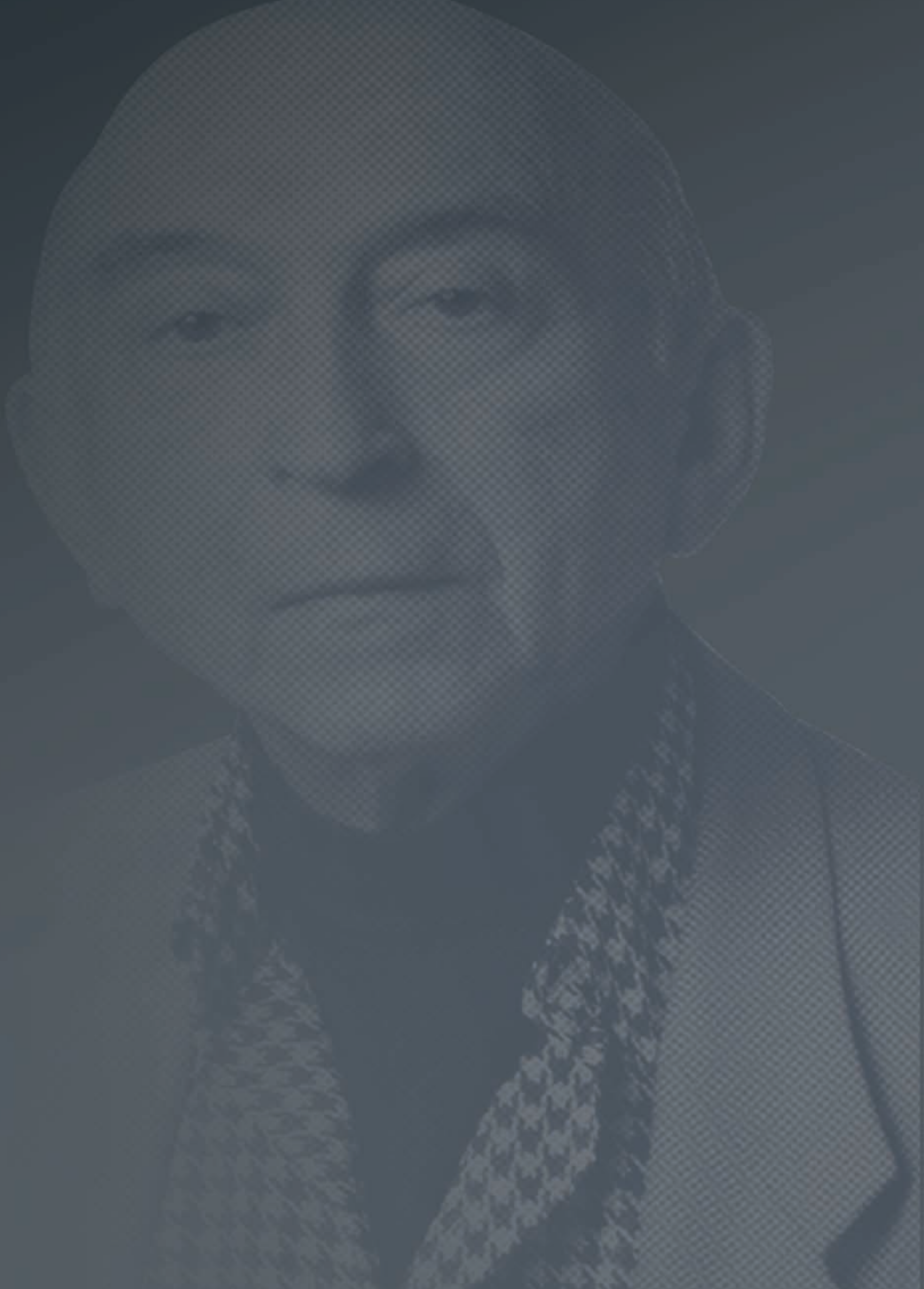
TFSS

Transactions on Fuzzy Sets and Systems

Vol. 2 No. 2 (November 2023)



Transactions on
Fuzzy Sets & Systems

Transactions on Fuzzy Sets and Systems

Transactions on Fuzzy Sets and Systems

About Journal

Transactions on Fuzzy Sets and Systems (TFSS) is an open access international scholarly journal. TFSS publishes new applied and pure articles related to fuzzy sets and systems as the semiannual journal and there is no charge for publishing an article in TFSS. All articles will be peer-review before publication. Manuscripts submitted to TFSS must be original and unpublished and not currently being considered for publication elsewhere.

The articles will be deposited immediately into the online repository, after the completion of the review processes. TFSS aims linking the ideas and techniques of fuzzy sets and systems with other disciplines to provide an international forum for refereed original research works in the theory and applications in all fields related to fuzzy science. Transactions on Fuzzy Sets and Systems (TFSS) is a semiannual international academic journal founded in 2022.

TFSS is an open access and free of charge and it follows the COPE publication ethics, also it follows the CC BY creative commons copyright license. TFSS aims to publish high-quality original articles that make a significant contribution to the research areas of both theoretical and applied mathematics in the field of fuzzy and all papers will be peer-review before publication.

Manuscripts submitted to the TFSS must be original and unpublished work and not currently being considered for publication elsewhere. TFSS aims to reflect the latest developments in fuzzy sets and systems and promote international academic exchanges. TFSS publishes 2 issues each year.

Aims and Scopes

TFSS aim is to present the development and new applications of fuzzy sets and systems to increase knowledge and help to publish the results of studies in the field of fuzzy logic, intelligent systems and other related topics. The scope of the journal includes fuzzy theory and its applications in every branch of science and technology. Also, TFSS publishes papers that use fuzzy in other fields, such as computer science and mathematics.

Instructions to Authors

The language of the journal is English. All accepted papers should also be prepared in LaTeX style and should be typed according to the style given in the journal homepage: https://tfss.journals.iau.ir/. Access to articles from this site is free: TFSS allows the author(s) to hold the copyright without restrictions.

We recommend that all authors review the "Guide to Authors" on the TFSS's website (https://tfss.journals.iau.ir/journal/authors.note).

Submission of Manuscripts

All papers should be prepared in LaTeX and the pdf file of the paper should be sent to the editorial office only through the electronic submission via "https://tfss.journals.iau.ir/". Manuscripts under consideration for the journal should not be published or submitted for publication elsewhere. TFSS doesn't have any submission and article processing charges.

We kindly recommend to all authors especially young ones to review publication ethics. There are many good sources for reminding publication ethics e.g., visit COPE.

Reviewers

Peer review and reviewers are at the heart of the academic publishing process. We would like to express our deepest gratitude for volunteering your time and expertise as a reviewer. Providing a voluntary service, such as peer review, under challenging circumstances is truly a dedication to your community and the advancement of research in your field. It is an honor for us to acknowledge your commitment by saying "thank you". On behalf of all our journal editorial members, we are indebted to the time you voluntarily dedicate to supporting our journal.

Cooperation with TFSS

Qualified specialists in various fields of Fuzzy Science issues who are willing to cooperate with the TFSS Journal as peer reviewers are welcomed and invited to send their CV along with their field(s) of interest to the Journal email address: tfssiauba@gmail.com.

Publication Office: Islamic Azad University, Bandar Abbas Branch, Bandar Abbas, Iran. Tel/fax: +98-76-3367-0243
Website: https://tfss.journals.iau.ir/, E-mails: tfssiauba@gmail.com; tfssiauba@yahoo.com, Online ISSN: 2821-0131

Transactions on Fuzzy Sets and Systems

(Vol.2, No.2, November 2023)

Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://tfss.journals.iau.ir/
Online ISSN: 2821-0131
Vol.2, No.2, (2023), 1-14
DOI: http://doi.org/10.30495/tfss.2023.1973510.1056

Article Type: Original Research Article

Sustainable Development Goals and Homelessness

John N. Mordeson , **Sunil Mathew*** , **Sujithra Puzhikunnath**

Abstract. The United Nation's Sustainable Development Goals encourage countries to solve many social problems. One of these problems is homelessness. We consider those goals which are most pertinent to homelessness according to [13]. We rank countries with respect to the achievement of these goals. We use fuzzy similarity measures to determine the degree of similarity between these rankings. We use three methods to rank the counties, namely, the Analytic Hierarchy Process, the Guiasu method, and the Yen method. Overall scores of categories in some basic research papers pertaining to Sustainable Development Goals were obtained by using multiplication of the scores of the category's targets. Multiplication was used to agree with the philosophy that in order for a high score to be obtained, all targets must have a high score. To support this philosophy in the decision process, we use the t-norms bounded difference, algebraic product, and standard intersection as experts. We also suggest a way the techniques used here can be extended to nonstandard analysis.

AMS Subject Classification 2020: 94D05; 03E72
Keywords and Phrases: Homelessness, Sustainable development goals, Analytic hierarchy process, Fuzzy similarity measures, Country rankings.

