max\{\mathrm{k}, e\}^\alpha + \max\{\mathrm{w}, \mathrm{k}\}^\alpha \max\{\mathrm{k}, e\}^\alpha].$$

Therefore,

$$\theta(\tau+\pi)\tau\pi + \tau\pi \max\{\mathrm{w}, e\}^\alpha \leq \theta(\tau+\pi)[\tau\pi + \pi \max\{\mathrm{w}, \mathrm{k}\}^\alpha + \tau \max\{\mathrm{k}, e\}^\alpha + \max\{\mathrm{w}, \mathrm{k}\}^\alpha \max\{\mathrm{k}, e\}^\alpha].$$

That is,

$$\tau\pi[\theta(\tau+\pi) + \max\{\mathrm{w}, e\}^\alpha] \leq \theta(\tau+\pi)[\tau + \max\{\mathrm{w}, \mathrm{k}\}^\alpha][\pi + \max\{\mathrm{k}, e\}^\alpha]$$

Hence,

$$\frac{\theta(\tau+\pi)}{\theta(\tau+\pi) + \max\{\mathrm{w}, e\}^\alpha} \geq \frac{\tau\pi}{[\tau + \max\{\mathrm{w}, \mathrm{k}\}^\alpha][\pi + \max\{\mathrm{k}, e\}^\alpha]}.$$

$$\frac{\theta(\tau+\pi)}{\theta(\tau+\pi) + \max\{\mathrm{w}, e\}^\alpha} \geq \frac{\tau}{\tau + \max\{\mathrm{w}, \mathrm{k}\}^\alpha}.$$

That is,

$$G(\mathrm{w}, e, \theta(\tau+\pi)) \geq G(\mathrm{w}, \mathrm{k}\tau) * G(\mathrm{k}, e, \pi).$$

$(B_\perp 10)$: One writes

$$\max\{\mathrm{w}, e\}^\alpha = \max\{\mathrm{w}, e\}^\alpha \max\left\{\frac{\max\{\mathrm{w}, \mathrm{k}\}^\alpha}{\max\{\mathrm{w}, \mathrm{k}\}^\alpha}, \frac{\max\{\mathrm{k}, e\}^\alpha}{\max\{\mathrm{k}, e\}^\alpha}\right\}.$$

Then

$$\max\{\mathrm{w}, e\}^\alpha \leq [\theta(\tau+\pi) + \max\{\mathrm{w}, e\}^\alpha] \max\left\{\frac{\max\{\mathrm{w}, \mathrm{k}\}^\alpha}{\max\{\mathrm{w}, \mathrm{k}\}^\alpha}, \frac{\max\{\mathrm{k}, e\}^\alpha}{\max\{\mathrm{k}, e\}^\alpha}\right\}.$$

That is,

$$\frac{\max\{\mathrm{w}, e\}^\alpha}{\theta(\tau+\pi) + \max\{\mathrm{w}, e\}^\alpha} \leq \max\left\{\frac{\max\{\mathrm{w}, \mathrm{k}\}^\alpha}{\tau + \max\{\mathrm{w}, \mathrm{k}\}^\alpha}, \frac{\max\{\mathrm{k}, e\}^\alpha}{\pi + \max\{\mathrm{k}, e\}^\alpha}\right\}.$$

Hence,

$$H(\mathrm{w}, e, \theta(\tau+\pi)) \leq H(\mathrm{w}, \mathrm{k}, \tau)\Delta H(\mathrm{k}, e, \pi).$$

Now, we show it's not an IFBM. Indeed, for $\pi = \tau = 1, \mathrm{w} = -1, \mathrm{k} = -\frac{1}{2}$ and $\alpha = 3$, (B4) and (B10) fail. □

Example 3.3. Let $\Xi = \mathbb{R}$ and define $\sigma * \theta = \sigma\theta$, $\sigma\Delta\theta = \min\{\sigma, \theta\}$ and $\perp$ by $\mathrm{w} \perp \mathrm{k}$ iff $\mathrm{w} + \mathrm{k} \geq 0$. Let

$$G(\mathrm{w}, \mathrm{k}, \tau) = \begin{cases} 1 \text{ if } \mathrm{w} = \mathrm{k}, \\ \left[e^{\frac{\max\{\mathrm{w}, \mathrm{k}\}^\alpha}{\tau}}\right]^{-1} \text{ otherwise.} \end{cases} \tag{3}$$

and

$$H(\mathrm{w},\mathrm{k},\tau)=\begin{cases}0 \text{ if } \mathrm{w}=\mathrm{k},\\ 1-\left[e^{\frac{\max\{\mathrm{w},\mathrm{k}\}^{\alpha}}{\tau}}\right]^{-1} \text{ otherwise.}\end{cases} \tag{4}$$

for all $\mathrm{w},\mathrm{k}\in\Xi,\tau>0$ with α belong to odd natural numbers.

Proof. $(B_{\perp}1)-(B_{\perp}3)$, $(B_{\perp}5)-(B_{\perp}9)$ and $(B_{\perp}11)$ are obvious. Here, we prove $(B_{\perp}4)$ and $(B_{\perp}10)$.
$(B_{\perp}4)$: for a random number $\theta\geq 1$, one writes

$$\max\{\mathrm{w},e\}^{\alpha}\leq\theta\left[\max\{\mathrm{w},\mathrm{k}\}^{\alpha}+\max\{\mathrm{k},e\}^{\alpha}\right].$$

Therefore,

$$\max\{\mathrm{w},e\}^{\alpha}\leq\theta\left[\frac{\tau+\pi}{\tau}\max\{\mathrm{w},\mathrm{k}\}^{\alpha}+\frac{\tau+\pi}{\pi}\max\{\mathrm{k},e\}^{\alpha}\right]$$

Then

$$\frac{\max\{\mathrm{w},e\}^{\alpha}}{\theta(\tau+\pi)}\leq\frac{\max\{\mathrm{w},\mathrm{k}\}^{\alpha}}{\tau}+\frac{\max\{\mathrm{k},e\}^{\alpha}}{\pi}$$

Since, e^{w} is an increasing function, one gets

$$e^{\frac{\max\{\mathrm{w},e\}^{\alpha}}{\theta(\tau+\pi)}}\leq e^{\frac{\max\{\mathrm{w},\mathrm{k}\}^{\alpha}}{\tau}}\cdot e^{\frac{\max\{\mathrm{k},e\}^{\alpha}}{\pi}}.$$

That is

$$\left[e^{\frac{\max\{\mathrm{w},e\}^{\alpha}}{\theta(\tau+\pi)}}\right]^{-1}\geq\left[e^{\frac{\max\{\mathrm{w},\mathrm{k}\}^{\alpha}}{\tau}}\right]^{-1}\cdot\left[e^{\frac{\max\{\mathrm{k},e\}^{\alpha}}{\pi}}\right]^{-1}.$$

Hence,

$$G(\mathrm{w},e,\theta(\tau+\pi))\geq G(\mathrm{w},\mathrm{k}\tau)*G(\mathrm{k},e,\pi).$$

$(B_{\perp}10)$: For a random $\theta\geq 1$, we write

$$\frac{\max\{\mathrm{w},e\}^{\alpha}}{\theta(\tau+\pi)}\leq\max\left\{\frac{\max\{\mathrm{w},\mathrm{k}\}^{\alpha}}{\tau},\frac{\max\{\mathrm{k},e\}^{\alpha}}{\pi}\right\}.$$

That is,

$$e^{\frac{\max\{\mathrm{w},e\}^{\alpha}}{\theta(\tau+\pi)}}\leq\max\left\{e^{\frac{\max\{\mathrm{w},\mathrm{k}\}^{\alpha}}{\tau}},e^{\frac{\max\{\mathrm{k},e\}^{\alpha}}{\pi}}\right\}.$$

Then,

$$\left[e^{\frac{\max\{\mathrm{w},e\}^{\alpha}}{\theta(\tau+\pi)}}\right]^{-1}\geq\max\left\{\left[e^{\frac{\max\{\mathrm{w},\mathrm{k}\}^{\alpha}}{\tau}}\right]^{-1},\left[e^{\frac{\max\{\mathrm{k},e\}^{\alpha}}{\pi}}\right]^{-1}\right\}.$$

That is,

$$1-\left[e^{\frac{\max\{\mathrm{w},e\}^{\alpha}}{\theta(\tau+\pi)}}\right]^{-1}\leq\max\left\{1-\left[e^{\frac{\max\{\mathrm{w},\mathrm{k}\}^{\alpha}}{\tau}}\right]^{-1},1-\left[e^{\frac{\max\{\mathrm{k},e\}^{\alpha}}{\pi}}\right]^{-1}\right\}.$$

Hence,

$$H(\mathrm{w},e,\theta(\tau+\pi))\leq H(\mathrm{w},\mathrm{k},\tau)\Delta H(\mathrm{k},e,\pi).\forall\mathrm{w},\mathrm{k},e\in\Xi,\forall\tau,\pi>0.$$

Now, we show it's not an IFBM. Indeed, for $\pi=\tau=1,\mathrm{w}=-1,\mathrm{k}=-\frac{1}{2},e=-2$ and $\alpha=3$, (B4) and (B10) is not satisfy. □

Example 3.4. Let $\Xi = \mathbb{R}$ and define $\sigma * \theta = \sigma\theta, \sigma\Delta\theta = \max\{\sigma, \theta\}$ and $\perp$ by $\mathrm{w} \perp \mathrm{k}$ iff $\mathrm{w} + \mathrm{k} \geq 0$. Suppose

$$G(\mathrm{w}, \mathrm{k}, \tau) = \frac{\tau + \min\{\mathrm{w}, \mathrm{k}\}^\alpha}{\tau + \max\{\mathrm{w}, \mathrm{k}\}^\alpha} \tag{5}$$

and

$$H(\mathrm{w}, \mathrm{k}, \tau) = 1 - \frac{\tau + \min\{\mathrm{w}, \mathrm{k}\}^\alpha}{\tau + \max\{\mathrm{w}, \mathrm{k}\}^\alpha} \tag{6}$$

for all $\mathrm{w}, \mathrm{k} \in \Xi, \tau > 0$ with α belong to odd natural numbers. Here, $(\Xi, G, H, *, \Delta, \perp)$ is an OIFBMS. It is not an IFBMS. Indeed, if it is the case, from (B4),

$$\frac{\theta(\tau + \pi) + \min\{\mathrm{w}, \mathrm{k}\}^\alpha}{\theta(\tau + \pi) + \max\{\mathrm{w}, \mathrm{k}\}^\alpha} \geq \frac{\tau + \min\{\mathrm{w}, \mathrm{k}\}^\alpha}{\tau + \max\{\mathrm{w}, \mathrm{k}\}^\alpha} \cdot \frac{\pi + \min\{\mathrm{w}, \mathrm{k}\}^\alpha}{\pi + \max\{\mathrm{w}, \mathrm{k}\}^\alpha}$$

and from case (B10)

$$1 - \frac{\theta(\tau + \pi) + \min\{\mathrm{w}, \mathrm{k}\}^\alpha}{\theta(\tau + \pi) + \max\{\mathrm{w}, \mathrm{k}\}^\alpha} \leq \max\left[1 - \frac{\tau + \min\{\mathrm{w}, \mathrm{k}\}^\alpha}{\tau + \max\{\mathrm{w}, \mathrm{k}\}^\alpha} \cdot 1 - \frac{\pi + \min\{\mathrm{w}, \mathrm{k}\}^\alpha}{\pi + \max\{\mathrm{w}, \mathrm{k}\}^\alpha}\right].$$

Then by taking $\mathrm{w} = \mathrm{k}, e = -2$ and $\alpha = \frac{1}{2}$, the above inequalities are not satisfied.

Remark 3.5. Every IFBMS is an OIFBMS, but the converse is not true. The above examples confirm this reverse statement.

Definition 3.6. An O-sequence $\{\mathrm{w}_n\}$ is an OIFBMS $(\Xi, G, H, *, \Delta, \perp)$ is called an orthogonal convergent (O-convergent) to $\mathrm{w} \in \Xi$, if

$$\lim_{n\to\infty} G(\mathrm{w}_n, \mathrm{w}, \tau) = 1, \forall \tau > 0,$$

and

$$\lim_{n\to\infty} H(\mathrm{w}_n, \mathrm{w}, \tau) = 0, \forall \tau > 0,$$

Definition 3.7. An O-sequence $\{\mathrm{w}_n\}$ is an OIFBMS $(\Xi, G, H, *, \Delta, \perp)$ is known to be an orthogonal Cauchy (O-Cauchy) if

$$\lim_{n\to\infty} G(\mathrm{w}_n, \mathrm{w}, \tau) = 1,$$

and

$$\lim_{n\to\infty} H(\mathrm{w}_n, \mathrm{w}, \tau) = 0,$$

for all $\tau > 0, p \geq 1$.

Definition 3.8. Let $\xi : \Xi \to \Xi$ is $\perp$-continuous at $\mathrm{w} \in \Xi$ is an OIFBMS $(\Xi, G, H, *, \Delta, \perp)$, whenever for each O-sequence w_n for all $n \in \mathbb{N}$ in Ξ if $\lim_{n\to\infty} G(\mathrm{w}_n, \mathrm{w}, \tau) = 1$ and $\lim_{n\to\infty} H(\mathrm{w}_n, \mathrm{w}, \tau) = 0$ for all $\tau > 0$, then $\lim_{n\to\infty} G(\xi\mathrm{w}_n, \xi\mathrm{w}, \tau) = 1$ and $\lim_{n\to\infty} H(\xi\mathrm{w}_n, \xi\mathrm{w}, \tau) = 0$ for all $\tau > 0$. Furthermore, ξ is $\perp$-continuous on Ξ if ξ $\perp$-continuous at each $\mathrm{w} \in \Xi$. Also, ξ is $\perp$- preserving if $\xi\mathrm{w} \perp \xi\mathrm{k}$, whence $\mathrm{w} \perp \mathrm{k}$.

Definition 3.9. An OIFBMS $(\Xi, G, H, *, \Delta, \perp)$ is known to be orthogonally complete (O-complete) if every O-Cauchy O-sequence is O- convergent.

Remark 3.10. It is necessary that the limit of an O-convergent O-sequence is unique in an OIFBMS.

Remark 3.11. It is necessary that the limit of an O-convergent O-sequence is O-Cauchy in an OIFBMS.

Lemma 3.12. *If for some* $v \in (0,1)$ *and* $\mathrm{w}, \mathrm{k} \in \Xi$,

$$G(\mathrm{w}, \mathrm{k}, \tau) \geq G\left(\mathrm{w}, \mathrm{k}, \frac{\tau}{v}\right), \tau > 0,$$

and

$$H(\mathrm{w}, \mathrm{k}, \tau) \leq H\left(\mathrm{w}, \mathrm{k}, \frac{\tau}{v}\right), \tau > 0,$$

then $\mathrm{w} = \mathrm{k}$.
Proof. *The proof is follows from [8].* □

Definition 3.13. Suppose $(\Xi, G, H, *, \Delta, \perp)$ be an OIFBMS. A mapping $\xi : \Xi \to \Xi$ is an orthogonal contraction ($\perp$-contraction) if there exists $\varrho \in (0,1)$ such that for every $\tau > 0$ and $\mathrm{w}, \mathrm{k} \in \Xi$ with $\mathrm{w} \perp \mathrm{k}$, we have

$$G(\xi\mathrm{w}, \xi\mathrm{k}, \varrho\tau) \geq G(\mathrm{w}, \mathrm{k}, \tau), \tag{7}$$

$$H(\xi\mathrm{w}, \xi\mathrm{k}, \varrho\tau) \leq H(\mathrm{w}, \mathrm{k}, \tau). \tag{8}$$

Theorem 3.14. *Let* $(\Xi, G, H, *, \Delta, \perp)$ *be an O-complete IFBMS such that*

$$\lim_{\tau\to\infty} G(\mathrm{w}, \mathrm{k}, \tau) = 1,$$

and

$$\lim_{\tau\to\infty} H(\mathrm{w}, \mathrm{k}, \tau) = 0.$$

for all $\mathrm{w}, \mathrm{k} \in \Xi$. *Suppose* $\xi : \Xi \to \Xi$ *be an* $\perp$*-continuous and* $\perp$*-preserving mapping. Thus,* ξ *has a unique FP, say* $\mathrm{w}_* \in \Xi$. *Furthermore,*

$$\lim_{\tau\to\infty} G(\xi^n\mathrm{w}, \mathrm{k}, \tau) = 1,$$

and

$$\lim_{\tau\to\infty} H(\xi^n\mathrm{w}, \mathrm{k}, \tau) = 0.$$

for all $\mathrm{w}, \mathrm{k} \in \Xi$.
Proof. *Let* $(\Xi, G, H, *, \Delta, \perp)$ *be an O-complete IFBMS, there exists* $\mathrm{w}_0 \in \Xi$ *such that* $\mathrm{w}_0 \perp \mathrm{k}$ *for all* $\mathrm{k} \in \Xi$, *that is,* $\mathrm{w}_0 \perp \xi\mathrm{w}_0$. *Take* $\mathrm{w}_n = \xi^n\mathrm{w}_0 = \xi\mathrm{w}_{n-1}$ *for all* $n \in \mathbb{N}$. *Since* ξ *is* $\perp$*-preserving,* $\{\mathrm{w}_n\}$ *is an O-sequence. From assumption that* ξ *is an* $\perp$*-contraction, we have*

$$G(\mathrm{w}_{n+1}, \mathrm{w}_n, \varrho\tau) = G(\xi\mathrm{w}_n, \xi\mathrm{w}_{n-1}, \varrho\tau) \geq G(\mathrm{w}_n, \mathrm{w}_{n-1}, \tau)$$

for all $n \in \mathbb{N}$ *and* $\tau > 0$. *Note that* G *is non-decreasing on* $(0, \infty)$. *By utilizing above inequality, we have*

$$G(\mathrm{w}_{n+1}, \mathrm{w}_n, \tau) \geq G(\mathrm{w}_{n+1}, \mathrm{w}_n, \varrho\tau) = G(\xi\mathrm{w}_{n+1}, \xi\mathrm{w}_n, \varrho\tau) \geq G(\mathrm{w}_n, \mathrm{w}_{n-1}, \tau)$$
$$= G(\xi\mathrm{w}_{n-1}, \xi\mathrm{w}_{n-2}, \tau) \geq G\left(\mathrm{w}_{n-1}, \mathrm{w}_{n-2}, \frac{\tau}{\varrho}\right) \geq \cdots \geq G\left(\mathrm{w}_1, \mathrm{w}_0, \frac{\tau}{\varrho^n}\right) \tag{9}$$

for all $n \in \mathbb{N}$ *and* $\tau > 0$. *Thus, from (9) and (B4), we deduce*

$$G(\mathrm{w}_n, \mathrm{w}_{n+m}, \tau) \geq G\left(\mathrm{w}_n, \mathrm{w}_{n+1}, \frac{\tau}{\theta}\right) * G\left(\mathrm{w}_{n+1}, \mathrm{w}_{n+m}, \frac{\tau}{\theta}\right)$$

$$\geq G\left(\mathrm{w}_n, \mathrm{w}_{n+1}, \frac{\tau}{\theta}\right) * G\left(\mathrm{w}_{n+1}, \mathrm{w}_{n+2}, \frac{\tau}{\theta^2}\right) * G\left(\mathrm{w}_{n+2}, \mathrm{w}_{n+3}, \frac{\tau}{\theta^3}\right) * \cdots * G\left(\mathrm{w}_{n+m-1}, \mathrm{w}_{n+m}, \frac{\tau}{\theta^{n+m}}\right)$$

$$\geq G\left(\mathrm{w}_1,\mathrm{w}_0,\frac{\tau}{\theta\varrho^n}\right)*G\left(\mathrm{w}_{n+1},\mathrm{w}_{n+2},\frac{\tau}{\theta^2\varrho^n}\right)*G\left(\mathrm{w}_{n+2},\mathrm{w}_{n+3},\frac{\tau}{\theta^3\varrho^n}\right)*\cdots*G\left(\mathrm{w}_{n+m-1},\mathrm{w}_{n+m},\frac{\tau}{\theta^{n+m}\varrho^n}\right) \tag{10}$$

We know that $\lim_{\tau\to\infty} G(\mathrm{w},\mathrm{k},\tau)=1$, *for all* $\mathrm{w},\mathrm{k}\in\Xi$ *and* $\tau>0$. *So, from (10), we have*

$$\lim_{\tau\to\infty} G(\mathrm{w}_n,\mathrm{w}_{n+m},\tau)\geq 1*1*\cdots*1=1. \tag{11}$$

Similarly,

$$H(\mathrm{w}_{n+1},\mathrm{w}_n,\varrho\tau)=H(\xi\mathrm{w}_n,\xi\mathrm{w}_{n-1},\varrho\tau)\leq H(\mathrm{w}_n,\mathrm{w}_{n-1},\tau)$$

for all $n\in\mathbb{N}$ *and* $\tau>0$. *By utilizing above inequality, we have*

$$H(\mathrm{w}_{n+1},\mathrm{w}_n,\tau)\leq H(\mathrm{w}_{n+1},\mathrm{w}_n,\varrho\tau)=H(\xi\mathrm{w}_{n+1},\xi\mathrm{w}_n,\varrho\tau)\leq H(\mathrm{w}_n,\mathrm{w}_{n-1},\tau)$$

$$=H(\xi\mathrm{w}_{n-1},\xi\mathrm{w}_{n-2},\tau)\leq H\left(\mathrm{w}_{n-1},\mathrm{w}_{n-2},\frac{\tau}{\varrho}\right)\leq\cdots\leq H\left(\mathrm{w}_1,\mathrm{w}_0,\frac{\tau}{\varrho^n}\right) \tag{12}$$

for all $n\in\mathbb{N}$ *and* $\tau>0$. *Thus, from (12) and (B10), we deduce*

$$H(\mathrm{w}_n,\mathrm{w}_{n+m},\tau)\leq H\left(\mathrm{w}_n,\mathrm{w}_{n+1},\frac{\tau}{\theta}\right)\Delta H\left(\mathrm{w}_{n+1},\mathrm{w}_{n+m},\frac{\tau}{\theta}\right)$$

$$\leq H\left(\mathrm{w}_n,\mathrm{w}_{n+1},\frac{\tau}{\theta}\right)\Delta H\left(\mathrm{w}_{n+1},\mathrm{w}_{n+2},\frac{\tau}{\theta^2}\right)\Delta H\left(\mathrm{w}_{n+2},\mathrm{w}_{n+3},\frac{\tau}{\theta^3}\right)\Delta\cdots\Delta H\left(\mathrm{w}_{n+m-1},\mathrm{w}_{n+m},\frac{\tau}{\theta^{n+m}}\right)$$

$$\leq H\left(\mathrm{w}_1,\mathrm{w}_0,\frac{\tau}{\theta\varrho^n}\right)\Delta H\left(\mathrm{w}_{n+1},\mathrm{w}_{n+2},\frac{\tau}{\theta^2\varrho^n}\right)\Delta H\left(\mathrm{w}_{n+2},\mathrm{w}_{n+3},\frac{\tau}{\theta^3\varrho^n}\right)\Delta\cdots\Delta H\left(\mathrm{w}_{n+m-1},\mathrm{w}_{n+m},\frac{\tau}{\theta^{n+m}\varrho^n}\right) \tag{13}$$

We know that $\lim_{\tau\to\infty} H(\mathrm{w},\mathrm{k},\tau)=0$, *for all* $\mathrm{w},\mathrm{k}\in\Xi$ *and* $\tau>0$. *So, from (13), we have*

$$\lim_{\tau\to\infty} H(\mathrm{w}_n,\mathrm{w}_{n+m},\tau)\leq 0\Delta0\Delta\cdots\Delta0=0. \tag{14}$$

So, $\{\mathrm{w}_n\}$ *is an O-sequence. The O-sequence. The O-completeness of the IFBMS* $(\Xi,\mathrm{w},\mathrm{k},*,\Delta,\perp)$ *ensure that there exists* $\mathrm{w}_*\in\Xi$ *such that* $G(\mathrm{w}_n,\mathrm{w}_*,\tau)\to 1$, *and* $H(\mathrm{w}_n,\mathrm{w}_*,\tau)\to 0$, *as* $n\to+\infty$ *for all* $\tau>0$. *Now, since* ξ *is an* $\perp$*-continuous mapping,* $G(\mathrm{w}_{n+1},\xi\mathrm{w}_*,\tau)=G(\xi\mathrm{w}_{n+},\xi\mathrm{w}_*,\tau)\to 1$ *and* $H(\mathrm{w}_{n+1},\xi\mathrm{w}_*,\tau)=H(\xi\mathrm{w}_{n+},\xi\mathrm{w}_*,\tau)\to 0$ *as* $n\to+\infty$. *Now, we have*

$$G(\mathrm{w}_*,\xi\mathrm{w}_*,\tau)\geq G\left(\mathrm{w}_*,\mathrm{w}_{n+1},\frac{\tau}{2\theta}\right)*G\left(\mathrm{w}_{n+1},\xi\mathrm{w}_*,\frac{\tau}{2\theta}\right),$$

$$H(\mathrm{w}_*,\xi\mathrm{w}_*,\tau)\leq H\left(\mathrm{w}_*,\mathrm{w}_{n+1},\frac{\tau}{2\theta}\right)\Delta H\left(\mathrm{w}_{n+1},\xi\mathrm{w}_*,\frac{\tau}{2\theta}\right).$$

Taking limit as $n\to\infty$, *we get* $G(\mathrm{w}_*,\xi\mathrm{w}_*,\tau)=1*1=1$ *and* $H(\mathrm{w}_*,\xi\mathrm{w}_*,\tau)=0\Delta0=0$*and hence* $\xi\mathrm{w}_*=\mathrm{w}_*$.

Uniqueness*:*

Let w_* *and* k_* *be two FPs of* ξ *such that* $\mathrm{w}_*\neq\mathrm{k}_*$. *We have* $\mathrm{w}_0\perp\mathrm{w}_*$ *and* $\mathrm{w}_0\perp\mathrm{k}_*$. *Since* T *is* $\perp$*-preserving, we have* $\xi\mathrm{w}_0\perp\xi^n\mathrm{w}_*$ *and* $\xi^n\mathrm{w}_0\perp\mathrm{k}_*$ *for all* $n\in\mathbb{N}$. *So from (7), we can drive*

$$G(\xi^n\mathrm{w}_0,\xi^n\mathrm{w}_*,\tau)\geq G(\xi^n\mathrm{w}_0,\xi^n\mathrm{w}_*,\varrho\tau)\geq G\left(\mathrm{w}_0,\mathrm{w}_*,\frac{\tau}{\varrho^n}\right)$$

and

$$G(\xi^n\mathrm{w}_0,\xi^n\mathrm{k}_*,\tau)\geq G(\xi^n\mathrm{w}_0,\xi^n\mathrm{k}_*,\varrho\tau)\geq G\left(\mathrm{w}_0,\mathrm{k}_*,\frac{\tau}{\varrho^n}\right)$$

Therefore,

$$G(\mathrm{w}_*,\mathrm{k}_*,\tau)=G(\xi^n\mathrm{w}_*,\xi^n\mathrm{k}_*,\tau)\geq G\left(\xi^n\mathrm{w}_0,\xi^n\mathrm{w}_*,\frac{\tau}{2\theta}\right)*G\left(\xi^n\mathrm{w}_0,\xi^n\mathrm{k}_*,\frac{\tau}{2\theta}\right)$$

$$\geq G\left(\mathrm{w}_0,\mathrm{w}_*,\frac{\tau}{2\theta\varrho^n}\right)*G\left(\mathrm{w}_0,\mathrm{k}_*,\frac{\tau}{2\theta\varrho^n}\right)\to 1$$

as $n\to\infty$ *So from (8), we can derive*

$$H(\xi^n\mathrm{w}_0,\xi^n\mathrm{w}_*,\tau)\leq H(\xi^n\mathrm{w}_0,\xi^n\mathrm{w}_*,\varrho\tau)\leq H\left(\mathrm{w}_0,\mathrm{w}_*,\frac{\tau}{\varrho^n}\right)$$

and

$$H(\xi^n\mathrm{w}_0,\xi^n\mathrm{k}_*,\tau)\leq H(\xi^n\mathrm{w}_0,\xi^n\mathrm{k}_*,\varrho\tau)\leq H\left(\mathrm{w}_0,\mathrm{k}_*,\frac{\tau}{\varrho^n}\right)$$

Therefore,

$$H(\mathrm{w}_*,\mathrm{k}_*,\tau)=H(\xi^n\mathrm{w}_*,\xi^n\mathrm{k}_*,\tau)\leq H\left(\xi^n\mathrm{w}_0,\xi^n\mathrm{w}_*,\frac{\tau}{2\theta}\right)*H\left(\xi^n\mathrm{w}_0,\xi^n\mathrm{k}_*,\frac{\tau}{2\theta}\right)$$

$$\leq H\left(\mathrm{w}_0,\mathrm{w}_*,\frac{\tau}{2\theta\varrho^n}\right)\Delta H\left(\mathrm{w}_0,\mathrm{k}_*,\frac{\tau}{2\theta\varrho^n}\right)\to 0$$

as $n\to\infty$ *So,* $\mathrm{w}_*=\mathrm{k}_*$, *hence* w_* *is the unique FP.* □

Corollary 3.15. *Suppose* $(\Xi,G,H,*,\Delta,\perp)$ *be an O-complete IFBMS. Assume* $\xi:\Xi\to\Xi$ *be* $\perp$*-contraction and* $\perp$*-preserving. Assume that if* $\{\mathrm{w}\}$ *is an O-sequence with* $\mathrm{w}_n\to\mathrm{w}\in\Xi$, *Then* $\mathrm{w}\perp\mathrm{w}_n$ *for all* $n\in\mathbb{N}$. *Then* ξ *has a unique FP, say* $\mathrm{w}_*\in\Xi$, *Moreover,* $\lim_{n\to\infty}G(\xi^n\mathrm{w},\mathrm{w}_*,\tau)=1$ *and* $\lim_{n\to\infty}H(\xi^n,\mathrm{w},\mathrm{w}_*,\tau)=0$, *for all* $\mathrm{w}\in\Xi$ *and* $\tau>0$.

Proof. *Follows from Theorem 2.1 that* w_n *is a O-Cauchy O-sequence and so it O-converges to* $\mathrm{w}_*\in\Xi$. *Hence* $\mathrm{w}_*\perp\mathrm{w}_n$ *for all* $n\in\mathbb{N}$ *from (7), we have*

$$G(\xi\mathrm{w}_*,\mathrm{w}_{n+1},\tau)=G(\xi\mathrm{w}_*,\xi\mathrm{w}_n,\tau)\geq G(\xi\mathrm{w}_*,\xi\mathrm{w}_n,\tau\varrho)\geq G(\mathrm{w}_*,\mathrm{w}_n,\tau)$$

and

$$\lim_{n\to\infty}G(\xi\mathrm{w}_*,\mathrm{w}_{n+1},\tau)=1.$$

Then, we can write

$$G(\mathrm{w}_*,\xi\mathrm{w}_*,\tau)\geq G\left(\mathrm{w}_*,\xi\mathrm{w}_{n+1},\frac{\tau}{2\theta}\right)*G\left(\mathrm{w}_{n+1},\xi\mathrm{w}_*,\frac{\tau}{2\theta}\right)$$

Taking limit as $n\to+\infty$, *We get* $G(\mathrm{w}_*,\xi\mathrm{w}_*,\tau)=1*1=1$
and from (8)

$$H(\xi\mathrm{w}_*,\mathrm{w}_{n+1},\tau)=H(\xi\mathrm{w}_*,\xi\mathrm{w}_n,\tau)\leq H(\xi\mathrm{w}_*,\xi\mathrm{w}_n,\tau\varrho)\leq H(\mathrm{w}_*,\mathrm{w}_n,\tau)$$

and

$$\lim_{n\to\infty}H(\xi\mathrm{w}_*,\mathrm{w}_{n+1},\tau)=0.$$

Then, we can write

$$H(\mathrm{w}_*,\xi\mathrm{w}_*,\tau)\leq H\left(\mathrm{w}_*,\xi\mathrm{w}_{n+1},\frac{\tau}{2\theta}\right)\Delta H\left(\mathrm{w}_{n+1},\xi\mathrm{w}_*,\frac{\tau}{2\theta}\right)$$

Taking limit as $n\to+\infty$, *We get* $H(\mathrm{w}_*,\xi\mathrm{w}_*,\tau)=0\Delta 0=0$, *So* $\xi\mathrm{w}_*=\mathrm{w}_*$. *Next follows from Theorem 3.13.*
□

Example 3.16. Let $\Xi = [-2, 2]$. We define $\perp$ by

$$\mathrm{w} \perp \mathrm{k} \Leftrightarrow \mathrm{w} + \mathrm{k} \in \{\mid \mathrm{w} \mid, \mid \mathrm{k} \mid \tag{15}$$

$$G(\mathrm{w}, \mathrm{k}, \tau) = \begin{cases} 1 \text{ if } \mathrm{w} = \mathrm{k}, \\ \left[e^{\frac{\max\{\mathrm{w},\mathrm{k}\}^{\alpha}}{\tau}} \right]^{-1} \text{ otherwise.} \end{cases} \tag{16}$$

and

$$H(\mathrm{w}, \mathrm{k}, \tau) = \begin{cases} 0 \text{ if } \mathrm{w} = \mathrm{k}, \\ 1 - \left[e^{\frac{\max\{\mathrm{w},\mathrm{k}\}^{\alpha}}{\tau}} \right]^{-1} \text{ otherwise.} \end{cases} \tag{17}$$

for all $\mathrm{w}, \mathrm{k} \in \Xi, \tau > 0$ with $\sigma \times \theta = \sigma \cdot \theta$ and $\sigma \Delta \theta = \max\{\sigma, \theta\}$. Then $(\Xi, G, *, \Delta, \perp)$ is an O-complete IFBMS. Define $\xi : \Xi \to \Xi$ by

$$\xi(\mathrm{w}) = \begin{cases} \frac{\mathrm{w}}{4}, \text{ if } \mathrm{w} \in [-2, 0] \\ 0, \text{ if } \mathrm{w} \in (0, 2]. \end{cases} \tag{18}$$

Then the below cases fulfilled:

1. if $\mathrm{w} \in [-2, 0]$ and $\mathrm{k} \in (0, 2]$, then $\xi(\mathrm{w}) = \frac{\mathrm{w}}{4}$ and $\xi(\mathrm{k}) = 0$,
2. if $\mathrm{w}, \mathrm{k} \in [-2, 0]$, then $\xi(\mathrm{w}) = \frac{\mathrm{w}}{4}$ and $\xi(\mathrm{k}) = \frac{\mathrm{k}}{4}$,
3. if $\mathrm{w}, \mathrm{k} \in (0, 2]$, then $\xi(\mathrm{w}) = 0$ and $\xi(\mathrm{k}) = 0$,
4. if $\mathrm{w} \in (0, 2]$ and $\mathrm{k} \in [-2, 0]$, then $\xi(\mathrm{w}) = 0$ and $\xi(\mathrm{k}) = \frac{\mathrm{k}}{4}$,

This is easy to see that $\xi((w)) + \xi(\mathrm{k}) \in \{\mid \xi(\mathrm{w}) \mid, \mid \xi(\mathrm{k}) \mid\}$. Hence, ξ is $\perp$-preserving. Let $\{\mathrm{w}_n\}$ be an arbitrary O-sequence in Ξ that $\{\mathrm{w}_n\}$ O-converges to $\mathrm{w} \in \Xi$. That is

$$\lim_{n\to\infty} G(\mathrm{w}_n, \mathrm{w}, \tau) = \lim_{n\to\infty} \left[e^{\frac{\max\{\mathrm{w}_n,\mathrm{w}\}^{\alpha}}{\tau}} \right]^{-1} = 1,$$

$$\lim_{n\to\infty} H(\mathrm{w}_n, \mathrm{w}, \tau) = 1 - \lim_{n\to\infty} \left[e^{\frac{\max\{\mathrm{w}_n,\mathrm{w}\}^{\alpha}}{\tau}} \right]^{-1} = 0.$$

We can easily see that if $\lim_{n\to\infty} G(\mathrm{w}_n, \mathrm{w}, \tau) = 1$, then $\lim_{n\to\infty} G(\xi\mathrm{w}_n, \xi\mathrm{w}, \tau) = 1$, and if $\lim_{n\to\infty} H(\mathrm{w}_n, \mathrm{w}, \tau) = 0$, then $\lim_{n\to\infty} H(\xi\mathrm{w}_n, \xi\mathrm{w}, \tau) = 0$, for all $\mathrm{w} \in \Xi$ and $\tau > 0$. That is, ξ is $\perp$-continuous. if $\mathrm{w} = \mathrm{k}$, then it is obvious. Suppose $\mathrm{w} \neq \mathrm{k}$, then there are following four cases for $\varrho \in [\frac{1}{2}, 1)$:
Case 1) if $\mathrm{w} \in [-2, 0]$ and $\mathrm{k} \in (0, 2]$, then $\xi\mathrm{w} = \frac{\mathrm{w}}{4}$ and $\xi\mathrm{k} = 0$. Here,

$$G(\xi\mathrm{w}_n, \xi\mathrm{w}, \varrho\tau) = G(\frac{\mathrm{w}}{4}, 0, \varrho\tau) = \left[e^{\frac{[\frac{\mathrm{w}}{4}]^{\alpha}}{\varrho\tau}} \right]^{-1} \geq \left[e^{\frac{\max\{\mathrm{w},\mathrm{k}\}^{\alpha}}{\tau}} \right]^{-1} = G(\mathrm{w}, \mathrm{k}, \tau),$$

$$H(\xi\mathrm{w}_n, \xi\mathrm{w}, \varrho\tau) = H(\frac{\mathrm{w}}{4}, 0, \varrho\tau) = 1 - \left[e^{\frac{[\frac{\mathrm{w}}{4}]^{\alpha}}{\varrho\tau}} \right]^{-1} \leq 1 - \left[e^{\frac{\max\{\mathrm{w},\mathrm{k}\}^{\alpha}}{\tau}} \right]^{-1} = H(\mathrm{w}, \mathrm{k}, \tau),$$

Case 2) If $\mathrm{w}, \mathrm{k} \in [-2, 0)$, then $\xi\mathrm{w} = \frac{\mathrm{w}}{4}$ and $\xi\mathrm{k} = \frac{\mathrm{k}}{4}$. We have

$$G(\xi\mathrm{w}_n, \xi\mathrm{w}, \varrho\tau) = G(\frac{\mathrm{w}}{4}, \frac{\mathrm{k}}{4}, \varrho\tau) = \left[e^{\frac{\max\left\{\frac{\mathrm{w}}{4},\frac{\mathrm{k}}{4}\right\}^{\alpha}}{\varrho\tau}} \right]^{-1} \geq \left[e^{\frac{\max\{\mathrm{w},\mathrm{k}\}^{\alpha}}{\tau}} \right]^{-1} = G(\mathrm{w}, \mathrm{k}, \tau),$$

$$H(\xi \mathrm{w}_n, \xi \mathrm{w}, \varrho\tau) = H(\frac{\mathrm{w}}{4}, \frac{\mathrm{k}}{4}, \varrho\tau) = 1 - \left[e^{\frac{\max\left\{\frac{\mathrm{w}}{4}, \frac{\mathrm{k}}{4}\right\}^{\alpha}}{\varrho\tau}}\right]^{-1} \leq 1 - \left[e^{\frac{\max\{\mathrm{w},\mathrm{k}\}^{\alpha}}{\tau}}\right]^{-1} = H(\mathrm{w}, \mathrm{k}, \tau),$$

Case 3) If $\mathrm{w}, \mathrm{k} \in (0, 2]$, then $\xi\mathrm{w} = 0$ and $\xi\mathrm{k} = 0$. Here,

$$G(\xi\mathrm{w}, \xi\mathrm{k}, \varrho\tau) = G(0, 0, \varrho\tau) = e^0 \geq \left[e^{\frac{\max\{\mathrm{w},\mathrm{k}\}^{\alpha}}{\tau}}\right]^{-1} = G(\mathrm{w}, \mathrm{k}, \tau),$$

$$H(\xi\mathrm{w}, \xi\mathrm{k}, \varrho\tau) = H(0, 0, \varrho\tau) = 1 - e^0 \leq 1 - \left[e^{\frac{\max\{\mathrm{w},\mathrm{k}\}^{\alpha}}{\tau}}\right]^{-1} = H(\mathrm{w}, \mathrm{k}, \tau),$$

Case 4) If $\mathrm{w} \in (0, 2]$ and $\mathrm{k} \in [-2, 0]$, then $\xi\mathrm{w} = 0$ and $\xi\mathrm{k} = \frac{\mathrm{k}}{4}$. We have

$$G(\xi\mathrm{w}, \xi\mathrm{k}, \varrho\tau) = G(0, \frac{\mathrm{k}}{4}, \varrho\tau) = \left[e^{\frac{\max\left\{0, \frac{\mathrm{k}}{4}\right\}^{\alpha}}{\varrho\tau}}\right]^{-1} \geq \left[e^{\frac{\max\{\mathrm{w},\mathrm{k}\}^{\alpha}}{\tau}}\right]^{-1} = G(\mathrm{w}, \mathrm{k}, \tau),$$

$$H(\xi\mathrm{w}, \xi\mathrm{k}, \varrho\tau) = H(0, \frac{\mathrm{k}}{4}, \varrho\tau) = 1 - \left[e^{\frac{\max\left\{0, \frac{\mathrm{k}}{4}\right\}^{\alpha}}{\varrho\tau}}\right]^{-1} \leq 1 - \left[e^{\frac{\max\{\mathrm{w},\mathrm{k}\}^{\alpha}}{\tau}}\right]^{-1} = H(\mathrm{w}, \mathrm{k}, \tau),$$

From all the above cases, We obtain that

$$G(\xi\mathrm{w}, \xi\mathrm{k}, \varrho\tau) \geq G(\mathrm{w}, \mathrm{k}, \tau), \tag{19}$$

$$G(\xi\mathrm{w}, \xi\mathrm{k}, \varrho\tau) \geq G(\mathrm{w}, \mathrm{k}, \tau), \tag{20}$$

Hence, ξ is an orthogonal contraction. But, ξ is not a contraction. In fact, let $\mathrm{w} = -1$ and $\mathrm{k} = -2$ and $\alpha = 3$, then

$$G(\xi\mathrm{w}, \xi\mathrm{k}, \varrho\tau) = \left[e^{\frac{\max\left\{\frac{\mathrm{w}}{4}, \frac{\mathrm{k}}{4}\right\}^{\alpha}}{\varrho\tau}}\right]^{-1} \geq 1,$$

$$H(\xi\mathrm{w}, \xi\mathrm{k}, \varrho\tau) = 1 - \left[e^{\frac{\max\left\{\frac{\mathrm{w}}{4}, \frac{\mathrm{k}}{4}\right\}^{\alpha}}{\varrho\tau}}\right]^{-1} \leq 0.$$

Which is not true. Hence, all assumptions of Theorem 3.13 are fulfilled and 0 is the unique FP of ξ. Also,

$$G(\mathrm{w}, \mathrm{w}, \tau) = G(0, 0, \tau) = e^0 = 1, \forall\tau > 0 \tag{21}$$

and

$$H(\mathrm{w}, \mathrm{w}, \tau) = H(0, 0, \tau) = 1 - e^0 = 0. \forall\tau > 0 \tag{22}$$

Theorem 3.17. *Suppose* $(\Xi, G, H, *, \Delta, \perp)$ *be an O-complete IFBMS such that* $\lim_{t\to\infty} G(\mathrm{w}, \mathrm{k}, \tau) = 1$, *and* $\lim_{t\to\infty} H(\mathrm{w}, \mathrm{k}, \tau) = 0, \forall \mathrm{w}, \mathrm{k} \in \Xi$ *and* $\tau > 0$. *Suppose* $\xi : \Xi \to \Xi$ *be* $\perp$*-continuous,* $\perp$*-contraction, and* $\perp$*-preserving. Suppose* $\varrho \in \left(0, \frac{1}{\theta}\right)$ *and* $\tau > 0$, *such that*

$$G(\xi\mathrm{w}, \xi\mathrm{k}, \varrho\tau) \geq \min\{G(\xi\mathrm{w}, \mathrm{w}, \tau), G(\xi\mathrm{k}, \mathrm{k}, \tau)\} \tag{23}$$

$$H(\xi\mathrm{w}, \xi\mathrm{k}, \varrho\tau) \leq \min\{H(\xi\mathrm{w}, \mathrm{w}, \tau), H(\xi\mathrm{k}, \mathrm{k}, \tau)\} \tag{24}$$

for all $\mathrm{w}, \mathrm{k} \in \Xi, \tau > 0$. *Then* ξ *has a unique FP, say* $\mathrm{w}_* \in \Xi$. *Moreover,* $\lim_{n\to\infty} G(\xi^n\mathrm{w}, \mathrm{w}_*, \tau) = 1$ *and* $\lim_{n\to\infty} H(\xi^n\mathrm{w}, \mathrm{w}_*, \tau) = 0$ *for all* $\mathrm{w} \in \Xi$ *and* $\tau > 0$.

Proof. *Let* $(\Xi, G, H, *, \Delta, \perp)$ *be an O-complete IFBMS, There exists* $\mathrm{w}_0 \in \Xi$ *such that*

$$\mathrm{w}_0 \perp \mathrm{k} \forall \mathrm{k} \in \Xi \tag{25}$$

Therefore, ξ is $\perp$-preserving, and $\{\mathrm{w}_n\}$ is an O-sequence. We have

$$G(\mathrm{w}_{n+1}, \mathrm{n}, \tau) \geq G(\mathrm{w}_{n+1}, \mathrm{n}, \varrho\tau) = G(\xi\mathrm{w}_n, \xi\mathrm{w}_{n-1}, \varrho\tau) \geq \min\{G(\xi\mathrm{w}_n, \mathrm{n}, \tau), G(\xi\mathrm{w}_{n-1}, \mathrm{w}_{n-1}, \tau)$$

$$H(\mathrm{w}_{n+1}, \mathrm{n}, \tau) \leq H(\mathrm{w}_{n+1}, \mathrm{n}, \varrho\tau) = H(\xi\mathrm{w}_n, \xi\mathrm{w}_{n-1}, \varrho\tau) \leq \min\{H(\xi\mathrm{w}_n, \mathrm{n}, \tau), H(\xi\mathrm{w}_{n-1}, \mathrm{w}_{n-1}, \tau)$$

Two cases arise.
Case 1: If $G(\mathrm{w}_{n+1}, \mathrm{n}, \tau) \geq G(\xi\mathrm{w}_n, \mathrm{w}_n, \tau)$, then

$$G(\mathrm{w}_{n+1}, \mathrm{w}_n, \tau) \geq G(\mathrm{w}_{n+1}, \mathrm{w}_n, \varrho\tau) \geq G(\xi\mathrm{w}_n, \mathrm{w}_n, \tau) = G(\mathrm{w}_{n+1}, \mathrm{w}_n, \tau)$$

and

$$H(\mathrm{w}_{n+1}, \mathrm{w}_n, \tau) \leq H(\mathrm{w}_{n+1}, \mathrm{w}_n, \varrho\tau) \leq H(\xi\mathrm{w}_n, \mathrm{w}_n, \tau) = H(\mathrm{w}_{n+1}, \mathrm{w}_n, \tau)$$

Then, by Lemma 3.12, $\mathrm{w}_n = \mathrm{w}_{n+1}$ for all $n \in \mathbb{N}$
Case 2): If $G(\mathrm{w}_{n+1}, \mathrm{n}, \tau) \geq G(\xi\mathrm{w}_{n-1}, \mathrm{w}_{n-1}, \tau)$, then

$$G(\mathrm{w}_{n+1}, \mathrm{w}_n, \tau) \geq G(\mathrm{w}_{n+1}, \mathrm{w}_n, \varrho\tau) \geq G(\xi\mathrm{w}_{n-1}, \mathrm{w}_{n-1}, \tau) \geq G(\mathrm{w}_n, \mathrm{w}_{n-1}, \tau)$$

and $H(\mathrm{w}_{n+1}, \mathrm{n}, \tau) \leq H(\xi\mathrm{w}_{n-1}, \mathrm{w}_{n-1}, \tau)$, then

$$H(\mathrm{w}_{n+1}, \mathrm{w}_n, \tau) \leq H(\mathrm{w}_{n+1}, \mathrm{w}_n, \varrho\tau) \leq H(\xi\mathrm{w}_{n-1}, \mathrm{w}_{n-1}, \tau) \leq H(\mathrm{w}_n, \mathrm{w}_{n-1}, \tau)$$

for all $n \in \mathbb{N}$ and $\tau > 0$. By utilizing Theorem 3.13, we have an O-Cauchy sequence. Since $(\Xi, G, H, *, \Delta, \perp)$ is complete, there exists $\mathrm{w}_* \in \Xi$, such that

$$\lim_{n\to\infty} G(\mathrm{w}_n, \mathrm{w}_*, \tau) = 1, \tag{26}$$

and

$$\lim_{n\to\infty} H(\mathrm{w}_n, \mathrm{w}_*, \tau) = 0, \tag{27}$$

for all $\tau > 0$. Science,ξ is an $\perp$-continuous, We have

$$\lim_{n\to\infty} G(\mathrm{w}_{n+1}, \mathrm{w}_*, \tau) = G(\xi\mathrm{w}_n, \xi\mathrm{w}_*, \tau) = 1,$$

and

$$\lim_{n\to\infty} H(\mathrm{w}_{n+1}, \mathrm{w}_*, \tau) = H(\xi\mathrm{w}_n, \xi\mathrm{w}_*, \tau) = 0,$$

Next, we examine that w_* is a FP of ξ. Let $\tau_1 \in (\varrho\theta, 1)$ and $\tau_2 = 1 - \tau_1$. then

$$G(\xi\mathrm{w}_*, \mathrm{w}_*, \tau) \geq G\left(\xi\mathrm{w}_*, \mathrm{w}_{n+1}, \frac{\tau\tau_1}{\theta}\right) * G\left(\xi\mathrm{w}_{n+1}, \mathrm{w}_*, \frac{\tau\tau_2}{\theta}\right),$$

$$= G\left(\xi\mathrm{w}_*, \xi\mathrm{w}_n, \frac{\tau\tau_1}{\theta}\right) * G\left(\xi\mathrm{w}_{n+1}, \mathrm{w}_*, \frac{\tau\tau_2}{\theta}\right),$$

$$\geq \min\left\{G\left(\xi\mathrm{w}_*, \mathrm{w}_*, \frac{\tau\tau_1}{\varrho\theta}\right) * G\left(\xi\mathrm{w}_n, \mathrm{w}_n, \frac{\tau\tau_2}{\varrho\theta}\right)\right\} * G\left(\xi\mathrm{w}_{n+1}, \mathrm{w}_*, \frac{\tau\tau_2}{\theta}\right)$$

$$= \min\left\{G\left(\xi\mathrm{w}_*, \mathrm{w}_*, \frac{\tau\tau_1}{\varrho\theta}\right) * G\left(\xi\mathrm{w}_{n+1}, \mathrm{w}_n, \frac{\tau\tau_2}{\varrho\theta}\right)\right\} * G\left(\xi\mathrm{w}_{n+1}, \mathrm{w}_*, \frac{\tau\tau_2}{\theta}\right)$$

Taking $n \to \infty$, We get

$$G(\xi\mathrm{w}_*, \mathrm{w}_*, \tau) \geq \min\left\{G\left(\xi\mathrm{w}_*, \mathrm{w}_*, \frac{\tau\tau_1}{\varrho\theta}\right), 1\right\} * 1,$$

$$G(\xi w_*, w_*, \tau) \geq G\left(\xi w_*, w_*, \frac{\tau}{\nu}\right) \tau > 0,,$$

and

$$H(\xi w_*, w_*, \tau) \leq H\left(\xi w_*, w_{n+1}, \frac{\tau\tau_1}{\theta}\right) \Delta H\left(\xi w_{n+1}, w_*, \frac{\tau\tau_2}{\theta}\right),$$

$$= H\left(\xi w_*, \xi w_n, \frac{\tau\tau_1}{\theta}\right) \Delta H\left(\xi w_{n+1}, w_*, \frac{\tau\tau_2}{\theta}\right),$$

$$\leq \min\left\{H\left(\xi w_*, w_*, \frac{\tau\tau_1}{\varrho\theta}\right) \Delta H\left(\xi w_n, w_n, \frac{\tau\tau_2}{\varrho\theta}\right)\right\} \Delta H\left(\xi w_{n+1}, w_*, \frac{\tau\tau_2}{\theta}\right)$$

$$= \min\left\{H\left(\xi w_*, w_*, \frac{\tau\tau_1}{\varrho\theta}\right) \Delta H\left(\xi w_{n+1}, w_n, \frac{\tau\tau_2}{\varrho\theta}\right)\right\} \Delta H\left(\xi w_{n+1}, w_*, \frac{\tau\tau_2}{\theta}\right)$$

Taking $n \to \infty$, *We get*

$$H(\xi w_*, w_*, \tau) \leq \min\left\{H\left(\xi w_*, w_*, \frac{\tau\tau_1}{\varrho\theta}\right), 0\right\} * 0,$$

$$H(\xi w_*, w_*, \tau) \leq H\left(\xi w_*, w_*, \frac{\tau}{\nu}\right) \tau > 0,,$$

There is $\nu = \frac{\theta\varrho}{\tau_1} \in (0,1)$, *and by utilizing Lemma 3.12, we get* $\xi w_* = w_*$.
Uniqueness : *Suppose* $w_* \neq k_*$ *are two FPs of* ξ. *We get* $w_0 \perp w_*$ *and* $w_0 \perp k_*$. *Therefore, since* ξ *is an* $\perp$*-preserving, we have* $\xi^n w_0 \perp \xi^n w_*$ *and* $\xi^n w_0 \xi^n \perp k_*$. *for all* $n \in \mathbb{N}$. *we can write*

$$G(\xi^n w_0, \xi^n w_*, \tau) \geq G(\xi^n w_0, \xi^n w_*, \varrho\tau) \geq \min\{G(\xi^n w_0, w_0, \tau), G(\xi^n w_*, w_*, \tau)\},$$

and

$$G(\xi^n w_0, \xi^n k_*, \tau) \geq G(\xi^n w_0, \xi^n k_*, \varrho\tau) \geq \min\{G(\xi^n w_0, w_0, \tau), G(\xi^n k_*, k_*, \tau)\},$$

Hence, we write that

$$G(w_0, k_*, \tau) = G(\xi^n w_*, \xi^n k_*, \tau) \geq \min\left\{G\left(\xi^n w_*, w_*, \frac{\tau}{\varrho}\right), G\left(\xi^n k_*, k_*, \frac{\tau}{\varrho}\right)\right\},$$

and

$$H(\xi^n w_0, \xi^n w_*, \tau) \leq G(\xi^n w_0, \xi^n w_*, \varrho\tau) \leq \min\{H(\xi^n w_0, w_0, \tau), H(\xi^n w_*, w_*, \tau)\},$$

and

$$H(\xi^n w_0, \xi^n k_*, \tau) \leq H(\xi^n w_0, \xi^n k_*, \varrho\tau) \leq \min\{H(\xi^n w_0, w_0, \tau), H(\xi^n k_*, k_*, \tau)\},$$

Hence, we write that

$$H(w_0, k_*, \tau) = H(\xi^n w_*, \xi^n k_*, \tau) \leq \min\left\{H\left(\xi^n w_*, w_*, \frac{\tau}{\varrho}\right), H\left(\xi^n k_*, k_*, \frac{\tau}{\varrho}\right)\right\},$$

for all $\tau > 0$. *Thus,* $w_* = k_*$. □

Corollary 3.18. *Suppose* $(\Xi, G, H, *, \Delta, \perp)$ *be a complete OIFBMS and* $\xi : \Xi \to \Xi$ *be an* $\perp$*-continuous and* $\perp$*-preserving. Let* $\varrho \in \left(0, \frac{1}{\theta}\right)$ *for all* $\tau > 0$*,with*

$$G(\xi w, \xi k, \varrho\tau) \geq \min\{G(\xi w, w, \tau), G(\xi k, k, \tau),$$

$$H(\xi w, \xi k, \varrho\tau) \leq \min\{H(\xi w, w, \tau), H(\xi k, k, \tau).$$

Then, ξ *has a unique FP. Furthermore,* $\lim_{n\to\infty} G(\xi^n w, w_*, \tau) = 1$ *and* $\lim_{n\to\infty} H(\xi^n w, w_*, \tau) = 0$, *for all* $w \in \Xi$ *and* $\tau > 0$.
Proof. *It is obvious from Theorem 3.14 and 3.17* □

Example 3.19. Suppose $\Xi = [-2, 2]$ and by $\perp$ by $\mathrm{w} \perp \mathrm{k} \Leftrightarrow \mathrm{w} + \mathrm{k} \geq 0$. Define G and H by

$$G(\mathrm{w}, \mathrm{k}, \tau) = \begin{cases} 1 \text{ ifw} = \mathrm{k}, \\ \frac{\tau}{\tau + \max\{\mathrm{w}, \mathrm{k}\}^\alpha} \text{ otherwise} \end{cases} \tag{28}$$

$$H(\mathrm{w}, \mathrm{k}, \tau) = \begin{cases} 0 \text{ ifw} = \mathrm{k}, \\ \frac{\max\{\mathrm{w}, \mathrm{k}\}^\alpha}{\tau + \max\{\mathrm{w}, \mathrm{k}\}^\alpha} \text{ otherwise} \end{cases} \tag{29}$$

for all $\mathrm{w}, \mathrm{k} \in \Xi$ and $\tau > 0$, with $\sigma * \theta = \sigma \cdot \theta$ and $\sigma \Delta \theta = \max\{\sigma, \theta\}$, Then $(\Xi, G, H, *, \Delta, \perp)$ is an O-complete IFBMS. Note that $\lim_{n\to\infty} G\mathrm{w}, \mathrm{k}, \tau = 1$ and $\lim_{n\to\infty} H\mathrm{w}, \mathrm{k}, \tau = 0$. Define $\xi : \Xi \to \Xi$ by

$$\xi(\mathrm{w}) = \begin{cases} \frac{\mathrm{w}}{4}, \ \mathrm{w} \in \left[-2, \frac{2}{3}\right], \\ 1 - \mathrm{w}, \ \mathrm{w} \in \left(\frac{2}{3}, 1\right], \\ \mathrm{w} - \frac{1}{2}, \ \mathrm{w} \in (1, 2]. \end{cases} \tag{30}$$

There are following four cases:

1. If $\mathrm{w}, \mathrm{k} \in \left[-2, \frac{2}{3}\right]$ then $\xi(\mathrm{w}) = \frac{\mathrm{w}}{4}$ and $\xi(\mathrm{k}) = \frac{\mathrm{k}}{4}$.
2. If $\mathrm{w}, \mathrm{k} \in \left(\frac{2}{3}, 1\right]$ then $\xi(\mathrm{w}) = 1 - \mathrm{w}$ and $\xi(\mathrm{k}) = 1 - \mathrm{k}$.
3. If $\mathrm{w}, \mathrm{k} \in (1, 2]$ then $\xi(\mathrm{w}) = \mathrm{w} - \frac{1}{2}$ and $\xi(\mathrm{k}) = \mathrm{k} - \frac{1}{2}$.
4. If $\mathrm{w}, \in \left[-2, \frac{2}{3}\right]$ and $\mathrm{k} \in \left(\frac{2}{3}, 1\right]$ then $\xi(\mathrm{w}) = \frac{\mathrm{w}}{4}$ and $\xi(\mathrm{k}) = \mathrm{k} - \frac{1}{2}$.
5. If $\mathrm{w}, \in \left[-2, \frac{2}{3}\right]$ and $\mathrm{k} \in (1, 2]$ then $\xi(\mathrm{w}) = \frac{\mathrm{w}}{4}$ and $\xi(\mathrm{k}) = \mathrm{k} - \frac{1}{2}$.
6. If $\mathrm{w}, \in \left(\frac{2}{3}, 1\right]$ and $\mathrm{k} \in \left(\frac{2}{3}, 1\right]$ then $\xi(\mathrm{w}) = 1 - \mathrm{w}$ and $\xi(\mathrm{k}) = \mathrm{k} - \frac{1}{2}$.
7. If $\mathrm{w}, \in (1, 2]$ and $\mathrm{k} \in \left(\frac{2}{3}, 1\right]$ then $\xi(\mathrm{w}) = \mathrm{w} - \frac{1}{2}$ and $\xi(\mathrm{k}) = 1 - \mathrm{k}$.
8. If $\mathrm{w} \in (1, 2]$ and $\mathrm{k} \in \left[-2, \frac{2}{3}\right]$ then $\xi(\mathrm{w}) = \mathrm{w} - \frac{1}{2}$ and $\xi(\mathrm{k}) = \frac{\mathrm{k}}{4}$.
9. If $\mathrm{w} \in \left(\frac{2}{3}, 1\right]$ and $\mathrm{k} \in \left[-2, \frac{2}{3}\right]$ then $\xi(\mathrm{w}) = 1 - \mathrm{w}$ and $\xi(\mathrm{k}) = \frac{\mathrm{k}}{4}$.

Because $\mathrm{w} \perp \mathrm{k} \Leftrightarrow \mathrm{w} + \mathrm{k} \geq 0$, it is clearly implies that $\xi\mathrm{w} + \mathrm{k} \geq 0$. that is, ξ is $\perp$-preserving. Suppose $\{\mathrm{w}_n\}$ be any O-sequence in Ξ that O-converges to $\mathrm{w} \in \Xi$. We get

$$\lim_{n\to\infty} G(\mathrm{w}_n, \mathrm{w}, \tau) = \lim_{n\to\infty} \frac{\tau}{\tau + \max\{\mathrm{w}_n, \mathrm{w}\}^3} = 1, \tag{31}$$

$$\lim_{n\to\infty} H(\mathrm{w}_n, \mathrm{w}, \tau) = \lim_{n\to\infty} \frac{\max\{\mathrm{w}_n, \mathrm{w}\}^3}{\tau + \max\{\mathrm{w}_n, \mathrm{w}\}^3} = 0, \tag{32}$$

Note that if $G(\mathrm{w}_n, \mathrm{w}, \tau) = 1$ and $H(\mathrm{w}_n, \mathrm{w}, \tau) = 0$, then $G(\xi\mathrm{w}_n, \xi\mathrm{w}, \tau) = 1$ and $H(\xi\mathrm{w}_n, \xi\mathrm{w}, \tau) = 0$ for all $\tau > 0$. that is, ξ is orthogonal continuous. For $\mathrm{w} = \mathrm{k}$, it is obvious. Assume $\mathrm{w} \neq \mathrm{k}$. We get

$$G(\xi\mathrm{w}, \xi\mathrm{k}, \varrho\tau) \geq \min\{G(\xi\mathrm{w}, \mathrm{w}, \tau), G(\xi\mathrm{k}, \mathrm{k}, \tau)\}$$

$$H(\xi\mathrm{w}, \xi\mathrm{k}, \varrho\tau) \leq \min\{H(\xi\mathrm{w}, \mathrm{w}, \tau), H(\xi\mathrm{k}, \mathrm{k}, \tau)\}.$$

It fulfilled above all cases. Now, we show that ξ is not a contraction. Suppose

$$\min\{G(\xi\mathrm{w}, \mathrm{w}, \tau), G(\xi\mathrm{k}, \mathrm{k}, \tau)\} = G(\xi\mathrm{w}, \mathrm{w}, \tau)$$

$$\min\{H(\xi\text{w},\text{w},\tau), H(\xi\text{k},\text{k},\tau)\} = H(\xi\text{w},\text{k},\tau).$$

then for $\text{w} = -1$ and $\text{k} = -2$, we have

$$G(\xi\text{w},\xi\text{k},\varrho\tau) = \frac{\varrho\tau}{\varrho\tau + \max\left\{\frac{\text{w}}{4},\frac{\text{k}}{4}\right\}^3} = \frac{64\varrho\tau}{64\varrho\tau - 1} \geq 1,$$

$$H(\xi\text{w},\xi\text{k},\varrho\tau) = \frac{\max\left\{\frac{\text{w}}{4},\frac{\text{k}}{4}\right\}^3}{\varrho\tau + \max\left\{\frac{\text{w}}{4},\frac{\text{k}}{4}\right\}^3} = \frac{-1}{64\varrho\tau - 1} \leq 0.$$

Which is not true. That is, all assumptions of Theorem 2.2 are fulfilled, and 0 is a unique FP of ξ.

Definition 3.20. Suppose $(\Xi, G, H, *, \Delta, \perp)$ be an OIFBMS. A mapping $\xi : \Xi \to \Xi$ is called a fuzzy $\theta-\perp$-contraction if their exists $\varrho \in (0,1)$ such that

$$\frac{1}{G(\xi\text{w},\xi\text{k},\tau)} - 1 \leq \varrho\left[\frac{1}{G(\text{w},\text{k},\tau)} - 1\right] \tag{33}$$

$$H(\xi\text{w},\xi\text{k},\tau) \leq \varrho H(\text{w},\text{k},\tau) \tag{34}$$

for all $\text{w},\text{k} \in \Xi$ and $\tau > 0$. Where ϱ is said to be an IFB-$\perp$-contractive constant of ξ.

Theorem 3.21. *Suppose* $(\Xi, G, H, *, \Delta, \perp)$ *be an OIFBMS. Such that*

$$\lim_{\tau\to\infty} G(\text{w},\text{k},\tau) = 1, \tag{35}$$

$$\lim_{\tau\to\infty} H(\text{w},\text{k},\tau) = 0, \forall \text{w},\text{k} \in \Xi. \tag{36}$$

Assume a mapping $\xi : \Xi \to \Xi$ *be a* $\perp$*-continuous, IFB-*$\perp$*-contraction and* $\perp$*-preserving mapping. Thus,* ξ *has a FP, call* $\nu \in \Xi$*. Moreover,* $G(\nu,\nu,\alpha) = 1$ *and* $H(\nu,\nu,\alpha) = 0$ *for all* $\alpha > 0$*.*
Proof. *Suppose* $(\Xi, G, H, *, \Delta, \perp)$ *be an O-complete IFBMS. For any point* $\text{w}_0 \in \Xi$, $\text{w}_0 \perp \text{k}$, *for all* $\text{k} \in \Xi$*. That is,* $\text{w}_0 \perp \xi\text{w}_0$*. Consider* $\text{w}_n = \xi^n\text{w}_0 = \xi\text{w}_{n-1}$ *for all* $n \in \mathbb{N}$*. Therefore,* ξ *is* $\perp$*-preserving and* $\{\text{w}_n\}$ *is an O-sequence. If* $\text{w}_n = \text{w}_{n-1}$ *for some* $n \in \mathbb{N}$ *then* w_n *is a FP of* ξ*. We suppose that* $\text{w}_n \neq \text{w}_{n-1}$ *for all* $n \in \mathbb{N}$*. For all* $\tau > 0$*,* $n \in \mathbb{N}$ *and utilizing (9), we have*

$$\frac{1}{G(\text{w}_n,\text{w}_{n+1},\tau)} - 1 = \frac{1}{G(\xi\text{w}_{n-1},\xi\text{w}_n,\tau)} - 1 \leq \varrho\left[\frac{1}{G(\text{w}_{n-1},\text{w},\tau)} - 1\right]$$

$$H(\text{w}_n,\text{w}_{n+1},\tau) = H(\xi\text{w}_{n-1},\xi\text{w}_n,\tau) \leq \varrho H(\text{w}_{n-1},\text{w}_n,\tau).$$

We have

$$\frac{1}{G(\text{w}_n,\text{w}_{n+1},\tau)} - 1 = \frac{\varrho}{G(\text{w}_{n-1},\text{w}_n,\tau)} + (1-\varrho), \forall\tau > 0$$

$$\frac{\varrho}{G(\xi\text{w}_{n-2},\xi\text{w}_{n-1},\tau)} + (1-\varrho) \leq \frac{\varrho^2}{G(\text{w}_{n-2},\text{w}_{n-1},\tau)} + \varrho(1-\varrho) + (1-\varrho).$$

Continuing in this way, we get

$$\frac{\varrho}{G(\text{w}_n,\text{w}_{n+1},\tau)} \leq \frac{\varrho^n}{G(\text{w}_0,\text{w}_1,\tau)} + \varrho^{n-1}(1-\varrho) + \varrho^{n-2}(1-\varrho) + \cdots + \varrho(1-\varrho) + (1-\varrho).$$

$$\leq \frac{\varrho^n}{G(\text{w}_0,\text{w}_1,\tau)} + (\varrho^{n-1} + \varrho^{n-2} + \cdots 1)(1-\varrho)$$

$$\leq \frac{\varrho^n}{G(\mathrm{w}_0, \mathrm{w}_1, \tau)} + (1 - \varrho^n)$$

We have

$$\frac{1}{\frac{\varrho^n}{G(\mathrm{w}_0,\mathrm{w}_1,\tau)} + (1 - \varrho^n)} \leq G(\mathrm{w}_n, \mathrm{w}_{n+1}, \tau), \forall \tau > 0, n \in \mathbb{N} \tag{37}$$

and

$$H(\mathrm{w}_n, \mathrm{w}_{n+1}, \tau) = H(\xi\mathrm{w}_{n-1}, \xi\mathrm{w}_n, \tau) \leq \varrho H(\mathrm{w}_{n-1}, \mathrm{w}_n, \tau) = \varrho H(\xi\mathrm{w}_{n-2}, \xi\mathrm{w}_{n-1}, \tau)$$

$$\leq \varrho^2 H(\mathrm{w}_{n-2}, \mathrm{w}_{n-1}, \tau) \leq \cdots \leq \varrho^n H(\mathrm{w}_0, \mathrm{w}_1, \tau) \forall \tau > 0, n \in \mathbb{N} \tag{38}$$

Now, for $m \geq 1$ and $n \in \mathbb{N}$, we have

$$G(\mathrm{w}_n, \mathrm{w}_{n+m}, \tau) \geq G\left(\mathrm{w}_n, \mathrm{w}_{n+1}, \frac{\tau}{\theta}\right) * G\left(\mathrm{w}_{n+1}, \mathrm{w}_{n+m}, \frac{\tau}{\theta}\right)$$

$$\geq G\left(\mathrm{w}_n, \mathrm{w}_{n+1}, \frac{\tau}{\theta}\right) * G\left(\mathrm{w}_{n+1}, \mathrm{w}_{n+2}, \frac{\tau}{\theta^2}\right) * G\left(\mathrm{w}_{n+2}, \mathrm{w}_{n+m}, \frac{\tau}{\theta^2}\right)$$

Again, continuing in this way, we get

$$G(\mathrm{w}_n, \mathrm{w}_{n+m}, \tau) \geq G\left(\mathrm{w}_n, \mathrm{w}_{n+1}, \frac{\tau}{\theta}\right) * G\left(\mathrm{w}_{n+1}, \mathrm{w}_{n+2}, \frac{\tau}{\theta^2}\right) * \cdots * G\left(\mathrm{w}_{n+m-1}, \mathrm{w}_{n+m}, \frac{\tau}{\theta^{m-1}}\right)$$

and

$$H(\mathrm{w}_n, \mathrm{w}_{n+m}, \tau) \leq H\left(\mathrm{w}_n, \mathrm{w}_{n+1}, \frac{\tau}{\theta}\right) \Delta H\left(\mathrm{w}_{n+1}, \mathrm{w}_{n+m}, \frac{\tau}{\theta}\right)$$

$$\leq H\left(\mathrm{w}_n, \mathrm{w}_{n+1}, \frac{\tau}{\theta}\right) \Delta H\left(\mathrm{w}_{n+1}, \mathrm{w}_{n+2}, \frac{\tau}{\theta^2}\right) \Delta H\left(\mathrm{w}_{n+2}, \mathrm{w}_{n+m}, \frac{\tau}{\theta^2}\right)$$

Continuing in this way, we get

$$H(\mathrm{w}_n, \mathrm{w}_{n+m}, \tau) \leq H\left(\mathrm{w}_n, \mathrm{w}_{n+1}, \frac{\tau}{\theta}\right) \Delta H\left(\mathrm{w}_{n+1}, \mathrm{w}_{n+2}, \frac{\tau}{\theta^2}\right) \Delta \cdots \Delta H\left(\mathrm{w}_{n+m-1}, \mathrm{w}_{n+m}, \frac{\tau}{\theta^{m-1}}\right)$$

By utilizing (37) in the above inequality, we get

$$G(\mathrm{w}_n, \mathrm{w}_{n+m}, \tau) \geq \frac{1}{\frac{\varrho^n}{G\left(\mathrm{w}_0,\mathrm{w}_1,\frac{\tau}{\theta}\right)} + (1 - \varrho^n)} * \frac{1}{\frac{\varrho^{n+1}}{G\left(\mathrm{w}_0,\mathrm{w}_1,\frac{\tau}{\theta^2}\right)} + (1 - \varrho^n)} * \cdots$$

$$* \frac{1}{\frac{\varrho^{n+m-1}}{G\left(\mathrm{w}_0,\mathrm{w}_1,\frac{\tau}{\theta^{m-1}}\right)} + (1 - \varrho^{n+m-1})}$$

$$\geq \frac{1}{\frac{\varrho^n}{G\left(\mathrm{w}_0,\mathrm{w}_1,\frac{\tau}{\theta}\right)}+1} * \frac{1}{\frac{\varrho^{n+1}}{G\left(\mathrm{w}_0,\mathrm{w}_1,\frac{\tau}{\theta^2}\right)}+1} * \cdots * \frac{1}{\frac{\varrho^{n+m-1}}{G\left(\mathrm{w}_0,\mathrm{w}_1,\frac{\tau}{\theta^{m-1}}\right)}+1}$$

Also, using (38), we have

$$H(\mathrm{w}_n, \mathrm{w}_{n+p}, \tau) \leq H\left(\mathrm{w}_n, \mathrm{w}_{n+1}, \frac{\tau}{\theta}\right) \Delta H\left(\mathrm{w}_{n+1}, \mathrm{w}_{n+2}, \frac{\tau}{\theta^2}\right) \Delta \cdots \Delta H\left(\mathrm{w}_{n+m-1}, \mathrm{w}_{n+m}, \frac{\tau}{\theta^{m-1}}\right)$$

As $\varrho \in (0,1)$, we have $\lim_{n\to\infty} G(\mathrm{w}_n, \mathrm{w}_{n+m}, \tau) = 1$ and $\lim_{n\to\infty} H(\mathrm{w}_n, \mathrm{w}_{n+m}, \tau) = 0$ for all $\tau > 0$, $m \geq 1$. Therefore, a sequence $\{\mathrm{w}\}$ is an O-Cauchy in $(\Xi, G, H, *, \Delta, \perp)$ is complete, and we have ξ is an $\perp$-continuous, there exist $\nu \in \Xi$ such that

$$\lim_{n\to\infty} G(\mathrm{w}_{n+1}, \nu, \tau) = \lim_{n\to\infty} G(\xi\mathrm{w}_n, \xi\nu, \tau) = 1, \forall \tau > 0, \tag{39}$$

$$\lim_{n\to\infty} H(\mathrm{w}_{n+1},\nu,\tau)=\lim_{n\to\infty} H(\xi\mathrm{w}_n,\xi\nu,\tau)=0,\forall\tau>0, \tag{40}$$

Now, we show that ν is a FP of ξ. By utilizing (33), we have

$$\frac{1}{G(\xi\mathrm{w},\xi\nu,\tau)}-1\le\varrho\left[\frac{1}{G(\mathrm{w}_n,\xi\nu,\tau)}-1\right]=\frac{\varrho}{G(\mathrm{w},\xi\nu,\tau)}-\varrho.$$

That is,

$$\frac{1}{G(\xi\mathrm{w},\xi\nu,\tau)+1-\varrho}\le G(\xi\mathrm{w}_n,\xi\nu,\tau).$$

Using the above inequality, we obtain

$$G(\nu,\xi\nu,\tau)\ge G\left(\nu,\mathrm{w}_{n+1},\frac{\tau}{2\theta}\right)*G\left(\mathrm{w}_{n+1},\xi\nu,\frac{\tau}{2\theta}\right)$$

$$=G\left(\nu,\mathrm{w}_{n+1},\frac{\tau}{2\theta}\right)*G\left(\xi\mathrm{w}_n,\xi\nu,\frac{\tau}{2\theta}\right)$$
$$\ge G\left(\nu,\mathrm{w}_{n+1},\frac{\tau}{2\theta}\right)*\frac{\varrho}{G\left(\mathrm{w}_n,\nu,\frac{\tau}{2\theta}\right)+1-\varrho}$$

and

$$H(\mathrm{w},\nu,\tau)=H(\xi\mathrm{w},\xi\nu,\tau)\le\varrho H(\mathrm{w},\nu,\tau)<H(\mathrm{w},\nu,\tau)$$
$$=H\left(\mathrm{w},\mathrm{w}_{n+1},\frac{\tau}{2\theta}\right)\Delta H\left(\xi\mathrm{w}_n,\xi\mathrm{w},\frac{\tau}{2\theta}\right)$$
$$\le H\left(\mathrm{w},\mathrm{w}_{n+1},\frac{\tau}{2\theta}\right)\Delta\varrho H\left(\mathrm{w}_n,\mathrm{w},\frac{\tau}{2\theta}\right)$$

Taking limit as $n\to\infty$ and using (39) and (40) in the above expression, we get $G(\nu,\xi\nu,\tau)=1$, that is, $\xi\nu=\nu$. Therefore, ν is a FP of ξ,and $G(\nu,\nu,\tau)=1$ and $H(\nu,\nu,\tau)=0$ for all $\tau>0$. □

Corollary 3.22. *Suppose $(\Xi,G,H,*,\Delta,\perp)$ be an O-complete IFBMS such that $\lim_{nto\infty}G(\mathrm{w},\mathrm{k},\tau)=1$ and $\lim_{nto\infty}H(\mathrm{w},\mathrm{k},\tau)=0$, for all $\mathrm{w},,\mathrm{k}\in\Xi$ and $\xi:\Xi to\Xi$ satisfy*

$$\frac{1}{G(\xi^n\mathrm{w},\xi^n\mathrm{k},\tau)}-1\le\varrho\left[\frac{1}{G(\mathrm{w},\mathrm{k},\tau)}-1\right] \tag{41}$$
$$H(\xi^n\mathrm{w},\xi^n\mathrm{k},\tau)\le\varrho H(\mathrm{w},\mathrm{k},\tau) \tag{42}$$

for all $n\in\mathbb{N},\mathrm{w},\mathrm{k}\in\Xi,\tau>0$,where $0<\varrho<1$. Then ξ has a FP, say $\nu\in\Xi$ and $G(\nu,\nu,\tau)=1$, for all $\tau>0$.
Proof. *$\nu\in\Xi$ is a unique FP of ξ^n by utilizing Theorem 3.22, and $G(\nu,\nu,\tau)=1$, for all $\tau>0$. $\xi\nu$ is also a FP of $\xi^n(\xi\nu)=\xi\nu$ from Theorem 3.22, $\xi\nu=\nu$. Hence, the FP of ξ is also a FP of ξ^n.* □

Example 3.23. Suppose $\Xi=[-1,2]$ and define $\perp$ by $\mathrm{w}\perp\mathrm{k}\Leftrightarrow\mathrm{w}+\mathrm{k}\ge0$. Define G,H as in Example 3.4 with $\alpha=3$,

$$G(\mathrm{w},\mathrm{k},\tau)=\frac{\tau+\min\{\mathrm{w},\mathrm{k}\}^3}{\tau+\min\{\mathrm{w},\mathrm{k}\}^3}\forall\mathrm{w},\mathrm{k}\in\Xi,\tau>0, \tag{43}$$

and

$$H(\mathrm{w},\mathrm{k},\tau)=1-\frac{\tau+\min\{\mathrm{w},\mathrm{k}\}^3}{\tau+\min\{\mathrm{w},\mathrm{k}\}^3}\forall\mathrm{w},\mathrm{k}\in\Xi,\tau>0, \tag{44}$$

with $\sigma*\theta=\sigma\cdot\theta$ and $\sigma\Delta\theta=\max\{\sigma,\theta\}$, then $(\Xi,G,H,*,\Delta,\perp)$ is an O-complete IFBMS. see that $\lim_{\tau\to\infty}G(\mathrm{w},\mathrm{k},\tau)=1$ and $\lim_{\tau\to\infty}H(\mathrm{w},\mathrm{k},\tau)=0$ for all $\mathrm{w},\mathrm{k}\in\Xi$. Define $\xi:\Xi\to\Xi$ by

$$G(\mathrm{w},\mathrm{k},\tau)=\begin{cases}2-\mathrm{w}\ \mathrm{w}\in[-1,1),\\1\ \mathrm{w}\in[1,2),\end{cases} \tag{45}$$

We have the following four cases:

1. if $\mathrm{w}, \mathrm{k} \in [-1, 1)$ then $\xi\mathrm{w} = 2 - \mathrm{w}$ and $\xi\mathrm{k} = 2 - \mathrm{k}$,
2. if $\mathrm{w}, \mathrm{k} \in [1, 2]$ then $\xi\mathrm{w} = \xi\mathrm{k} = 1$,
3. if $\mathrm{w}, \in [-1, 1)$ and $\mathrm{k} \in [1, 2]$ then $\xi\mathrm{w} = 2 - \mathrm{w}$ and $\xi\mathrm{k} = 1$,
3. if $\mathrm{w}, \in [1, 2]$ and $\mathrm{k} \in [-1, 1)$ then $\xi\mathrm{w} = 1$ and $\xi\mathrm{k} = 2 - \mathrm{k}$,

Because $\mathrm{w} \perp \mathrm{k} \Leftrightarrow \mathrm{w} + \mathrm{k} \geq 0$, it is clearly implies that $\xi(\mathrm{w}) + \xi(\mathrm{k}) \geq 0$. That is, ξ is $\perp$-preserving. Suppose $\{\mathrm{w}_n\}$ be any O-sequence in Ξ that O-converges to $\mathrm{w} \in \Xi$. we get

$$\lim_{n\to\infty} G(\mathrm{w}, \mathrm{k}, \tau) = \lim_{n\to\infty} \frac{\tau + \min\{\mathrm{w}, \mathrm{k}\}^3}{\tau + \min\{\mathrm{w}, \mathrm{k}\}^3} = 1 \forall \mathrm{w}, \mathrm{k} \in \Xi, \tau > 0,$$

and

$$\lim_{n\to\infty} H(\mathrm{w}, \mathrm{k}, \tau) = 1 - \lim_{n\to\infty} \frac{\tau + \min\{\mathrm{w}, \mathrm{k}\}^3}{\tau + \min\{\mathrm{w}, \mathrm{k}\}^3} = 0 \forall \mathrm{w}, \mathrm{k} \in \Xi, \tau > 0,$$

we can easily see that if $\lim_{n\to\infty} G(\mathrm{w}_n, \mathrm{w}, \tau) = 1$, and $\lim_{n\to\infty} H(\mathrm{w}_n, \mathrm{w}, \tau) = 0$, then $\lim_{n\to\infty} G(\xi\mathrm{w}_n, \xi\mathrm{w}, \tau) = 1$ and $\lim_{n\to\infty} H(\xi\mathrm{w}_n, \xi\mathrm{w}, \tau) = 0$ for all $\tau > 0$. That is, ξ is orthogonal continuous. For $\mathrm{w} = \mathrm{k}$, it is obvious.

$$\frac{1}{G(\xi\mathrm{w}, \xi\mathrm{k}, \tau)} - 1 \leq \varrho \left[\frac{1}{G(\mathrm{w}, \mathrm{k}, \tau)} - 1\right]$$

$$H(\xi\mathrm{w}, \xi\mathrm{k}, \tau) leq \varrho H(\mathrm{w}, \mathrm{k}, \tau).$$

All conditions of Theorem 3.21 are satisfied and 1 is a FP of ξ

4 An Application to an Integeal Equation

Let $\Xi = C([\sigma, \theta], \mathbb{R})$ be the set of all continuous real valued functions defined on $[\sigma, \theta]$. Now, we consider the Fredholm type integral equation of fiest kind:

$$\mathrm{w}(\eta) = \int_\sigma^\theta F(\eta, j)\mathrm{w}(\eta)\mathrm{k}j, for\ \eta, j \in [\sigma, \theta] \tag{46}$$

Where, $F \in \Xi$. Define G as in Example 3.2, That is

$$G(\mathrm{w}(\eta), \mathrm{k}(\eta), \tau) = \sup_{\eta\in[\sigma,\theta]} \begin{cases} 1 \text{ if } \mathrm{w} = \mathrm{k}, \\ \left[e^{\frac{\max\{\mathrm{w}(\eta), \mathrm{k}(\eta)\}^\alpha}{\tau}}\right]^{-1} & \text{otherwise,} \end{cases} \tag{47}$$

and

$$H(\mathrm{w}(\eta), \mathrm{k}(\eta), \tau) = \sup_{\eta\in[\sigma,\theta]} \begin{cases} 0 \text{ if } \mathrm{w} = \mathrm{k}, \\ 1 - \left[e^{\frac{\max\{\mathrm{w}(\eta), \mathrm{k}(\eta)\}^\alpha}{\tau}}\right]^{-1} & \text{otherwise,} \end{cases} \tag{48}$$

for all $\mathrm{w}, \mathrm{k} \in \Xi$ and $\tau > 0$. Then $(\Xi, G, H, *, \Delta, \perp)$ is an O-complete IFBMS.

Theorem 4.1. *Assume that* $\max\{F(\eta, j)\mathrm{w}(\eta), F(\eta, j)\mathrm{k}(\eta)\} \leq \varrho \max\{\mathrm{w}(\eta), \mathrm{k}(\eta)\} for$ $\mathrm{w}, \mathrm{k} \in \Xi, \varrho \in (0, 1)$ *and* $\eta, j \in [\sigma, \theta]$*. Also, consider* $\int_\sigma^\theta \mathrm{k}j = 1$*. Then the Fredholm type integral equation of first kind in equation (46) has a unique solution.*

Proof. *Define* $\xi : \Xi \to \Xi$ *by* $\mathrm{w}(\eta) = \int_\sigma^\theta F(\eta, j)\mathrm{w}(\eta)\mathrm{k}j, for\ \eta, j \in [\sigma, \theta]$*. Define Orthogonality as:* $\mathrm{w}(\eta) \perp \mathrm{k}(\eta) \Leftrightarrow \mathrm{w}(\eta)\mathrm{k}(\eta) \in \{|\mathrm{w}(\eta)|, |\mathrm{k}(\eta)|\}$*. We see that* $\mathrm{w}(\eta)$ *and* $\xi\mathrm{w}(\eta)$ *belong to* Ξ*. So, observe that if* $\mathrm{w}(\eta) \perp \mathrm{k}(\eta)$*,* *then must be* $\xi \mathrm{w}(\eta) \perp \xi \mathrm{k}(\eta)$*. Observe that the existence of a FP of the operator* ξ *is equivalent to the existance of a solution of the Fredholm type integral equation* (46)*. Now, for* $\mathrm{w}(\eta) = \mathrm{k}(\eta)$*, the contraction condition holds. While for* $\mathrm{w} \neq \mathrm{k}$*, We have*

$$G(\xi \mathrm{w}(\eta), \xi \mathrm{k}(\eta), \varrho\tau) = \left[e^{\frac{\max\{\mathrm{w}(\eta),\mathrm{k}(\eta)\}^{\alpha}}{\varrho\tau}}\right]^{-1}$$

$$= \left[e^{\frac{\max\left\{\int_{\sigma}^{\theta} F(\eta,j)\mathrm{w}(\eta)\mathrm{k}j, \int_{\sigma}^{\theta} F(\eta,j)\mathrm{k}(\eta)\mathrm{k}j\right\}^{\alpha}}{\varrho\tau}}\right]^{-1}$$

$$= \left[e^{\frac{\left(\int_{\sigma}^{\theta} \max\{F(\eta,j)\mathrm{w}(\eta)\mathrm{k}j, F(\eta,j)\mathrm{k}(\eta)\mathrm{k}j\}\right)^{\alpha}}{\varrho\tau}}\right]^{-1}$$

$$\geq \left[e^{\frac{\left(\int_{\sigma}^{\theta} \max\{\mathrm{w}(\eta)\mathrm{k}j, \mathrm{k}(\eta)\mathrm{k}j\}\right)^{\alpha}}{\varrho\tau}}\right]^{-1}$$

$$\geq \sup_{\eta\in[\sigma,\theta]} \left[e^{\frac{(\varrho \max\{\mathrm{w}(\eta)\mathrm{k}j, \mathrm{k}(\eta)\})^{\alpha}\left(\int_{\sigma}^{\theta} \mathrm{k}j\right)^{\alpha}}{\varrho\tau}}\right]^{-1}$$

$$= \sup_{\eta\in[\sigma,\theta]} \left[e^{\frac{(\max\{\mathrm{w}(\eta)\mathrm{k}j, \mathrm{k}(\eta)\})^{\alpha}}{\tau}}\right]^{-1}$$

$$= G(\mathrm{w}(\eta), \mathrm{k}(\eta), \tau),$$

and

$$H(\xi \mathrm{w}(\eta), \xi \mathrm{k}(\eta), \varrho\tau) = 1 - \left[e^{\frac{\max\{\mathrm{w}(\eta),\mathrm{k}(\eta)\}^{\alpha}}{\varrho\tau}}\right]^{-1}$$

$$= 1 - \left[e^{\frac{\max\left\{\int_{\sigma}^{\theta} F(\eta,j)\mathrm{w}(\eta)\mathrm{k}j, \int_{\sigma}^{\theta} F(\eta,j)\mathrm{k}(\eta)\mathrm{k}j\right\}^{\alpha}}{\varrho\tau}}\right]^{-1}$$

$$1 - \left[e^{\frac{\left(\int_{\sigma}^{\theta} \max\{F(\eta,j)\mathrm{w}(\eta)\mathrm{k}j, F(\eta,j)\mathrm{k}(\eta)\mathrm{k}j\}\right)^{\alpha}}{\varrho\tau}}\right]^{-1}$$

$$\leq 1 - \left[e^{\frac{\left(\int_{\sigma}^{\theta} \max\{\mathrm{w}(\eta)\mathrm{k}j, \mathrm{k}(\eta)\mathrm{k}j\}\right)^{\alpha}}{\varrho\tau}}\right]^{-1}$$

$$\leq 1 - \sup_{\eta\in[\sigma,\theta]} \left[e^{\frac{(\varrho \max\{\mathrm{w}(\eta)\mathrm{k}j, \mathrm{k}(\eta)\})^{\alpha}\left(\int_{\sigma}^{\theta} \mathrm{k}j\right)^{\alpha}}{\varrho\tau}}\right]^{-1}$$

$$= 1 - \sup_{\eta\in[\sigma,\theta]} \left[e^{\frac{(\max\{\mathrm{w}(\eta)\mathrm{k}j, \mathrm{k}(\eta)\})^{\alpha}}{\tau}}\right]^{-1}$$

$$= H(\mathrm{w}(\eta), \mathrm{k}(\eta), \tau),$$

Hence, ξ *is an* $\perp$*-contraction. Let* $\{\mathrm{w}_n\}$ *be an O-sequence in* Ξ *O-converging to* $\mathrm{w} \in \Xi$*. Because* ξ *is an* $\perp$*-preserving, then* $\{\xi \mathrm{w}_n\}$ *is an O-sequence for each* $n \in \mathbb{N}$*. We have*

$$G(\xi \mathrm{w}_n(\eta), \xi \mathrm{w}, \varrho\tau) \geq G(\mathrm{w}_n(\eta), \mathrm{w}(\eta), \tau) \tag{49}$$

and

$$H(\xi \mathrm{w}_n(\eta), \xi \mathrm{w}, \varrho\tau) \leq H(\mathrm{w}_n(\eta), \mathrm{w}(\eta), \tau) \tag{50}$$

As $\lim_{n\to\infty} G(\xi \mathrm{w}_n(\eta), \xi \mathrm{w}, \varrho\tau) = 1$ *and* $\lim_{n\to\infty} H(\xi \mathrm{w}_n(\eta), \xi \mathrm{w}, \varrho\tau) = 0$ *for all* $\tau > 0$*, it is clear that*

$$\lim_{n\to\infty} G(\xi \mathrm{w}_n(\eta), \xi \mathrm{w}, \varrho\tau) = 1, \tag{51}$$

$$\lim_{n\to\infty} H(\xi \mathrm{w}_n(\eta), \xi \mathrm{w}, \varrho\tau) = 0, \tag{52}$$

Hence, ξ is $\perp$-continuous. Therefore, all conditions of Theorem 3.13 are satisfied. Hence, the operator ξ has a unique FP. That is, the Fredholm type integral equation (46) has a unique solution. □

5 Conclusion

In this study, we established the concept of an OIFBMS as a generalization of an IFBMS. We established some fixed point theorems and solved some non-trivial examples with an application to Fredholm integral equations. This work is extendable in the structure of orthogonal neutrosophic b-metric spaces,and orthogonal inutionistic fuzzy controlled metric spaces and we can increase self mappings to get new results.

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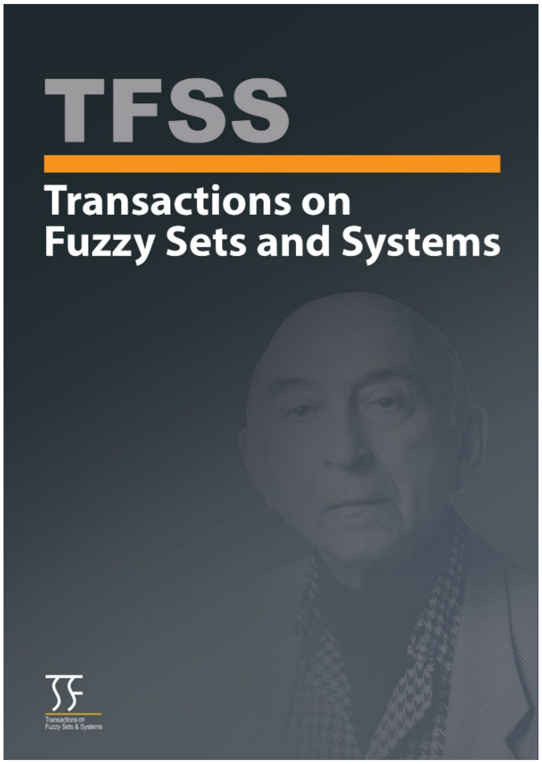

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Fuzzy Logistic Regression Analysis Using the Least Squares Method

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Abstract. One of the most efficient statistical tools for modeling the relationship between a dependent variable and several independent variables is regression. In practice, observations relating to one or more variables, or the relationship between variables, may be vague or non-specific. In such cases, classic regression methods will not have enough capability to model data, and one of the alternative methods is regression in a fuzzy environment. The fuzzy logistic regression model provides a framework in the fuzzy environment to investigate the relationship between a binary response variable and a set of covariates. The purpose of this paper is to attempt to develop a fuzzy model that is based on the idea of the possibility of success. These possibilities are characterized by several linguistic phrases, including low, medium, and high, among others. Next, we use a set of precise explanatory variable observations to model the logarithm transformation of "possibilistic odds." We assume that the model's parameters are triangular fuzzy numbers. We use the least squares method in fuzzy linear regression to estimate the parameters of the provided model. We compute three types of goodness-of-fit criteria to evaluate the model. Ultimately, we model suspected cases of Systemic Lupus Erythematosus (SLE) disease based on significant risk factors to identify the model's application. We do this due to the widespread use of logistic regression in clinical studies and the prevalence of ambiguous observations in clinical diagnosis. Furthermore, to assess the prevalence of diabetes in the community, we will collect a sample of plasma glucose levels, measured two hours after a meal, from each participant in a clinical survey. The proposed model has the potential to rationally replace an ordinary model in modeling the clinically ambiguous condition, according to the findings.

AMS Subject Classification 2020: 62J86; 62J07
Keywords and Phrases: Least square, Distance measure, Logistic regression.

1 Introduction

Regression is one of the most efficient statistical tools for modeling the relationship between a dependent variable and one or more independent variables. Regression analysis primarily aims to identify the functional relationship between the dependent variable and the independent variable, enabling control over the dependent variable's values or future prediction. The standard model for statistical linear regression is as follows:

$$V_i = w_0 + w_1 u_{i1} + \cdots + w_n u_{in} + \epsilon_i, \quad i = 1, \cdots, p \tag{1}$$

where V_i is the dependent variable for the $i-$th observation and u_{ij} is the value of the $j-$th independent variable in the $i-$th sample observation and w_j are the coefficients of the independent variables in the regression function or model parameters. These parameters are based on a sample of observations and The basis of statistical methods is estimated. In practice, observations of variables or their relationships can be

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vague or non-precise. In such cases, classical regression methods will not have enough capability to model the data. In such cases, one of the alternative methods of classical regression is fuzzy regression, or, in other words, regression in a fuzzy environment.
While linear regression models have dominated most existing studies in this field, sometimes the relationships between variables are too complex to model and analyse using a linear relationship. In the category of non-linear models, some models are inherently linear. In other words, appropriate transformations can linearise the relationship between model variables. One of these models is the logistic regression. We use this to model the relationship between a binary dependent variable and one or more independent variables. When defining the dependent variable's classes, we code the desired condition with the number one and the opposite class with the number zero. This model has many applications in various scientific fields, including health and medical studies. For instance, it models disease status (sick or healthy) and patient survival (death or survival).
Many scientific studies use imprecise observations, but the logistic regression model, like other statistical models, uses precise observations to fit the model. It is impossible to verify model assumptions with imprecise observations or small sample sizes. Any violation of these assumptions makes using the logistic model unreasonable. The previous discussion introduced us to the concept of modeling in a fuzzy environment. Because fuzzy modeling has more flexibility, it works well, especially when the sample size is small or the observations and relationships between variables are imprecise and approximate.
In 1965, Zadeh first introduced fuzzy sets [1]. Subsequently, Tanaka and his colleagues [2] engaged in a debate on the subject of fuzzy regression. Tanaka assumed that the data consisted of triangular fuzzy integers and proceeded to estimate the regression coefficients by minimising a fuzzy index. Tanaka based his work on mathematical programming methodologies. In the same year, Yager [3], with a different approach, predicted the value of the dependent variable, the simplest form of fuzzy regression, in her contract with fuzzy observations. Jajoga [4] calculated the linear regression coefficients using a generalised version of the least squares method, while until then most of the fuzzy regression models were analysed using the mathematical programming method. Celmins [5] proposed a method for fitting a multivariate fuzzy model by minimising a least squares objective function and presented a least squares method for fuzzy regression models. Diamond [6] introduced a distance measure on the set of fuzzy numbers and used it to define the least squares criterion. In general, there are three main methods for analysing fuzzy linear regression models:

- Fuzzy least squares methods,
- Mathematical programming methods,
- Numerical methods (simulation or iteration).

Pourahmad et al. [7, 8] investigated fuzzy logistic regression from two perspectives: possibility and least squares. Namdari et al. [9] conducted a study on using fuzzy logistic regression models to analyze data with crisp input and fuzzy output. The study assessed the imprecision of the dependent variable using linguistic terminology. Their study primarily focused on the development of the least absolute deviations approach for modeling, followed by a comparison of the obtained findings to those derived from the least squares estimate method. In their work, the authors of reference [10] proposed a method for calculating the integral distance of cut sets. Additionally, they introduced a fuzzy adjustment term to reduce the likelihood of significant fuzzy errors in the fuzzy output, mainly when representing the independent variables as crisp integers. The least squares approach yields the parameters of the fuzzy logistic regression model. Mustafa et al. [11] proposed a fuzzy probabilistic logistic model that utilizes trapezoidal membership functions. Salmani et al. [12] proposed a fuzzy regression model that integrates fuzzy covariates to address the issue of erroneous binary-based response variables. The researchers used a least-squares methodology to estimate the model's parameters, and then used a bootstrap technique to compute confidence intervals and test hypotheses about the model parameters. Salmani et al. [13] suggested three ways to measure the goodness-of-fit in logistic regression models: the Mean Squared Error (MSE), the Akaike Information Criterion (AIC), and C_p. The authors created a forward model selection method for fuzzy logistic regression that takes into account fuzzy sets' efficiency level and mean squared error.

Logistic regression analysis is one of the famous non-linear methods used to model the binary response variable based on ordinary explanatory variables. This method is particularly appropriate for models involving disease state (diseased or healthy), patient survival (alive or dead), and decision-making (yes or no). Therefore, studies in the health sciences widely use it (for more details refer to Bagley et al. [14]). Classical logistic regression encounters problems such as (1) Violation of distribution assumptions (Bernoulli probability distribution for the binary response variable, uncorrelated explanatory variables, independence, and identically distributed error terms). (2) Low sample size. (3) Vagueness in the relationship between variables that do not follow the random error patterns in logistic regression models; and (4) Non-precise observations. In fact, non-precise or vague observations, which occur frequently in practice, may cause the other difficulties. Take clinical research as an example; certain diseases lack biological examinations, and their diagnosis relies on well-defined and widely accepted criteria. To distinguish patients with these diseases, cases with some of those defined criteria (but not all of them) have a vague status. Lupus 1 and Behcet 2 are examples in this field [15]. In the case of hypertension, it is not rational to use a blood pressure threshold of 3 as a precise borderline to identify the patient. Furthermore, linguistic terms such as low, medium, and high describe some variables, such as pain severity or disease severity.

The main contributions of this paper are the creation of a fuzzy multiple linear least squares logistic regression model, the sharing of computational formulas for figuring out regression parameters, and the addition of a similarity measure between LR-type fuzzy numbers to test how well the proposed model works. We structure the remaining sections of this paper as follows: In Section 2, we provide some established findings about LR-type fuzzy numbers. We talk in depth about the suggested distance measurements between LR-type fuzzy numbers and show how to use computers to find regression parameters in Section 3. In Section 4, we show how the suggested model performs with two numerical instances. In the last section, we briefly summarize our results and provide directions for further research.

2 Preliminaries of Fuzzy Arithmetic

In 1965, Professor Zadeh proposed the concept of fuzzy sets and partial membership for sets whose boundaries are not completely clear. He introduced the concept of a fuzzy set as a collection of objects that belong to the set with a degree between 0 and 1, where degree 1 indicates complete membership and degree 0 indicates complete non-membership in the set. The membership function, which assigns a number from the interval $[0, 1]$ to each object, served as the basis for this definition.

Definition 2.1. The fuzzy set $\tilde{A}$ of $\mathbf{R}$ is called a fuzzy number if it applies in the following three conditions:

- $\tilde{A}$ is normal, it means that there exists exactly one $x \in \mathbf{R}$ such that $\tilde{A}(x) = 1$.
- $\tilde{A}$ is upper semicontinuous, that is, all $\alpha-$cuts of that interval are closed.
- The support $\tilde{A}$ is bounded.

Definition 2.2. A fuzzy number $\tilde{A}$ is called an LR fuzzy number if the membership function of $\tilde{A}$ is as follows:

$$A(x) = \begin{cases} L(\frac{m-x}{s_l}), & x < m, \\ R(\frac{x-m}{s_r}), & x \geq m \end{cases}$$

where s_l, $s_r > 0$ and $L, R : [0, \infty) \to [0, 1]$ are continuous, decreasing and invertible functions on $[0, 1]$ and also $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$. We call m, s_l and s_r the center, left width, and right width of the fuzzy number $\tilde{A}$, respectively. For simplicity, we denote $\tilde{A}$ by $\tilde{A} = (m, s_l, s_r)$. If $L = R$ and $s = s_l = s_r$, $\tilde{A}$ is called a symmetric fuzzy number and we denote it by $\tilde{A} = (m, s_l, s_r)$. A fuzzy number with reference functions $L(x) = R(x) = max\{0, 1 - x\}$ is called a triangular fuzzy number and we denote it by $\tilde{A} = (m, s_l, s_r)$.

Definition 2.3. For two fuzzy numbers $\tilde{A} = (m, s_l, s_r)$ and $\tilde{B} = (n, t_l, t_r)$, we will have:

- $\tilde{A} + \tilde{B} = (m + n, s_l + t_l, s_r + t_r)$.
- $$\lambda\tilde{A} = \begin{cases} (\lambda m, \lambda s_l, \lambda s_r), & \lambda > 0, \\ (\lambda m, \lambda s_r, \lambda s_l), & \lambda < 0. \end{cases}$$

Since one of the methods of solving regression models is to use the least squares estimator, and in this method we need to calculate the distance between two fuzzy numbers, we must define the measure of the distance between two fuzzy numbers. Researchers in this field have so far expressed different measures to calculate the distance between two fuzzy numbers, which can be referred to [16] for further study. Here, we improve the distance measure that Li et al. [16] stated so that the distance between two fuzzy numbers can be calculated at different levels of decision-making. One of the benefits of this improved interval is that we can have a model suitable for the same level of decision-making for the problem data by choosing the appropriate parameter values. The distance measure that was defined by Li et al. [16] for two fuzzy numbers $\tilde{A} = (m, s_l, s_r)$ and $\tilde{B} = (n, t_l, t_r)$,is as follows:

$$D(\tilde{A}, \tilde{B})^2 = \alpha_0(m - n)^2 + \alpha_1(s_l - t_l)^2 + \alpha_2(s_r - t_r)^2 + 2(m - n)(\alpha_3(s_r - t_r) - \alpha_4(s_l - t_l)), \tag{2}$$

Its specific modes

$$\alpha_0 = 3,\ \alpha_1 = \lambda^2,\ \alpha_2 = \rho^2,\ \alpha_3 = \rho \, and \, \alpha_4 = \lambda$$

and

$$\alpha_0 = 1,\ \alpha_1 = \lambda_2,\ \alpha_2 = \rho_2,\ \alpha_3 = \rho_1 \, and \, \alpha_4 = \lambda_1,$$

that results in the measures of the distance defined by Yang and Ko [17] and Diamond and Korner [18], respectively. where

$$\lambda = \int_0^1 L^{-1}(q)dq, \qquad \lambda_1 = \frac{1}{2}\int_0^1 |L^{-1}(q)|dq, \qquad \lambda_2 = \frac{1}{2}\int_0^1 |L^{-1}(q)|^2 dq,$$
$$\rho = \int_0^1 R^{-1}(q)dq, \qquad \rho_1 = \frac{1}{2}\int_0^1 |R^{-1}(q)|dq, \qquad \rho_2 = \int_0^1 |R^{-1}(q)|^2 dq$$

The least squares method uses minimizing the sum of squared errors as a fit criterion. Fuzzy least squares methods are also based on the lowest degree of difference between the observed values and the fitted values. In the following, we use the least squares method to estimate the parameters of the logistic regression model. In this method, we use the meter introduced in relation 2 to measure the error sentences and the distance between the observed and fitted fuzzy numbers. This meter is an extended version of the previous meters Yang and Ko [17] and Diamond and Körner [18] talked about here.

3 Fuzzy Logistic Regression

Consider a regression model in which the dependent variable has a binary state, such as illness or health, death or life, buying or not buying, going bankrupt or not going bankrupt, etc. Initially, medical applications utilized this model primarily to predict the likelihood of a disease's occurrence. Today, it finds widespread use across all scientific fields. Logistic regression can be a suitable model for such situations.
The logistic regression model can be considered a generalized linear model that uses the logit function as a link function, and its error follows a polynomial distribution. When the response variable follows a binomial distribution, we use binary logistic regression as a statistical method. This approach models a linear combination of explanatory variables using a function known as the "logit". The logit function is defined as the natural logarithm of the ratio of the probability of success (π) to the probability of failure ($1-\pi$). The following mathematical representation can express the association between the independent and dependent variables in the context of logistic regression:

$$V_i = logit(\pi_i) = Ln\left(\frac{\pi_i}{1-\pi_i}\right) = w_0 + w_1 u_{i1} + \cdots + w_n u_{in}, \quad i = 1, \cdots, p \tag{3}$$

This study will primarily examine a scenario where the explanatory factors represent crisp values, but the dependent variable is imprecise and quantified using language phrases. The definition of "possibilistic odds," as provided by Pourahmad et al. [7], will be presented in the subsequent definition.

Definition 3.1. Let μ_i represent the probability of seeing feature 1 or success, denoted as $V_i = 1$, for the ith example in a sample of size n. The eventuality of achieving success for the selected feature is determined by a linguistic word, $\mu_i \in \{\cdots, low, average, high, \cdots\}$. We can use expert-defined fuzzy numbers to accurately represent each term of a linguistic variable. It is important to provide precise definitions for these words in a manner that ensures the collective range of their respective supports encompasses the whole of the interval $(0,1)$. The ratio $\frac{\mu_i}{1-\mu_i}$ is regarded as the possibilistic odds of the ith scenario, indicating the eventuality of success in relation to the eventuality of failure.

For instance, triangular fuzzy numbers, which are designed to represent the eventuality of success as $\mu = (Verylow, Low, Medium, High, Veryhigh)$, are provided in equation (4) and visually shown in Figure 1.

$$\begin{aligned} &Very\,low = (0.01, 0.02, 0.18),\ Low = (0.1, 0.25, 0.40),\ Medium = (0.35, 0.50, 0.65), \\ &High = (0.6, 0.75, 0.90),\ Very\ High = (0.8, 0.98, 0.90) \end{aligned} \tag{4}$$

3.1 Introducing the Model

The logistic regression model is a generalized linear model that uses the logit function as the dependent variable and a binomial distribution for the error sentences. The following diagram illustrates this model:

$$v_i = w_0 + w_1 u_{i1} + \cdots + w_k u_{ik} + \epsilon_i, \quad i = 1, \cdots, n \tag{5}$$

Remark 3.2. According to the second part of Definition 2.3, there are differences when the coefficients of fuzzy numbers are positive or negative. Therefore, according to this definition and applying changes, calculations for negative coefficients can also be considered, but in this article, calculations based on positive coefficients are considered.

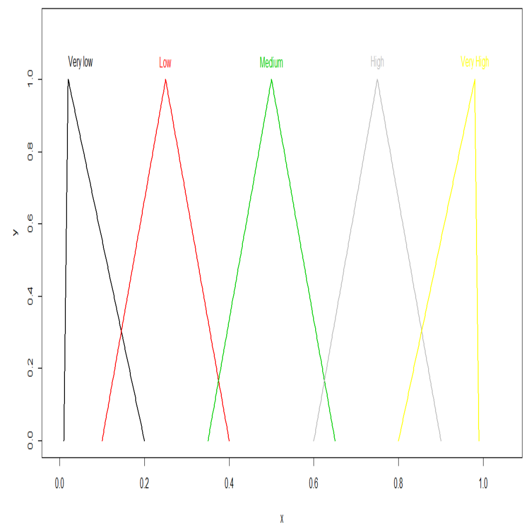

Figure 1: The membership functions of triangular fuzzy numbers represent the eventuality of success as $\mu = (Verylow, Low, Medium, High, Veryhigh)$

For the fuzzy model, consider the set of observations $U_i = (u_{i1}, u_{i2}, ..., u_{ik})$, where U_i is the non-fuzzy observation vector of covariates for the ith case. We indicate the observation of the corresponding answer with v_i, which is a number between 0 and 1, and it shows the possibility of having a desirable characteristic for the ith case. Consequently, we present the fuzzy logistic regression model with fuzzy coefficients as follows:

$$\tilde{v}_i = \tilde{w}_0 + \tilde{w}_1 u_{i1} + \cdots + \tilde{w}_k u_{ik} + \epsilon_i, \quad i = 1, \cdots, n \tag{6}$$

$\tilde{w}_j$, $j = 0, 1, ..., k$ are the parameters of the model, which are assumed to be triangular in fuzzy number $\tilde{w}_j = (w_j, l_j, r_j)_T$ calculations for simplicity. $\tilde{v}_i = ln\dfrac{\mu_i}{1-\mu_i}$ is the probability logarithmic transformation estimator, so based on the properties of addition and subtraction of triangular fuzzy numbers, $\tilde{V}_i$ will also be a triangular fuzzy number in the form of $\tilde{V}_i = (f_{ic}(u), f_{il}(u), f_{ir}(u))$, which:

$$\begin{aligned} f_{ic}(u) &= w_0 + w_1 u_{i1} + \cdots + w_k u_{ik}, \\ f_{il}(u) &= l_0 + l_1 u_{i1} + \cdots + l_k u_{ik}, \\ f_{ir}(u) &= r_0 + r_1 u_{i1} + \cdots + r_k u_{ik}, \end{aligned} \tag{7}$$

Therefore, the fuzzy estimated output membership function is obtained as follows:

$$\tilde{V}_i(v_i) = \begin{cases} 1 - \dfrac{f_{ic}(u) - v_i}{f_{il}(u)}, & f_{ic}(u) - f_{il}(u) \leq v_i \leq f_{ic}(u) \\ 1 - \dfrac{v_i - f_{ic}(u)}{f_{ir}(u)}, & f_{ic}(u) \leq v_i \leq f_{ic}(u) + f_{ir}(u) \end{cases} \tag{8}$$

As mentioned, $\tilde{V}_i$ is the natural logarithm of the probability of having the desired property for the observed ith case. According to the expansion principle, if $\tilde{N}$ is a fuzzy number with the membership function $\tilde{N}(x)$

and $f(x) = \exp(x)$, then $f(\tilde{N}) = \exp(\tilde{N})$ is a fuzzy number with the following membership function:

$$\exp(\tilde{N}(x)) = \begin{cases} \tilde{N}(ln(x)), & x > 0 \\ 0, & o.w. \end{cases} \tag{9}$$

Therefore, after estimating the coefficients of the model, the probability membership function of $\exp(\tilde{V}_i(x))$, $x > 0$ can be defined as follows:

$$\exp(\tilde{V}_i(u)) = \tilde{V}_i(ln(u)) = \begin{cases} 1 - \dfrac{f_{ic}(u) - ln(u)}{f_{il}(u)}, & f_{ic}(u) - f_{il}(u) \leq ln(u) \leq f_{ic}(u) \\ 1 - \dfrac{ln(u) - f_{ic}(u)}{f_{ir}(u)}, & f_{ic}(u) \leq ln(u) \leq f_{ic}(u) + f_{ir}(u) \end{cases} \tag{10}$$

Therefore, for a new fuzzy observed case, its probability is predicted as a fuzzy number using the odds model.

3.2 Estimation of Parameters

We consider the regression model to be a logistic model with fuzzy output, regression coefficients, and a non-fuzzy input vector (independent variables). To estimate the coefficients, we use the least squares error method, which uses the distance measure introduced in Equation 2. To achieve this goal, we will estimate the parameters by minimizing the following relationship:

$$S(\tilde{w}) = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} D^2(\tilde{v}_i, \tilde{V}(u_i)) \tag{11}$$

which is defined as following:

$$\sum_{i=1}^{n} \left(\alpha_0 \left(v_i - w_0 - \sum_{j=1}^{k} w_j u_{ji} \right)^2 + \alpha_1 \left(v_{l_i} - l_0 - \sum_{j=1}^{k} l_j u_{ji} \right)^2 + \alpha_2 \left(v_{r_i} - r_0 - \sum_{j=1}^{k} r_j u_{ji} \right)^2 \right)$$
$$+ \sum_{i=1}^{n} \left(2 \left(v_i - w_0 - \sum_{j=1}^{k} w_j u_{ji} \right) \left(\alpha_3 \left(v_{r_i} - r_0 - \sum_{j=1}^{k} r_j u_{ji} \right) - \alpha_4 \left(v_{l_i} - l_0 - \sum_{j=1}^{k} l_j u_{ji} \right) \right) \right)$$

In order to minimize $S(\tilde{w})$, the partial derivatives of S with respect to the primal variables w_j, r_j, l_j, $j = 1, 2, ..., k$ have to vanish for optimality. To compress the above relationships, we use the matrix symbol as follows.

$$\begin{aligned} S(\tilde{w}) &= \sum_{i=1}^{n} \epsilon_i^2 = \epsilon' \epsilon \\ &= \big(\alpha_0 (V - XW)'(V - XW) + \alpha_1 (L - XS)'(L - XS) + \alpha_2 (R - XP)'(R - XP)\big) \\ &+ \big(2(V - XW)' \left(\alpha_3 (R - XP) - \alpha_4 (L - XS)'\right)\big) \end{aligned}$$

where in

$$V = \begin{bmatrix} V_1 \\ V_2 \\ . \\ . \\ . \\ V_n \end{bmatrix} \quad R = \begin{bmatrix} r_1 \\ r_2 \\ . \\ . \\ . \\ r_n \end{bmatrix} \quad L = \begin{bmatrix} l_1 \\ l_2 \\ . \\ . \\ . \\ l_n \end{bmatrix} \quad X = \begin{bmatrix} 1 & x_{11} & x_{12} & \cdots & x_{1k} \\ 1 & x_{21} & x_{22} & \cdots & x_{2k} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ 1 & x_{n1} & x_{n2} & \cdots & x_{nk} \end{bmatrix}$$

$$W = \begin{bmatrix} w_0 \\ w_1 \\ . \\ . \\ . \\ w_k \end{bmatrix} \quad S = \begin{bmatrix} w_{l_0} \\ w_{l_1} \\ . \\ . \\ . \\ w_{l_k} \end{bmatrix} \quad P = \begin{bmatrix} w_{r_0} \\ w_{r_1} \\ . \\ . \\ . \\ w_{r_k} \end{bmatrix}$$

where W and S, P are the central values and the left and right bounds for the unknown parameters. The X-matrix is a matrix of observations, and the first column is one. We have also observed the V, L, and R vectors, which represent central values, left width, and right width, respectively, for possible odds values. The following is the estimate of the least squares of the unknown parameters obtained by deriving the above expression:

$$\begin{aligned} \hat{P} &= (X'X)^{-1}X' \left(d(\alpha_2 - \alpha_1\alpha_3)R + ((\alpha_1\alpha_0 - \alpha_4^2) - d\alpha_3)Y\right)/b \\ \hat{W} &= (X'X)^{-1}X' \left((\alpha_2 R - \alpha_3 Y) - \alpha_2\hat{P}\right)/-\alpha_3 \\ \hat{S} &= (X'X)^{-1}X' \left((\alpha_1 L + \alpha_4 Y) - \alpha_4\hat{W}\right)/\alpha_1 \end{aligned} \tag{12}$$

where $d = (\alpha_1\alpha_0 - \alpha^2)/\alpha_3$ and $b = \alpha_2 d - \alpha_3\alpha_1$.

3.3 Goodness of Fit Criteria

To evaluate the model, there are many criteria for the goodness of fit. In this article, we will use the following two criteria to evaluate the model:

$$S = \frac{1}{n}\sum_{i=1}^{n}\frac{\int min\{\hat{v}_i(t), \tilde{\hat{v}}_i(t)\}dt}{\int max\{\hat{v}_i(t), \tilde{\hat{v}}_i(t)\}dt}, \quad E_1 = \frac{1}{n}\sum_{i=1}^{n}\int |\hat{v}_i(t), \tilde{\hat{v}}_i(t)|dt, \quad E_2 = \frac{1}{n}\sum_{i=1}^{n}\frac{\int |\hat{v}_i(t), \tilde{\hat{v}}_i(t)|dt}{\int \hat{v}_i(t)dt}. \tag{13}$$

Many authors (e.g., [19, 20, 21]) commonly use these criteria for model evaluation. That S is the similarity criterion and the closer it is to one, the better. For the two criteria E_1 and E_2, the smaller value indicates a better model.

4 Numerical Example

This part uses two real-life examples from the field of medicine and clinical issues to show how well the suggested method works for estimating parameters, testing hypotheses, and figuring out confidence intervals in fuzzy logistic regression models.

Example 4.1. The data includes information about 15 people suspected of having lupus who are aged 18 to 40 years. Lupus is a chronic disease where the body's immune system, for unknown reasons, produces antibodies While the body defends itself against bacteria and viruses, it also targets its healthy organs. These attacks cause symptoms such as pain and muscle cramps. Several body organs, such as the skin, joints, kidneys, heart, and nervous system, are involved in this type of disease at the same time [9]. This disease takes several months or even several years to show its symptoms. Therefore, there is no specific test to identify it. Doctors must gather the required information from various sources, such as a person's medical history, laboratory test results, and some external symptoms. This disease is diagnosed based on its symptoms. Early detection accelerates treatment and prevents disease progression. Generally, lupus disease is defined as a set of 11 symptoms, and a person with at least four symptoms is considered a patient. Here, we categorize the degree of illness in the patient group based on the quantity of symptoms.

This study aims to model the status of people suspected of having lupus based on several important risk factors. The fitted model estimates each person's potential risk of contracting the disease. Past research has identified risk factors such as exposure to sunlight, family history, and various laboratory tests like ANA and DNA-Anti. In addition, in ESR, we use these special blood tests to diagnose lupus. We can summarise the introduction of these tests by stating that the nucleus of living cells consists of a significant quantity of chemicals known as RNA and DNA. The term ANA, or anti-nuclear antibody, literally translates to "anti-nuclear substance of the cell." These substances can damage and destroy cells and tissues. DNA-Anti also means special anti-DNA immune cells. For these two tests, the unit of measurement is defined by the number of these substances per millilitre of blood (ml/u). Their normal value is also considered to be less than 25 ml/u. In addition, ESR is a sign of inflammation. It is uncertain whether ESR increases in lupus patients, and it may be higher in women and elderly individuals. About 95 to 98% of lupus patients have a high value in the ANA test, and the amount of DNA - Anti in the blood of lupus patients increases. However, the high results of these tests alone do not indicate the presence of disease [8]. In order to model the relationship

Table 1: Doubtful cases of lupus and its risk factors

No.	Family History	Sun exposure	ANA test	Anti DNA test	ESR	Possibility of disease
1	1	1	112	105	1	High
2	0	1	80	23	0	Medium
3	0	1	115	15	0	High
4	0	1	105	107	1	High
5	0	0	89	150	1	Medium
6	1	1	160	110	1	Very High
7	0	1	100	23	0	Medium
8	0	0	100	85	1	High
9	0	1	48	83	0	Low
10	1	0	15	19	1	Very Low
11	0	0	50	91	0	Low
12	0	1	59	200	1	Medium
13	0	1	83	20	1	Low
14	0	0	15	200	0	Low
15	1	0	85	15	1	Medium

between the possibility of lupus and the risk factors mentioned in Table 1, the following model is used:

$$\tilde{y}_i = \ln \frac{\tilde{\pi}_i}{1-\tilde{\pi}_i} = \tilde{w}_0 + \tilde{w}_1 u_{i1} + \tilde{w}_2 u_{i2} + \tilde{w}_3 u_{i3} + \tilde{w}_4 u_{i4} + \tilde{w}_5 u_{i5},\ i = 1, 2, ..., 15. \tag{14}$$

We estimate the model's parameters using the least squares method with the meter introduced in the previous section. Ultimately, we calculate the parameter estimation as follows: (with $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = (3, 0.25, 0.25, 0.5, 0.5)$)

$$\begin{aligned}\hat{\tilde{y}}_i =& (-4.0885, -3.3528, -0.5554)_T + (-0.7017, -0.3554, 0.4145)_T\, ESR \\ &+ (0.01033, 0.0042, -0.0045)_T\, Anti\, DNA\, test + (0.0451, 0.0168, -0.0123)_T\, ANA\, test \\ &+ (-0.1405, -0.0853, -0.0235)_T\, Sun.. + (0.2975, -0.4653, -0.8175)_T\, Fam.\end{aligned}$$

The outputs of the fitted log-odds model estimate each suspect's probability of developing lupus. We can calculate the possibility of infection for each suspected person using the principle of expansion of possible odds. For example, for the 6th person studied with the variables $Family\,History = 1$, $Sun\,exposure = 1$, $ANA\,test = 160\,Anti\,DNA\,test = 23\,ESR = 0$ based on the estimated model, we calculate the logarithm of potential odds as follows:

$$\begin{aligned}\hat{\tilde{y}}_i =&(-4.0885, -3.3528, -0.5554)_T + (0.01033, 0.0042, -0.0045)_T\,23 + (0.0451, 0.0168, -0.0123)_T 160\\&+ (-0.1405, -0.0853, -0.0235)_T + (0.2975, -0.4653, -0.8175)_T\end{aligned}$$

So we will have, $\hat{\tilde{V}}_6(3.72, -1.11, -3.45)_T$. This implies that the model has calculated the likelihood of lupus disease in the sixth individual as follows:

$$(0.98, 0.25, 0.03)_T$$

This is extremely close to the table's actual value. We can also use this model to predict the likelihood of disease in a new case. For example, if a person suspected of lupus presents with the following information, we can use the model to predict the likelihood of disease: She (he) has a family history of $u5 = 1$; she (he) has not been exposed to sunlight $u_4 = 0$; the results of the ANA, Anti-DNA, and ESR tests for this person were $u_3 = 110$, $u_2 = 87$, $u_1 = 0$, respectively. Given the characteristics as mentioned above, we estimate the likelihood of a disease and calculate the logarithm of its potential odds as follows: $\hat{\tilde{V}}_{new}(2.07, -1.61, -3.12)_T$ and $(0.89, 0.17, 0.04)_T$.
The fitted values for the possibility of disease, as well as the logarithm of odds, were calculated and recorded in Table 2 using the estimated model. The values of the goodness of fit indices introduced in Equation 13

Table 2: Prediction of the logarithm values of the odds possibility and the possibility of contracting lotus disease for the data in Table 1.

No.	Possibility of disease	The predicted of the logarithm odds disease	The predicted of possibility of lupus
1	High	$(1.50, -1.94, -2.83)_T$	$(0.82, 0.12, 0.05)_T$
2	Medium	$(-0.38, -2.00, -1.67)_T$	$(0.40, 0.12, 0.16)_T$
3	High	$(1.11, -1.44, --2.06)_T$	$(0.75, 0.20, 0.11)_T$
4	High	$(0.91, -1.58, -1.94)_T$	$(0.71, 0.17, 0.12)_T$
5	Medium	$(0.77, -1.59, -1.91)_T$	$(0.68, 0.17, 0.13)_T$
6	Very High	$(3.72, -1.11, -3.45)_T$	$(0.98, 0.25, 0.03)_T$
7	Medium	$(0.52, -1.66, -1.91)_T$	$(0.63, 0.16, 0.13)_T$
8	High	$(0.60, -1.67, -1.75)_T$	$(0.64, 0.16, 0.15)_T$
9	Low	$(-1.21, -2.29, -1.54)_T$	$(0.23, 0.09, 0.18)_T$
10	Very Low	$(-3.62, -3.84, -1.23)_T$	$(0.026, 0.02, 0.23)_T$
11	Low	$(-0.89, -2.13, -1.58)_T$	$(0.29, 0.10, 0.17)_T$
12	Medium	$(-0.20, -1.97, -1.79)_T$	$(0.45, 0.12, 0.14)_T$
13	Low	$(-0.98, -2.32, -1.27)_T$	$(0.27, 0.09, 0.22)_T$
14	Low	$(-1.34, -2.27, -1.64)_T$	$(0.21, 0.09, 0.16)_T$
15	Medium	$(-0.50, -2.68, -2.07)_T$	$(0.38, 0.06, 0.11)_T$

are equal to:

$$S = 0.8156 \quad E1 = 0.0194 \quad E2 = 0.1725.$$

These criteria have been calculated by using the estimated values and actual values for the response variable and placing them in Equation 13.

Example 4.2. We will use a sample of each community member's two-hour postprandial plasma glucose levels from a clinical survey to assess their diabetes condition. We discovered that 15 instances fell within the range of 140–200 (mg/dl), using a cut-off point of 200 (mg/dl). To guess how likely it was that these people had diabetes, we added extra information like their gender (female), age (in years), BMI (body mass index, which is weight in kilograms divided by height in meters squared), family history (including father, mother, sister, and brother), and two-hour plasma glucose levels (measured in milligrams per decilitre), all of which have been linked to a higher risk of diabetes (see Table 4). We asked an expert to assign a probability of illness to each instance. Two-hour postprandial plasma glucose (THPPG)

Table 3: The values of associated risk variables and fuzzy binary observations in SLE disease

No.	Sex	THPPG (mg/dl)	Age(year)	Family history	BMI(kg/m2)	π
1	1	145	40	0	24	$(0.1, 0.74)_T$
2	1	147	42	0	25	$(0.15, 0.74)_T$
3	0	150	45	1	21	$(0.35, 0.82)_T$
4	0	155	37	1	23	$(0.42, 0.83)_T$
5	0	157	59	1	25	$(0.49, 0.83)_T$
6	1	160	44	0	20	$(0.50, 0.72)_T$
7	1	160	38	1	26	$(0.60, 0.90)_T$
8	1	165	52	0	33	$(0.60, 0.77)_T$
9	0	182	50	0	31	$(0.70, 0.64)_T$
10	1	187	55	1	33	$(0.85, 0.91)_T$
11	0	190	53	1	35	$(0.90, 0.86)_T$
12	0	192	62	1	30	$(0.97, 0.85)_T$
13	0	195	57	0	32	$(0.95, 0.65)_T$
14	1	195	50	0	34	$(0.95, 0.77)_T$
15	1	196	60	1	35	$(0.99, 0.92)_T$

$$\hat{\hat{V}}_i = (-16.566, 0.018)_T + (0.476, 0.581)_T\ \times 0 + 0.102\ \times 150 + 0.031\ \times 45 \\ + (0.680, 1.13)_T \times 1 + (-0.0727, 0.019)_T\ \times 21$$

This means that $\hat{\hat{V}}_3 = (0.32, 0, 82)_T$, and the logarithm of possibilistic odds for case 3 is about $(-0.77, 1.54)_T$ This model is capable of estimating the possibility odds of diabetes in a case that is suspected of having the condition. Please be aware that the estimated probability odds for each case are provided in a fuzzy format. For example, suppose we want to predict the possible disease odds for the case number 3 in Table 4. We have:

$$\hat{\hat{V}}_i = (-16.566, 0.018)_T + (0.476, 0.581)_T\, Sex + 0.102\, THPPG + 0.031\, Age \\ + (0.680, 1.13)_T Family\, history + (-0.0727, 0.019)_T\, BMI$$

The predicted values for the possibility of disease, as well as the logarithm of odds, were calculated and recorded in Table 2 using this estimated model. To assess the model, we use the three criteria suggested in Section 3.3, namely, S, E_1, and E_2.

$$S = 0.9991 \quad E1 = 0.0011 \quad E2 = 0.00087$$

Table 4: The values of associated risk variables and fuzzy binary observations in SLE disease are significant.

No.	π	The predicted of the logarithm odds disease	The predicted of possibility of disease
1	$(0.1, 0.74)_T$	$(-1.85, 1.05)_T$	$(0.13, 0.74)_T$
2	$(0.15, 0.74)_T$	$(-1.66, 1.07)_T$	$(0.16, 0.74)_T$
3	$(0.35, 0.82)_T$	$(-0.77, 1.54)_T$	$(0.32, 0.82)_T$
4	$(0.42, 0.83)_T$	$(-0.65, 1.58)_T$	$(0.340.83)_T$
5	$(0.49, 0.83)_T$	$(0.08, 1.62)_T$	$(0.52, 0.83)_T$
6	$(0.50, 0.72)_T$	$(0.09, 0.97)_T$	$(0.52, 0.72)_T$
7	$(0.60, 0.90)_T$	$(0.14, 2.22)_T$	$(0.54, 0.90)_T$
8	$(0.60, 0.77)_T$	$(-0.10, 1.22)_T$	$(0.47, 0.77)_T$
9	$(0.70, 0.64)_T$	$(1.23, 0.60)_T$	$(0.77, 0.64)_T$
10	$(0.85, 0.91)_T$	$(2.90, 2.35)_T$	$(0.95, 0.91)_T$
11	$(0.90, 0.86)_T$	$(2.53, 1.81)_T$	$(0.93, 0.86)_T$
12	$(0.97, 0.85)_T$	$(3.37, 1.71)_T$	$(0.97, 0.85)_T$
13	$(0.95, 0.65)_T$	$(2.70, 0.62)_T$	$(0.94, 0.65)_T$
14	$(0.95, 0.77)_T$	$(2.81, 1.24)_T$	$(0.94, 0.77)_T$
15	$(0.99, 0.92)_T$	$(3.83, 2.39)_T$	$(0.98, 0.92)_T$

5 Conclusion

Typically, the actual conditions of the data do not fully align with the assumed distributional properties of theoretical statistical models. This encourages academics to use fuzzy models as a means of simulating data within a more adaptable framework that closely resembles the actual circumstances of the observations. Researchers have extensively researched these models and implemented them in various fields. Undoubtedly, fuzzy models are more intricate than conventional ones regarding computation and interpretation. However, the assumptions of standard statistical models limit their utility. When the data does not meet the model assumptions, applying standard procedures is not logical because it introduces bias in the findings. Note that you cannot substitute conventional and fuzzy models for one another because of their distinct uses. Typically, it is not possible to use both of these models on the same dataset concurrently, therefore making it impossible to compare their respective outcomes.

When observations are not accurate, we advise using fuzzy modeling methods. Clinical investigations frequently reveal these characteristics. Occasionally, clinical measurement devices may exhibit mistakes. Furthermore, this research includes some ethical issues. Often, the precise magnitude of variables remains unmeasurable in such instances, leading to the reporting of observations based on approximations. The diagnosis of illness, which determines a condition based on established criteria, presents another ambiguous scenario in clinical investigations. We classify an individual as a patient if they exhibit all the signs of an illness. On the other hand, we classify an individual as healthy if they show no symptoms. What is the outcome when an individual experiences only a subset of these symptoms? The physician is unsure whether to start treatment. Furthermore, clinical laboratory tests do not provide a clear-cut threshold to distinguish between patients and healthy individuals. It implies that all people near the cut-off point have ambiguous status. To identify the primary risk factors that contribute to the disease's progression of the disease, it is not logical to rely on vague observations in the typical modeling analysis. Disregarding or neglecting these observations in the analysis is not rational. For this situation, fuzzy models appear to be suitable methods. Fuzzy logistic regression provides a framework in a fuzzy environment for investigating the relationship between a binary response variable and a set of covariates. To date, researchers have presented two general methods to estimate the parameters in fuzzy logistic regression models: the least squares error method and the probability method, both of which use the definition of probability to estimate the parameters. The term ”possible odds” refers to the ratio between the possibility of having the desired feature and not having it. This paper presents a method for estimating the parameters of the fuzzy logistic regression model using the least squares method.

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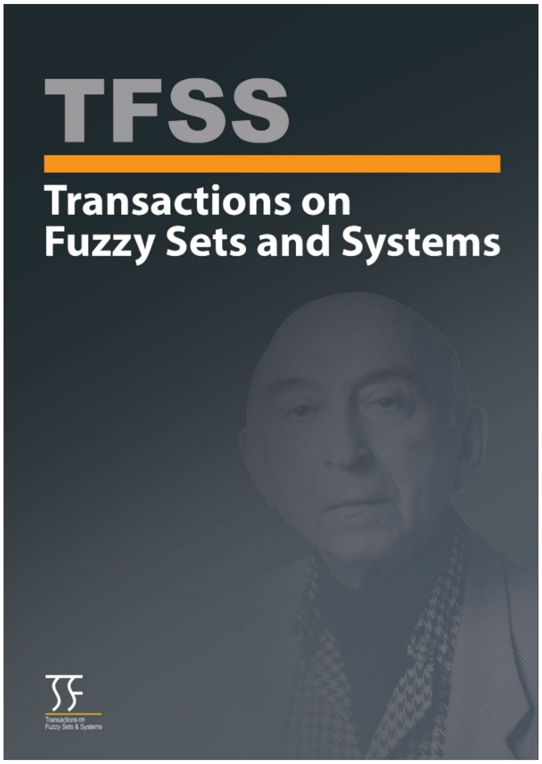

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Nonlinear Contraction Mappings in b-metric Space and Related Fixed Point Results with Application

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Nonlinear Contraction Mappings in b-metric Space and Related Fixed Point Results with Application

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Abstract. The paper aims to introduce some fixed point results in the setting of sequential compact b-metric spaces to prove Eldeisten-Suzuki-type contraction for self-mappings. These contributions extend the existing literature on fixed point for ordered metric spaces and fixed point theory. Through illustrative examples, we showcase the practical applicability of our proposed notions and results, demonstrating their effectiveness in real-world scenarios.

AMS Subject Classification 2020: 47H10; 54H25
Keywords and Phrases: Fixed point, Coincidence point, Eldeisten-Suzuki-type contraction, b-metric space.

1 Introduction

The Banach fixed point theorem, originally proved by Stefan Banach [1] in 1922, is one of the most foundational and influential results in the field of fixed point theory. It states that a contractive mapping on a complete metric space has a unique fixed point. Since its introduction, the theorem has been generalized and extended in many ways, with applications in a variety of scientific disciplines. In particular, generalized metric spaces have been shown to be an extremely useful tool for studying fixed points in Banach spaces. The Banach contraction principle has been the subject of much research in recent years, with many different extensions and generalizations being explored. For example, Ran and Reurings [2] considered the existence of fixed points for mappings in partially ordered metric spaces, while Nieto and Lopez [3] extended this result to non-decreasing mappings. Another notable result is that of solving partial differential equations with periodic boundary conditions. Since its introduction, the Banach contraction mapping principle has been generalized and refined in numerous ways, leading to a wealth of articles dedicated to its improvement [4, 5, 6, 7, 8, 9, 10]. Czerwik's introduction of b-metric spaces [11] was a significant development in the field of generalized metric spaces. He weakened the triangle inequality in a metric space, which led to a generalized form of the Banach contraction principle. Building on this work, Boriceanu [12] provided concrete examples of b-metric spaces and investigated the fixed-point properties of set-valued operators in these spaces. Furthermore, Hussain et al. [13] introduced a new type of generalized metric space known as a dislocated b-metric space, which further extends the possibilities of the Banach contraction principle.
In 1962, mathematician Martin Edelstein [14] proved a generalization of the Banach contraction principle, a

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fundamental result in fixed point theory. Edelstein's generalization is sometimes referred to as the Edelstein fixed point theorem. It states that if a generalized metric space satisfies certain conditions, then a mapping that is contractive in the generalized metric has a unique fixed point. Motivated by the work of Banach and Edelstein, mathematician Suzuki [15] proved a refinement of their fixed point theorems, known as Suzuki's fixed point theorem. This result states that if a metric space satisfies certain additional conditions, a contractive mapping on the space has a unique fixed point. Many authors have proposed variants of Suzuki's theorem, such as those in [16, 17, 18, 19].
Based on the above insight, we present some fixed point results in the setting of sequential compact b-metric spaces to prove Eldeisten-Suzuki-type contraction for self-mappings and apply our main results to establish the existence of fixed point for ordered metric spaces. Through illustrative examples, we showcase the practical applicability of our proposed notions and results, demonstrating their effectiveness in real-world scenarios.

The common notations and terminology used in nonlinear analysis are utilized throughout this work.

2 Preliminaries

We begin this section by outlining a few fundamental definitions.

Definition 2.1. [20] Assume that $d : X \times X \to [0, +\infty)$ and X are non-empty sets. (X, d) is a symmetric space (also known as an E-space) if and only if it meets the requirements listed below:
i. $d(x, y) = 0$ if and only if $x = y$;
ii. $d(x, y) = d(y, x)$ for all $x, y \in X$.

Remark 2.2. [20] In the absence of triangle inequality, symmetric spaces are different from more practical metric spaces. However, a lot of concepts have definitions that are comparable to those in metric spaces.

Definition 2.3. [20] A sequence $\{x_n\}$ has a limit point in a symmetric space (X, d) defined by $\lim_{n\to+\infty} d(x_n, x) = 0$ if and only if $\lim_{n\to+\infty} x_n = x$.

Definition 2.4. [20] If, for every given $\epsilon > 0$, there exists a positive integer $n(\epsilon)$ such that $d(x_m, x_n) < \epsilon$ for all $m, n > n(\epsilon)$, then a sequence $\{x_n\} \subset X$ is a Cauchy sequence.

Definition 2.5. [20] If every Cauchy sequence in a symmetric space (X, d) converges to a point x in X, then the space is considered complete.

Definition 2.6. [21] Let $s \geq 1$ be a given real integer and let X be a nonempty set. A function $d : X \times X \to [0, +\infty)$ is considered a b-metric if and only if each of the subsequent requirements holds for any $x, y, z \in X$:
i. $d(x, y) = 0$ if and only if $x = y$;
ii. $d(x, y) = d(y, x)$;
iii. $d(x, z) \leq s[d(x, y) + d(y, z)]$.
A triplet (X, d, s) is called a b-metric space.

Remark 2.7. [21] The definitions of complete space, Cauchy sequence, and convergent sequence are defined as in symmetric spaces.

Definition 2.8. [5] If there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to a point x in X for each sequence $\{x_n\}$ in X, then a b-metric space (X, d, s) is sequentially compact.

Example 2.9. [20] Let $d : X \times X \to [0, +\infty)$ and $X = [0, 1]$ be defined by $d(x, y) = (x - y)^2$, for all $x, y \in X$. Obviously, $(X, d, 2)$ is a b-metric space.

Definition 2.10. [17] Assume that X is a non-empty set. $(X, \preccurlyeq)$ is referred to as an ordered b-metric space if $(X, d, s, \preccurlyeq)$ is a b-metric space and $(X, \preccurlyeq)$ is a partially ordered set. When $x \preccurlyeq y$ or $y \preccurlyeq x$ holds, then $x, y \in X$ are referred to as comparable.

Definition 2.11. [22, 23] If $(X, \preccurlyeq)$ is a partially ordered set, then a self-mappings f is dominated if and only if $x \preccurlyeq fx$ for all x in X and $fx \preccurlyeq x$ for all x in X.

Definition 2.12. [20] A sequential limit comparison property of an ordered b-metric space $(X, d, s, \preccurlyeq)$ exist if, for each decreasing sequence $\{x_n\}$ in X such that $x_n \to x \in X$, then $x \prec x_n$.

Definition 2.13. [24, 25] Let $f, g : X \to X$ and (X, d) be a metric space. If $fx = gx$, then there is a coincidence point at $x \in X$ for a pair of self mappings f and g. Additionally, if $fx = gx = x$, then a point $x \in X$ is a common fixed point of f and g.

Definition 2.14. [20] For every sequence $\{(x_n, y_n)\} \subset [0, +\infty) \times [0, +\infty)$, then $F : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is referred to as upper semi-continuous from the right if and only if $\lim\limits_{n\to+\infty} x_n = x^+$ and $\lim\limits_{n\to+\infty} y_n = y^+$, then

$$\lim_{n\to+\infty} \sup F(x_n, y_n) \leq F(x, y).$$

We represent Ψ the collection of all the functions $\phi : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ satisfying the following conditions:
(ϕ_1) ϕ admits upper semi-continuous from the right;
(ϕ_2) $\phi(t, 0) \leq t$ for all $t \geq 0$.

Definition 2.15. [20] Consider the b-metric space (X, d, s). The collection of all the functions $\alpha_L : X \times X \to [0, +\infty)$ satisfying the following assertions is also denoted by Ψ_L.
(α_1) if $\{x_n\}$ and $\{y_n\}$ are two sequences in (X, d, s) such that $x_n \to x$ and $y_n \to y$, then

$$\lim_{n\to+\infty} \sup \alpha_L(x_n, y_n) \leq \alpha_L(x, y),$$

(α_2) $\alpha_L(x, y) = 0$ when $x = y$.

3 Main Results

Here is the Theorem that we use to start this section.

Theorem 3.1. *Let f be a self mapping on X and $(X,\ d, s)$ be a sequential compact b-metric space. Suppose that*

$$d(fx, fy) < a_1 d(x, y) + a_2 \frac{d(x, fx)d(x, fy) + d(y, fx)d(y, fy)}{d(y, fx) + d(x, fy)} + a_3 \frac{d(x, fx)d(y, fy)}{d(x, y)} + \frac{a_4}{s} d(x, fy) + Ld(y, fx) \tag{3.1}$$

for all $x, y \in X$, $x \neq y$, where $a_1 + a_2 + a_3 + 2a_4 = 1$, $a_3 \neq 1$, $L \geq 0$ and satisfies the following conditions:
i. If f and d are continuous,
then f possesses a fixed point in X.
Additionally,
ii. If $a_1 + \frac{a_4}{s} + L \leq 1$;
then f possesses a unique fixed point.

Proof. Let us take any arbitrary point $x_0 \in X$ and let $\{x_n\}$ in X be defined as $x_n = f^n x_0 = fx_{n-1}$. If $x_n = x_{n-1}$ for some $n \geq 1$, then x_n is a fixed point of f and the proof is finished.

Now, let $d_n = d(x_n, x_{n+1})$ and $d_{n-1} = d(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$. Assume that $x_n \neq x_{n+1}$, for all $n \geq 1$. From condition (3.1) with $x = x_{n-1}$ and $y = x_n$, we get

$$\begin{aligned} d_n =& d(x_n, x_{n+1}) = d(fx_{n-1}, fx_n) \\ & < a_1 d(x_{n-1}, x_n) + a_2 \frac{d(x_{n-1}, fx_{n-1})d(x_{n-1}, fx_n) + d(x_n, fx_{n-1})d(x_n, fx_n)}{d(x_n, fx_{n-1}) + d(x_{n-1}, fx_n)} \\ & \quad + a_3 \frac{d(x_{n-1}, fx_{n-1})d(x_n, fx_n)}{d(x_{n-1}, x_n)} + \frac{a_4}{s} d(x_{n-1}, fx_n) + Ld(x_n, fx_{n-1}) \\ & = a_1 d(x_{n-1}, x_n) + a_2 \frac{d(x_{n-1}, x_n)d(x_{n-1}, x_{n+1}) + d(x_n, x_n)d(x_n, x_{n+1})}{d(x_n, x_n) + d(x_{n-1}, x_{n+1})} \\ & \quad + a_3 \frac{d(x_{n-1}, x_n)d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} + \frac{a_4}{s} d(x_{n-1}, x_{n+1}) + Ld(x_n, x_n) \\ & = a_1 d_{n-1} + a_2 d_{n-1} + a_3 d_n + \frac{a_4}{s} d(x_{n-1}, x_{n+1}) \\ & \leq a_1 d_{n-1} + a_2 d_{n-1} + a_3 d_n + a_4 [d_{n-1} + d_n]. \end{aligned} \tag{3.2}$$

From (3.2), we get $[1 - (a_3 + a_4)]d_n < (a_1 + a_2 + a_4)d_{n-1}$. Since $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$, we have $[1 - (a_3 + a_4)] > 0$ and so

$$d_n < \frac{a_1 + a_2 + a_4}{1 - (a_3 + a_4)} d_{n-1} = d_{n-1}.$$

Consequently, $\{d_n\}$ is a decreasing sequence of positive real numbers and hence there exists $d^* \geq 0$ such that $\lim_{n\to\infty} d_n = d^*$. By using the sequentially compactness of X, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to x^* \in X$ as $i \to +\infty$. Again, by using the continuity of d and f, we have

$$d_{n_i} = d(x_{n_i}, x_{n_i+1}) = d(x_{n_i}, fx_{n_i}) \to d(x^*, fx^*) \text{ as } i \to +\infty$$

Similarly,

$$d_{n_i+1} = d(x_{n_i+1}, x_{n_i+2}) = d(fx_{n_i}, ffx_{n_i}) \to d(fx^*, ffx^*) \text{ as } i \to +\infty.$$

If $x^* = fx^*$, then f has a fixed point. Assume that $x^* \neq fx^*, d^* = d(x^*, fx^*) > 0$, with $x = x^*$ and $y = fx^*$ in (3.2), we have

$$\begin{aligned} d^* &= d(fx^*, ffx^*) \\ & < a_1 d(x^*, fx^*) + a_2 \frac{d(x^*, fx^*)d(x^*, ffx^*) + d(fx^*, fx^*)d(fx^*, ffx^*)}{d(fx^*, fx^*) + d(x^*, ffx^*)} \\ & \quad + a_3 \frac{d(x^*, fx^*)d(fx^*, ffx^*)}{d(x^*, fx^*)} + \frac{a_4}{s} d(x^*, ffx^*) + Ld(fx^*, fx^*) \\ & \leq (a_1 + a_2 + a_3)d^* + \frac{a_4}{s} d(x^*, ffx^*) \\ & \leq (a_1 + a_2 + a_3)d^* + a_4 [d(x^*, fx^*) + d(fx^*, ffx^*)] \\ & = (a_1 + a_2 + a_3 + 2a_4)d^* = d^*, \end{aligned}$$

a contradiction. Hence, $d^* = d(x^*, fx^*) = 0$, that is $x^* = fx^*$. Thus, x^* represent a fixed point of f.

To prove the uniqueness of the fixed point, suppose z is another fixed point of f different from x^*, so that $d(z,x^*) > 0$. Using $x = z$ and $y = x^*$ in (3.1), we have

$$\begin{aligned} d(z,x^*) &= d(fz,fx^*) \\ &< a_1 d(z,x^*) + a_2 \frac{d(z,fz)d(z,fx^*) + d(x^*,fz)d(x^*,fx^*)}{d(x^*,fz) + d(z,fx^*)} \\ &\quad + a_3 \frac{d(z,fz)d(x^*,fx^*)}{d(z,x^*} + \frac{a_4}{s} d(z,fx^*) + Ld(x^*,fz) \\ &= \left(a_1 + \frac{a_4}{s} + L\right) d(z,x^*) \\ &\leq d(z,x^*), \end{aligned}$$

a contradiction and hence $z = x^*$. □

Example 3.2. Consider $X = [0,1]$ and assume $d : X \times X \to [0,+\infty)$. endowed with $d(x,y) = (x-y)^2$, for all $x,y \in X$. Then, let $f : X \to X$ be defined as

$$f(x) = \frac{1}{4(x^2+1)}.$$

Clearly, $(X,d,2)$ represent a sequentially compact b-metric space. Since

$$d(fx,fy) = \left|\frac{x+y}{4(x^2+1)(y^2+1)}\right|^2 |x-y|^2 < |x-y|^2 = d(x,y) \text{ for all } x,y \in X, x \neq y.$$

Thus, all the hypotheses of Theorem 3.1 are verified, with $a_1 = 1, a_2 = a_3 = a_4 = L = 0$ and hence f has a unique fixed point.

Corollary 3.3. *Let f be a self mapping on X and (X, d, s) be a sequential compact b-metric space. Suppose that*

$$d(fx,fy) < a_1 d(x,y) + a_2 \frac{d(x,fx)d(y,fy)}{d(x,y)} + \frac{a_3}{s} d(x,fy) + Ld(y,fx) \tag{3.3}$$

for all $x,y \in X$, $x \neq y$, where $a_1 + a_2 + a_3 + 2a_4 = 1$, $a_2 \neq 1$, $L \geq 0$ and satisfies the following conditions:
i. If f and d are continuous,
then f possesses a fixed point in X.
Additionally,
ii. If $a_1 + \frac{a_4}{s} + L \leq 1$;
then f possesses a unique fixed point.
Proof. Theorem 3.1 provides the basis for the Corollary's proof. □

Corollary 3.4. *Let f be a self mapping on X and (X, d, s) be a sequential compact b-metric space. Suppose that*

$$d(fx,fy) < a_1 \frac{d(x,fx)d(x,fy) + d(y,fx)d(y,fy)}{d(y,fx) + d(x,fy)} + a_2 \frac{d(x,fx)d(y,fy)}{d(x,y)} + \frac{a_3}{s} d(x,fy) + Ld(y,fx) \tag{3.4}$$

for all $x,y \in X$, $x \neq y$, where $a_1 + a_2 + 2a_3 = 1$, $a_2 \neq 1$, $L \geq 0$ and satisfies the following conditions:
i. If f and d are continuous,
then f possesses a fixed point in X.
Additionally,
ii. If $a_1 + \frac{a_3}{s} + L \leq 1$;
then f possesses a unique fixed point.
Proof. It is evident that the proof of the Corollary follows from Theorem 3.1. □

Corollary 3.5. *Let f be a self mapping on X and $(X,\ d,s)$ be a sequential compact b-metric space. Suppose that*

$$d(fx,fy) < a_1 d(x,y) + a_2 \frac{d(x,fx)d(x,fy)+d(y,fx)d(y,fy)}{d(y,fx)+d(x,fy)} + a_3 \frac{d(x,fx)d(y,fy)}{d(x,y)} + \frac{a_4}{s} d(x,fy) \tag{3.5}$$

for all $x,y \in X$, $x \neq y$, where $a_1+a_2+a_3+2a_4 = 1$, $a_3 \neq 1$, $L \geq 0$ and satisfies the following conditions:
i. If f and d are continuous,
then f possesses a fixed point in X.
Additionally,
ii. If $a_1 + \frac{a_4}{s} + L \leq 1$;
then f possesses a unique fixed point.
Proof. Theorem 3.1 provides the proof of the Corollary in the case where $L = 0$. □

The next theorem is the Suzuki type fixed point result.

Theorem 3.6. *Let f be a self mapping on X and (X,d,s) be a sequential compact b-metric space. Suppose that*

$$\frac{1}{2s} d(x,fx) < d(x,y)$$

implies

$$d(fx,fy) < \varphi\left(d(y,fx), \frac{d(x,fx)d(x,fy)+d(y,fx)d(y,fy)}{d(y,fx)+d(x,fy)}\right) + \alpha_L d(y,fx) \tag{3.6}$$

for all $x,y \in X$ and d is continuous, then f has a fixed point.