1 Introduction

The United Nation's Sustainable Development Goals provide a mechanism for encouraging nations to make progress towards shared goals. They generate collaboration, funding, definition, targeting, and measurement for many social problems such as poverty and sanitation for all, [15]. However, homelessness is not explicitly mentioned in the Sustainable Development Goals, [1]. The United Nations Human Settlement Program estimates that 1.6 billion people live in inadequate housing, and the best data available suggest that more than 100 million people have no housing at all. Related works can be seen in [2], [3] and [14].

In this paper, we consider four SDGs as seen by [13] as pertinent to homelessness. We rank countries with respect to their achievement of these goals. We then use fuzzy similarity measures to determine the degree of similarity between these rankings and the ranking of countries with respect to the number of people, per 10,000 who are homeless, [5]. We determine measures of similarity of these rankings using the techniques of fuzzy similarity relations developed in [8]. For the similarity measure M, if the value is between 0 and 0.2, we say the similarity is very low, between 0.2 and 0.4, we say the similarity is low, between 0.4 and 0.6 medium, between 0.6 and 0.8 high, between 0.8 and 1 very high. We find that the similarity of the four rankings is medium. A similar interpretation can be made for the similarity relation S. The rankings and similarity measures are done for various regions of the world. We find that the similarity measures are very high. The results can be found in detail in Sections 4, 5, and 6. We also determine the similarity measure between a

*Corresponding author: Sunil Mathew, Email: sm@nitc.ac.in, ORCID: 0000-0001-8478-9245
Received: 24 November 2022; **Revised:** 12 December 2022; **Accepted:** 26 January 2023; **Available Online**: 29 January 2023; **Published Online:** 7 November 2023.

How to cite: Mordeson JN, Mathew S and Sujithra P. Sustainable Development Goals and Homelessness. *Trans. Fuzzy Sets Syst.* 2023; 2(2): 1-14. DOI: http://doi.org/10.30495/tfss.2023.1973510.1056

ranking of a country's number of homelessness and the ranking of countries according to their achievement of the SDGs. We found that similarity ranged from medium to high depending on the region involved.

We use three methods to rank countries with respect to their achievement of the SDGs pertinent to homelessness. The Analytic Hierarchy Process (AHP) is a multicriteria decision method introduced in [11] and [12]. We consider a factor to be studied by the examination of subfactors of the factor. In our case, each expert $E_j, j = 1, ..., n$, assigns a number w_{ij} to each subfactor, $i = 1, ..., m$, of the factor, as to its importance with respect to the overarching goal. The row average, w_i, of each row of the matrix $[w_{ij}]$ is determined to form a matrix R whose ij-th element is w_i/w_j. The columns of R are then normalized in order to form the $m \times n$ matrix N whose ij-th element is $(w_i/w_j)/\sum_{i=1}^{m} w_i/w_j = w_i/\sum_{i=1}^{m} w_i, i = 1, ..., m$. The row vector yields the weights for the subfactors for the linear equation of the overarching goal, the dependent variable, in terms of the subfactors, the independent variables.

If the matrix W already has its columns normalized, then $w_i = \sum_{j=1}^{n} w_{ij}/n, i = 1, ..., m$. Since $\sum_{=1}^{m} w_{ij} = 1, j = 1, ..., n$, it follows that $\sum_{i=1}^{m} w_i = 1$. Hence $w_i/\sum_{i=1}^{m} w_i = w_i$, i.e., w_i is the weight for the i-th subfactor in the linear equation, $i = 1, ..., m$. It thus follows that if the columns of W are already normal, then the Guiasu method (with probabilistic assignments) and the analytic hierarchy process yield the same weights. However, in general, the Guiasu weights and the AHP weights can have quite different weights [9].

Yen's method addresses the issue of managing imprecise and vague information in evidential reasoning by combining the Dempster-Shafer theory with fuzzy set theory, [16]. Several researchers have extended the Dempster-Shafer theory to deal with vague information, but their extensions did not preserve an important principle that the belief and plausibility measures are lower and upper probabilities. Yen's method preserves this principle. Nevertheless, we use various measures of subsethood to determine belief functions. We do this to compare the results of the beliefs with Yen's method.

Yen's method is developed under the assumption that the focal elements are normalized. If the focal elements are not normal, he normalizes them.

We let $\mathbb{N}$ denote the positive integers. If X is a set, we let $\mathcal{FP}(X)$ denote the set of all fuzzy subsets of X. We let $\vee$ denote supremum or maximum and $\wedge$ denote infimum or minimum.

2 Preliminary Results

Proposition 2.1. *Let T denote an $m \times n$ matrix whose entries are from the closed interval $[0, 1]$. Let C_j denote the sum of the entries from column $j, j = 1, ..., n$. If $C_1 = ... = C_n$, then the AHP and the Guiasu weights are the same.*

Proof. Let $C = C_1 = ... = C_n$. Let R_i denote the sum of the elements in row $i, i = 1, ..., m$. Then in the AHP matrix, the row averages are $R_i/n, i = 1, ..., m$. Hence the coefficients for the AHP equation are $(R_i/n)/(R_1 + ... + R_m)/n = R_i/(R_1 + ... + R_m), i = 1, ..., m$. The Guiasu matrix is obtained from the AHP matrix by dividing each entry in its column by that column sum which by assumption is C. Thus the row average of the i-th row is $R_i/nC, i = 1, ..., m$. Hence the coefficients of the Guiasu equation is are $(R_i/nC)/(R_1 + ... + R_m)/nC) = R_i/(R_1 + ... + R_n), i = 1, ..., m$. □

Proposition 2.2. *Let M denote the $m \times n$ Guiasu matrix. Let m_j^* denote the maximum entry in column $j, j = 1, ..., n$. Suppose there exists m^* such that $m_1^* = ... = m_n^* = m^*$. Then the Guiasu and the Yen weights are the same.*

Proof. The entries of the columns of M add to 1. It follows that the row average column entries are $\frac{1}{n}R_i, i = 1, ..., m$, and so the Guiasu weights are $\frac{R_i}{R_1+...+R_m}, i = 1, ..., m$. The entries of the Yen matrix are $\frac{a_{ij}}{m^*} i = i, ..., m; j = 1, ..., n$. Hence the entries of the Yen row average column are $\frac{1}{n}\frac{R_i}{m^*}, i = 1, ..., n$. Hence the Yen weights are $(\frac{1}{n}\frac{R_i}{m^*})/(\frac{1}{n}\frac{R_1+...+R_m}{m^*}) = \frac{R_i}{R_1+...+R_n}, i = 1, ..., m$. □

Proposition 2.1 suggests that if the column sums are nearly equal, then the AHP and Guiasu weights will be nearly equal. We examine this in a nonstandard analysis setting. This examination suggests a possible extension of the paper to nonstandard analysis, [7]. First, we review some basic concepts from nonstandard analysis. Let $\mathbb{R}$ denote the real numbers. Let $\mathbb{R}^*$ denote the field of hyperreals which includes infinitesimal numbers and infinite numbers. Let $\mathbb{R}_{fin}$ denote the set of those elements of $\mathbb{R}^*$ which are not infinite. Then $\mathbb{R}_{fin}$ is a local ring with unique maximal ideal M, where M denotes the set of all infinitesimal elements, [7]. It follows that the relation $\approx$ defined on $\mathbb{R}_{fin}$ by for all $x, y \in \mathbb{R}_{fin}, x \approx y$ if and only if $x - y \in M$ is an equivalence relation.

Proposition 2.3. *Let* $a, c \in \mathbb{R}_{fin} \backslash M$ *(set difference) and* $b, d \in \mathbb{R}_{fin}$ *be such that* $a \approx b$ *and* $c \approx d$*. Then* $\frac{a}{c} \approx \frac{b}{d}$.

Proof. Since $a \approx b$ and $c \approx d$, there exists $m, m' \in M$ such that $b = a + m$ and $d = c + m'$. Thus $a(c + m') - c(a + m) = m' - m \in M$. Since $a, c \notin M$ and $\mathbb{R}_{fin} is$ a local ring, $\frac{1}{c} \in \mathbb{R}_{fin}$. Since M is an ideal in $\mathbb{R}_{fin}$, $\frac{a}{c}(c + m') - (a + m) \in M$. Now $\frac{1}{c+m'} \in \mathbb{R}_{fin}$ since $\mathbb{R}_{fin}$ is a local ring. Thus $\frac{a}{c} - \frac{a+m}{c+m'} \in M$. Hence $\frac{a}{c} - \frac{b}{d} \in M$. That is, $\frac{a}{c} \approx \frac{b}{d}$. □

To see how this applies to our situation, consider the situation where the $m \times n$ matrix has entries a_{ij} from $\mathbb{R}_{fin}$ and are positive. Let C_j denote the sum of the a_{ij} in column $j, j = 1, ..., n$. Suppose there exists $C \in R_{fin}$ and $\in_j \in M$, such that $C_j = C + \in_j, j = 1, ..., n$. Then the weights of the AHP equation are $\sum_{j=1}^{n} a_{ij} / \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}$. The weights of the corresponding Guiasu equation are $(\sum_{j=1}^{n} a_{ij}/(C + \in_j))/(\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}/(C + \in_j)) \approx (\sum_{j=1}^{n} a_{ij}/(C))/(\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}/(C)) = \sum_{j=1}^{n} a_{ij} / \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}.$, where we have $\approx$ holding by Proposition 2.3 and by noting that $C + \in_j \approx C$.