Proof. Let $r = \inf d(x,fx) : x \in X$. We define a sequence $\{x_n\}$ in X be

$$\lim_{n\to\infty} d(x_n, fx_n) = r. \tag{3.7}$$

Since X is sequentially compact, we assume that $x_n \to u$ and $fx_n \to v$ with $u,v \in X$. Now we prove that $r = 0$. Assume on the contrary that $r > 0$. Using the continuity of d, we have

$$\lim_{n\to+\infty} d(x_n, v) = (u,v) = \lim_{n\to+\infty} d(x_n, fx_n) = r \tag{3.8}$$

and

$$\lim_{n\to+\infty} d(u, fx_n) = (u,v) = \lim_{n\to+\infty} d(x_n, fx_n) = r. \tag{3.9}$$

Hence, there exists $n_1 \in \mathbb{N}$ such that

$$\frac{2}{3s} < d(x_n, v) \text{ and } d(x_n, fx_n) < \frac{4}{3} f, \text{ for all } n \geq n_1.$$

For all $n \geq n_1$, we have

$$\frac{1}{2s} d(x_n, fx_n) < \frac{1}{2s}\frac{4}{3} r = \frac{1}{s}\frac{2}{3} r < \frac{1}{s} d(x_n, v) \leq d(x_n, v),$$

and by (3.6), we get

$$d(fx_n, fv) < \varphi\left(d(v,fx_n), \frac{d(x_n,fx_n)d(x_n,fv)+d(v,fx_n)d(v,fv)}{d(v,fx_n)+d(x_n,fv)}\right) + \alpha_L d(v, fx_n). \tag{3.10}$$

Taking the *limsup* as $n \to +\infty$ in (3.10), we get

$$\begin{aligned} d(v, fv) &= \limsup_{n\to+\infty} d(fx_n, fv) \\ &\leq \limsup_{n\to+\infty} \varphi\left(d(v, fx_n), \frac{d(x_n, fx_n)d(x_n, fv) + d(v, fx_n)d(v, fv)}{d(v, fx_n) + d(x_n, fv)}\right) \\ &\quad + \limsup_{n\to+\infty} \alpha_L d(v, fx_n) \\ &\leq \varphi(0, d(u, v)) + \alpha_L d(v, v) \leq d(u, v) = r. \end{aligned} \tag{3.11}$$

Thus, from (3.11), we have $d(v, fv) = r$. Since $r > 0, v \neq fv$. So

$$\frac{1}{2s} d(v, fv) < d(v, fv).$$

And by condition (3.6), we get

$$d(fv, ffv) < \varphi\left(d(fv, fv), \frac{d(v, fv)d(v, ffv) + d(fv, fv)d(fv, ffv)}{d(fv, fv) + d(v, ffv)}\right) + \alpha_L d(fv, fv)$$

implies

$$d(fv, ffv) < d(v, fv) = r, \tag{3.12}$$

a contradiction with the given definition of r. Thus, $r = 0$ and hence $u = v$. Now, we prove by contradiction. Assume on the contrary that f does not have fixed points. Since

$$\frac{1}{2s} d(x_n, fx_n) < d(x_n, fx_n), \text{ for all } n \geq 1,$$

by condition (3.6), we have

$$\begin{aligned} d(fx_n, ffx_n) < \varphi&\left(d(fx_n, fx_n), \frac{d(x_n, fx_n)d(x_n, ffx_n) + d(fx_n, fx_n)d(fx_n, ffx_n)}{d(fx_n, fx_n) + d(x_n, ffx_n)}\right) \\ &+ \alpha_L d(fx_n, fx) \end{aligned}$$

implies

$$d(fx_n, ffx_n) < d(x_n, fx_n), \text{ for all } n \geq 1. \tag{3.13}$$

From

$$d(u, ffx_n) \leq s[d(u, fx_n) + d(fx_n, ffx_n)] \leq s[d(u, fx_n) + d(x_n, fx_n)],$$

as $n \to +\infty$, we have $f^2 x_n \to u$ and $fx_n \to u$. Suppose that there exists $n \geq 1$ such that

$$\frac{1}{2s} d(x_n, fx_n) \geq d(x_n, u) \text{ and } \frac{1}{2s} d(fx_n, ffx_n) \geq d(fx_n, u),$$

then by (3.13), we get

$$\begin{aligned} d(x_n, fx_n) &\leq s[d(x_n, u) + d(u, fx_n)] \\ &\leq s\frac{1}{2s} d(x_n, fx_n) + s\frac{1}{2s} d(fx_n, ffx_n) \\ &\leq \frac{1}{2} d(x_n, fx_n) + \frac{1}{2} d(x_n, fx_n) \\ &= d(x_n, fx_n), \end{aligned}$$

a contradiction. Hence, for every $n \geq 1$, we have

$$\frac{1}{2s}d(x_n, fx_n) < d(x_n, u), \text{ or } \frac{1}{2s}d(fx_n, ffx_n) < d(fx_n, u).$$

By (3.6) for each $n \geq 1$,

$$d(fx_n, fu) < \varphi\left(d(u, fx_n), \frac{d(x_n, fx_n)d(x_n, fu) + d(u, fx_n)d(u, fu)}{d(u, fx_n) + d(x_n, fu)}\right) + \alpha_L d(u, fx_n) \tag{3.14}$$

or

$$\begin{aligned} d(ffx_n, fu) < \varphi &\left(d(u, ffx_n), \frac{d(fx_n, ffx_n)d(fx_n, fu) + d(u, ffx_n)d(u, fu)}{d(u, ffx_n) + d(fx_n, fu)}\right) \\ &+ \alpha_L d(u, ffx_n) \end{aligned} \tag{3.15}$$

(3.14) and(3.15) hold.
Assume that (3.14) holds for every $n \in J \subset \mathbb{N}$. If J is infinite set, then

$$\begin{aligned} d(u, fu) &= \limsup_{n\to+\infty, n\in J} d(fx_n, fu) \\ &\leq \limsup_{n\to+\infty, n\in J} \varphi\left(d(u, fx_n), \frac{d(x_n, fx_n)d(x_n, fu) + d(u, fx_n)d(u, fu)}{d(u, fx_n) + d(x_n, fu)}\right) \\ &\quad + \limsup_{n\to+\infty, n\in J} \alpha_L d(u, fx_n) \\ &\leq 0, \end{aligned}$$

Thus, $u = fu$. The same conclusion satisfies if $\mathbb{N} \setminus J$ represents an infinite set, in this case we use condition (3.15). In (3.14) and (3.15), we have shown that u is a fixed point of f. □

Corollary 3.7. *Let f be a self mapping on X and (X, d, s) be a sequential compact b-metric space. Suppose that*

$$\frac{1}{2s}d(x, fx) < d(x, y)$$

implies

$$d(fx, fy) < \varphi\left(d(x, y), d(y, fx)\right) + \alpha_L d(y, fx) \tag{3.16}$$

for all $x, y \in X$ and d is continuous, then f has a fixed point.

Proof. Clearly, the proof of the corollary follows from Theorem 3.2. □

If we take $\alpha_L d(y, fx)$ in Theorem 3.2 to be $L \min\{d(y, fx), d(x, fx), d(y, fy)\}$ with $L \geq 0$, we have

Corollary 3.8. *Let f be a self mapping on X and (X, d, s) be a sequential compact b-metric space. Suppose that*

$$\frac{1}{2s}d(x, fx) < d(x, y),$$

implies

$$\begin{aligned} d(fx, fy) < \varphi &\left(d(y, fx), \frac{d(x, fx)d(x, fy) + d(y, fx)d(y, fy)}{d(y, fx) + d(x, fy)}\right) \\ &+ L \min\{d(y, fx), d(x, fx), d(y, fy)\} \end{aligned} \tag{3.17}$$

for all $x, y \in X$ and d is continuous, then f has a fixed point.

Proof. The proof of the Corollary follows from Theorem 3.2 if $\alpha_L d(y, fx) = L\min\{d(y, fx), d(x, fx), d(y, fy)\}$ with $L \geq 0$. $\square$

Corollary 3.9. *Let f be a self mapping on X and (X, d, s) be a sequential compact b-metric space. Suppose that*

$$\frac{1}{2s} d(x, fx) < d(x, y),$$

implies

$$d(fx, fy) < \varphi\left(d(x, y), d(y, fx)\right) + L\min\{d(y, fx), d(x, fx), d(y, fy)\} \tag{3.18}$$

for all $x, y \in X$ and d is continuous with $L \geq 0$, then f has a fixed point.

Proof. Clearly the proof of the Corollary follows from Theorem 3.2. $\square$

Corollary 3.10. *Let f be a self mapping on X and (X, d, s) be a sequential compact b-metric space. Suppose that*

$$\frac{1}{2s} d(x, fx) < d(x, y),$$

implies

$$d(fx, fy) < d(x, y) + L\min\{d(y, fx), d(x, fx), d(y, fy)\} \tag{3.19}$$

for all $x, y \in X$ and d is continuous with $L \geq 0$, then f has a fixed point.

4 Application

Ran and Reurings pioneered the study of fixed point results on partially ordered sets in their paper [22], where they explored the applications of these results to the solution of matrix equations. Nieto and Rodrıguez-Lopez continued this research direction in their paper [26], in which they provided several applications to differential equations.

We obtain the subsequent theorems in partially ordered metric spaces through the application of our previously demonstrated results.

Theorem 4.1. *Assume that d is continuous in the ordered b-metric space $(X, d, s, \preccurlyeq)$ and let $f, g : X \to X$ be such that $f(X) \subset g(X)$, $g(X)$ represents a sequentially compact subspace of X, f a dominated mapping and g a dominating mapping. Suppose that*

$$d(fx, fy) < a_1 d(gx, gy) + a_2\frac{d(gx, fx)d(gx, fy) + d(gy, fx)d(gy, fy)}{d(gy, fx) + d(gx, fy)} + a_3\frac{d(gx, fx)d(gy, fy)}{d(gx, gy)} + \frac{a_4}{s} d(gx, fy) + Ld(gy, fx) \tag{4.1}$$

for every comparable elements $x, y \in X$, $gx \neq gy$, where $a_1 + a_2 + a_3 + 2a_4 = 1$, $a_3 \neq 1$, $L \geq 0$ and satisfies the following conditions:
(i) X possesses a sequential limit comparison property, then g and f possesses a coincidence point in X. Additionally,
(ii) If $a_1 + \frac{a_4}{s} + L \leq 1$,
then the points of coincidence of g and f is well ordered if and only if g and f possesses one and only one point of coincidence.

Proof. Let us take any arbitrary point $x_0 \in X$ and let $\{x_n\}$ in X be defined as

$$g_{n+1} = fx_n, \text{ for all } n \geq 0.$$

Since the range of g contains the range of f. If $d(gx_n, gx_{n+1}) = 0$ for some $n \geq 0$, then $gx_n = gx_{n+1} = fx_n$ and so x_n is a coincidence point of f and g. Assume that $d(gx_n, gx_{n+1} > 0$ for all $n \geq 0$. On using the property of the mappings f and g, we have

$$x_{n+1} \preccurlyeq gx_{n+1} = fx_n \preccurlyeq gx_n \text{ for all } n \geq 0.$$

Then x_n and x_{n+1} are comparable for all $n \geq 0$. Since $d(gx_n, gx_{n+1} > 0$, we get that $gx_{n+1} \prec gx_n$, for all $n \geq 0$. Thus $\{gx_n\}$ is a decreasing sequence. Using the hypothesis that $g(X)$ is a sequentially compact subspace of X, we can assume that $gx_n \to gu$ for some $u \in X$. Now, condition (i) guarantees that $gu \prec gx_n$, for all $n \geq 0$. Now, we prove that $fu = gu$. We have

$$\begin{aligned} d(gu, fu) &= \lim_{n\to+\infty} d(gx_{n+1}, fu) = \lim_{n\to+\infty} d(fx_n, fu) \\ &\leq \lim_{n\to+\infty} \Bigg[a_1 d(gx_n, gu) + a_2 \frac{d(gx_n, fx_n)d(gx_n, fu) + d(gu, fx_n)d(gu, fu)}{d(gu, fx_n) + d(gx_n, fu)} \\ &\quad + a_3 \frac{d(gx_n, fx_n)d(gu, fu)}{d(gx_n, gu)} + \frac{a_4}{s} d(gx_n, fu) + Ld(gu, fx_n)\Bigg] \\ &= a_3 d(gy, fu) + \frac{a_4}{s} d(gu, fu) \\ &= \left(a_3 + \frac{a_4}{s}\right) d(gu, fu) \\ &< d(gu, fu) \end{aligned}$$

a contradiction. That is, $d(gu, fu) = 0$ and hence $fu = gu$. Therefore, u is a coincidence point of f and g. Now, suppose that the set of points of coincidence of f and g is well ordered. We claim that the point of coincidence of f and g is unique. Assume on the contrary that there exists another point v in X such that $fv = gv$ with $gu \neq gv$. Assume that $gu \prec gv$, then $u \preccurlyeq gu \prec gv = fv \preccurlyeq v$ and u, v are comparable. Now, using the condition (4.1), we get

$$\begin{aligned} d(fu, fv) &< a_1 d(gu, gv) + a_2 \frac{d(gu, fu)d(gu, fv) + d(gv, fu)d(gv, fv)}{d(gv, fu) + d(gu, fv)} \\ &\quad + a_3 \frac{d(gu, fu)d(gv, fv)}{d(gu, gv)} + \frac{a_4}{s} d(gu, fv) + Ld(gv, fu) \\ &= \left(a_1 + \frac{a_4}{s} + L\right) d(fu, fv) \\ &\leq d(fu, fv), \end{aligned}$$

a contradiction and hence $gu = gv$. The same holds if $gv \prec gu$. Therefore $fu = gu = z$ is the unique point of coincidence of f and g in X. Conversely, if f and g have one and only one point of coincidence, then the set of points of coincidence of f and g being singleton is well ordered. □

Theorem 4.2. *Consider all the hypotheses of Theorem 4.1 with the following assertions:*
(ii) If $\{gx_n\}$ possess a decreasing sequence that converges to gu for some $u \in X$, then $ggu \preccurlyeq gu$;
(iii) g and f possess a weakly compatible;
then g and f possesses a common fixed point in X.
Additionally, g and f possesses a unique common fixed point in X if coincidence of g and f is well ordered.

Proof. Let us take any arbitrary point $x_0 \in X$ and let $\{x_n\}$ in X be defined as

$$gx_{n+1} = fx_n \text{ for all } n \geq 0.$$

Continuing as in the proof of Theorem 4.1, we deduce that $\{gx_n\}$ is a decreasing sequence that converges to gu for some $u \in X$ and $gu = fu = z$. Using condition (ii), we have $gz \preccurlyeq gu$. Since, the mappings f and g are weakly compatible we obtain that $fz = fgu = gfu = gz$. If $gz = gu = z$, then z is a common fixed point of f and g. If $gz \prec gu$, then u, z are comparable and using the condition (4.1), we get $gz = gu$. So z is a common fixed point of f and g. If the set of points of coincidence of f and g is well ordered, then f and g have a unique point of coincidence and so z is a unique common fixed point of f and g. □

Corollary 4.3. *Assume that d is continuous in the ordered b-metric space $(X, d, s, \preccurlyeq)$ and let $f : X \to X$ be such that $f(X) \subset X$ possess a sequentially compact subspace of X, f a dominated mapping. Suppose that*

$$d(fx, fy) < a_1 d(x, y) + a_2 \frac{d(x, fx)d(x, fy) + d(y, fx)d(y, fy)}{d(y, fx) + d(x, fy)} + a_3 \frac{d(x, fx)d(y, fy)}{d(x, y)} + \frac{a_4}{s} d(x, fy) + Ld(y, fx) \tag{4.2}$$

for every comparable elements $x, y \in X$, $x \neq y$, where $a_1 + a_2 + a_3 + 2a_4 = 1$, $a_3 \neq 1$, $L \geq 0$ and satisfies the following conditions:
(i) X possess a sequential limit comparison property,
then f possesses a fixed point in X.
Additionally,
(ii) If $a_1 + \frac{a_4}{s} + L \leq 1$,
then f possesses a unique fixed point.

5 Conclusion

The main findings of this study demonstrate applicability of sequential compact b-metric spaces in establishing fixed point theorems for Eldeisten-Suzuki-type contraction mappings. This study provides significant advancements in the understanding of sequential compact b-metric spaces, with potential applications in differential equations and nonlinear integral equation.

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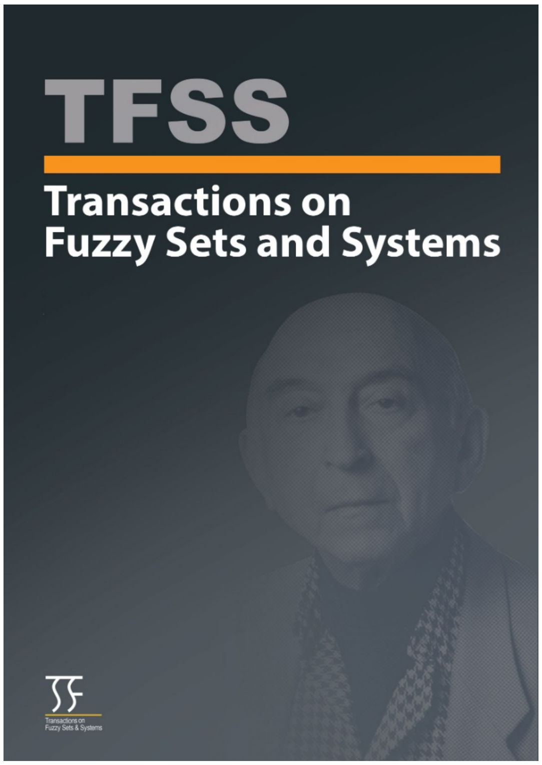

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Fuzzy Metric Spaces and Corresponding Fixed Point Theorems for Fuzzy Type Contraction

Musa Adeku Ibrahim, **Muhammed Raji***, **Kamilu Rauf**

Abstract. In this paper, we present innovative concepts of fuzzy type contractions and leverage them to establish fixed point theorems for fuzzy mappings within the framework of fuzzy metric spaces. The results of this article are applied to multivalued mappings and fuzzy mappings for contractive fuzzy type mappings. Through illustrative examples, we showcase the practical applicability of our proposed notions and results, demonstrating their effectiveness in real-world scenarios.

AMS Subject Classification 2020: 47H10; 54H25; 46S40
Keywords and Phrases: Fixed point, Fuzzy fixed point, Contractive type mapping, Hausdorff fuzzy metric space, fuzzy mapping.

1 Introduction

Throughout the years, Banach's fixed point theorems for contraction mappings have emerged as pivotal findings in the realm of mathematical analysis. These results, particularly Banach's contraction principle [1], have greatly contributed to the evolution of metric fixed point theory. By offering a reliable framework, this principle and its variations serve as invaluable tools in ensuring both the existence and uniqueness of solutions to nonlinear problems, including integral equations, differential equations, variational inequalities, and optimization problems. Numerous mathematicians have dedicated extensive efforts to refine and broaden this principle from various angles. Below, we delve into some of these noteworthy contributions. By reducing the triangle inequality constraint of the standard metric spaces, Czerwik [2] established the idea of b-metric spaces. The fixed-point properties of set-valued operators in b-metric spaces were then examined by Boriceanu [3], who also gave some specific instances of b-metric spaces. The notion of dislocated b-metric space, which is a generalization of b-metric spaces, was further developed by Hussain et al. [4]. They also proved certain fixed-point findings for four mappings that meet the generalized weak contractive conditions in a partially ordered dislocated b-metric space. The idea of fuzzy sets was first introduced by Zadeh [5], who also laid the groundwork for further studies in fuzzy mathematics. Weiss [6] explored fuzzy mappings and obtained multiple fixed point findings, expanding on Zadeh's work. Heilpern [7] introduced the idea of fuzzy contraction mappings, which is a further development of fuzzy mappings. Similar to Nadler's [8] fixed point theorem for multivalued mappings, he established a fixed point theorem for fuzzy contraction mappings. Later, in order to establish some common fixed point results for fuzzy mappings obeying a new rational F-contraction of Ciric type, Shahzad et al. [9] introduced the concept of F-contractions. The presence of fuzzy fixed

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points of set-valued fuzzy mappings in metric and fuzzy metric spaces that meet the Ciric type contraction in complete metric spaces was recently studied by Kanwal et al. [10]. In this direction several authors obtained further results, some of which can be found in [11, 12, 13, 14, 15, 16] . Considering the insights, we aim to introduce novel concepts of fuzzy type contractions and subsequently establish fixed point results for fuzzy mappings within the framework of fuzzy metric spaces. To bolster our findings, we offer illustrative examples demonstrating the practical application of the presented results and concepts. In addition, we present applications of our main results to multivalued mappings and fuzzy mappings.
Throughout our discourse, we let $CB(X)$ denote the family of all closed and bounded subsets of the metric space (X, d).

2 Preliminaries

In this section, we will introduce some definitions and lemmas that will be used in the rest of this work.

Definition 2.1. [17, 18] A function with X as its domain and the interval $[0, 1]$ as its range is called a fuzzy set in X. $F(X)$ represents the set of all fuzzy sets in X. The degree of membership of x in A is denoted by the value $A(x)$, given a fuzzy set A and a point x in X. A fuzzy set A's α-level set is represented by $[A]_\alpha$ and has the following definition:

$$[A]_\alpha = \{x : A(x) \geq \alpha\} \ \ where\ \alpha \in (0, 1), [A]_0 = \{x : A(x) > 0\}$$

Definition 2.2. [19, 20] Let Y be a metric space and X a nonempty set. If a mapping T is a mapping from X into $F(Y)$, the set of all fuzzy sets on Y, then it is referred to as a fuzzy mapping. The degree to which y is a member of $T(x)$ is the membership function of a fuzzy mapping T, represented as $T(x)(y)$. Stated differently, $T(x)(y)$ represents y's degree of membership in the fuzzy set $T(x)$. Instead of using $[T(x)]_\alpha$ to denote the α-level set of $T(x)$, we will simply use $[Tx]_\alpha$.

Definition 2.3. [21, 22, 23] A fuzzy fixed point of a fuzzy mapping $T : X \to F(X)$ is defined as a point $x \in X$ where $\alpha \in (0, 1]$ and $x \in [Tx]_\alpha$.

Definition 2.4. [24] Let (X, d) be a metric space. Hausdorff metric H on $CB(X)$ induced by d is defined as

$$H(A, B) = max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\}, for all A, B \in CB(X),$$

where $d(a, B) = \inf\{d(a, b) : b \in B\}$.

Lemma 2.5. [25] *Assume that A and B are bounded, nonempty subsets of a metric space (X, d). If $a \in A$, then*

$$d(a, B) \leq H(A, B)$$

Lemma 2.6. [25] *Assume that A and B are bounded, nonempty subsets of a metric space (X, d) and $0 < \sigma \in R$. Then for $a \in A$, there exists $b \in B$ such that*

$$d(a, b) \leq H(A, B) + \sigma.$$

Lemma 2.7. [10] *If $A, B \in CB(X)$ with $H(A, B) < \varepsilon$, then for all $a \in A$, there exists $b \in B$ such that*

$$d(a, b) < \varepsilon.$$

Lemma 2.8. [10] *Let $\mu \in X$ and $A \in CB(X)$, $d(\mu, A) \leq d(\mu, v)$ for all $v \in A$.*

3 Main Results

Here are the definitions that we use to start this section.

Definition 3.1. Let $T : X \to F(X)$ be a fuzzy mapping and $(X,\ d)$ be a complete metric space. Assume that $\alpha(x) \in (0,1]$, and that the closed, bounded subsets of X are $[Tx]_{\alpha(x)}$ and $[Ty]_{\alpha(y)}$, respectively, non-empty. Then, T is considered fuzzy type I if it satisfies the following requirement for all x, y in X and $a_1, a_2, a_3, a_4 \geq 0$ with $a_1 + 2a_2 + a_3 + a_4 < 1$:

$$H([Tx]_{\alpha(x)}, [Ty]_{\alpha(y)}) \leq a_1 d(x,y) + a_2[d(x,[Tx]_{\alpha(x)}) + d(y,[Ty]_{\alpha(y)})] + a_3 \frac{d(x,[Tx]_{\alpha(x)})d(y,[Ty]_{\alpha(y)})}{d(x,y) + d(x,[Ty]_{\alpha(y)}) + d(y,[Tx]_{\alpha(x)})}$$
$$+ a_4 \frac{d(x,[Tx]_{\alpha(x)})d(x,[Ty]_{\alpha(y)}) + d(y,[Tx]_{\alpha(x)})d(y,[Ty]_{\alpha(y)})}{d(x,[Ty]_{\alpha(y)}) + d(y,[Tx]_{\alpha(x)})} \tag{3.1}$$

Definition 3.2. Let $T : X \to F(X)$ be a fuzzy mapping and $(X,\ d)$ be a complete metric space. Assume that $\alpha(x) \in (0,1]$, and that the closed, bounded subsets of X are $[Tx]_{\alpha(x)}$ and $[Ty]_{\alpha(y)}$, respectively, non-empty. Then, T is considered fuzzy type II if it satisfies the following requirement for all x, y in X and $a_1, a_2, a_3, a_4 \geq 0$ with $a_1 + 2a_2 + a_3 + a_4 < 1$:

$$H([Tx]_{\alpha(x)}, [Ty]_{\alpha(y)}) \leq a_1 d(x,y) + a_2[d(x,[Ty]_{\alpha(y)}) + d(y,[Tx]_{\alpha(x)})] + a_3 \frac{d(x,[Tx]_{\alpha(x)})d(y,[Ty]_{\alpha(y)})}{d(x,y) + d(x,[Ty]_{\alpha(y)}) + d(y,[Tx]_{\alpha(x)})}$$
$$+ a_4 \frac{d(x,[Tx]_{\alpha(x)})d(x,[Ty]_{\alpha(y)}) + d(y,[Tx]_{\alpha(x)})d(y,[Ty]_{\alpha(y)})}{d(x,[Ty]_{\alpha(y)}) + d(y,[Tx]_{\alpha(x)})} \tag{3.2}$$

Theorem 3.3. *Let $(X,\ d)$ be a complete metric space and $T : X \to F(X)$ be a fuzzy type I contraction mapping. Then, T has a fixed point in X.*

Proof. Let $x_0 \in X$ be any arbitrary point in X and $[Tx_0]_{\alpha(x_0)} \neq 0$ be a closed and bounded subsets of X. Let $x_1 \in [Tx_0]_{\alpha(x_0)}$. Since $[Tx_1]_{\alpha(x_1)} \neq 0$ is a closed and bounded subsets of X and by using Lemma 2.6, there exists $x_2 \in [Tx_1]_{\alpha(x_1)}$ such that

$$d(x_1, x_2) \leq H([Tx_0]_{\alpha(x_0)}, [Tx_1]_{\alpha(x_1)}) + \sigma \tag{3.3}$$

Again, $[Tx_0]_{\alpha(x_0)} \neq 0$ is a closed and bounded subsets of X and by using Lemma 2.6, there exists $x_3 \in [Tx_2]_{\alpha(x_2)}$ such that

$$d(x_2, x_3) \leq H([Tx_1]_{\alpha(x_1)}, [Tx_2]_{\alpha(x_2)}) + \sigma^2 \tag{3.4}$$

Continuing in this manner, we create a sequence x_n of points in X such that $x_n \in [Tx_{n-1}]_{\alpha(x_{n-1})}$, we can choose $x_{n+1} \in [Tx_{n-1}]_{\alpha(x_{n-1})}$ such that

$$d(x_n, x_{n+1}) \leq H([Tx_{n-1}]_{\alpha(x_{n-1})}, [Tx_n]_{\alpha(x_n)}) + \sigma n. \tag{3.5}$$

Now, from (3.3),

$$d(x_1, x_2) \leq H([Tx_0]_{\alpha(x_0)}, [Tx_1]_{\alpha(x_1)}) + \sigma,$$

using (3.1), we get

$$d(x_1, x_2) \leq H([Tx_0]_{\alpha(x_0)}, [Tx_1]_{\alpha(x_1)}) + \sigma,$$

$$\leq a_1 d(x_0, x_1) + a_2[d(x_0,[Tx_0]_{\alpha(x_0)}) + d(x_1,[Tx_1]_{\alpha(x_1)})] + a_3 \frac{d(x_0,[Tx_0]_{\alpha(x_0)})d(x_1,[Tx_1]_{\alpha(x_1)})}{d(x_0,x_1) + d(x_0,[Tx_1]_{\alpha(x_1)}) + d(x_1,[Tx_0]_{\alpha(x_0)})}$$

$$+ a_4 \frac{d(x_0,[Tx_0]_{\alpha(x_0)})d(x_0,[Tx_1]_{\alpha(x_1)}) + d(x_1,[Tx_0]_{\alpha(x_0)})d(x_1,[Tx_1]_{\alpha(x_1)})}{d(x_0,[Tx_1]_{\alpha(x_1)}) + d(x_1,[Tx_0]_{\alpha(x_0)})} + \sigma \tag{3.6}$$

$$d(x_1,x_2) \leq a_1 d(x_0,x_1) + a_2[d(x_0,x_1) + d(x_1,x_2)] + a_3 d(x_0,x_1) + a_4 d(x_0,x_1) + \sigma$$

$$d(x_1,x_2) \leq \frac{a_1+a_2+a_3+a_4}{1-a_2} d(x_0,x_1) + \frac{\sigma}{1-a_2}. \tag{3.7}$$

Let $\sigma = \frac{a_1+a_2+a_3+a_4}{1-a_2}$. Since $a_1 + 2a_2 + a_3 + a_4 < 1$ implies that $\frac{a_1+a_2+a_3+a_4}{1-a_2} < 1$. Hence,

$$d(x_1,x_2) \leq \sigma\, d(x_0,x_1) + \frac{\sigma}{1-a_2} \ \ for\ \ all\ \ n \in N. \tag{3.8}$$

Now, from (3.4),

$$d(x_2,x_3) \leq H([Tx_1]_{\alpha(x_1)},[Tx_2]_{\alpha(x_2)}) + \sigma^2,$$

using (3.1), we get

$$d(x_2,x_3) \leq H([Tx_1]_{\alpha(x_1)},[Tx_2]_{\alpha(x_2)}) + \sigma^2,$$

$$\leq a_1 d(x_1,x_1) + a_2[d(x_1,[Tx_1]_{\alpha(x_1)}) + d(x_2,[Tx_2]_{\alpha(x_2)})] + a_3 \frac{d(x_1,[Tx_1]_{\alpha(x_1)})d(x_2,[Tx_2]_{\alpha(x_2)})}{d(x_1,x_2) + d(x_1,[Tx_2]_{\alpha(x_2)}) + d(x_2,[Tx_1]_{\alpha(x_1)})}$$

$$+ a_4 \frac{d(x_1,[Tx_1]_{\alpha(x_1)})d(x_1,[Tx_2]_{\alpha(x_2)}) + d(x_2,[Tx_1]_{\alpha(x_1)})d(x_2,[Tx_2]_{\alpha(x_2)})}{d(x_1,[Tx_2]_{\alpha(x_2)}) + d(x_2,[Tx_1]_{\alpha(x_1)})} + \sigma^2 \tag{3.9}$$

$$d(x_2,x_3) \leq a_1 d(x_1,x_2) + a_2[d(x_1,x_2) + d(x_2,x_3)] + a_3 d(x_1,x_2) + a_4 d(x_1,x_2) + \sigma^2$$

$$d(x_2,x_3) \leq \frac{a_1+a_2+a_3+a_4}{1-a_2} d(x_1,x_2) + \frac{\sigma^2}{1-a_2}. \tag{3.10}$$

Let $\sigma = \frac{a_1+a_2+a_3+a_4}{1-a_2}$. Since $a_1 + 2a_2 + a_3 + a_4 < 1$ implies that $\frac{a_1+a_2+a_3+a_4}{1-a_2} < 1$. Hence,

$$d(x_2,x_3) \leq \sigma\, d(x_1,x_2) + \frac{\sigma^2}{1-a_2} \ \ for\ \ all\ \ n \in N. \tag{3.11}$$

$$d(x_2,x_3) \leq \sigma\, [\sigma\, d(x_0,x_1) + \frac{\sigma}{1-a_2}\ \ ] + \frac{\sigma^2}{1-a_2}$$

$$d(x_2,x_3) \leq \sigma^2 d(x_0,x_1) + \frac{\sigma^2}{1-a_2} + \frac{\sigma^2}{1-a_2}$$

$$d(x_2,x_3) \leq \sigma^2 d(x_0,x_1) + \frac{2\sigma^2}{1-a_2} \tag{3.12}$$

Now,

$$d(x_3,x_4) \leq H([Tx_2]_{\alpha(x_2)},[Tx_3]_{\alpha(x_3)}) + \sigma^3,$$

using (3.1), we get

$$d(x_3,x_4) \leq H([Tx_2]_{\alpha(x_2)},[Tx_3]_{\alpha(x_3)}) + \sigma^3,$$

$$\leq a_1 d(x_2,x_2) + a_2[d(x_2,[Tx_2]_{\alpha(x_2)}) + d(x_3,[Tx_3]_{\alpha(x_3)})] + a_3 \frac{d(x_2,[Tx_2]_{\alpha(x_2)})d(x_3,[Tx_3]_{\alpha(x_3)})}{d(x_2,x_3) + d(x_2,[Tx_3]_{\alpha(x_3)}) + d(x_3,[Tx_2]_{\alpha(x_2)})}$$

$$+ a_4 \frac{d(x_2,[Tx_2]_{\alpha(x_2)})d(x_2,[Tx_3]_{\alpha(x_3)}) + d(x_3,[Tx_2]_{\alpha(x_2)})d(x_3,[Tx_3]_{\alpha(x_3)})}{d(x_2,[Tx_3]_{\alpha(x_3)}) + d(x_3,[Tx_2]_{\alpha(x_2)})} + \sigma^3 \tag{3.13}$$

$$d(x_3, x_4) \leq a_1 d(x_2, x_3) + a_2[d(x_2, x_3) + d(x_3, x_4)] + a_3 d(x_2, x_3) + a_4 d(x_2, x_3) + \sigma^3$$

$$d(x_3, x_4) \leq \frac{a_1 + a_2 + a_3 + a_4}{1 - a_2} d(x_2, x_3) + \frac{\sigma^3}{1 - a_2}. \tag{3.14}$$

Let $\sigma = \frac{a_1+a_2+a_3+a_4}{1-a_2}$. Since $a_1 + 2a_2 + a_3 + a_4 < 1$ implies that $\frac{a_1+a_2+a_3+a_4}{1-a_2} < 1$. Hence,

$$d(x_3, x_4) \leq \sigma\, d(x_2, x_3) + \frac{\sigma^3}{1 - a_2} \;\; for \;\; all \;\; n \in N. \tag{3.15}$$

$$d(x_3, x_4) \leq \sigma\, [\sigma^2 d(x_0, x_1) + \frac{2\sigma^2}{1 - a_2} \;] + \frac{\sigma^3}{1 - a_2}$$

$$d(x_3, x_4) \leq \sigma^3 d(x_0, x_1) + \frac{2\sigma^3}{1 - a_2} + \frac{\sigma^3}{1 - a_2}$$

$$d(x_3, x_4) \leq \sigma^3 d(x_0, x_1) + \frac{3\sigma^3}{1 - a_2} \tag{3.16}$$

Again, continuing in this fashion, we have

$$d(x_n, x_{n+1}) \leq \sigma^n d(x_0, x_1) + \frac{n\sigma^n}{1 - a_2} \tag{3.17}$$

If $n > m$ and $n, m \in N$, then we have

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m). \tag{3.18}$$

Applying (3.17) in (3.18) , we get

$$d(x_n, x_m) \leq \sigma^n d(x_0, x_1) + \frac{n\sigma^n}{1 - a_2} + \sigma^{n+1} d(x_0, x_1) + \frac{(n+1)\sigma^{n+1}}{1 - a_2} + \cdots + \sigma^{m-1} d(x_0, x_1) + \frac{(m-1)\sigma^{m-1}}{1 - a_2}$$

$$d(x_n, x_m) \leq \sigma^n d(x_0, x_1)(1 + \sigma + \sigma^2 + \sigma^3 + \cdots + \sigma^{m-n-1}) + \sum_{i=n}^{m-1} \frac{i\sigma^i}{1 - a_2}$$

$$d(x_n, x_m) \leq \sigma^n d(x_0, x_1)(\frac{1 - \sigma^{m-n}}{1 - \sigma}) + \sum_{i=n}^{m-1} \frac{i\sigma^i}{1 - a_2} \tag{3.19}$$

On taking $m, n \to \infty$ in (3.19), we get

$$d(x_n, x_m) = 0. \tag{3.20}$$

This proves that the sequence $\{x_n\}$ is a Cauchy sequence in X. Since X is a complete metric space, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Now,

$$d(x^*, [Tx^*]_{\alpha(x^*)}) \leq [d(x^*, x_n) + d(x_n, [Tx^*]_{\alpha(x^*)})],$$

using (3.1), we get

$$d(x^*, [Tx^*]_{\alpha(x^*)}) \leq d(x^*, x_n) + a_1 d(x_{n-1}, x^*) + a_2[d(x_{n-1}, [Tx_{n-1}]_{\alpha(x_{n-1})}) + d(x^*, [Tx^*]_{\alpha(x^*)})] +$$
$$a_3 \frac{d(x_{n-1}, [Tx_{n-1}]_{\alpha(x_{n-1})}) d(x^*, [Tx^*]_{\alpha(x^*)})}{d(x_{n-1}, x^*) + d(x_{n-1}, [Tx^*]_{\alpha(x^*)}) + d(x^*, [Tx_{n-1}]_{\alpha(x_{n-1})})} +$$

$$a_4\frac{d(x_{n-1},[Tx_{n-1}]_{\alpha(x_{n-1})})d(x_{n-1},[Tx^*]_{\alpha(x^*)})+d(x^*,[Tx_{n-1}]_{\alpha(x_{n-1})})d(x^*,[Tx^*]_{\alpha(x^*)})}{d(x_{n-1},[Tx^*]_{\alpha(x^*)})+d(x^*,[Tx_{n-1}]_{\alpha(x_{n-1})})}$$

$$\begin{aligned} d(x^*,[Tx^*]_{\alpha(x^*)}) \le d(x^*,x_n)+a_1d(x_{n-1},x^*)+a_2[d(x_{n-1},x_n)+d(x^*,[Tx^*]_{\alpha(x^*)})]+\\ a_3\frac{d(x_{n-1},x_n)d(x^*,[Tx^*]_{\alpha(x^*)})}{d(x_{n-1},x^*)+d(x_{n-1},[Tx^*]_{\alpha(x^*)})+d(x^*,x_n)}+\\ a_4\frac{d(x_{n-1},x_n)d(x_{n-1},[Tx^*]_{\alpha(x^*)})+d(x^*,x_n)d(x^*,[Tx^*]_{\alpha(x^*)})}{d(x_{n-1},[Tx^*]_{\alpha(x^*)})+d(x^*,x_n)} \end{aligned} \tag{3.21}$$

On taking $n\to\infty$ in (3.21), we get

$$\begin{aligned} d(x^*,[Tx^*]_{\alpha(x^*)}) \le d(x^*,x^*)+a_1d(x^*,x^*)+a_2[d(x^*,x^*)+d(x^*,[Tx^*]_{\alpha(x^*)})]+\\ a_3\frac{d(x^*,x^*)d(x^*,[Tx^*]_{\alpha(x^*)})}{d(x^*,x^*)+d(x^*,[Tx^*]_{\alpha(x^*)})+d(x^*,x^*)}+\\ a_4\frac{d(x^*,x^*)d(x^*,[Tx^*]_{\alpha(x^*)})+d(x^*,x^*)d(x^*,[Tx^*]_{\alpha(x^*)})}{d(x^*,[Tx^*]_{\alpha(x^*)})+d(x^*,x^*)} \end{aligned}$$

implies

$$(1-a_2)d(x^*,[Tx^*]_{\alpha(x^*)})\le 0 \tag{3.22}$$

Since $a_1+2a_2+a_3+a_4<1$ implies $a_1+a_2+a_3+a_4<1-a_2$, that is, $1-a_2\neq 0$. Hence,

$$d(x^*,[Tx^*]_{\alpha(x^*)})=0.$$

Implies

$$x^*\in[Tx^*]_{\alpha(x^*)}.$$

Thus, $x^*\in X$ is the fixed point. □

Theorem 3.4. *Let $(X,\ d)$ be a complete metric space and $T:X\to F(X)$ be a fuzzy type II contraction mapping. Then, T has a fixed point in X.*

Proof. Let $x_0\in X$ be any arbitrary point in X and $[Tx_0]_{\alpha(x_0)}\neq 0$ be a closed and bounded subsets of X. Let $x_1\in[Tx_0]_{\alpha(x_0)}$. Since $[Tx_1]_{\alpha(x_1)}\neq 0$ is a closed and bounded subsets of X and by using Lemma 2.6, there exists $x_2\in[Tx_1]_{\alpha(x_1)}$ such that

$$d(x_1,x_2)\le H([Tx_0]_{\alpha(x_0)},[Tx_1]_{\alpha(x_1)})+\sigma. \tag{3.23}$$

Again, $[Tx_0]_{\alpha(x_0)}\neq 0$ is a closed and bounded subsets of X and by using Lemma 2.6, there exists $x_3\in[Tx_2]_{\alpha(x_2)}$ such that

$$d(x_2,x_3)\le H([Tx_1]_{\alpha(x_1)},[Tx_2]_{\alpha(x_2)})+\sigma^2 \tag{3.24}$$

Continuing in this manner, we create a sequence x_n of points in X such that $x_n\in[Tx_{n-1}]_{\alpha(x_{n-1})}$, we can choose $x_{n+1}\in[Tx_{n-1}]_{\alpha(x_{n-1})}$ such that

$$d(x_n,x_{n+1})\le H([Tx_{n-1}]_{\alpha(x_{n-1})},[Tx_n]_{\alpha(x_n)})+\sigma^n. \tag{3.25}$$

Now, from (3.23),

$$d(x_1,x_2)\le H([Tx_0]_{\alpha(x_0)},[Tx_1]_{\alpha(x_1)})+\sigma,$$

using (3.2), we get

$$d(x_1, x_2) \leq H([Tx_0]_{\alpha(x_0)}, [Tx_1]_{\alpha(x_1)}) + \sigma,$$

$$\leq a_1 d(x_0, x_1) + a_2[d(x_0, [Tx_1]_{\alpha(x_1)}) + d(x_1, [Tx_0]_{\alpha(x_0)})] + a_3 \frac{d(x_0, [Tx_0]_{\alpha(x_0)})d(x_1, [Tx_1]_{\alpha(x_1)})}{d(x_0, x_1) + d(x_0, [Tx_1]_{\alpha(x_1)}) + d(x_1, [Tx_0]_{\alpha(x_0)})}$$

$$+ a_4 \frac{d(x_0, [Tx_0]_{\alpha(x_0)})d(x_0, [Tx_1]_{\alpha(x_1)}) + d(x_1, [Tx_0]_{\alpha(x_0)})d(x_1, [Tx_1]_{\alpha(x_1)})}{d(x_0, [Tx_1]_{\alpha(x_1)}) + d(x_1, [Tx_0]_{\alpha(x_0)})} + \sigma \tag{3.26}$$

$$d(x_1, x_2) \leq a_1 d(x_0, x_1) + a_2[d(x_0, x_2) + d(x_1, x_1)] + a_3 d(x_0, x_1) + a_4 d(x_0, x_1) + \sigma.$$

By triangular inequality, we have

$$d(x_1, x_2) \leq \frac{a_1 + a_2 + a_3 + a_4}{1 - a_2} d(x_0, x_1) + \frac{\sigma}{1 - a_2}. \tag{3.27}$$

Let $\sigma = \frac{a_1+a_2+a_3+a_4}{1-a_2}$. Since $a_1 + 2a_2 + a_3 + a_4 < 1$ implies that $\frac{a_1+a_2+a_3+a_4}{1-a_2} < 1$. Hence,

$$d(x_1, x_2) \leq \sigma\, d(x_0, x_1) + \frac{\sigma}{1 - a_2} \;\; for \;\; all \;\; n \in N. \tag{3.28}$$

Now, from (3.24),

$$d(x_2, x_3) \leq H([Tx_1]_{\alpha(x_1)}, [Tx_2]_{\alpha(x_2)}) + \sigma^2,$$

using (3.2), we get

$$d(x_2, x_3) \leq H([Tx_1]_{\alpha(x_1)}, [Tx_2]_{\alpha(x_2)}) + \sigma^2,$$

$$\leq a_1 d(x_1, x_1) + a_2[d(x_1, [Tx_2]_{\alpha(x_2)}) + d(x_2, [Tx_1]_{\alpha(x_1)})] + a_3 \frac{d(x_1, [Tx_1]_{\alpha(x_1)})d(x_2, [Tx_2]_{\alpha(x_2)})}{d(x_1, x_2) + d(x_1, [Tx_2]_{\alpha(x_2)}) + d(x_2, [Tx_1]_{\alpha(x_1)})}$$

$$+ a_4 \frac{d(x_1, [Tx_1]_{\alpha(x_1)})d(x_1, [Tx_2]_{\alpha(x_2)}) + d(x_2, [Tx_1]_{\alpha(x_1)})d(x_2, [Tx_2]_{\alpha(x_2)})}{d(x_1, [Tx_2]_{\alpha(x_2)}) + d(x_2, [Tx_1]_{\alpha(x_1)})} + \sigma^2 \tag{3.29}$$

$$d(x_2, x_3) \leq a_1 d(x_1, x_2) + a_2[d(x_1, x_3) + d(x_2, x_2)] + a_3 d(x_1, x_2) + a_4 d(x_1, x_2) + \sigma^2$$

Again, by triangular inequality, we obtain

$$d(x_2, x_3) \leq \frac{a_1 + a_2 + a_3 + a_4}{1 - a_2} d(x_1, x_2) + \frac{\sigma^2}{1 - a_2}. \tag{3.30}$$

Let $\sigma = \frac{a_1+a_2+a_3+a_4}{1-a_2}$. Since $a_1 + 2a_2 + a_3 + a_4 < 1$ implies that $\frac{a_1+a_2+a_3+a_4}{1-a_2} < 1$. Hence,

$$d(x_2, x_3) \leq \sigma\, d(x_1, x_2) + \frac{\sigma^2}{1 - a_2} \;\; for \;\; all \;\; n \in N. \tag{3.31}$$

$$d(x_2, x_3) \leq \sigma\, [\sigma\, d(x_0, x_1) + \frac{\sigma}{1 - a_2} \;] + \frac{\sigma^2}{1 - a_2}$$

$$d(x_2, x_3) \leq \sigma^2 d(x_0, x_1) + \frac{\sigma^2}{1 - a_2} + \frac{\sigma^2}{1 - a_2}$$

$$d(x_2, x_3) \leq \sigma^2 d(x_0, x_1) + \frac{2\sigma^2}{1 - a_2} \tag{3.32}$$

Now,

$$d(x_3, x_4) \leq H([Tx_2]_{\alpha(x_2)}, [Tx_3]_{\alpha(x_3)}) + \sigma^3,$$

using (3.2), we get

$$d(x_3, x_4) \leq H([Tx_2]_{\alpha(x_2)}, [Tx_3]_{\alpha(x_3)}) + \sigma^3,$$

$$\leq a_1 d(x_2, x_2) + a_2[d(x_2, [Tx_3]_{\alpha(x_3)}) + d(x_3, [Tx_2]_{\alpha(x_2)})] + a_3 \frac{d(x_2, [Tx_2]_{\alpha(x_2)}) d(x_3, [Tx_3]_{\alpha(x_3)})}{d(x_2, x_3) + d(x_2, [Tx_3]_{\alpha(x_3)}) + d(x_3, [Tx_2]_{\alpha(x_2)})}$$

$$+ a_4 \frac{d(x_2, [Tx_2]_{\alpha(x_2)}) d(x_2, [Tx_3]_{\alpha(x_3)}) + d(x_3, [Tx_2]_{\alpha(x_2)}) d(x_3, [Tx_3]_{\alpha(x_3)})}{d(x_2, [Tx_3]_{\alpha(x_3)}) + d(x_3, [Tx_2]_{\alpha(x_2)})} + \sigma^3 \tag{3.33}$$

$$d(x_3, x_4) \leq a_1 d(x_2, x_3) + a_2[d(x_2, x_4) + d(x_3, x_3)] + a_3 d(x_2, x_3) + a_4 d(x_2, x_3) + \sigma^3$$

By triangular inequality, we get

$$d(x_3, x_4) \leq \frac{a_1 + a_2 + a_3 + a_4}{1 - a_2} d(x_2, x_3) + \frac{\sigma^3}{1 - a_2}. \tag{3.34}$$

Let $\sigma = \frac{a_1+a_2+a_3+a_4}{1-a_2}$. Since $a_1 + 2a_2 + a_3 + a_4 < 1$ implies that $\frac{a_1+a_2+a_3+a_4}{1-a_2} < 1$. Hence,

$$d(x_3, x_4) \leq \sigma\, d(x_2, x_3) + \frac{\sigma^3}{1 - a_2} \;\; for \;\; all \;\; n \in N. \tag{3.35}$$

$$d(x_3, x_4) \leq \sigma\, [\sigma^2 d(x_0, x_1) + \frac{2\sigma^2}{1 - a_2}\;\;] + \frac{\sigma^3}{1 - a_2}$$

$$d(x_3, x_4) \leq \sigma^3 d(x_0, x_1) + \frac{2\sigma^3}{1 - a_2} + \frac{\sigma^3}{1 - a_2}$$

$$d(x_3, x_4) \leq \sigma^3 d(x_0, x_1) + \frac{3\sigma^3}{1 - a_2} \tag{3.36}$$

Again, continuing in this fashion, we have

$$d(x_n, x_{n+1}) \leq \sigma^n d(x_0, x_1) + \frac{n\sigma^n}{1 - a_2} \tag{3.37}$$

If $n > m$ and $n, m \in N$, then we have

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \tag{3.38}$$

Applying (3.37) in (3.38) , we get

$$d(x_n, x_m) \leq \sigma^n d(x_0, x_1) + \frac{n\sigma^n}{1 - a_2} + \sigma^{n+1} d(x_0, x_1) + \frac{(n+1)\sigma^{n+1}}{1 - a_2} + \cdots + \sigma^{m-1} d(x_0, x_1) + \frac{(m-1)\sigma^{m-1}}{1 - a_2}$$

$$d(x_n, x_m) \leq \sigma^n d(x_0, x_1)(1 + \sigma + \sigma^2 + \sigma^3 + \cdots + \sigma^{m-n-1}) + \sum_{i=n}^{m-1} \frac{i\sigma^i}{1 - a_2}$$

$$d(x_n, x_m) \leq \sigma^n d(x_0, x_1)(\frac{1 - \sigma^{m-n}}{1 - \sigma}) + \sum_{i=n}^{m-1} \frac{i\sigma^i}{1 - a_2} \tag{3.39}$$

On taking $m, n \to \infty$ in (3.19), we get

$$d(x_n, x_m) = 0. \tag{3.40}$$

This proves that the sequence $\{x_n\}$ is a Cauchy sequence in X. Since X is a complete metric space, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Now,

$$d(x^*, [Tx^*]_{\alpha(x^*)}) \leq [d(x^*, x_n) + (x_n, [Tx^*]_{\alpha(x^*)})],$$

Using (3.2), we get

$$d(x^*, [Tx^*]_{\alpha(x^*)}) \leq d(x^*, x_n) + a_1 d(x_{n-1}, x^*) + a_2[d(x_{n-1}, [Tx^*]_{\alpha(x^*)}) + d(x^*, [Tx_{n-1}]_{\alpha(x_{n-1})})]+$$
$$a_3 \frac{d(x_{n-1}, [Tx_{n-1}]_{\alpha(x_{n-1})}) d(x^*, [Tx^*]_{\alpha(x^*)})}{d(x_{n-1}, x^*) + d(x_{n-1}, [Tx^*]_{\alpha(x^*)}) + d(x^*, [Tx_{n-1}]_{\alpha(x_{n-1})})}+$$
$$a_4 \frac{d(x_{n-1}, [Tx_{n-1}]_{\alpha(x_{n-1})}) d(x_{n-1}, [Tx^*]_{\alpha(x^*)}) + d(x^*, [Tx_{n-1}]_{\alpha(x_{n-1})}) d(x^*, [Tx^*]_{\alpha(x^*)})}{d(x_{n-1}, [Tx^*]_{\alpha(x^*)}) + d(x^*, [Tx_{n-1}]_{\alpha(x_{n-1})})}$$

$$d(x^*, [Tx^*]_{\alpha(x^*)}) \leq d(x^*, x_n) + a_1 d(x_{n-1}, x^*) + a_2[d(x_{n-1}, [Tx^*]_{\alpha(x^*)}) + d(x^*, x_n)]+$$
$$a_3 \frac{d(x_{n-1}, x_n) d(x^*, [Tx^*]_{\alpha(x^*)})}{d(x_{n-1}, x^*) + d(x_{n-1}, [Tx^*]_{\alpha(x^*)}) + d(x^*, x_n)}+$$
$$a_4 \frac{d(x_{n-1}, x_n) d(x_{n-1}, [Tx^*]_{\alpha(x^*)}) + d(x^*, x_n) d(x^*, [Tx^*]_{\alpha(x^*)})}{d(x_{n-1}, [Tx^*]_{\alpha(x^*)}) + d(x^*, x_n)} \tag{3.41}$$

On taking $n \to \infty$ in (3.41), we get

$$d(x^*, [Tx^*]_{\alpha(x^*)}) \leq d(x^*, x^*) + a_1 d(x^*, x^*) + a_2[d(x^*, [Tx^*]_{\alpha(x^*)}) + d(x^*, x^*)]+$$
$$a_3 \frac{d(x^*, x^*) d(x^*, [Tx^*]_{\alpha(x^*)})}{d(x^*, x^*) + d(x^*, [Tx^*]_{\alpha(x^*)}) + d(x^*, x^*)}+$$
$$a_4 \frac{d(x^*, x^*) d(x^*, [Tx^*]_{\alpha(x^*)}) + d(x^*, x^*) d(x^*, [Tx^*]_{\alpha(x^*)})}{d(x^*, [Tx^*]_{\alpha(x^*)}) + d(x^*, x^*)}$$

implies

$$(1 - a_2) d(x^*, [Tx^*]_{\alpha(x^*)}) \leq 0 \tag{3.42}$$

Since $a_1 + 2a_2 + a_3 + a_4 < 1$ implies $a_1 + a_2 + a_3 + a_4 < 1 - a_2$, that is, $1 - a_2 \neq 0$. Hence,

$$d(x^*, [Tx^*]_{\alpha(x^*)}) = 0.$$

Implies

$$x^* \in [Tx^*]_{\alpha(x^*)}.$$

Thus, $x^* \in X$ is the fixed point. □

Example 3.5. Consider $X = [0, 2]$ the usual metric space which is complete and $T : X \to F(X)$ be a fuzzy type mapping such that $T(x) \in F(X)$, where $x \in X$ and $T(x) : X \to [0, 1]$ is a function defined by

$$T(x)(t) = \begin{cases} \frac{1}{2}, & 0 \leq t \leq \frac{1}{2} \\ \frac{1}{3}, & \frac{1}{2} < t < 1 \\ 0, & 1 \leq t \leq 2 \end{cases}$$

If for all $x \in X$, there exist $\alpha(x) = \frac{1}{2}$ such that

$$[Tx]_{\frac{1}{2}} = \left\{ t : Tx(t) \geq \frac{1}{2} \right\},$$

$$[Tx]_{\frac{1}{2}} = \left[0, \frac{1}{2}\right] \, and \, \ [Ty]_{\frac{1}{2}} = \left[0, \frac{1}{2}\right].$$

Now,

$$H\left([Tx]_{\frac{1}{2}}, [Ty]_{\frac{1}{2}}\right) = \max\left\{ \sup_{x \in [Tx]_{\frac{1}{2}}} d\left(x, [Ty]_{\frac{1}{2}}\right), \sup_{y \in [Ty]_{\frac{1}{2}}} d\left(y, [Tx]_{\frac{1}{2}}\right) \right\},$$

$$H\left([Tx]_{\frac{1}{2}}, [Ty]_{\frac{1}{2}}\right) = 0.$$

$$H(x, y) = |x - y|.$$

$$H\left([Ty]_{\frac{1}{2}}, y\right) = \begin{cases} 0 \ \text{ if } \ \ y \in [Ty]_{\frac{1}{2}} \\ \text{Otherwise nonzero} \end{cases}$$

$$H\left([Tx]_{\frac{1}{2}}, x\right) = \begin{cases} 0 \ \text{ if } \ \ x \in [Tx]_{\frac{1}{2}} \\ \text{Otherwise nonzero} \end{cases}$$

$$H\left([Tx]_{\frac{1}{2}}, y\right) = \begin{cases} 0 \ \text{ if } \ \ y \in [Tx]_{\frac{1}{2}} \\ \text{Otherwise nonzero} \end{cases}$$

$$H\left([Ty]_{\frac{1}{2}}, x\right) = \begin{cases} 0 \ \text{ if } \ \ x \in [Ty]_{\frac{1}{2}} \\ \text{Otherwise nonzero} \end{cases}$$

let $a_1 = \frac{1}{50}, a_2 = \frac{1}{10}, a_3 = \frac{1}{20}, a_4 = \frac{1}{30}$. Then,

$$H([Tx]_{\frac{1}{2}}, [Ty]_{\frac{1}{2}}) \leq \frac{1}{50} d(x, y) + \frac{1}{10} [d(x, [Tx]_{\frac{1}{2}}) + d(y, [Ty]_{\frac{1}{2}})] +$$

$$\frac{1}{20} \frac{d(x, [Tx]_{\frac{1}{2}}) d(y, [Ty]_{\frac{1}{2}})}{d(x, y) + d(x, [Ty]_{\frac{1}{2}}) + d(y, [Tx]_{\frac{1}{2}})}$$

$$+ \frac{1}{30} \frac{d(x, [Tx]_{\frac{1}{2}}) d(x, [Ty]_{\frac{1}{2}}) + d(y, [Tx]_{\frac{1}{2}}) d(y, [Ty]_{\frac{1}{2}})}{d(x, [Ty]_{\frac{1}{2}}) + d(y, [Tx]_{\frac{1}{2}})}$$

$$0 \leq \frac{1}{50} |x - y| + \frac{1}{10} [d(x, [Tx]_{\frac{1}{2}}) + d(y, [Ty]_{\frac{1}{2}})] +$$

$$\frac{1}{20} \frac{d(x, [Tx]_{\frac{1}{2}}) d(y, [Ty]_{\frac{1}{2}})}{d(x, y) + d(x, [Ty]_{\frac{1}{2}}) + d(y, [Tx]_{\frac{1}{2}})}$$

$$+ \frac{1}{30} \frac{d(x, [Tx]_{\frac{1}{2}}) d(x, [Ty]_{\frac{1}{2}}) + d(y, [Tx]_{\frac{1}{2}}) d(y, [Ty]_{\frac{1}{2}})}{d(x, [Ty]_{\frac{1}{2}}) + d(y, [Tx]_{\frac{1}{2}})}$$

As all the conditions of Theorem 3.3 are satisfied, we can conclude that T has a fixed point in X. Similarly, for fuzzy type II contraction mapping, we have

$$H([Tx]_{\frac{1}{2}},[Ty]_{\frac{1}{2}}) \leq \frac{1}{50}d(x,y) + \frac{1}{10}[d(x,[Ty]_{\frac{1}{2}}) + d(y,[Tx]_{\frac{1}{2}})]+$$
$$\frac{1}{20}\frac{d(x,[Tx]_{\frac{1}{2}})d(y,[Ty]_{\frac{1}{2}})}{d(x,y)+d(x,[Ty]_{\frac{1}{2}})+d(y,[Tx]_{\frac{1}{2}})}$$
$$+\frac{1}{30}\frac{d(x,[Tx]_{\frac{1}{2}})d(x,[Ty]_{\frac{1}{2}})+d(y,[Tx]_{\frac{1}{2}})d(y,[Ty]_{\frac{1}{2}})}{d(x,[Ty]_{\frac{1}{2}})+d(y,[Tx]_{\frac{1}{2}})}$$
$$0 \leq \frac{1}{50}|x-y| + \frac{1}{10}[d(x,[Ty]_{\frac{1}{2}}) + d(y,[Tx]_{\frac{1}{2}})]+$$
$$\frac{1}{20}\frac{d(x,[Tx]_{\frac{1}{2}})d(y,[Ty]_{\frac{1}{2}})}{d(x,y)+d(x,[Ty]_{\frac{1}{2}})+d(y,[Tx]_{\frac{1}{2}})}$$
$$+\frac{1}{30}\frac{d(x,[Tx]_{\frac{1}{2}})d(x,[Ty]_{\frac{1}{2}})+d(y,[Tx]_{\frac{1}{2}})d(y,[Ty]_{\frac{1}{2}})}{d(x,[Ty]_{\frac{1}{2}})+d(y,[Tx]_{\frac{1}{2}})}$$

As all of the conditions of Theorem 3.4 are satisfied, we can conclude that T has a fixed point in X.

Corollary 3.6. *Let (X,d) be a complete metric space and $T: X \to CB(X)$ be a fuzzy mapping. Suppose there exists $\alpha(x) \in (0,1]$, with $[Tx]_{\alpha(x)}$ and $[Ty]_{\alpha(y)}$ a closed, bounded, non-empty subsets of X. Then T has a fixed point in X, if for all $x,y \in X$ and $a_1,a_2,a_3,a_4,a_5 \geq 0$ with $a_1+2a_2+2a_3+a_4+a_5 < 1$ satisfying the following condition:*

$$H([Tx]_{\alpha(x)},[Ty]_{\alpha(y)}) \leq a_1d(x,y) + a_2[d(x,[Tx]_{\alpha(x)}) + d(y,[Ty]_{\alpha(y)})]+$$
$$a_3[d(x,[Ty]_{\alpha(y)}) + d(y,[Tx]_{\alpha(x)})] + a_4\frac{d(x,[Tx]_{\alpha(x)})d(y,[Ty]_{\alpha(y)})}{d(x,y)+d(x,[Ty]_{\alpha(y)})+d(y,[Tx]_{\alpha(x)})}$$
$$+ a_5\frac{d(x,[Tx]_{\alpha(x)})d(x,[Ty]_{\alpha(y)})+d(y,[Tx]_{\alpha(x)})d(y,[Ty]_{\alpha(y)})}{d(x,[Ty]_{\alpha(y)})+d(y,[Tx]_{\alpha(x)})} \tag{3.43}$$

4 Application

In this section, we explore a specific application of our results. We demonstrate how Theorems 3.3 and 3.4 can be applied to show the existence of fixed points for multivalued mappings in metric spaces. To begin, we start with the following definitions

Definition 4.1. Let $R: X \to CB(X)$ be a multivalued mapping and (X,d) be a complete metric space. Let us assume that $\alpha(x) \in (0,1]$, where $R(x)$ and $R(y)$ are closed, bounded, non-empty subsets of X. Then R is said to be fuzzy type I contraction if for all $x,y \in X$ and $a_1,a_2,a_3,a_4 \geq 0$ with $a_1+2a_2+a_3+a_4 < 1$ satisfying the following requirement:

$$H(R(x),R(y)) \leq a_1d(x,y) + a_2[d(x,R(x)) + d(y,R(y))]$$
$$+ a_3\frac{d(x,R(x))d(y,R(y))}{(d(x,y)+d(x,R(y))+d(y,R(x)))} + a_4\frac{d(x,R(x))d(x,R(y))+d(y,R(x))d(y,R(y))}{(d(x,R(y))+d(y,R(x)))} \tag{4.1}$$

Definition 4.2. Let $R : X \to CB(X)$ be a multivalued mapping and (X, d) be a complete metric space. Let us assume that $\alpha(x) \in (0, 1]$, where $R(x)$ and $R(y)$ are closed, bounded, non-empty subsets of X. Then R is said to be fuzzy type II contraction if for all $x, y \in X$ and $a_1, a_2, a_3, a_4 \geq 0$ with $a_1 + 2a_2 + a_3 + a_4 < 1$ satisfying the following requirement:

$$H(R(x), R(y)) \leq a_1 d(x, y) + a_2[d(x, R(y)) + d(y, R(x))]$$

$$+ a_3 \frac{d(x, R(x))d(y, R(y))}{(d(x, y) + d(x, R(y)) + d(y, R(x))} + a_4 \frac{d(x, R(x))d(x, R(y)) + d(y, R(x))d(y, R(y))}{(d(x, R(y)) + d(y, R(x))} \tag{4.2}$$

Theorem 4.3. *Let (X, d) be a complete metric space and $R : X \to CB(X)$ be a fuzzy type I contraction mapping. Then, T has a fixed point in X.*

Proof. Let $\alpha : X \to (0, 1]$ be an arbitrary mapping. Consider a fuzzy mapping $R : X \to F(X)$ defined by

$$(Tx)(t) = \begin{cases} \alpha(x), & t \in Rx, \\ 0, & t \notin Rx. \end{cases}$$

We have that

$$[Tx]_{\alpha(x)} = \{t : Tx(t) \geq \alpha(X)\} = Rx.$$

Hence, condition (4.1) becomes condition (3.1) in Theorem 3.3. It implies that there exists $x^* \in X$ such that $x^* \in [Tx^*]_{\alpha(x^*)} = Rx^*$. $\square$

Theorem 4.4. *Let (X, d) be a complete metric space and $R : X \to CB(X)$ be a fuzzy type II contraction mapping. Then, T has a fixed point in X.*

Proof. Let $\alpha : X \to (0, 1]$ be an arbitrary mapping. Consider a fuzzy mapping $R : X \to F(X)$ defined by

$$(Tx)(t) = \begin{cases} \alpha(x), & t \in Rx, \\ 0, & t \notin Rx. \end{cases}$$

We have that

$$[Tx]_{\alpha(x)} = \{t : Tx(t) \geq \alpha(x)\} = Rx.$$

Hence, condition (4.2) becomes condition (3.2) in in Theorem 3.4. It implies that there exists $x^* \in X$ such that $x^* \in [Tx^*]_{\alpha(x^*)} = Rx^*$. $\square$

Corollary 4.5. *Let (X, d) be a complete metric space and $R : X \to CB(X)$ be a multivalued mapping. Suppose there exists $\alpha(x) \in (0, 1]$, with $R(x)$ and $R(y)$ a closed, bounded, non-empty subsets of X. Then R has a fixed point in X, if for all $x, y \in X$ and $a_1, a_2, a_3, a_4, a_5 \geq 0$ with $a_1 + 2a_2 + 2a_3 + a_4 + a_5 < 1$ satisfying the following condition:*

$$H(R(x), R(y)) \leq a_1 d(x, y) + a_2[d(x, R(x)) + d(y, R(y))] + a_3[d(x, R(y)) + d(y, R(x))]$$

$$+ a_4 \frac{d(x, R(x))d(y, R(y))}{(d(x, y) + d(x, R(y)) + d(y, R(x))} + a_5 \frac{d(x, R(x))d(x, R(y)) + d(y, R(x))d(y, R(y))}{(d(x, R(y)) + d(y, R(x))} \tag{4.3}$$

Proof. It follows from the logic of the proof of Theorem 4.3 and Theorem 4.4. $\square$

5 Conclusion

The main findings of this study demonstrate applicability of fuzzy type contractions in establishing fixed point theorems for fuzzy mappings. This study provides significant advancements in the understanding of fuzzy metric spaces, with potential applications in differential equations and nonlinear Fredholm integral equation.

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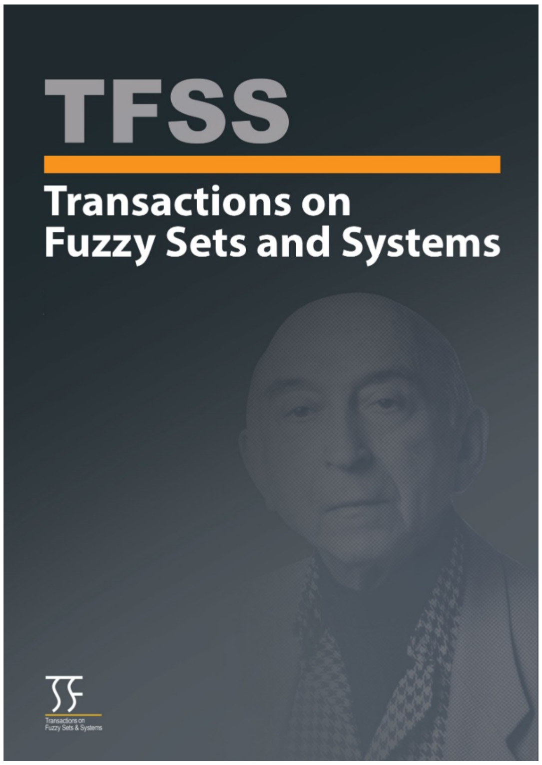

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Novel Generalisation of Some Fixed Point Results Using a New Type of Simulation Function

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Novel Generalisation of Some Fixed Point Results Using a New Type of Simulation Function

Mohd Hasan*

Abstract. I am utilizing a brand-new simulation function that has previously been developed by eminent mathematicians and that uses fuzzy metric-like spaces to establish new fixed point theorems. Here, this is demonstrated that the current conclusion is unquestionably a unified one that can generalize earlier current results. To further demonstrate the relevance of my findings, a few additional theorems and corollaries are demonstrated. Additionally, several excellent examples are provided to show how useful my findings are. I provide an application of my major finding in the conclusion.

AMS Subject Classification 2020: MSC 54H25; MSC 47H10
Keywords and Phrases: Metric space, Fuzzy metric space, Fuzzy metric-like space, α-admissible MA-simulation function.

1 Introduction

In 1951, Menger pioneered the idea of a metric which is statistical metric; see [1]. Kramosil and Michalek initiated the concept of a new metric called fuzzy metric in 1975([2]), building on the idea of a statistical metric. This idea is what is known as in a short form KM(Kramosil and Michalek)-fuzzy metric. In some ways, a KM(Kramosil and Michalek)- fuzzy metric is comparable to a metric based on statistics, but there are important distinctions in how they are explained and clarified. George and Veeramani [3], who are cited in [3, 4], inconsistently altered the fundamental idea of a KM(Kramosil and Michalek)- fuzzy metric; this improvement is known as a GV(George and Veeramani)-fuzzy metric. This improvement enables a number of realistic examples(some of them are very natural) of fuzzy metrics in unique fuzzy metrics established from measures. GV(George and Veeramani)-fuzzy metrics surface to be much more practical for looking at induced topological structures as well, in addition fuzzy metrics have sparked interest in between experts working in a variety of applied feilds of mathematics in addition to the main zest of many mathematicians based on theory phase of the principle of particularly fuzzy metrics, their topological and sequential components, their completeness, fixed points on maps, etc.

The Banach contraction principle guarantees the existence and uniqueness of a fixed point for a specific type of function. Fuzzy mathematics uses the Banach contraction principle to prove the existence and uniqueness of solutions to some fuzzy equations.

According to fuzzy mathematics, the Banach contraction principle is as follows:

There exists a constant $\alpha \in (0, 1)$ such that if (X, M) is a fuzzy metric space and $T : X \to X$ is a fuzzy contraction

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For every $x, y \in X$, $M(Tx, Ty) \leq \alpha M(x, y)$.

Then, in X, T has a fixed point which is unique.

A fuzzy contraction is a function that maps items to itself in a fuzzy metric space, therefore decreasing the fuzzy distances between those objects. The constant is known as the contraction constant. The Banach contraction principle states that a fuzzy contraction needs to have a clear fixed point in order to exist.

The map $\varsigma : [0, \infty) \times [0, \infty) \to \Re$ supposed to be a function which is simulation , it meets the given requirements:

(ς_1) ς(0, 0) = 0;

(ς_2) ς(r, w) <r - tw ∀t,r,w > 0;

(ς_3) if {r$_n$} and {w$_n$} re-orders(in (0, ∞) s.t.

$$0 < \lim_{n\to\infty} r_n = \lim_{n\to\infty} w_n,$$

if so

$$0 > \limsup_{n\to\infty} \varsigma(r_n, w_n).$$

The notion of simulation function was extended and modified, along with other concepts like b and θ-metric spaces, to produce the fixed point findings. By removing (ς_1), Argoubi [5] refined this idea in the same way, and Roldan et al. [6] simultaneously enhanced condition (ς_3) as follows:

(ς_3)‘ let two sequences {r$_n$} and {w$_n$} in (0, ∞) s.t.

$$\lim_{n\to\infty} w_n = \lim_{n\to\infty} r_n > 0, \quad \& \quad w_n > r_n, \quad \forall \quad n \in \mathbb{N}$$

if so

$$0 > \limsup_{n\to\infty} \varsigma(r_n, w_n).$$

By including α-admissible mappings, Karapinar [7] demonstrated a more broadly applicable version of the finding of Khojasteh [8].

In this paper, I prove a new type fixed point theorem in fuzzy metric-like spaces using the recently created MA-simulation function, a novel simulation function proposed by Perveen and Imdad [9] (see also [10]). Furthermore, I show that our results can be used more widely to synthesize several current conclusions from the literature and develop a few new findings as corollaries. I also offer a solid illustration to back up our conclusion. As an application of my theorems, I finally give the Fredholm nonlinear integral equation, which has an existential solution.

2 Preliminaries

Definition 2.1. [11] A t-norm which is continouos of a mapping(binary operation) $\star : (-\infty, 1] \cap [0, \infty) \times (-\infty, 1] \cap [0, \infty) \to (-\infty, 1] \cap [0, \infty)$ if the subsequent circumstances holds:

(I) $\star$ is continuous evrywhere;

(II) $\star$ is associative & commutative;

(III) for all $r \in [0, 1]$, $\quad r \star 1 = a$;

(IV) $\forall \quad r,s,t,u \in [0,1]$ $r \star s \leq t \star u$ whenever $r \leq t$ and $s \leq u$.

For further details on continuous t-norms and their classical instances, consider the t-norms of maximum, product, and minimum, which are represented by the symbols $T_l(r,s), = \max(r+s-1, 0)$, $T_p(r,s) = rs$ & $T_m(r,s) = \min(r,s)$, respectively.

The definition below was provided in 1994 by George and Veeramani ([3]), who also made major changes to Kramosil and Michalek's definition ([2]).

Definition 2.2. [3] Given that X is arbitrary and M be a fuzzy set, and $\star$ be a t-norm which is a continuous on this triplet, it is called to be a FMS that meets the criteria listed below, $s,t > 0$ & $\forall \quad x,z,y \in X$:

(I) $M(x,y,t)$ greator than zero;

(II) $M(x,y,t)$ equals to 1, for all $t > 0$ if and only if x and y are same;

(III) $M(x,y,t)$ is commutative:

(IV) $M(x,y,t)$ is holds traingular inequality i.e. $M(x,y,t) \star M(y,z,s) \leq M(x,z,t+s)$;

(V) $M(x,y,t)$ is continouos defined as $M(x,y,.) : (0,\infty) \to [0,1]$.

If x is not equal to y, $M(x,y,t)$ is greater than 0 and less than 1, as shown by (1) and (2) (cf. [12]), for all $t > 0$. It is clear that $M(x,y,.)$ is an increasing function for any $x,y \in X$. See the following works for further details: Citations for George and Veeramani [3], Gregori et al. [12], Roldan et al. [13] and Sapena [14].

Remark 2.3. Remark 2.3.[15] $M(x,y,.)$ be a non-decreasing function on $\forall \ \ x,y \in X$ & $\Re \cap (0,\infty)$.

Definition 2.4. [16] Let $\star$ is a continuous t-norm on the triplet $(X, \mathbb{F}, \star)$, here $\mathbb{F}$ is a fuzzy set and the set X be an arbitrary set. This triplet is referred to as a fuzzy metric-like space if it meets the conditions listed below $t,s > 0$ & $\forall \quad x,y,z \in X$.

(I) $\mathbb{F}(x,y,t)$ is greator than 0;

(II) If $\mathbb{F}(x,y,t)$ is equals to 1, then $x = y$, $\forall\ t > 0$,;

(III) $\mathbb{F}(x,y,t)$ is commutative;

(IV) $\mathbb{F}(x,y,t) \star F(y,z,s) \leq \mathbb{F}(x,z,t+s)$;

(V) $\mathbb{F}$ is continuous where $\mathbb{F}(x,y,.) : \Re \cap (0,\infty) \to [0,1]$.

In this case, $\mathbb{F}$ (fitted with $\star$) is described as a fuzzy metric-like on X.

Remark 2.5. This fuzzy metric-like space has an additional constraint, which is that $\mathbb{F}(x,x,t)$ may be smaller than 1 for all $t > 0$ for all (or may be some) $x \in X$. Shukla et al. [16] to make this argument. Additionally, for all $t > 0$ and for all $x \in X$, any fuzzy metric space is the same as a fuzzy metric-like space when $F(x,x,t) = 1$.

The fact that the value of $\mathbb{F}(x,x,t)$ may be less than 1 indicates that the definition above is usable when the degree of proximity between y and x is not the same, whereas this is not the case for the George and Veeramani [3] definition.

Example 2.6. Let this $(X, \mathbb{F}, \star_l)$ is a fuzzy metric-like space, with $X = \Re \cap [0,1]$, then, the $\mathbb{F}$ be a fuzzy set is defined like this;

$$\mathbb{F}(x,y,t) = \begin{cases} 1 & \text{if} \quad \text{x and y are same and equal to 0;} \\ \frac{x+y}{2} & \text{if} \quad \text{else} \end{cases},$$

$\forall \ \ t > 0.$

The following propositions can be used to identify different examples of triplet $(X, \mathbb{F}, \star)$ (fuzzy metric-like spaces).

Proposition 2.7. *[16] Let metric-like space be (X, σ) (see Harandi [17]. The fuzzy set $\mathbb{F}$ is provided by, and $(X, \mathbb{F}, \star_p)$ is a fuzzy metric-like space*

$$\mathbb{F}(x,y,t) = \frac{kt^n}{kt^n + m\sigma(x,y)}$$

$\forall\ x, y \in X,\ t > 0,\ m > 0$ *and* $n \geq 1$ *where* $k \in \Re$.

Remark 2.8. [16] Given that $k = n = m = 1$ in standard metric-like space induces a fuzzy metric-like space, this fuzzy metric-like space is known as standard fuzzy metric-like space. This fuzzy metric-like space is where

$$F_\sigma(x,y,t) = \frac{t}{t + \sigma(x,y)}$$

$\forall \ \ t > 0, \ \ x, y \in X.$

Proposition 2.9. *[16] Let's say that the fuzzy set F is defined as $\mathbb{F}(x,y,t) = e^{-\frac{\sigma(x,y)}{t^n}}$, where $n \geq 1$ (here, (X, σ) is metric-like sapce) is true for any $x, y \in X$ $, t > 0$. Then $(X, \mathbb{F}, \star_p)$ is a space that resembles a fuzzy metric.*

Example 2.10. Let $\mathbb{F}$ be a fuzzy set in $X^2 \times \Re \cap (0, \infty)$ by $\mathbb{F}(x,y,t) = \frac{1}{e^{\max\{x,y\}/t}}$ and X be a natural numbers. Here we take prduct t-norm(i.e.. $a \star b = ab$) and $\forall \ \ x, y \in X, t > 0$. Therefore, according to Proposition 2.9, the triplet $(X, \mathbb{F}, \star)$ is not a fuzzy metric space but rather a fuzzy metric-like space since $\sigma(x,y) = \max(x,y)$, for any $x, y \in X$, is a fuzzy metric-like on X as $\mathbb{F}(x,x,t) = \frac{1}{e^{x/t}} \neq 1, \quad \forall \ \ x > 0$ and $t > 0$.

Example 2.11. ([16]) Let $\mathbb{F}$ be a fuzzy set in $X^2 \times (0, \infty)$ by

$$\mathbb{F}(x,y,t) = \begin{cases} \frac{x}{y^3} & \text{if } x \leq y; \\ \frac{y}{x^3} & \text{if } y \leq x \end{cases},$$

for all $x, y \in X, t > 0$. and $X = \Re \cap [0,1]$. Define t-norm by product norm($a \star b = ab$). Then triplet $(X, \mathbb{F}, \star)$ is a fuzzy metric-like space.

Even if we use the minimum $t-$norm $\star_m(a \star b = \min\{a, b\}$ instead of the product $t-$norm $a \star b = ab$ (see [16]), the Propositions 2.7 and 2.9 are still valid.