Similar comments concerning Proposition 2.2 can be made.

3 SDGs and Homelessness

In the following table, the G_i denote a particular Sustainable Development Goal. Here G_1 denotes End poverty in all its forms everywhere, G_8 denotes Promote sustained, inclusive and sustainable economic growth, full and productive employment and decent work for all, G_{10} denotes Reduce inequality within and among countries, and G_{11} denotes Make cities and human settlements inclusive, safe, resilient, and sustainable. The scores of the assessors were used to obtain an average for each category. Then these category averages were multiplied to obtain an overall average score for each target. Multiplication was used to agree with the philosophy that in order for a high score, all categories must have a high score. To support this philosophy, we use the t-norms bounded difference, algebraic product, and standard intersection. These t-norms are considered as experts when we apply the methods known as AHP, Guiasu and Yen. The entries of the Target values are taken from [10] and then divided by 2 so that the values will be in the closed interval $[0, 1]$. The Goal values are obtained by averaging the Target values. Applicability: In the opinion of the assessor is the target relevant, suitable and/or appropriate to developed countries; Implementable: In the opinion of the assessor will a reasonable allocation of resources result in the achievement of the goal/target in developed countries; Transformationalism: In the opinion of the assessor will the achievement of the goal/target require significant and additional policy action beyond what is currently in place and/or planned.

Table 1: t-norms as Decision Makers

Goal/Target	Applicable	Implementable	Transformative
G_1	0.575	0.85	0.325
1.4	0.5	0.85	0.15
1.5	0.65	0.85	0.5

Table 1: t-norms as Decision Makers (cont.)

Goal/Target	Applicable	Implementable	Transformative
G_8	0.85	0.85	0.65
8.5	0.85	0.85	0.65
G_{10}	0.667	0.9	0.617
10.2	0.5	0.85	0.5
10.3	0.5	0.85	0.5
10.4	1.0	1.0	0.85
G_{11}	0.5	0.85	0.5
11.1	0.5	0.85	0.5

The equations determined below are used to determine how well countries are doing in achieving the SDGs pertinent to homelessness. The entries in Table 2 below are obtained from Table 1. Recall that bounded difference is defined as $0 \vee (a+b-1)$ for all $a, b \in [0,1]$, see [4]. Consider G_1. For Bounded Difference, we get $0 \vee (0.575+0.85-1) = 0.425$ and $0 \vee (0.425+0.325-1) = 0$ or equivalently $0 \vee (0.425+0.85+0.325-2) = 0$.

Table 2: AHP Method

AHP	Bounded Difference	Algebraic Product	Standard Intersection	Row Average
G_1	0	0.159	0.325	0.161
G_8	0.350	0.470	0.650	0.490
G_{10}	0.184	0.370	0.617	0.390
G_{11}	0	0.213	0.500	0.238
Col Sum	0.534	1.212	2.092	1.279

$H_1 = 0.126G_1 + 0.383G_8 + 0.305G_{10} + 0.186G_{11}$.

Table 3: Guiasu Method

Guiasu	Bounded Difference	Algebraic Product	Standard Intersection	Row Average
G_1	0	0.130	0.155	0.095
G_8	0.655	0.388	0.311	0.451
G_{10}	0.345	0.306	0.295	0.315
G_{11}	0	0.176	0.239	0.138
Col Sum				0.999

$H_2 = 0.095G_1 + 0.451G_8 + 0.315G_{10} + 0.138G_{11}$.

Table 4 below is determined from Table 3 by dividing each entry in the column by the maximum entry of that column.

Table 4: Yen Method

Yen	Bounded Difference	Algebraic Product	Standard Intersection	Row Average
G_1	0	0.335	0.498	0.278

Table 4: Yen Method(cont.)

Yen	Bounded Difference	Algebraic Product	Standard Intersection	Row Average
G_8	1.000	1.000	1.000	1.000
G_{10}	0.527	0.789	0.949	0.755
G_{11}	0	0.454	0.768	0.407
Col Sum				2.440

$H_3 = 0.114G_1 + 0.410G_8 + 0.309G_{10} + 0.167G_{11}$.

4 Country Rankings

The values that state how well a country is achieving the *SDG*s are given in [15]. We do not present them here. These values are substituted into the variables $G_1, G_8, G_{10,}$ and G_{11} in the above equations to determine the values provided in Tables 5-10.

OECD

Table 5: OECD Ranks

Country	AHP / rank	Guiasu / rank	Yen / rank
Australia	0.820 / 24	0.814 / 25	0.818 / 25
Austria	0.865 / 13	0.859 / 13	0.863 / 13
Belgium	0.875 / 10	0.870 / 10	0.873 / 10
Canada	0.837 / 23	0.833 / 19	0.835 / 20
Chile	0.667 / 33	0.656 / 33	0.663 / 34
Czech Rep.	0.899 / 7	0.893 / 7	0.897 / 7
Denmark	0.909 / 4	0.902 / 4	0.906 / 4
Estonia	0.839 / 20	0.830 / 21	0.835 / 21
Finland	0.905 / 5	0.899 / 5	0.902 / 5
France	0.847 / 15	0.837 / 17	0.843 / 15
Germany	0.872 / 11	0.864 / 12	0.869 / 11
Greece	0.671 / 32	0.650 / 35	0.663 / 33
Hungary	0.830 / 23	0.822 / 23	0.827 / 24
Iceland	0.913 / 2	0.906 / 3	0.911 / 3
Ireland	0.877 / 9	0.875 / 9	0.876 / 9
Israel	0.753 / 29	0.747 / 30	0.750 / 30
Italy	0.775 / 26	0.770 / 27	0.773 / 27
Japan	0.838 / 21	0.840 / 15	0.839 / 17
Korea Rep.	0.868 / 12	0.867 / 11	0.868 / 12
Latvia	0.837 / 22	0.830 / 20	0.835 / 22
Lithuania	0.738 / 31	0.720 / 32	0.734 / 32
Luxembourg	0.839 / 19	0.820 / 24	0.831 / 23
Mexico	0.585 / 36	0.571 / 36	0.580 / 36
Netherlands	0.902 / 6	0.894 / 6	0.899 / 6
N. Zealand	0.841 / 17	0.839 / 16	0.840 / 16
Norway	0.891 / 8	0.883 / 8	0.888 / 8
Poland	0.759 / 28	0.754 / 29	0.757 / 29

Table 5: OECD Ranks (cont.)

Country	AHP / rank	Guiasu / rank	Yen / rank
Portugal	0.771 / 27	0.763 / 28	0.768 / 28
Slovak Rep.	0.840 / 18	0.834 / 18	0.838 / 19
Slovenia	0.913 / 3	0.911 / 2	0.913 / 2
Spain	0.788 / 25	0.774 / 26	0.783 / 26
Sweden	0.918 / 1	0.911 / 1	0.915 / 1
Switzerland	0.858 / 14	0.843 / 14	0.852 / 14
Turkey	0.665 / 35	0.655 / 34	0.661 / 35
U. K.	0.843 / 16	0.830 / 22	0.838 / 18
U. S.	0.750 / 30	0.743 / 31	0.747 / 31

Some countries in the following are not ranked due to insufficient data.

East and South Asia

Table 6: East and South Asia Ranks

Country	AHP / rank	Guiasu / rank	Yen / rank
Bangladesh	0.698 / 11	0.716 / 8	0.705 / 11
Bhutan	0.745 / 6	0.735 / 6	0.742 / 6
Brunei Dar			
Cambodia	0.770 / 4	0.757 / 5	0.765 / 4
China	0.779 / 3	0.779 / 3	0.779 / 3
India	0.653 / 14	0.669 / 13	0.659 / 14
Indonesia	0.616 / 16	0.616 / 16	0.616/ 16
Korean Dem. Rep.			
Lao PDR	0.709 / 9	0.713 / 9	0.710 / 9
Malaysia	0.717 / 8	0.706 / 11	0.713 / 8
Maldives	0.809 / 1	0.796 / 1	0.804 / 1
Mongolia	0.725 / 7	0.732 / 7	0.727 / 7
Myanmar	0.708 / 10	0.706 / 12	0.708 / 10
Nepal	0.695 / 12	0.712 / 10	0.702 / 12
Pakistan	0.621 / 15	0.623 / 15	0.622 / 15
Philippines	0.614 / 17	0.610 / 17	0.612 / 17
Singapore			
Sri Lanka	0.693 / 13	0.687 / 14	0.691 / 13
Thailand	0.767 / 5	0.758 / 4	0.763 / 5
Timor Leste			
Vietnam	0.787 / 2	0.780 / 2	0.784 / 2

Eastern Europe and Central Asia

Table 7: Eastern Europe and Central Asia Ranks

Country	AHP / rank	Guiasu / rank	Yen / rank
Afghanistan			
Albania	0.689 / 17	0.670 / 17	0.682 / 17

Table 7: Eastern Europe and Central Asia Ranks(cont.)

Country	AHP / rank	Guiasu / rank	Yen / rank
Andorra			
Armenia	0.635 / 21	0.623 / 21	0.630 / 21
Azerbaijan	0.750 / 12	0.733 / 13	0.743 / 12
Belarus	0.834 / 3	0.827 / 3	0.831 / 3
Bosnia & Herzegovina	0.748 / 13	0.734 / 12	0.743 / 13
Bulgaria	0.770 / 8	0.762 / 8	0.767 / 8
Croatia	0.778 / 6	0.762 / 7	0.775 / 6
Cyprus	0.792 / 5	0.783 / 5	0.787 / 5
Georgia	0.646 / 19	0.632 / 19	0.640 / 19
Kazakhstan	0.755 / 10	0.745 / 10	0.751 / 10
Kyrgz Rep.	0.778 / 7	0.766 / 6	0.773 / 7
Liecheristan			
Malta	0.903 / 1	0.902 / 1	0.903 / 1
Moldova	0.840 / 2	0.831 / 2	0.836 / 2
Monaco			
Montenegro	0.701 / 16	0.690 / 16	0.697 / 16
North Macedonia	0.643 / 20	0.629 / 20	0.638 / 20
Romania	0.675 / 18	0.664 / 18	0.67 / 18
Russian Federation	0.733 / 14	0.720 / 15	0.728 / 14
San Marino			
Serbia	0.753 / 11	0.745 / 11	0.750 / 11
Tajikistan	0.730 / 15	0.720 / 14	0.726 / 15
Turkmenistan			
Ukraine	0.831 / 4	0.821 / 4	0.827 / 4
Uzbekistan	0.770 / 9	0.762 / 9	0.767 / 9

Latin America and the Caribbean

Table 8: Latin America and Caribbean Ranks

Country	AHP / rank	Guiasu / rank	Yen / rank
Antigua & Barbuda			
Argentina	0.675 / 7	0.659 / 10	0.669 / 7
Bahamas			
Barbados			
Belize			
Bolivia	0.713 / 2	0.732 / 1	0.710 / 2
Brazil	0.610 / 13	0.599 / 13	0.606 / 13
Columbia	0.601 / 14	0.587 / 15	0.596 / 14
Costa Rica	0.695 / 4	0.679 / 5	0.689 / 4
Cuba			
Dominica			
Dominican Rep.	0.670 / 9	0.659 / 9	0.666 / 9
Ecuador	0.676 / 6	0.661 / 6	0.670 / 6
El Salvador	0.668 / 10	0.650 / 11	0.661 / 11

Table 8: Latin America and Caribbean Ranks(cont.)

Country	AHP / rank	Guiasu / rank	Yen / rank
Grenada			
Guatemala	0.599 / 15	0.589 / 14	0.595 / 15
Guyana			
Haiti	0.540 / 18	0.555 / 18	0.546 / 18
Honduras	0.584 / 16	0.580 / 16	0.582 / 16
Jamacia	0.708 / 3	0.695 / 3	0.703 / 3
Nicaragua	0.670 / 8	0.661 / 7	0.666 / 8
Panama	0.657 / 12	0.641 / 12	0.651 / 12
Paraguay	0.690 / 5	0.682 / 4	0.687 / 5
Peru	0.666 / 11	0.660 / 8	0.664 / 10
St Kitts and Nevis			
St. Lucia			
St Vincent and the Grenadines			
Suriname			
Uruguay	0.734 / 1	0.721 / 2	0.729 / 1
Venezuela	0.541 / 17	0.556 / 17	0.547 / 17

Middle East and North Africa

Table 9: Middle East and North Africa Ranks

Country	AHP / rank	Guiasu / rank	Yen / rank
Algeria	0.785 / 1	0.779 / 1	0.783 / 1
Bahrain			
Egypt	0.583 / 7	0.573 / 7	0.579 / 7
Iran	0.722 / 3	0.710 / 3	0.717 / 3
Iraq	0.740 / 2	0.738 / 2	0.739 / 2
Jordan	0.659 / 6	0.645 / 6	0.654 / 6
Kuwait			
Lebanon	0.707 / 4	0.701 / 4	0.705 / 4
Libya			
Morocco	0.700 / 5	0.688 / 5	0.695 / 5
Oman			
Qatar			
Saudi Arabia			
Syria			
Tunisia			
UAE			
Yemen			

Sub-Saharan Africa

Table 10: Sub-Saharan Africa Ranks

Country	AHP / rank	Guiasu / rank	Yen / rank
Angola	0.546 / 20	0.557 / 20	0.551 / 20

Table 10: Sub-Saharan Africa Ranks(cont.)

Country	AHP / rank	Guiasu / rank	Yen / rank
Benin	0.502 / 25	0.523 / 25	0.510 / 25
Botswana	0.468 / 33	0.455 / 35	0.463 / 33
Burkino Faso	0.641 / 5	0.662 / 4	0.649 / 5
Burundi	0.482 / 30	0.491 / 31	0.485 / 31
Cabo Verde	0.612 / 9	0.611 / 12	0.612 / 10
Cameroon	0.526 / 23	0.543 / 22	0.533 / 23
Central African Rep.	0.254 / 42	0.269 / 42	0.260 / 42
Chad	0.473 / 32	0.490 / 32	0.480 / 32
Comoros	0.544 / 21	0.530 / 24	0.538 / 21
Congo Dem. Rep.	0.494 / 28	0.517 / 26	0.503 / 26
Congo Rep.	0.427 / 38	0.438 / 38	0.431 / 38
Cote d'lvoire	0.594 / 14	0.609 / 13	0.560 / 19
Djibouti	0.601 / 12	0.599 / 17	0.600 / 12
Equatorial Guinea			
Eritrea			
Eswatini	0.357 / 40	0.342 / 41	0.351 / 40
Ethiopia	0.632 / 6	0.649 / 6	0.639 / 6
Gabon	0.592 / 15	0.588 / 19	0.591 / 17
Gambia	0.599 / 13	0.601 / 16	0.600 / 13
Ghana	0.652 / 3	0.665 / 5	0.657 / 3
Guinea	0.651 / 4	0.667 / 3	0.657 / 4
Guinea-Bissau			
Kenya	0.533 / 22	0.546 / 21	0.538 / 22
Lesotho	0.345 / 41	0.344 / 40	0.345 / 41
Liberia	0.584 / 18	0.617 / 10	0.597 / 14
Madagascar	0.453 / 35	0.470 / 33	0.460 / 34
Malawi	0.496 / 26	0.512 / 28	0.502 / 27
Mali	0.624 / 7	0.642 / 7	0.631 / 7
Mauritania	0.609 / 10	0.606 / 14	0.608 / 11
Mauritius	0.700 / 2	0.682 / 2	0.693 / 2
Mozambique	0.495 / 27	0.501 / 29	0.497 / 29
Namibia	0.460 / 34	0.450 / 36	0.456 / 35
Niger	0.606 / 11	0.630 / 9	0.616 / 9
Nigeria	0.358 / 39	0.382 / 39	0.367 / 39
Rwanda	0.481 / 31	0.498 / 30	0.488 / 30
Sao Tome & Principe	0.736 / 1	0.740 / 1	0.738 / 1
Senegal	0.586 / 17	0.604 / 15	0.593 / 16
Seychelles			
Sierra Leone	0.568 / 19	0.588 / 18	0.576 / 18
Somalia			
South Africa	0.442 / 37	0.431 / 37	0.438 / 37
South Sudan			
Sudan	0.523 / 24	0.535 / 23	0.528 / 24
Tanzania	0.616 / 8	0.635 / 8	0.624 / 8
Togo	0.490 / 29	0.519 / 27	0.501 / 28

Table 10: Sub-Saharan Africa Ranks(cont.)

Country	AHP / rank	Guiasu / rank	Yen / rank
Uganda	0.587 / 16	0.612 / 11	0.597 / 15
Zambia	0.443 / 36	0.456 / 34	0.448 / 36
Zimbabwe			

5 Fuzzy Similarity Measures and Conclusions

In this section, we briefly consider the fuzzy similarity measures we will be using.

Definition 5.1. *Let S be a function of $\mathcal{FP}(X)\times\mathcal{FP}(X)$ into $[0,1]$. Then S is called a* ***fuzzy similarity measure*** *on $\mathcal{FP}(X)$ if the following properties hold $\forall\mu,\nu,\rho\in\mathcal{FP}(X)$:*
(1) $S(\mu,\nu)=S(\nu,\mu)$;
(2) $S(\mu,\nu)=1$ *if and only if* $\mu=\nu$;
(3) *If* $\mu\subseteq\nu\subseteq\rho$, *then* $S(\mu,\rho)\le S(\mu,\nu)\wedge S(\nu,\rho)$;
(4) *If* $S(\mu,\nu)=0$, *then* $\forall x\in X,\mu(x)\wedge\nu(x)=0$.