Proposition 2.12. *If $K > 0$ exists and $\sigma(x,y) \leq K$ for all u, v in X, then (X, σ) is the bounded metric-like space and the fuzzy set $\mathbb{F}$ is defined by $\mathbb{F}(u,v,t) = 1 - \frac{\sigma(u,v)}{K+t}$, where $t > 0$ for all u, v in X. A fuzzy metric-like space is thus represented by the triplet $(X, \mathbb{F}, \star_l)$.*

Proof. The characteristics (i)-(iii) and (v) (defined in Definition 2.4) are clear and simple to prove. For (iv)(Definition 2.4), let $t > 0$ and $u, v, w \in X$, then since $\sigma(u,w) \leq \sigma(u,v) + \sigma(v,w)$, we have

$$1 - \frac{\sigma(u,w)}{K+t} \geq 1 - \frac{\sigma(u,v) + \sigma(v,w)}{K+t}.$$

From the above inequality it follows that

$$\max\left\{1 - \frac{\sigma(u,v) + \sigma(v,w)}{K+t}, 0\right\} \leq 1 - \frac{\sigma(u,v)}{K+t}.$$

This demonstrates that (iv) was met.

□

I will now determine Cauchy sequences, completeness, and convergence in fuzzy metric-like spaces.

Definition 2.13. [16] Let $\{u_n\}$ be a sequence in any X and the triplet $(X, \mathbb{F}, \star)$ be a fuzzy metric-like space. Then

(a) A u is referred to be the limit of a u_n sequence, and a u_n sequence is referred to as convergent to $u \in X$ if for all $t > 0$,

$$\lim_{n\to\infty} \mathbb{F}(u_n, u, t) = \mathbb{F}(u, u, t)$$

(b) The limit $\lim_{n\to\infty} \mathbb{F}(u_{n+p}, u_n, t)$ exists if $\forall\ t > 0$ and each $p > 1$. The sequence u_n is then referred to as Cauchy.

(c) if every Cauchy sequence u_n in any X converges to a particular u point in X. The triplet $(X, \mathbb{F}, \star)$ is therefore said to be complete if and only if

$\lim_{n\to\infty} \mathbb{F}(u_n, u, t) = \mathbb{F}(u, u, t) = \lim_{n\to\infty} \mathbb{F}(u_{n+p}, u_n, t)$, for each $p \geq 1$ and $\forall\ t > 0$.

Lemma 2.14. *[15] The mappings in the fuzzy metric-like space $(X, \mathbb{F}, \star)$ are continuous on $X \times X \times (0, \infty)$.*

In the debate that follows, the following may be necessary.

Definition 2.15. [18] Let triplet $(X, \mathbb{F}, \star)$ is a fuzzy metric-like space. A mapping $h : X \to X$ is said to be α-admissible if $\exists$ a function $\alpha : X \times X \times \Re \cap (0, \infty) \to \Re \cap [0, \infty)$ such that for all $t > 0$

$$u, v \in X, \alpha(u, v, t) \geq 1 \text{ implies } \alpha(hu, hv, t) \geq 1.$$

Definition 2.16. [19] Let the space $(X, \mathbb{F}, \star)$ represent a fuzzy metric-like. If $\forall\ t > 0$, a triangular α-admissile mapping $h : X \to X$ is said to exist.

$$u, v, w \in X, \alpha(u, v, t) \geq 1 \quad \text{and} \quad \alpha(v, w, t) \geq \implies \alpha(u, w, t) \geq 1.$$

Lemma 2.17. *[19] Assume that the triplet $(X, \mathbb{F}, \star)$ is a fuzzy metric-like space and that the mapping $h : X \to X$ is α-admissible. Assume there is a point u_0 in X where $\alpha(u_0, hu_0, t)$ is true. Define a sequence$u_0 \subseteq X$ by $u_n = fu_{n-1}$, $\forall\ n \in \mathbb{N}$. Then comes*

$$\alpha(u_n, u_m, t) \geq 1, \quad n < m, \quad \textit{for all} \quad m, n \in \mathbb{N},.$$

3 Results

A novel simulation function, the MA-simulation function, is introduced by Khojasteh et al. [8], Parveen and Imdad [9]. Using this function, I have created a new sort of contraction called the α-admissible $\Gamma_M A$-contraction, which will be used to deduce several new findings while also establishing a new result that unifies numerous results from the literature already in existence.

Definition 3.1. [9] If a mapping $\gamma : (-\infty, 1] \cap (0, \infty) \times (-\infty, 1] \cap (0, \infty) \to \Re$ satisfies the following criteria, it is said to be an MA-simulation function:

(γ_1) $\gamma(r, w) < \frac{1}{r} - \frac{1}{w}, \quad \forall \quad r, w \in (0, 1)$;

(γ_2) if $\{r_n\}$ and $\{w_n\}$ are given sequences lies in $(0, 1]$ such that $\lim_{n\to\infty} r_n = \lim_{n\to\infty} w_n = l \in (0, 1)$ and $r_n < w_n, \quad \forall \quad n \in \Re$ then

$$\limsup_{n\to\infty} \gamma(r_n, w_n) < 0.$$

The set of all MA-simulation functions represented by the notation Γ_{MA}.

I provide several instances of MA-simulation function in the lines that follow.

Example 3.2. Suppose $\gamma : (-\infty, 1] \cap (0, \infty) \times (-\infty, 1] \cap (0, \infty) \to \Re$ having a clear valuet as

$$\gamma(r, w) = c\Big(\frac{1}{r} - 1\Big) - \Big(\frac{1}{w} - 1\Big),$$

$\forall \quad r, w \in (0, 1]$ and $c \in (0, 1)$.

Example 3.3. Suppose $\gamma : (-\infty, 1] \cap (0, \infty) \times (-\infty, 1] \cap (0, \infty) \to \Re$ having a clear valuet as

$$\gamma(r, w) = \psi\Big(\frac{1}{r} - 1\Big) - \Big(\frac{1}{w} - 1\Big),$$

$\forall \quad r, w \in (0, 1]$ where ψ is self mapping at the interval $[0, \infty)$ and $\forall \quad r > 0, \psi(r)r$ are right continuous functions.

Example 3.4. Suppose $\gamma : (-\infty, 1] \cap (0, \infty) \times (-\infty, 1] \cap (0, \infty) \to \Re$ having a clear value as

$$\gamma(r, w) = \Big(\frac{1}{r} - 1\Big) - \psi\Big(\frac{1}{r} - 1\Big) - \Big(\frac{1}{w} - 1\Big),$$

$\forall \quad r, w \in (0, 1]$ where ψ is a self-mapped variable at the range $[0, \infty)$ and$r > 0,$, $\psi(r) > 0$, and $\psi(0) = 0$.

Example 3.5. Suppose $\gamma : (-\infty, 1] \cap (0, \infty) \times (-\infty, 1] \cap (0, \infty) \to \Re$ having a clear value as

$$\gamma(r, w) = w - \psi(r), \quad \forall \quad r, w \in (0, 1]$$

where $\psi(r) > r$,for all r in $(0, 1)$ and $\psi : (0, 1] \to (0, 1]$ are left-continuous and non-decreasing, respectively.

Example 3.6. Suppose $\gamma : (-\infty, 1] \cap (0, \infty) \times (-\infty, 1] \cap (0, \infty) \to \Re$ having a clear value as

$$\gamma(r, w) = \Big(\frac{1}{r} - 1\Big)\psi\Big(\frac{1}{w} - 1\Big) - \Big(\frac{1}{r} - 1\Big),$$

$\forall \quad r, w \in (0, 1]$ where $\psi : \Re \cap [0, \infty) \to \Re \cap (0, 1)$ is a function that is specified so that $\forall \quad R > 0, \quad \lim_{r\to R^+} \psi(r) < 1$.

Example 3.7. Suppose $\gamma : (-\infty, 1] \cap (0, \infty) \times (-\infty, 1] \cap (0, \infty) \to \Re$ having a clear value as

$$\gamma(r, w) = \Big(\frac{1}{r} - 1\Big) - \int_0^{\frac{1}{w}-1} \psi(w)dw,$$

$\forall \quad r, w \in (0, 1]$ and $\forall \quad s > 0$ where ψ is a self-mapped variable at the range,$[0, \infty)$ and $\int_0^s \psi(w)dw > s$, respectively.

Now, I am able present the notion for fuzzy metric-like space called it α-admissible Γ_{MA}-contraction.

Definition 3.8. The triplet $(X, \mathbb{M}, \star)$ is a fuzzy metric-like space, and a self mapping h on set X is said to be a α-admissible Γ_{MA}-contraction defind on this triplet. If a $\gamma \in \Gamma_{MA}$ exists and is such that for any $t > 0$, it fulfills the following

$$x, y \in X, \quad \alpha(x, y, t) \geq 1 \Rightarrow \gamma\Big(M(x, y, t), M(hx, hy, t)\Big) \geq 0, \tag{3.1.1}$$

I am prepared to offer our primary finding right here.

Theorem 3.9. *If h is a self-mapping on X a α-admissible Γ_{MA}-contraction in respect of γ, then $(X, \mathbb{M}, \star)$ is a fcomplete fuzzy metric-like space. Assume the following circumstances are true:*

(i) $\exists\ x_0 \in X$ *like that* $\alpha(x_0, hx_0, t) \geq 1$*;*

(ii) *h to be triangular α-admissible;*

(iii) *h to be continuous*
or
if $\forall \quad n \in N, \quad t > 0$ *and* $\{x_n\} \to x$*, such that* $\alpha(x_n, x_{n+1}, t) \geq 1$*, where* $\{x_n\}$ *is a sequence in* X *for some* $x \in X$*,* $\exists$ *a subsequence* $\{x_{n_k}\} \in \{x_n\}$ *such that* $\alpha(x_{n_k}, x, t) \geq 1, \quad$ *for all* $\quad k$ *is natural number and* t *greater than* 0*.*

Next, h maintain a fixed point.

Proof. Assume that $x_0 \in X$ is a random point. Explain the Picard sequence. $\{x_n = h^n x_0\}$. Suppose $\exists$ some $m_0 \in \mathbb{N}$ such that $h^{m_0}(x_0) = h^{m_0+1}x_0$, i.e., $x_{m_0} = x_{m_0+1}$, then x_{m_0} is a fixed point of h. Now, suppose that $h_{n-1}x_0 \neq h_n x_0$, $\forall \quad n \in \mathbb{N}$. Using Lemma 2.2, we then have

$$\alpha(x_n, x_m, t) \geq 1, \quad \text{for all} \quad \text{m,n be are natural numbers}, \quad n < m, \tag{3.1.2.}$$

In light of (3.1.2) and (3.1.1), for $y = x_n$ and $x = x_{n-1}$ I obtain

$$0 \leq \gamma\Big(\mathbb{M}(x_{n-1}, x_n, t), \mathbb{M}(hx_{n-1}, hx_n, t)\Big) = \gamma\Big(\mathbb{M}(x_{n-1}, x_n, t), \mathbb{M}(x_n, x_{n+1}, t)\Big)$$

$$< \frac{1}{\mathbb{M}(x_{n-1}, x_n, t)} - \frac{1}{\mathbb{M}(x_n, x_{n+1}, t)},$$

which implies

$$\mathbb{M}(x_{n-1}, x_n, t) < \mathbb{M}(x_n, x_{n+1}, t)$$

Therefore, $\{\mathbb{M}(x_n, x_{n+1}, t)\}$ is non-decreasing(an increasing) sequence of $\Re_+$ in $\Re \cap (0, 1]$. Let $\lim_n \to \infty \mathbb{M}(x_n, x_n + 1, t) = r(t)$. I claim that $r(t) = 1$, for every $t > 0$. On the other hand, suppose that for some $t_0 > 0$, $r(t_0) < 1$. Then, as $\{r_n = \mathbb{M}(x_{n-1}, x_n, t_0)\} \to r(t_0)$ and $\{w_n = \mathbb{M}(x_n, x_{n+1}, t_0)\} \to s(t_0)$ so using (γ_2), I obtain

$$0 \leq \limsup_{n\to\infty} \gamma\Big(\mathbb{M}(x_{n-1}, x_n, t_0), \mathbb{M}(x_n, x_{n+1}, t_0)\Big) < 0.$$

a contradiction, thus, we obtain $(\forall t > 0)$ from the expression $r(t) = 1, \forall t > 0$.

$$\lim_{n\to\infty} \mathbb{M}(x_n, x_{n+1}, t) = 1 \qquad (3.1.3.)$$

The next step is to demonstrate that x_n is a Cauchy sequence. Let's say it's not true, then $\exists\ 0 < \epsilon_0 < 1,\quad t_0 > 0$ and 2 sub-sequences $\{\{x_{n_k}\}, \{x_{m_k}\}\}$ of $\{x_n\}$ such that $m(k) > n(k) \geq k$ and

$$\mathbb{M}(x_{n(k)}, x_{m(k)}, t_0) \leq 1 - \epsilon_0.$$

From the Remark 2.3, we have

$$\mathbb{M}\Big(x_{n(k)}, x_{m(k)}, \frac{t_0}{2}\Big) \leq 1 - \epsilon_0 \qquad (3.1.4.)$$

Let's now assume that $m(k)$ is the smallest integer that can be used to represent $n(k)$ and yet fulfill (3.1.4). Then comes

$$\mathbb{M}\Big(x_{n(k)}, x_{m(k)-1}, \frac{t_0}{2}\Big) \leq 1 - \epsilon_0. \qquad (3.1.5)$$

Now, using condition ((iv of Definition 2.5), (3.1.4) and (3.1.5), we obtain

$$1 - \epsilon_0 \geq \mathbb{M}(x_{n(k)}, x_{m(k)}, t_0)$$

$$\geq \mathbb{M}\Big(x_{n(k)}, x_{m(k)-1}, \frac{t_0}{2}\Big) \star \mathbb{M}\Big(x_{m(k)-1}, x_{m(k)}, \frac{t_0}{2}\Big)$$

$$> (1 - \epsilon_0) \star \mathbb{M}\Big(x_{m(k)-1}, x_{m(k)}, \frac{t_0}{2}\Big)$$

Applying the t-norm and allowing $k \to \infty$, it produces

$$1 - \epsilon_0 \geq \mathbb{M}(x_{n(k)}, x_{m(k)}, t_0) \geq 1 - \epsilon_0$$

and hence

$$\lim_{n\to\infty} \mathbb{M}(x_{n(k)}, x_{m(k)}, t_0) = 1 - \epsilon_0. \qquad (3.1.6)$$

Also, again by (3.1.1) and (γ_2), for $x = x_{n_k-1}$, $\ y = x_{m_k-1}$ and $t = t_0$, we get

$$0 \leq \gamma\Big(\mathbb{M}(x_{n(k)-1}, x_{m(k)-1}, t_0), \mathbb{M}(x_{n(k)}, x_{m(k)}, t_0)\Big)$$

$$< \frac{1}{\mathbb{M}(x_{n(k)-1}, x_{m(k)-1}, t_0)} - \frac{1}{\mathbb{M}(x_{n(k)}, x_{m(k)}, t_0)},$$

so that

$$\mathbb{M}(x_{n(k)}, x_{m(k)}, t_0) > \mathbb{M}(x_{n(k)-1}, x_{m(k)-1}, t_0)$$

$$\geq \mathbb{M}\Big(x_{n(k)-1}, x_{n(k)}, \frac{t_0}{2}\Big) \star \mathbb{M}\Big(x_{n(k)}, x_{m(k)-1}, \frac{t_0}{2}\Big)$$

$$> \mathbb{M}\Big(x_{n(k)-1}, x_{n(k)}, \frac{t_0}{2}\Big) \star (1 - \epsilon_0)$$

which on letting $k \to \infty$ and using t-norm yields

$$1 - \epsilon_0 > \lim_{k\to\infty} \mathbb{M}(x_{n(k)-1}, x_{m(k)-1}, t_0) \geq 1 - \epsilon_0.$$

Hence, we have

$$\lim_{k\to\infty} \mathbb{M}(x_{n(k)-1}, x_{m(k)-1}, t_0) = 1 - \epsilon_0. \tag{3.1.7}$$

As a result, according to (3.1.2), we obtain $\alpha(x_{n(k)-1}, x_{m(k)-1}, t_0) \geq 1$, assuming $\{r_k = \mathbb{M}(x_{n(k)-1}, x_{m(k)-1}, t_0)\}$ and $\{w_k = \mathbb{M}(x_{n(k)}, x_{m(k)}, t_0)\}$ and applying (γ_2), we obtain

$$0 \leq \lim\sup_{k\to\infty} \gamma\Big(\mathbb{M}(x_{n(k)-1}, x_{m(k)-1}, t_0), \mathbb{M}(x_{n(k)}, x_{m(k)}, t_0)\Big) < 0,$$

a contradiction. Thus, $(X, \mathbb{M}, \star)$ has a Cauchy sequence (x_n). Now, due to X's completeness, $\{x_n\} \to x$ exists within X. If h is continuous, then we have $\{hx_n\} \to hx$, which implies that $hx = x$ by the uniqueness of the limit.

□

We now give the example below, which illustrates how Theorem 3.1 can be used.

Example 3.10. Let $X = [0, 1]$. Define $* : [0, 1] \times [0, 1] \to [0, 1]$be a t-norm as $p * q = \min\{p, q\}$. Define fuzzy metric-like space $\mathbb{M}$ by

$$\mathbb{M}(x, y, t) = \frac{t}{\sigma(x, y) + t},$$

where $\sigma(x, y) = x^2 + y^2$ is metric-like space. This is $(X, \mathbb{M}, .)$ a complete fuzzy metric-like space. A mapping with the definitions of $h : X \to X$ and $\alpha : X \times X \times \Re_+ \to [0, \infty)$ is as follows:

$$\alpha(x, y, t) = \begin{cases} 1 & \text{if } x, y \in [0, \frac{1}{2}]; \\ 0 & \text{if } otherwise \end{cases},$$

and

$$hx = \begin{cases} \frac{ax}{1+x} & \text{if } x \in [0, \frac{1}{2}]; \\ x & \text{if } otherwise \end{cases},$$

in where $a \in (0, 1)$. Then, there is ($\forall \quad x, y \in X$ and $t > 0$)

$$\frac{1}{\mathbb{M}(x, y, t)} - 1 = \frac{t + \sigma(x, y)}{t} - 1 = \frac{\sigma(x, y)}{t} = \frac{x^2 + y^2}{t}$$

Also, for $x, y \in X$ such that $\alpha(z, y, t) \geq 1$, we have

$$\frac{1}{\mathbb{M}(hx, hy, t)} - 1 = \frac{t + \sigma(hx, hy)}{t} - 1 = \frac{\sigma(hx, hy)}{t}$$

$$= \frac{(hx)^2 + (hy)^2}{t} = \frac{(\frac{ax}{1+x})^2 + (\frac{ay}{1+y})^2}{t}$$

$$= \frac{\frac{a^2x^2}{(1+x)^2} + \frac{a^2y^2}{(1+y)^2}}{t}.$$

Then, using the formula $\gamma(t, s) = k(\frac{1}{t} - 1) - (frac1s - 1)$, we can obtain (for $x, y \in X$) for any $k \in [a, 1]$ and $a \in (0, 1)$.

$$\alpha(x,y,t) \geq 1 \Rightarrow \xi\Big(\mathbb{M}(x,y,t), \mathbb{M}(hx,hy,t)\Big)$$
$$= k\Big(\frac{x^2+y^2}{t}\Big) - \Big(\frac{\frac{a^2x^2}{(1+x)^2} + \frac{a^2y^2}{(1+y)^2}}{t}\Big)$$
$$= \frac{x^2}{t}\Big(k - \frac{a^2}{(1+x)^2}\Big) + \frac{y^2}{t}\Big(k - \frac{a^2}{(1+y)^2}\Big) \geq 0$$

$\forall \quad t > 0$. Thus, Theorem 3.1's prerequisites are all met, and the theorem's conclusionthat h has a unique fixed point, namely $x = 0$. However, the Gregori and Sapena [12] result cannot be applied. In fact, there is no k in $(0,1)$ such that (1.1) is met for any $x, y \in (\frac{1}{2}, 1]$.

Next theorem shows the uniqueness of fixed point.

Theorem 3.11. *Theorem 3.9's premise is met. along with one extra following observation is fulfilled:*

(iv) for each $x, y \in Fix(h)$, $\exists \ \ w \in X$ like that $1 \leq \alpha(y,w,t)$, and $\alpha(x,w,t) \geq 1$ for all $t > 0$,

then $h(x) = x$ is unique.

Proof. Theorem above follows the existence portion. In order to determine if a fixed point is unique, let's suppose that x and $x^\star$ are two separate fixed points of h. Then, according to condition (iv), there is a point $w \in X$ where $\forall \quad t > 0$, $\alpha(x^*\star w, t) \geq 1$ and $\alpha(x,w,t) \geq 1$.

Create a sequence $w_n \subseteq X$ by setting $w_n + 1 = Tw_n$ and $w_0 = w$ and , for every $n \in \mathbb{N} \cup \{0\}$. Triangular α-admissibility provides us with

$$\alpha(x^*, w_n, t) \geq 1 \quad \text{and} \quad \alpha(x, w_n, t) \geq 1, \quad \forall \quad t > 0 \quad \text{and} \quad n \in \mathbb{N} \cup \{0\} \tag{3.1.8}$$

Using 3.1.8 and 3.1.1 (for $x = x$ and $y = w_n$), we can now deduce

$$M(x, w_{n+1}, t) > M(x, w_n, t), \quad \forall \quad t > 0 \quad \text{and} \quad n \in \mathbb{N} \cup \{0\} \tag{3.1.9}$$

which demonstrates that the sequence $\{M(x, w_n, t)\}$ is an increasing series of positive real numbers in the range $\lim_{n\to\infty} M(x, w_n, t) = L(t)$. Our contention is that $\forall \quad t > 0$ gives $L(t) = 1$, . On the other hand, suppose that certain $t_0 > 0$ exist and that $L(t_0)1$. As a result, for $\{t_n = M(x, w_n, t_0)\}$ and $\{s_n = M(x, w_n + 1, t_0)\}$, we obtain (γ_2) by using 3.1.1.

$$0 \leq \lim_{n\to\infty} \gamma\Big(M(x, w_{n+1}, t_0), M(x, w_n, t_0)\Big) < 0,$$

a contradiction. As a result, $L(t) = 1$ and for all$t > 0$. As a result, $\lim_{n\to\infty} w_n = x$ from $\lim_{n\to\infty} M(x, w_n, t) = 1$, for all $t > 0$. The same pattern allows us to demonstrate that $\lim_{n\to\infty} w_n = x*$. We get to $x = x*$ through the uniqueness of the limit.

□

Next example below, which illustrates how Theorem 3.11 where fixed point is unique.

Example 3.12. Let $X = [0,1]$. Define $* : [0,1] \times [0,1] \to [0,1]$be a t-norm as $p * q = \min\{p, q\}$. A mapping with the definitions of $h : X \to X$ and $\alpha : X \times X \times \Re_+ \to [0, \infty)$ is as follows:

$$hx = \begin{cases} \frac{ax}{1+x} & \text{if } x \in [0, \frac{1}{2}]; \\ x & \text{if } \textit{otherwise} \end{cases},$$

in where $a \in (0,1)$. and

$$\alpha(x,y,t) = \begin{cases} 1 & \text{if } x,y \in [0,\frac{1}{2}]; \\ 0 & \text{if } otherwise \end{cases},$$

Define fuzzy metric-like space $\mathbb{M}$ by

$$\mathbb{M}(x,y,t) = \frac{t}{\sigma(x,y)+t},$$

where $\sigma(x,y) = x^2 + y^2$ is metric-like space. This is $(X,\mathbb{M},\star)$ a complete fuzzy metric-like space. One can easily varify this exapmle on lines of Example 3.10, in this example fixed point is unique.

4 Consequences

I now derive a few corollaries for fuzzy metric-like spaces as a result of Theorem 3.1, starting with the one that follows.

Corollary 4.1. *([20] type) Assume that $(X,,M,\star)$ is a complete fuzzy metric-like space and h is a satisfied self mapping on X.*

$$x,y \in X \quad \alpha(x,y,t) \geq 1 \Rightarrow \frac{1}{\mathbb{M}(hx,hy,t)} - 1 \leq k\Big(\frac{1}{\mathbb{M}(x,y,t)} - 1\Big),$$

Both $k \in (0,1)$ and $\forall \ t > 0$. After that, h has a unique fixed point.

Proof. One can proof this corollary from Theorem 3.9 and Example 3.2. □

Corollary 4.1 may be reduced to the following result by assuming that $\alpha(x,y,t) = 1$, for any $x,y \in X$ and $t > 0$ by Gregori and Sapena [12].

Corollary 4.2. *Let triplet $(X,\mathbb{M},\star)$ be a complete fuzzy metric-like space, and let $h : X \to X$ be a satisfied .*

$$k\Big(\frac{1}{\mathbb{M}(x,y,t)} - 1\Big) \geq \frac{1}{\mathbb{M}(hx,hy,t)} - 1,$$

$\forall \ k \in (0,1)$ and $t > 0$, $\ x,y \in X$,. After that, $hx = x$ i.e. h has a fixed point which is unique.

The Boyd and Wong [21] type result for fuzzy metric-like spaces will be presented in the following corollary.

Corollary 4.3. *If h is a satisfied self mapping on X, then triplet $(X,\mathbb{M},\star)$ is a complete fuzzy metric-like space. Then*

$$\alpha(x,y,t) \geq 1 \Rightarrow \frac{1}{\mathbb{M}(hx,hy,t)} - 1 \leq \psi\Big(\frac{1}{\mathbb{M}(x,y,t)} - 1\Big),$$

$\forall \ x,y \in X$ and $t > 0$, where $\psi : \Re \cap [0,\infty) \to \Re \cap [0,\infty)$ is a given function like that $\psi(r) < r$, $\psi(0) = 0$ and $\forall \ r > 0$. After that, $hx = x$ i.e. h has a fixed point which is unique.

Proof. The conclusion arises from Theorem 3.9 and Example 3.3. □

The fixed point result from Abbas et al. [22] is shown below.

Corollary 4.4. *Suppose triplet $(X, \mathbb{M}, \star)$ is a complete fuzzy metric-like space and $h : X \to X$ is satisfying*

$$\alpha(x,y,t) \geq 1 \Rightarrow \frac{1}{\mathbb{M}(hx,hy,t)} - 1 \leq \Big(\frac{1}{\mathbb{M}(x,y,t)} - 1\Big) - \psi\Big(\frac{1}{\mathbb{M}(x,y,t)} - 1\Big),$$

$\forall\ t > 0$ and $x, y \in X$, where $\psi : \Re \cap [0,\infty) \to \Re \cap [0,\infty)$ is a function such that $\psi(0) = 0$, and $\psi(r) > 0$ for all $r > 0$. After that, h has a fixed point which is unique.

Proof. The conclusion follows in light of Theorem 3.9 and Example 3.4. $\square$

The results that follow are known in some natural settings but appear novel in the fuzzy context.

Corollary 4.5. *Suppose triplet $(X, \mathbb{M}, \star)$ is a complete fuzzy metric-like space and $h : X \to X$ is satisfying*

$$\alpha(x,y,t) \geq 1 \Rightarrow \mathbb{M}(hx,hy,t) \geq \psi(\mathbb{M}(x,y,t)),$$

$\forall\ t > 0$ and $x, y \in X$, where $\psi : \Re \cap (0,1] \to \Re \cap (0,1]$ is a left-continuousfunction and nondecreasing such that $\forall\ r \in \Re \cap (0,1]$, $\psi(r) > r$,. After that, h has a fixed point which is unique.

Proof. Theorem 3.9 and Example 3.5 lead to the proof. $\square$

Corollary 4.6. *Suppose triplet $(X, \mathbb{M}, \star)$ is a complete fuzzy metric-like space and $h : X \to X$ is satisfying*

$$x,y \in X \quad \alpha(x,y,t) \geq 1 \Rightarrow \frac{1}{\mathbb{M}(hx,hy,t)} - 1 \leq \Big(\frac{1}{\mathbb{M}(x,y,t)} - 1\Big).\psi\Big(\frac{1}{\mathbb{M}(x,y,t)} - 1\Big),$$

$\forall\ t > 0$ and $x, y \in X$, where $\psi : \Re \cap [0,\infty) \to \Re \cap [0,\infty)$ is a given function such that $\lim_{r \to s+} \psi(r) > 0$, $\forall\ r > 0$. After that, h has a fixed point which is unique.

Proof. The conclusion is inferred from Example 3.6 and Theorem 3.9. $\square$

5 An Application

Many authors have recently used various sufficient conditions to determine the existence and uniqueness of integral equation solutions in various contexts. Here, I focus on a Fredholm nonlinear integral equation and use our established finding for fuzzy metric-like spaces to identify the problem's one and only solution. I see that by using Theorem 3.1, this Fredholm non-linear integral equation has a unique solution under particular circumstances, and that if these circumstances are not met, I am unable to use our findings to obtain the unique solution.

To illustrate this, I take into account the following:

$$x(t) = \int_a^b K(t,s)h(x(w))dw + g(r), \tag{5.1}$$

$\forall\ t \in \Omega = [a,b](a, b \in \Re)$, $\quad g, h \in C(\Omega, \Re) \quad K \in C(\Omega \times \Omega, \Re)$.

Let Phi represent the collection of all mappings from $\phi : \Re \cap [0,\infty) \to \Re \cap [0,\infty)$ that meet the criteria listed below:

(ϕ_1) $\forall\ t \in [o,\infty)$, $\qquad \phi(t) \leq t$;

(ϕ_2) ϕ is non-decreasing.

I can now state our theorem as follows in this section:

Theorem 5.1. *The following requirements must be met for the integral equation (5.1) with the variables* $K \in C(\Omega \times \Omega, \Re)$ *and* $g \in C(\Omega, \Re)$ *to be valid:*

(i) $\exists$ *a +ve number* $\phi \in \Phi$ *and* λ *such that the following is true for any* $x, y \in C(\Omega, \mathbb{R})$*:*

$$h(x) - h(y) \leq \lambda\phi(x - y) \qquad (5.2);$$

(ii) $\lambda \sup_{t\in\Omega} \int_a^b |K(r, w)|dr \leq \frac{1}{2}$.

Then, $C(\Omega, \Re)$ *is the ounique solution to equation (5.1).*

Proof. Be aware that $X = C(\Omega, \Re)$ is a complete metric space in terms of its sup-metric.

$$\sigma(x, y) = \sup_{t\in\Omega}(|x(t)| + |y(t)| + a).$$

Additionally, the space $(X, \mathbb{M}, \star)$

$$\forall \quad t > 0 \quad \text{and} \quad x, y \in X \quad \mathbb{M}(x, y, t) = \frac{t}{t + \sigma(x, y)},$$

be a complete fuzzy metric-like space with product t-norm.

Now we define a mapping $h : X \to X$as:

$$Sx(r) = \int_a^b K(r, w)h(x(w))dw + g(r) \qquad (5.3)$$

$\forall \quad r \in \Omega$. Using (5.2) and (5.3), we have

$$hx(r) - hy(r) = \int_a^b K(r, w)[h(x(w)) - h(y(w))]dw$$

$$\leq \lambda \int_a^b K(r, w)\phi(x(w) - y(w))dw \qquad (5.4).$$

Using(ϕ_1), we have

$$\phi(x(w) - y(w) \leq \phi(\sup(|x(w)| + |y(w)| + a)) = \phi(\sigma(x, y)). \qquad (5.5).$$

Applying (5.5) in (5.4), we obtain

$$\phi(x(w) - y(w) \leq \lambda \int_a^b K(r, w)\phi(\sigma(x, y))dw.$$

Taking supremum over $r \in \Omega$, using conditions (II) and (ϕ_2), we get

$$\sigma(hx, hy) \leq \lambda\phi(\sigma(x, y)) \int_a^b |K(r, w)|dw$$

$$\leq \frac{1}{2}\phi(\sigma(x, y)) \leq \frac{1}{2}(\sigma(x, y)). \qquad (5.6).$$

Now, we have

$$\frac{1}{\mathbb{M}(hx,hy,t)} - 1 = \frac{\sigma(hx,hy)}{t}$$
$$\leq \frac{\sigma(x,y)}{2t} = \frac{1}{2}\Big(\frac{1}{\mathbb{M}(x,y,t)} - 1\Big)$$

By using the formula $\gamma(r,w) = \frac{1}{2}, \Big(\frac{1}{r} - 1\Big) - \Big(\frac{1}{w} - 1\Big)$ and $\alpha(x,y,t) = 1$ for all $t > 0$ and $x, y \in X$ satisfies all the criteria of Theorem 3.1 for every $r, w \in (0,1])$. Theorems 3.1 and 3.2's results lead to the conclusion that $C(\Omega, \Re)$ is the only solution to equation (5.1). □

6 Conclusion

In this paper, motivated by the work of Khojasteh et al. [8], Perveen and Imdad [9] and Karapinar citeKarapinar, we propose the idea of a new contraction called the α-admissible Γ_{MA}-contraction and use it to prove fixed point results, ensuring the existence and uniqueness of fixed points. We also introduce a new simulation function. Additionally, we show through a few corollaries that our main finding is broad enough to encompass a number of findings from the body of literature already in existence. Finally, we demonstrate the utility of our primary result by showing an application.

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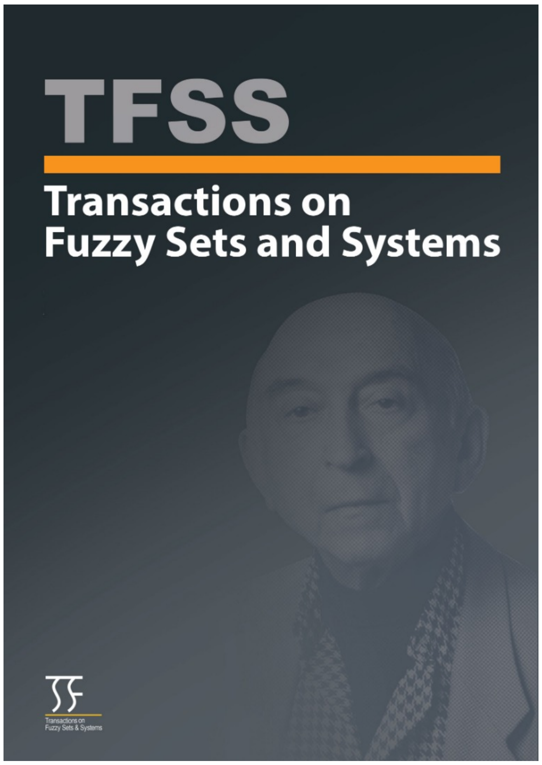

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Fuzzy Cone Metric Spaces and Fixed Point Theorems for Fuzzy Type Contraction

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Abstract. The paper aims to introduce novel concepts of fuzzy type contractions and establish fixed point theorems for fuzzy mappings within the framework of fuzzy cone metric spaces. These contributions extend the existing literature on fuzzy mappings and fixed point theory. Through illustrative examples, we showcase the practical applicability of our proposed notions and results, demonstrating their effectiveness in real-world scenarios.

AMS Subject Classification 2020: 54H25; 47H10
Keywords and Phrases: Fuzzy cone metric spaces, Fixed point, Fuzzy mapping, Real Banach space.

1 Introduction

Banach's fixed point theorem for contraction mappings has been one of the most influential results in mathematical analysis. Banach's contraction principle [1] has been instrumental in the development of metric fixed point theory, and has been used to solve a wide range of problems, including differential equations, integral equations, optimization problems, and variational inequalities. Since its introduction, the Banach contraction mapping principle has been generalized and refined in numerous ways, leading to a wealth of articles dedicated to its improvement [2, 3, 4].

Guang and Xian [5] extended the notion of metric spaces by considering a real Banach space as the range set, thereby introducing the concept of cone metric spaces. Through their exploration of cone metric spaces, they uncovered significant properties that led to the derivation of several fixed point theorems, some of which can be found in [6, 7, 8].

Zadeh [9] pioneered the concept of fuzzy sets, laying the foundation for subsequent research in fuzzy mathematics. Building upon Zadeh's work, Weiss [10] delved into fuzzy mappings and derived numerous fixed point results. Heilpern [11] further expanded upon fuzzy mappings by introducing the concept of fuzzy contraction mappings. He established a fixed point theorem for fuzzy contraction mappings akin to Nadler's fixed point theorem for multivalued mappings. Moreover, Bag [12] introduced the innovative notion of fuzzy cone metric spaces, leveraging this framework to derive fixed point results for fuzzy T-Kannan contraction and fuzzy T-Chatterjea contraction mappings. Recently, Raji and Ibrahim [13] proved some fixed point results for fuzzy mappings in a complete dislocated b-metric space.

Based on the above insight, we introduce novel concepts of fuzzy type contractions and subsequently establish fixed point results for fuzzy mappings within the framework of fuzzy cone metric spaces. To bolster our findings, we offer illustrative examples demonstrating the practical application of the presented results and

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concepts.
Throughout our discourse, we denote by E a fuzzy real Banach space, by $\mathcal{F}$ a fuzzy cone in E with a non-empty interior, and by $\leq$ a partial ordering with respect to $\mathcal{F}$.

2 Preliminaries

A fuzzy cone metric space integrates concepts from fuzzy metric space and cone metric space, offering a broader and more adaptable approach to handle uncertainty and fuzziness in distance measurements. We begin this section with a few key definitions.

Definition 2.1. [14, 15] A function with X as its domain and the interval $[0,1]$ as its range is called a fuzzy set in X. $\mathcal{F}(X)$ represents the set of all fuzzy sets in X. The degree of membership of x in A is denoted by the value $A(x)$, given a fuzzy set A and a point x in X. A fuzzy set A's α-level set is represented by $[A]_\alpha$ and has the following definition:

$$[A]_\alpha = \{x : A(x) \geq \alpha\} \text{ where } \alpha \in (0,1), [A]_0 = \{x : A(x) > 0\}$$

Definition 2.2. [16, 17] Let Y be a metric space and X a nonempty set. If a mapping T is a mapping from X into $\mathcal{F}(Y)$, the set of all fuzzy sets on Y, then it is referred to as a fuzzy mapping. The degree to which y is a member of $T(x)$ is the membership function of a fuzzy mapping T, represented as $T(x)(y)$. Stated differently, $T(x)(y)$ represents y's degree of membership in the fuzzy set $T(x)$. Instead of using $[T(x)]_\alpha$ to denote the α-level set of $T(x)$, we will simply use $[Tx]_\alpha$.

Definition 2.3. [18, 19] A fuzzy fixed point of a fuzzy mapping $T : X \to \mathcal{F}(X)$ is defined as a point $x \in X$ where $\alpha \in (0,1]$ and $x \in [Tx]_\alpha$.

Definition 2.4. [20] Consider the fuzzy real Banach space $(E, \|.\|)$, where $\|\Box\| : E \to R(I)$. Use $E^*(I)$ to indicate the range of $\|.\|$,Thus, $E^*(I) \subset R^*(I)$.

Definition 2.5. [21] An interior point is defined as member $\eta \in A \subset R^*(I)$ if there exists $r > 0$ such that

$$S(\eta, r) = \{\delta \in R^*(I) : \delta \ominus \eta < \bar{r}\} \subset A$$

set of all interior points of A is called interior A.

Definition 2.6. [11] Fuzzy closed subset $\mathcal{F}$ of $E^*(I)$ is defined as follows: for each sequence $\{\eta_n\}$, such that

$$\lim_{n\to\infty} \eta_n = \eta \quad \text{implies} \quad \eta \in \mathcal{F}.$$

Definition 2.7. [22] A fuzzy cone is defined as a subset $\mathcal{F}$ of $E^*(I)$ if
i. $\mathcal{F}$ is fuzzy closed,nonempty and $\mathcal{F} \neq \{\bar{0}\}$,
ii. $a, b \in R, a, b \geq 0, \eta, \delta \in \mathcal{F} \implies a\eta \oplus b\delta \in \mathcal{F}$

Definition 2.8. [22] A mapping $x : R \mapsto [0,1]$ over the set R of all real numbers is called a fuzzy real number

Definition 2.9. [22] A fuzzy real number x is convex if $x(t) \geq \wedge (x(s), x(r))$ where $s \leq t \leq r$.

Definition 2.10. [9] α-level set of fuzzy real number x is defined by $\{t \in R : x(t) \geq \alpha\}$ where $\alpha \in (0,1]$. If there exists a $t_0 \in R$ such that $x(t_0) = 1$, then x called normal. For $0 < \alpha \leq 1$, α-level set of an upper semi continuous convex normal fuzzy real number η denoted by $[\eta]_\alpha$, serves as a closed interval $[a_\alpha, b_\alpha]$, where $a_\alpha = -\infty$ and $b_\alpha = +\infty$ are admissible.

Definition 2.11. [22] Given a fuzzy cone $\mathcal{F} \in E^*(I)$ define a partial odering $\leq$ with respect to $\mathcal{F}$ by $\eta \leq \delta$ iff $\delta \ominus \eta \in \mathcal{F}$ and $\eta < \delta$ indicates that $\eta \leq \delta$ but $\eta \neq \delta$ while $\eta << \delta$ will stand for $\delta \ominus \eta \in \ Int\mathcal{F}$ where $Int\ \mathcal{F}$ denote the interior of $\mathcal{F}$.

Definition 2.12. [22] The fuzzy cone $\mathcal{F}$ is called normal if there exists a number $K > 0$ such tha for all $x, y \in E$ with $\bar{0} \leq \|x\| \leq \|y\|$ implies $\|x\| \leq K\|y\|$. The least positive number satisfying above is called the normal constant of $\mathcal{F}$.

Definition 2.13. [22] If every growing sequence that is bounded from above is convergent, the fuzzy cone $\mathcal{F}$ is said to be regular. That is $\{x_n\}$ is a sequence in E such that
$\|x_1\| \leq \|x_2\| \leq \cdots \leq \|y\|$ for some $y \in E$, then there exists $x \in E$ such that $\|x_n - x\| \to \bar{0}$ as $n \to \infty$

Definition 2.14. [22] Let X be a nonempty set. Suppose the mapping $d : X \times X \mapsto E^*(I)$ satisfies
$(fd1)$ $d(x, y) \ \geq \bar{0}$ and $d(x, y) = \bar{0}$ iff $x = y$;
$(fd2)$ $d(x, y) = d(y, x)$;
$(fd3)$ $d(x, y) \leq d(x, z) \oplus d(z, y)$ for all $x, y, z \in X$.
Then, the d is called a fuzzy cone metric and the pair (X, d) is called a fuzzy cone metric space.

Definition 2.15. [22] Let (X, d) be a fuzzy cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $\bar{0} << \|c\|$, there is a positive integer N such that for all $n > N$, $d(x_n, x) << \|c\|$, then, $\{x_n\}$ is said to be convergent and converges to x and x is called the limit of $\{x_n\}$. Denoted by

$$\lim_{n\to\infty} x_n = x$$

Definition 2.16. [22] Let $\{x_n\}$ be a sequence in X and (X, d) be a fuzzy cone metric space. $\{x_n\}$ is referred to as a Cauchy sequence in X if, for any $c \in E$ with $\bar{0} << \|c\|$, there exists a natural integer N such that, for any $m, n > N, d(x_n, x_m) << \|c\|$.

Definition 2.17. [22] Let (X, d) be a metric space with fuzzy cones. X is referred to as a complete fuzzy cone metric space if every Cauchy sequence is convergent in it.

Definition 2.18. [22] Let $\{x_n\}$ be a sequence in X and (X, d) be a fuzzy cone metric space with normal fuzzy cone. Then
i. $\{x_n\}$ converges to x if and only if $d(x_n, x) \to \bar{0}$ as $n \to \infty$
ii. $\{x_n\}$ is a Cauchy sequence if and only if $d(x_m, x_n) \to \bar{0}$ as $m, n \to \infty$

3 Main Results

We start this section with the definitions that follow.

Definition 3.1. Suppose $(X,\ d)$ is a fuzzy cone metric space. Let $T, S : X \to X$ be two functions. Then S is said to be fuzzy cone T-type I contraction if for all $x, y \in X$,$Tx \neq Ty$ and $a_1, a_2, a_3, a_4 \geq 0$ with $2a_1 + a_2 + a_3 + a_4 < 1$ satisfying the following condition:

$$d\left(TSx, TSy\right) \leq a_1 \left[d(Tx, TSx) \oplus d(Ty, TSy)\right] \oplus a_2 \frac{d(Tx, TSx)d(Ty, TSy)}{d(Tx, Ty)}$$
$$\oplus a_3 \frac{d(Tx, TSx)d(Ty, TSy)}{d(Tx, Ty) \oplus d(Tx, TSy) \oplus d(Ty, TSx)} \oplus a_4 \frac{d(Tx, TSx)d(Tx, TSy) \oplus d(Ty, TSx)d(Ty, TSy)}{d(Tx, TSy) \oplus d(Ty, TSx)} \tag{3.1}$$

Definition 3.2. Suppose $(X,\ d)$ is a fuzzy cone metric space. Let $T, S : X \to X$ be two functions. Then S is said to be fuzzy cone T-type II contraction if for all $x, y \in X, Tx \neq Ty$ and $a_1, a_2, a_3, a_4 \geq 0$ with $2a_1 + a_2 + a_3 + a_4 < 1$ satisfying the following condition:

$$\begin{aligned} d\left(TSx, TSy\right) \leq a_1\left[d\left(Tx, TSy\right) \oplus d\left(Ty, TSx\right)\right] \oplus a_2 \frac{d\left(Tx, TSx\right) d\left(Ty, TSy\right)}{d\left(Tx, Ty\right)} \\ \oplus a_3 \frac{d\left(Tx, TSx\right) d\left(Ty, TSy\right)}{d\left(Tx, Ty\right) \oplus d\left(Tx, TSy\right) \oplus d\left(Ty, TSx\right)} \\ \oplus a_4 \frac{d\left(Tx, TSx\right) d\left(Tx, TSy\right) \oplus d\left(Ty, TSx\right) d\left(Ty, TSy\right)}{d\left(Tx, TSy\right) \oplus d\left(Ty, TSx\right)} \end{aligned} \tag{3.2}$$

Theorem 3.3. *Suppose $(X,\ d)$ is a complete fuzzy cone metric space, $\mathcal{F}$ be a normal fuzzy cone with normal constant K. Let $T : X \to X$ be a one-one continuous function and $S : X \to X$ be a fuzzy cone T -type I contraction mapping. Then, the following conditions are satisfied:*
i. for every $x_0 \in X$, $\lim_{n\to\infty} d(TS^n x_0, TS^{n+1}x_0) = \bar{0}$;
ii. there exists $v \in X$ such that $\lim_{n\to\infty} TS^n x_0 = v$;
iii. if T is sequentially convergent, then $\{S^n x_0\}$ has a convergent subsequence;
iv. there is a unique $u \in X$ such that $Su = u$;
v. if T is sequentially convergent, then for each $x_0 \in X$ the iterate sequence $\{S^n x_0\}$ converges to u.

Proof. Let $x_0 \in X$ be any arbitrary point in X. Define the iterate sequence $\{x_n\}$ by $x_{n+1} = Sx_n = S^n x_0$, Now, by using (3.1), we get

$$\begin{aligned} (Tx_n, Tx_{n+1}) &= d\left(TSx_{n-1}, TSx_n\right) \\ &\leq a_1\left[d\left(Tx_{n-1}, TSx_{n-1}\right) \oplus d\left(Tx_n, TSx_n\right)\right] \\ &\oplus a_2 \frac{d\left(Tx_{n-1}, TSx_{n-1}\right) d\left(Tx_n, TSx_n\right)}{d\left(Tx_{n-1}, Tx_n\right)} \\ &\oplus a_3 \frac{d\left(Tx_{n-1}, TSx_{n-1}\right) d\left(Tx_n, TSx_n\right)}{d\left(Tx_{n-1}, Tx_n\right) \oplus d\left(Tx_{n-1}, TSx_n\right) \oplus d\left(Tx_n, TSx_{n-1}\right)} \\ &\oplus a_4 \frac{d\left(Tx_{n-1}, TSx_{n-1}\right) d\left(Tx_{n-1}, TSx_n\right) \oplus d\left(Tx_n, TSx_{n-1}\right) d\left(Tx_n, TSx_n\right)}{d\left(Tx_{n-1}, TSx_n\right) \oplus d\left(Tx_n, TSx_{n-1}\right)} \\ &= a_1\left[d\left(Tx_{n-1}, Tx_n\right) \oplus d\left(Tx_n, Tx_{n+1}\right)\right] \oplus a_2 \frac{d\left(Tx_{n-1}, Tx_n\right) d\left(Tx_n, Tx_{n+1}\right)}{d\left(Tx_{n-1}, Tx_n\right)} \\ &\oplus a_3 \frac{d\left(Tx_{n-1}, Tx_n\right) d\left(Tx_n, Tx_{n+1}\right)}{d\left(Tx_{n-1}, Tx_n\right) \oplus d\left(Tx_{n-1}, Tx_{n+1}\right) \oplus d\left(Tx_n, Tx_n\right)} \\ &\oplus a_4 \frac{d\left(Tx_{n-1}, Tx_n\right) d\left(Tx_{n-1}, Tx_{n+1}\right) \oplus d\left(Tx_n, Tx_n\right) d\left(Tx_n, Tx_{n+1}\right)}{d\left(Tx_{n-1}, Tx_{n+1}\right) \oplus d\left(Tx_n, Tx_n\right)} \end{aligned}$$

$$\begin{aligned} d\left(Tx_n, Tx_{n+1}\right) &\leq a_1 d\left(Tx_{n-1}, Tx_n\right) \oplus a_1 d\left(Tx_n, Tx_{n+1}\right) \oplus a_2 d\left(Tx_n, Tx_{n+1}\right) \\ &\oplus a_3 d\left(Tx_{n-1}, Tx_n\right) \oplus a_4 d\left(Tx_{n-1}, Tx_n\right) \\ d\left(Tx_n, Tx_{n+1}\right) &\leq \frac{a_1 + a_3 + a_4}{1 - (a_1 + a_2)} d\left(Tx_{n-1}, Tx_n\right) \end{aligned} \tag{3.3}$$

Let $\lambda = \frac{a_1+a_3+a_4}{1-(a_1+a_2)}$. Since $2a_1 + a_2 + a_3 + a_4 < 1$ implies that $\frac{a_1+a_3+a_4}{1-(a_1+a_2)} < 1$. Hence,

$$d\left(Tx_n, Tx_{n+1}\right) \leq \lambda d\left(Tx_{n-1}, Tx_n\right) \ \forall \ n \in \mathbb{N} \tag{3.4}$$

Then, by repeated application of (3.4), we have

$$d\left(TS^n x_0, TS^{n+1} x_0\right) \leq \lambda^n d\left(Tx_0, TSx_0\right) \ \forall \ n \in \mathbb{N} \tag{3.5}$$

Since $\mathcal{F}$ is a normal cone with constant K, we have from (3.5),

$$d\left(TS^n x_0, TS^{n+1} x_0\right) \leq \lambda^n K d\left(Tx_0, TSx_0\right) \ \forall \ n \in \mathbb{N} \tag{3.6}$$

Implies

$$d\left(TS^n x_0, TS^{n+1} x_0\right) \leq \lambda^n K d_\alpha^i\left(Tx_0, TSx_0\right) \ for \ i = 1, 2 \tag{3.7}$$

On taking the limit in (3.7), we have

$$\lim_{n\to\infty} d_\alpha^i\left(TS^n x_0, TS^{n+1} x_0\right) = 0 \text{ for } i = 1, 2, \alpha \in (0, 1] \ \left(\text{since } \frac{a_1 + a_3 + a_4}{1 - (a_1 + a_2)} < 1\right)$$

Hence,

$$\lim_{n\to\infty} d\left(TS^n x_0, TS^{n+1} x_0\right) = 0 \tag{3.8}$$

For any $m > n$ where $m, n \in \mathbb{N}$, we have,

$$\begin{aligned} d\left(Tx_n, Tx_m\right) &\leq d\left(Tx_n, Tx_{n+1}\right) \oplus d\left(Tx_{n+1}, Tx_{n+2}\right) \oplus \cdots \oplus d\left(Tx_{m-1}, Tx_m\right) \\ \leq \left[\left(\frac{a_1 + a_3 + a_4}{1 - (a_1 + a_2)}\right)^n + \left(\frac{a_1 + a_3 + a_4}{1 - (a_1 + a_2)}\right)^{n+1} + \cdots + \left(\frac{a_1 + a_3 + a_4}{1 - (a_1 + a_2)}\right)^{m-1}\right] d\left(Tx_0, TSx_0\right) \end{aligned}$$

$$\begin{aligned} &\leq \left[\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}\right] d\left(Tx_0, TSx_0\right) \\ &\leq \lambda^n \frac{1}{1-\lambda} d\left(Tx_0, TSx_0\right) \end{aligned} \tag{3.9}$$

So

$$d\left(TS^n x_0, TS^m x_0\right) \leq \lambda^n \frac{1}{1-\lambda} d\left(Tx_0, TSx_0\right) \tag{3.10}$$

Since $\mathcal{F}$ is normal, we get

$$d\left(TS^n x_0, TS^m x_0\right) \leq \lambda^n \frac{k}{1-\lambda} d\left(Tx_0, TSx_0\right) \tag{3.11}$$

Taking the limit as $m, n \to \infty$, we get

$$\lim_{m,n\to\infty} d\left(TS^n x_0, TS^m x_0\right) = \bar{0} \ \left(\text{since } \frac{a_1 + a_2 + a_3 + a_4}{1 - (a_2 + a_3)} < 1\right) \tag{3.12}$$

This proves that $\{(TS^n x_0\}$ is Cauchy sequence in X. Since X is a complete metric space, there exists $v \in X$ such that

$$\lim_{n\to\infty} (TS^n x_0 = v). \tag{3.13}$$

Now if T is subsequentially convergent, $\{S^n x_0\}$ has a convergent subsequence.So there exists $u \in X$ and $\{n_i\}$ such that

$$\lim_{i\to\infty} S^{n_i} x_0 = u. \tag{3.14}$$

Since T is continuous by (3.13), we get

$$\lim_{i\to\infty} TS^{n_i} x_0 = Tu. \tag{3.15}$$

Considering (3.14) and (3.15), we get $Tu = u$.
Now

$$d(TSu, Tu) \leq d(TSu, TS^{n_i}(x_0)) \oplus d\left(TS^{n_i}(x_0), TS^{n_i+1}(x_0)\right) \oplus d\left(TS^{n_i+1}(x_0), Tu\right).$$

$$d(TSu, Tu) \leq a_1\left[d(Tu, TSu) \oplus d\left(TS^{n_i-1}(x_0), TS^{n_i}(x_0)\right)\right] \oplus a_2 \frac{d(Tu, TSu)\, d\left(TS^{n_i-1}(x_0), TS^{n_i}(x_0)\right)}{d(Tu, TS^{n_i-1}(x_0))}$$
$$\oplus a_3 \frac{d(Tu, TSu)\, d\left(TS^{n_i-1}(x_0), TS^{n_i}(x_0)\right)}{d(Tu, TS^{n_i-1}(x_0)) \oplus d(Tu, TS^{n_i}(x_0)) \oplus d(TS^{n_i-1}(x_0), TSu)}$$
$$\oplus a_4 \frac{d(Tu, TSu)\, d(Tu, TS^{n_i}(x_0)) \oplus d\left(TS^{n_i-1}(x_0), TSu\right) d\left(TS^{n_i-1}(x_0), TS^{n_i}(x_0)\right)}{d(Tu, TS^{n_i-1}(x_0)) \oplus d(TS^{n_i-1}(x_0), TSu)}$$
$$\oplus \lambda^{n_i} d(Tx_0, TSx_0) \oplus d\left(TS^{n_i+1}(x_0), Tu\right)$$

So

$$(TSu, Tu) \leq \lambda d\left(TS^{n_i-1}(x_0), TS^{n_i}(x_0)\right) \oplus \frac{1}{1-\lambda}\lambda^n d(Tx_0, TSx_0) \oplus \frac{1}{1-\lambda} d\left(TS^{n_i+1}(x_0), Tu\right)$$

Since $\mathcal{F}$ is normal cone with normal constant K, we have

$$(TSu, Tu) \leq \lambda K d\left(TS^{n_i-1}(x_0), TS^{n_i}(x_0)\right) \oplus \frac{k}{1-\lambda}\lambda^n d(Tx_0, TSx_0) \oplus \frac{k}{1-\lambda} d\left(TS^{n_i+1}(x_0), Tu\right)$$

Taking the llimit $i \to \infty$, using (3.15) and $\lambda < 1$, we get
$d_\alpha^i(TSu, Tu) = 0$ for all $\alpha \in (0,1]$ and $i = 1, 2$,
Hence,

$$d(TSu, Tu) = \bar{0} \tag{3.17}$$

So that $TSu = Tu$.
Since T is one-one, we get $Su = u$. So S has a fixed point.
if v is another fixed point of S, then $Sv = v$. Since S is type I contraction, we obtain

$$d(TSu, TSv) \leq a_1\left[d(Tu, TSu) \oplus d(Tv, TSv)\right] \oplus a_2 \frac{d(Tu, TSu)\, d(Tv, TSv)}{d(Tu, Tv)} \oplus$$
$$a_3 \frac{d(Tu, TSu)\, d(Tv, TSv)}{d(Tu, Tv) \oplus d(Tu, TSv) \oplus d(Tv, TSu)} \oplus a_4 \frac{d(Tu, TSu)\, d(Tu, TSv) \oplus d(Tv, TSu)\, d(Tv, TSv)}{d(Tu, TSv) \oplus d(Tv, TSu)} \tag{3.18}$$

$$= a_1\left[d(Tu, Tu) \oplus d(Tv, Tv)\right] \oplus a_2 \frac{d(Tu, Tu)\, d(Tv, Tv)}{d(Tu, Tv)} \oplus$$
$$a_3 \frac{d(Tu, Tu)\, d(Tv, Tv)}{d(Tu, Tv) \oplus d(Tu, Tv) \oplus d(Tv, Tu)} \oplus a_4 \frac{d(Tu, Tu)\, d(Tu, Tv) \oplus d(Tv, Tu)\, d(Tv, Tv)}{d(Tu, Tv) \oplus d(Tv, Tu)}$$

Implies

$$d(TSu, TSv) = \bar{0}$$

So that $TSu = TSv$. Since S is injective, we get $u = v$. Thus,S has a unique fixed point.
Lastly, if T is sequentially convergent, by replacing n for n_i, we get that

$$\lim_{n\to\infty} S^n x_0 = u.$$

Thus, $\{S^n x_0\}$ is convergent to the fixed point u.
□

Theorem 3.4. *Suppose $(X,\ d)$ is a complete fuzzy cone metric space, $\mathcal{F}$ be a normal fuzzy cone with normal constant K. Let $T : X \to X$ be a one-one continuous function and $S : X \to X$ be a fuzzy cone T -type II contraction mapping. Then, the following conditions are satisfied:*
i. for every $x_0 \in X$, $\lim_{n\to\infty} d(TS^n x_0, TS^{n+1}x_0) = \bar{0}$;
ii. there exists $v \in X$ such that $\lim_{n\to\infty} TS^n x_0 = v$;
iii. if T is sequentially convergent, then $\{S^n x_0\}$ has a convergent subsequence;
iv. there is a unique $u \in X$ such that $Su = u$;
v. if T is sequentially convergent, then for each $x_0 \in X$ the iterate sequence $\{S^n x_0\}$ converges to u.

Proof. Let $x_0 \in X$ be any arbitrary point in X. Define the iterate sequence x_n by $x_{(n+1)} = Sx_n = S^n x_0$. Now, by using (3.2), we get

$$\begin{aligned}
(Tx_n, Tx_{n+1}) &= d(TSx_{n-1}, TSx_n) \\
&\leq a_1 \left[d(Tx_{n-1}, TSx_n) \oplus d(Tx_n, TSx_{n-1})\right] \\
&\oplus a_2 \frac{d(Tx_{n-1}, TSx_{n-1})\, d(Tx_n, TSx_n)}{d(Tx_{n-1}, Tx_n)} \\
&\oplus a_3 \frac{d(Tx_{n-1}, TSx_{n-1})\, d(Tx_n, TSx_n)}{d(Tx_{n-1}, Tx_n) \oplus d(Tx_{n-1}, TSx_n) \oplus d(Tx_n, TSx_{n-1})} \\
&\oplus a_4 \frac{d(Tx_{n-1}, TSx_{n-1})\, d(Tx_{n-1}, TSx_n) \oplus d(Tx_n, TSx_{n-1})\, d(Tx_n, TSx_n)}{d(Tx_{n-1}, TSx_n) \oplus d(Tx_n, TSx_{n-1})} \\
&= a_1 \left[d(Tx_{n-1}, Tx_{n+1}) \oplus d(Tx_n, Tx_n)\right] \oplus a_2 \frac{d(Tx_{n-1}, Tx_n)\, d(Tx_n, Tx_{n+1})}{d(Tx_{n-1}, Tx_n)} \\
&\oplus a_3 \frac{d(Tx_{n-1}, Tx_n)\, d(Tx_n, Tx_{n+1})}{d(Tx_{n-1}, Tx_n) \oplus d(Tx_{n-1}, Tx_{n+1}) \oplus d(Tx_n, Tx_n)} \\
&\oplus a_4 \frac{d(Tx_{n-1}, Tx_n)\, d(Tx_{n-1}, Tx_{n+1}) \oplus d(Tx_n, Tx_n)\, d(Tx_n, Tx_{n+1})}{d(Tx_{n-1}, Tx_{n+1}) \oplus d(Tx_n, Tx_n)}
\end{aligned}$$

$$\begin{aligned}
d(Tx_n, Tx_{n+1}) &\leq a_1 d(Tx_{n-1}, Tx_n) \oplus a_1 d(Tx_n, Tx_{n+1}) \oplus a_2 d(Tx_n, Tx_{n+1}) \\
&\oplus a_3 d(Tx_{n-1}, Tx_n) \oplus a_4 d(Tx_{n-1}, Tx_n) \\
d(Tx_n, Tx_{n+1}) &\leq \frac{a_1 + a_3 + a_4}{1 - (a_1 + a_2)} d(Tx_{n-1}, Tx_n) \qquad (3.20)
\end{aligned}$$

Let $\lambda = \frac{a_1+a_3+a_4}{1-(a_1+a_2)}$. Since $2a_1 + a_2 + a_3 + a_4 < 1$ implies that $\frac{a_1+a_3+a_4}{1-(a_1+a_2)} < 1$. Hence,

$$d(Tx_n, Tx_{n+1}) \leq \lambda d(Tx_{n-1}, Tx_n)\ \ \forall\ n \in \mathbb{N} \qquad (3.21)$$

Then, by repeated application of (3.21), we have

$$d\left(TS^n x_0, TS^{n+1}x_0\right) \leq \lambda^n d\left(Tx_0, TSx_0\right) \ \forall \ n \in \mathbb{N} \tag{3.22}$$

Since $\mathcal{F}$ is a normal cone with constant K, we have from (3.22),

$$d\left(TS^n x_0, TS^{n+1}x_0\right) \leq \lambda^n K d\left(Tx_0, TSx_0\right) \ \forall \ n \in \mathbb{N} \tag{3.23}$$

Implies

$$d\left(TS^n x_0, TS^{n+1}x_0\right) \leq \lambda^n K d_\alpha^i\left(Tx_0, TSx_0\right) \ for \ i = 1, 2 \tag{3.24}$$

On taking the limit in (3.24), we have

$$\lim_{n\to\infty} d_\alpha^i\left(TS^n x_0, TS^{n+1}x_0\right) = 0 \text{ for } i = 1, 2, \alpha \in (0, 1] \ \left(\text{since } \frac{a_1 + a_3 + a_4}{1 - (a_1 + a_2)} < 1\right)$$

Hence,

$$\lim_{n\to\infty} d\left(TS^n x_0, TS^{n+1}x_0\right) = 0 \tag{3.25}$$

For any $m > n$ where $m, n \in \mathbb{N}$, we have,

$$\begin{aligned} d\left(Tx_n, Tx_m\right) &\leq d\left(Tx_n, Tx_{n+1}\right) \oplus d\left(Tx_{n+1}, Tx_{n+2}\right) \oplus \cdots \oplus d\left(Tx_{m-1}, Tx_m\right) \\ &\leq \left[\left(\frac{a_1 + a_3 + a_4}{1 - (a_1 + a_2)}\right)^n + \left(\frac{a_1 + a_3 + a_4}{1 - (a_1 + a_2)}\right)^{n+1} + \cdots + \left(\frac{a_1 + a_3 + a_4}{1 - (a_1 + a_2)}\right)^{m-1}\right] d\left(Tx_0, TSx_0\right) \\ &\leq \left[\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}\right] d\left(Tx_0, TSx_0\right) \\ &\leq \lambda^n \frac{1}{1-\lambda} d\left(Tx_0, TSx_0\right) \end{aligned} \tag{3.26}$$

So

$$d\left(TS^n x_0, TS^m x_0\right) \leq \lambda^n \frac{1}{1-\lambda} d\left(Tx_0, TSx_0\right) \tag{3.27}$$

Since $\mathcal{F}$ is normal, we get

$$d\left(TS^n x_0, TS^m x_0\right) \leq \lambda^n \frac{k}{1-\lambda} d\left(Tx_0, TSx_0\right) \tag{3.28}$$

Taking the limit as $m, n \to \infty$, we get

$$\lim_{m,n\to\infty} d\left(TS^n x_0, TS^m x_0\right) = \bar{0} \ \left(\text{since } \frac{a_1 + a_2 + a_3 + a_4}{1 - (a_2 + a_3)} < 1\right) \tag{3.29}$$

This proves that $\{(TS^n x_0\}$ is Cauchy sequence in X. Since X is a complete metric space, there exists $v \in X$ such that

$$\lim_{n\to\infty} (TS^n x_0 = v). \tag{3.30}$$

Now if T is subsequentially convergent, $\{S^n x_0\}$ has a convergent subsequence.So there exists $u \in X$ and $\{n_i\}$ such that

$$\lim_{i\to\infty} S^{n_i} x_0 = u. \tag{3.31}$$

Since T is continuous by (3.31), we get

$$\lim_{i\to\infty} TS^{n_i}x_0 = Tu. \tag{3.32}$$

Considering (3.31) and (3.32), we get $Tu = u$.
Now

$$d\left(TSu,Tu\right) \le d\left(TSu,TS^{n_i}(x_0)\right) \oplus d\left(TS^{n_i}(x_0),TS^{n_i+1}(x_0)\right) \oplus d\left(TS^{n_i+1}(x_0),Tu\right).$$

$$\begin{aligned} d\left(TSu,Tu\right) \le a_1\left[d\left(Tu,TS^{n_i}(x_0)\right) \oplus d\left(TS^{n_i-1}(x_0),TSu\right)\right] \oplus a_2\frac{d\left(Tu,TSu\right)d\left(TS^{n_i-1}(x_0),TS^{n_i}(x_0)\right)}{d\left(Tu,TS^{n_i-1}(x_0)\right)} \\ \oplus a_3\frac{d\left(Tu,TSu\right)d\left(TS^{n_i-1}(x_0),TS^{n_i}(x_0)\right)}{d\left(Tu,TS^{n_i-1}(x_0)\right)\oplus d\left(Tu,TS^{n_i}(x_0)\right)\oplus d\left(TS^{n_i-1}(x_0),TSu\right)} \\ \oplus a_4\frac{d\left(Tu,TSu\right)d\left(Tu,TS^{n_i}(x_0)\right)\oplus d\left(TS^{n_i-1}(x_0),TSu\right)d\left(TS^{n_i-1}(x_0),TS^{n_i}(x_0)\right)}{d\left(Tu,TS^{n_i-1}(x_0)\right)\oplus d\left(TS^{n_i-1}(x_0),TSu\right)} \\ \oplus\lambda^{n_i}d\left(Tx_0,TSx_0\right)\oplus d\left(TS^{n_i+1}(x_0),Tu\right) \end{aligned}$$

So

$$(TSu,Tu) \le \lambda d\left(TS^{n_i-1}(x_0),TS^{n_i}(x_0)\right) \oplus \frac{1}{1-\lambda}\lambda^n d\left(Tx_0,TSx_0\right) \oplus \frac{1}{1-\lambda}d\left(TS^{n_i+1}(x_0),Tu\right)$$

Since $\mathcal{F}$ is normal cone with normal constant K, we have

$$(TSu,Tu) \le \lambda K d\left(TS^{n_i-1}(x_0),TS^{n_i}(x_0)\right) \oplus \frac{k}{1-\lambda}\lambda^n d\left(Tx_0,TSx_0\right) \oplus \frac{k}{1-\lambda}d\left(TS^{n_i+1}(x_0),Tu\right)$$

Taking the llimit $i\to\infty$, using (3.33) and $\lambda<1$, we get
$d^i_\alpha\left(TSu,Tu\right)=0$ for all $\alpha\in(0,1]$ and $i=1,2,$
Hence,

$$d\left(TSu,Tu\right)=\bar{0} \tag{3.34}$$

So that $TSu = Tu$.
Since T is one-one, we get $Su = u$. So S has a fixed point.
if v is another fixed point of S, then $Sv = v$. Since S is type I contraction, we obtain

$$\begin{aligned} d\left(TSu,TSv\right) \le a_1\left[d\left(Tu,TSv\right)\oplus d\left(Tv,TSu\right)\right]\oplus a_2\frac{d\left(Tu,TSu\right)d\left(Tv,TSv\right)}{d\left(Tu,Tv\right)}\oplus \\ a_3\frac{d\left(Tu,TSu\right)d\left(Tv,TSv\right)}{d\left(Tu,Tv\right)\oplus d\left(Tu,TSv\right)\oplus d\left(Tv,TSu\right)}\oplus a_4\frac{d\left(Tu,TSu\right)d\left(Tu,TSv\right)\oplus d\left(Tv,TSu\right)d\left(Tv,TSv\right)}{d\left(Tu,TSv\right)\oplus d\left(Tv,TSu\right)} \end{aligned} \tag{3.35}$$

$$\begin{aligned} = a_1\left[d\left(Tu,Tv\right)\oplus d\left(Tv,Tu\right)\right]\oplus a_2\frac{d\left(Tu,Tu\right)d\left(Tv,Tv\right)}{d\left(Tu,Tv\right)}\oplus \\ a_3\frac{d\left(Tu,Tu\right)d\left(Tv,Tv\right)}{d\left(Tu,Tv\right)\oplus d\left(Tu,Tv\right)\oplus d\left(Tv,Tu\right)}\oplus a_4\frac{d\left(Tu,Tu\right)d\left(Tu,Tv\right)\oplus d\left(Tv,Tu\right)d\left(Tv,Tv\right)}{d\left(Tu,Tv\right)\oplus d\left(Tv,Tu\right)} \end{aligned}$$

Implies

$$\begin{aligned} d\left(TSu,TSv\right) &\le 2a_1 d\left(Tu,Tv\right) \\ &< d\left(Tu,Tv\right) \text{ as } 2a_1<1, \end{aligned}$$

this is a contradiction. So that $TSu = TSv$. Since S is injective, we get $u = v$. Thus,S has a unique fixed point.
Lastly, if T is sequentially convergent, by replacing n for n_i, we get that

$$\lim_{n\to\infty} S^n x_0 = u.$$

Thus, $\{S^n x_0\}$ is convergent to the fixed point u.
□

Example 3.5. Consider $E = C[0,1]$ and $\mathcal{F} = \{\eta \in E^*(I) : \eta \geq \bar{0}\}$ and $X = R$. Let $d : X \times X \mapsto E^*(I)$ be a fuzzy mapping define by

$$d(x,y)(t) = \begin{cases} \frac{|x-y|e^{k_0}}{t}, & \text{if } t \geq |x-y|e^{k_0} \\ 0, & \text{if } t < |x-y|e^{k_0} \end{cases}$$

Where k_0 is a fixed number in $[0,1]$. Now,

$$\frac{|x-y|e^{k_0}}{t} \geq \alpha \implies t \leq \frac{|x-y|e^{k_0}}{\alpha}$$

Thus, α-level set of $d(x,y)$ are given by

$$[d(x,y)]_\alpha = \left[|x-y|e^{k_0}.\frac{|x-y|e^{k_0}}{\alpha}\right],\ \alpha \in (0,1].$$

Choose the ordering " $\leq$ " as " $\preceq$ ", then it is easy to verify that,
$(Fd1) d(x,y) \succeq \bar{0}$ and $d(x,y) = \bar{0}$ iff $x = y$;
$(Fd2) d(x,y) = d(y,x)$;
$(Fd3) d(x,y) \preceq d(x,z) \oplus d(z,y)$ forall $x,y,z \in X$.
Then, the pair (X,d) is completely fuzzy cone metric space.
Now, show that (X,d) is a complete fuzzy cone metric space.
Let $\{x_n\}$ be a Cauchy sequence in (X,d). Then $(x_n,x_m) \to \bar{0}$ as $m,n \to \infty$, that is $d^1_\alpha(x_n,x_m) \to \bar{0}$ as $m,n \to \infty$ forall $\alpha \in (0,1]$. So $\{x_n\}$ is Cauchy sequence in $X(R)$. Since X is complete, there exists $x \in X$ such that

$$|x_n - x| \to \bar{0} \text{ as } n \to \infty.$$

Thus, (X,d) is complete. Since for any $\eta,\mu \in E^*(I), \eta \leq \mu \implies \eta \leq 1.\mu$, then,$\mathcal{F}$ is a fuzzy normal cone with normal constant 1.

Now consider the functions $T,S : X \mapsto X$ defined by $Tx = x^2$ and $Sx = \frac{1}{2}$. Let $a_1 = \frac{1}{50}$,$a_2 = \frac{1}{20}$,$a_3 = \frac{1}{30}$,$a_4 = \frac{1}{40}$. Then, we have.

$$d^1_\alpha(TSx,TSy) = |TSx - TSy|e^{k_0} = \left|\frac{x^2}{4} - \frac{y^2}{4}\right| e^{k_0}$$

$$d^1_\alpha(TSx,TSy) \leq \frac{1}{50}\left[|Tx - TSx| + |Ty - TSy|\right]e^{k_0} + \frac{1}{20}\frac{\left[|Tx - TSx||Ty - TSy|\right]e^{k_0}}{|Tx - Ty|e^{k_0}}$$

$$+\frac{1}{30}\frac{\left[|Tx - TSx||Ty - TSy|\right]e^{k_0}}{\left[|Tx - Ty| + |Tx - TSy| + |Ty - TSx|\right]e^{k_0}} + \frac{1}{40}\frac{\left[|Tx - TSx||Tx - TSy| + |Ty - TSx||Ty - TSy|\right]e^{k_0}}{\left[|Tx - TSy| + |Ty - TSx|\right]e^{k_0}}$$

$$d_\alpha^1\left(TSx, TSy\right) \leq \frac{1}{50}\left[d_\alpha^1(Tx,TSx)+d_\alpha^1(Ty,TSy)\right]+\frac{1}{20}\frac{d_\alpha^1(Tx,TSx)d_\alpha^1(Ty,TSy)}{d_\alpha^1(Tx,Ty)}$$
$$+\frac{1}{30}\frac{d_\alpha^1(Tx,TSx)d_\alpha^1(Ty,TSy)}{d_\alpha^1(Tx,Ty)+d_\alpha^1(Tx,TSy)+d_\alpha^1(Ty,TSx)}+\frac{1}{40}\frac{d_\alpha^1(Tx,TSx)d_\alpha^1(Tx,TSy)+d_\alpha^1(Ty,TSx)d_\alpha^1(Ty,TSy)}{d_\alpha^1(Tx,TSy)+d_\alpha^1(Ty,TSx)} \tag{3.36}$$

Also,

$$d_\alpha^2\left(TSx, TSy\right) \leq \frac{1}{50}\left[\frac{|Tx-TSx}{\alpha}+\frac{|Ty-TSy}{\alpha}\right]e^{k_0}+\frac{1}{20}\frac{\left[\frac{|Tx-TSx|}{\alpha}+\frac{|Ty-TSy|}{\alpha}\right]e^{k_0}}{\frac{|Tx-TSx|}{\alpha}e^{k_0}}$$
$$+\frac{1}{30}\frac{\left[\frac{|Tx-TSx|}{\alpha}\frac{|Ty-TSy|}{\alpha}\right]e^{k_0}}{\left[\frac{|Tx-Ty|}{\alpha}+\frac{|Tx-TSy|}{\alpha}+\frac{|Ty-TSx|}{\alpha}\right]}+$$
$$\frac{1}{40}\frac{\left[\frac{|Tx-TSx|}{\alpha}\frac{|Tx-TSy|}{\alpha}+\frac{|Ty-TSx|}{\alpha}\frac{|Ty-TSy|}{\alpha}\right]e^{k_0}}{\left[\frac{|Tx-TSy|}{\alpha}+\frac{|Ty-TSx|}{\alpha}\right]e^{k_0}}$$

$$d_\alpha^2\left(TSx, TSy\right) \leq \frac{1}{50}\left[d_\alpha^2(Tx,TSx)+d_\alpha^2(Ty,TSy)\right]+\frac{1}{20}\frac{d_\alpha^2(Tx,TSx)d_\alpha^2(Ty,TSy)}{d_\alpha^2(Tx,Ty)}$$
$$+\frac{1}{30}\frac{d_\alpha^2(Tx,TSx)d_\alpha^2(Ty,TSy)}{d_\alpha^2(Tx,Ty)+d_\alpha^2(Tx,TSy)+d_\alpha^2(Ty,TSx)}+\frac{1}{40}\frac{d_\alpha^2(Tx,TSx)d_\alpha^2(Tx,TSy)+d_\alpha^2(Ty,TSx)d_\alpha^2(Ty,TSy)}{d_\alpha^2(Tx,TSy)+d_\alpha^2(Ty,TSx)} \tag{3.37}$$

From (3.36) and (3.37), we have

$$d\left(TSx, TSy\right) \leq \frac{1}{50}\left[d(Tx,TSx)+d(Ty,TSy)\right]+\frac{1}{20}\frac{d(Tx,TSx)d(Ty,TSy)}{d(Tx,Ty)}$$
$$+\frac{1}{30}\frac{d(Tx,TSx)d(Ty,TSy)}{d(Tx,Ty)+d(Tx,TSy)+d(Ty,TSx)}+\frac{1}{40}\frac{d(Tx,TSx)d(Tx,TSy)+d(Ty,TSx)d(Ty,TSy)}{d(Tx,TSy)+d(Ty,TSx)}$$

Thus, S is a fuzzy cone T-type I contraction for $2a_1+a_2+a_3+a_4<1$.
Now, to show that S is a fuzzy T-type II contraction, Let $a_1=\frac{1}{30}$,$a_2=\frac{1}{10}$,$a_3=\frac{1}{20}$,$a_4=\frac{1}{40}$.