We apply fuzzy similarity measures to rankings of a finite set. Suppose that X is a finite set with n elements. Let A be a one-to-0ne function of X into $\{1,2,...,n\}$. Then A is called a ranking of X. Define the fuzzy subset μ_A of X as follows: $\forall x\in X,\mu_A(x)=A(x)/n$. We wish to consider the similarity of two rankings of X by using fuzzy similarity measures. We use the two fuzzy similarity measures provided in the following Example.

Example 5.2. Let μ_A and μ_B be the fuzzy subsets of X associated with two rankings A and B, respectively. Then M and S below are fuzzy similarity measures.

$$M(\mu_A,\mu_B)=\frac{\sum_{x\in X}\mu_A(x)\wedge\mu_B(x)}{\sum_{x\in X}\mu_A(x)\vee\mu_B(x)};$$
$$S(\mu_A,\mu_B)=1-\frac{\sum_{x\in X}|\mu_A(x)-\mu_B(x)|}{\sum_{x\in X}(\mu_A(x)+\mu_B(x))}.$$

Theorem 5.3. (See [6]) *Let $n\in\mathbb{N}$ and*
(1) *Let n be even. Then the smallest value $M(\mu_A,\mu_B)$ can be is $\frac{n+2}{3n+2}$.*
(2) *Let n be odd. Then the smallest value $M(\mu_A,\mu_B)$ can be is $\frac{n+1}{3n-1}$.*
(3) *Let n be even. Then the smallest value $S(\mu_A,\mu_B)$ can be is $\frac{n/2+1}{n+1}$.*
(4) *Let n be odd. Then the smallest value $S(\mu_A,\mu_B)$ can be is $\frac{1}{2}+\frac{1}{2n}$.*

It follows that the quantity, the value of M minus the smallest value it can be, divided by the quantity 1 minus the smallest value M can be, is the percentage of the way M is from 0 to 1.

Let $n\in\mathbb{N},n\ge 2$, and let X be a set. Let $\mathcal{FP}^n(X)=\{(\mu_1,...,\mu_n)|\mu_i\in\mathcal{FP}(X),i=1,...,n\}$.

Definition 5.4. (See [8]) *Let $\widehat{S}$ be a function of $\mathcal{FP}^n(X)$ into $[0,1]$. Then $\widehat{S}$ is called an n-**dimensional fuzzy similarity measure** on $\mathcal{FP}(X)$ if the following properties hold:*
(1) $\widehat{S}(\mu_1,...,\mu_n)=\widehat{S}(\mu_{\pi(1)},...\mu_{\pi(n)})$ *for any permutation* π *of* $\{1,...,n\}$;
(2) $\widehat{S}(\mu_1,...,\mu_n)=1$ *if and only if* $\mu_1=...=\mu_n$;
(3) *If* $\mu_{i_1}\subseteq\mu_{i_2}\subseteq\mu_{i_3}$, *then* $\widehat{S}(...,\mu_{i_1},...,\mu_{i_3},...)\le\widehat{S}(...,\mu_{i_1},...,\mu_{i_2},...)\wedge\widehat{S}(...,\mu_{i_2},...,\mu_{i_3},...)$;
(4) *If* $\widehat{S}(\mu_1,...,\mu_n)=0$, *then for all* $x\in X$, *there exists* $i\in\{1,...,n\}$ *such that* $\mu_i(x)=0$.

Example 5.5. (See [8]) Let $\mu_1, ..., \mu_n$ be fuzzy subsets of X. Then $\widehat{M}$ and $\widehat{S}$ are n-similarity fuzzy similarity measures, where

$$\begin{aligned}\widehat{M}(\mu_1, ..., \mu_n) &= \frac{\sum_{x\in X}\mu_1(x)\wedge ...\wedge \mu_n(x)}{\sum_{x\in X}\mu_1(x)\vee ...\vee \mu_n(x)};\\ \widehat{S}(\mu_1, ..., \mu_n) &= 1-\frac{\sum_{x\in X}(\vee\{\mu_j(x)|j=1,...,n\}-\wedge\{\mu_j(x)|j=1,...,n\})}{\sum_{x\in X}(\vee\{\mu_j(x)|j=1,...,n\}+\wedge\{\mu_j(x)|j=1,...,n\})}.\end{aligned}$$

Suppose we consider n elements and that they have been ranked twice 1 through n with no ties. We wish to consider their rankings using the above similarity operations. We can accomplish this by mapping the elements to their rank divided by n. For example, let X denote a set of n elements and if x is ranked i, then we define the fuzzy subset μ of X by $\mu(x) = \frac{i}{n}$. Let μ and ν be two such fuzzy subsets of X. Then

$$\widehat{M}(\mu,\nu) = \frac{\sum\mu(x_i)\wedge\nu(x_i)}{\sum\mu(x_i)\vee\nu(x_i)} = \frac{\sum n\mu(x_i)\wedge n\nu(x_i)}{\sum n\mu(x_i)\vee n\nu(x_i)}.$$

Consequently, there is no loss in generality in assuming that we are measuring the similarity of two rankings using the integers, $1, ..., n$. The notion can be extended from 2 rankings to any finite number of rankings.

Let m and n be positive integers such that $2 \leq m \leq n$. Then there exist positive integers q and r such that $n = qm + r$, where $0 \leq r < m$.

Theorem 5.6. (See [8]) *The smallest value $\widehat{M}$ can be is* $\frac{m(\frac{(q+1)q}{2})+r(q+1)}{m\frac{2qn+q-q^2}{2}+r(n-q)}$.

Theorem 5.7. (See [8]) $\widehat{S} = \frac{2\widehat{M}}{1+\widehat{M}}$.

Corollary 5.8. (See [8]) *The smallest value $\widehat{S}$ can be is $\frac{2a}{1+a}$, where a is the smallest value $\widehat{M}$ can be.*

Let $\widehat{m} = 3$. It is shown in [8] that the values for $\widehat{M}$ and $\widehat{S}$ can be converted to the case where $m = 2$ by the following formulas

$$\begin{aligned}M &= \frac{5}{6}\widehat{M}+\frac{1}{6},\\ S &= \frac{3}{4}\widehat{S}+\frac{1}{4}.\end{aligned}$$

We next provide the similarity measures for the regions.μ_1, μ_2,and μ_3 denote AHP, Guiasu, and Yen, respectively.

For OECD, $\widehat{M}(\mu_1,\mu_2,\mu_3) = \frac{639}{686} = 0.931$ and $\widehat{S}(\mu_1,\mu_2,\mu_3) = 1-\frac{47}{1325} = 0.965$. Here $n = 36, m = 3, q = 12$, and $r = 0$. The smallest $\widehat{M}$ can be is $[\frac{m(q+1)q}{2}+r(q+1)]/[\frac{m(2qn+q-q^2)}{2}+r(n-q)] = \frac{(13)(12)}{2(12)(36)+12-144} = \frac{156}{732} = 0.213$. The smallest $\widehat{S}$ can be is $\frac{2(0.213)}{1+0.213} = 0.351$. Now $\frac{\widehat{M}-0.213}{1-0.213} = \frac{0.931-0.213}{1-0.213} = \frac{0.718}{0.787} = 0.912$ and $\frac{\widehat{S}-0.351}{1-0.351} = \frac{0.965-0.351}{1-0.351} = \frac{0.614}{0.649} = 0.946$.

For East and South Asia, $\widehat{M}(\mu_1,\mu_2,\mu_3) = \frac{146}{160} = 0.9125$ and $\widehat{S}(\mu_1,\mu_2,\mu_3) = 1-\frac{14}{306} = 0.954$. Here $n = 17, m = 3, q = 5$, and $r = 2$. The smallest $\widehat{M}$ can be is $[\frac{m(q+1)q}{2}+r(q+1)]/[\frac{m(2qn+q-q^2)}{2}+r(n-q)] = \frac{45+12}{225+24} = 0.229$. The smallest $\widehat{S}$ can be is $\frac{2(0.229)}{1+0.229} = 0.373$. Now $\frac{\widehat{M}-0.229}{1-0.229} = \frac{0.912-0.229}{1-0.229} = \frac{0.683}{0.771} = 0.886$ and $\frac{\widehat{S}-0.373}{1-0.373} = \frac{0.954-0.373}{1-0.373} = \frac{0.581}{0.627} = 0.927$.