Then, we have

$$d_\alpha^1(TSx,TSy) = |TSx - TSy|e^{k_0} = \left|\frac{x^2}{4} - \frac{y^2}{4}\right| e^{k_0}$$

$$\begin{aligned} d_\alpha^1(TSx,TSy) \leq \frac{1}{30}\left[\left|\left(x^2 - \frac{y^2}{4}\right) - \left(y^2 - \frac{x^2}{4}\right)\right|\right] e^{k_0} + \frac{1}{10}\frac{\left[\left|\left(x^2 - \frac{x^2}{4}\right)\left(y^2 - \frac{y^2}{4}\right)\right|\right] e^{k_0}}{|x^2 - y^2|e^{k_0}} \\ + \frac{1}{20}\frac{\left[\left|\left(x^2 - \frac{x^2}{4}\right)\left(y^2 - \frac{y^2}{4}\right)\right|\right] e^{k_0}}{\left[|x^2 - y^2| + \left|\left(x^2 - \frac{y^2}{4}\right) - \left(y^2 - \frac{x^2}{4}\right)\right|\right] e^{k_0}} \\ + \frac{1}{40}\frac{\left[\left|\left(x^2 - \frac{x^2}{4}\right)\left(x^2 - \frac{y^2}{4}\right) - \left(y^2 - \frac{x^2}{4}\right)\left(y^2 - \frac{y^2}{4}\right)\right|\right] e^{k_0}}{\left[\left|\left(x^2 - \frac{y^2}{4}\right) - \left(y^2 - \frac{x^2}{4}\right)\right|\right] e^{k_0}} \end{aligned}$$

$$\begin{aligned} d_\alpha^1(TSx,TSy) \leq \frac{1}{30}[|Tx - TSy| + |Ty - TSx|]e^{k_0} + \frac{1}{10}\frac{[|Tx - TSx||Ty - TSy|]e^{k_0}}{[|Tx - Ty|]e^{k_0}} \\ + \frac{1}{20}\frac{[|Tx - TSx||Ty - TSy|]e^{k_0}}{[[|Tx - Ty|]e^{k_0} + |Tx - TSy| + |Ty - TSx|]\, e^{k_0}} \\ + \frac{1}{40}\frac{[|Tx - TSx||Tx - TSy| + |Ty - TSx||Ty - TSy|]\, e^{k_0}}{[|Tx - TSy| + |Ty - TSx|]\, e^{k_0}} \end{aligned}$$

Implies

$$\begin{aligned} d_\alpha^1(TSx,TSy) \leq \frac{1}{30}\left[d_\alpha^1(Tx,TSy) + d_\alpha^1(Ty,TSx)\right] + \frac{1}{10}\frac{d_\alpha^1(Tx,TSx)\, d_\alpha^1(Ty,TSy)}{d_\alpha^1(Tx,Ty)} \\ + \frac{1}{20}\frac{d_\alpha^1(Tx,TSx)\, d_\alpha^1(Ty,TSy)}{d_\alpha^1(Tx,Ty) + d_\alpha^1(Tx,TSy) + d_\alpha^1(Ty,TSx)} \\ + \frac{1}{40}\frac{d_\alpha^1(Tx,TSx)\, d_\alpha^1(Tx,TSy) + d_\alpha^1(Ty,TSx)\, d_\alpha^1(Ty,TSy)}{d_\alpha^1(Tx,TSy) + d_\alpha^1(Ty,TSx)} \end{aligned} \tag{3.38}$$

Also,

$$\begin{aligned} d_\alpha^2(TSx,TSy) \leq \frac{1}{30}\left[d_\alpha^2(Tx,TSy) + d_\alpha^2(Ty,TSx)\right] + \frac{1}{10}\frac{d_\alpha^2(Tx,TSx)\, d_\alpha^2(Ty,TSy)}{d_\alpha^2(Tx,Ty)} \\ + \frac{1}{20}\frac{d_\alpha^2(Tx,TSx)\, d_\alpha^2(Ty,TSy)}{d_\alpha^2(Tx,Ty) + d_\alpha^2(Tx,TSy) + d_\alpha^2(Ty,TSx)} \\ + \frac{1}{40}\frac{d_\alpha^2(Tx,TSx)\, d_\alpha^2(Tx,TSy) + d_\alpha^2(Ty,TSx)\, d_\alpha^2(Ty,TSy)}{d_\alpha^2(Tx,TSy) + d_\alpha^2(Ty,TSx)} \end{aligned} \tag{3.39}$$

From (3.38) and (3.39), we have

$$\begin{aligned} d\left(TSx,TSy\right) \le \frac{1}{30}\left[d\left(Tx,TSy\right)+d\left(Ty,TSx\right)\right]+\frac{1}{10}\frac{d\left(Tx,TSx\right)d\left(Ty,TSy\right)}{d\left(Tx,Ty\right)} \\ +\frac{1}{20}\frac{d\left(Tx,TSx\right)d\left(Ty,TSy\right)}{d\left(Tx,Ty\right)+d\left(Tx,TSy\right)+d\left(Ty,TSx\right)} \\ +\frac{1}{40}\frac{d\left(Tx,TSx\right)d\left(Tx,TSy\right)+d\left(Ty,TSx\right)d\left(Ty,TSy\right)}{d\left(Tx,TSy\right)+d\left(Ty,TSx\right)} \end{aligned}$$

Thus, S is a fuzzy cone T-type II contraction for $2a_1+a_2+a_3+a_4<1$.

Example 3.6. Consider $E=C[0,1]$ and $\mathcal{F}=\{\eta\in E^*(I):\eta\ge\bar{0}\}\subset E^*(I)$, $X=R$. Let $d:X\times X\mapsto E^*(I)$ be a fuzzy mapping define by

$$d(x,y)=|x-y|e^{k_0}\ ,e^{k_0}\in E$$

Where k_0 is a fixed number in $[0,1]$ and the α-level set of $d(x,y)$ are given by

$$[d(x,y)]_\alpha=\left[|x-y|e^{k_0}.\frac{|x-y|e^{k_0}}{\alpha}\right],\ \alpha\in(0,1].$$

Then, the pair (X,d) is called a fuzzy cone metric space as in Example 3.5 and consider the functions $T,S:X\mapsto X$ defined by $Tx=x$ and $Sx=\frac{x}{2}$. Clearly, T is one-one and continuous. Then, by Theorem 3.4, $v=0$ is the unique fixed point of S in X.

Corollary 3.7. *Suppose $(X,\ d)$ is a complete fuzzy cone metric space, $\mathcal{F}$ be a normal fuzzy cone with normal constant K. Let $T:X\to X$ be a one-one continuous function and $S:X\to X$ be a fuzzy cone T -type I contraction mapping, that is, for all $x,y\in X$,$Tx\ne Ty$ and $a_1,a_2,a_3\ge 0$ with $2a_1+a_2+a_3<1$ satisfying the following*

$$\begin{aligned} d\left(TSx,TSy\right)\le a_1\left[d(Tx,TSx)\oplus d(Ty,TSy)\right]\oplus a_2\frac{d(Tx,TSx)d(Ty,TSy)}{d(Tx,Ty)} \\ \oplus a_3\frac{d(Tx,TSx)d(Ty,TSy)}{d(Tx,Ty)\oplus d(Tx,TSy)\oplus d(Ty,TSx)} \end{aligned} \tag{3.40}$$

Then, the following conditions are satisfied:
i. for every $x_0\in X$, $\lim_{n\to\infty}d(TS^nx_0,TS^{n+1}x_0)=\bar{0}$;
ii. there exists $v\in X$ such that $\lim_{n\to\infty}TS^nx_0=v$;
iii. if T is sequentially convergent, then $\{S^nx_0\}$ has a convergent subsequence;
iv. there is a unique $u\in X$ such that $Su=u$;
v. if T is sequentially convergent, then for each $x_0\in X$ the iterate sequence $\{S^nx_0\}$ converges to u.

Proof. Let $a_4=0$ in Theorem 3.3, we get the result immediately. □

Corollary 3.8. *Suppose $(X,\ d)$ is a complete fuzzy cone metric space, $\mathcal{F}$ be a normal fuzzy cone with normal constant K. Let $T:X\to X$ be a one-one continuous function and $S:X\to X$ be a fuzzy cone T -type I contraction mapping, that is, for all $x,y\in X$,$Tx\ne Ty$ and $a_1,a_2,a_3\ge 0$ with $2a_1+a_2+a_3<1$ satisfying* *the following*

$$d\left(TSx,TSy\right)\leq a_1\left[d(Tx,TSx)\oplus d(Ty,TSy)\right]\oplus a_2\frac{d(Tx,TSx)d(Ty,TSy)}{d(Tx,Ty)}$$
$$\oplus a_3\frac{d(Tx,TSx)d(Tx,TSy)\oplus d(Ty,TSx)d(Ty,TSy)}{d(Tx,TSy)\oplus d(Ty,TSx)} \tag{3.41}$$

Then, the following conditions are satisfied:
i. for every $x_0\in X$, $\lim_{n\to\infty}d(TS^nx_0,TS^{n+1}x_0)=\bar{0}$;
ii. there exists $v\in X$ *such that* $\lim_{n\to\infty}TS^nx_0=v$;
iii. if T *is sequentially convergent, then* $\{S^nx_0\}$ *has a convergent subsequence;*
iv. there is a unique $u\in X$ *such that* $Su=u$;
v. if T *is sequentially convergent, then for each* $x_0\in X$ *the iterate sequence* $\{S^nx_0\}$ *converges to* u.

Proof. Let $a_3=0$ in Theorem 3.3, we get the result immediately. □

Corollary 3.9. *Suppose* $(X,\ d)$ *is a complete fuzzy cone metric space,* $\mathcal{F}$ *be a normal fuzzy cone with normal constant* K. *Let* $T:X\to X$ *be a one-one continuous function and* $S:X\to X$ *be a fuzzy cone* T *-type I contraction mapping, that is, for all* $x,y\in X$,$Tx\neq Ty$ *and* $a_1,a_2,a_3\geq 0$ *with* $2a_1+a_2+a_3<1$ *satisfying the following*

$$d\left(TSx,TSy\right)\leq a_1\left[d(Tx,TSx)\oplus d(Ty,TSy)\right]$$
$$\oplus a_2\frac{d(Tx,TSx)d(Ty,TSy)}{d(Tx,Ty)\oplus d(Tx,TSy)\oplus d(Ty,TSx)}\oplus a_3\frac{d(Tx,TSx)d(Tx,TSy)\oplus d(Ty,TSx)d(Ty,TSy)}{d(Tx,TSy)\oplus d(Ty,TSx)} \tag{3.42}$$

Then, the following conditions are satisfied:
i. for every $x_0\in X$, $\lim_{n\to\infty}d(TS^nx_0,TS^{n+1}x_0)=\bar{0}$;
ii. there exists $v\in X$ *such that* $\lim_{n\to\infty}TS^nx_0=v$;
iii. if T *is sequentially convergent, then* $\{S^nx_0\}$ *has a convergent subsequence;*
iv. there is a unique $u\in X$ *such that* $Su=u$;
v. if T *is sequentially convergent, then for each* $x_0\in X$ *the iterate sequence* $\{S^nx_0\}$ *converges to* u.

Proof. Let $a_2=0$ in Theorem 3.3, we get the result immediately. □

Corollary 3.10. *Suppose* $(X,\ d)$ *is a complete fuzzy cone metric space,* $\mathcal{F}$ *be a normal fuzzy cone with normal constant* K. *Let* $T:X\to X$ *be a one-one continuous function and* $S:X\to X$ *be a fuzzy cone* T *-type I contraction mapping, that is, for all* $x,y\in X$,$Tx\neq Ty$ *and* $a_1,a_2,a_3\geq 0$ *with* $2a_1+a_2+a_3<1$ *satisfying the following*

$$d\left(TSx,TSy\right)\leq a_1\frac{d(Tx,TSx)d(Ty,TSy)}{d(Tx,Ty)}$$
$$\oplus a_2\frac{d(Tx,TSx)d(Ty,TSy)}{d(Tx,Ty)\oplus d(Tx,TSy)\oplus d(Ty,TSx)}\oplus a_3\frac{d(Tx,TSx)d(Tx,TSy)\oplus d(Ty,TSx)d(Ty,TSy)}{d(Tx,TSy)\oplus d(Ty,TSx)} \tag{3.43}$$

Then, the following conditions are satisfied:
i. for every $x_0\in X$, $\lim_{n\to\infty}d(TS^nx_0,TS^{n+1}x_0)=\bar{0}$;

ii. there exists $v \in X$ *such that* $\lim_{n\to\infty} TS^n x_0 = v$;
iii. if T *is sequentially convergent, then* $\{S^n x_0\}$ *has a convergent subsequence;*
iv. there is a unique $u \in X$ *such that* $Su = u$;
v. if T *is sequentially convergent, then for each* $x_0 \in X$ *the iterate sequence* $\{S^n x_0\}$ *converges to* u.

Proof. Let $a_1 = 0$ in Theorem 3.3, we get the result immediately. □

Corollary 3.11. *Suppose* $(X,\ d)$ *is a complete fuzzy cone metric space,* $\mathcal{F}$ *be a normal fuzzy cone with normal constant* K. *Let* $T : X \to X$ *be a one-one continuous function and* $S : X \to X$ *be a fuzzy cone* T *-type II contraction mapping, that is, for all* $x, y \in X, Tx \neq Ty$ *and* $a_1, a_2, a_3 \geq 0$ *with* $2a_1 + a_2 + a_3 < 1$ *satisfying the following*

$$\begin{aligned} d(TSx, TSy) \leq a_1 \left[d(Tx, TSy) \oplus d(Ty, TSx) \right] \oplus a_2 \frac{d(Tx, TSx)\, d(Ty, TSy)}{d(Tx, Ty)} \\ \oplus a_3 \frac{d(Tx, TSx)\, d(Ty, TSy)}{d(Tx, Ty) \oplus d(Tx, TSy) \oplus d(Ty, TSx)} \end{aligned} \tag{3.44}$$

Then, the following conditions are satisfied:
i. for every $x_0 \in X$, $\lim_{n\to\infty} d(TS^n x_0, TS^{n+1} x_0) = \bar{0}$;
ii. there exists $v \in X$ *such that* $\lim_{n\to\infty} TS^n x_0 = v$;
iii. if T *is sequentially convergent, then* $\{S^n x_0\}$ *has a convergent subsequence;*
iv. there is a unique $u \in X$ *such that* $Su = u$;
v. if T *is sequentially convergent, then for each* $x_0 \in X$ *the iterate sequence* $\{S^n x_0\}$ *converges to* u.

Proof. Let $a_4 = 0$ in Theorem 3.4, we get the result immediately. □

Corollary 3.12. *Suppose* $(X,\ d)$ *is a complete fuzzy cone metric space,* $\mathcal{F}$ *be a normal fuzzy cone with normal constant* K. *Let* $T : X \to X$ *be a one-one continuous function and* $S : X \to X$ *be a fuzzy cone* T *-type II contraction mapping, that is, for all* $x, y \in X, Tx \neq Ty$ *and* $a_1, a_2, a_3 \geq 0$ *with* $2a_1 + a_2 + a_3 < 1$ *satisfying the following*

$$\begin{aligned} d(TSx, TSy) \leq a_1 \left[d(Tx, TSy) \oplus d(Ty, TSx) \right] \oplus a_2 \frac{d(Tx, TSx)\, d(Ty, TSy)}{d(Tx, Ty)} \\ \oplus a_3 \frac{d(Tx, TSx)\, d(Tx, TSy) \oplus d(Ty, TSx)\, d(Ty, TSy)}{d(Tx, TSy) \oplus d(Ty, TSx)} \end{aligned} \tag{3.45}$$

Then, the following conditions are satisfied:
i. for every $x_0 \in X$, $\lim_{n\to\infty} d(TS^n x_0, TS^{n+1} x_0) = \bar{0}$;
ii. there exists $v \in X$ *such that* $\lim_{n\to\infty} TS^n x_0 = v$;
iii. if T *is sequentially convergent, then* $\{S^n x_0\}$ *has a convergent subsequence;*
iv. there is a unique $u \in X$ *such that* $Su = u$;
v. if T *is sequentially convergent, then for each* $x_0 \in X$ *the iterate sequence* $\{S^n x_0\}$ *converges to* u.

Proof. Let $a_3 = 0$ in Theorem 3.4, we get the result immediately. □

Corollary 3.13. *Suppose* $(X,\ d)$ *is a complete fuzzy cone metric space,* $\mathcal{F}$ *be a normal fuzzy cone with normal constant* K. *Let* $T : X \to X$ *be a one-one continuous function and* $S : X \to X$ *be a fuzzy cone* T

-type II contraction mapping, that is, for all $x, y \in X$,$Tx \neq Ty$ and $a_1, a_2, a_3 \geq 0$ with $2a_1 + a_2 + a_3 < 1$ satisfying the following

$$d(TSx, TSy) \leq a_1 [d(Tx, TSy) \oplus d(Ty, TSx)] \oplus a_2 \frac{d(Tx, TSx)\, d(Ty, TSy)}{d(Tx, Ty) \oplus d(Tx, TSy) \oplus d(Ty, TSx)} \oplus a_3 \frac{d(Tx, TSx)\, d(Tx, TSy) \oplus d(Ty, TSx)\, d(Ty, TSy)}{d(Tx, TSy) \oplus d(Ty, TSx)} \tag{3.46}$$

Then, the following conditions are satisfied:
i. for every $x_0 \in X$, $\lim_{n\to\infty} d(TS^n x_0, TS^{n+1} x_0) = \bar{0}$;
ii. there exists $v \in X$ such that $\lim_{n\to\infty} TS^n x_0 = v$;
iii. if T is sequentially convergent, then $\{S^n x_0\}$ has a convergent subsequence;
iv. there is a unique $u \in X$ such that $Su = u$;
v. if T is sequentially convergent, then for each $x_0 \in X$ the iterate sequence $\{S^n x_0\}$ converges to u.

Proof. Let $a_2 = 0$ in Theorem 3.4, we get the result immediately. □

4 Conclusion

The main findings of this study demonstrate applicability fuzzy cone metric spaces in establishing fixed point theorems for fuzzy mappings. This study provides significant advancements in the understanding of fuzzy cone metric spaces, with potential applications in differential equations and nonlinear Fredholm integral equation. Future work could also explore the extension of this results to other types of fuzzy mappings and their applications in real-world problems.

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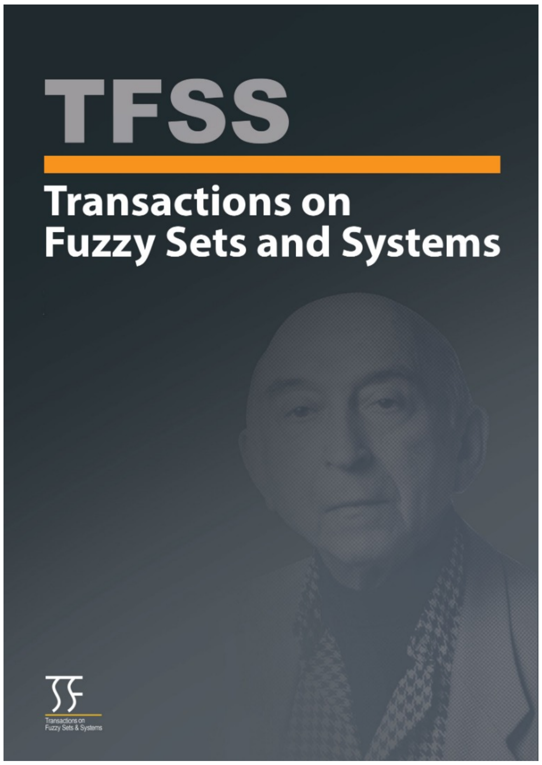

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Some Result on Fuzzy Integration

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Abstract. In this article, we introduce a new concept of fuzzy measurement, the space of fuzzy measurable functions and fuzzy integral, which has a dynamic position and is different from previous approaches. With this concept, we create a new version of measurement theory and fuzzy integral. The main goal of this paper is to define the fuzzy integral in the fuzzy size space. First, we introduce fuzzy measurable functions and L^+ essential and related concepts in fuzzy space. In the continuation of the work, with the help of fuzzy measurable functions, we define the fuzzy integral in the fuzzy measurement space and examine the theorems related to it and the relationship between them in the fuzzy measurement space. The next step is to establish one of the fundamental convergence theorems with the uniform convergence theorem in the fuzzy measurement space and prove it. Finally, we prove Fatou's lemma as an application of the theorems raised in the fuzzy measurement space.

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1 Introduction

Here, we let $\Xi = [0,1]$, $\Upsilon = [0,\infty]$ and $\mho = (0,\infty)$.

Let us assume $\star : \Xi \times \Xi \to \Xi$ is a topological monoid with unit 1 and $\varrho \star \gamma \leq \varsigma \star \delta$ whenever $\varrho \leq \gamma$ and $\varsigma \leq \delta \quad (\varrho, \varsigma, \gamma, \delta \in \Xi)$. In this case, $\star$ is called a continuous t-norm and $\star_{i=1}^{\infty} = \lim_{n\to\infty} \star_{i=1}^{n}$. For some examples $\varrho \star \varsigma = \varrho.\varsigma$ and $\varrho \star \varsigma = \wedge(\varrho, \varsigma)$ are continuous t-norms.

Let us consider $U \neq \emptyset$, $\star$ is a continuous t-norm and ρ is a fuzzy set on $U^2 \times \mho$. Then $(U, \rho, \star)$ is said to be a fuzzy metric space where for arbitrary $\varepsilon, \upsilon, \eta \in U$ and $\tau, \theta > 0$,

(FM1) $\rho(\varepsilon, \upsilon, \tau) = 1$ for every $\tau \in \mho$ iff $\varepsilon = \upsilon$;

(FM2) $\rho(\varepsilon, \upsilon, \tau) = \rho(\upsilon, \varepsilon, \tau), \forall \varepsilon, \upsilon \in U, \quad \forall \tau \in \mho$;

(FM3) $\rho(\varepsilon, \eta, \tau + \theta) \geq \rho(\varepsilon, \upsilon, \tau) \star \rho(\upsilon, \eta, \theta), \quad \forall \varepsilon, \upsilon, \eta \in U, \quad \forall \tau, \theta \in \mho$;

(FM4) $\rho(\varepsilon, \upsilon, \cdot) : \mho \to (0,1]$ is continuous.([1, 2, 3, 4, 5, 6, 7, 8])

Definition 1.1. Suppose that $\mathfrak{X} \neq \emptyset$ and $\mathcal{C} \subseteq 2^{\mathfrak{X}}$ such that

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(i) $\emptyset \in \mathcal{C}$ and $\mathfrak{X} \in \mathcal{C}$;

(ii) if $\mathfrak{A} \in \mathcal{C}$, then $\mathfrak{A}^c \in \mathcal{C}$;

(iii) if $\mathfrak{A}_1, \cdots, \mathfrak{A}_n \in \mathcal{C}$, then $\cup_{i=1}^{n} \mathfrak{A}_i$ and $\cap_{i=1}^{n} \mathfrak{A}_i$ are in $\mathcal{C}$.

(iv) whenever $\mathfrak{A}_1, \mathfrak{A}_2, \cdots$ are in $\mathcal{C}$, then $\cup_{i=1}^{\infty} \mathfrak{A}_i$ and $\cap_{i=1}^{\infty} \mathfrak{A}_i$ are in $\mathcal{C}$.

So $\mathcal{C}$ is called a σ-algebra and $(\mathfrak{X}, \mathcal{C})$ is called a measurable space.

Definition 1.2. Let us assume that $\mathfrak{X} \neq \emptyset$ and $\mathcal{C} \subseteq 2^{\mathfrak{X}}$ a σ-algebra. A fuzzy function $\nu : \mathcal{C} \times \mho \to \Xi$ such that

(i) $\nu(\emptyset, \tau) = 1, \quad \forall \tau \in \mho$;

(ii) if $\mathfrak{A}_i \in \mathcal{C}$, for $i = 1, 2, \cdots$ are pairwise disjoint,

$$\nu(\cup_{i=1}^{\infty} \mathfrak{A}_i^{\infty}, \tau) = \star_{i=1}^{\infty} \nu(\mathfrak{A}_i, \tau), \quad \forall \tau \in \mho.$$

So ν is said to be a fuzzy measure and $(\mathfrak{X}, \mathcal{C}, \nu, \star)$ is said to be a fuzzy measure space.

Definition 1.3. Consider $(\mathfrak{X}, \mathcal{M})$ and $(\mathfrak{Y}, \mathcal{N})$ measurable spaces, a mapping $\jmath : \mathfrak{X} \to \mathfrak{Y}$ is called $(\mathcal{M}, \mathcal{N})$-measurable if $\jmath^{-1}(\mathfrak{E}) \in \mathcal{M}$ for all $\mathfrak{E} \in \mathcal{N}$. We know $\mathcal{B}_{\mathbb{R}}$ as σ-algebra on $\mathbb{R}$.

2 Main Results

Theorem 2.1. *Consider $\mathfrak{X}$ is a metric (or topological) space, then every cotinuous $\jmath : \mathfrak{X} \to \mathbb{R}$ is $(\mathcal{B}_{\mathfrak{X}}, \mathcal{B}_{\mathbb{R}})$-measurable.*

Proof. $\jmath$ is continuouse iff $\jmath^{-1}(U)$ in $\mathfrak{X}$ for every $U \subseteq \mathbb{R}$. $\square$

Theorem 2.2. *Consider $(\mathfrak{X}, \mathcal{M})$, $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ measurable spaces and $\jmath : \mathfrak{X} \to \mathbb{R}$, then the following statements are equivalence.*

(i) $\jmath$ *is* $\mathcal{M}$*-measurable;*

(ii) $\jmath^{-1}((q, \infty]) \in \mathcal{M}, \quad \forall q \in \mathbb{R}$;

(iii) $\jmath^{-1}([q, \infty]) \in \mathcal{M}, \quad \forall q \in \mathbb{R}$.

Lemma 2.3. *Suppose $\jmath, \imath : \mathfrak{X} \to \mathbb{R}$ are $\mathcal{M}$-measurable so $\mathfrak{F} : \mathfrak{X} \to \mathbb{R} \times \mathbb{R}$ with $\mathfrak{F}(p) = (\jmath(d), \imath(p))$ is $\mathcal{M}$-measurable.*

Proof. We know $\mathcal{B}_{\mathbb{R}\times\mathbb{R}} = \mathcal{B}_{\mathbb{R}} \bigotimes \mathcal{B}_{\mathbb{R}}$. So $\mathfrak{F}$ is a $(\mathcal{M}, \mathcal{B}_{\mathbb{R}\times\mathbb{R}})$-measurable. $\square$

Theorem 2.4. *If $\jmath, \imath : \mathfrak{X} \to \mathbb{R}$ are $\mathcal{M}$-measurable, then $\jmath + \imath : \mathfrak{X} \to \mathbb{R}$ with $(\jmath + \imath)(p) = \jmath(p) + \imath(p)$ is a $\mathcal{M}$-measurable.*

Proof. Define $\mathfrak{F} : \mathfrak{X} \to \mathbb{R} \times \mathbb{R}$ with $\mathfrak{F}(p) = (\jmath(p), \imath(p))$, $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with $\phi(z, w) = z + w$. Since $\mathcal{B}_{\mathbb{R}\times\mathbb{R}} = \mathcal{B}_{\mathbb{R}} \bigotimes \mathcal{B}_{\mathbb{R}}$, $\mathfrak{F}$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}\times\mathbb{R}})$-measurable, wherease ϕ is $(\mathcal{B}_{\mathbb{R}\times\mathbb{R}}, \mathcal{B}_{\mathbb{R}})$-measurable by theorem 2.1. Thus $\jmath + \imath = \phi \circ \mathfrak{F}$ is $\mathcal{M}$-measurable. $\square$

Theorem 2.5. *If $\jmath_j$ is a sequence of $\mathbb{R}$-valued measurable functions on $(\mathfrak{X},\mathcal{M})$, then the functions $\imath_n(p) = \sup_{j\geq n} \jmath_j(p)$, $\imath(p) = \limsup \jmath_j(p)$, $\mathfrak{h}_n(p) = \inf_{j\geq n} \jmath_j(p)$, $\mathfrak{h}(p) = \liminf \jmath_j(p)$ are $\mathcal{M}$-measurable. If $\jmath(p) = \lim_{j\to\infty} \jmath_j(p)$ exists for every $p \in \mathfrak{X}$, then $\jmath$ is $\mathcal{M}$-measurable.*

Proof. We have $\imath_n^{-1}((q,\infty]) = \cup_{j=n}^{\infty}\jmath_j^{-1}((q,\infty]) \in \mathcal{M}$, $\mathfrak{h}_n^{-1}((q,\infty]) = \cap_{j=n}^{\infty}\jmath_j^{-1}((q,\infty]) \in \mathcal{M}$, more generally, if $\mathfrak{h}_k(p) = \sup_{j\geq n} \jmath_j(p)$ then $\mathfrak{h}_k$ is measurable for each k, so $\imath(p) = \inf_{k\geq 1} \mathfrak{h}_k(p)$ is measurable, and likewise for $\mathfrak{h}$. Finally, $\jmath$ exists then $\jmath = \imath = \mathfrak{h}$, so $\jmath$ is measurable. □

Corollary 2.6. *If $\jmath, \imath : \mathfrak{X} \to \mathbb{R}$ are fuzzy measurable, then so* $\max\{\jmath,\imath\}$ *and* $\min\{\jmath,\imath\}$.

Proof. We now discuss the functions that are building blocks for the theory of integration. Let $(\mathfrak{X},\mathcal{M})$ be a measurable space. If $\mathfrak{E} \subseteq \mathfrak{X}$, the characteristic function $\chi_{\mathfrak{E}}$ of $\mathfrak{E}$ is given as

$$\chi_{\mathfrak{E}}(p) = \begin{cases} 1 & p \in \mathfrak{E}, \\ 0 & p \notin \mathfrak{E}. \end{cases}$$

$\chi_{\mathfrak{E}}$ is measurable iff $\mathfrak{E} \in \mathcal{M}$. A simple function on $\mathfrak{X}$ is a finite linear combination, with coefficients in $\mathbb{R}$, of characteristic functions of sets in $\mathcal{M}$. So $\jmath : \mathfrak{X} \to \mathbb{R}$ is simple iff $\jmath$ is measurable and the range of $\jmath$ is a finite subset of $\mathbb{R}$. Indeed, we have $\jmath = \sum_{j=1}^{n} z_j \chi_{\mathfrak{E}_j}$, where $\mathfrak{E}_j = \jmath^{-1}(\{z_j\})$ and $range(\jmath) = \{z_1, z_2, \cdots, z_n\}$, we call this the standard representation of $\jmath$. □

Theorem 2.7. *If $\jmath, \imath : \mathfrak{X} \to \mathbb{R}$ are simple functions, so then $\jmath + \imath$.*

Theorem 2.8. *Consider $(\mathfrak{X},\mathcal{M})$ measurable space. If $\jmath : \mathfrak{X} \to \mathbb{R}$ is fuzzy measurable, there exist a sequence $\{\phi_n\}$ of fuzzy simple functions such that $0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq \jmath$ pointwise, and $\phi_n \to \jmath$ uniformly on $\mathfrak{X}$.*

Definition 2.9. Consider $(\mathfrak{X},\mathcal{C},\nu)$ measure space, define

$$L^+ = \{\jmath : \mathfrak{X} \times \mho \to [0,\infty) \mid \jmath \text{ is measurable function and increase with second component}\}.$$

Consider $\phi \in L^+$ a simple function by $\phi = \sum_{i=1}^{n} q_i \chi_{\mathfrak{E}_i}$, for every $q_i \geq 0$, we define the fuzzy integral of ϕ with respect to ν by

$$\int_{\mathfrak{X}} \phi(p) d\nu(p,\tau) = \star_{i=1}^{n} \nu(\mathfrak{E}_i, \frac{\tau}{q_i}),$$

for every $\tau \in \mho$. If $\mathfrak{A} \in \mathcal{C}$, then $\phi\chi_{\mathfrak{A}}$ is also simple function and define $\int_{\mathfrak{A}} \phi(p) d\nu(p,\tau)$ to be $\int_{\mathfrak{X}} (\phi\chi_{\mathfrak{A}})(p) d\nu(p,\tau)$.

Theorem 2.10. *Consider ϕ and ψ simple functions in L^+.*

(i) *If $c \in \Xi$ then $\int_{\mathfrak{X}} (c\phi)(p) d\nu(p,\tau) \geq c \int_{\mathfrak{X}} \phi(p) d\nu(p,\tau)$, if $c \in (1,\infty)$ then $\int_{\mathfrak{X}} (c\phi)(p) d\nu(p,\tau) \leq c \int_{\mathfrak{X}} \phi(p) d\nu(p,\tau)$, $\forall \tau \in \mho$;*

(ii) $\int_{\mathfrak{X}} (\phi + \psi)(p) d\nu(p,\tau) \geq \left(\int_{\mathfrak{X}} \phi(p) d\nu(p,\tau) \star \int_{\mathfrak{X}} \psi(p) d\nu(p,\tau)\right)$, $\quad \forall \tau \in \mho$;

(iii) *If $\phi \leq \psi$, then $\int_{\mathfrak{X}} \phi(p) d\nu(p,\tau) \geq \int_{\mathfrak{X}} \psi(p) d\nu(p,\tau)$,* $\quad \forall \tau \in \mho$;

(iv) *The map $\mathfrak{A} \to \int_{\mathfrak{A}} \phi(p) d\nu(p,\tau)$ is a measure on $\mathcal{C}$,* $\quad \forall \tau \in \mho$.

Proof.

(i) : If $c \in \Xi$ then

$$\begin{aligned} & \int_{\mathfrak{X}} (c\phi)(p) d\nu(p,\tau) \\ = & \int_{\mathfrak{X}} \left(\sum_{i=1}^{n} c a_i \chi_{\mathfrak{E}_i}(p) \right) d\nu(p,\tau) \\ = & \star_{i=1}^{n} \nu \left(\mathfrak{E}_i, \frac{\tau}{c a_i} \right) \\ \geq & \star_{i=1}^{n} \nu \left(\mathfrak{E}_i, \frac{\tau}{q_i} \right) \\ = & \int_{\mathfrak{X}} (c\phi)(p) d\nu(p,\tau) \\ \geq & c \int_{\mathfrak{X}} \phi(p) d\nu(p,\tau), \end{aligned}$$

if If $c \in (1, \infty)$ then

$$\begin{aligned} & \int_{\mathfrak{X}} (c\phi)(p) d\nu(p,\tau) \\ = & \int_{\mathfrak{X}} \left(\sum_{i=1}^{n} c a_i \chi_{\mathfrak{E}_i}(p) \right) d\nu(p,\tau) \\ = & \star_{i=1}^{n} \nu \left(\mathfrak{E}_i, \frac{\tau}{c a_i} \right) \\ \leq & \star_{i=1}^{n} \nu \left(\mathfrak{E}_i, \frac{\tau}{q_i} \right) \\ = & \int_{\mathfrak{X}} (c\phi)(p) d\nu(p,\tau) \\ \leq & c \int_{\mathfrak{X}} \phi(p) d\nu(p,\tau). \end{aligned}$$

(ii) :

$$\begin{aligned} & \int_{\mathfrak{X}} (\phi + \psi)(p) d\nu(p,\tau) \qquad (1) \\ = & \int_{\mathfrak{X}} \left(\left(\sum_{i=1}^{n} q_i \chi_{\mathfrak{E}_i}(p) \right) + \left(\sum_{j=1}^{m} b_j \chi_{\mathfrak{F}_j}(p) \right) \right) d\nu(p,\tau) \\ = & \int_{\mathfrak{X}} \left(\sum_{i,j} (q_i + b_j) \chi_{\mathfrak{E}_i \cap \mathfrak{F}_j}(p) \right) d\nu(p,\tau) \\ = & \star_{i=1}^{n} \star_{j=1}^{m} \nu \left((\mathfrak{E}_i \cap \mathfrak{F}_j), \frac{\tau}{(q_i + b_j)} \right) \end{aligned}$$

Since

$$
\begin{aligned}
& \left(\int_{\mathfrak{X}} \phi(p) d\nu(p,\tau) \star \int_{\mathfrak{X}} \psi(p) d\nu(p,\tau)\right) \\
= & \left(\int_{\mathfrak{X}} \left(\sum_{i=1}^{n} q_i \chi_{\mathfrak{E}_i}(p)\right) d\nu(p,\tau)\right) \star \left(\int_{\mathfrak{X}} \left(\sum_{j=1}^{m} b_j \chi_{\mathfrak{F}_j}(p)\right) d\nu(p,\tau)\right) \\
= & \left(\star_{i=1}^{n} \star_{j=1}^{m} \nu\left(\left(\mathfrak{E}_i \cap \mathfrak{F}_j\right), \frac{\tau}{q_i}\right)\right) \star \left(\star_{j=1}^{m} \star_{i=1}^{n} \nu\left(\left(\mathfrak{E}_i \cap \mathfrak{F}_j\right), \frac{\tau}{b_j}\right)\right) \\
= & \star_{i=1}^{n} \star_{j=1}^{m} \left(\nu\left(\left(\mathfrak{E}_i \cap \mathfrak{F}_j\right), \frac{\tau}{q_i}\right) \star \nu\left(\left(\mathfrak{E}_i \cap \mathfrak{F}_j\right), \frac{\tau}{b_j}\right)\right) \\
\leq & \star_{i=1}^{n} \star_{j=1}^{m} \left(\nu\left(\left(\mathfrak{E}_i \cap \mathfrak{F}_j\right), \frac{\tau}{(q_i+b_j)}\right) \star \nu\left(\left(\mathfrak{E}_i \cap \mathfrak{F}_j\right), \frac{\tau}{(q_i+b_j)}\right)\right) \\
\leq & \star_{i=1}^{n} \star_{j=1}^{m} \left(\nu\left(\left(\mathfrak{E}_i \cap \mathfrak{F}_j\right), \frac{\tau}{(q_i+b_j)}\right)\right)
\end{aligned}
\tag{2}
$$

(iii) :If $\phi \leq \psi$, then $q_i \leq b_j$ whenever $\mathfrak{E}_i \cap \mathfrak{E}_j = \emptyset$, so

$$
\begin{aligned}
& \int_{\mathfrak{X}} \phi(p) d\nu(p,\tau) \\
= & \int_{\mathfrak{X}} \left(\sum_{i=1}^{n} q_i \chi_{\mathfrak{E}_i}(p)\right) d\nu(p,\tau) \\
= & \int_{\mathfrak{X}} \left(\sum_{i=1}^{n} q_i \chi_{\mathfrak{E}_i \cup (\cap_{j=1}^{\infty} \mathfrak{E}_j)}(p)\right) d\nu(p,\tau) \\
= & \int_{\mathfrak{X}} \left(\sum_{i=1}^{n} \sum_{j=1}^{m} q_i \chi_{\mathfrak{E}_i \cap \mathfrak{E}_j}(p)\right) d\nu(p,\tau) \\
= & \star_{i=1}^{n} \star_{j=1}^{m} \nu\left(\left(\mathfrak{E}_i \cap \mathfrak{F}_j\right), \frac{\tau}{q_i}\right) \\
\geq & \star_{i=1}^{n} \star_{j=1}^{m} \nu\left(\left(\mathfrak{E}_i \cap \mathfrak{F}_j\right), \frac{\tau}{b_j}\right) \\
\geq & \star_{j=1}^{m} \nu\left(\mathfrak{F}_j, \frac{\tau}{b_j}\right) \\
= & \int_{\mathfrak{X}} \psi(p) d\nu(p,\tau).
\end{aligned}
$$

(iv) : Assume that $\mathfrak{A}_k \in \mathcal{C}$ is a disjoint sequence and $\mathfrak{A} = \cup_{k=1}^{\infty}\mathfrak{A}_k$ where,

$$\begin{aligned}
&\int_{\mathfrak{A}} \phi(p)d\nu(p,\tau) \\
=\ & \int_{\mathfrak{X}} (\phi\chi_{\mathfrak{A}})(p)d\nu(p,\tau) \\
=\ & \int_{\mathfrak{X}} \left(\sum_{i=1}^{n} q_i\chi_{\mathfrak{E}_i}\right)\chi_{\mathfrak{A}}(p)d\nu(p,\tau) \\
=\ & \int_{\mathfrak{X}} \left(\sum_{i=1}^{n} q_i\chi_{\cup_{k=1}^{\infty}(\mathfrak{A}_k\cap\mathfrak{E}_i)}\right)(p)d\nu(p,\tau) \\
=\ & \int_{\mathfrak{X}} \left(\sum_{i=1}^{n} q_i\sum_{k=1}^{\infty}\chi_{\mathfrak{A}_k\cap\mathfrak{E}_i}\right)(p)d\nu(p,\tau) \\
=\ & \int_{\mathfrak{X}} \left(\sum_{i,k} q_i\chi_{\mathfrak{A}_k\cap\mathfrak{E}_i}\right)(p)d\nu(p,\tau) \\
=\ & \star_{i=1}^{n}\star_{k=1}^{\infty}\,\nu\left((\mathfrak{E}_i\cap\mathfrak{A}_k),\frac{\tau}{q_i}\right) \\
=\ & \star_{k=1}^{\infty}\int_{\mathfrak{X}} (\phi\chi_{\mathfrak{A}_k})(p)d\nu(p,\tau) \\
=\ & \star_{k=1}^{\infty}\int_{\mathfrak{A}_k} \phi(p)d\nu(p,\tau).
\end{aligned}$$

Now we will have the definition of integral with expansion for all functions $\jmath \in L^+$ as follows

$$\int_{\mathfrak{X}} \jmath(p)d\nu(p,\tau) = \inf\left\{\int_{\mathfrak{X}} \phi(p)d\nu(p,\tau) \mid \quad 0 \le \phi \le \jmath, \quad \phi \text{ is a simple}\right\}. \tag{3}$$

□

It is obvious from the definition that theorem 2.10 satisfying for every $\jmath, \imath \in L^+$.

Theorem 2.11. *Consider $\jmath_n \in L^+$ such that $\jmath_i \le \jmath_{i+1}$ for every i, and $\jmath = \lim \jmath_n (= \sup_{n\in\mathbb{N}} \jmath_n)$, then $\int_{\mathfrak{X}} \jmath(p)d\nu(p,\tau) = \lim_{n\to\infty}\int_{\mathfrak{X}} \jmath_n(p)d\nu(p,\tau)$.*

Proof. The sequence $\{\jmath_n(p)\}$, for every $p \in \mathfrak{X}$, is an increasing sequence of numbers, therefore $\lim_{n\to\infty}\jmath_n(p) = \jmath(p)$, moreover $\jmath_n(p) \le \jmath_n(p)$, for every $n \in \mathbb{N}$, so

$$\int_{\mathfrak{X}} \jmath(p)d\nu(p,\tau) \le \int_{\mathfrak{X}} \jmath_n(p)d\nu(p,\tau),$$

then

$$\lim_{n\to\infty}\int_{\mathfrak{X}} \jmath_n(p)d\nu(p,\tau) \ge \int_{\mathfrak{X}} \jmath(p)d\nu(p,\tau). \tag{4}$$

Now, Consider ϕ a simple function with $0 \le \phi \le \jmath$ and $\mathfrak{E}_n = \{p \mid \jmath_n(p) \ge \phi(p)\}$, then $\mathfrak{E}_n$ is a measurable set. We claim $\mathfrak{E}_n \subseteq \mathfrak{E}_{n+1}$ and $\cup_{n\in\mathcal{N}}\mathfrak{E}_n = \mathfrak{X}$, sine for any $p \in \mathfrak{E}_n$

$$\phi(p) \le \jmath_n(p) \le \jmath_{n+1}(p),$$

so $p \in \mathfrak{E}_{n+1}$. If $p \in \mathfrak{X}$ and $p \notin \cup_{n\in\mathcal{N}}\mathfrak{E}_n$, we have $\jmath_n(p) \leq \phi(p)$, for all $n \in \mathcal{N}$, so $\jmath(p) \leq \phi(p)$, it is a contradiction. Then $\cup_{n\in\mathcal{N}}\mathfrak{E}_n = \mathfrak{X}$ and there is $m \in \mathcal{N}$ such that $p \in \mathfrak{E}_m$, we have

$$\int_{\mathfrak{X}} \jmath_n(p)d\nu(p,\tau) \leq \int_{\mathfrak{E}_n} \jmath_n(p)d\nu(p,\tau) \leq \int_{\mathfrak{E}_n} \phi(p)d\nu(p,\tau).$$

$\lim_{n\to\infty} \int_{\mathfrak{E}_n} \phi(p)d\nu(p,\tau) = \int_{\mathfrak{X}} \phi(p)d\nu(p,\tau)$ and hence $\lim_{n\to\infty} \int_{\mathfrak{X}} \jmath_n(p)d\nu(p,\tau) \leq \int_{\mathfrak{X}} phi(p)d\nu(p,\tau)$. By taking the infimum over all simple $0 \leq \phi \leq \jmath$, we get

$$\lim_{n\to\infty} \int_{\mathfrak{X}} \jmath_n(p)d\nu(p,\tau) \leq \int_{\mathfrak{X}} \jmath(p)d\nu(p,\tau). \tag{5}$$

Frome (4) and (5) we have

$$\lim_{n\to\infty} \int_{\mathfrak{X}} \jmath_n(p)d\nu(p,\tau) = \int_{\mathfrak{X}} \lim_{n\to\infty} \jmath_n(p)d\nu(p,\tau) = \int_{\mathfrak{X}} \jmath(p)d\nu(p,\tau).$$

□

3 Application

In this section, as an application of the theorems raised in the previous section, we prove Fatou's lemma fuzzy measure space.

Theorem 3.1. *If $\jmath_n$ is any sequence in L^+, then*

$$\liminf \int_{\mathfrak{X}} \jmath_n(p)d\nu(p,\tau) \leq \int_{\mathfrak{X}} \liminf \jmath_n(p)d\nu(p,\tau).$$

Proof. For each $k \geq 1$, $\inf_{n\geq k} \jmath_n \leq \jmath_j$ for $j \geq k$, hence

$$\int_{\mathfrak{X}} \inf_{n\geq k} \jmath_n(p)d\nu(p,\tau) \geq \int_{\mathfrak{X}} \jmath_j(p)d\nu(p,\tau),$$

for $j \geq k$, hence

$$\int_{\mathfrak{X}} \inf_{n\geq k} \jmath_n(p)d\nu(p,\tau) \geq \inf_{j\geq k} \int_{\mathfrak{X}} \jmath_j(p)d\nu(p,\tau).$$

Now let $k \to \infty$ and apply the monotone convergence theorem

$$\lim \int_{\mathfrak{X}} \inf_{n\geq k} \jmath_n(p)d\nu(p,\tau) = \int_{\mathfrak{X}} \liminf \jmath_n(p)d\nu(p,\tau) \geq \liminf \int_{\mathfrak{X}} \jmath_n(p)d\nu(p,\tau).$$

□

4 Conclusion

We worked on a new concept of fuzzy measurement. We define a new type of fuzzy measure with distance functions. With this concept, we introduced a new version of measurement theory and fuzzy integral and addressed theorems about it. As a continuation of this research, by defining the fuzzy outer measure, a new concept of fuzzy measurement can be defined and using it, new theorems in the fuzzy measure theory can be proposed.

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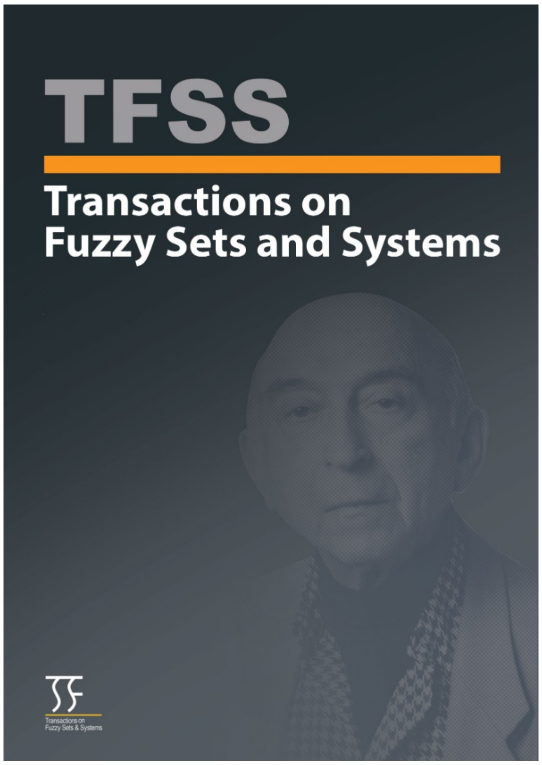

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Fermatean Fuzzy CRADIS Approach Based on Triangular Divergence for Selecting Online Shopping Platform

Mijanur Rahaman Seikh*, **Arnab Mukherjee**

Abstract. In the evolving landscape of e-commerce, selecting an optimal online shopping platform is crucial for businesses aiming to enhance customer experience and operational efficiency. This paper introduces a novel approach that combines the Fermatean fuzzy set theory with the triangular divergence distance measure in Compromise Ranking of Alternatives from Distance to Ideal Solution (CRADIS) method to streamline the decision-making process in online platform selection. By applying the CRADIS method, businesses can systematically evaluate and select an online shopping platform that best meets their operational needs and strategic goals, thereby enhancing their e-commerce effectiveness and customer satisfaction. Through a comprehensive example, we illustrate the application of this approach in evaluating and ranking four distinct online shopping platforms based on multiple criteria. This result shows that Myntra ($\widehat{\chi}_4$) is the best choice. Through this integrated approach, decision-makers can gain valuable insights into the relative merits of each online shopping platform, allowing them to make informed choices aligned with their preferences and requirements. Furthermore, by accommodating uncertainty and imprecision, the Fermatean fuzzy set theory enhances the robustness of the decision-making process, minimizing the risk of making sub-optimal decisions. Overall, this paper demonstrates the practical applicability of Fermatean fuzzy set theory in decision support systems for online platform selection. To demonstrate the proposed method's applicability, we have compared the results with existing Multi-attribute decision making (MADM) methods. To establish its stability, we conducted a sensitivity analysis. By leveraging the CRADIS method alongside Fermatean fuzzy set theory, decision-makers can navigate the complex landscape of online shopping platforms with greater confidence and efficiency, ultimately leading to more satisfactory outcomes for both consumers and businesses alike.

AMS Subject Classification 2020: 90B50; 03E72; 62A86
Keywords and Phrases: Multi-attribute decision-making, Fermatean fuzzy set, Distance measure, CRADIS, Triangular divergence, Online shopping platform selection.

1 Introduction

Distance measures play an important role in handling fuzzy information. Several distance measures have been developed for different fuzzy environments over the years. Recently, Ganie et al. [1] define an innovative picture fuzzy distance measure and novel multi-attribute decision-making method. Puška et al. [2] proposed a comprehensive decision framework for selecting distribution center locations: a hybrid improved fuzzy SWARA and fuzzy compromise ranking of alternatives from distance to ideal solution (CRADIS) approach. Deng et al. [3] proposed a new distance measure in Fermatean fuzzy sets (FFSs). Palanikumar et al. [4] present the novelty of different distance approaches for multi-criteria decision-making challenges using q-rung vague sets. Robot sensors process based on generalized Fermatean normal different aggregation operators

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framework is proposed by Palanikumar et al. [5]. Some recent decision-making problems can be found in various fuzzy environments [6, 7, 8, 9].

There are a few distance measures for Fermatean fuzzy sets in the literature. Senapati et al. [10] proposed the general-Euclidean distance measure (GEDM) for FFSs and used Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) to solve some illustrative Multi-attribute decision making (MADM) problems. Onyeke and Ejegwa [11] defined a modified distance measure for FFSs to fulfill the axiomatic description of the distance function. These distances are not completely competent for calculating accurate distances between FFSs. FFSs, being more advanced and efficient in depicting fuzzy information, automatically call for the development of modified distance measures for better decision-making methods. Recently, the triangular divergence measure, a method generally used in probability distributions, has been researched to develop distance measures based on it. We find instances where the existing distance measures fail to evaluate the distances between FFSs accurately. The triangular divergence measure, proposed by Yehudayoff [12], has been extended to form distance measures by some researchers. Liu [13] defines a distance measure of Fermatean fuzzy sets based on triangular divergence and its application in medical diagnosis. Sahoo [14] uses similarity measures for Fermatean fuzzy sets and its applications in group decision-making. Mandal and Seikh [15] explain the interval-valued Fermatean fuzzy (TOPSIS) method and its application to sustainable development programs. In recent times, FFSs have been utilized in various decision-making problems [16, 17, 18, 19, 20]. Seikh and Chatterjee [21] establish the determination of the best renewable energy sources in India using SWARA-ARAS in a confidence level-based interval-valued Fermatean fuzzy environment. Seikh and Mandal [22] mentioned interval-valued Fermatean fuzzy Dombi aggregation operators and SWARA-based PROMETHEE II method for bio-medical waste management.

Several MADM methods use the distance measure to identify the best alternative. One such method is CRADIS. CRADIS is a relatively new method proposed by Puška et al. [23] in 2022. Hence, it has relatively fewer applications in decision-making problems and fewer extensions in other fuzzy environments. It identifies the best alternatives more comprehensively and simply by using the merits of MARCOS, ARAS, and TOPSIS. Yuan et al. [24] proposed a novel distance measure and CRADIS method in picture fuzzy environment. Further, Puška et al. [25] clarify fuzzy multi-criteria analysis on green supplier selection in an agri-food company. Krishankumar et al. [26] select the IoT service provider for sustainable transport using q-rung orthopair fuzzy CRADIS and unknown weights. Most of these studies utilized the GEDM or the Hamming distance measure (HDM) in the CRADIS method. From the thorough review of the literature, it is observed that Fermatean fuzzy numbers (FFNs) are efficient in expressing fuzzy information and are a popular research area. Also, triangular divergence distance measure forming effective distance measures has few studies on them. Moreover, the CRADIS method is a recently developed and strong method combining the merits of various decision-making methods that have been applied to solve a variety of decision-making problems. Hence, modification of the CRADIS method would eventually make it better and stronger. In this study, the triangular divergence-based distance measure (TDDM) for FFSs is proposed. To improve the existing distance measure, the hesitancy degree of FFSs is included in the distance formula. We further employ it in the CRADIS method to improve the existing CRADIS method.

There are several motivations for this study. They are as follows:

- FFSs have two popular distance measures, but they are not entirely competent in calculating distances between all FFNs. It leads to the requirement of a new and better distance measure for achieving more accurate results in decision-making problems.

- Triangular divergence measure is a popular classical method, mostly used in probability distributions. It has been utilized for distance measures for FFSs and Interval-valued intuitionistic fuzzy sets (IVIFSs). FFNs are better at expressing fuzzy information than FFSs and IVIFSs. Hence, extending the triangular divergence-based distance measure to FFNs will be more beneficial and realistic.

- The previous application of triangular divergence-based measure does not include the hesitancy degree. It leaves out certain fuzzy information from the calculated data and thus may cause discrepancies in results. Hence, including the hesitancy degree in the distance measure will make it more precise.

The following are some significant contributions of this study.

- A new triangular divergence-based distance measure for FFSs is proposed and its properties are discussed.
- The hesitancy degree is also included in the triangular divergence-based distance measure for FFSs to overcome the loss of information.
- The proposed distance measure is utilized in the CRADIS method, eventually modifying the method to give better results to decision-making problems.
- The proposed method is used to solve a real-life decision-making problem of selection of the best online shopping platform.

The study has been organized in the following way: Section 2, consists of the preliminaries. Section 3 has the newly proposed distance measure with its properties and we establish the superiority of the proposed triangular divergence-based distance measure. In Section 4, we iterate the modified-CRADIS method used to solve an illustrative MADM problem. In Section 5, we use an example to apply our proposed method and solve it followed by a comparative and sensitivity analysis. Lastly, Section 6 has the conclusion, research implications, limitations, and future research scopes. Table 1 presents the list of abbreviations used in the manuscript.

Table 1: List of abbreviation.

Abbreviation	Full-form
CRADIS	Compromise ranking of alternatives from distance to ideal solution
TOPSIS	Technique for order performance by similarity to ideal solution
GEDM	General-Euclidean distance measure
HDM	Hamming distance measure
VIKOR	VlseKriterijumska Optimizacija I Kompromisno Resenje
MADM	Multi-attribute decision making
FFSs	Fermatean fuzzy sets
FFNs	Fermatean fuzzy numbers
IoT	Internet of things
SWARA	Stepwise weight assessment ratio analysis
ARAS	Additive ratio assessment method
MARCOS	Measurement of alternatives and ranking according to the compromise solution
IVIFSs	Interval valued intuitionistic fuzzy sets
TDDM	Triangular divergence-based distance measure
LVS	Linguistic variables
PDM	Positive distance matrix
NDM	Negative distance matrix

2 Preliminaries

In this section, some basic definitions and preliminaries are recalled. Throughout the manuscript, the universal set is consistently denoted as Υ.

Definition 2.1. [10] *Let $\Re$ be an FFS over Υ and is defined as follows:*

$$\Re = \{\langle\ \hbar, \alpha_{\Re}(\hbar), \beta_{\Re}(\hbar)\rangle | \hbar \in \Upsilon\}$$

introducing this condition

$$0 \leq (\alpha_{\Re}(\hbar))^3 + (\beta_{\Re}(\hbar))^3 \leq 1.$$

For all $\hbar \in \Upsilon$, the numbers $\alpha_{\Re}(\hbar)$ and $\beta_{\Re}(\hbar)$ denote the degree of membership and the degree of non membership.

Where, $\alpha_{\Re} : \Upsilon \longrightarrow [0,1]$ and $\beta_{\Re} : \Upsilon \longrightarrow [0,1]$.

For any Fermatean fuzzy set $\Re$ and $\hbar \in \Upsilon$.

$$\gamma_{\Re}(\hbar) = \sqrt[3]{1 - (\alpha_{\Re}(\hbar))^3 - (\beta_{\Re}(\hbar))^3}$$

is define as the degree of indeterminacy of $\hbar$ to $\Re$.

Definition 2.2. [10] *Let $\Re = (\alpha_{\Re}, \beta_{\Re})$, $\Re_1 = (\alpha_{\Re_1}, \beta_{\Re_1})$ and $\Re_2 = (\alpha_{\Re_2}, \beta_{\Re_2})$ be three FFNs, then some operation are defined as below:*

1. $\Re_1 \bigcap \Re_2 = (min\{\alpha_{\Re_1}, \beta_{\Re_2}\}, max\{\beta_{\Re_1}, \beta_{\Re_2}\})$.
2. $\Re_1 \bigcup \Re_2 = (max\{\alpha_{\Re_1}, \beta_{\Re_2}\}, min\{\beta_{\Re_1}, \beta_{\Re_2}\})$.
3. $\Re^c = (\beta_{\Re}, \alpha_{\Re})$.

Definition 2.3. [10] *Let $\Re = (\alpha_{\Re}, \beta_{\Re})$, $\Re_1 = (\alpha_{\Re_1}, \beta_{\Re_1})$ and $\Re_2 = (\alpha_{\Re_2}, \beta_{\Re_2})$ be three FFNs and $\lambda >0$, then some mathematical operations are formulated as below:*

1. $\Re_1 \boxplus \Re_2 = (\sqrt[3]{\alpha^3_{\Re_1} + \alpha^3_{\Re_2} - \alpha^3_{\Re_1}\alpha^3_{\Re_2}}, \beta_{\Re_1}\beta_{\Re_2})$.
2. $\Re_1 \boxtimes \Re_2 = (\alpha_{\Re_1}\alpha_{\Re_2}, \sqrt[3]{\beta^3_{\Re_1} + \beta^3_{\Re_2} - \beta^3_{\Re_1}\beta^3_{\Re_2}})$.
3. $\lambda\Re = (\sqrt[3]{1 - (1 - \alpha^3_{\Re})^{\lambda}}, \beta^{\lambda}_{\Re})$.
4. $\Re^{\lambda} = (\alpha^{\lambda}_{\Re}, \sqrt[3]{1 - (1 - \beta^3_{\Re})^{\lambda}})$.

Definition 2.4. *Let $\Re_1$=$(\alpha_{\Re_1}, \beta_{\Re_1})$ and $\Re_2$=$(\alpha_{\Re_2}, \beta_{\Re_2})$ be two FFNs. Then the Euclidean distance measure [10] $d_e(\Re_1, \Re_2)$ and the Hamming distance measure [3] $d_h(\Re_1, \Re_2)$ between $\Re_1$ and $\Re_2$ are defined as follow:*

$$d_e(\Re_1, \Re_2) = \sqrt{\frac{1}{2}[(\alpha^3_{\Re_1} - \alpha^3_{\Re_2})^2 + (\beta^3_{\Re_1} - \beta^3_{\Re_2})^2 + (\gamma^3_{\Re_1} - \gamma^3_{\Re_2})^2]} \tag{1}$$

$$d_h(\Re_1, \Re_2) = \frac{1}{2}|[(\alpha^3_{\Re_1} - \alpha^3_{\Re_2})^2 + (\beta^3_{\Re_1} - \beta^3_{\Re_2})^2 + (\gamma^3_{\Re_1} - \gamma^3_{\Re_2})^2]|. \tag{2}$$

Definition 2.5. [12] *The set $\Xi_n = \{M = (m_1, m_2, ..., m_n) | m_i > 0, \sum\limits_{i=1}^{n} m_i = 1\}$, with $n \geq 2$, represents a collection of finite discrete probability distributions. For $\forall\ M, P \in \Xi_n$, the classical triangular divergence measure between M and P is defined as follows:*

$$\Delta(M, P) = \sum_{i=1}^{n} \frac{(m_i - p_i)^2}{m_i + p_i}.$$

Greater triangular divergence indicates greater difference between the probability distributions M and P. Using the above-mentioned equation, the square root of the triangular divergence is presented in the following manner:

$$d(M,P) = \sqrt{\sum_{i=1}^{n} \frac{(m_i - p_i)^2}{m_i + p_i}}$$

where, by convention, $0/0 = 0$.

3 Distance Measure Based on Triangular Divergence for Fermatean Fuzzy Sets

In this section, we proposed the new distance measure for Fermatean fuzzy sets based on triangular divergance measure.

Definition 3.1. *Let* $\Re_i = \langle\ \hbar, \alpha_{\Re_i}(\hbar), \beta_{\Re_i}(\hbar)\rangle$ *for* $i = 1, 2$ *be two FFSs in* $\Upsilon = \{\hbar_1, \hbar_2\}$*, then the triangular divergence-based modified distance measure(TDDM) between FFSs* $\Re_1$ *and* $\Re_2$ *denoted by* d_T *is given by*

$$d_T(\Re_1, \Re_2) = \sqrt{\frac{1}{2n}\sum_{j=1}^{n}\left[\frac{\left(\alpha^3_{\Re_1}(\hbar_j) - \alpha^3_{\Re_2}(\hbar_j)\right)^2}{\alpha^3_{\Re_1}(\hbar_j) + \alpha^3_{\Re_2}(\hbar_j)} + \frac{\left(\beta^3_{\Re_1}(\hbar_j) - \beta^3_{\Re_2}(\hbar_j)\right)^2}{\beta^3_{\Re_1}(\hbar_j) + \beta^3_{\Re_2}(\hbar_j)} + \frac{\left(\gamma^3_{\Re_1}(\hbar_j) - \gamma^3_{\Re_2}(\hbar_j)\right)^2}{\gamma^3_{\Re_1}(\hbar_j) + \gamma^3_{\Re_2}(\hbar_j)}\right]}. \quad (3)$$

Theorem 3.2. *The distance measure* $d_T(\Re_1, \Re_2)$*, between the two FFSs* $\Re_1$ *and* $\Re_2$*, follows the following properties. Here* $\Re_1$*,* $\Re_2$ *and* $\Re_3$ *are FFSs.*

I. $d_T(\Re_1, \Re_2) = 0 \Leftrightarrow \Re_1 = \Re_2$;

II. $d_T(\Re_1, \Re_2) = d_T(\Re_2, \Re_1)$;

III. $0 \leq d_T(\Re_1, \Re_2) \leq 1$;

IV. If $\Re_1 \leq \Re_2 \leq \Re_3$*, then* $d_T(\Re_1, \Re_2) \leq d_T(\Re_1, \Re_3)$ *and* $d_T(\Re_2, \Re_3) \leq d_T(\Re_1, \Re_3)$.

Proof. I. Let $d_T(\Re_1, \Re_2) = 0$ for any $\hbar \in \Upsilon$. Then we can say that

$$d_T(\Re_1, \Re_2) = \sqrt{\frac{1}{2n}\sum_{j=1}^{n}\left[\frac{\left(\alpha^3_{\Re_1}(\hbar_j) - \alpha^3_{\Re_2}(\hbar_j)\right)^2}{\alpha^3_{\Re_1}(\hbar_j) + \alpha^3_{\Re_2}(\hbar_j)} + \frac{\left(\beta^3_{\Re_1}(\hbar_j) - \beta^3_{\Re_2}(\hbar_j)\right)^2}{\beta^3_{\Re_1}(\hbar_j) + \beta^3_{\Re_2}(\hbar_j)} + \frac{\left(\gamma^3_{\Re_1}(\hbar_j) - \gamma^3_{\Re_2}(\hbar_j)\right)^2}{\gamma^3_{\Re_1}(\hbar_j) + \gamma^3_{\Re_2}(\hbar_j)}\right]} = 0.$$

Then

$$\frac{\left(\alpha^3_{\Re_1}(\hbar_j) - \alpha^3_{\Re_2}(\hbar_j)\right)^2}{\alpha^3_{\Re_1}(\hbar_j) + \alpha^3_{\Re_2}(\hbar_j)} = \frac{\left(\beta^3_{\Re_1}(\hbar_j) - \beta^3_{\Re_2}(\hbar_j)\right)^2}{\beta^3_{\Re_1}(\hbar_j) + \beta^3_{\Re_2}(\hbar_j)} = \frac{\left(\gamma^3_{\Re_1}(\hbar_j) - \gamma^3_{\Re_2}(\hbar_j)\right)^2}{\gamma^3_{\Re_1}(\hbar_j) + \gamma^3_{\Re_2}(\hbar_j)} = 0.$$

That is,

$$\left(\alpha^3_{\Re_1}(\hbar_j) - \alpha^3_{\Re_2}(\hbar_j)\right)^2 = \left(\beta^3_{\Re_1}(\hbar_j) - \beta^3_{\Re_2}(\hbar_j)\right)^2 = \left(\gamma^3_{\Re_1}(\hbar_j) - \gamma^3_{\Re_2}(\hbar_j)\right)^2.$$

Again we know that

$$0 \leq \alpha_{\Re_1}, \alpha_{\Re_2}, \beta_{\Re_1}, \beta_{\Re_2}, \gamma_{\Re_1}, \gamma_{\Re_2} \leq 1.$$

Hence, we have $\alpha_{\Re_1}(\hbar_j) = \alpha_{\Re_2}(\hbar_j), \beta_{\Re_1}(\hbar_j) = \beta_{\Re_2}(\hbar_j), \gamma_{\Re_1}(\hbar_j) = \gamma_{\Re_2}(\hbar_j)$.
Therefore,

$$\Re_1 = \Re_2.$$

Conversely, when $\Re_1 = \Re_2$, one has

$$\alpha_{\Re_1}(\hbar_j) = \alpha_{\Re_2}(\hbar_j), \beta_{\Re_1}(\hbar_j) = \beta_{\Re_2}(\hbar_j), \gamma_{\Re_1}(\hbar_j) = \gamma_{\Re_2}(\hbar_j).$$

Then, we can obtain

$$d_T(\Re_1, \Re_2) = \sqrt{\frac{1}{2n}\sum_{j=1}^{n}\left[\frac{\left(\alpha^3_{\Re_1}(\hbar_j) - \alpha^3_{\Re_2}(\hbar_j)\right)^2}{\alpha^3_{\Re_1}(\hbar_j) + \alpha^3_{\Re_2}(\hbar_j)} + \frac{\left(\beta^3_{\Re_1}(\hbar_j) - \beta^3_{\Re_2}(\hbar_j)\right)^2}{\beta^3_{\Re_1}(\hbar_j) + \beta^3_{\Re_2}(\hbar_j)} + \frac{\left(\gamma^3_{\Re_1}(\hbar_j) - \gamma^3_{\Re_2}(\hbar_j)\right)^2}{\gamma^3_{\Re_1}(\hbar_j) + \gamma^3_{\Re_2}(\hbar_j)}\right]} = 0.$$

Hence, property I holds.

II. Next we prove that $d_T(\Re_1, \Re_2) = d_T(\Re_2, \Re_1)$. We know that

$$\begin{aligned} d_T(\Re_1, \Re_2) &= \sqrt{\frac{1}{2n}\sum_{j=1}^{n}\left[\frac{\left(\alpha^3_{\Re_1}(\hbar_j) - \alpha^3_{\Re_2}(\hbar_j)\right)^2}{\alpha^3_{\Re_1}(\hbar_j) + \alpha^3_{\Re_2}(\hbar_j)} + \frac{\left(\beta^3_{\Re_1}(\hbar_j) - \beta^3_{\Re_2}(\hbar_j)\right)^2}{\beta^3_{\Re_1}(\hbar_j) + \beta^3_{\Re_2}(\hbar_j)} + \frac{\left(\gamma^3_{\Re_1}(\hbar_j) - \gamma^3_{\Re_2}(\hbar_j)\right)^2}{\gamma^3_{\Re_1}(\hbar_j) + \gamma^3_{\Re_2}(\hbar_j)}\right]} \\ &= \sqrt{\frac{1}{2n}\sum_{j=1}^{n}\left[\frac{\left(\alpha^3_{\Re_2}(\hbar_j) - \alpha^3_{\Re_1}(\hbar_j)\right)^2}{\alpha^3_{\Re_2}(\hbar_j) + \alpha^3_{\Re_1}(\hbar_j)} + \frac{\left(\beta^3_{\Re_2}(\hbar_j) - \beta^3_{\Re_1}(\hbar_j)\right)^2}{\beta^3_{\Re_2}(\hbar_j) + \beta^3_{\Re_1}(\hbar_j)} + \frac{\left(\gamma^3_{\Re_2}(\hbar_j) - \gamma^3_{\Re_1}(\hbar_j)\right)^2}{\gamma^3_{\Re_2}(\hbar_j) + \gamma^3_{\Re_1}(\hbar_j)}\right]} \\ &= d_T(\Re_2, \Re_1). \end{aligned}$$

Hence, property II holds.

III. We then prove that $0 \le d_T(\Re_1, \Re_2) \le 1$. From Definition 2.1, it is obvious that $0 \le d_T(\Re_1, \Re_2)$ and we observe that, $0 \le \alpha^3_{\Re_1}(\hbar) + \beta^3_{\Re_1}(\hbar) \le 1$, $0 \le \alpha^3_{\Re_2}(\hbar) + \beta^3_{\Re_2}(\hbar) \le 1$. So, the following inequality holds

$$\left(\alpha^3_{\Re_1}(\hbar) - \alpha^3_{\Re_2}(\hbar)\right)^2 \le \left(\alpha^3_{\Re_1}(\hbar) + \alpha^3_{\Re_2}(\hbar)\right)^2 \text{ and } \left(\beta^3_{\Re_1}(\hbar) - \beta^3_{\Re_2}(\hbar)\right)^2 \le \left(\beta^3_{\Re_1}(\hbar) + \beta^3_{\Re_2}(\hbar)\right)^2.$$

Then,

$$\begin{aligned} d_T(\Re_1, \Re_2) &= \sqrt{\frac{1}{2n}\sum_{j=1}^{n}\left[\frac{\left(\alpha^3_{\Re_1}(\hbar_j) - \alpha^3_{\Re_2}(\hbar_j)\right)^2}{\alpha^3_{\Re_1}(\hbar_j) + \alpha^3_{\Re_2}(\hbar_j)} + \frac{\left(\beta^3_{\Re_1}(\hbar_j) - \beta^3_{\Re_2}(\hbar_j)\right)^2}{\beta^3_{\Re_1}(\hbar_j) + \beta^3_{\Re_2}(\hbar_j)} + \frac{\left(\gamma^3_{\Re_1}(\hbar_j) - \gamma^3_{\Re_2}(\hbar_j)\right)^2}{\gamma^3_{\Re_1}(\hbar_j) + \gamma^3_{\Re_2}(\hbar_j)}\right]} \\ &\le \sqrt{\frac{1}{2n}\sum_{j=1}^{n}\left[\frac{\left(\alpha^3_{\Re_1}(\hbar_j) + \alpha^3_{\Re_2}(\hbar_j)\right)^2}{\alpha^3_{\Re_1}(\hbar_j) + \alpha^3_{\Re_2}(\hbar_j)} + \frac{\left(\beta^3_{\Re_1}(\hbar_j) + \beta^3_{\Re_2}(\hbar_j)\right)^2}{\beta^3_{\Re_1}(\hbar_j) + \beta^3_{\Re_2}(\hbar_j)} + \frac{\left(\gamma^3_{\Re_1}(\hbar_j) + \gamma^3_{\Re_2}(\hbar_j)\right)^2}{\gamma^3_{\Re_1}(\hbar_j) + \gamma^3_{\Re_2}(\hbar_j)}\right]} \\ &= \sqrt{\frac{1}{2n}\sum_{j=1}^{n}\left[\alpha^3_{\Re_1}(\hbar_j) + \alpha^3_{\Re_2}(\hbar_j) + \beta^3_{\Re_1}(\hbar_j) + \beta^3_{\Re_2}(\hbar_j) + \gamma^3_{\Re_1}(\hbar_j) + \gamma^3_{\Re_2}(\hbar_j)\right]} \\ &= \sqrt{\frac{1}{2n}\sum_{j=1}^{n}2} \\ &= 1. \end{aligned}$$

Hence, property III holds.

IV. Lastly, we prove that if $\Re_1 \le \Re_2 \le \Re_3$, then $d_T(\Re_1, \Re_2) \le d_T(\Re_1, \Re_3)$ and
$d_T(\Re_2, \Re_3) \le d_T(\Re_1, \Re_3)$).

When $\Re_1 \leq \Re_2 \leq \Re_3$, we have

$$\alpha^3_{\Re_1} \leq \alpha^3_{\Re_2} \leq \alpha^3_{\Re_3} \text{ and } \beta^3_{\Re_3} \leq \beta^3_{\Re_2} \leq \beta^3_{\Re_1}$$

for $0 \leq \eta_k \leq 1(k = 1, 2)$ and $0 \leq \eta_1 + \eta_2 \leq 1$, a function $h(\hbar_1, \hbar_2)$ could be establish below

$$h(\hbar_1, \hbar_2) = \sum_{k=1}^{2} \frac{(\hbar_k - \eta_k)^2}{\hbar_K + \eta_K}, \hbar_k \in [0, 1]$$

then the partial derivative of the function $h(\hbar_1, \hbar_2)$ in term of $\hbar_i$ will be calculated as follow

$$\frac{\delta h}{\delta \hbar_K} = \frac{(\hbar_k - \eta_k)(\hbar_k + 3\eta_k)}{(\hbar_k + \eta_k)^2}. \tag{4}$$

From the partial derivation function of Equation (4) one has

$$\begin{cases} \frac{\delta h}{\delta \hbar_k} \geq 0, & 0 \leq \eta_k \leq \hbar_k \leq 1, \\ \frac{\delta h}{\delta \hbar_k} < 0, & 0 \leq \hbar_k \leq \eta_k \leq 1. \end{cases}$$

Therefore, when $\hbar_k \geq \eta_k$, $h(\hbar_1, \hbar_2)$ is monotonically increasing function for $\hbar_k$ and when $\hbar_k \geq \eta_k$, $h(\hbar_1, \hbar_2)$ is a monotonically decreasing function for $\hbar_k$.