For Eastern Europe and Central Asia, $\widehat{M}(\mu_1,\mu_2,\mu_3) = \frac{228}{234} = 0.974$ and $\widehat{S}(\mu_1,\mu_2,\mu_3) = 1 - \frac{6}{462} = 0.987$. Here $n = 21, m = 3, q = 7$, and $r = 0$. The smallest $\widehat{M}$ can be is $[\frac{m(q+1)q}{2} + r(q+1)]/[\frac{m(2qn+q-q^2)}{2} + r(n-q)] = \frac{8(7)}{14(21)+7-49} = \frac{56}{252} = 0.222$. The smallest $\widehat{S}$ can be is $\frac{2(0.222)}{1+0.222} = 0.363$. Now $\frac{\widehat{M}-0.222}{1-0.222} = \frac{0.974-0.222}{1-0.222} = \frac{0.752}{0.778} = 0.967$ and $\frac{\widehat{S}-0.363}{1-0.363} = \frac{0.987-0.363}{1-0.363} = \frac{0.624}{0.637} = 0.980$.

For Latin America and the Caribbean, $\widehat{M}(\mu_1,\mu_2,\mu_3) = \frac{164}{178} = 0.921$ and $\widehat{S}(\mu_1,\mu_2,\mu_3) = 1 - \frac{14}{342} = 0.959$. Here $n = 18, m = 3, q = 6$, and $r = 0$. The smallest $\widehat{M}$ can be is $[\frac{m(q+1)q}{2} + r(q+1)]/[\frac{m(2qn+q-q^2)}{2} + r(n-q)] = \frac{7(6)}{216-30} = 0.226$. The smallest $\widehat{S}$ can be is $\frac{2(0.226)}{1+0.226} = 0.367$. Now $\frac{\widehat{M}-0.226}{1-0.266} = \frac{0.921-0.226}{1-0.226} = \frac{0.695}{0.774} = 0.898$ and $\frac{\widehat{S}-0.367}{1-0.367} = \frac{0.959-0.367}{1-0.367} = \frac{0.592}{0.633} = 0.935$.

For Middle East and North Africa, there wasn't sufficient data available.

For Sub-Sahran Africa, $\widehat{M}(\mu_1,\mu_2,\mu_3) = \frac{869}{942} = 0.923$ and $\widehat{S}(\mu_1,\mu_2,\mu_3) = 1 - \frac{73}{1811} = 0.960$. Here $n = 42, m = 3, q = 13$, and $r = 0$. The smallest $\widehat{M}$ can be is $[\frac{m(q+1)q}{2} + r(q+1)]/[\frac{m(2qn+q-q^2)}{2} + r(n-q)] = \frac{15(14)}{1176-182} = \frac{210}{994} = 0.211$. The smallest $\widehat{S}$ can be is $\frac{2(0.211)}{1+0.221} = 0.346$. Now $\frac{\widehat{M}-0.211}{1-0.211} = \frac{0.923-0.211}{1-0.211} = \frac{0.712}{0.789} = 0.902$ and $\frac{\widehat{S}-0.346}{1-0.346} = \frac{0.960-0.346}{1-0.346} = \frac{0.614}{0.654} = 0.939$.

6 SDG Achievement vs Number of Homeless

In [5], the number of homeless people per country was given. We ranked the countries according to homeless per $10,000$. The fewer the homeless the higher the rank. We do not present the rankings here. We then found the similarity between this ranking and the ranking of countries according to their achievement of the SDGs given in the above tables.

For OECD, $M(SDG,H) = \frac{398}{724} = 0.550$ and $S(SDG,H) = 1 - \frac{328}{1122} = 0.708$. Here $n = 33$. The smallest M can be is $\frac{n+1}{3n-1} = \frac{34}{98} = 0.347$ and the smallest S can be is $\frac{1}{2} + \frac{1}{2n} = \frac{1}{2} + \frac{1}{66} = 0.515$. Now $\frac{M-0.347}{1-0.347} = \frac{0.550-0.347}{1-0.347} = \frac{0.203}{0.653} = 0.311$ and $\frac{S-0.515}{1-0515} = \frac{0.708-0.515}{1-0.515} = \frac{0.193}{0.485} = 0.398$.

For East and South Asia, $M(SDG,H) = \frac{34}{56} = 0.607$ and $S(SDG,H) = 1 - \frac{22}{90} = 0.756$. Here $n = 9$. The smallest M can be is $\frac{n+1}{3n-1} = \frac{10}{28} = 0.357$ and the smallest S can be is $\frac{1}{2} + \frac{1}{2n} = \frac{1}{2} + \frac{1}{18} = 0.556$. Now $\frac{M-0.357}{1-0.357} = \frac{0.607-0.357}{1-0.357} = \frac{0.250}{0.643} = 0.389$ and $\frac{S-0.556}{1-0.556} = \frac{0.756-0.556}{1-0.556} = \frac{0.200}{0.444} = 0.450$.

For Eastern Europe and Central Asia, $M(SDG,H) = \frac{26}{46} = 0.565$ and $S(SDG,H) = 1 - \frac{18}{72} = 0.750$. Here $n = 8$. The smallest M can be is $\frac{n+2}{3n+2} = \frac{10}{26} = 0.385$ and the smallest S can be is $\frac{n/2+1}{n+1} = \frac{5}{9} = 0.556$. Now $\frac{M-0.385}{1-0.385} = \frac{0.565-0.385}{1-0.385} = \frac{0.180}{0.615} = 0.293$ and $\frac{S-0.556}{1-0.556} = \frac{0.750-556}{1-0.556} = \frac{0.194}{0.444} = 0.437$.

For Latin America and the Caribbean, $M(SDG,H) = \frac{19}{23} = 0.828$ and $S(SDG,H) = 1 - \frac{4}{42} = 0.901$. Here $n = 6$. The smallest M can be is $\frac{n+2}{3n+2} = \frac{8}{20} = 0.400$ and the smallest S can be is $\frac{n/2+1}{n+1} = \frac{4}{7} = 0.571$. Now $\frac{M-0.400}{1-0.400} = \frac{0.828-0.400}{1-0.400} = \frac{0.428}{0.600} = 0.713$ and $\frac{S-0.571}{1-0.571} = \frac{0.901-0.571}{1-0.571} = \frac{0.330}{0.430} = 0.767$.

For the Middle East and North Africa, there wasn't sufficient data available.

For Sub-Saharan Africa, $M(SDG,H) = \frac{106}{166} = 0.639$ and $S(SDG,H) = 1 - \frac{60}{272} = 0.779$. Here $n = 16$. The smallest M can be is $\frac{n+2}{3n+2} = \frac{18}{50} = 0.360$ and the smallest S can be is $\frac{n/2+1}{n+1} = \frac{9}{17} = 0.529$. Now $\frac{M-0.360}{1-0.360} = \frac{0.639-0.360}{1-0.360} = \frac{0.279}{0.640} = 0.436$ and $\frac{S-0.529}{1-0.529} = \frac{0.779-0.529}{1-0.529} = \frac{0.250}{0.471} = 0.531$.

7 Conclusion

In this paper, we considered those Sustainable Development Goals which are most pertinent to homelessness. We ranked countries with respect to the achievement of these goals. We used fuzzy similarity measures to determine the degree of similarity between these rankings. We used three methods to rank the counties, namely, the Analytic Hierarchy Process, the Guiasu method, and the Yen method. We found that the similarity measures were very high. We also determined the similarity measure between a ranking of a country's number of homelessness and the ranking of countries according to their achievement of the SDGs. We found that similarity ranged from medium to high depending on the region involved.

Conflict of Interest: The authors declare that there are no conflict of interest.

References

[1] Casey C, Stazen L. Seeing homelessness through the Sustainable Development Goals. *European Journal of Homelessness.* 2021; 15(3): 63-71.

[2] Dempster AP. Upper and lower probabilities induced by muiltivaluedmappings. *Ann. Math. Stat.* 1967; 38(2): 325-528. DOI: http://doi.org/10.1214/aoms/1177698950

[3] Dempster AP. Upper and lower probability inferences based on a sample from finite univariant population. *Biometrica.* 1967; 54(3-4): 515-528. DOI: http://doi.org/10.2307/2335042

[4] Yaun B, Klir GJ. Fuzzy Sets and Fuzzy Logic, Theory and Applications. *Prentice Hall, Upper Saddle Creek River, NJ.* 1995.

[5] List of countries by homeless population, Wikipedia (https://en.wikipedia.org/wiki/List-of-sovereign-states-by-homeless-population).

[6] Mordeson JN, Mathew S. Studies in Systems, Decision and Control. *Mathematics of Uncertainty for Coping with World Challenges, Climate Change, World Hunger, Modern Slavery, Coronavirus, Human Trafficking.* 2021; 353. DOI: http://doi.org/10.1007/978-3-030-68684-0

[7] Mordeson JN, Mathew S. Fuzzy mathematics and nonstandard analysis: Application to the theory of relativity. *Transactions on Fuzzy Sets and Systems.* 2021; 1(1): 143-154. DOI: http://doi.org/10.30495/TFSS.2022.1953823.1014

[8] Mordeson JN, Mathew S. Similarity of Country Rankings on Sustainability Performance. *Transactions on Fuzzy Sets and Systems.* 2023; 2(1): to appear. DOI: http://doi.org/10.30495/tfss.2022.1963756.1042

[9] Wething HC, Mordeson JN, Clark TC. A fuzzy mathematical model of nuclear stability. *New Mathematics and Natural Computation.* 2010; 6: 119-140. DOI: http://doi.org/10.1142/51793005710001669

[10] Cutter A, Osborn D, Ullah F. Universal Sustainable Development Goals, Understanding the Transformational Challenge for Developed Countries. *Report of a Study by Stakeholder Forum.* 2015; 1-26.

[11] Saaty TL. A scaling method for priorities in hierarchical structure. *J. Math. Psychol.* 1977; 15: 234-281. DOI: http://doi.org/10.1016/0022-2496(77)90033-5

[12] Saaty TL. The Analytic Hierarchy Process. *McGraw Hill, New York.* 1980. p.11-21.

[13] Salcedo J. Homelessness and the SDGs, United Nations Settlements Programme. *UN Habitat.* 2019.

[14] Shafer G. A Mathematical Theory of Evidence. *Princeton University Press, Princeton.* 1976.

[15] Sustainable Development Report. Transformation to achieve the Sustainable Developments Goals, Includes the SDG Index and Dashboards. *Bertelamann Stiftung.* 2019.

[16] Yen J. Generalizing the Dempster-Shafer theory to fuzzy sets, In Wang, Z. Klir, G. J. (eds.) Fuzzy Measure Theory Ch. 7. *Plenum Press, New York.* 1992; 257-283. DOI: http://doi.org/10.1007/978-3-540-44792-4_21

John N. Mordeson
Department of Mathematics
Creighton University
USA
E-mail: mordes@creighton.edu

Sunil Mathew
Associate Professor
Department of Mathematics
National Institute of Technology
Calicut, India
E-mail: sm@nitc.ac.in

Sujithra Puzhikunnath
Research Scholar
Department of Mathematics
National Institute of Technology
Calicut, India
E-mail: sujisrigopal95@gmail.com

Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://tfss.journals.iau.ir/
Online ISSN: 2821-0131
Vol.2, No.2, (2023), 15-38
DOI: http://doi.org/10.30495/TFSS.2023.1979163.1063

Article Type: Original Research Article

Fuzzy Ordinary and Fractional General Sigmoid Function Activated Neural Network Approximation

George A. Anastassiou*

Abstract. Here we research the univariate fuzzy ordinary and fractional quantitative approximation of fuzzy real valued functions on a compact interval by quasi-interpolation general sigmoid activation function relied on fuzzy neural network operators. These approximations are derived by establishing fuzzy Jackson type inequalities involving the fuzzy moduli of continuity of the function, or of the right and left Caputo fuzzy fractional derivatives of the involved function. The approximations are fuzzy pointwise and fuzzy uniform. The related feed-forward fuzzy neural networks are with one hidden layer. We study in particular the fuzzy integer derivative and just fuzzy continuous cases. Our fuzzy fractional approximation result using higher order fuzzy differentiation converges better than in the fuzzy just continuous case.

AMS Subject Classification 2020: 26A33; 26E50; 41A17; 41A25; 41A30; 41A36; 47S40.
Keywords and Phrases: General sigmoid activation function, Neural network fuzzy fractional approximation, Fuzzy quasi-interpolation operator, Fuzzy modulus of continuity, Fuzzy derivative and fuzzy fractional derivative.

1 Introduction

The author in [1] and [2], see chapters 2-5, was the first to derive quantitative neural network approximations to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He studied there both the univariate and multivariate cases. The defining of these operators "bell-shaped" and "squashing" functions are assumed to be of compact support.

The author inspired by [23], continued his studies on neural networks approximation by introducing and using the proper quasi-interpolation operators of sigmoidal and hyperbolic tangent type which resulted in [10], [13] - [22], by treating both the univariate and multivariate cases.

Continuation of the author's works ([17], [18] and [19], Chapter 20) is this article where fuzzy neural network approximation based on a general sigmoid activation function is taken at the fractional and ordinary levels resulting in higher rates of approximation. We involve the fuzzy ordinary derivatives and the right and left Caputo fuzzy fractional derivatives of the fuzzy function under approximation and we establish tight fuzzy Jackson type inequalities. An extensive background is given on fuzzyness, fractional calculus and neural networks, all needed to present our work.

Our fuzzy feed-forward neural networks (FFNNs) are with one hidden layer. About neural networks in general study [29], [32], [33].

*Corresponding Author: George A. Anastassiou, Email: ganastss@memphis.edu, ORCID: 0000-0002-3781-9824
Received: 1 February 2023; **Revised**: 8 May 2023; **Accepted**: 11 May 2023; **Available Online**: 11 May 2023; **Published Online**: 7 November 2023.

How to cite: Anastassiou GA. Fuzzy Ordinary and Fractional General Sigmoid Function Activated Neural Network Approximation. *Trans. Fuzzy Sets Syst.* 2023; 2(2): 15-38. DOI: http://doi.org/10.30495/TFSS.2023.1979163.1063

2 Fuzzy Fractional Mathematical Analysis Basics

(see also [19], pp. 432-444)

We need the following basic background

Definition 2.1. (see [36]) Let $\mu : \mathbb{R} \to [0,1]$ with the following properties:

(i) is normal, i.e., $\exists\, x_0 \in \mathbb{R}$; $\mu(x_0) = 1$.

(ii) $\mu(\lambda x + (1-\lambda) y) \geq \min\{\mu(x), \mu(y)\}$, $\forall\, x, y \in \mathbb{R}$, $\forall\, \lambda \in [0,1]$ (μ is called a convex fuzzy subset).

(iii) μ is upper semicontinuous on $\mathbb{R}$, i.e. $\forall\, x_0 \in \mathbb{R}$ and $\forall\, \varepsilon > 0$, $\exists$ neighborhood $V(x_0) : \mu(x) \leq \mu(x_0) + \varepsilon$, $\forall\, x \in V(x_0)$.

(iv) The set $\overline{\operatorname{supp}(\mu)}$ is compact in $\mathbb{R}$ (where $\operatorname{supp}(\mu) := \{x \in \mathbb{R} : \mu(x) > 0\}$).

We call μ a fuzzy real number. Denote the set of all μ with $\mathbb{R}_{\mathcal{F}}$.

E.g. $\chi_{\{x_0\}} \in \mathbb{R}_{\mathcal{F}}$, for any $x_0 \in \mathbb{R}$, where $\chi_{\{x_0\}}$ is the characteristic function at x_0.

For $0 < r \leq 1$ and $\mu \in \mathbb{R}_{\mathcal{F}}$ define

$$[\mu]^r := \{x \in \mathbb{R} : \mu(x) \geq r\}$$

and

$$[\mu]^0 := \overline{\{x \in \mathbb{R} : \mu(x) \geq 0\}}.$$

Then it is well known that for each $r \in [0,1]$, $[\mu]^r$ is a closed and bounded interval on $\mathbb{R}$ ([28]).

For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r, \quad [\lambda \odot u]^r = \lambda [u]^r, \quad \forall\, r \in [0,1],$$

where

$[u]^r + [v]^r$ means the usual addition of two intervals (as substes of $\mathbb{R}$) and

$\lambda [u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see, e.g. [36]).

Notice $1 \odot u = u$ and it holds

$$u \oplus v = v \oplus u, \; \lambda \odot u = u \odot \lambda.$$

If $0 \leq r_1 \leq r_2 \leq 1$ then

$$[u]^{r_2} \subseteq [u]^{r_1}.$$

Actually $[u]^r = \left[u_-^{(r)}, u_+^{(r)}\right]$, where $u_-^{(r)} \leq u_+^{(r)}$, $u_-^{(r)}$, $u_+^{(r)} \in \mathbb{R}$, $\forall\, r \in [0,1]$.

For $\lambda > 0$ one has $\lambda u_\pm^{(r)} = (\lambda \odot u)_\pm^{(r)}$, respectively.

Define $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ by

$$D(u,v) := \sup_{r \in [0,1]} \max\left\{\left|u_-^{(r)} - v_-^{(r)}\right|, \left|u_+^{(r)} - v_+^{(r)}\right|\right\},$$

where

$$[v]^r = \left[v_-^{(r)}, v_+^{(r)}\right]; \;\; u, v \in \mathbb{R}_{\mathcal{F}}.$$

We have that D is a metric on $\mathbb{R}_{\mathcal{F}}$.

Then $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space, see [36], [37].

Here $\overset{*}{\sum}$ stands for fuzzy summation and $\widetilde{o} := \chi_{\{0\}} \in \mathbb{R}_{\mathcal{F}}$ is the neutral element with respect to $\oplus$, i.e.,

$$u \oplus \widetilde{0} = \widetilde{0} \oplus u = u, \;\; \forall\, u \in \mathbb{R}_{\mathcal{F}}.$$

Denote

$$D^*(f,g) = \sup_{x \in X \subseteq \mathbb{R}} D(f,g),$$

where $f, g : X \to \mathbb{R}_{\mathcal{F}}$.