Let, $\eta_1 = \alpha^3_{\Re_1}, \eta_2 = \beta^3_{\Re_1}$

when $\Re_1 \leq \Re_2 \leq \Re_3$

$$\eta_1 = \alpha^3_{\Re_1} \leq \alpha^3_{\Re_2} \leq \alpha^3_{\Re_3},$$
$$\beta^3_{\Re_3} \leq \beta^3_{\Re_2} \leq \beta^3_{\Re_1} = \eta_2.$$

Because, $h(\hbar_1, \hbar_2)$ is monotonically increasing when $\hbar_1 \geq \eta_1$ if $\alpha^3_{\Re_3} \geq \alpha^3_{\Re_2}$ one has

$$h(\alpha^3_{\Re_3}, \beta^3_{\Re_3}) \geq h(\alpha^3_{\Re_2}, \beta^3_{\Re_3}). \tag{5}$$

Meanwhile, because $h(\hbar_1, \hbar_2)$ is monotonically decreasing when $\hbar_3 \leq \eta_3$ if $\beta^3_{\Re_3} \leq \beta^3_{\Re_2}$ one has

$$h(\alpha^3_{\Re_2}, \beta^3_{\Re_3}) \geq h(\alpha^3_{\Re_2}, \beta^3_{\Re_2}). \tag{6}$$

Combining (5) and (6) one has

$$h(\alpha^3_{\Re_3}, \beta^3_{\Re_3}) \geq h(\alpha^3_{\Re_2}, \beta^3_{\Re_2})$$

that is,

$$\frac{(\alpha^3_{\Re_2} - \alpha^3_{\Re_1})^2}{\alpha^3_{\Re_2} + \alpha^3_{\Re_1}} + \frac{(\beta^3_{\Re_2} - \beta^3_{\Re_1})^2}{\beta^3_{\Re_2} + \beta^3_{\Re_1}} \leq \frac{(\alpha^3_{\Re_3} - \alpha^3_{\Re_1})^2}{\alpha^3_{\Re_3} + \alpha^3_{\Re_1}} + \frac{(\beta^3_{\Re_3} - \beta^3_{\Re_1})^2}{\beta^3_{\Re_3} + \beta^3_{\Re_1}}.$$

Consequently, we have

$$\begin{aligned} d_T(\Re_1, \Re_2) &= \sqrt{\frac{1}{2n} \sum_{j=1}^{n} \left[\frac{\left(\alpha^3_{\Re_2}(\hbar_j) - \alpha^3_{\Re_1}(\hbar_j)\right)^2}{\alpha^3_{\Re_2}(\hbar_j) + \alpha^3_{\Re_1}(\hbar_j)} + \frac{\left(\beta^3_{\Re_2}(\hbar_j) - \beta^3_{\Re_2}(\hbar_j)\right)^2}{\beta^3_{\Re_2}(\hbar_j) + \beta^3_{\Re_1}(\hbar_j)} \right]} \\ &\leq \sqrt{\frac{1}{2n} \sum_{j=1}^{n} \left[\frac{\left(\alpha^3_{\Re_3}(\hbar_j) - \alpha^3_{\Re_1}(\hbar_j)\right)^2}{\alpha^3_{\Re_3}(\hbar_j) + \alpha^3_{\Re_1}(\hbar_j)} + \frac{\left(\beta^3_{\Re_3}(\hbar_j) - \beta^3_{\Re_2}(\hbar_j)\right)^2}{\beta^3_{\Re_3}(\hbar_j) + \beta^3_{\Re_1}(\hbar_j)} \right]} \\ &= d_T(\Re_1, \Re_3). \end{aligned}$$

Since, the hesitancy degree is dependent on the membership and non-membership degree, we can say that $d_T(\Re_1, \Re_2) \leq d_T(\Re_1, \Re_3)$. Similarly, we can proved that $d_T(\Re_2, \Re_3) \leq d_T(\Re_1, \Re_3)$. □

Next, we utilize the following example to establish the superiority of the proposed triangular divergence-based distance measure for FFNs.

Example 3.3. Let there be three FFNs, $\bar{F}_1 = (0.65, 0.8321)$, $\bar{F}_2 = (0.85, 0.6831)$ and $\bar{F}_3 = (0.85, 0.6849)$. Clearly $\bar{F}_2 \neq \bar{F}_3$, so distance between $(\bar{F}_1, \bar{F}_2)$ and $(\bar{F}_1, \bar{F}_3)$ should not be equal. We now calculate the GEDM (proposed by Senapati [10]), HDM (proposed by Deng. [3]) and TDDM (proposed) between $(\bar{F}_1, \bar{F}_2)$ and $(\bar{F}_1, \bar{F}_3)$ using Equations (1), (2) and (3) respectively, to establish superiority of the proposed distance measure. Table 2 gives the values of the calculated distances.

Table 2: Comparison of distance measure for FFNs.

FFN Pair	GEDM [10]	HDM [3]	TDDM (proposed)
$(\bar{F}_1, \bar{F}_2)$	0.306	0.339	**0.342**
$(\bar{F}_1, \bar{F}_3)$	0.306	0.339	**0.343**

Thus we see that even though the GEDM and HDM give equal distances for the pairs, the proposed TDDM gives different distances for the given pair, thus establishing the superiority of the proposed distance measure.

From Figure 1, we notice that there is a significant difference in the distance measures between the given pairs in the case of TDDM, whereas the existing distance measures fail to distinguish between them.

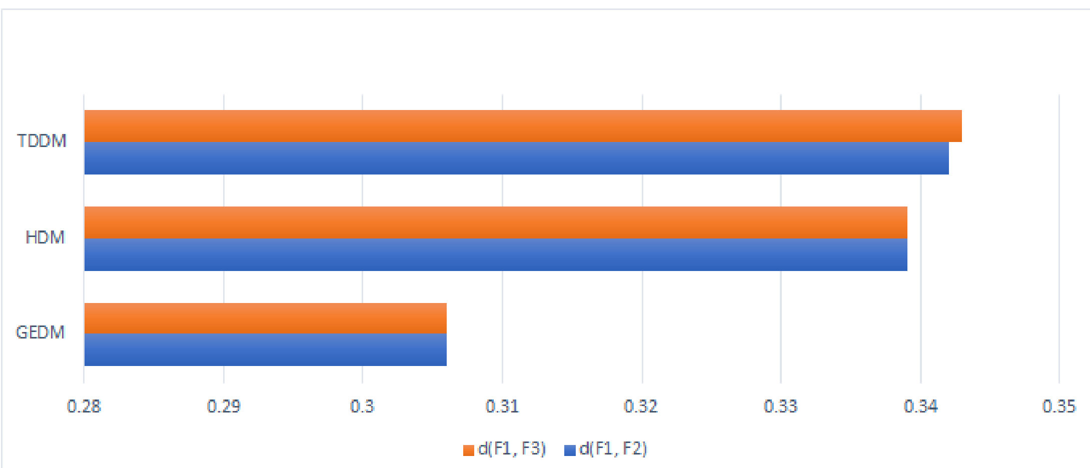

Figure 1: Superiority of the proposed distance measure.

4 MADM Process Using Modified Distance Based FFNs-CRADIS Method

In this section, we propose a MADM process utilizing a modified version of the CRADIS method by implementing the proposed distance measure for FFNs.

Let the set of alternatives be $\widehat{\chi} = \{\widehat{\chi}_1, \widehat{\chi}_2, ..., \widehat{\chi}_m\}$ such that there are "m" alternatives and "n" criteria such that $SC = \{SC_1, SC_2, ..., SC_n\}$ be the set of criteria where their weights are $\varpi_1, \varpi_2, ..., \varpi_n$ respectively. Here $0 \leq \varpi_j \leq 1$ and $\sum_{j=1}^{n} \varpi_j = 1$.

General the initial decision matrices is $B = (b_{ij})_{m\times n}$. A panel of specialists has been invited to offer their assessments in order to achieve the desired ranking of the "m" alternatives regarding "n" attributes. Then construct the Fermatean fuzzy evaluation matrix as $B = (b_{ij})_{m\times n}$, where $i = 1, 2..., m$ and $j = 1, 2, ..., n$ also $b_{ij} = (\alpha_{ij}, \beta_{ij})$. The algorithm shown in Figure 2.

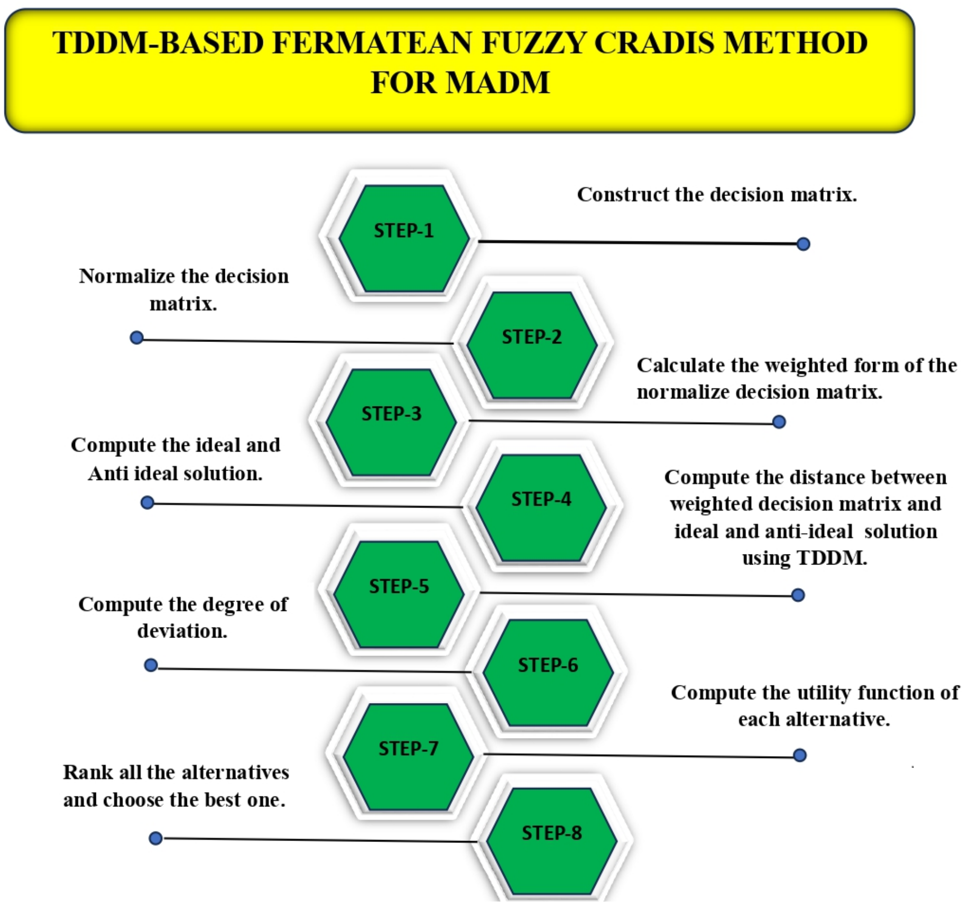

Figure 2: CRADIS-based MADM process using the triangular divergence distance measure.

The triangular distance-based CRADIS method is used for ranking the alternatives. The algorithm of the method is as follows:

Step 1: Construct the decision matrix for the FFNs.

Step 2: Now normalized the decision matrix. The two group are cost type and benefit type. The normalized

Fermatean fuzzy decision matrix, $D = (d_{ij})_{m\times n}$ is as follows:

$$D = (d_{ij})_{m\times n} = \begin{cases} b_{ij} = (\alpha_{ij}, \beta_{ij}), & SC_j \text{ is a benefit attribute,} \\ b^c_{ij} = (\alpha_{ij}, \beta_{ij}), & SC_j \text{ is a cost attribute.} \end{cases}$$

Step 3: Set up the Fermatean fuzzy weighted decision matrix.

$T = (t_{ij})_{m\times n}$ based on the subsequent formula

$$\begin{aligned} T &= d_{ij}\varpi_j \\ &= (\sqrt[3]{1-(1-\alpha^3_{d_{ij}})^{\varpi_j}}, \beta^{\varpi_j}_{d_{ij}}) \end{aligned} \tag{7}$$

where, d_{ij} is the component of normalized decision matrix D and ϖ_j is the attribute weight SC_j.

Step 4: Find out the ideal and anti-ideal solution.

Ideal alternatives ,

$$\begin{aligned} \widehat{\chi}_0 &= \{t_{01}, t_{02}, ..., t_{0n}\} \\ t_{0j} &= (\alpha_{t_{oj}}, \beta_{t_{oj}}) \\ &= \{\max_{1\le i\le m} \alpha_{t_{ij}}, \min_{1\le i\le m} \beta_{t_{ij}}\}, j = 1, 2, ..., n \end{aligned}$$

and

anti-ideal alternatives ,

$$\begin{aligned} \widehat{\chi}_{m+1} &= (t_{m+11}, ..., t_{m+1n}) \\ t_{m+1j} &= (\alpha_{tm+ij}, \beta_{tm+ij}) \\ &= \{\max_{1\le i\le m} \alpha_{t_{ij}}, \min_{1\le i\le m} \beta_{t_{ij}}\}, j = 1, 2, ..., n. \end{aligned} \tag{8}$$

Step 5: Deviation are obtain by TDDM

$$d^+_T = d(t_{ij}, t_{0j}) \text{ for } i = 1, 2..., m \text{ and } j = 1, 2, ..., n. \tag{9}$$

$$d^-_T = d(t_{ij}, t_{m+ij} \text{ for } i = 1, 2..., m \text{ and } j = 1, 2, ..., n. \tag{10}$$

Using the formula (3).

Step 6: Determine the degree of deviation of every option from the ideal and undesirable solution

$$s^+_i = \sum_{j=1}^{n} d^+_j, s^-_i = \sum_{j=1}^{n} d^-_j.$$

Step 7: Analysis of every alternative utility function concerning its deviation from ideal option

$$k^+_i = \frac{s^+_0}{s^+_i}, k^-_i = \frac{s^-_i}{s^-_0}$$

where, s^-_0 is the optimal choice that is situated at the greatest distance from anti-ideal solution and s^+_0 is the best option that is the closet to the ideal solution.

Step 8: Ranking possible option. Finding the average departure of the option from the degree of value yield the final ranking

$$Q_i = \frac{(k^+_i + k^-_i)}{2}. \tag{11}$$

The selection possessing the greatest numerical magnitude is the ideal choice Q_i.

5 Illustrative Example

This section illustrates the proposed MADM technique by solving a numerical problem. Suppose there are four online platforms for shopping. There are four alternatives Amazon($\widehat{\chi}_1$); Flipkart($\widehat{\chi}_2$); Meesho($\widehat{\chi}_3$); Myntra($\widehat{\chi}_4$). The decision expert will assess the alternative online shopping platform based on the following eight criteria: SC_1 is the price competitiveness; SC_2 is the delivery charge; SC_3 is the subscription; SC_4 is the customer reviews and rating; SC_5 is the fast delivery; SC_6 is the return policy; SC_7 is the product quality; SC_8 is the customer support. Among the criteria SC_1, SC_2 and SC_3 are the "cost criteria" and SC_4, SC_5, SC_6, SC_7 and SC_8 are "benefit criteria".

The linguistic variables (LVs) used to rate the importance of decision experts, and criteria and evaluate the alternatives are given in Table 3. The framework is shown in Figure 3.

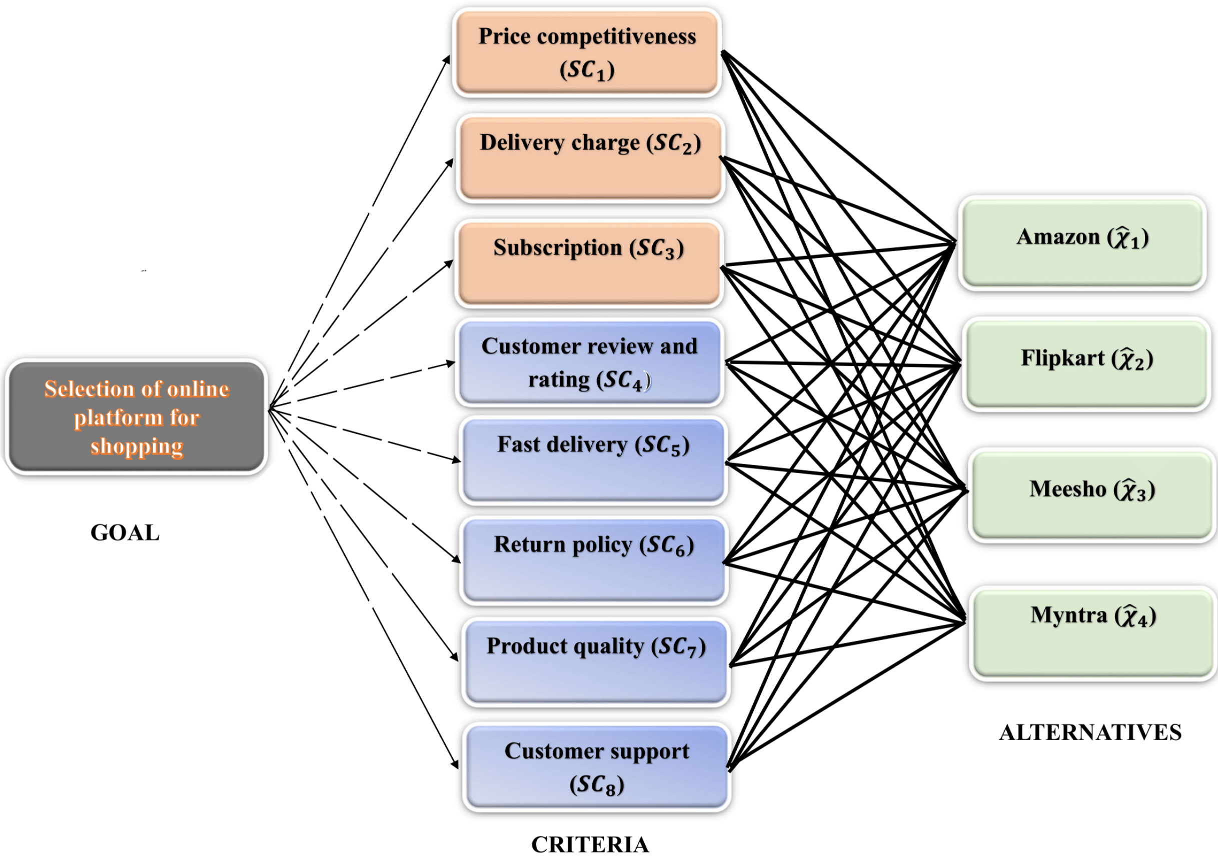

Figure 3: Framework for selecting online shopping platform alternatives.

Table 3: LVs for assessing importance of DEs, criteria and alternatives.

LVs	FFNs
Extremely good (EG)	(0.98,0.3)
Good (G)	(0.9,0.6)
Medium (M)	(0.85,0.7)
Bad (B)	(0.78,0.8)
Very bad (VB)	(0.3,0.98)

The weight of the criteria's are $\varpi_1 = 0.15, \varpi_2 = 0.22, \varpi_3 = 0.18, \varpi_4 = 0.22, \varpi_5 = 0.12, \varpi_6 = 0.08, \varpi_7 = 0.15, \varpi_8 = 0.08$ for SC_1 to SC_8 respectively.

The next and last step is the application of the proposed triangular divergence distance measure-based CRADIS method for ranking of the alternatives. The initial decision matrix made by the decision maker is given in Table 4.

Table 4: Decision matrix for in terms of LVs.

Alternative	SC_1	SC_2	SC_3	SC_4	SC_5	SC_6	SC_7	SC_8
$\widehat{\chi}_1$	(G)	(M)	(G)	(B)	(VB)	(M)	(G)	(EG)
$\widehat{\chi}_2$	(EG)	(M)	(G)	(VB)	(B)	(M)	(EG)	(B)
$\widehat{\chi}_3$	(G)	(B)	(M)	(VB)	(EG)	(B)	(M)	(G)
$\widehat{\chi}_4$	(B)	(M)	(VB)	(EG)	(B)	(M)	(G)	(G)

Step 1 The decision matrix made by the decision experts is given below in Table 5.

Table 5: Decision matrix.

Alternative	SC_1	SC_2	SC_3	SC_4	SC_5	SC_6	SC_7	SC_8
$\widehat{\chi}_1$	(0.9,0.6)	(0.85,0.7)	(0.9,0.6)	(0.78,0.8)	(0.3,0.98)	(0.85,0.7)	(0.9,0.6)	(0.98,0.3)
$\widehat{\chi}_2$	(0.98,0.3)	(0.85,0.7)	(0.9,0.6)	(0.3,0.98)	(0.78,0.8)	(0.85,0.7)	(0.98,0.3)	(0.78,0.8)
$\widehat{\chi}_3$	(0.9,0.6)	(0.78,0.8)	(0.85,0.7)	(0.3,0.98)	(0.98,0.3)	(0.78,0.8)	(0.85,0.7)	(0.9,0.6)
$\widehat{\chi}_4$	(0.78,0.8)	(0.85,0.7)	(0.3,0.98)	(0.98,0.3)	(0.78,0.8)	(0.85,0.7)	(0.9,0.6)	(0.9,0.6)

Step 2 Construct the normalized decision matrix for the FFNs which is given in Table 6.

Table 6: Normalized decision matrices.

Alternative	SC_1	SC_2	SC_3	SC_4	SC_5	SC_6	SC_7	SC_8
$\widehat{\chi}_1$	(0.6,0.9)	(0.7,0.85)	(0.6,0.9)	(0.78,0.8)	(0.3,0.98)	(0.85,0.7)	(0.9,0.6)	(0.98, 0.3)
$\widehat{\chi}_2$	(0.3,0.98)	(0.7,0.85)	(0.6,0.9)	(0.3,0.98)	(0.78,0.8)	(0.85,0.7)	(0.98,0.3)	(0.78,0.8)
$\widehat{\chi}_3$	(0.6,0.9)	(0.8,0.78)	(0.7,0.85)	(0.3,0.98)	(0.98,0.3)	(0.78,0.8)	(0.85,0.7)	(0.9,0.6)
$\widehat{\chi}_4$	(0.8,0.78)	(0.7,0.85)	(0.98,0.3)	(0.98,0.3)	(0.78,0.8)	(0.85,0.7)	(0.9,0.6)	(0.9,0.6)

Step 3 We calculate the weighted decision matrix using Equation (7). The weighted decision matrix is given in Table 7.

Table 7: Weighted decision matrix.

Alternative	SC_1	SC_2	SC_3	SC_4	SC_5	SC_6	SC_7	SC_8
$\widehat{\chi}_1$	(0.330,0.984)	(0.445,0.965)	(0.349,0.981)	(0.234,0.995)	(0.148,0.997)	(0.419,0.972)	(0.562,0.926)	(0.587,0.908)
$\widehat{\chi}_2$	(0.160,0.997)	(0.445,0.965)	(0.349,0.981)	(0.081,0.999)	(0.420,0.974)	(0.419,0.972)	(0.702,0.835)	(0.369,0.982)
$\widehat{\chi}_3$	(0.330,0.984)	(0.526,0.947)	(0.418,0.971)	(0.082,0.999)	(0.660,0.865)	(0.369,0.982)	(0.511,0.948)	(0.463,0.960)
$\widehat{\chi}_4$	(0.467,0.963)	(0.445,0.965)	(0.736,0.805)	(0.380,0.976)	(0.420,0.973)	(0.419,0.972)	(0.562,0.926)	(0.462,0.959)

Step 4 Establish the ideal and anti-ideal solution which is given in Table 8 using Equation (8).

Table 8: Ideal and anti-ideal solution.

ideal and anti ideal	SC_1	SC_2	SC_3	SC_4	SC_5	SC_6	SC_7	SC_8
(t_{0j})	(0.467,0.963)	(0.527,0.947)	(0.736,0.805)	(0.380,0.976)	(0.660,0.865)	(0.419,0.972)	(0.702,0.835)	(0.587,0.908)
(t_{m+ij})	(0.160,0.997)	(0.445,0.965)	(0.350,0.981)	(0.0812,0.999)	(0.148,0.997)	(0.369,0.982)	(0.510,0.948)	(0.369,0.982)

Step 5 Next, we find the distance of every alternative from both the ideal and anti-ideal using the proposed TDDM given in Equations (9) and (10). The positive distance matrix (PDM) and negative distance matrix (NDM) are recorded in Table 9.

Table 9: PDM and NDM.

Alternative	SC_1		SC_2		SC_3		SC_4		SC_5		SC_6		SC_7		SC_8	
	D^+	D^-	D^+	D^-	D^+	D^-	D^+	D^-	D^+	D^-	D^+	D^-	D^+	D^-	D^+	D^-
$\widehat{\chi}_1$	0.136	0.118	0.098	0	0.478	0	0.143	0.075	0.449	0	0	0.067	0.231	0.077	0	0.281
$\widehat{\chi}_2$	0.218	0	0.098	0	0.478	0	0.185	0	0.339	0.184	0	0.067	0	0.302	0.281	0
$\widehat{\chi}_3$	0.136	0.118	0	0.098	0.438	0.064	0.185	0	0	0.449	0.067	0	0.302	0	0.177	0.121
$\widehat{\chi}_4$	0	0.218	0.098	0	0	0.478	0	0.185	0.339	0.184	0	0.067	0.231	0.077	0.177	0.121

Step 6 In this step, we calculate the degree of deviation using Equation (11). The deviation values of alternatives are enlisted in Table 10.

Table 10: Deviation of alternatives.

Alternative	S_i^+	S_i^-
$\widehat{\chi}_1$	1.536	0.619
$\widehat{\chi}_2$	1.600	0.553
$\widehat{\chi}_3$	1.306	0.851
$\widehat{\chi}_4$	0.845	1.332

Step 7 We now compute utility function of each alternative using Equation (11) which are given in table 11.

Step 8 In the last step, we rank the alternatives using Equation (11) as depicted in Table 11.

Table 11: Ranking.

Alternative	K_i^+	K_i^-	Q_i	*Ranking*
$\widehat{\chi}_1$	0.550	0.465	0.507	3
$\widehat{\chi}_2$	0.528	0.416	0.471	4
$\widehat{\chi}_3$	0.647	0.639	0.643	2
$\widehat{\chi}_4$	1	1	1	1

After ranking of all alternatives we get that $\widehat{\chi}_4$ is the best option.

5.1 Comparative Analysis

In this section, the proposed modified CRADIS method is compared with the existing methods and distance measures. The comparison has been performed with the TOPSIS and VIKOR methods. The algorithm of the TOPSIS method as given by Kirisci [27], is used to solve the illustrative problem. In place of the distance measure used by Kirisci [27], we have used GEDM, HDM, and the proposed TDDM. To compare with the VIKOR method, the algorithm given by Gül [28] is used where, the GEDM, HDM, and proposed TDDM are used to calculate distances of alternatives from an ideal solution.

The TDDM-based CRADIS method gives the ranking $\widehat{\chi}_4 > \widehat{\chi}_3 > \widehat{\chi}_1 > \widehat{\chi}_2$. First, we compare the TDDM-based CRADIS method with the GEDM-based CRADIS method which is given in Equation (1) to evaluate the distance between the alternatives and the PIS and NIS and we see that the ranking order is $\widehat{\chi}_4 > \widehat{\chi}_3 > \widehat{\chi}_2 > \widehat{\chi}_1$. Next, we used HDM-based CRADIS which is given in Equation (2) in place of TDDM-based CRADIS for comparison. The outcome remains unchanged in the case of the GEDM-based CRADIS method, i.e., $\widehat{\chi}_4 > \widehat{\chi}_3 > \widehat{\chi}_2 > \widehat{\chi}_1$.

Now the comparison is done by the VlseKriterijumska Optimizacija I Kompromisno Resenje (VIKOR) [29] method. The comparison is performed by using TDDM-based VIKOR in place of the modified TDDM-based CRADIS method and the result obtained is $\widehat{\chi}_4 > \widehat{\chi}_3 > \widehat{\chi}_1 > \widehat{\chi}_2$, which is the same as the proposed method. We also use GEDM-based VIKOR and HDM-based VIKOR in place of the modified TDDM-based CRADIS method for comparison. In both cases, we see that the ranking is equivalent to the proposed method, i.e., $\widehat{\chi}_4 > \widehat{\chi}_3 > \widehat{\chi}_1 > \widehat{\chi}_2$.

Lastly, a comparison is done with a novel TOPSIS [27] method. For comparison, we use TDDM-based TOPSIS in place of TDDM-based CRADIS and here the ranking result is $\widehat{\chi}_4 > \widehat{\chi}_3 > \widehat{\chi}_1 > \widehat{\chi}_2$ which is the same as TDDM-based CRADIS method. We also compare GEDM-based TOPSIS and we get the ranking $\widehat{\chi}_3 > \widehat{\chi}_4 > \widehat{\chi}_1 > \widehat{\chi}_2$. Comparing TDDM-based CRADIS method with HDM-based TOPSIS, it gives the ranking $\widehat{\chi}_4 > \widehat{\chi}_3 > \widehat{\chi}_1 > \widehat{\chi}_2$ which is the same as TDDM-based CRADIS method.

Given in Table 12 are the ranking orders using different distance measures. Therefore, it can be stated that the proposed TDDM method is superior as well as reliable.

Table 12: Comparison of ranking results using different distance-based MADM methods.

MADM method	Alternatives ranked based on	Distance measure	Ranking result
CRADIS (proposed)	Degree of value yield	GEDM	$\widehat{\chi}_4 > \widehat{\chi}_3 > \widehat{\chi}_2 > \widehat{\chi}_1$
		HDM	$\widehat{\chi}_4 > \widehat{\chi}_3 > \widehat{\chi}_2 > \widehat{\chi}_1$
		TDDM	$\widehat{\chi}_4 > \widehat{\chi}_3 > \widehat{\chi}_1 > \widehat{\chi}_2$
VIKOR [28]	Compromise measure	GEDM	$\widehat{\chi}_4 > \widehat{\chi}_3 > \widehat{\chi}_1 > \widehat{\chi}_2$
		HDM	$\widehat{\chi}_4 > \widehat{\chi}_3 > \widehat{\chi}_1 > \widehat{\chi}_2$
		TDDM	$\widehat{\chi}_4 > \widehat{\chi}_3 > \widehat{\chi}_1 > \widehat{\chi}_2$
TOPSIS [27]	Closeness coefficient	GEDM	$\widehat{\chi}_3 > \widehat{\chi}_4 > \widehat{\chi}_1 > \widehat{\chi}_2$
		HDM	$\widehat{\chi}_4 > \widehat{\chi}_3 > \widehat{\chi}_1 > \widehat{\chi}_2$
		TDDM	$\widehat{\chi}_4 > \widehat{\chi}_3 > \widehat{\chi}_1 > \widehat{\chi}_2$

Thus, we see for all the cases, the best alternative is $\widehat{\chi}_4$, except GEDM based TOPSIS.

Figure 4 shows the comparative analysis concerning different distance measures.

5.2 Sensitivity Analysis

In this particular subsection, a sensitivity analysis is conducted to ascertain the stability of the method put forth. Four attributes were utilized in the case study. To conduct the sensitivity analysis, six different sets of attribute weights are employed, which are derived from the rearrangement of the initially computed attribute weights. Through the examination of the model's reaction to various weighting methods, it is possible to pinpoint the attributes that carry the greatest influence on the outcomes and detect probable sources of uncertainty.

Through this iterative process, sensitivity analysis improves decision-making by elucidating the resilience of the model across various scenarios, directing stakeholders towards better informed and adaptable decisions.

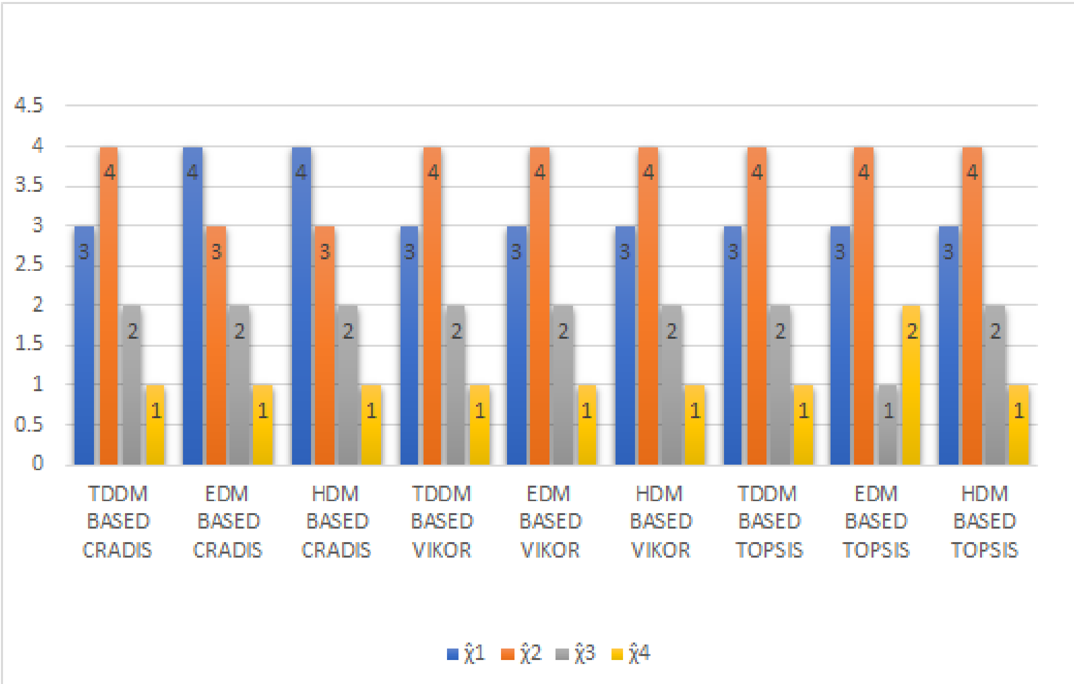

Figure 4: Comparison analysis.

The criteria weights for six sets are presented in Table 13. These particular sets of criteria weights are employed in the suggested approach, leading to alterations in the prioritization of the alternatives. Table 13 presents the selection of weight sets utilized for conducting sensitivity analysis. It is observed that in sets 2 and 4, the variable SC_1 holds the maximum weight, while for sets 4 and 6, SC_7 is assigned the minimum weight.

Table 13: Set of weight of criteria.

Sets	SC_1	SC_2	SC_3	SC_4	SC_5	SC_6	SC_7	SC_8
Set 1	0.15	0.22	0.18	0.02	0.12	0.08	0.15	0.08
Set 2	0.22	0.18	0.15	0.12	0.08	0.02	0.08	0.15
Set 3	0.18	0.02	0.12	0.08	0.15	0.08	0.15	0.22
Set 4	0.22	0.18	0.15	0.08	0.08	0.15	0.02	0.12
Set 5	0.12	0.08	0.15	0.08	0.15	0.22	0.18	0.02
Set 6	0.08	0.15	0.08	0.15	0.22	0.18	0.02	0.12

From the sensitivity analysis results presented in Table 14, it is evident that the ordering of alternatives remains consistent across all six sets of attribute weights, with the exception of set 5. The rankings consistently place Myntra ($\widehat{\chi}_4$) in the first position, followed by Meesho ($\widehat{\chi}_3$), Amazon ($\widehat{\chi}_1$) and Flipkart ($\widehat{\chi}_2$). However, in set 5, there is a deviation where Flipkart ($\widehat{\chi}_2$) is ranked third and Amazon ($\widehat{\chi}_1$) is ranked fourth. This shift in rankings can be attributed to variations in the weights assigned to the criteria, leading to a reduction in the overall utility of the alternatives. Therefore, the consistency observed in the ranking outcomes indicates that the proposed methodology exhibits significant stability and effectiveness across different configurations of criterion weights.

From Figure 5 we show the graphical demonstration of the sensitivity analysis.

Table 14: Sensitivity of proposed method.

Sets	Q_1	Q_2	Q_3	Q_4	Ranking
Set 1	0.507	0.472	0.643	1	$\widehat{\chi}_4 > \widehat{\chi}_3 > \widehat{\chi}_1 > \widehat{\chi}_2$
Set 2	0.501	0.338	0.521	1	$\widehat{\chi}_4 > \widehat{\chi}_3 > \widehat{\chi}_1 > \widehat{\chi}_2$
Set 3	0.563	0.430	0.589	1	$\widehat{\chi}_4 > \widehat{\chi}_3 > \widehat{\chi}_1 > \widehat{\chi}_2$
Set 4	0.487	0.309	0.519	1	$\widehat{\chi}_4 > \widehat{\chi}_3 > \widehat{\chi}_1 > \widehat{\chi}_2$
Set 5	0.436	0.463	0.541	1	$\widehat{\chi}_4 > \widehat{\chi}_3 > \widehat{\chi}_2 > \widehat{\chi}_1$
Set 6	0.513	0.377	0.625	1	$\widehat{\chi}_4 > \widehat{\chi}_3 > \widehat{\chi}_1 > \widehat{\chi}_2$

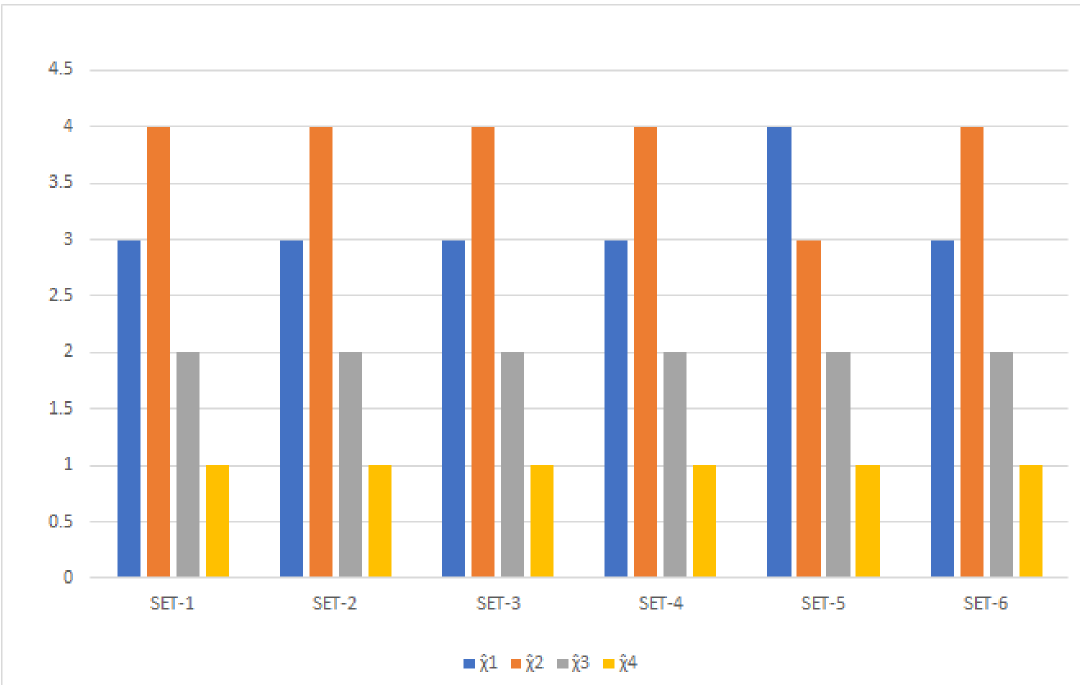

Figure 5: Sensitivity analysis.

6 Conclusion

This study proposes a modified CRADIS based on triangular divergence distance measure in the Fermatean Fuzzy sets. The FFNs are more efficient at accommodating fuzzy information compared to fuzzy extensions. Given the shortcomings of the existing distance measures for FFNs, a triangular divergence-based distance measure is proposed. To prevent any loss of information, the proposed triangular divergence-based distance measure includes the hesitancy degree of FFNs. The superiority of the proposed TDDM is established by an example where the proposed TDDM successfully distinguishes the distances between two given pairs of FFNs. To validate the proposed modified CRADIS method's practical applicability, it is used to solve a numerical problem. To check the applicability of the proposed modified CRADIS method, it has been compared with existing methods and distance measures. The comparative analysis suggests the superiority and reliability of the proposed method. We also conduct a sensitivity analysis to check its stability.

However, every research endeavor inevitably encounters certain constraints. In this paper, only the CRADIS method has been modified using the proposed TDDM. Utilizing the proposed TDDM in other distance-based MADM methods can establish the practicality of the proposed distance measure. Another limitation is that we have addressed only one numerical problem using the proposed modified CRADIS method. We have involved only one decision expert. Including more decision experts in the decision-making process can give better results. Also, we have assumed the weight of the criteria to maintain the simplicity of the method. Using different criteria weight determination methods can improve the model significantly.

There are numerous avenues for future investigation. The proposed distance measure can be extended to other fuzzy environments like 3,4-quasirung fuzzy sets [30], p,q-quasirung orthopair fuzzy sets [31], hesitant fuzzy sets [32], neutrosophic fuzzy sets [33], linear diophantine fuzzy sets [34], q-rung linear diophantine fuzzy hypersoft fuzzy set [35]. Also, the proposed TDDM can be applied to other distance-based MADM methods. The proposed TDDM-based modified CRADIS can be applied to other real-life MADM problems to establish further applicability of the method.

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Conflict of Interest: The authors declare no conflict of interest.

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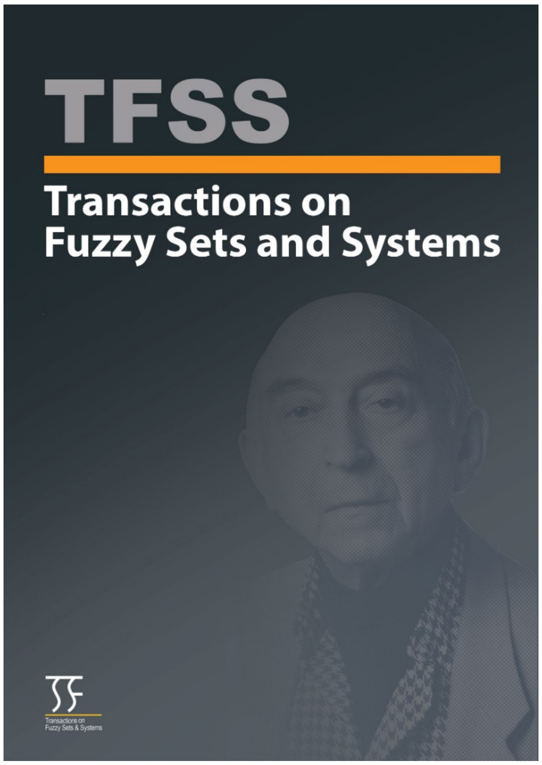

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Approximate Solution of Complex LR Fuzzy Linear Matrix Equation †

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Approximate Solution of Complex LR Fuzzy Linear Matrix Equation †

Xiaobin Guo*, **Xiangyang Fan**, **Hangru Lin**

Abstract. This paper aims at solving a class of linear LR complex fuzzy matrix equations $A\widetilde{X}B = \widetilde{C}$ using a matrix approach. By using the basic operation of LR fuzzy number matrix, the original complex fuzzy matrix equation is transformed into a clear matrix equation group. Two new and simplified models for calculating fuzzy solutions are designed in detail, and sufficient conditions for strong fuzzy solutions are analyzed. Finally, two examples are given to illustrate the feasibility and effectiveness of the proposed method. Now that the complex fuzzy numbers can describe uncertain factors more vivid and reasonable than the real fuzzy numbers sometimes and the wide application of matrix equations under uncertain conditions, our research work enriches the fuzzy linear systems theory.

AMS Subject Classification 2020: 15A30; 15A24
Keywords and Phrases: Complex fuzzy numbers, Matrix analysis, Fuzzy matrix equations, Approximate solutions.

1 Introduction

There are a large number of phenomena and events in the real world that we can not find a definite classification standard to judgment them. We call this kind of property of things as fuzziness and it is difficult to be accurately measured and described by classical mathematics. In 1965, the American cybernetics expert Professor Zadeh [1] proposed the concept of fuzzy sets wich made the birth of the new subject of fuzzy mathematics. In the past half century, the development of fuzzy mathematics has shown extraordinary vitality, its theoretical research involves fuzzy analysis, fuzzy algebra, fuzzy topology and other disciplines, and its application practice covers many fields such as artificial intelligence, cluster analysis, expert system, fault diagnosis, system evaluation, social sciences, big data processing and so on. As we all know, no matter in statistical analysis or in management science, only the linear system that theory is relatively mature and easy to calculate. If this uncertainty is expressed and calculated by fuzzy numbers, the description of the problem will be more reasonable and accurate, and analysis and decision of the problem will be convenient. Therefore, it is of practical significance to study uncertain linear systems based on fuzzy numbers. Fuzzy number is a special kind of fuzzy set, which is a generalization of one kind of natural real numbers.[2, 3, 4]. In 1998, Friedman[5] et al. proposed a general model for solving fuzzy linear systems, and studied the fuzzy linear system $A\widetilde{x} = \widetilde{b}$ by using the embedding method. After this, T. Allahviranloo et al. and B. Zheng et al. studied some other forms of fuzzy linear systems such as DFLS, GFLS, cffls, DFFLS, and GDFLS

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[6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. In recent years, new studies on fuzzy numbers and various types of fuzzy linear systems have emerged in an endless stream. [16, 17, 18].

It is well known that matrix systems play a crucial role in the vast field of scientific computing. These systems often need to deal with situations that contain some or all of the parameter uncertainty, which is particularly common when modeling and predicting complex phenomena. In 2009 and 2018, respectively, Allahviranloo et al.[19] And AmirfakhrianIn et al.[20] used different methods to study and solve the fuzzy linear matrix equation of the form $A\widetilde{X}B = \widetilde{C}$. Different forms of fuzzy matrix equations have been systematically studied by Guo et al. in the last decade. [21] − [22]. For complex fuzzy linear systems, few researchers have proposed research methods in recent decades. The concept of fuzzy complex numbers was first introduced by J.J. Buckley[23] in 1989. In 2000, Qiu et al.[24] restudied fuzzy complex sequences and their convergence properties by studying $n \times n$ fuzzy complex linear systems.In 2009, Rahgooy et al.[25] applied fuzzy composite linear equations to circuit analysis problems. In 2014, Behera and Chakraverty used the embedding method to analyze and discuss fuzzy complex systems of linear equations, and improved the arithmetic operations of complex fuzzy numbers[16].In 2018, Guo et al.[26] introduced the complex fuzzy matrix equation $\widetilde{Z}C = \widetilde{W}$ and proposed a general model for complex LR fuzzy solutions. Recently, Wu et al.[25] established a method for calculating generalized fuzzy solutions of the semi-complex fuzzy matrix equation AX=B by meabs of the MPwg inverse of a crisp matrix.

In this paper, a matrix model for solving fuzzy matrix equation $A\widetilde{X}B = \widetilde{C}$ is proposed.Compare with the present work, this paper has three mathematical contribution, that is, (1) semi complex LR fuzzy matrix equation $AXB = C$ is firstly investigated by a matrix method. Through giving basic operations of complex LR fuzzy matrices; (2) two new and simple computing models that is a system of linear matrix equations are constructed; (3) Two sufficient condition of strong complex fuzzy solution condition are analyzed and provided. The content structure is as follows:

In Section 2, we review the concept of complex LR fuzzy numbers, based on which we introduce the concept of complex LR fuzzy linear matrix equation. In Section 3, we construct a detailed model of the LR complex fuzzy matrix equation, solve the equation by using the generalized inverse of the coefficient matrix, and at the same time explore the existence conditions of strong fuzzy solutions and their properties in depth. In order to verify the effectiveness and practicability of the method, some numerical examples are given. Section 4 is the summary and conclusion refinement of the whole paper, and Section 5 is the prospect of future research directions, and puts forward the topics and potential research areas for further exploration.

2 Preliminaries

The concepts of fuzzy numbers and fuzzy matrices have the following definitions. [2, 3, 4]

Definition 2.1. *A fuzzy number is a special kind of fuzzy set, denoted as a map* $\widetilde{u} : R \to I = [0, 1]$*, which has the following four conditions::*

(1) $\widetilde{u}$ *is upper semi continuous,*

(2) $\widetilde{u}$ *is fuzzy convex, i.e.,* $\widetilde{u}(\lambda x + (1 - \lambda)y) \geq \min\{\widetilde{u}(x), \widetilde{u}(y)\}$ *for all* $x, y \in R, \lambda \in [0, 1]$*,*

(3) $\widetilde{u}$ *is normal, i.e., there exists* $x_0 \in R$ *such that* $\widetilde{u}(x_0) = 1$*,*

(4) supp$\widetilde{u} = \{x \in R \mid \widetilde{u}(x) > 0\}$ *is the support of the* $\widetilde{u}$*, and its closure cl(supp*$\widetilde{u}$*) is compact.*

Let E^1 *be the set of all fuzzy numbers on R.*

Definition 2.2. *We represent an arbitrary fuzzy number* $(\underline{u}(r), \overline{u}(r))$, $0 \leq r \leq 1$, *by a set of ordered pairs of functions satisfying the following conditions:*

(1) $\underline{u}(r)$ *is a bounded monotonic increasing left continuous function,*

(2) $\overline{u}(r)$ *is a bounded monotonic decreasing left continuous function,*

(3) r *in the interval* $[0,1]$,$\underline{u}(r)$ *is always less than or equal to overlineu*(r).

A crisp number x *can be represented as a fuzzy number by setting both* $\underline{u}(r)$ *and* $\overline{u}(r)$ *to* x *with* $0 \leq r \leq 1$. *By introducing a proper definition, the space of fuzzy numbers* $\{(\underline{u}(r), \overline{u}(r))\}$ *forms a convex cone* E^1. *This convex cone can be embedded in a Banach space in an isomorphic and metric consistent manner.*

Definition 2.3. *A fuzzy number* $\widetilde{M}$ *is said to be a LR fuzzy number if*

$$\mu_{\widetilde{M}}(x) = \begin{cases} L(\frac{m-x}{m_\alpha}), & x \leq m, \quad \alpha > 0, \\ R(\frac{x-m}{m_\beta}), & x \geq m, \quad \beta > 0, \end{cases}$$

Here m *is the principal mean of* $\widetilde{M}$, m_α *is the left extension,* m_β *is the right extension, and the function* $L(\cdot)$, *we will make it a left-shaped function that satisfies the following conditions:*

(1) $L(x) = L(-x)$,

(2) $L(0) = 1$ *and* $L(1) = 0$,

(3) $L(x)$ *is non increasing on* $[0, \infty)$.

The definition of a right shape function $R(\cdot)$ *is similar to that of* $L(\cdot)$.

Clearly, when two LR fuzzy numbers $\widetilde{M} = (m, m_\alpha, m_\beta)_{LR}$ *and* $\widetilde{N} = (n, n_\alpha, n_\beta)_{LR}$ *are equal, if and only if* $m = n, m_\alpha = n_\alpha, m_\beta = n_\beta$. *Similarly, if* $\widetilde{M}$ *is positive (negative), if and only if* $m - m_\alpha > 0 (m + m_\beta < 0)$.

Definition 2.4. *We have for any LR fuzzy numbers* $\widetilde{M} = (m, m_\alpha, m_\beta)_{LR}$ *and* $\widetilde{N} = (n, n_\alpha, n_\beta)_{LR}$, *the following.*

(1) Addition

$$\widetilde{M} \oplus \widetilde{N} = (m, m_\alpha, m_\beta)_{LR} \oplus (n, n_\alpha, n_\beta)_{LR} = (m + n, m_\alpha + n_\alpha, m_\beta + n_\beta)_{LR}.$$

(2) Subtraction

$$\widetilde{M} - \widetilde{N} = (m, m_\alpha, m_\beta)_{LR} - (n, n_\alpha, n_\beta)_{LR} = (m - n, m_\alpha + n_\beta, m_\beta + n_\alpha)_{LR}.$$

(3) Scalar multiplication

$$\lambda \otimes \widetilde{M} = \lambda \otimes (m, m_\alpha, m_\beta)_{LR} \cong \begin{cases} (\lambda m, \lambda m_\alpha, \lambda m_\beta)_{LR}, & \lambda \geq 0, \\ (\lambda m, -\lambda m_\beta, -\lambda m_\alpha)_{RL}, \lambda < 0. \end{cases}$$

Definition 2.5. *The LR complex fuzzy number consists of real part and imaginary part. An arbitrary complex LR fuzzy number could be represented as* $\widetilde{x} = \widetilde{p} + i\widetilde{q}$, where $\widetilde{p} = (p, p^l, p^r), \widetilde{q} = (q, q^l, q^r)$. In this case, $\widetilde{x}$ can be written as

$$\widetilde{x} = \widetilde{p} + i\widetilde{q} = (p, p^l, p^r) + i(q, q^l, q^r).$$

Definition 2.6. *Each element of A complex LR fuzzy matrix* $\widetilde{A} = (\widetilde{a}_{ij})$ *is a matrix constructed from complex fuzzy numbers. Let* $\widetilde{A} = (\widetilde{a}_{ij}) = ([m, m^l, m^r] + i[n, n^l, n^r])_{ij}, i, j = 1, 2, \cdots, n$, *the complex LR fuzzy matrix* $\widetilde{A}$ *can be represented by* $\widetilde{A} = (M, M^l, M^r) + i(N, N^l, N^r)$.

Definition 2.7. *Given two arbitrary complex LR fuzzy matrices* $\widetilde{X}$ *and* $\widetilde{Y}$, *consisting of real parts* $\widetilde{P}$ *and* $\widetilde{U}$, *and imaginary parts* $\widetilde{Q}$ *and* $\widetilde{V}$, *respectively, where these real and imaginary parts are LR fuzzy number matrices. The arithmetic operation rules between these two complex LR fuzzy matrices are defined as follows.*

(1) $\widetilde{X} + \widetilde{Y} = (\widetilde{P} + \widetilde{U}) + i(\widetilde{Q} + \widetilde{V})$,

(2) $\widetilde{X} - \widetilde{Y} = (\widetilde{P} - \widetilde{U}) + i(\widetilde{Q} - \widetilde{V})$,

(3) $k\widetilde{X} = k\widetilde{P} + ik\widetilde{Q}, k \in R$,

(4) $\widetilde{X} \times \widetilde{Y} = (\widetilde{P} \times \widetilde{U} - \widetilde{Q} \times \widetilde{V}) + i(\widetilde{P} \times \widetilde{V} + \widetilde{Q} \times \widetilde{U})$.

Definition 2.8. *The matrix system*

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{12} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} \widetilde{x}_{11} & \widetilde{x}_{12} & \cdots & \widetilde{x}_{1n} \\ \widetilde{x}_{21} & \widetilde{x}_{12} & \cdots & \widetilde{x}_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \widetilde{x}_{n1} & \widetilde{x}_{n2} & \cdots & \widetilde{x}_{nn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{12} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} \widetilde{c}_{11} & \widetilde{c}_{12} & \cdots & \widetilde{c}_{1m} \\ \widetilde{c}_{21} & \widetilde{c}_{12} & \cdots & \widetilde{c}_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \widetilde{c}_{n1} & \widetilde{c}_{n2} & \cdots & \widetilde{c}_{nn} \end{pmatrix}, \tag{2.1}$$

where $a_{ij}, b_{ij}, 1 \leq i, j \leq n$ are crisp numbers and $\widetilde{c}_{ij}, 1 \leq i, j \leq n$ are complex LR fuzzy numbers, is called a LR complex fuzzy linear matrix equations(CLRFLMEs).

Using matrix notation, we have

$$A\widetilde{X}B = \widetilde{C}, \tag{2.2}$$

A complex LR fuzzy numbers matrix

$$\widetilde{X} = (M, M^l, M^r) + i(N, N^l, N^r)$$

$$= (m, m^l, m^r) + i(n, n^l, n^r), 1 \leq i, j \leq n$$

is said to the solution of the general dual complex fuzzy matrix equation (2.1) *if $\widetilde{X}$ satisfies the Eqs.*(2.2).

3 Solving the CLRFLMEs

Theorem 3.1. *Given a complex fuzzy linear matrix system $A\widetilde{X}B = \widetilde{C}$, it can be equivalently expressed as a series of linear matrix equations, as follows.*

$$\begin{cases} A[MN] \bigotimes B = [UV], \\ A \bigotimes \begin{pmatrix} M^l & M^r \\ N^l & N^r \end{pmatrix} \begin{pmatrix} B^+ & -B^- \\ -B^- & B^+ \end{pmatrix} = \begin{pmatrix} U^l & U^r \\ V^l & V^r \end{pmatrix} \end{cases}, \tag{3.1}$$

where

$$\widetilde{X} = (M, M^l, M^r) + i(N, N^l, N^r), \widetilde{C} = (U, U^l, U^r) + i(V, V^l, V^r). \tag{3.2}$$

And the elements b^+_{ij} of matrix B^+ and b^-_{ij} of matrix B^- are determined by this way: if $b_{ij} \geq 0, b^+_{ij} = b_{ij}$ else $b^+_{ij} = 0, 1 \leq i, j \leq n$; if $b_{ij} < 0, b^-_{ij} = b_{ij}$ else $b^-_{ij} = 0, 1 \leq i, j \leq n$.

Proof. Let $\widetilde{C} = [U, U^l, U^r] + i[V, V^l, V^r] = ([u, u^l, u^r] + i[v, v^l, v^r])_{n\times n}$ and the unknown complex fuzzy matrix $\widetilde{X} = [M, M^l, M^r] + i[N, N^l, N^r] = ([m, m^l, m^r] + i[n, m^l, m^r])_{n\times n}$. We also let $A = A^+ + A^-$ in which the elements a^+_{ij} of matrix A^+ and a^-_{ij} of matrix A^+ are determined by the following way: if $a_{ij} \geq 0, a^+_{ij} = a_{ij}$ else $a^+_{ij} = 0, 1 \leq i, j \leq n$; if $a_{ij} < 0, a^-_{ij} = a_{ij}$ else $a^-_{ij} = 0, 1 \leq i, j \leq n$ and also let $B = B^+ + B^-$ in the same way.

For complex fuzzy matrix equation $A\widetilde{X}B = \widetilde{C}$, i.e.,

$$A([M, M^l, M^r] + i[N, N^l, N^r])B = [U, U^l, U^r] + i[V, V^l, V^r].$$

Supposing $A = A^+ + A^-$ and $B = B^+ + B^-$, we have

$$(A^+ + A^-)([M, M^l, M^r] + i[N, N^l, N^r])(B^+ + B^-) = [U, U^l, U^r] + i[V, V^l, V^r]. \tag{3.3}$$

Since

$$\widetilde{m}_{ij}k = \begin{cases} (k\underline{m}_{ij}(r), k\overline{m}_{ij}(r)), & k \geq 0, \\ (k\overline{m}_{ij}(r), k\underline{m}_{ij}(r)), & k < 0, \end{cases}$$

and

$$\widetilde{M}B = \begin{cases} (\underline{M}(r)B, \overline{M}(r)B), & B \geq 0, \\ (\overline{M}(r)B, \underline{M}(r)B), & B < 0, \end{cases}$$

so the Eqs.(3.3) can be rewritten as

$$\begin{gathered} A^+[M, M^l, M^r]B^+ + A^+[M, M^l, M^r]B^- + A^-[M, M^l, M^r]B^+ + A^-[M, M^l, M^r]B^- + \\ i(A^+[N, N^l, N^r]B^+ + A^+[N, N^l, N^r]B^- + A^-[N, N^l, N^r]B^+ + A^-[N, N^l, N^r]B^-) \\ = [U, U^l, U^r] + i[V, V^l, V^r]. \end{gathered}$$

In comparison with the coefficients of i, we get

$$A^+[M, M^l, M^r]B^+ + A^+[M, M^l, M^r]B^- + A^-[M, M^l, M^r]B^+ + A^-[M, M^l, M^r]B^- = [U, U^l, U^r],$$

and

$$A^+[N, N^l, N^r]B^+ + A^+[N, N^l, N^r]B^- + A^-[N, N^l, N^r]B^+ + A^-[N, N^l, N^r]B^- = [V, V^l, V^r],$$

i.e.,

$$\begin{cases} A^+MB^+ + A^+MB^- + A^-MB^+ + A^-MB^- = U, \\ A^+M^lB^+ - A^+M^rB^- - A^-M^rB^+ + A^-M^lB^- = U^l, \\ A^+M^rB^+ - A^+M^lB^- - A^-M^lB^+ + A^-M^rB^- = U^l, \\ A^+NB^+ + A^+NB^- + A^-NB^+ + A^-NB^- = V, \\ A^+N^lB^+ - A^+N^rB^- - A^-N^rB^+ + A^-N^lB^- = V^l, \\ A^+N^rB^+ - A^+N^lB^- - A^-N^lB^+ + A^-N^rB^- = V^l, \end{cases} \tag{3.4}$$

Denoting in matrix form, they can be written as

$$\begin{cases} AMB = U, \\ A\left(\; M^l, M^r \; \right) \begin{pmatrix} B^+ & -B^- \\ -B^- & B^+ \end{pmatrix} = \left(\; U^l, U^r, \; \right), \end{cases} \tag{3.5}$$

and

$$\begin{cases} ANB = V, \\ A\left(\; M^l, M^r \; \right) \begin{pmatrix} B^+ & -B^- \\ -B^- & B^+ \end{pmatrix} = \left(\; V^l, V^r, \; \right). \end{cases} \tag{3.6}$$

From Eqs.(3.5) and (3.6), we obtain the Eqs.(3.1). and (3.2) as follows:

$$\begin{cases} A[MN] \bigotimes B = [UV], \\ A \bigotimes \begin{pmatrix} M^l & M^r \\ N^l & N^r \end{pmatrix} \begin{pmatrix} B^+ & -B^- \\ -B^- & B^+ \end{pmatrix} = \begin{pmatrix} U^l & U^r \\ V^l & V^r \end{pmatrix}, \end{cases}$$

where $\bigotimes$ is the Kronecker product of matrices and

$$\widetilde{X} = (M, M^l, M^r) + i(N, N^l, N^r), \widetilde{C} = (U, U^l, U^r) + i(V, V^l, V^r).$$

Similarly, we can derivd another model for solving the Eqs.(2.2). □

Theorem 3.2. *The complex LR fuzzy linear matrix system $A\widetilde{X}B = \widetilde{C}$ can be converted into the following system of linear matrix equations*

$$\begin{cases} A \bigotimes \begin{pmatrix} M \\ N \end{pmatrix} B = \begin{pmatrix} U \\ V \end{pmatrix}, \\ \begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix} \begin{pmatrix} M^l & N^l \\ M^r & N^r \end{pmatrix} \bigotimes B = \begin{pmatrix} U^l & V^l \\ U^r & V^r \end{pmatrix}, \end{cases} \tag{3.7}$$

where

$$\widetilde{X} = (M, M^l, M^r) + i(N, N^l, N^r), \widetilde{C} = (U, U^l, U^r) + i(V, V^l, V^r). \tag{3.8}$$

And the elements a_{ij}^+ of matrix A^+ and a_{ij}^- of matrix A^- are determined by the following way: if $a_{ij} \geq 0, a_{ij}^+ = a_{ij}$ else $a_{ij}^+ = 0, 1 \leq i, j \leq n$; if $a_{ij} < 0, a_{ij}^- = a_{ij}$ else $a_{ij}^- = 0, 1 \leq i, j \leq n$.

Proof. The proof is similar with the above Theorem 3.1. □

Theorem 3.3. [27]*Given a matrix S belong to $R^{m\times n}$, T belong to $R^{p\times q}$, and C belong to $R^{m\times q}$, there exists a minimal solution X^* to the matrix equation $SXT = C$, which can be expressed as follows.*

$$X^* = S^\dagger C T^\dagger.$$

In order to find a solution to the fuzzy matrix equation (2.2), we first need to compute the system of linear equations (3.1) or (3.7). We obtain the minimum solution of the linear system(2.2) as follows.

$$\begin{cases} [MN] = A^\dagger [UV] \bigotimes B^\dagger, \\ \begin{pmatrix} M^l & M^r \\ N^l & N^r \end{pmatrix} == A^\dagger \bigotimes \begin{pmatrix} U^l & U^r \\ V^l & V^r \end{pmatrix} \begin{pmatrix} B^+ & -B^- \\ -B^- & B^+ \end{pmatrix}^\dagger \end{cases} \tag{3.9}$$

or

$$\begin{cases} \begin{pmatrix} M \\ N \end{pmatrix} = A^\dagger \bigotimes \begin{pmatrix} U \\ V \end{pmatrix} B^\dagger, \\ \begin{pmatrix} M^l & N^l \\ M^r & N^r \end{pmatrix} = \begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix}^\dagger \begin{pmatrix} U^l & V^l \\ U^r & V^r \end{pmatrix} \bigotimes B^\dagger, \end{cases} \tag{3.10}$$

where $(.)^\dagger$ is the Moore-Penrose generalized inverse of matrix $(.)$.

It seems that we obtained the complex fuzzy solution matrix $\widetilde{X} = (M, M^l, M^r) + i(N, N^l, N^r)$ as the above expression (3.9) or (3.10). However, the solution matrix may still not be an appropriate LR fuzzy numbers matrix except for that both $\widetilde{M} = (M, M^l, M^r)$ and $\widetilde{N} = (N, N^l, N^r)$ are appropriate LR fuzzy matrices. So we give the definition of LR fuzzy solution to the Eq.(2.2) as follows.

Definition 3.4. *Let $\widetilde{X} = (M, M^l, M^r) + i(N, N^l, N^r)$. If $((M, M^l, M^r)$ and (N, N^l, N^r) is the minimal solution of Eqs.(3.1) or (3.7), such that $M^l \geq O, M^r \geq O$ and $N^l \geq O, N^r \geq O$, we said that $\widetilde{X} = (M, M^l, M^r) + i(N, N^l, N^r)$ is a strong LR complex fuzzy minimal solution of fuzzy matrix equation(2.2). Otherwise, the $\widetilde{X} = (M, M^l, M^r) + i(N, N^l, N^r)$ is said to a weak LR complex fuzzy fuzzy minimal solution of fuzzy matrix equation(2.2) given by*

$$\widetilde{X} = \widetilde{m}_{ij} + i\widetilde{n}_{ij}$$

where

$$\widetilde{m}_{ij} = \begin{cases} (m_{ij}, m^l_{ij}, m^r_{ij}), & m^l_{ij} > 0, \quad m^r_{ij} > 0, \\ (m_{ij}, 0, \max\{-m^l_{ij}, m^r_{ij}\}), & m^l_{ij} < 0, \quad m^r_{ij} > 0, \\ (m_{ij}, \max\{m^l_{ij}, -m^r_{ij}\}, 0,), & m^l_{ij} > 0, \quad m^r_{ij} < 0, \\ (m_{ij}, -m^l_{ij}, -m^r_{ij}), & m^l_{ij} < 0, \quad m^r_{ij} < 0. \end{cases} \quad i, j = 1, \dots, n. \tag{3.11}$$

and

$$\widetilde{n}_{ij} = \begin{cases} (n_{ij}, n^l_{ij}, n^r_{ij}), & n^l_{ij} > 0, \quad n^r_{ij} > 0, \\ (n_{ij}, 0, \max\{-n^l_{ij}, n^r_{ij}\}), & n^l_{ij} < 0, \quad n^r_{ij} > 0, \\ (n_{ij}, \max\{n^l_{ij}, -n^r_{ij}\}, 0,), & n^l_{ij} > 0, \quad n^r_{ij} < 0, \\ (n_{ij}, -n^l_{ij}, -n^r_{ij}), & n^l_{ij} < 0, \quad n^r_{ij} < 0. \end{cases} \quad i, j = 1, \dots, n. \tag{3.12}$$

Theorem 3.5. *Let*

$$S^\dagger = \begin{pmatrix} E & F \\ F & E \end{pmatrix},$$

where

$$S = \begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix}.$$

Then

$$\begin{cases} E = \frac{1}{2}((A^+ - A^-)^\dagger + (A^+ + A^-)^\dagger), \\ F = \frac{1}{2}((A^+ - A^-)^\dagger - (A^+ + A^-)^\dagger), \end{cases} \tag{3.13}$$

where $(A^+ + A^-)^\dagger, (A^+ - A^-)^\dagger$ are Moore-Penrose inverses of matrices $A^+ + A^-$ and $A^+ - A^-$, respectively.

For the model Eqs.(3.7)*, we have the following result.*

Theorem 3.6. *If*

$$A^\dagger \geq O,$$

$$(B^+ - B^-)^\dagger + (B^+ + B^-)^\dagger) \geq O, (B^+ - B^-)^\dagger - (B^+ + B^-)^\dagger) \geq O,$$

the fuzzy matrix equation (2.2) *has a strong LR complex fuzzy minimal solution as follows:*

$$\widetilde{X} = (M, M^l, M^r) + i(N, N^l, N^r],$$

where

$$\begin{cases} [MN] = A^\dagger [UV] \bigotimes B^\dagger, \\ M^l = A^\dagger U^l E + A^\dagger U^r F, \\ M^r = A^\dagger U^l F + A^\dagger U^r E, \\ N^l = A^\dagger V^l E + A^\dagger V^r F, \\ N^r = A^\dagger V^l F + A^\dagger V^r E, \\ E = \frac{1}{2}((B^+ - B^-)^\dagger + (B^+ + B^-)^\dagger), \\ F = \frac{1}{2}((B^+ - B^-)^\dagger - (B^+ + B^-)^\dagger). \end{cases} \tag{3.14}$$

Proof. Since U^l and U^r are the left and right extensions of the fuzzy matrix $\widetilde{U}$, respectively, $C^l \geq O$ and $C^r \geq O$, this indicates that $\begin{pmatrix} U^l, U^r \end{pmatrix}$ is a nonnegative matrix. Similarly, for the fuzzy matrix $\widetilde{\widetilde{V}} = (V, V^l, V^r)$, the properties are the same.

Let

$$S^\dagger = \begin{pmatrix} E & F \\ F & E \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (B^+ - B^-)^\dagger + (B^+ + B^-)^\dagger & (B^+ - B^-)^\dagger - (B^+ + B^-)^\dagger \\ (B^+ - B^-)^\dagger - (B^+ + B^-)^\dagger & (B^+ - B^-)^\dagger + (B^+ + B^-)^\dagger \end{pmatrix}.$$

The nonnegativity of the condition $S^\dagger$, is equivalent to the fact that both matrices E and F satisfy the nonnegativity condition.

Now that $E \geq O$ and $F \geq O$, the product of three non negative matrices

$$\begin{pmatrix} M^l, M^r \\ N^l, N^r \end{pmatrix} = A^\dagger \bigotimes \begin{pmatrix} U^l, U^r \\ V^l, V^r \end{pmatrix} \begin{pmatrix} B^+ & -B^- \\ -B^- & B^+ \end{pmatrix}^\dagger$$

$$= A^\dagger \begin{pmatrix} C^l, C^r \end{pmatrix} \begin{pmatrix} E & F \\ F & E \end{pmatrix} = \begin{pmatrix} A^\dagger U^l E + A^\dagger U^r F, A^\dagger U^l F + A^\dagger U^r E \\ A^\dagger V^l E + A^\dagger V^r F, A^\dagger V^l F + A^\dagger V^r E \end{pmatrix} \geq O$$

is non negative in nature. It means that $M^l \geq O, M^r \geq O$ and $N^l \geq O, N^r \geq O$.

For the model Eqs.(3.1), we have the following result similarly. □

Theorem 3.7. *If*

$$B^\dagger \geq O,$$
$$(A^+ - A^-)^\dagger + (A^+ + A^-)^\dagger) \geq O, (A^+ - A^-)^\dagger - (A^+ + A^-)^\dagger) \geq O,$$

the fuzzy matrix equation (2.2) *has a strong LR fuzzy minimal solution as follows:*

$$\widetilde{X} = (M, M^l, M^r) + i(N, N^l, N^r],$$

where

$$\begin{cases} \begin{pmatrix} M \\ N \end{pmatrix} = A^\dagger \bigotimes \begin{pmatrix} U \\ V \end{pmatrix} B^\dagger, \\ M^l = EU^l B^\dagger + FU^r B^\dagger, \\ M^r = FU^l B^\dagger + EU^r B^\dagger, \\ N^l = EV^l B^\dagger + FV^r B^\dagger, \\ N^r = FV^l B^\dagger + EV^r B^\dagger, \\ E = \frac{1}{2}((A^+ - A^-)^\dagger + (A^+ + A^-)^\dagger), \\ F = \frac{1}{2}((A^+ - A^-)^\dagger - (A^+ + A^-)^\dagger). \end{cases} \tag{3.15}$$

Proof. The proof is straight forward. □

4 Numerical Examples

Example 4.1. Consider the following complex LR fuzzy linear matrix equation

$$\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \widetilde{x}_{11} & \widetilde{x}_{12} \\ \widetilde{x}_{21} & \widetilde{x}_{22} \end{pmatrix} \begin{pmatrix} -3 & 4 \\ 1 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} (3,2,1)_{LR} & (4,1,1)_{LR} \\ (5,2,2)_{LR} & (3,1,2)_{LR} \end{pmatrix} + i \begin{pmatrix} (4,1,2)_{LR} & (2,1,1)_{LR} \\ (5,3,2)_{LR} & (3,2,1)_{LR} \end{pmatrix}.$$

By the Theorem 3.2., the original fuzzy matrix equation is extended into the following a system of linear matrix equations (3.7)

$$\begin{cases} A \bigotimes \begin{pmatrix} M \\ N \end{pmatrix} B = \begin{pmatrix} U \\ V \end{pmatrix}, \\ \begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix} \begin{pmatrix} M^l & N^l \\ M^r & N^r \end{pmatrix} \bigotimes B = \begin{pmatrix} U^l & V^l \\ U^r & V^r \end{pmatrix}, \end{cases}$$

where

$$\widetilde{X} = (M, M^l, M^r) + i(N, N^l, N^r), \widetilde{C} = (U, U^l, U^r) + i(V, V^l, V^r).$$

and

$$A^{+} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, A^{-} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix},$$

$$U = \begin{pmatrix} 3 & 4 \\ 5 & 3 \end{pmatrix}, U^{l} = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}, U^{r} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix},$$

$$V = \begin{pmatrix} 4 & 2 \\ 5 & 3 \end{pmatrix}, V^{l} = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}, V^{r} = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix},$$

From the Eqs. (3.14), the solution of the computing model is

$$\begin{pmatrix} M \\ N \end{pmatrix} = A^{\dagger} \bigotimes \begin{pmatrix} U \\ V \end{pmatrix} B^{\dagger}$$

$$= \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}^{\dagger} \begin{pmatrix} 3 & 4 \\ 5 & 3 \\ 4 & 5 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -3 & 4 \\ 1 & 0 \end{pmatrix}^{\dagger} = \begin{pmatrix} 1.3750 & 9.6250 \\ 0.3750 & 3.6250 \\ 0.8570 & 9.1250 \\ 0.3750 & 3.6250 \end{pmatrix}$$

and

$$\begin{pmatrix} M^{l} & N^{l} \\ M^{r} & N^{r} \end{pmatrix} = \begin{pmatrix} A^{+} & -A^{-} \\ -A^{-} & A^{+} \end{pmatrix}^{\dagger} \begin{pmatrix} U^{l} & V^{l} \\ U^{r} & V^{r} \end{pmatrix} \bigotimes B^{\dagger}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}^{\dagger} \begin{pmatrix} 2 & 1 & 1 & 1 \\ 2 & 1 & 3 & 2 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} -3 & 4 \\ 1 & 0 \end{pmatrix}^{\dagger}$$

$$= \begin{pmatrix} 0.3750 & 1.5000 & 0.1250 & 0.8750 \\ 0.1250 & 0.2500 & 0.2500 & 2.2500 \\ 0.5000 & 0.8750 & 0.0000 & 0.5000 \\ 0.2500 & 0.6250 & 0.1250 & 0.8750 \end{pmatrix}.$$

It means

$$\widetilde{M} = \begin{pmatrix} (1.3750, 0.3750, 0.5000) & (9.6250, 1.5000, 0.8750) \\ (0.3750, 0.1250, 0.2500) & (3.6250, 0.2500, 0.6250) \end{pmatrix}$$

and

$$\widetilde{N} = \begin{pmatrix} (0.8570, 0.1250, 0.0000) & (9.1250, 0.8750, 0.5000) \\ (0.3750, 0.2500, 0.1250) & (3.6250, 2.2500, 0.8750) \end{pmatrix}.$$

Since M^{l}, M^{r} and N^{l}, N^{r} are nonnegative matrices and $M - M^{l} > O, N - N^{l} > O$, the solution we obtained is an appropriate LR complex fuzzy matrix

$$\widetilde{X} = (M, M^{l}, M^{r}) + i(N, N^{l}, N^{r})$$

$$= \begin{pmatrix} (1.3750, 0.3750, 0.5000) & (9.6250, 1.5000, 0.8750) \\ (0.3750, 0.1250, 0.2500) & (3.6250, 0.2500, 0.6250) \end{pmatrix}$$

$$+ i \begin{pmatrix} (0.8570, 0.1250, 0.0000) & (9.1250, 0.8750, 0.5000) \\ (0.3750, 0.2500, 0.1250) & (3.6250, 2.2500, 0.8750) \end{pmatrix},$$

which admits a nonnegative strong LR complex fuzzy solution of the original fuzzy matrix system.

Example 4.2. Consider another complex LR fuzzy linear matrix equation

$$\begin{pmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \widetilde{x}_{11} & \widetilde{x}_{12} \\ \widetilde{x}_{21} & \widetilde{x}_{22} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (2,2,1)_{LR} & (3,2,1)_{LR} \\ (3,1,1)_{LR} & (2,1,2)_{LR} \\ (1,1,1)_{LR} & (3,2,1)_{LR} \end{pmatrix} + i \begin{pmatrix} (5,1,3)_{LR} & (2,1,2)_{LR} \\ (3,2,1)_{LR} & (3,1,2)_{LR} \\ (2,1,2)_{LR} & (1,1,1)_{LR} \end{pmatrix}.$$

Suppose

$$\widetilde{X} = (M, M^l, M^r) + i(N, N^l, N^r),$$

$$A = A^+ + A^- = \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix},$$

$$B = B^+ + B^- = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

and

$$\widetilde{U} = \left(U, U^l, U^r\right) = \left(\begin{pmatrix} 2 & 3 \\ 3 & 2 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 1 \end{pmatrix} \right),$$

$$\widetilde{V} = \left(V, V^l, V^r\right) = \left(\begin{pmatrix} 5 & 2 \\ 3 & 3 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 1 & 2 \\ 2 & 1 \end{pmatrix} \right).$$

By the Theorem 3.1., the original fuzzy matrix equation is extended into the following a system of linear matrix equations(3.5)

$$\begin{cases} A[MN] \bigotimes B = [UV], \\ A \bigotimes \begin{pmatrix} M^l & M^r \\ N^l & N^r \end{pmatrix} \begin{pmatrix} B^+ & -B^- \\ -B^- & B^+ \end{pmatrix} = \begin{pmatrix} U^l & U^r \\ V^l & V^r \end{pmatrix}, \end{cases}$$

where

$$\widetilde{X} = (M, M^l, M^r) + i(N, N^l, N^r), \widetilde{C} = (U, U^l, U^r) + i(V, V^l, V^r).$$

From the Eqs.(3.13), the solution of the computing model is

$$[MN] = A^\dagger [UV] \bigotimes B^\dagger$$

$$= \begin{pmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}^\dagger \begin{pmatrix} 2 & 3 & 5 & 2 \\ 3 & 2 & 3 & 3 \\ 1 & 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}^\dagger$$

$$= \begin{pmatrix} 0.0179 & -0.0179 & 0.2500 & 0.2500 \\ -0.3393 & 0.3393 & 0.2500 & 0.2500 \end{pmatrix},$$

$$\begin{pmatrix} M^l & M^r \\ N^l & N^r \end{pmatrix} = A^\dagger \bigotimes \begin{pmatrix} U^l & U^r \\ V^l & V^r \end{pmatrix} \begin{pmatrix} B^+ & -B^- \\ -B^- & B^+ \end{pmatrix}^\dagger$$

$$= \begin{pmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}^{\dagger} \begin{pmatrix} 2 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 3 & 2 \\ 2 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}^{\dagger}$$

$$= \begin{pmatrix} 0.4464 & 0.3571 & 0.3571 & 0.4464 \\ 0.0179 & 0.2143 & 0.2143 & 0.0179 \\ 0.5179 & 0.4464 & 0.4464 & 0.5179 \\ -0.0893 & 0.2679 & 0.2679 & -0.0893 \end{pmatrix}.$$

It means

$$\widetilde{M} = \begin{pmatrix} (0.0179, 0.4464, 0.3571)_{LR} & (-0.0179, 0.3571, 0.4464)_{LR} \\ (-0.3393, 0.0179, 0.2143)_{LR} & (0.3393, 0.4464, 0.0179)_{LR} \end{pmatrix}$$

and

$$\widetilde{N} = \begin{pmatrix} (0.2500, 0.5179, 0.4464)_{LR} & (0.2500, 0.4464, 0.5179)_{LR} \\ (0.2500, -0.0893, 0.2679)_{LR} & (0.2500, 0.2679, -0.0893)_{LR} \end{pmatrix}.$$

Since $\widetilde{M}$ is an appropriate LR complex fuzzy matrix, but $\widetilde{N}$ is not an appropriate one, the solution we obtained is

$$\widetilde{X} = (M, M^l, M^r) + i(N, N^l, N^r)$$

$$= \begin{pmatrix} (0.0179, 0.4464, 0.3571)_{LR} & (-0.0179, 0.3571, 0.4464)_{LR} \\ (-0.3393, 0.0179, 0.2143)_{LR} & (0.3393, 0.4464, 0.0179)_{LR} \end{pmatrix}$$

$$+i \begin{pmatrix} (0.2500, 0.5179, 0.4464)_{LR} & (0.2500, 0.4464, 0.5179)_{LR} \\ (0.2500, 0.0000, 0.2679)_{LR} & (0.2500, 0.2679, 0.0000)_{LR} \end{pmatrix},$$

which admits a weak complex LR fuzzy solution of the original fuzzy matrix system by the by Definition 3.5.

5 Conclusion

In this paper, two models are proposed to solve the LR complex fuzzy linear matrix equation $A\widetilde{X}B = \widetilde{C}$, where A and B are crisp matrices for $m \times m$ and $n \times n$, respectively, and $\widetilde{C}$ is an arbitrary matrix of LR fuzzy numbers for $m \times n$.We obtained the complex fuzzy approximate solutions of fuzzy linear matrix equations by solving a crisp linear matrix equation system. In addition, we also discussed the two existence conditions of strongly complex fuzzy solutions. We demonstrated two numerical examples to show effectiveness of the proposed method. Next, we will consider the case where matrices A and B are complex matrices, and apply the algorithm to other types of linear matrix equations. This method is not limited to a specific type of fuzzy matrix equations, it has a wide range of applicability. To some extent, our study enriches the computational theory of fuzzy linear systems.

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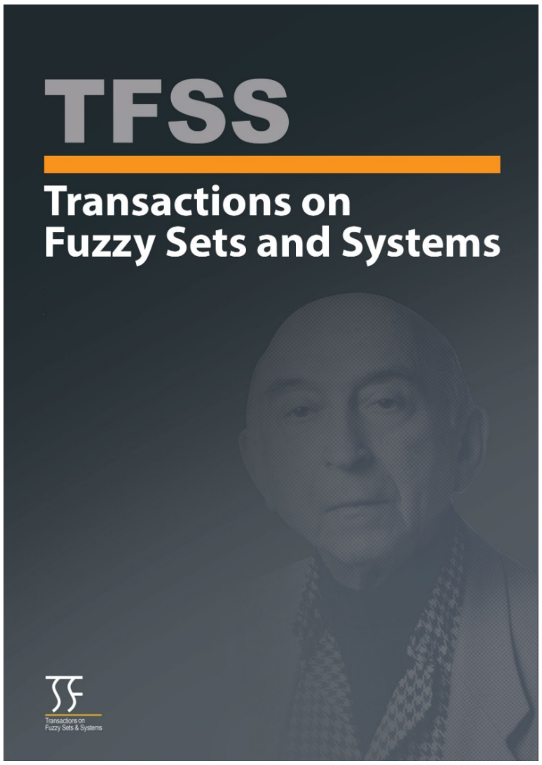

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A Stochastic-Process Methodology for Detecting Anomalies at Runtime in Embedded Systems

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A Stochastic-Process Methodology for Detecting Anomalies at Runtime in Embedded Systems

Alfredo Cuzzocrea*, **Enzo Mumolo**, **Islam Belmerabet**, **Abderraouf Hafsaoui**

Abstract. *Embedded computing systems* are very vulnerable to *anomalies* that can occur during execution of deployed software. Anomalies can be due, for example, to faults, bugs or deadlocks during executions. These anomalies can have very dangerous consequences on the systems controlled by embedded computing devices. Embedded systems are designed to perform autonomously, i.e., without any human intervention, and thus the possibility of debugging an application to manage the anomaly is very difficult, if not impossible. Anomaly detection algorithms are the primary means of being aware of anomalous conditions. In this paper, we describe a novel approach for detecting an anomaly during the execution of one or more applications. The algorithm exploits the differences in the behavior of *memory reference sequences* generated during executions. *Memory reference sequences* are monitored in real-time using the *PIN tracing tool*. The memory reference sequence is divided into randomly-selected blocks and spectrally described with the *Discrete Cosine Transform* (DCT) [1]. Experimental analysis performed on various benchmarks shows very low error rates for the anomalies tested.

AMS Subject Classification 2020: 62H30; 62M02
Keywords and Phrases: Anomaly detection, Embedded systems, Stochastic processes, Inference models.

1 Introduction

Nowadays, *embedded computing systems* are extensively diffused and their *uses* include automotive applications, consumer applications and particular domains such as industrial subsystems or military applications. Embedded systems share some important properties, namely the fact that their failures often result in severe consequences (whose degree of gravity depends on the specific application), and interacting with them is difficult, if not impossible, and the number of concurrent executions is limited and frequently known in advance . Embedded system failures may be caused by software errors (bugs), faults, or the injection of new applications, including those deliberately designed to cause failures (malware), possibly coming from the network to which some embedded systems could be connected [2]. All of these events could result in *runtime anomalies*. The ability to automatically detect these anomalies may prevent failures in embedded systems and, hence, avoid damage to the controlled systems.

Anomalies are events that differ from some standard or reference events. They can be detected explicitly, i.e., through pattern recognition, which aims to classify patterns using a-priori knowledge or statistical information extracted from patterns [3, 4, 5]. *Anomaly detection* is a key application of Machine Learning, focusing on identifying data points that deviate from the norm and understanding why this occurs. Its uses

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are numerous, ranging from noise reduction and data cleaning to security-related tasks such as fault detection, fraud prevention, predictive maintenance, and social security.

Our anomaly detection technique establishes the behavior of the normal executions under examination, compares the observed behavior with the normal behavior, and signals when the observed behavior differs significantly from its normal profile. Since anomaly detection techniques signal all anomalies, false alarms are expected when anomalies are caused by behavioral irregularities. Therefore, this realizes a methodology based on *Stochastic Processes* (e.g., [6]).

Following these considerations, in this paper we propose a technique (and related algorithm) to build a profile of program behavior and to detect deviations from this profile. The profile is based on a *statistical model* of the *memory references* generated during the execution [7]. Our technique is designed to operate, for the detection phase, on embedded devices. Its computational complexity is low, and hence the overhead on the embedded device is limited. In particular, our prototypical implementation on an embedded device currently introduces an overhead lower than 35%. However, it can easily be speeded-up.

Our approach uses the *memory address sequences* generated by the applications during their execution, since these sequences contain a lot of information about the running applications. After an initial time period where the applications perform initialization tasks, we train, for each application, a *Hidden Markov Model* (HMM) of the execution. Then, we compute the likelihood that the sequences observed during the following execution are consistent with the HMM models, and we use this figure to detect the anomalies.

This paper significantly extends our previous conference paper [8], where we have first introduced the preliminary concepts of our research. Here, with respect to the previous paper, we made the following contributions:

- we provide a extended concepts on the methodology we proposed, along with a better organization and linkage with the results introduced in our original work;
- we provide a clearer description about the paper organization;
- we extend related work analysis, as to include other emerging initiatives dealing with the anomaly detection research problem;
- we extend our proposed methodology with several algorithms that clearly describe our main proposal for anomaly detection in embedded systems;
- we extend the experimental evaluation and analysis of our proposed framework, by introducing novel experimental metrics;
- we provide an innovative case study that clearly describes runtime anomaly detection using our methodology, within the context of embedded systems.

The remaining part of this paper is organized as follows. Section 2 provides a summary of relevant work in anomaly detection for embedded systems. In Section 3, we provide the theoretical foundations of our research. Section 4 presents a detailed description of our methodology in the case of offline executions. In Section 4.5, we experimentally demonstrate that the analysis framework performs well in classification tasks. Section 5 focuses on the runtime analysis and anomaly detection in an embedded system, thus leading to our innovative algorithm, including its experimental validation. In Section 6, we introduce an innovative case study on runtime anomaly detection using our methodology, within the context of embedded systems. Finally, Section 7 provides conclusions and possible future work.

2 Related Work

In this Section, we provide an overview on research efforts that are related to our work.

Different aspects of a program execution may be used for describing its behavior and, hence, analyzing it in order to detect anomalies. *A system call trace* is a common type of measure for detecting anomalies [9]. A system call trace is an ordered sequence of system calls that a process performs during its execution. Other systems use measures based on the use of resources [10], such as CPU, memory or I/O. The traces collected during a normal execution are classified with *standard pattern matching tools* such as HMM [11, 12, 13, 14, 15], *Embedded Hidden Markov Model* (EHMM) [16], *Neural Networks* and *Genetic Programming* [17], *Support Vector Machine* (SVM) [18], and *rule-based* classifiers [19]. Anomaly detection, also called *intrusion detection* in networked systems, is a very important problem that has been widely studied in different areas and applications. Markovian techniques are one of the best methods for detecting anomalies in a sequence of discrete symbols [20]. Training a Markov model means fine-tuning the parameters of a probabilistic model of a sequence without anomalies; after training, the likelihood of unknown sequences are computed given the parameters of the trained model.

[3] presents two methods for detecting anomalies in embedded systems, namely Markov and *Sequence Time Delay Embedding* (Stide) . The Markov approach evaluates the probabilities of the transitions between events in a training set and uses these probabilities to see if they correspond to the transitions of the test set. The Stide approach builds templates of normal executions and compares the templates with unknown sequences. Other approaches, such as [21], use Markov Models of system call sequences. In some cases, enhanced models can be obtained with Hidden Markov Models, which are widely used for *sequence modeling*.

In [22], authors report a survey of *HMM-based techniques for intrusion detection*. Despite their power, there are few papers dealing with the use of HMM for anomaly detection in embedded systems . Sugaya et al. describe in [23] an anomaly detection system based on HMM modeling of resource consumption, such as CPU, memory and network. In [24], Zandrahimi et al. propose two methods, a *buffer-based* and a *probabilistic detector*. The buffer-based detector builds a cache formed with events considered as normal. During test stage, the method counts the cache misses. However, the probabilistic detector employs the probability of events to evaluate the testing sequence. The approaches are suitable for embedded systems, as they require a smaller memory size and can be easily implemented in hardware. Some authors, for example [25, 16, 26], consider the discrete sequences as signals and use *signal processing techniques* to analyze them.

[27] presents a significant contribution to the field of anomaly detection by developing a novel methodology for *acquiring reliable performance results* for *frequency-based anomaly detectors*. By identifying and characterizing key aspects of the data environment, such as the frequency distribution of data, the paper constructs a synthetic data environment specifically tailored to assess detector performance comprehensively. Through a systematic series of experiments, this approach effectively maps out the performance landscape of the anomaly detector, by highlighting its strengths and exposing areas of weakness. Furthermore, the study demonstrates the practical applicability and extensibility of the insights gained from synthetic data to real-life scenarios, providing valuable guidance for improving anomaly detection techniques.

In [28], authors introduce an *innovative network transmission model* and localization algorithm designed to detect and rank anomalies using only *coarse-grained information* from *network endpoints*. The research addresses the critical challenge of anomaly detection in distributed systems (e.g. [29, 30, 31]), where the lack of sufficient sensors impedes monitoring and timely detection of traffic flow irregularities across interconnected nodes. By developing a novel metric to accurately rank anomalies, the study surpasses traditional statistical models that rely on *standard deviation measures*. The experimental results demonstrate that the proposed algorithm effectively identifies and ranks anomalies, and aligns well with transportation events reported on social media, thereby improving overall system reliability and performance.

[32] provides a *formal runtime security model* that enhances *anomaly-based malware detection* in network-

connected embedded systems. By defining normal system behavior, including execution sequences and timing, and leveraging on-chip hardware to non-intrusively monitor system execution via the *processor trace port*. This approach addresses significant limitations of existing anomaly-based methods, which often suffer from performance overheads and susceptibility to mimicry attacks. The detection method is evaluated on a *network-connected pacemaker* benchmark, which is prototyped in *FPGA*, and simulated in *SystemC*, which highlights its effectiveness against various impression attacks at different.

In [33], authors advance the field of complex system design and analysis by introducing *wrappings integration infrastructure*, a novel *knowledge-based* approach that enhances *system-level* verification beyond *component-level* analysis. This research demonstrates how the integration infrastructure utilizes *domain-specific knowledge* to effectively manage system resources, detect anomalies, and monitor behavior, thereby improving the reliability and robustness of complex systems. The infrastructure provides a flexible framework for incorporating anomaly detection algorithms originally developed for verification and validation of knowledge-based systems, facilitating both offline and online evaluation studies. This contribution not only bridges the gap between component-level and system-level verification but also empowers system developers with tools for better anomaly detection and system monitoring, which leads to more dependable and efficient system designs.

With respect to classical state-of-the-art proposals, our method has the specific merit of addressing embedded systems, which is very relevant at the moment . With respect to similar approaches that make use of HMM for anomaly detection, our main contribution consists in specifically pointing memory references as the input of our analysis, contrary to others that make use of other parameters like CPU and network flows.

3 Preliminaries

In this Section, we summarize the fundamental concepts used through this paper, namely *spectral description of the virtual memory sequences* and program representation by means of HMM. Here, we consider a spectral representation of memory sequences using *Discrete Cosine Transform* (DCT) as described in [16].

3.1 Spectral Description of Memory References

The *Short-Time Fourier Transform* (STFT) is a *Fourier-related transform* used to determine the *sinusoidal frequency* and phase content of local sections of a signal as it changes over time. It describes how the energy is distributed over a spectral range.

We show hereafter that memory references can be described with spectral parameters. In fact, important parts of a program are composed of loops that become peaks in the spectral domain, as we will point out shortly. Let us consider, for example, a simple cycle of this type:

```
i=0;
while(i<N) {
    i++;
}
```

The virtual memory reference sequence generated during the execution of this loop can be modeled with a *sawtooth signal*, as shown in Figure 1 . Calling $F(\omega)$ the *amplitude spectrum* of a single ramp, the analytic form of the *sawtooth spectrum* is defined as follows:

$$F(\omega) = \sum_n \delta(n - N) \tag{1}$$

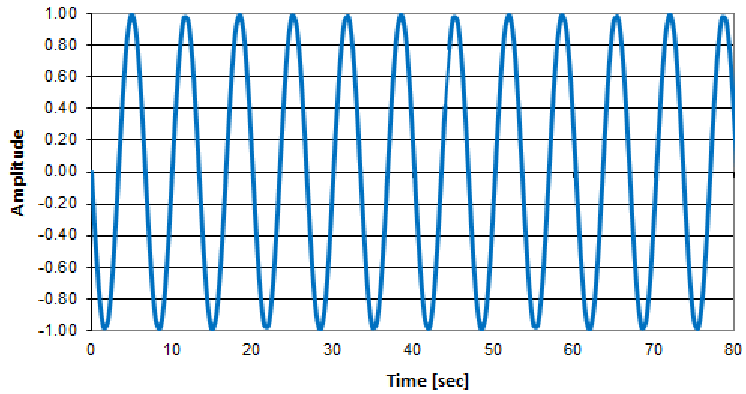

Figure 1: Sawtooth model of a loop

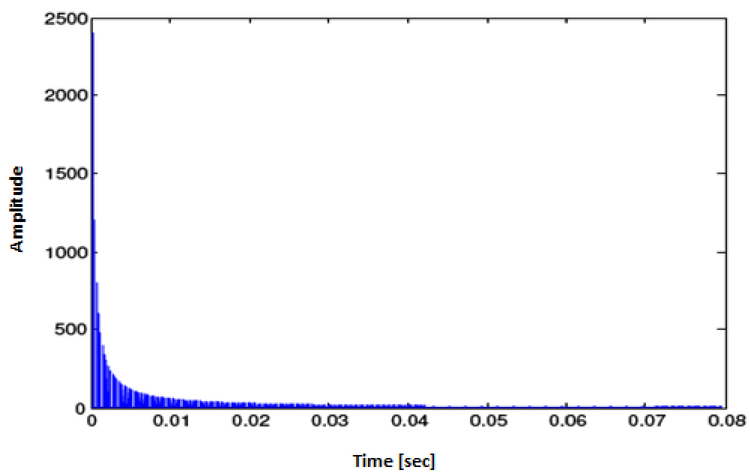

Figure 2: Sawtooth in the amplitude spectrum domain

where N is the number of iterations of the loop. The spectrum is therefore composed by a periodic series of peaks with decreasing amplitudes whose period is related to the loop width N (see Figure 2) .

As a more practical example, let us consider the code fragment reported in Figure 3, which represents a *bubble sort algorithm* . After acquiring the virtual memory sequence and performing its spectral analysis, the STFT described in Equation (2) is applied:

$$X(n) = \sum_{-\infty}^{\infty} x(n)w(n-m)e^{-j\omega n} \tag{2}$$

The sequence of memory addresses is divided into *chunks* or frames, (which usually overlap each other, to reduce artifacts at the boundary). Each chunk is Fourier transformed, and the amplitude spectrum over time is reported in Figure 4. The spectral patterns can be used to characterize the executions. We obtain spectral information with *Fast Discrete Cosine Transform*.

Algorithm 1 `Bubble Sort Algorithm`

Input: Integer vet, Integer N
Output: Integer vet

Begin
  $i \leftarrow 0$;
  $j \leftarrow 0$;
  $temp \leftarrow 0$;
  for $(i = 0; i < (N - 1); i++)$ **do**
    for $(j = i + 1; j > 0; j--)$ **do**
      if $(vet[j] < vet[j - 1])$ **then**
        $temp \leftarrow vet[j]$;
        $vet[j] \leftarrow vet[j - 1]$;
        $vet[j - 1] \leftarrow temp$;
      end if
    end for
  end for
  return vet;
End

Figure 3: Bubble sort algorithm

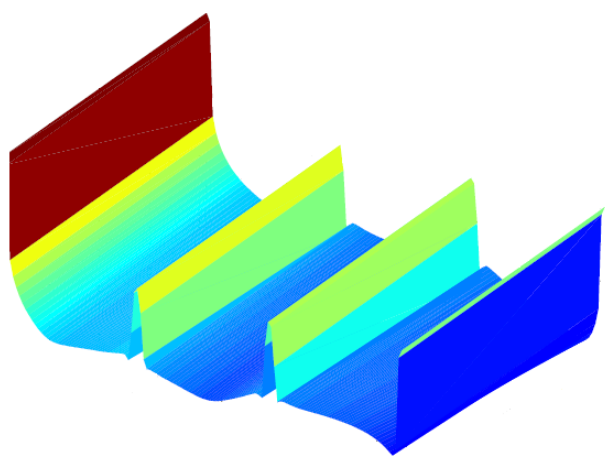

Figure 4: STFT spectrum of the bubble sort routine

3.2 Discrete Cosine Transform Representation

Discrete Cosine Transform is a method to obtain spectral information, and it is used in this work instead of STFT because it is fast and has a good *energy compaction* capability. Energy compaction means the capability of the transform to redistribute *signal energy* into a small number of transform coefficients. It can be characterized by the fraction of the total number of signal transform coefficients that carry a certain (substantial) percentage of the signal energy. The lower this fraction is for a given energy percentage, the better the transform energy compaction capability is.

$$X(n) = \sum_{i=0}^{N-1} x_i \cos\left(\frac{k\pi}{N}\left(i + \frac{1}{2}\right)\right),\ n = 0, \ldots, N-1 \tag{3}$$

The principle advantage of DCT transformation is the removal of redundancy between neighboring addresses. This leads to uncorrelated transform coefficients, which can be processed independently. The effectiveness of a transformation scheme can be directly gauged by its ability to pack input data into as few coefficients as possible. This allows the quantizer to discard coefficients with relatively small amplitudes without introducing visual distortion in the reconstructed image. DCT exhibits excellent energy compaction.

3.3 Hidden Markov Modeling

A standard Hidden Markov model with N states and M possible observation symbols can be denoted as follows:

$$\lambda = (A, B, \pi) \tag{4}$$

such that, A matrix gives the probability of each transition from one state to another, B matrix gives the probability of observing each symbol in each state, and π vector specifies the initial state distribution.

The *Baum-Welch algorithm* [34] is typically used to learn the state transition (i.e., A matrix) and observation symbol probability distributions (i.e., B matrix) of an HMM. The well-known backward and forward procedures can be used to iteratively estimate the model parameters with a space and time complexity of $O(N^2T)$, where T is the length of the sequence of events. The quality of a model can be evaluated using the forward procedure to find the probability that a sequence O was generated by the model λ for all possible paths, namely the likelihood $P(O|\lambda)$.

4 Anomaly Detection Methodology: The Offline Case

In this Section, we describe in details our proposed methodology in the offline case, including its preliminary experimental validation. Later, we introduce our algorithm for supporting the detection at runtime, particularly in embedded systems.

We define the formulation of our problem as follows. Here, we assume that we have a group of N *concurrent processes* running on behalf of a given user. Our objective is to detect anomalies during the execution of the processes due, for example, to a malware attack or a bug in the code. We considered the following two situations.

In the first considered situation, we assume that all but one of the N processes are prone to anomalies because there are well-known services that we know for sure cannot generate any anomalies. In this case, there is only one process that can introduce an anomaly. Our algorithm exploits the effects that the anomaly introduces by measuring the distance of the execution that can contain anomaly with all the other executions.

In the other considered situation, we use the same concept of measuring distances among executions and detect anomalies when all the distances change. The execution that introduces anomalies is found using an *argmax* criterion. The current limit of our approach is that we detect only one anomaly at a time, or, in other words, we detect the execution that introduces the biggest anomaly, in the sense that we will explain shortly.

The methodology applies the following steps:

- employing a `pseudo2D-HMM` model [35] to characterize the workloads of the target system;
- applying HMM to analyze the executions within the target system;

- classifying the target executions by means of the previous HMM analysis.

In the next Sections, we describe all these steps in details.

We used *trace-driven simulations* [36] to test the proposed approach. The traces were a subset of the *SPEC2000* benchmark suite [37]. In particular, we consider the following six workloads: `bzip2` and `gzip`, which are two popular *compression programs*; `eon`, which is a *ray traces program*; `gcc`, `perl` and `vpr`, which are *FPGA place & route programs*. CPU address traces have been obtained by running the applications with different input data. As a result, several executions of each application have been considered. The programs reported above run on an *Intel(R) Core(TM)* $i7-10700$CPU @ 2.90GHz, 16GB of RAM with Windows 10 Pro as operating system . The benchmarks were downloaded from *traces.byu.edu*.

4.1 Pseudo-2D Hidden Markov Models for Workloads Classification

The basic Markov model is the Markov chain, which is represented with a graph composed by a set of N states. This graph describes the fact that the probability of the next event depends on the previous event. The current state is temporally linked to k states in the past via a set of N^k transition probabilities.

In our approach, we use Hidden Markov Models (e.g. [38, 39, 39, 40]) to describe the dynamic behavior of workloads. However, since a program exhibits different behaviors during its execution, different HMMs should be used to model memory references for each behavior. This led us to a `pseudo2D-HMM` structure [35], in which each state in the model, called a *superstate*, represents another HMM. Specifically, `pseudo2D-HMM` is a machine learning approach that requires a learning phase to estimate its parameters. In particular, the goal of the parameter re-estimation is to estimate the parameters of the `pseudo2D-HMM` λ that maximize $P(O|\lambda)$, the probability that the observed sequence O is produced by the model λ. Thus, in the training phase, the memory reference sequence related to a given workload is uniformly divided into segments, on which a DCT is applied. The Discrete Cosine Transform is a well-known signal processing operation with important properties [41]. For example, it is useful in reducing signal redundancy since it places as much energy as possible in as few coefficients as possible (energy compaction). The greatest DCT coefficients are given as input to the `pseudo2D-HMM`.

The `pseudo2D-HMM` is incrementally trained on the segments pertaining to a single memory reference workload type. Each different workload type is modeled using a different `pseudo2D-HMM`.

4.2 HMM Analysis of Executions

In this Section, we state the validity of the analysis methodology we use in the anomaly detection algorithm. Given N executions running in our system, we have N HMM models of the form $\lambda = (A, B, \pi)$, which we call λ_1, λ_2, ..., λ_N. Together with the HMM models, we also have N observations, O_1, O_2, ..., O_N, which are the sequences of symbols estimated from the memory reference sequence with *vector quantization* and from which we estimate the HMM models. The final goal of the proposed methodology is the computation of likelihood matrices computed with these models and observations.

Our basic assumption is that with a HMM model, we capture a high-level description of each observation. As a result, we perform the HMM training with blocks of memory references randomly extracted from an initial part of the address sequence. For the same reason, HMM testing, i.e., the computation of $P(O|\lambda)$, is performed by randomly selecting blocks of data and averaging the results. In the following, we describe how we divide the sequence of memory references.

The data used for the HMM analysis is taken as follows: the sequence of memory references is divided into 1024 address blocks ; on these blocks, a spectral vector is computed with Discrete Cosine Transform and from each vector, the first sixteen coefficients are extracted. The following step is to vector quantize the 16-dimension vectors into 64 centroids; the final result is that each address block is represented by a discrete symbol from 0 to 63. The symbols are used to train a discrete Hidden Markov Model. It is worth noting that the data for HMM training and testing is different. As regards the HMM training, we consider nine thousand blocks randomly chosen within the first 20000 blocks (each block is of 1024 addresses), and we use the Baum-Welch algorithm. In conclusion, we use 20 million addresses for each execution to train a HMM. It is worth noting that the first million addresses are omitted from the training procedure because they are generally related to an initialization phase.

As regards HMM testing, namely the procedure to obtain the $P(O|\lambda)$ likelihoods, we use the forward algorithm. The data used for testing is chosen into subsequent 10^9 virtual addresses in the following way: each test is performed on 100 blocks chosen randomly into the section of 10^9 addresses. The final value of likelihood is computed by averaging all the computed likelihoods.

Finally, the sequence of memory references is divided into sections of one billion addresses, which we call *EPOCHS*. For each epoch, we compute the likelihoods by averaging the values obtained on 100 blocks of memory reference, each block is composed of 1024 addresses, chosen randomly. The analysis algorithm does not work continuously: as said before, during each epoch, the algorithm acquires addresses, computes DCT coefficients and vector quantization, and computes likelihoods by randomly sampling portions of data.

While the models are obtained from the first 20 million address references, a block of about three million references is extracted every million from the sequence of address references. On these blocks, the likelihoods $P(O|\lambda)$ are computed.

In this way, during the execution of the programs, a series of likelihood matrices , one for each epoch, are computed as follows:

$$\begin{vmatrix} P(O_1^1|\lambda_1) & P(O_2^1|\lambda_1) & \dots & P(O_N^1|\lambda_1) \\ P(O_1^1|\lambda_2) & P(O_2^1|\lambda_2) & \dots & P(O_N^1|\lambda_2) \\ \dots & & & \\ P(O_1^1|\lambda_N) & P(O_2^1|\lambda_N) & \dots & P(O_N^1|\lambda_N) \end{vmatrix}$$

$$\dots$$

$$\begin{vmatrix} P(O_1^N|\lambda_1) & P(O_2^N|\lambda_1) & \dots & P(O_N^N|\lambda_1) \\ P(O_1^N|\lambda_2) & P(O_2^N|\lambda_2) & \dots & P(O_N^N|\lambda_2) \\ \dots & & & \\ P(O_1^N|\lambda_N) & P(O_2^N|\lambda_N) & \dots & P(O_N^N|\lambda_N) \end{vmatrix}$$

where $P(O_i^k|\lambda_j)$ is the likelihood that the j-th model generates the j-th observation, at the k-th time epoch.

Our monitoring algorithm is based upon the differences between HMM models. Usually, the distance is a measure of how well model λ_1 matches observations generated by model λ_2, relative to how well model λ_2 matches observations generated by itself. In other words, this distance is computed by means of Equation (5) :

$$d(\lambda_1, \lambda_2) = |P(O_2|\lambda_1) - P(O_2|\lambda_2)| \tag{5}$$

This definition, however, does not take into account how the other observations behave with reference to λ_1 and λ_2. We therefore consider the two vectors $\Lambda_1 = |P(O_1|\lambda_1), P(O_2|\lambda_1), \dots, P(O_N|\lambda_1)|$ and $\Lambda_2 = |P(O_1|\lambda_2), P(O_2|\lambda_2), \dots, P(O_N|\lambda_2)|$ to better represent the models λ_1 and λ_2. We extend the distance measure between models reported in Equation (5) in the following way:

$$d(\lambda_1, \lambda_2) = \sum_{i=1}^{N} |P(O_i|\lambda_1) - P(O_i|\lambda_2)| \tag{6}$$

The distance measure reported in Equation (6) allows us to better separate the different benchmarks than the distance reported in Equation (5).

4.3 Tools Developed for Offline Analysis: *Valgrind* Plugin

Before developing the detection algorithm, we studied the validity of the described analysis framework by performing several experiments, as we will report shortly. It was impossible to perform this study using stored address traces, because of the extremely large amount of data required to perform the experiments, and the high processing time due to the reading operations. It is worth noting that an experiment may require analyzing several hundred of GBs. For this reason, we developed a tool based on *Valgrind*, which we called *Tracehmm*, to perform all the described processing online.

Valgrind tool [42] is a tracing framework designed to give the possibility to perform dynamic analysis of software, i.e., analysis performed during the code execution. *Valgrind* is distributed with several tools designed to perform common analysis of memory, threading, etc. *Valgrind* is available under the GNU GPLv2 license; it is possible to modify the available tools and to modify also the code of the framework itself, i.e. Coregrind. Coregrind [43] has been designed to analyze already-developed execution code. When the name of the executable file is given as input to *Valgrind*, the code itself is loaded in memory together with the related libraries. The instructions are translated into instructions of a RISC-like language, called VEX IR, and then executed on a virtual CPU.

The *Tracehmm* tool performs the DCT analysis and the HMM training directly using the memory addresses generated during execution. The direct porting of the off-line analysis tools cannot be performed because Coregrind cannot use any library. For this reason, we rewrite all the writing/reading, memory allocation and memory copy functions of the standard library. Moreover, we rewrite also some necessary mathematical functions, namely $cos()$, $sqrt()$ and $log()$. The tool is called from the command line as follows:

```
valgrind --tool=tracehmm [opt] prog & args
```

The *Tracehmm* tool's structure is reported in Figure 5. With this tool, it is possible to perform the training of a new HMM model or the re-estimation of an already computed HMM model. It also performs the *Viterbi test* on a previously trained HMM model for computing the likelihood that the model λ may generate the execution sequence O.

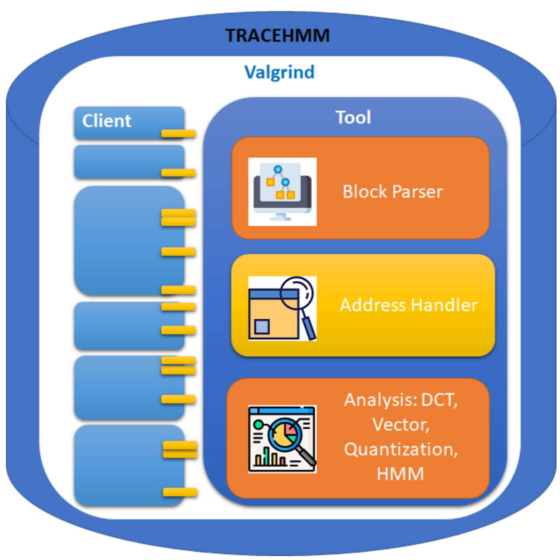

Figure 5: Structure of the *Tracehmm* tool

4.4 Classification of Executions

Before extensively using *Tracehmm*, we perform a limited classification experiment with only four SPEC benchmarks to explore the performance of Neural Networks using the following standard configuration: *feed-forward topology* , weights computation with the *back-propagation algorithm*, six hundred inputs and one output, and 256 elements in the hidden layer. Then we used HMM for the same classification task. We first stored the address memory sequences for the four benchmarks (`bzip`, `gcc`, `go`, and `perl`), we compute the DCT coefficients and vector quantization, and perform classification with Neural Networks and HMM. The classification results are reported in Figure 6 and Figure 7.

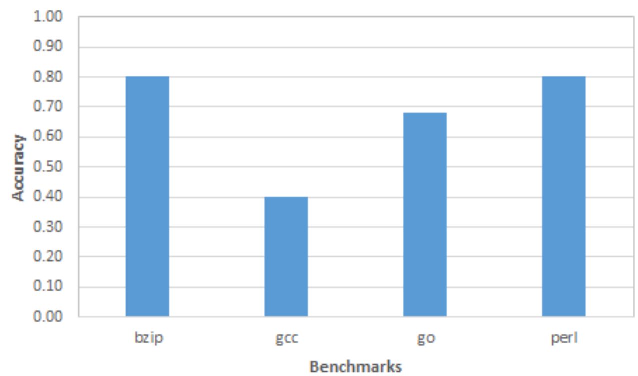

Figure 6: Classification results with neural network model

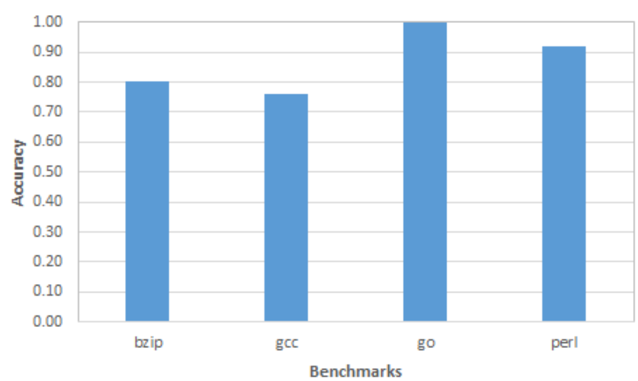

Figure 7: Classification results with HMM

From this comparison, it is clear the superiority of HMM. Then, we performed more extensive classification experiments with the *Valgrind* tool *Tracehmm*. Classification results are reported in Figure 8 for all the SPEC benchmarks: `h264`, `gcc`, `perl`, `bzip`, `go`, `mcf`, `hmmer`, `sjeng`, `quantum`, `omnet`, `astar`, and `xalanc`.

Moreover, in order to evaluate the classification results for different `pseudo2D-HMM` orders, an extensive test has been performed for different number of superstates and states. The results have been averaged for all the workloads. In Figure 9, a graphical representation of the mean classification of all the sequences over the number of states and superstates is depicted. From this data, it comes out that the best `pseudo2D-HMM` orders are six superstates and nine states. As Figure 9 represents average values, we have considered the results for each workload, i.e., averaging the results for all the traces belonging to a given workload type.

The good results reported in the previous Section say that the executions are well discriminated in the

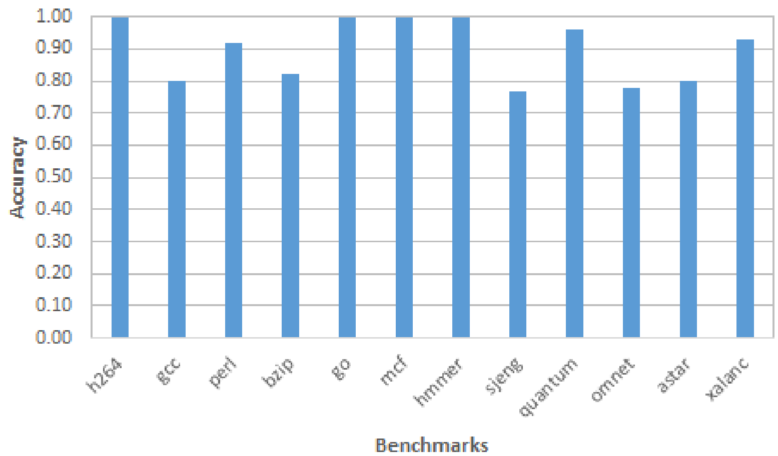

Figure 8: HMM classification result for all the twelve SPEC benchmarks

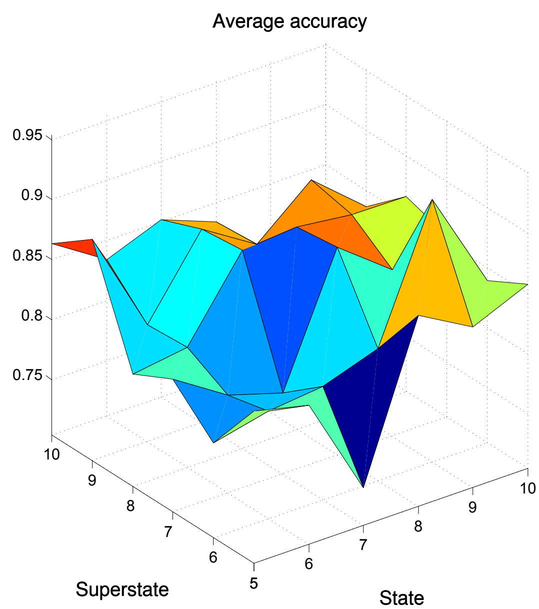

Figure 9: Average recognition rate for all the memory reference sequences over the number of states and superstates of the `pseudo2D-HMM` models

HMM likelihoods space. For this reason, we perform the following experiment: for each epoch, compute the distance matrix between models where the elements are computed according to Equation (6), and find a $3D$ distribution of points whose inter-point Euclidean distance closely resembles this distance matrix. This operation, called *Multidimensional Scaling*, may be used to represent in graphical form a distance matrix. By repeating this operation among several epochs, we obtain the $3D$ graph reported in Figure 10, which shows a good separation among the benchmarks. The line connected to each benchmark represents the dynamic over the first five epochs.

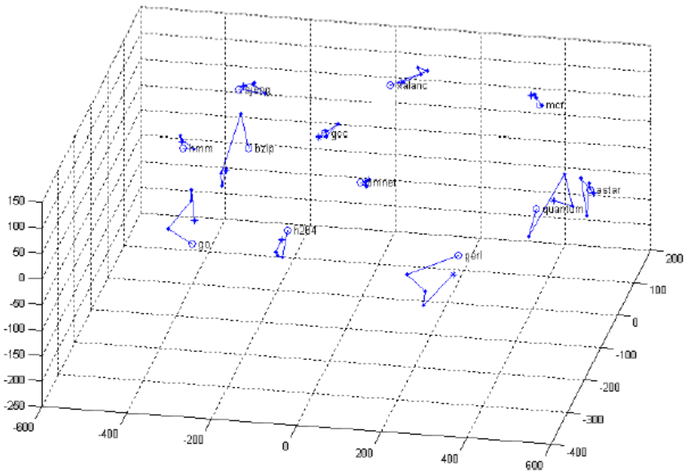

Figure 10: 3D visualization of the distance matrix between models of all the twelve SPEC benchmarks

4.5 Validation of the HMM-based Machine Learning Approach

Before performing experimental evaluations of the detection tool [44, 45], we performed extensive classification experiments to verify the described analysis framework [46]. We assume that the HMM models of the applications and the computation of the likelihood matrices are able to well describe the executions. To explore the validity of this assumption, we classify the executions.

Namely, the HMM models computed from each observation are used to see if the observations with the highest likelihood correspond to the related model. To perform this classification, we use the programs contained in the suite CINT2006 of the SpecCPU2006 benchmark. The suite is composed by twelve programs with different inputs. Of course, a program with a different input generates different observations.

In other words, if the model λ is computed from the observation O, can λ be used to verify that $P(O|\lambda)$ is the maximum for the observation used to compute that model? Of course, numerous observations are generated from the same application.

Classification results are reported in Figure 11 for all the SPEC benchmarks: `h264`, `gcc`, `perl`, `bzip`, `go`, `mcf`, `hmmer`, `sjeng`, `quantum`, `omnet`, `astar`, `xalanc`.

It was impossible to perform this analysis using stored address traces, because of the extremely large amount of data required to perform the experiments, and the high processing time due to the reading operations. In fact, a single classification experiment may require analyzing several billion bytes. For this reason, we developed a *Valgrind* tool, that we called *Tracehmm*, to perform all the described processing online .

5 Runtime Anomalies Detection Algorithm in Embedded Systems

In this Section, we demonstrate how our proposed methodology can be effectively and efficiently used in the context of runtime anomaly detection in embedded systems. Particularly, we provide an algorithm that makes use of the proposed theoretical framework, which has been described in the offline case by Section 4.

We use the programs contained in the suite *CINT2006* of the *SpecCPU2006* benchmark. The suite is composed by twelve programs with different inputs. CPU2006 [47] is *SPEC's CPU-intensive benchmark suite*, stressing a system's processor, memory subsystem and compiler. SPEC designed CPU2006 to provide a comparative measure of compute-intensive performance across the widest practical range of hardware using

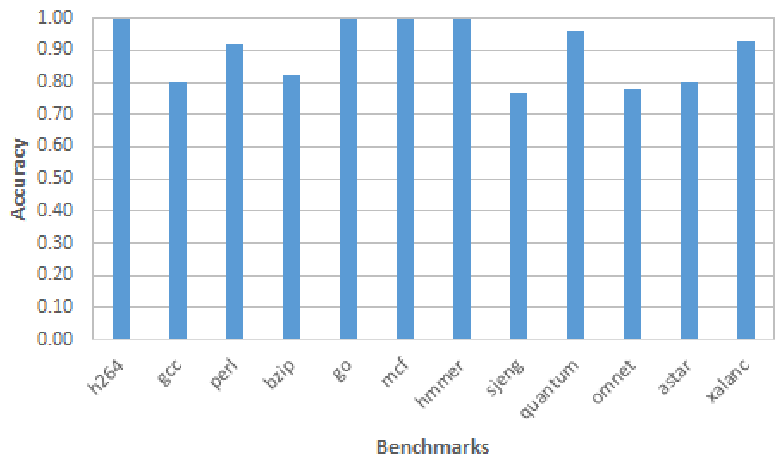

Figure 11: Average classification rate for all the considered workloads

workloads developed from real user applications [48, 49]. All the benchmarks are provided as source code. The programs included in the benchmark suite are the following:

`401.bzip2:` It is derived from the *bzip2 compressor*. The inputs are different files to compress.

`445.gobmk:` It is a version of the *Gnu GO* game. The input data is related to different positions in the board. The output is the following movement in the board.

`458.sjeng:` It is a modified version of the chess game Sjeng 11.2. The inputs is composed by different board positions and different game depths.

`403.gcc:` The program is based on the version 3.2 of GNU Gcc, and it is configured to generate assembler code for the processor $X86-64$.

`h264ref:` This program is the *Karsten Sfuhring* implementation of the H.264 compression standard. The program implements only the compression, but not the decompression.

`400.perlbench:` It is the Perl version 5.8.7.

`429.mcf:` It is derived from a scheduling software used for public transportation. The software has been modified to reduce the number of cache miss and therefore to increment the impact on the CPU than the memory. The inputs are different paths to optimize.

`462.libquantum:` It is a simulator of a quantum computer. It executes the *Peter Shor* algorithm. The inputs are different numbers and the algorithms find the numbers of their factors.

`456.hmmer:` This program analyzes *DNA sequences* using HMM. Its inputs are different reference models and it perform the searching of the correct sequence.

`471.omnetpp:` It simulates an Internet network using the *OMNeT++* simulation system. The inputs are the networks and hosts configurations.

`473.astar:` It implements an algorithm for finding the minimum cost path.

`483.xalancbmk`: It derives from *Xalan-C++*, which is a system that processes documents XML and transform them using style sheets.

Each program with a different input generates a different process. We run each process, and a sequence of virtual memory addresses is generated. So, for the program *bzip2* of the benchmark suite, for example, a different memory reference sequence is generated for each input. With each sequence of memory references, an HMM model is estimated.

Our work is based on the assumption that by iteratively re-estimating the HMM parameters with each sequence, we obtain an HMM model that describes the program itself, not the execution sequence. With the trained HMM's and other memory reference sequences, we derive the likelihood that the sequences are generated by the model. It is worth noting that the data for HMM training and testing is different.

Furthermore, the runtime anomaly detection algorithm based on the proposed execution model is presented. It has been tested on the memory references captured from an ARM9 Linux based embedded device. Using the PIN tracing tool [50], we have developed a runtime monitoring tool that has the structure described in Figure 12.

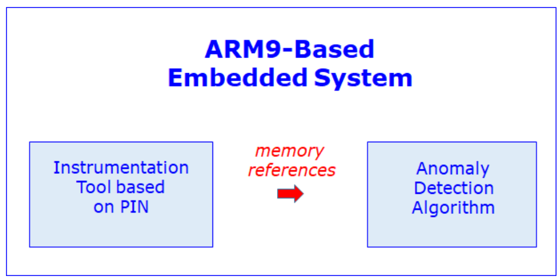

Figure 12: Runtime anomalies detection diagram

5.1 Parametrization of Memory Reference Sequences

The initial part of the executions is normally devoted to initialization tasks and is very different from the *steady-state phase* of the programs. For this reason, we simply blindly fast forward for 1 billion instructions before starting data analysis.

After that, each sequence of memory address references is divided into 1024 address blocks; on these blocks, a spectral vector is computed with Discrete Cosine Transform, and from each vector, the first sixteen coefficients are extracted. The following step is to perform vector quantization with 64 centroids [51] to reduce the 16-dimension vectors into 1 symbol; the final result is that each block of 1024 addresses is represented by a discrete symbol from 0 to 63. The sequences of memory addresses are then transformed into sequences of symbols, which are called *Observations*.

Given N programs running in our system, we have thus N observations, O_1, O_2, ..., O_N, which are the sequences of symbols estimated from the memory reference sequences with DCT analysis and vector quantization.

The N observations are used to train a Hidden Markov Model of each application, called λ_1, λ_2, ..., λ_N in the following. The training of the HMM models is performed as follows. For each observation, nine thousand blocks are randomly chosen within the first 20000 symbols (recall that each symbol corresponds to a block, which is composed of 1024 addresses) and the HMM parameters are computed with the Baum-Welch algorithm. Thus, we use 20 million addresses of each execution to train a HMM. It is worth recalling that the first billion of addresses is omitted from the training procedure because it is generally related to an initialization phase.

After HMM modeling, the observations are used to compute, with the forward-back-word algorithm [52], the $P(O|\lambda)$ likelihoods. The data used for computing the likelihoods is chosen into the subsequent billion memory addresses in the following way: twenty sub-sequences of 100 symbols are chosen randomly within the sequence corresponding to the billion addresses. The final value of likelihood is obtained by averaging all the computed likelihoods. It is worth recalling that the observations used to compute the $P(O|\lambda)$ are formed by sequences of 100 symbols.

In conclusion, the sequence of memory references is divided in sections of one billion addresses that we call *epochs*. For each epoch we compute the likelihoods by averaging the values obtained on twenty sets of 100 blocks of memory reference, each block is of 1024 addresses, chosen randomly. The analysis algorithm does not work continuously: as said before, during each epoch the algorithm acquires addresses, computes DCT coefficients and vector quantization and computes likelihoods by randomly sampling portions of data. The initial memory references are thrown away to avoid the initial transient. After that, the N observations are used to train a Hidden Markov Model of each observation, called λ_1, λ_2, ..., λ_N in the following.

A standard HMM with N states and M possible observation symbols can be denoted $\lambda = (A, B, \pi)$. The A matrix gives the probability of each transition from one state to another, the B matrix gives the probability of observing each symbol in each state, and the π vector specifies the initial state distribution. The Baum-Welch algorithm [34] is typically used to learn the state transition (A matrix) and observation symbol probability distributions (B matrix) of an HMM. The well known forward-backward procedures is used to evaluate the quality of models λ_i, with a space and time complexity of $O(N^2T)$, where T is the length of the input sequence. The quality of a model is the probability that the observation sequence O is generated by the model λ for all possible paths, and is referred to the likelihood $P(O|\lambda)$.

The ultimate goal of the proposed methodology is the computation of likelihood matrices computed with these models and observations. Our basic assumption is that with a HMM model we capture a high-level description of each observation. For this reason we perform the HMM training with blocks of 1024 memory references randomly extracted from an initial part of the address sequence. For the same reason, the HMM testing, i.e. the computation of $P(O|\lambda)$, is performed by randomly selecting blocks of data and averaging the results. While the models are obtained from the first 20 million address references, a block of about three million references is extracted every million from the sequence of address references. On these blocks the likelihoods $P(O|\lambda)$ are computed.

5.2 Anomaly Detection Algorithm

The process running on the embedded device is instrumented and the memory references are collected. The time sequence of memory references is divided in epochs used to train the corresponding Hidden Markov Model of the execution. After the training, each new epoch of the time sequence is used to compute the new matrix $|P(O_i|\lambda_j)|$, as described in Section 4.2. From this matrix, the symmetric matrix M^k of the distances between the models is obtained:

$$\begin{array}{c|cccc} & \lambda_1 & \lambda_2 & \lambda_3 & . \\ \hline \lambda_1 & 0 & m_{1,2} & m_{1,3} & . \\ \lambda_2 & m_{2,1} & 0 & m_{2,3} & . \\ \lambda_3 & m_{3,1} & m_{3,2} & 0 & . \\ . & . & . & . & 0 \end{array} \tag{7}$$

where $m_{i,j}$ is the Euclidean norm of the difference between the row i and the row j of the $|P(O_i|\lambda_j)|$ matrix.

At epoch k, the difference matrix D^k is computed as follows:

$$D^k = \sqrt{|M^k - M^{k-1}|} \tag{8}$$

where k is the current epoch and $k-1$ is the previous one.

Then, the ev^k vector containing the eigenvalues of D^k is computed:

$$ev^k = eigenvalues(D^k). \tag{9}$$

Finally, to detect whether an anomaly is occurred in the current epoch, the *Nuclear Norm*, NN of the eigenvalue vector ev^k is computed as follows :

$$\text{NN} = \sum_{i=1}^{N} \left| ev_i^k \right|. \tag{10}$$

It is worth recalling that NN is an excellent energy index [53]. The percentage variation of the nuclear norm is compared to a given threshold THR in order to detect if in the current epoch an anomaly is occurring.

The meaning of the described algorithm is that when an observation varies due to an anomaly, the distance between the corresponding likelihood among different epochs accordingly changes. Moreover, if an anomaly is detected, we can determine which model has produced the anomaly, i.e. which execution has changed its behavior. The anomaly detection algorithm is summarized Figure 13.

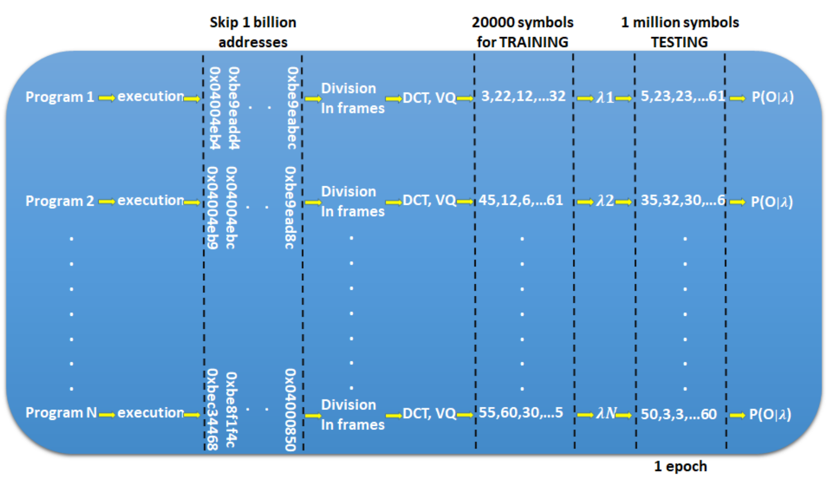

Figure 13: Schematic view of the anomalies detection algorithm

On the other hand, by focusing on more algorithmic details, Figure 14 show the pseudo code of the proposed anomaly detection algorithm presented in Figure 13.

5.3 Runtime Implementation using the PIN Tool

Tools capable of extracting and processing memory traces from running processes have been developed to monitor systems in real-time. We chose Intel Pin as tracing framework, as it is freely available on both x86, x86_64 and ARM architectures under a Linux environment. Using the API of PIN [50], we developed a tool that attach to a running process, track memory accesses until requested then detach and let the monitored application continue its execution unharmed. Provided a rule set to detect processes to monitor (e.g. processes listening on a specific port, processes running as a specific user), standard system tools can be used to find matching *Process IDs* (PID). PIDs will be used by training and tracking PIN tools to attach to each process and produce models or test workload resemblance as requested [54].

Algorithm 3 `Anomalies Detection Algorithm`

Input: Memory References M_r, Integer THR
Output: Integer $Process_{ID}$

```
Begin
  ANOMALY ← FALSE;
  Process_ID ← 0;
  epoch ← divideMemory(M_r);
  TS ← trainHMM(epoch);
  while (TRUE) do
    P ← computeLikelihoodMatrix(TS);
    M ← computeDistanceMatrix(P);
    D ← computeDifferenceMatrix(M);
    NN ← computeNuclearNorm(D);
    if (NN > THR) then
      ANOMALY ← TRUE;
      Process_ID ← findAnomalyExecution();
      return Process_ID;
    end if
  end while
End
```

Figure 14: Pseudo code for the anomalies detection algorithm

A new group of applications that have been chosen to be monitored requires an (automated) step of tuning, where training and testing parameters are optimized to suite the group of applications chosen [55, 56]. Once parameters are defined, a PIN tool is used to generate Markov model and centroids files of each application. Multiple input sources are used to train the model in order to obtain a good generalization of the application model. Another PIN tool provides testing capabilities: it requires the PID of the process to track and the Markov model of the application it claims to belong to. A logarithmic affinity probability is then produced.

A tracking system daemon is devoted to run the testing tool at regular time intervals: the tool attaches to target process, dumps memory references and then detaches, so target process is able to keep running without any more overhead. Memory addresses are then processed and output is used to classify the tracked process as belonging to the claimed application or not. The main tasks to be performed can be summarized in the following steps:

1. find the process to track;
2. use PIN to extract memory references of the process;
3. train the DCT-HMM description of the process;
4. compute the matrix $P(O_i|\lambda_j)$ and check the difference of error matrix to detect anomalies.

In Figure 15, a detailed pseudo code of the real-time monitoring tool is reported.

5.4 Experimental Results

The tool based on PIN described above has been tested in three different experiments, as we will show shortly. In all the tests of this Section , an embedded device equipped with an ARM9 at 500 MHz, 128 Mb RAM and

Algorithm 2 Anomalies Detection Tools

Input: Integer THR, Integer N, Integer $EPOCHE\_SIZE$

Output: Boolean $Anomaly$

```
Begin
  P_ID ← findProcess();
  attachPIN(P_ID);

  PIN TOOL:
  COUNT ← 0;
  while (COUNT < EPOCHE_SIZE) do
     traceMemoryAccesses(N);
     writeToBuffer();
     COUNT++;
     skipMemoryAccess();
  end while

  TRACKING DAEMON:
  while (TRUE) do
     waiteForBufferToFill(N);
     COUNT++;
     spawnThread(analyzeTrace());
     if (COUNT < EPOCH_SIZE) then
        continue;
     else
        sleepUntilNextFill();
     end if
  end while

  EPOCH DAEMON:
  EveryXminutes :
  if (newEpochExists()) then
     for all (epoch in previousEpochs) do
        Δ_i ← computeMatrix(epoch);
        if (Δ_i > THR) then
           detectAnomaly();
           Anomaly ← TRUE;
           return Anomaly;
        end if
     end for
  end if
End
```

Figure 15: Pseudo code for the anomalies detection tools

a Debian Linux has been used. An important issue related to any monitoring system applyied to embedded systems is overhead. Embedded systems like the ARM based computers, in fact, have low computational power and limited power supplies, thus constraints in terms of computational and power requirements are particularly important. Our prototype implemented PIN with processor instructions, therefore only computational constraints have been experimentally evaluated. Computational constraints have been evaluated in terms of the slowdown that the embedded system suffers due to the instrumentation of the code to extract the memory references, since most of the computational load can be assigned to a different system. Our tool each time a memory address is accessed by the instrumented application executes a callback function in order to save the memory reference, hence introducing an unavoidable overhead to the normal execution of the application. This is done for the strictly necessary number of references to provide good values of accuracy, then tool detaches from the application, which can continue its execution normally. Those memory references are shared with a tracking process, typically on a different machine on the same network, which will begin the analysis of the trace avoiding the tracing tool to slow down the application even more. In our prototype the tracing caused an about 33% slow down. However, using other instrumentation approaches the slowdown can be greatly reduced.

5.4.1 Experiment 1 - Malware Detection

In this first experimental campaign, the PIN tool has been attached to two different processes, and the anomaly detector changes artificially its input at a given epoch. This test aims at simulating a malware affecting a process that suddenly change its behavior becoming a different process. For this test 8 different models of *SPEC CPU2006* benchmark have been used, namely `sjeng`, `omnet`, `astar`, `h264`, `xalanc`, `mcf`, `perl`, `quantum` , changing this 8 execution suddenly into `gcc`, `hmm`, `bzip_0`, `bzip_1`, `bzip_2`, `bzip_3`, `go_0`, `go_1`, `go_2`, `go_3`, where for bzip and go different execution have been considered. Thus, 80 different anomalies have been tested. For example, in this test, at a given epoch, the execution of `astar` suddenly becomes `gcc` (or `bzip_0`, etc.). This behavioral change can be detected by the proposed algorithm.

In Figure 16, the corresponding False Positive versus False Negative behavior for anomaly epoch determination has been obtained.

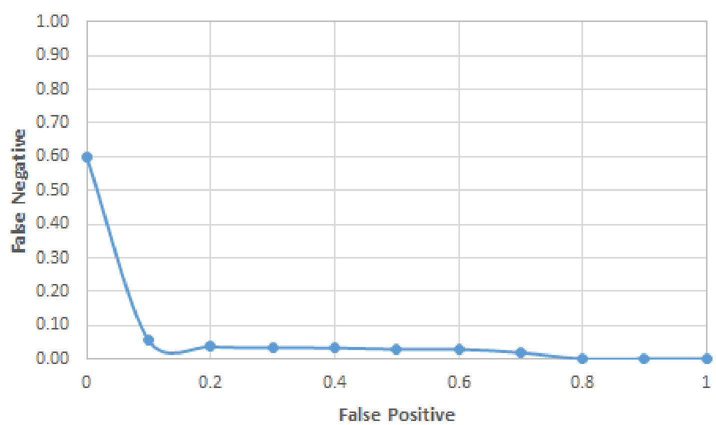

Figure 16: Performances of the detection algorithm for change behavior anomalies

The experimental results show that the algorithm can determine in an accurate way the epoch that has produced the error, with an equal error rate below 0.8%.

5.4.2 Experiment 2 - Loop Bug Detection

In this experiment, an infinite loop substitutes the normal execution of the benchmark at a given epoch. This experiment shows if the proposed algorithm is capable to detect anomalies in programs that remain blocked in loops.

Table 1: Loop bug detection results

SNR	Correct Detections
0 dB	all
5 dB	all
10 dB	all
15 dB	all
20 dB	all
25 dB	all
30 dB	all
35 dB	all
40 dB	none
45 dB	none
50 dB	none

This test has been conducted using the following benchmarks: `omnet`, `astar`, `h264`, `xalanc`, `mcf`, `perl`, `gcc`, `bzip`. In all the 8 tested cases, the epoch in which the anomaly has occurred has been always correctly determined. In this test, the execution that introduces the anomaly has also been detected with an accuracy of 89.5%.

5.4.3 Experiment 3 - Random Error in Memory References Detection

In this third experiment, the memory trace gathered using PIN has been modified by adding a *white gaussian noise*. This experiment shows what energy differences in memory reference are detected as anomaly by the proposed program.

The results have been conducted on the same 8 benchmark of Experiment 2, namely: `omnet`, `astar`, `h264`, `xalanc`, `mcf`, `perl`, `gcc`, `bzip` , and the noise has been added to one benchmark at a time, resulting in 8 tests for each value of SNR, using a threshold $THR = 100$.

This experiment shows that the proposed method is capable of detecting random modifications of the memory references when this behavior is evident, starting from 35 dB, and it becomes easier and easier as the SNR is decreasing.

6 Case Study: Runtime Anomaly Detection in Embedded Systems

In this Section, we introduce an innovative case study on runtime anomaly detection using our methodology, within the context of embedded systems.

Embedded devices have seen a remarkable surge in popularity recently, paralleling a significant increase in the variety and sophistication of anomalies that can affect their performance and security. Consequently, both industry and academia have focused extensively on addressing the unique challenges of anomaly detection in embedded systems.

Anomaly detection is crucial for these systems due to their widespread use in critical applications where reliability is paramount. To tackle this, advanced behavior-based anomaly detection systems have been developed, leveraging machine learning algorithms to identify and analyze deviations in data such as sensor readings, communication patterns, and power consumption. Hence, we provide in this Section a detailed case study demonstrating how our proposed approach can be employed for anomaly detection in embedded systems and IoT devices. By examining real-world scenarios, we illustrate the effectiveness of our method in identifying and mitigating anomalies, thereby enhancing the reliability and security of these systems. This case study showcases how our approach can detect irregularities in various parameters, such as sensor data, communication protocols, and energy consumption, ensuring the robust operation of embedded and IoT devices.

In recent years, the significance of anomaly detection in embedded systems and IoT devices has escalated due to the increasing integration of these technologies in critical applications. Statistical data from various reports highlight the scale and impact of anomaly detection efforts.

For instance, a comprehensive survey on anomaly detection strategies for *cyber-physical systems* (CPS) indicates that advanced methods such as edge and edge-Cloud computing are becoming pivotal. These methods are designed to handle large volumes of high-dimensional data generated by IoT devices, with one study reporting a significant reduction in detection latency and an improvement in overall accuracy through the use of these techniques [57, 58].

Furthermore, the adoption of machine learning models in anomaly detection has shown promising results. A study on *industrial IoT* (IIoT) networks revealed that using a hybrid machine learning approach can enhance the detection rate of anomalies while maintaining low false-positive rates. This is particularly important in environments where real-time response is critical [57]. Additionally, [58] focused on *distributed online one-class support vector machines* (SVM) demonstrated that such approaches are effective for anomaly detection in networked embedded systems. This method allows for continuous learning and adaptation, crucial for maintaining security and operational efficiency in dynamic environments.

Monitoring data in embedded systems is crucial for assessing system health, identifying workload patterns, and defining metric spaces, which are subsequently used to detect anomalies. To ensure the efficacy of anomaly detection, various fault scenarios can be simulated and analyzed, including:

- **Sensitive Sensor Interactions**. Embedded systems often interact with sensitive sensors and actuators. Anomalies can be detected by monitoring unusual patterns in these interactions, which may indicate tampering or misuse. For instance, unauthorized access to a temperature sensor in an industrial control system could signal an attempt to disrupt normal operations.

- **CPU-Intensive Loops**. Faulty or malicious code can introduce infinite loops or heavy computational tasks that exhaust CPU resources, leading to performance degradation or system crashes. By analyzing CPU usage patterns and detecting abnormal spikes, these issues can be identified and mitigated. For example, an embedded system might exhibit abnormal CPU usage due to a spin lock fault, causing a significant slowdown.

- **Memory Leaks**. Memory leaks occur when a system continuously allocates memory without releasing it, eventually exhausting available memory and causing crashes. In embedded systems, this can be particularly problematic due to limited memory resources. Anomaly detection algorithms can monitor memory usage over time to identify and address leaks before they cause significant harm. For instance, a gradual increase in memory usage without a corresponding release could indicate a memory leak.

- **Disk I/O Errors**. Disk I/O operations in embedded systems often follow predictable patterns. Anomalies in these patterns, such as unexpected spikes in read/write operations, can indicate hardware failures or malicious activity. By continuously monitoring disk I/O performance, potential issues can be detected early. For example, a sudden increase in disk access could be a sign of a disk I/O error caused by malware.

- **Network Anomalies**. Embedded systems are increasingly networked, making them vulnerable to various network-based attacks. Unusual network traffic patterns, such as unexpected bursts of data transmission or communication with unknown hosts, can indicate network anomalies. Monitoring network traffic for these patterns helps in identifying and responding to potential threats. For instance, abnormal outgoing network traffic could suggest a denial-of-service attack or data theft attempt.

In Figure 17, a comprehensive IoT workflow is shown where various stages are involved to ensure efficient monitoring and anomaly detection:

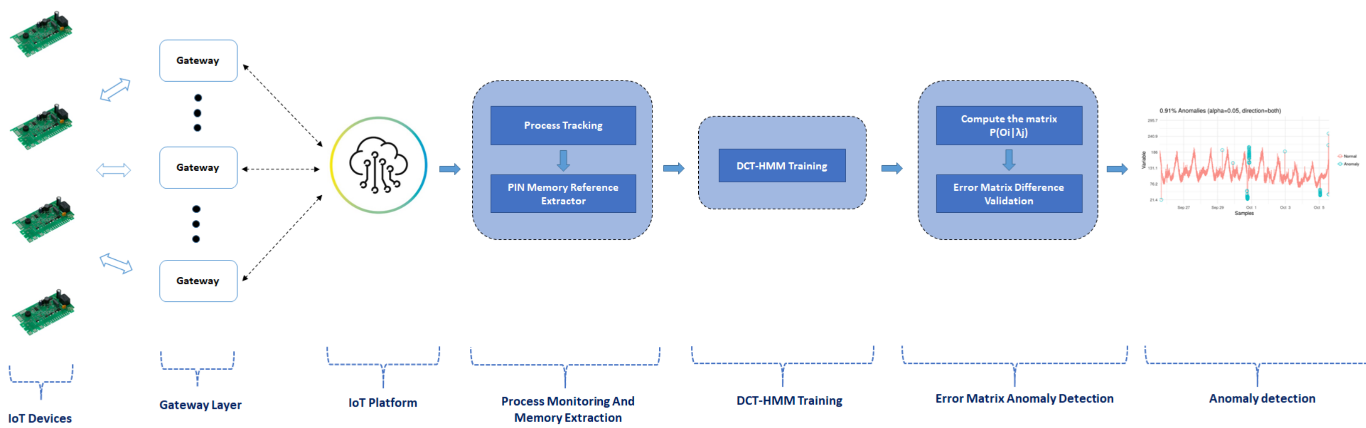

Figure 17: Workflow for monitoring and anomaly detection in IoT systems using DCT-HMM

- **IoT Platform Setup**: The workflow begins with the establishment of an IoT platform, which includes IoT gateways and devices. The platform facilitates seamless communication and data aggregation from numerous IoT devices deployed in the field.
- **Process Monitoring**: A specific process is selected for monitoring. This process involves tracking various performance metrics and data points critical to the operation of the IoT devices.
- **Memory Reference Extraction**: Using the PIN tool, memory references of the selected process are extracted. PIN is a dynamic binary instrumentation framework that allows the collection of detailed memory access patterns and other relevant information from running applications.
- **Training the DCT-HMM Model**: The next step involves training a *Discrete Cosine Transform-Hidden Markov Model* (DCT-HMM) using the extracted memory references. This model helps in understanding the normal behavior of the process by analyzing the patterns in the data.
- **Computing Matrix** $P(O|\lambda)$: Once the model is trained, the matrix $P(O|\lambda)$, which represents the transition probabilities between different states in the HMM, is computed. This matrix is crucial for detecting deviations from normal behavior.
- **Anomaly Detection**: Finally, anomalies are detected by comparing the difference in the error matrix. This involves calculating the error matrix during real-time monitoring and checking it against the learned model. Significant differences indicate potential anomalies, prompting further investigation or corrective actions.

7 Conclusion

In this paper, we propose a technique to determine anomaly behaviors in programs based on a model built from memory reference sequences. We present a detailed modeling techniques based on spectral representation of memory reference sequences and Hidden Markov Models, and show that the execution epochs of each program can be clustered and represented using multidimensional scaling. This modeling technique is the basis of the proposed algorithm for anomaly detection, which is capable of accurately determine the epoch where an anomaly has occurred [59, 60, 61, 62, 63]. It is also capable to determine the program subject to the anomaly.

The main limitation of the technique is that it considers a single thread of execution. In a multi-threaded program, this technique should be extended to consider an aggregate model of all the different threads. In the scenario of monitoring programs running on a given embedded system, this is not a problem, as typically the processes in such environments are single-threaded.

The experimental evaluations of the algorithm reported in the paper is obviously preliminary, but, at the same, it has provided a clear vision of the potentialities offered by our proposed framework, and its reliability in effectively and efficiently supporting anomaly detection. Indeed, we obviously plan to further experimentally investigate the algorithm through extensive experiments with different faults and anomaly injections. In addition to this, we plan to extend our proposed framework to other classes of data, such as *streaming data* (e.g., [64]), and *data compression* (e.g., [65, 66, 67]) and *privacy issues* (e.g., [68]) in order to catch other advanced features that may return to be useful in *emerging Big Data environments* (e.g., [69, 70]).

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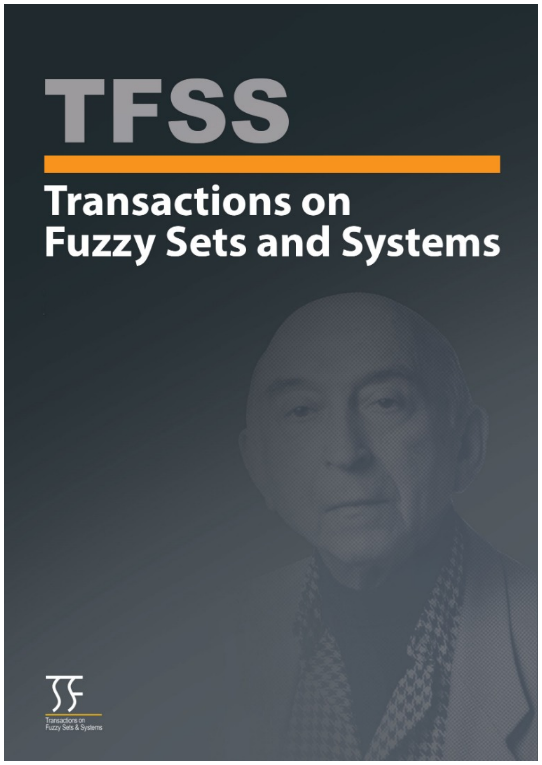

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Logical Entropy of Partitions for Interval-Valued Intuitionistic Fuzzy Sets

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Abstract. In this study, we present the ideas of logical entropy and logical conditional entropy for partitions in interval-valued intuitionistic fuzzy sets, and we establish their fundamental properties. First, we establish the definitions of logical entropy and logical conditional entropy, demonstrating their key characteristics and relationships. We then define logical mutual information and explore its properties, providing a comprehensive understanding of its behavior within the context of interval-valued intuitionistic fuzzy sets. Additionally, we propose the concept of logical divergence of states defined on interval-valued intuitionistic fuzzy sets and examine its properties in detail, including its application and implications for understanding state transitions within these fuzzy sets. Finally, we extend our study to dynamical systems, introducing the logical entropy of such systems when modeled with interval-valued intuitionistic fuzzy sets. We present several results related to this extension, highlighting the applicability and relevance of logical entropy in analyzing and understanding the behavior of dynamical systems. Overall, this paper offers a thorough exploration of logical entropy, mutual information, and divergence within the framework of interval-valued intuitionistic fuzzy sets, providing new insights and potential applications in various fields.

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1 Introduction

Entropy is a crucial concept in numerous scientific disciplines, Including fields like physics, computer science, systems theory, information theory, statistics, sociology, and various others. It was initially introduced in the dynamical systems theory by Kolmogorov in 1958 [1]. Sinai later extended this concept by defining entropy for a dynamical system with a probability space as the state space [2]. Shannon conceptualized entropy in information theory [3], and more recently, Ellerman introduced logical entropy based on logical partitioning [4]. Several authors have recently defined entropy and logical entropy for dynamical systems with an algebraic structure [5, 6, 7, 8, 9, 10, 11].
Fuzzy generalizations of dynamical systems and their Shannon entropy have also been studied [6, 7, 8]. In 1975, Zadeh introduced interval-valued fuzzy sets ($IVFS$) as an extension of fuzzy sets [12]. Subsequently, in 1989, Atanassov and Gargov proposed the concept of interval-valued intuitionistic fuzzy sets ($IVIFS(X)$) as an extension of interval-valued fuzzy sets [13].
This paper aims to explore logical entropy for interval-valued intuitionistic fuzzy sets in a non-empty set X and to introduce the logical entropy of dynamical systems in $IVIFS(X)$. The structure of the paper is as follows:

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Section 2 presents the concepts of logical entropy for partitions in $IVIFS(X)$ and its logical conditional entropy, along with an investigation of their properties. In Section 3 delves into the concepts of logical mutual information and their properties. Then in Section 4 defines the logical divergence of states on $IVIFS(X)$ moreover examines its properties. Finally, Section 5 focuses on the study of logical entropy of dynamical systems in $IVIFS(X)$.

The subsequent sections offer essential information that will be used throughout the paper. The concepts mentioned below are all derived from references [13, 10]. For further details, please refer to [13, 10].

Consider $C[0,1]$ as the set comprising all closed subintervals of $[0,1]$. An interval-valued intuitionistic fuzzy set I within a universe X is described as $I = \{\langle x, \alpha_I(x), \beta_I(x)\rangle \mid x \in X\}$, where $\alpha_I : X \to C[0,1]$ and $\beta_I : X \to C[0,1]$, and these functions satisfy the condition $\alpha_{IU}(x) + \beta_{IU}(x) \leq 1$ for every $x \in X$. The intervals $\alpha_I(x)$ and $\beta_I(x)$indicate the degrees of membership and non-membership of an element x in I, respectively. Expressly, $\alpha_{IL}(x)$ and $\alpha_{IU}(x)$ represent the minimum and maximum levels of membership of x in I, respectively, and these values satisfy $0 \leq \alpha_{IL}(x) \leq \alpha_{IU}(x) \leq 1$.

For convenience, the collection of all interval-valued intuitionistic fuzzy sets over X is denoted as $IVIFS(X)$. In this discussion, we will write

$$I = \langle [\alpha_{IL}, \alpha_{IU}], [\beta_{IL}, \beta_{IU}]\rangle,$$

instead of

$$\{\langle x, [\alpha_{IL}(x), \alpha_{IU}(x)], [\beta_{IL}(x), \beta_{IU}(x)] \mid x \in X\rangle\}.$$

Two partial binary operations $\oplus$ and $\cdot$ on $IVIFS(X)$ are defined as follows: for any $I = \langle [\alpha_{IL}, \alpha_{IU}], [\beta_{IL}, \beta_{IU}]\rangle$ and $J = \langle [\alpha_{JL}, \alpha_{JU}], [\beta_{JL}, \beta_{JU}]\rangle \in IVIFS(X)$,

$$I \oplus J = \langle [\alpha_{IL} + \alpha_{JL}, \alpha_{IU} + \alpha_{JU}], [\beta_{IL} + \beta_{JL} - 1, \beta_{IU} + \beta_{JU} - 1]\rangle,$$

whenever $\alpha_{IL} + \alpha_{JL} \leq 1, \alpha_{IU} + \alpha_{JU} \leq 1, \beta_{IL} + \beta_{JL} \geq 1$, and $\beta_{IU} + \beta_{JU} \geq 1$, and

$$I \cdot J = \langle [\alpha_{IL} \cdot \alpha_{JL}, \alpha_{IU} \cdot \alpha_{JU}], [\beta_{IL} + \beta_{JL} - \beta_{IL} \cdot \beta_{JL}, \beta_{IU} + \beta_{JU} - \beta_{IU} \cdot \beta_{JU}]\rangle.$$

See the properties of these partial binary operations in [10].

A function $\mathbf{m} : IVIFS(X) \to [0,1]$ is termed a state on $IVIFS(X)$ if it satisfies certain conditions that allow it to measure or evaluate the interval-valued intuitionistic fuzzy sets over X within the range from 0 to 1.

To be specific, the function $\mathbf{m}$ must meet the following criteria:

1. Normalization: The state assigns the value 1 to the specific fuzzy set $\langle [1,1], [0,0]\rangle$. This particular fuzzy set represents an element that is entirely a member (with a membership degree interval of $[1,1]$) and not at all a non-member (with a non-membership degree interval of $[0,0]$). Mathematically, this is expressed as:

$$\mathbf{m}(\langle [1,1], [0,0]\rangle) = 1.$$

2. Additivity: The state is additive for the operation $\oplus$, which is a binary operation defined for combining two interval-valued intuitionistic fuzzy sets. For any $I, J \in IVIFS(X)$, if $I \oplus J$ is defined, the state of the combined set $I \oplus J$ is equal to the sum of the states of the individual sets I and J. Formally, this is written as:

$$\mathbf{m}(I \oplus J) = \mathbf{m}(I) + \mathbf{m}(J).$$

A finite collection $\mathcal{F} = \{I_1, ..., I_n\}$ of elements of $IVIFS(X)$ is said to be a partition if

$$\oplus_{i=1}^{n} I_i = \langle [1,1], [0,0]\rangle.$$

Thus, the relation between a state $\mathbf{m}$ and a partition $\mathcal{F} = \{I_1, ..., I_n\}$ is

$$\mathbf{m}(\oplus_{i=1}^n I_i) = \sum_{i=1}^n \mathbf{m}(I_i).$$

Let $\mathcal{F}_1 = \{I_1, ..., I_n\}$ and $\mathcal{F}_2 = \{J_1, ..., J_m\}$. The partition $\mathcal{F}_2 = \{J_1, ..., J_m\}$ is called a refinement of $\mathcal{F}_1 = \{I_1, ..., I_n\}$, written as $\mathcal{F}_1 \preceq \mathcal{F}_2$, if there exists a partition $k(1), ..., k(n)$ of the set $\{1, ..., m\}$ such that

$$\mathbf{m}(I_i) = \sum_{h \in k(i)} \mathbf{m}(J_h),$$

for every $i = 1, ..., n$. The collection

$$\mathcal{F}_1 \vee \mathcal{F}_2 = \{I_i \cdot J_j : i = 1, ..., n, j = 1, ..., m\},$$

which is a partition.

2 Logical Entropy of Partitions in Interval-Valued Intuitionistic Fuzzy Sets

Logical entropy and logical conditional entropy provide more refined tools for measuring and managing uncertainty in $IVIFS$, where both fuzziness and hesitation need to be considered. These concepts extend classical entropy ideas to work better within the richer structure of $IVIFS$. For this purpose, in this section, we introduce the concepts of logical entropy and logical conditional entropy for partitions within $IVIFS$ and explore their properties.

Definition 2.1. Let $\mathcal{F} = \{I_1, \cdots, I_n\}$ be a partition in $IVIFS(X)$,, and let $\mathbf{m} : IVIFS(X) \to [0, 1]$ be a state. The logical entropy of $\mathcal{F}$ for state $\mathbf{m}$ is defined as follows:

$$H^l_{\mathbf{m}}(\mathcal{F}) = \sum_{i=1}^n \mathbf{m}(I_i)(1 - \mathbf{m}(I_i)). \tag{1}$$

Remark 2.2. The logical entropy $H^l_{\mathbf{m}}(A)$ is always non-negative. Given that $\sum_{i=1}^n \mathbf{m}(I_i) = \mathbf{m}(\oplus_{i=1}^n I_i) = \mathbf{m}(\langle [1, 1], [0, 0] \rangle) = 1$, equation (1) can also be expressed in the form shown below:

$$H^l_{\mathbf{m}}(\mathcal{F}) = 1 - \sum_{i=1}^n (\mathbf{m}(I_i))^2. \tag{2}$$

Example 2.3. $M = \{\langle [1, 1], [0, 0] \rangle\}$ represents a partition of $IVIFS(X)$, and for every partition B of $IVIFS(X)$, it holds that $B \preceq M$. If we set $M = \{\langle [1, 1], [0, 0] \rangle\}$, then $H^l_{\mathbf{m}}(M) = 0$.

Example 2.4. Suppose that $\langle [\alpha_{IL}, \alpha_{IU}], [\beta_{IL}, \beta_{IU}] \rangle \in IVIFS(X)$. Then, $\mathcal{F} = \{I_1 = \langle [\alpha_{IL}, \alpha_{IU}], [\beta_{IL}, \beta_{IU}] \rangle, I_2 = \langle [1 - \alpha_{IL}, 1 - \alpha_{IU}], [1 - \beta_{IL}, 1 - \beta_{IU}] \rangle\}$ forms a partition of $IVIFS(X)$.

If $\mathbf{m}$ is a state such that $\mathbf{m}(\langle [\alpha_{IL}, \alpha_{IU}], [\beta_{IL}, \beta_{IU}] \rangle) = s < 1$ and $\mathbf{m}(\langle [1 - \alpha_{IL}, 1 - \alpha_{IU}], [1 - \beta_{IL}, 1 - \beta_{IU}] \rangle) = 1 - s$, then $H^l_{\mathbf{m}}(\mathcal{F}) = 2s(1 - s)$.

Definition 2.5. Consider $\mathcal{F}_1 = \{I_1, \ldots, I_n\}$ and $\mathcal{F}_2 = \{J_1, \ldots, J_m\}$ as two partitions of $IVIFS(X)$. The logical conditional entropy of $\mathcal{F}_1$ given $\mathcal{F}_2$ is defined as follows:

$$H^l_{\mathbf{m}}(\mathcal{F}_1|\mathcal{F}_2) = \sum_{i=1}^{n}\sum_{j=1}^{m} \mathbf{m}(I_i \cdot J_j)\left(\mathbf{m}(J_j) - \mathbf{m}(I_i \cdot J_j)\right). \tag{3}$$

Proposition 2.6. *([10]) Consider $\mathcal{F} = \{I_1, ..., I_n\}$ as a partition of $IVIFS(X)$, and let $K \in IVIFS(X)$. then*

$$\mathbf{m}(K) = \sum_{i=1}^{n} \mathbf{m}(I_i \cdot K).$$

Remark 2.7. According to Proposition 2.6, we have $\sum_{i=1}^{n} \mathbf{m}(I_i \cdot I_j) = \mathbf{m}(I_j)$. Therefore, equation (3) can be rewritten in the following form:

$$H^l_{\mathbf{m}}(\mathcal{F}_1|\mathcal{F}_2) = \sum_{j=1}^{m}(J_j)^2 - \sum_{i=1}^{n}\sum_{j=1}^{m}(\mathbf{m}(I_i \cdot J_j))^2. \tag{4}$$

Remark 2.8. Since $\mathbf{m}(I_i \cdot J_j) \leq \mathbf{m}(J_j)$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$, the logical conditional entropy $H^l_{\mathbf{m}}(\mathcal{F}_1 \mid \mathcal{F}_2)$ is always nonnegative. Suppose $M = \{\langle [1,1], [0,0] \rangle\}$. It is straightforward to verify that $H^l_{\mathbf{m}}(\mathcal{F} \mid M) = H^l_{\mathbf{m}}(\mathcal{F})$ for any partition $\mathcal{F}$ of $IVIFS(X)$.

Theorem 2.9. *For any arbitrary partitions $\mathcal{F}_1$ and $\mathcal{F}_2$ of $IVIFS(X)$, the following property hold.*

$$H^l_{\mathbf{m}}(\mathcal{F}_1 \vee \mathcal{F}_2) = H^l_{\mathbf{m}}(\mathcal{F}_1) + H^l_{\mathbf{m}}(\mathcal{F}_2 \mid \mathcal{F}_1). \tag{5}$$

Proof. Suppose that $\mathcal{F}_1 = \{I_1, \cdots, I_n\}$ and $\mathcal{F}_2 = \{J_1, \cdots, J_m\}$. Then By equations (2) and (3) we derive:

$$\begin{aligned} H^l_{\mathbf{m}}(\mathcal{F}_1) + H^l_{\mathbf{m}}(\mathcal{F}_2 \mid \mathcal{F}_1) &= 1 - \sum_{i=1}^{n}(\mathbf{m}(I_i)^2 + \sum_{i=1}^{n}(\mathbf{m}(I_i)^2 \\ &- \sum_{i=1}^{n}\sum_{j=1}^{m}(\mathbf{m}(I_i \cdot J_j))^2 \\ &= 1 - \sum_{i=1}^{n}\sum_{j=1}^{m}(\mathbf{m}(I_i \cdot J_j)) \\ &= H^l_{\mathbf{m}}(\mathcal{F}_1 \vee \mathcal{F}_2). \end{aligned}$$

□

Remark 2.10. Let $\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_n$ be partitions of $IVIFS(X)$. Using induction, we obtain the following generalization of equation (5):

$$\begin{aligned} H^l_{\mathbf{m}}(\mathcal{F}_1 \vee \mathcal{F}_2 \vee \cdots \vee \mathcal{F}_n) &= H^l_{\mathbf{m}}(P_1) \\ &+ \sum_{i=2}^{n} H^l_{\mathbf{m}}(\mathcal{F}_i \mid \mathcal{F}_1 \vee \cdots \vee \mathcal{F}_{i-1} \vee \mathcal{F}_{i+1} \cdots \vee \mathcal{F}_n). \end{aligned} \tag{6}$$

Remark 2.11. For any arbitrary partitions $\mathcal{F}_1$ and $\mathcal{F}_2$ of $IVIFS(X)$, the following relationship is easily obtained:

$$H^l_{\mathbf{m}}(\mathcal{F}_1 \vee \mathcal{F}_2) = H^l_{\mathbf{m}}(\mathcal{F}_1) + H^l_{\mathbf{m}}(\mathcal{F}_2 \mid \mathcal{F}_1) = H^l_{\mathbf{m}}(\mathcal{F}_2) + H^l_{\mathbf{m}}(\mathcal{F}_1 \mid \mathcal{F}_2). \tag{7}$$

Theorem 2.12. *For any partitions $\mathcal{F}_1$ and $\mathcal{F}_2$ of $IVIFS(X)$, the following assertions hold:*

(i) $H^l_{\mathbf{m}}(\mathcal{F}_1 \mid \mathcal{F}_2) \leq H^l_{\mathbf{m}}(\mathcal{F}_1)$.

(ii) $H^l_{\mathbf{m}}(\mathcal{F}_1 \vee \mathcal{F}_2) \leq H^l_{\mathbf{m}}(\mathcal{F}_1) + H^l_{\mathbf{m}}(\mathcal{F}_2)$.

Proof. Assume that $\mathcal{F}_1 = \{I_1, \ldots, I_n\}$ and $\mathcal{F}_2 = \{J_1, \ldots, J_m\}$. (i) Given that proposition 2.6 establishes that $\sum_{j=1}^m \mathbf{m}(I_i \cdot J_j) = \mathbf{m}(I_i)$, it follows that:

$$\begin{aligned}
\sum_{j=1}^m \mathbf{m}(I_i \cdot J_j)(\mathbf{m}(J_j) - \mathbf{m}(I_i \cdot J_j)) &\leq (\sum_{j=1}^m \mathbf{m}(I_i \cdot J_j))(\sum_{j=1}^m (\mathbf{m}(J_j) - \mathbf{m}(I_i \cdot J_j)) \\
&= \mathbf{m}(I_i)(\sum_{j=1}^m \mathbf{m}(J_j) - \sum_{j=1}^m \mathbf{m}(I_i \cdot J_j) \\
&= \mathbf{m}(I_i)(1 - \mathbf{m}(I_i)).
\end{aligned}$$

So

$$\begin{aligned}
H^l_{\mathbf{m}}(\mathcal{F}_1 \mid \mathcal{F}_2) &= \sum_{i=1}^n \sum_{j=1}^m \mathbf{m}(I_i \cdot J_j)(\mathbf{m}(J_j) - \mathbf{m}(I_i \cdot J_j)) \\
&\leq \sum_{i=1}^n \mathbf{m}(I_i)(1 - \mathbf{m}(I_i)) \\
&= H^l_{\mathbf{m}}(\mathcal{F}_1).
\end{aligned}$$

(ii) Based on equation (5) and property (i), property (ii) is derived. □

Proposition 2.13. *([10]) If $\mathcal{F}_1 = \{I_1, \ldots, I_n\}$ and $\mathcal{F}_2 = \{J_1, \ldots, J_m\}$ are both partitions of $IVIFS(X)$, then $\mathcal{F}_1 \vee \mathcal{F}_2$ also constitutes a partition. Furthermore $\mathcal{F}_1 \preceq \mathcal{F}_1 \vee \mathcal{F}_2$.*

Theorem 2.14. *For any partitions $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}_3$ of $IVIFS(X)$, the following properties apply:*

(i) $\mathcal{F}_1 \preceq \mathcal{F}_2$ *implies* $H^l_{\mathbf{m}}(\mathcal{F}_1) \leq H^l_{\mathbf{m}}(\mathcal{F}_2)$;

(ii) $H^l_{\mathbf{m}}(\mathcal{F}_1 \vee \mathcal{F}_2) \geq max[H^l_{\mathbf{m}}(\mathcal{F}_1), H^l_{\mathbf{m}}(\mathcal{F}_2)]$.

(iii) $\mathcal{F}_1 \preceq \mathcal{F}_2$ *implies* $H^l_{\mathbf{m}}(\mathcal{F}_1 \mid \mathcal{F}_3) \leq H^l_{\mathbf{m}}(\mathcal{F}_2 \mid \mathcal{F}_3)$.

Proof. (i) Assume $\mathcal{F}_1 = \{I_1, \ldots, I_n\}$ and $\mathcal{F}_2 = \{J_1, \ldots, J_m\}$. Under the premise $\mathcal{F}_1 \preceq \mathcal{F}_2$, a partition $\{k(1), \ldots, k(n)\}$ of the set $\{1, 2, \ldots, m\}$ exists such that $I_i = \sum_{j\in k(i)} J_j$ for all $i = 1, \ldots, n$. Therefore,

$$\begin{aligned} H^l_{\mathbf{m}}(\mathcal{F}_1) &= 1 - \sum_{i=1}^{n}(\mathbf{m}(I_i))^2 \\ &= 1 - \sum_{i=1}^{n}(\mathbf{m}(\sum_{j\in k(i)} J_j))^2 \\ &= 1 - \sum_{i=1}^{n}(\sum_{j\in k(i)} \mathbf{m}(J_j))^2 \\ &\leq 1 - \sum_{i=1}^{n}\sum_{j\in k(i)} (\mathbf{m}(J_j))^2 \\ &= 1 - \sum_{j=1}^{m}(\mathbf{m}(J_j))^2 \\ &= H^l_{\mathbf{m}}(\mathcal{F}_2). \end{aligned}$$

The inequality mentioned previously arises from the inequality $(a_1 + a_2 + \cdots + a_n)^2 \geq a_1^2 + a_2^2 + \cdots + a_n^2$ which holds for all nonnegative real numbers $a_1, \ldots, a_n$.
(ii) Since $\mathcal{F}_1 \preceq \mathcal{F}_1 \vee \mathcal{F}_2$ and $\mathcal{F}_2 \preceq \mathcal{F}_1 \vee \mathcal{F}_2$, property (ii) follows as a result of property (i).
(iii) Assuming $\mathcal{F}_1 \preceq \mathcal{F}_2$, Proposition 2.13 indicates that $\mathcal{F}_1 \vee \mathcal{F}_3 \preceq \mathcal{F}_2 \vee \mathcal{F}_3$. Consequently, using equation (5) and property (i), we can deduce that:

$$H^l_{\mathbf{m}}(\mathcal{F}_1 \mid \mathcal{F}_3) = H^l_{\mathbf{m}}(\mathcal{F}_1 \vee \mathcal{F}_3) - H^l_{\mathbf{m}}(\mathcal{F}_3) \leq H^l_{\mathbf{m}}(\mathcal{F}_2 \vee \mathcal{F}_3) - H^l_{\mathbf{m}}(\mathcal{F}_3) = H^l_{\mathbf{m}}(\mathcal{F}_2 \mid \mathcal{F}_3).$$

□

The set of all states defined on $IVIFS(X)$ is represented by $\mathbf{M}(IVIFS(X))$. In the subsequent theorem, we demonstrate that $\mathbf{M}(IVIFS(X))$ forms a convex set.

Theorem 2.15. *If $\mathbf{m}_1, \mathbf{m}_2 \in \mathbf{M}(IVIFS(X))$, then for any t within the interval $[0, 1]$, the combination $t\mathbf{m}_1 + (1-t)\mathbf{m}_2$ belongs to $\mathbf{M}(IVIFS(X))$.*

Proof. This proof is straightforward. □

The theorem below establishes that logical entropy is a convex function on $\mathbf{M}(IVIFS(X))$.

Theorem 2.16. *Given a partition $\mathcal{F}$ of $IVIFS(X)$, it is true that for any $\mathbf{m}_1, \mathbf{m}_2 \in \mathbf{M}(IVIFS(X))$ and for any t within the interval $[0, 1]$, the following holds:*

$$tH^l_{\mathbf{m}_1}(\mathcal{F}) + (1-t)H^l_{\mathbf{m}_2}(\mathcal{F}) \leq H^l_{t\mathbf{m}_1+(1-t)\mathbf{m}_2}(\mathcal{F}).$$

Proof. Assume $\mathcal{F} = \{I_1, \ldots, I_n\}$. Given that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is convex for all

$x \in \mathbb{R}$, it follows that for any t in the interval $[0, 1]$, we derive:

$$\begin{aligned} tH^l_{\mathbf{m}1}(\mathcal{F}) + (1-t)H^l_{\mathbf{m}_2}(\mathcal{F}) &= t\left[1 - \sum_{i=1}^{n}(\mathbf{m}_1(I_i))^2\right] \\ &+ (1-t)\left[1 - \sum_{i=1}^{n}(\mathbf{m}_2(I_i))^2\right] \\ &= 1 - t\sum_{i=1}^{n}(\mathbf{m}_1(I_i))^2 - (1-t)\sum_{i=1}^{n}(\mathbf{m}_2(I_i))^2 \\ &\leq 1 - \sum_{i=1}^{n}((t\mathbf{m}_1(I_i) + (1-t)\mathbf{m}_2(I_i))^2 \\ &= 1 - \sum_{i=1}^{n}(t\mathbf{m}_1 + (1-t)\mathbf{m}_2)(I_i))^2 \\ &= H^l_{t\mathbf{m}_1+(1-t)\mathbf{m}_2}(\mathcal{F}). \end{aligned}$$

□

3 Logical Mutual Information in Interval-Valued Intuitionistic Fuzzy Sets

In this section, the concept of logical mutual information for partitions in $IVIFS$ is introduced. The introduction of logical mutual information for partitions in $IVIFS$ is aimed at quantifying the interdependence between different fuzzy partitions in a way that takes into account both fuzziness and hesitation due to intuitionistic uncertainty. This measure provides a nuanced way to understand how one fuzzy concept can reduce uncertainty about another in complex, real-world decision-making and data analysis tasks where uncertainty is an inherent challenge.

Definition 3.1. The logical mutual information of partitions $\mathcal{F}_1$ and $\mathcal{F}_2$ in $IVIFS(X)$ is defined as follows:

$$\mathcal{I}^l_{\mathbf{m}}(\mathcal{F}_1, \mathcal{F}_2) = H^l_{\mathbf{m}}(\mathcal{F}_1) - H^l_{\mathbf{m}}(\mathcal{F}_1 \mid \mathcal{F}_2). \tag{8}$$

Remark 3.2. Since $H^l_{\mathbf{m}}(\mathcal{F}_1 \mid \mathcal{F}_2) \leq H^l_{\mathbf{m}}(\mathcal{F}_1)$, this implies that the logical mutual information $I^l_{\mathbf{m}}(\mathcal{F}_1, \mathcal{F}_2)$ is consistently nonnegative.

Theorem 3.3. *The logical mutual information of partitions $\mathcal{F}_1$ and $\mathcal{F}_2$ in $IVIFS(X)$ exhibits the following properties:*

(i) $\mathcal{I}^l_{\mathbf{m}}(\mathcal{F}_1, \mathcal{F}_2) = H^l_{\mathbf{m}}(\mathcal{F}_1) + H^l_{\mathbf{m}}(\mathcal{F}_2) - H^l_{\mathbf{m}}(\mathcal{F}_1 \vee \mathcal{F}_2)$;

(ii) $\mathcal{I}^l_{\mathbf{m}}(\mathcal{F}_1, \mathcal{F}_2) = \mathcal{I}^l_{\mathbf{m}}(\mathcal{F}_2, \mathcal{F}_1)$;

(iii) $\mathcal{I}^l_{\mathbf{m}}(\mathcal{F}_1, \mathcal{F}_2) \leq min\left[H^l_{\mathbf{m}}(\mathcal{F}_1), H^l_{\mathbf{m}}(\mathcal{F}_2)\right]$.

Proof. (i) Based on equation 5, we have $H^l_{\mathbf{m}}(\mathcal{F}_1 \mid \mathcal{F}_2) = H^l_{\mathbf{m}}(\mathcal{F}_1) - H^l_{\mathbf{m}}(\mathcal{F}_1 \vee \mathcal{F}_2)$. Consequently, utilizing equation (8), the following identities are established:

$$\mathcal{I}^l_{\mathbf{m}}(\mathcal{F}_1, \mathcal{F}_2) = H^l_{\mathbf{m}}(\mathcal{F}_1) + H^l_{\mathbf{m}}(\mathcal{F}_2) - H^l_{\mathbf{m}}(\mathcal{F}_1 \vee \mathcal{F}_2). \tag{9}$$

(ii) This property is derived from equation (9).
(iii) According to part (iii) of Theorem 2.14, $H^l_{\mathbf{m}}(\mathcal{F}_1) \leq H^l_{\mathbf{m}}(\mathcal{F}_1 \vee \mathcal{F}_2)$, which implies that $\mathcal{I}^l_{\mathbf{m}}(\mathcal{F}_1, \mathcal{F}_2) \leq min\left[H^l_{\mathbf{m}}(\mathcal{F}_1), H^l_{\mathbf{m}}(\mathcal{F}_2)\right]$. □

Theorem 3.4. *If partitions $\mathcal{F}_1$ and $\mathcal{F}_2$ are statistically independent, then:*

(i) $\mathcal{I}^l_{\mathbf{m}}(\mathcal{F}_1, \mathcal{F}_2) = H^l_{\mathbf{m}}(\mathcal{F}_1).H^l_{\mathbf{m}}(\mathcal{F}_2)$;

(ii) $1 - H^l_{\mathbf{m}}(\mathcal{F}_1 \vee \mathcal{F}_2) = \left(1 - H^l_{\mathbf{m}}(\mathcal{F}_1)\right) . \left(1 - H^l_{\mathbf{m}}(\mathcal{F}_2)\right)$.

Proof. (i) Assume $\mathcal{F}_1 = \{I_1, \ldots, I_n\}$ and $\mathcal{F}_2 = \{J_1, \ldots, J_m\}$. Based on equations (2) and (9), we derive:

$$\begin{aligned}
\mathcal{I}^l_{\mathbf{m}}(\mathcal{F}_1, \mathcal{F}_2) &= 1 - \sum_{i=1}^{n} (\mathbf{m}(I_i))^2 + 1 - \sum_{j=1}^{m} (\mathbf{m}(J_j)^2 - 1 \\
&+ \sum_{i=1}^{n} \sum_{j=1}^{m} (\mathbf{m}(I_i \cdot J_j))^2 \\
&= \left(1 - \sum_{i=1}^{n} (\mathbf{m}(I_i))^2\right) . \left(1 - \sum_{j=1}^{m} (\mathbf{m}(J_j))^2\right) \\
&= H^l_{\mathbf{m}}(\mathcal{F}_1).H^l_{\mathbf{m}}(\mathcal{F}_2).
\end{aligned}$$

(ii) Utilizing item (i) and equation (9), we arrive at:

$$\begin{aligned}
(1 - H^l_{\mathbf{m}}(\mathcal{F}_1)).(1 - H^l_{\mathbf{m}}(\mathcal{F}_2)) &= 1 - H^l_{\mathbf{m}}(\mathcal{F}_1) - H^l_{\mathbf{m}}(\mathcal{F}_2) + H^l_{\mathbf{m}}(\mathcal{F}_1).H^l_{\mathbf{m}}(\mathcal{F}_2) \\
&= 1 - H^l_{\mathbf{m}}(\mathcal{F}_1) - H^l_{\mathbf{m}}(\mathcal{F}_2) + I^l_{\mathbf{m}}(\mathcal{F}_1, \mathcal{F}_2) \\
&= 1 - H^l_{\mathbf{m}}(\mathcal{F}_1) - H^l_{\mathbf{m}}(\mathcal{F}_2) + H^l_{\mathbf{m}}(\mathcal{F}_1) + H^l_{\mathbf{m}}(\mathcal{F}_2) \\
&- H^l_{\mathbf{m}}(\mathcal{F}_1 \vee \mathcal{F}_2) \\
&= 1 - H^l_{\mathbf{m}}(\mathcal{F}_1 \vee \mathcal{F}_2).
\end{aligned}$$

□

4 Logical Divergence in Interval-Valued Intuitionistic Fuzzy Sets

In this section, we introduce the concept of logical divergence entropy within $IVIFS$. The introduction of this concept is motivated by the need to measure the divergence or difference between fuzzy partitions that include both fuzziness and intuitionistic hesitation (due to interval-valued uncertainty). This measure captures not only the imprecision in membership but also the hesitation in decision-making processes. It is useful for various applications, such as decision-making, pattern recognition, and data analysis, providing a more nuanced way to compare fuzzy sets in uncertain environments.

Definition 4.1. Assume $\mathcal{F} = \{I_1, \ldots, I_n\}$ is a partition of $IVIFS(X)$ and $\mathbf{m}_1, \mathbf{m}_2 \in \mathbf{M}(IVIFS(X))$. The logical divergence of states $\mathbf{m}_1$ and $\mathbf{m}_2$ for $\mathcal{F}$ is defined as follows:

$$\mathcal{D}^l_{\mathcal{F}}(\mathbf{m}_1 \| \mathbf{m}_2) = \frac{1}{2} \sum_{i=1}^{n} (\mathbf{m}_1(I_i) - \mathbf{m}_2(I_i))^2.$$

Theorem 4.2. *Assume $\mathcal{F} = \{I_1, \ldots, I_n\}$ is a partition of $IVIFS(X)$ and $\mathbf{m}_1, \mathbf{m}_2 \in \mathbf{M}(IVIFS(X))$. The logical divergence of states $\mathbf{m}_1$ and $\mathbf{m}_2$ for $\mathcal{F}$ fulfills the following conditions:*

(i) $\mathcal{D}^l_{\mathcal{F}}(\mathbf{m}_1 \| \mathbf{m}_2) = \mathcal{D}^l_{\mathcal{F}}(\mathbf{m}_2 \| \mathbf{m}_1)$.

(ii) $\mathcal{D}^l_{\mathcal{F}}(\mathbf{m}_1 \| \mathbf{m}_2) \geq 0$, *where equality holds if and only if the states $\mathbf{m}_1$ and $\mathbf{m}_2$ are identical over $\mathcal{F}$.*

Proof. Based on the definition provided above, this proof is straightforward. □ In the example below, it is demonstrated that logical divergence does not qualify as a distance metric because it does not fulfill the triangle inequality.

Example 4.3. In Example 2.4, assume $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$ are three states on $IVIFS(X)$ where $\mathbf{m}_1(I_1) = s_1$, $\mathbf{m}_2(I_1) = s_2$, and $\mathbf{m}_3(I_1) = s_3$, with s_1, s_2, s_3 each in the interval (0,1). Consequently, $\mathbf{m}_1(I_2) = 1 - s_1$, $\mathbf{m}_2(I_2) = 1 - s_2$, and $\mathbf{m}_3(I_2) = 1 - s_3$. Thus, we derive:

$$\mathcal{D}^l_{\mathcal{F}}(\mathbf{m}_1 \| \mathbf{m}_2) = \frac{1}{2}(\mathbf{m}_1(I_1) - \mathbf{m}_2(I_1))^2 + \frac{1}{2}(\mathbf{m}_1(I_2) - \mathbf{m}_2(I_2))^2 = (s_1 - s_2)^2.$$

Similarly, we have:

$$\mathcal{D}^l_{\mathcal{F}}(\mathbf{m}_1 \| \mathbf{m}_3) = (s_1 - s_3)^2, \text{ and } \mathcal{D}^l_{\mathcal{F}}(\mathbf{m}_2 \| \mathbf{m}_3) = (s_2 - s_3)^2.$$

Set $s_1 = \frac{1}{3}$, $s_2 = \frac{1}{4}$, $s_3 = \frac{1}{5}$. Clearly,

$$\mathcal{D}^l_{\mathcal{F}}(\mathbf{m}_1 \| \mathbf{m}_3) \geq \mathcal{D}^l_{\mathcal{F}}(\mathbf{m}_1 \| \mathbf{m}_2) + \mathcal{D}^l_{\mathcal{F}}(\mathbf{m}_2 \| \mathbf{m}_3).$$

This outcome indicates that the triangle inequality does not generally hold for logical divergence in $IVIFS(X)$.

Theorem 4.4. *Assume $\mathcal{F} = \{I_1, \ldots, I_n\}$ is a partition of $IVIFS(X)$. Then, for every pair of states $\mathbf{m}_1$ and $\mathbf{m}_2$ defined on $IVIFS(X)$, the following is true:*

$$\mathcal{D}^l_{\mathcal{F}}(\mathbf{m}_1 \| \mathbf{m}_2) = \left(\sum_{i=1}^{n} \mathbf{m}_1(I_i)(1 - \mathbf{m}_2(I_i))\right) - \left[\frac{1}{2}(H^l_{\mathbf{m}_1}(\mathcal{F}) + H^l_{\mathbf{m}_2}(\mathcal{F}))\right].$$

Proof. Assume $\mathcal{F} = \{I_1, \ldots, I_n\}$. Let's proceed with the calculation:

$$\left(\sum_{i=1}^{n} \mathbf{m}_1(I_i)(1 - \mathbf{m}_2(I_i))\right) - \left[\frac{1}{2}(H^l_{\mathbf{m}_1}(\mathcal{F}) + H^l_{\mathbf{m}_2}(\mathcal{F}))\right]$$

$$= 1 - \sum_{i=1}^{n} \mathbf{m}_1(I_i)\mathbf{m}_2(I_i) - \frac{1}{2}(1 - \sum_{i=1}^{n}(\mathbf{m}_1(I_i))^2) - \frac{1}{2}(1 - \sum_{i=1}^{n}(\mathbf{m}_2(I_i))^2)$$

$$= \frac{1}{2}(1 - \sum_{i=1}^{n}(\mathbf{m}_1(I_i) - \mathbf{m}_2(I_i))^2) = \mathcal{D}^l_{\mathcal{F}}(\mathbf{m}_1 \| \mathbf{m}_2).$$

□

5 The Logical Entropy of Dynamical System in $IVIFS(X)$

The concept of logical entropy for a dynamical system within the framework of IVIFS(X) is introduced to measure and track the evolving uncertainty and distinctions in systems characterized by both fuzziness and intuitionistic uncertainty. Logical entropy allows for a more nuanced analysis of dynamical systems where both imprecision and hesitation are present, providing deeper insights into the complexity and unpredictability of the systems behavior over time.

Definition 5.1. [10] A dynamical system in $IVIFS(X)$ consists of the triple $(IVIFS(X), \mathbf{m}, \psi)$, where $\mathbf{m} : IVIFS(X) \to [0, 1]$ is a state function on $IVIFS(X)$ and $\psi : IVIFS(X) \to IVIFS(X)$ is a mapping that meets the following criteria:

1. If $I \cdot J = \langle [0,0],[1,1] \rangle$, then $\psi(I) \cdot \psi(J) = \langle [1,1],[0,0] \rangle$ and $\psi(I \oplus J) = \psi(I) \oplus \psi(J)$, for any $I, J \in IVIFS(X)$.
2. $\psi(\langle [1,1],[0,0] \rangle) = \langle [1,1],[0,0] \rangle$;
3. $s(\psi(I)) = \mathbf{m}(I)$ for any $I \in IVIFS(X)$.

Theorem 5.2. *Consider $(IVIFS(X), \mathbf{m}, \psi)$ as a dynamical system in $IVIFS(X)$, with $\mathcal{F}_1$ and $\mathcal{F}_2$ as partitions within $IVIFS(X)$. The following assertions hold:*

(i) $\psi(\mathcal{F}_1 \vee \mathcal{F}_2) = \psi(\mathcal{F}_1) \vee \psi(\mathcal{F}_2)$.

(ii) $\mathcal{F}_1 \preceq \mathcal{F}_2$ *implies* $\psi(\mathcal{F}_1) \preceq \psi(\mathcal{F}_2)$.

Proof. The proof of (i) is derived from condition (ii) of Definition 5.1.
Consider $\mathcal{F}_1 = \{I_1, \ldots, I_n\}$ and $\mathcal{F}_2 = \{J_1, \ldots, J_m\}$, with $\mathcal{F}_1 \preceq \mathcal{F}_2$. Consequently, there is a partition $\{k(1), \ldots, k(n)\}$ of the set $\{1, 2, \ldots, m\}$ such that $I_i = \sum_{j \in k(i)} J_j$ for each $i = 1, 2, \ldots, n$. Therefore, according to condition (i) of Definition 5.1, it follows:

$$\psi(I_i) = \psi(\sum_{j \in k(i)} J_j) = \sum_{j \in k(i)} \psi(J_j),$$

for $i = 1, 2, \ldots, n$. This implies that $\psi(\mathcal{F}_1) \preceq \psi(\mathcal{F}_2)$. □

Theorem 5.3. *Consider $(IVIFS(X), \mathbf{m}, \psi)$ as a dynamical system within $IVIFS(X)$, with $\mathcal{F}_1$ and $\mathcal{F}_2$ as partitions of $IVIFS(X)$. Then, for any non-integer n, the following assertions are valid:*

(i) $H^l_{\mathbf{m}}(\psi^n(\mathcal{F}_1)) = H^l_{\mathbf{m}}(\mathcal{F}_1)$;

(ii) $H^l_{\mathbf{m}}(\psi^n(\mathcal{F}_1) | \psi^n(\mathcal{F}_2)) = H^l_{\mathbf{m}}(\mathcal{F}_1 | \mathcal{F}_2)$.

Proof. Suppose that $\mathcal{F}_1 = \{I_1, \ldots, I_n\}$ and $\mathcal{F}_2 = \{J_1, \ldots, J_m\}$.

(i) Since for any non-negative integer n and for each $i = 1, \ldots, k$, it is true that $\mathbf{m}(\psi^n(I_i)) = \mathbf{m}(I_i)$, we conclude:

$$H^l_{\mathbf{m}}(\psi^n(\mathcal{F}_1)) = \sum_{i=1}^{n} \mathbf{m}(\psi^n(I_i) - \mathbf{m}(\psi^n(I_i))^2 = \sum_{i=1}^{n} \mathbf{m}(I_i) - \mathbf{m}(I_i)^2 = H^l_{\mathbf{m}}(\mathcal{F}_1).$$

(ii) Based on the same argument, we have:

$$H^l_{\mathbf{m}}(\psi^n(\mathcal{F}_1) | \psi^n(\mathcal{F}_2)) = \sum_{j=1}^{m} \mathbf{m}(\psi^n(J_i)^2 - \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbf{m}(\psi^n(I_i \cdot J_j))^2$$

$$= \sum_{j=1}^{m} \mathbf{m}(J_i)^2 - \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbf{m}(I_i \cdot J_j)^2 = H^l_{\mathbf{m}}(\mathcal{F}_1 | \mathcal{F}_2).$$

□

Theorem 5.4. *Take $(IVIFS(X), \mathbf{m}, \psi)$ as a dynamical system, where $\mathcal{F}$ is a partition of $IVIFS(X)$. Then,*

$$\lim_{n \longrightarrow \infty} \frac{1}{n} H^l_s(\bigvee_{i=0}^{n-1} \psi^i(\mathcal{F})).$$

Proof. Suppose $a_n = H^l_{\mathbf{m}}\left(\bigvee_{i=0}^{n-1}\psi^i(\mathcal{F})\right)$ for $n = 1, 2, \ldots$. Then the sequence $\{a_n\}_{n=1}^{\infty}$ consists of non-negative real numbers and satisfies the property $a_{s+r} \leq a_s + a_r$ for any natural numbers s and r. According to property (i) of Theorem 5.3 and using the sub-additivity of logical entropy, we have:

$$\begin{aligned} a_{s+r} &= H^l_{\mathbf{m}}(\bigvee_{i=0}^{s+r-1}\psi^i(\mathcal{F})) \\ &\leq H^l_{\mathbf{m}}(\bigvee_{i=0}^{s-1}\psi^i(\mathcal{F}) + H^l_{\mathbf{m}}(\bigvee_{i=s}^{s+r-1}\psi^i(\mathcal{F})) \\ &= a_s + H^l_{\mathbf{m}}(\psi^s(\bigvee_{i=0}^{r-1}\psi^i(\mathcal{F})) \\ &= a_s + H^l_{\mathbf{m}}(\bigvee_{i=0}^{r-1}\psi^i(\mathcal{F}) = a_s + a_r. \end{aligned}$$

Therefore, by Theorem 4.9 from [14], $\lim_{n\to\infty}\frac{1}{n}a_n$ exists. □

Definition 5.5. Consider $(IVIFS(X), \mathbf{m}, \psi)$ as a dynamical system, with $\mathcal{F}$ being a partition of $IVIFS(X)$. We then define the logical entropy of ψ relative to $\mathcal{F}$ as follows:

$$H^l_{\mathbf{m}}(\psi, \mathcal{F}) = \lim_{n\longrightarrow\infty}\frac{1}{n}H^l_{\mathbf{m}}(\bigvee_{i=0}^{n-1}\psi^i(\mathcal{F})).$$

Remark 5.6. Let $(IVIFS(X), \mathbf{m}, \psi)$ be a dynamical system in $IVIFS(X)$ and let $\mathcal{F} = \{\langle[1,1],[0,0]\rangle\}$. Then $\bigvee_{i=0}^{n-1}\psi^i(\mathcal{F}) = \mathcal{F}$, and

$$H^l_{\mathbf{m}}(\psi, \mathcal{F}) = \lim_{n\to\infty}\frac{1}{n}H^l_{\mathbf{m}}\left(\bigvee_{i=0}^{n-1}\psi^i(\mathcal{F})\right) = \lim_{n\to\infty}\frac{1}{n}H^l_{\mathbf{m}}(\mathcal{F}) = 0.$$

Theorem 5.7. *Consider $(IVIFS(X), \mathbf{m}, \psi)$ as a dynamical system, with $\mathcal{F}$ being a partition of $IVIFS(X)$. Then, for every non-negative integer k, the following holds:*

$$H^l_{\mathbf{m}}(\psi, \mathcal{F}) = H^l_{\mathbf{m}}(\psi, \bigvee_{i=0}^{k}\psi^i(\mathcal{F})).$$

Proof. Using Definition 5.5, we derive:

$$\begin{aligned} H^l_{\mathbf{m}}(\psi, \bigvee_{i=0}^{k}\psi^i(\mathcal{F})) &= \lim_{n\longrightarrow\infty}\frac{1}{n}H^l_{\mathbf{m}}(\bigvee_{j=0}^{n-1}\psi^j(\bigvee_{i=0}^{k}\psi^i(\mathcal{F})) \\ &= \lim_{n\longrightarrow\infty}\frac{k+n}{n}\frac{1}{k+n}H^l_{\mathbf{m}}(\bigvee_{j=0}^{k+n-1}\psi^j(\mathcal{F})) \\ &= \lim_{n\longrightarrow\infty}\frac{1}{k+n}H^l_{\mathbf{m}}(\bigvee_{j=0}^{k+n-1}\psi^j(\mathcal{F})) = H^l_{\mathbf{m}}(\psi, \mathcal{F}). \end{aligned}$$

□

Theorem 5.8. *Consider $(IVIFS(X), \mathbf{m}, \psi)$ as a dynamical system, with $\mathcal{F}_1, \mathcal{F}_2$ being two partitions of $IVIFS(X)$ such that $\mathcal{F}_1 \preceq \mathcal{F}_2$. Then $H^l_{\mathbf{m}}(\psi, \mathcal{F}_1) \leq H^l_{\mathbf{m}}(\psi, \mathcal{F}_2)$.*

Proof. Suppose that $\mathcal{F}_1 \preceq \mathcal{F}_2$. By Theorems 5.2 and 5.3, we have $\bigvee_{i=0}^{n-1} \psi^i(\mathcal{F}_1) \preceq \bigvee_{i=0}^{n-1} \psi^i(\mathcal{F}_2)$ for $n = 1, 2, \ldots$. Therefore, by a property of logical entropy, we get:

$$H^l_{\mathbf{m}}(\bigvee_{i=0}^{n-1} \psi^i(\mathcal{F}_1)) \leq H^l_{\mathbf{m}}(\bigvee_{i=0}^{n-1} \psi^i(\mathcal{F}_2)).$$

Taking the limit as $n \to \infty$, we obtain $H^l_{\mathbf{m}}(\psi, \mathcal{F}_1) \leq H^l_{\mathbf{m}}(\psi, \mathcal{F}_2)$. □

Definition 5.9. Let $(IVIFS(X), \mathbf{m}, \psi)$ be a dynamical system in $IVIFS(X)$. The logical entropy of $(IVIFS(X), \mathbf{m}, \psi)$ is defined as:

$$H^l_{\mathbf{m}}(\psi) = \sup\{H^l_{\mathbf{m}}(\psi, \mathcal{F}) \mid \mathcal{F} \text{ is a partition of } IVIFS(X)\}.$$

Theorem 5.10. *Let $(IVIFS(X), \mathbf{m}, \psi)$ be a dynamical system in $IVIFS(X)$. Then, for every natural number n, $H^l_{\mathbf{m}}(\psi^n) = n \cdot H^l_{\mathbf{m}}(\psi)$.*

Proof. Suppose that P be a partition in $IVIFS(X)$. Then for every $n \in \mathbb{N}$, we have:

$$\begin{aligned} H^l_{\mathbf{m}}(\psi^n, \bigvee_{i=0}^{n-1} \psi^i(\mathcal{F})) &= \lim_{k\to\infty} \frac{1}{k} H^l_{\mathbf{m}}(\bigvee_{j=0}^{k-1} (\psi^n(\mathcal{F}))^j)(\bigvee_{i=0}^{n-1} \psi^i(\mathcal{F})) \\ &= \lim_{k\to\infty} \frac{1}{k} H^l_{\mathbf{m}}(\bigvee_{j=0}^{k-1} \bigvee_{i=0}^{n-1} (\psi^{nj+i}(\mathcal{F})) \\ &= \lim_{k\to\infty} \frac{kn}{k} \frac{1}{kn} H^l_{\mathbf{m}}(\bigvee_{i=0}^{kn-1} \psi^i(\mathcal{F})) = n.H^l_{\mathbf{m}}(\psi, \mathcal{F}). \end{aligned}$$

Therefore

$$\begin{aligned} n.H^l_{\mathbf{m}}(\psi) &= n.\sup\{H^l_{\mathbf{m}}(\psi, \mathcal{F});\ \mathcal{F} \text{ is a partition } in\ IVIFS(X)\} \\ &= \sup\{H^l_{\mathbf{m}}(\psi^n, \bigvee_{i=0}^{n-1} \psi^i(\mathcal{F})); \mathcal{F} \text{ is a partition } in\ IVIFS(X)\} \\ &\leq \sup\{H^l_{\mathbf{m}}(\psi^n, \mathcal{G}); \mathcal{G} \text{ is a partition } in\ IVIFS(X)\} = H^l_{\mathbf{m}}(\psi^n). \end{aligned}$$

On the other hand, Since $\mathcal{F} \preceq \bigvee_{i=0}^{n-1} \psi^i(\mathcal{F})$, by Theorem 5.8, we obtain:

$$H^l_{\mathbf{m}}(\psi^n, \mathcal{F}) \leq H^l_{\mathbf{m}}(\psi^n, \bigvee_{i=0}^{n-1} \psi^i(\mathcal{F})) = n.H^l_{\mathbf{m}}(\psi, \mathcal{F}).$$

Thus

$$\begin{aligned} H^l_{\mathbf{m}}(\psi^n) &= \sup\{H^l_{\mathbf{m}}(\psi^n, \mathcal{F}); \mathcal{F} \text{ is a partition in } IVIFS(X)\} \\ &\leq n.\sup\{H^l_{\mathbf{m}}(\psi, \mathcal{F}); \mathcal{F} \text{ is a partition in } IVIFS(X)\} \\ &= n.H^l_{\mathbf{m}}(\psi). \end{aligned}$$

□

6 Conclusion

This paper offers an in-depth examination of logical entropy and its associated measures in the context of interval-valued intuitionistic fuzzy sets (IVIFS). It begins by introducing the core concepts of logical entropy for partitions in IVIFS, alongside logical conditional entropy, and thoroughly explores their properties. The discussion then expands to cover logical mutual information, highlighting its importance in measuring shared information between fuzzy partitions. The concept of logical divergence is introduced next, providing a detailed analysis of state divergence in IVIFS and exploring the properties of these measures. The study concludes by applying logical entropy to dynamical systems in IVIFS, focusing on how evolving uncertainty and distinctions can be measured in such systems. Collectively, the paper presents a comprehensive theoretical framework for understanding and quantifying uncertainty in complex environments characterized by both fuzziness and intuitionistic hesitation, with potential applications in decision-making, control theory, and systems analysis.

Conflict of Interest: The authors declare no conflict of interest.

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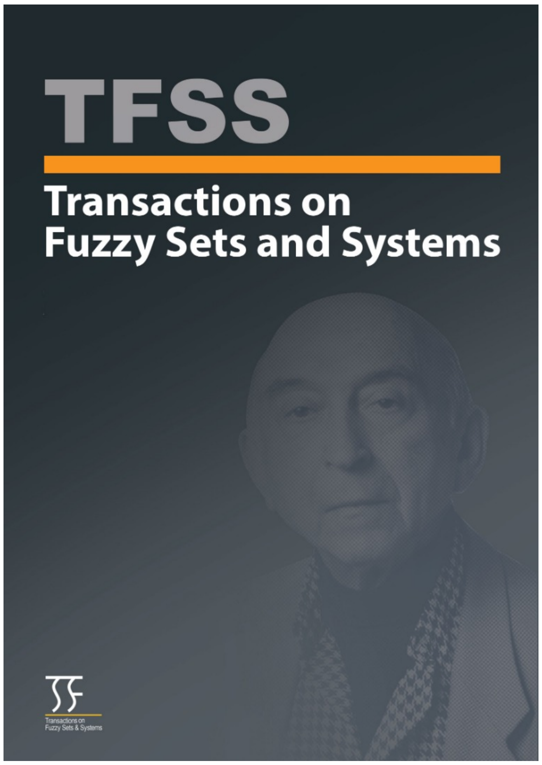

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A Method for Finding LR Fuzzy Eigenvectors of Real Symmetric Matrix †

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A Method for Finding LR Fuzzy Eigenvectors of Real Symmetric Matrix †

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Abstract. In this paper, the calculation methods of the real eigenvalues and LR fuzzy eigenvectors of clear real symmetry matrices are deeply considered. The original fuzzy feature problem is extended by using the arithmetic algorithm of LR fuzzy numbers into a simple feature problem with a high-order clear real symmetry matrix. We discuss two cases: (a) λ is a non-negative unknown eigenvalue; (b) λ is a negative unknown eigenvalue. We established two computational models and proposed an algorithm for finding the fuzzy eigenvectors of the true symmetry matrix. Some numerical examples are used to illustrate our proposed method.

AMS Subject Classification 2020: 15A30; 15A24
Keywords and Phrases: Fuzzy eigenvector, Fuzzy number, Fuzzy linear system, Matrix computation.

1 Introduction

Compared with the phenomenon of certainty, there are a large number of uncertain events exist and occur in real life. Fuzzy mathematics is born from this and happens to be one of the best tools to describe and analyze these uncertain phenomena. Some descriptive processes of motion and change often have uncertainty of all and part of the parameters. Sometime the uncertainty of the parameters is represented and computed by the fuzzy numbers. The concept of fuzzy numbers and their arithmetic operations were first coined and studied by Zadeh [1, 2], Dubois et al.[3] and Nahmias [4]. In the past half century, many scholars at home and abroad have paid more and more attention to a series of studies based on algebraic equations of fuzzy numbers. A new method is proposed to study fuzzy numbers and fuzzy number spatial structures by Puri and Ralescu [5] Goetschell et al.[6] and Wu Congxin et al.[7, 8].In the past two decades, more scholars have studied some more general and complex fuzzy linear systems based on Friedman et al. [9]'s 1998 embedding method to discuss a class of semi-fuzzy linear systems $A\widetilde{x} = \widetilde{b}$, such as dual fuzzy linear systems, generalized fuzzy linear systems, complex fuzzy linear systems, dual fully fuzzy linear systems, and general dual fuzzy linear systems, see [10, 11, 12, 13, 14].Through the joint efforts of many scholars, new theories and methods have been proposed in recent years, enriching fuzzy numbers and fuzzy linear systems [15, 16, 17]. The combination of fuzzy numbers and many mathematical problems has become a new research direction for many scholars. Guo et al. related linear matrix equations to fuzzy numbers and did some research [18, 19, 20, 14, 21, 22].

In recent years, the problem of fuzzy eigenvalues and eigenvectors has attracted the attention of many scholars. The reason is that the problem of finding the eigenvalues and eigenvectors of a matrix is widely used in problems in many fields such as engineering, management, physics and finance, but many of the

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parameters are uncertain. This uncertainty has been linked to fuzzy theory by scholars, and the methods of solving fuzzy eigenvalues and fuzzy eigenvectors have been used to solve the problem.The fuzzy eigenvalues and the generalized fuzzy eigenvalues of the form $\tilde{A}\tilde{x} = \tilde{\lambda}\tilde{B}\tilde{x}$ were studied by Buckley [23] and Chiao [24] using the same method, respectively. After that, Thoedorou al. [25] obtained a fuzzy eigenvector based on trigonometric fuzzy numbers by using a two-step method. In 2010, Tian founded the real matrix fuzzy eigenvector and studied the relationship between the real eigenvectors and the fuzzy eigenvectors [26], and he also studied the structure of the fuzzy eigenspace with some results. In 2013, Allahviranloo et al. [12] studied how to obtain the required price difference by deriving the maximum and minimum eigenvalues and general fuzzy eigenvalues under different conditions.

Most of the matrices involved in many studies today are symmetry matrices, and the role of symmetry matrices in many aspects is undoubtedly very large. We are not help guessing that which fuzzy vectors under symmetric linear transformation are simply just scaling This paper focuses on the study of the real eigenvalues and fuzzy eigenvectors of clear real matrices, which are based on the extension of LR fuzzy numbers and the universal application of symmetric matrices. Here's how this article is structured:

In Section 2, we will review some of the relevant definitions and arithmetic operations of LR fuzzy numbers, and in Section 3 we will deepen and extend the initial fuzzy vector eigenvalue problem to make it a simple eigenvalue problem for high-order clear real symmetry matrices, and we will propose an algorithm to solve the fuzzy eigenvectors of real symmetric matrices. In Section 4, we will give a few representative examples to illustrate. In Section 5, we draw some conclusions and conduct in-depth investigation and outlook.

2 Preliminaries

The concept of fuzzy numbers can be defined in some ways(see [3, 4, 1]).

Definition 2.1. *Let X be a non-empty set. Let's put a fuzzy set $\widetilde{A}$ in X like this*

$$\mu_{\widetilde{A}} : X \to [0, 1],$$

each of these elements $x \in X$, is associated with a real number in the closed interval $[0, 1]$, where the value $\mu_{\widetilde{A}}$ represents the degree of membership of x in fuzzy set $\widetilde{A}$, the function $\mu_{\widetilde{A}} : X \to [0, 1]$ is called the membership function of $\widetilde{A}$. A fuzzy set $\widetilde{A}$ is represented by the set of ordered pairs of element x and grade $\mu_{\widetilde{A}}$ which can be written as

$$\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x)) | x \in X\}.$$

Definition 2.2. *A fuzzy number is a fuzzy set like $t : R \to I = [0, 1]$ which satisfaction is as follows:*
(1) t is upper semi-continuous,
(2) t is fuzzy convex, i.e. $t(\lambda x + (1 - \lambda)y) \geq \min\{t(x), t(y)\}$ for all $x, y \in R, \lambda \in [0, 1]$,
(3) t is normal, i.e. there exists $x_0 \in R$ such that $u(x_0) = 1$,
(4) $suppt = \{x \in R \mid u(x) > 0\}$ is the support of the t, and its closure cl(suppt) is compact.
Let E^1 be the set of all fuzzy numbers on R.

Definition 2.3. *If a fuzzy number $\widetilde{t}$ satisfies the following conditions, then $\widetilde{t}$ is called a LR fuzzy number:*

$$\mu_{\widetilde{t}}(x) = \begin{cases} L(\frac{t-x}{\alpha}), & x \leq t, \quad \alpha > 0, \\ R(\frac{t-m}{\beta}), & x \geq t, \quad \beta > 0, \end{cases}$$

where t, α and β are called the mean value, left and right spreads of $\widetilde{t}$, respectively. The left shape function $L(\cdot)$, satisfies:

(1) $L(x) = L(-x)$,
(2) $L(0) = 1$ *and* $L(1) = 0$,
(3) $L(x)$ *is non increasing on* $[0, \infty)$.
Under the similar conditions, the right shape function $R(\cdot)$, *satisfies:*
(1) $R(x) = R(-x)$,
(2) $R(0) = 1$ *and* $R(1) = 0$,
(3) $R(x)$ *is non decreasing on* $(-\infty, 0]$.

So we can get a situation like this, when two LR fuzzy numbers $\widetilde{t} = (t, \alpha, \beta)_{LR}$ *and* $\widetilde{u} = (u, \gamma, \delta)_{LR}$ *are equal, if and only if* $t = u, \alpha = \gamma$ *and* $\beta = \delta$.

Definition 2.4. *For arbitrary LR fuzzy numbers* $\widetilde{t} = (t, \alpha, \beta)_{LR}$ *and* $\widetilde{u} = (u, \gamma, \delta)_{LR}$, *we have*

(1) Addition

$$\widetilde{t} + \widetilde{u} = (t, \alpha, \beta)_{LR} + (u, \gamma, \delta)_{LR} = (t + u, \alpha + \gamma, \beta + \delta)_{LR}.$$

(2) Subtraction

$$\widetilde{t} - \widetilde{u} = (t, \alpha, \beta)_{LR} - (u, \gamma, \delta)_{LR} = (t - u, \alpha - \delta, \beta - \gamma)_{LR}.$$

(3) Scalar multiplication

$$\lambda\widetilde{t} = \lambda(t, \alpha, \beta)_{LR} \cong \begin{cases} (\lambda t, \lambda\alpha, \lambda\beta)_{LR}, & \lambda \geq 0, \\ (\lambda t, -\lambda\beta, -\lambda\alpha)_{RL}, \lambda < 0. \end{cases}$$

2.1 Fuzzy Eigenvector of Real Matrix

Definition 2.5. *If the mean value of a LR fuzzy number is* 0 *and the left and right spread values are* α *and* β *where* $0 \leq \alpha, \beta < 1$, *this fuzzy number is called LR zero fuzzy number and denoted by* $\widetilde{0} = (0, \alpha, \beta)$.

A fuzzy vector $\widetilde{x} = (\widetilde{x}_i), i = 1, \cdots, n$ *is called a LR zero fuzzy vector, if each element* $\widetilde{x}_i$ *of* $\widetilde{x}$ *is a LR zero fuzzy number.*

Definition 2.6. *Let* A *to be a* $n \times n$ *real matrix. If the real number* λ *and the non zero fuzzy vector* $\widetilde{x}$ *satisfies the following linear system*

$$A\widetilde{x} = \lambda\widetilde{x}, \tag{2.1}$$

i.e.,

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{12} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \\ \vdots \\ \widetilde{x}_n \end{pmatrix} = \lambda \begin{pmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \\ \vdots \\ \widetilde{x}_n \end{pmatrix}, \tag{2.2}$$

in this case, λ *is a real eigenvalue of the real matrix* A *and the fuzzy eigenvector belonging to the real matrix* A *with the eigenvalue* λ *is* $\widetilde{x}$.

3 Finding the Fuzzy Eigenvectors

We mainly study the problem of how to obtain fuzzy eigenvectors of real matrices by calculation in this paper. We first assume that A is a symmetric matrix, which makes the problem and calculation become simple and universal. On this basis, we consider the case of eigenvalues λ which are non-negative and negative respectively.

3.1 Extended Models and Its Solution

On the basis of the LR fuzzy number multiplication algorithm (by Dubois et al.), we can get the following conclusions.

(a) When a non-negative eigenvalue of matrix A is λ.

Theorem 3.1. *Suppose A a real matrix. In the case of $\lambda \geq 0$, the fuzzy feature problem* (2.1) *can be extended into a clear linear system as follows:*

$$\begin{cases} Ax = \lambda x, \\ \begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix} \begin{pmatrix} x^l \\ x^r \end{pmatrix} = \lambda \begin{pmatrix} x^l \\ x^r \end{pmatrix}, \end{cases} \tag{3.1}$$

where

$$\widetilde{x} = (x, x^l, x^r), A = A^+ + A^-. \tag{3.2}$$

And the elements a_{ij}^+ of matrix A^+ and a_{ij}^- of matrix A^- are determined by this way: if $a_{ij} \geq 0, a_{ij}^+ = a_{ij}$ else $a_{ij}^+ = 0, 1 \leq i, j \leq n$; if $a_{ij} < 0, a_{ij}^- = a_{ij}$ else $a_{ij}^- = 0, 1 \leq i, j \leq n$.

Proof. Let $A = A^+ + A^-, \widetilde{x} = (x, x^l, x^r)$. The elements a_{ij}^+ of matrix A^+ and a_{ij}^- of matrix A^- are determined by this way: if $a_{ij} \geq 0, a_{ij}^+ = a_{ij}$ else $a_{ij}^+ = 0, 1 \leq i, j \leq n$; if $a_{ij} < 0, a_{ij}^- = a_{ij}$ else $a_{ij}^- = 0, 1 \leq i, j \leq n$.

Firstly

$$\begin{aligned} A\widetilde{x} &= (A^+ + A^-)(x, x^l, x^r) = (A^+x, A^+x^l, A^+x^r) + (A^-x, -A^-x^r, -A^-x^l) \\ &= (A^+x + A^-x, A^+x^l - A^-x^r, A^+x^r - A^-x^l), \end{aligned} \tag{3.3}$$

On the other hand,

$$\lambda\widetilde{x} = (\lambda x, \lambda x^l, \lambda x^r), \lambda \geq 0. \tag{3.4}$$

From $A\widetilde{x} = \lambda\widetilde{x}$, we have

$$\begin{cases} A^+x + A^-x = \lambda x, \\ A^+x^l - A^-x^r = \lambda x^l, \\ A^+x^r - A^-x^l = \lambda x^r. \end{cases} \tag{3.5}$$

By the matrix multiplication, the Eqs.(3.5) can be written as

$$\begin{cases} Ax = \lambda x, \\ \begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix} \begin{pmatrix} x^l \\ x^r \end{pmatrix} = \lambda \begin{pmatrix} x^l \\ x^r \end{pmatrix}, \end{cases}$$

where

$$\widetilde{x} = (x, x^l, x^r), A = A^+ + A^-.$$

The proof was completed. □

(b) When a negative eigenvalue of matrix A is λ.

Theorem 3.2. *Suppose A a real matrix. In the case of $\lambda < 0$, the fuzzy feature problem (*2.1*) can be extended into a clear linear system as follows:*

$$\begin{cases} Ax = \lambda x, \\ \begin{pmatrix} -A^- & A^+ \\ A^+ & -A^- \end{pmatrix} \begin{pmatrix} x^r \\ x^l \end{pmatrix} = \lambda \begin{pmatrix} x^r \\ x^l \end{pmatrix}, \end{cases} \tag{3.6}$$

where

$$\widetilde{x} = (x, x^l, x^r), A = A^+ + A^-. \tag{3.7}$$

And the elements a^+_{ij} of matrix A^+ and a^-_{ij} of matrix A^- are determined by this way: if $a_{ij} \geq 0, a^+_{ij} = a_{ij}$ else $a^+_{ij} = 0, 1 \leq i, j \leq n$; if $a_{ij} < 0, a^-_{ij} = a_{ij}$ else $a^-_{ij} = 0, 1 \leq i, j \leq n$.

Proof. Let $A = A^+ + A^-, \widetilde{x} = (x, x^l, x^r)$. The elements a^+_{ij} of matrix A^+ and a^-_{ij} of matrix A^- are determined by this way: if $a_{ij} \geq 0, a^+_{ij} = a_{ij}$ else $a^+_{ij} = 0, 1 \leq i, j \leq n$; if $a_{ij} < 0, ba^-_{ij} = a_{ij}$ else $a^-_{ij} = 0, 1 \leq i, j \leq n$.

Firstly

$$\begin{aligned} A\widetilde{x} &= (A^+ + A^-)(x, x^l, x^r) = (A^+x, A^+x^l, A^+x^r) + (A^-x, -A^-x^r, -A^-x^l) \\ &= (A^+x + A^-x, A^+x^l - A^-x^r, A^+x^r - A^-x^l), \end{aligned} \tag{3.8}$$

On the other hand,

$$\lambda\widetilde{x} = (\lambda x, -\lambda x^r, -\lambda x^l), \lambda < 0. \tag{3.9}$$

From $A\widetilde{x} = \lambda\widetilde{x}$, we have

$$\begin{cases} A^+x + A^-x = \lambda x, \\ A^+x^l - A^-x^r = -\lambda x^r, \\ A^+x^r - A^-x^l = -\lambda x^l. \end{cases} \tag{3.10}$$

By the matrix multiplication, the Eqs.(3.10) can be written as

$$\begin{cases} \qquad\qquad Ax = \lambda x, \\ \begin{pmatrix} -A^- & A^+ \\ A^+ & -A^- \end{pmatrix} \begin{pmatrix} x^r \\ x^l \end{pmatrix} = -\lambda \begin{pmatrix} x^r \\ x^l \end{pmatrix}, \end{cases}$$

where

$$\widetilde{x} = (x, x^l, x^r), A = A^+ + A^-.$$

The proof was completed. □

3.2 Solving the Extended Model

With the above preparations, let's consider the calculation of the model (3.1) and (3.9).

In the first step, under the condition of $Ax = \lambda x$, i.e., we need to compute all the eigenvalues and eigenvectors of the real matrix $A = A^+ + A^-$.

Then we solve the roots to the equation about λ

$$f(\lambda) = det(\lambda I - A) = 0, \tag{3.11}$$

and the nonzero solution of homogeneous group of linear equations

$$(\lambda I - A)x = O, \tag{3.12}$$

The above method can help us solve all the eigenvalues and eigenvectors of the real symmetry matrix A. Since the eigenvalues of the symmetry matrix A are all real numbers, we can sort these eigenvalues by magnitude. The assumptions are shown below

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{j-1} \geq 0 \geq \lambda_{j+1} \geq \cdots \geq \lambda_n, \tag{3.13}$$

and all eigenvectors belonging to the real symmetry matrix A are

$$x_1, x_2, \cdots, x_n, \tag{3.14}$$

each x_i is an eigenvector of the eigenvector of the real matrix A that belongs to the eigenvalue λ_i.

In the second step, we have obtained our eigenvalues, and we naturally start to solve the eigenvectors of the real matrix S.

We solve homogeneous group nonzero solutions of linear equations for every $\lambda_i, i = 1, 2, \cdots, j$ as follows:

$$(\lambda_i I - S)\begin{pmatrix} x^l \\ x^r \end{pmatrix} = O, \tag{3.15}$$

where

$$S = \begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix}.$$

We solve homogeneous group nonzero solutions of linear equations for every $\lambda_i, i = j+1, \cdots, n$ as follows:

$$(\lambda_i I + S)\begin{pmatrix} x^r \\ x^l \end{pmatrix} = O, \tag{3.16}$$

where

$$S = \begin{pmatrix} -A^- & A^+ \\ A^+ & -A^- \end{pmatrix}.$$

In the last step, we have the solution to the models (3.1) and (3.9) is as follows:

$$\begin{aligned} &\textit{eigenvalues} : \lambda_1, \lambda_2, \cdots, \lambda_n, \\ &\textit{eigenvectors} : \widehat{x}_1, \widehat{x}_2, \cdots, \widehat{x}_n, \end{aligned} \tag{3.17}$$

where $\widehat{x}_i = (x_i, x_i^l, x_i^r), i = 1, \cdots, n$.

Remark 3.3. *We know that the eigenvalues of symmetric positive-definite matrices are positive, then when matrix A is a symmetric positive-definite matrix*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0,$$

and the eigenvectors

$$\widehat{x}_1, \widehat{x}_2, \cdots, \widehat{x}_n$$

*of real matrix S are determined by the model (*3.1*).*

When matrix A is symmetric negative definite matrix, its eigenvalues are all negative.

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n < 0,$$

and the eigenvectors

$$\widehat{x}_1, \widehat{x}_2, \cdots, \widehat{x}_n$$

*of real matrix S are determined by the model (*3.9*).*

3.3 Fuzzy Eigenvector

But we find that the LR fuzzy number solution vector $\widehat{x}$ obtained from the above may still be inappropriate, except for $x^l \geq 0, x^r \geq 0$. Therefore, we give the definition of the LR fuzzy feature vector to the problem (2.1) as follows:

Definition 3.4. *Let $\widehat{x} = (x, x^l, x^r)$. If (x, x^l, x^r) is the minimal solution of the Eqs.(3.1) or (3.9),such that $x^l \geq 0, x^r \geq 0$, we define $\widetilde{x} = \widehat{x} = (x, x^l, x^r)$ is a strong LR fuzzy eigenvector of fuzzy eigen problem (2.1) . Otherwise, the $\widetilde{x} = (x, x^l, x^r)$ is defined as a weak LR fuzzy eigenvector of fuzzy eigen problem (2.1) given by*

$$\widetilde{x} = \widetilde{x}_i,$$

where

$$\widetilde{x}_i = \begin{cases} (x_i, x_i^l, x_i^r), & x_i^l > 0, \quad x_i^r > 0, \\ (x_i, 0, \max\{-x_i^l, x_i^r\}), & x_i^l < 0, \quad x_i^r > 0, \\ (x_i, \max\{x_i^l, -x_i^r\}, 0,), & x_i^l > 0, \quad x_i^r < 0, \\ (x_i, -x_i^l, -x_i^r), & x_i^l < 0, \quad x_i^r < 0. \end{cases} \quad i = 1, \ldots, n. \tag{3.18}$$

The solution matrix can be a strong LR fuzzy solution only if $\widetilde{x} = (x, x^l, x^r)$ is an LR fuzzy vector, that is, every element in $\widetilde{x} = (x, x^l, x^r)$ is an LR fuzzy number.

Here we give a specific algorithm for how to find fuzzy eigenvectors of real symmetric matrices.

Algorithm 3.1

Step 1: Decomposing the matrix A with $A = A^+ + A^-$.

Step 2: By calculating the equation $Ax = \lambda x$,i.e, all the eigenvalues and eigenvectors of the real symmetry matrix A are obtained,

$$eigenvalues : \lambda_1, \lambda_2, \cdots, \lambda_n,$$

$$eigenvectors : x_1, x_2, \cdots, x_n.$$

Step 3: Solving the left and right spread values of fuzzy eigenvectors of real matrix A.

If $\lambda_i \geq 0$, then we can calculate the fuzzy eigen problem (2.1) by

$$\begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix} \begin{pmatrix} x^l \\ x^r \end{pmatrix} = \lambda_i \begin{pmatrix} x^l \\ x^r \end{pmatrix},$$

If $\lambda < 0$, then we can calculate the fuzzy eigen problem ((2.1) by

$$\begin{pmatrix} -A^- & A^+ \\ A^+ & -A^- \end{pmatrix} \begin{pmatrix} x^r \\ x^l \end{pmatrix} = \lambda_i \begin{pmatrix} x^r \\ x^l \end{pmatrix},$$

Step 4: Arrange the fuzzy eigenvectors of the real symmetric matrix A that we derive, i.e,

$$eigenvalues : \lambda_1, \lambda_2, \cdots, \lambda_n,$$

$$eigenvectors : \widetilde{x}_1, \widetilde{x}_2, \cdots, \widetilde{x}_n,$$

where $\widetilde{x}_i = (x_i, x_i^l, x_i^r), i = 1, \cdots, n$.

Step 5: Judge the strong LR fuzzy feature vector and take it as

$$\widetilde{x} = (x_i, x_i^l, x_i^r)$$

or a weak LR fuzzy eigenvector

$$\widetilde{x}_i = \begin{cases} (x_i, x_i^l, x_i^r), & x_i^l > 0, \quad x_i^r > 0, \\ (x_i, 0, \max\{-x_i^l, x_i^r\}), & x_i^l < 0, \quad x_i^r > 0, \\ (x_i, \max\{x_i^l, -x_i^r\}, 0,), & x_i^l > 0, \quad x_i^r < 0, \\ (x_i, -x_i^l, -x_i^r), & x_i^l < 0, \quad x_i^r < 0. \end{cases} \quad i = 1, \ldots, n.$$

by the Definition 3.2.

Based on the above, we also need to discuss the existence conditions of strong fuzzy eigenvectors, so we re-analyze the equation (3.1) and (3.9).

Firstly, we can rewrite the equation (3.1) as

$$Sy = \lambda y, \tag{3.19}$$

where

$$S = \begin{pmatrix} A & O & O \\ O & A^+ & -A^- \\ O & -A^- & A^+ \end{pmatrix}, y = \begin{pmatrix} x \\ x^l \\ x^r \end{pmatrix}, \tag{3.20}$$

when $\lambda \geq 0$.

Also, we can rewrite the equation (3.9) as

$$Ty = (-\lambda)y, \tag{3.21}$$

where

$$T = \begin{pmatrix} -A & O & O \\ O & -A^- & A^+ \\ O & A^+ & -A^- \end{pmatrix}, y = \begin{pmatrix} x \\ x^r \\ x^l \end{pmatrix}, \tag{3.22}$$

when $\lambda < 0$.

Therefore, the matrix S is a higher order non-negative symmetric matrix if and only if matrix A is a non-negative symmetric matrix in equation (3.19). Similarly, matrix T is a higher order non-negative symmetric matrix if and only if A is an non-positive symmetric matrix in equation (3.21).

Theorem 3.5. [27] *We assume that $G \in R^{m\times m}$ is a non-negative matrix with the eigenvalue λ, and that there exists a non-negative vector $z \in R^m, z \geq 0$, which is subject to*

$$Gz = \lambda z.$$

Now, by using the generalized Perron theorem for non-negative matrices, we give a sufficient condition to prove that strong fuzzy eigenvectors of real symmetric matrices exist.

Theorem 3.6. *Suppose the crisp matrix S and matrix T, when matrix A is a non negative one in the Eqs.*(3.19) *or matrix A is a non positive one in the Eqs.(*3.21*), the strong LR fuzzy eigenvector must exist in the real symmetric matrix A.*

Proof. According to the structure of matrix S or T and the Theorem 3.3, the proof of Theorem 3.5 is straightforward. The proof is completed. □

4 Numerical Examples

Example 4.1. Consider the fuzzy eigenvector of the following real symmetric matrix

$$A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix}.$$

Let

$$A = A^+ + A^- = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & -4 \\ -2 & -4 & 0 \end{pmatrix}.$$

From $Ax = \lambda x$, i.e., for λ, we solve the root of the equation respect to λ

$$f(\lambda) = det(\lambda I - A) = 0,$$

and the nonzero solution of homogeneous group of linear equations

$$(\lambda I - A)x = O,$$

we can get

$$\lambda_1 = \lambda_2 = 1, \lambda_3 = 10,$$

$$x_1 = \begin{pmatrix} -0.2981 \\ -0.5963 \\ -0.7455 \end{pmatrix}, x_2 = \begin{pmatrix} 0.8944 \\ -0.4472 \\ 0.0000 \end{pmatrix}, x_3 = \begin{pmatrix} 0.3333 \\ 0.6667 \\ 0.0000 \end{pmatrix},$$

the above $x_i (i = 1, 2, 3)$ and $\lambda_i (i = 1, 2, 3)$ can be used as all the eigenvalues and eigenvectors of the real symmetric positive definite matrix A.

For $\lambda_i = 1 > 0, i = 1, 2$, we solve non zero solutions to the homogeneous systems of linear equation

$$(2I - S) \begin{pmatrix} x^l \\ x^r \end{pmatrix} = O, S = \begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix},$$

and obtain

$$\begin{pmatrix} x^l \\ x^r \end{pmatrix} = \begin{pmatrix} -0.4149 & 0.2981 \\ 0.2075 & 0.5963 \\ -0.5491 & 0,0000 \\ 0.6598 & 0.0000 \\ 0.2192 & 0.0000 \\ 0.0000 & -0.7454 \end{pmatrix}.$$

For $\lambda_3 = 10 > 0$, we solve non zero solutions to the homogeneous systems of linear equation

$$(10I - S) \begin{pmatrix} x^r \\ x^l \end{pmatrix} = O, S = \begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix},$$

and get

$$\begin{pmatrix} x^r \\ x^l \end{pmatrix} = \begin{pmatrix} 0.3333 \\ 0.66674 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ 0.6667 \end{pmatrix}.$$

Based on the above, we can get all the real eigenvalues and fuzzy eigenvectors of the real symmetric matrix A, i.e

$$\lambda_1 = \lambda_2 = 1, \lambda_3 = 10,$$

$$\widetilde{x}_1 = \begin{pmatrix} (-0.2981, 0.0000, 0.6598) \\ (-0.5963, 0.2075, 0.2192) \\ (-0.7455, 0.0000, 0.0000) \end{pmatrix}, \widetilde{x}_2 = \begin{pmatrix} (0.8944, 0.2981, 0.0000) \\ (0.4472, 0.5963, 0.0000) \\ (0.0000, 0.0000, 0.0000) \end{pmatrix},$$

$$\widetilde{x}_3 = \begin{pmatrix} (0.3333, 0.3333, 0.0000) \\ (0.6667, 0.6667, 0.0000) \\ (-0.6667, 0.0000, 0.6667) \end{pmatrix},$$

among them, the eigenvectors $\widetilde{x}_1$ and $\widetilde{x}_2$ corresponding to the eigenvalues $\lambda_{1,2}$ of the original real matrix A are two weak LR fuzzy eigenvectors, and the eigenvectors $\widetilde{x}_3$ corresponding to the eigenvalues λ_3 of the original real matrix A is a strong LR fuzzy eigenvector.

Example 4.2. Consider the fuzzy eigenvector of the following real matrix

$$A = \begin{pmatrix} 1 & -2 & 2 \\ -2 & -2 & 4 \\ 2 & 4 & -2 \end{pmatrix},$$

Let

$$A = A^+ + A^- = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 4 \\ 2 & 4 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

From $Ax = \lambda x$, i.e., we solve the roots of the equation respect to λ

$$f(\lambda) = det(\lambda I - A) = 0,$$

and the nonzero solution of homogeneous group of linear equations

$$(\lambda I - A)x = O,$$

and get

$$\lambda_1 = \lambda_2 = 2, \lambda_3 = -7,$$

$$x_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, x_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, x_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

For $\lambda_{1,2} = 2 > 0$, we solve non zero solutions to the homogeneous systems of linear equation

$$(2I - S)\begin{pmatrix} x^l \\ x^r \end{pmatrix} = O, S = \begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix},$$

and obtain

$$\begin{pmatrix} x^l \\ x^r \end{pmatrix} = \begin{pmatrix} 0.6667 & 0.0002 \\ -0.6668 & 0.5000 \\ 0.1665 & 0,5000 \\ -0.6667 & -0.0002 \\ 0.6668 & -0.5000 \\ -0.1665 & -0,5000 \end{pmatrix}.$$

For $\lambda_3 = -7 < 0$, we solve non zero solutions to the homogeneous systems of linear equation

$$(-7I - S)\begin{pmatrix} x^r \\ x^l \end{pmatrix} = O, S = \begin{pmatrix} -A^- & A^+ \\ A^+ & -A^- \end{pmatrix},$$

and obtain

$$\begin{pmatrix} x^r \\ x^l \end{pmatrix} = \begin{pmatrix} 0.2357 \\ 0.4714 \\ -0.4714 \\ -0.2357 \\ -0.4714 \\ 0.4714 \end{pmatrix}.$$

Through the above operation, we can obtain all the real eigenvalues and fuzzy eigenvectors of the real symmetric matrix A, which are respectively

$$\lambda_1 = \lambda_2 = 2, \lambda_3 = -7,$$

$$\widetilde{x}_1 = \begin{pmatrix} (-2, 0.6667, 0.0000) \\ (1, 0.0000, 0.6668) \\ (0, 0.1665, 0.0000) \end{pmatrix}, \widetilde{x}_2 = \begin{pmatrix} (2, 0.0002, 0.0000) \\ (0, 0.5000, 0.0000) \\ (1, 0.5000, 0.0000) \end{pmatrix},$$

$$\widetilde{x}_3 = \begin{pmatrix} (1, 0.0000, 0.2357) \\ (1, 0.0000, 0.4714) \\ (-2, 0.4714, 0.0000) \end{pmatrix}.$$

According to the Definition 3.2.,we can draw the conclusion that the eigenvectors $\widetilde{x}_1$ and $\widetilde{x}_2$ corresponding to the eigenvalues $\lambda_{1,2}$ of the original real matrix A are two weak LR fuzzy eigenvectors, and the eigenvector $\widetilde{x}_3$ corresponding to the eigenvalue λ_3 of the original real matrix A is also a weak LR fuzzy eigenvector.

5 Conclusion

In this paper, we study the LR fuzzy eigenvector problem of fuzzy matrices, and propose two computational models and algorithms for real symmetric matrices, which can solve the non-negative or negative LR fuzzy eigenvectors. Clear and straightforward mathematical derivations are used in the proof process, which is easy to understand. The practical application value of the algorithm is illustrated by example. We can consider extending the algorithm to the complex number field to solve more complex problems, and we can also try to further explore other properties of fuzzy linear systems, such as stability, so as to better enrich the theory of fuzzy linear systems.

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