We mention

Definition 2.2. Let $f : X \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$, X interval, we define the (first) fuzzy modulus of continuity of f by

$$\omega_1^{(\mathcal{F})}(f,\delta)_X = \sup_{x,y\in X,\ |x-y|\leq\delta} D(f(x), f(y)), \quad \delta > 0.$$

When $g : X \subseteq \mathbb{R} \to \mathbb{R}$, we define

$$\omega_1(g,\delta) = \omega_1(g,\delta)_X = \sup_{x,y\in X,\ |x-y|\leq\delta} |g(x) - g(y)|.$$

We define by $C_{\mathcal{F}}^U(\mathbb{R})$ the space of fuzzy uniformly continuous functions from $\mathbb{R} \to \mathbb{R}_{\mathcal{F}}$, also $C_{\mathcal{F}}(\mathbb{R})$ is the space of fuzzy continuous functions on $\mathbb{R}$, and $C_b(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$ is the fuzzy continuous and bounded functions.

We mention

Proposition 2.3. *([5]) Let $f \in C_{\mathcal{F}}^U(X)$. Then $\omega_1^{(\mathcal{F})}(f,\delta)_X < \infty$, for any $\delta > 0$.*

By [9], p. 129 we have that $C_{\mathcal{F}}^U([a,b]) = C_{\mathcal{F}}([a,b])$, fuzzy continuous functions on $[a,b] \subset \mathbb{R}$.

Proposition 2.4. *([5]) It holds*

$$\lim_{\delta\to 0}\omega_1^{(\mathcal{F})}(f,\delta)_X = \omega_1^{(\mathcal{F})}(f,0)_X = 0,$$

iff $f \in C_{\mathcal{F}}^U(X)$, where X is a compact interval.

Proposition 2.5. *([5]) Here $[f]^r = \left[f_-^{(r)}, f_+^{(r)}\right]$, $r \in [0,1]$. Let $f \in C_{\mathcal{F}}(\mathbb{R})$. Then $f_\pm^{(r)}$ are equicontinuous with respect to $r \in [0,1]$ over $\mathbb{R}$, respectively in $\pm$.*

Note 2.6. It is clear by Propositions 2.4, 2.5, that if $f \in C_{\mathcal{F}}^U(\mathbb{R})$, then $f_\pm^{(r)} \in C_U(\mathbb{R})$ (uniformly continuous on $\mathbb{R}$). Also if $f \in C_b(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$ implies $f_\pm^{(r)} \in C_b(\mathbb{R})$ (continuous and bounded functions on $\mathbb{R}$).

Proposition 2.7. *Let $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$. Assume that $\omega_1^{\mathcal{F}}(f,\delta)_X$, $\omega_1\left(f_-^{(r)},\delta\right)_X$, $\omega_1\left(f_+^{(r)},\delta\right)_X$ are finite for any $\delta > 0$, $r \in [0,1]$, where X any interval of $\mathbb{R}$.*

Then

$$\omega_1^{(\mathcal{F})}(f,\delta)_X = \sup_{r\in[0,1]} \max\left\{\omega_1\left(f_-^{(r)},\delta\right)_X, \omega_1\left(f_+^{(r)},\delta\right)_X\right\}.$$

Proof. Similar to Proposition 14.15, p. 246 of [9]. □

We need

Remark 2.8. ([3]). Here $r \in [0,1]$, $x_i^{(r)}, y_i^{(r)} \in \mathbb{R}$, $i = 1, ..., m \in \mathbb{N}$. Suppose that

$$\sup_{r\in[0,1]} \max\left(x_i^{(r)}, y_i^{(r)}\right) \in \mathbb{R}, \text{ for } i = 1, ..., m.$$

Then one sees easily that

$$\sup_{r\in[0,1]} \max\left(\sum_{i=1}^m x_i^{(r)}, \sum_{i=1}^m y_i^{(r)}\right) \leq \sum_{i=1}^m \sup_{r\in[0,1]} \max\left(x_i^{(r)}, y_i^{(r)}\right). \tag{1}$$

We need

Definition 2.9. Let $x, y \in \mathbb{R}_{\mathcal{F}}$. If there exists $z \in \mathbb{R}_{\mathcal{F}} : x = y \oplus z$, then we call z the *H*-difference on x and y, denoted $x - y$.

Definition 2.10. ([35]) Let $T := [x_0, x_0 + \beta] \subset \mathbb{R}$, with $\beta > 0$. A function $f : T \to \mathbb{R}_{\mathcal{F}}$ is H-differentiable at $x \in T$ if there exists an $f'(x) \in \mathbb{R}_{\mathcal{F}}$ such that the limits (with respect to D)

$$\lim_{h\to 0+} \frac{f(x+h) - f(x)}{h}, \quad \lim_{h\to 0+} \frac{f(x) - f(x-h)}{h} \tag{2}$$

exist and are equal to $f'(x)$.

We call f' the H-derivative or fuzzy derivative of f at x.

Above is assumed that the H-differences $f(x+h) - f(x)$, $f(x) - f(x-h)$ exists in $\mathbb{R}_{\mathcal{F}}$ in a neighborhood of x.

Higher order H-fuzzy derivatives are defined the obvious way, like in the real case.

We denote by $C^N_{\mathcal{F}}(\mathbb{R})$, $N \geq 1$, the space of all N-times continuously H-fuzzy differentiable functions from $\mathbb{R}$ into $\mathbb{R}_{\mathcal{F}}$, similarly is defined $C^N_{\mathcal{F}}([a,b])$, $[a,b] \subset \mathbb{R}$.

We mention

Theorem 2.11. *([30]) Let $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be H-fuzzy differentiable. Let $t \in \mathbb{R}$, $0 \leq r \leq 1$. Clearly*

$$[f(t)]^r = \left[f(t)^{(r)}_-, f(t)^{(r)}_+ \right] \subseteq \mathbb{R}.$$

Then $(f(t))^{(r)}_{\pm}$ are differentiable and

$$[f'(t)]^r = \left[\left(f(t)^{(r)}_- \right)', \left(f(t)^{(r)}_+ \right)' \right].$$

I.e.

$$(f')^{(r)}_{\pm} = \left(f^{(r)}_{\pm} \right)', \quad \forall\, r \in [0,1].$$

Remark 2.12. ([4]) Let $f \in C^N_{\mathcal{F}}(\mathbb{R})$, $N \geq 1$. Then by Theorem 2.11 we obtain

$$\left[f^{(i)}(t) \right]^r = \left[\left(f(t)^{(r)}_- \right)^{(i)}, \left(f(t)^{(r)}_+ \right)^{(i)} \right],$$

for $i = 0, 1, 2, ..., N$, and in particular we have that

$$\left(f^{(i)} \right)^{(r)}_{\pm} = \left(f^{(r)}_{\pm} \right)^{(i)},$$

for any $r \in [0,1]$, all $i = 0, 1, 2, ..., N$.

Note 2.13. ([4]) Let $f \in C^N_{\mathcal{F}}(\mathbb{R})$, $N \geq 1$. Then by Theorem 2.11 we have $f^{(r)}_{\pm} \in C^N(\mathbb{R})$, for any $r \in [0,1]$.

Items 11-13 are valid also on $[a,b]$.

By [9], p. 131, if $f \in C_{\mathcal{F}}([a,b])$, then f is a fuzzy bounded function.

We need also a particular case of the Fuzzy Henstock integral ($\delta(x) = \frac{\delta}{2}$), see [36].

Definition 2.14. ([27], p. 644) Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$. We say that f is Fuzzy-Riemann integrable to $I \in \mathbb{R}_{\mathcal{F}}$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any division $P = \{[u,v]; \xi\}$ of $[a,b]$ with the norms $\Delta(P) < \delta$, we have

$$D\left(\sum_P^{*} (v-u) \odot f(\xi), I \right) < \varepsilon.$$

We write

$$I := (FR) \int_a^b f(x)\, dx. \tag{3}$$

We mention

Theorem 2.15. *([28]) Let $f:[a,b]\to\mathbb{R}_{\mathcal{F}}$ be fuzzy continuous. Then*

$$(FR)\int_a^b f(x)\,dx$$

exists and belongs to $\mathbb{R}_{\mathcal{F}}$, furthermore it holds

$$\left[(FR)\int_a^b f(x)\,dx\right]^r=\left[\int_a^b (f)_-^{(r)}(x)\,dx,\int_a^b (f)_+^{(r)}(x)\,dx\right],$$

$\forall\ r\in[0,1]$.

For the definition of general fuzzy integral we follow [31] next.

Definition 2.16. Let (Ω,Σ,μ) be a complete σ-finite measure space. We call $F:\Omega\to R_{\mathcal{F}}$ measurable iff $\forall$ closed $B\subseteq\mathbb{R}$ the function $F^{-1}(B):\Omega\to[0,1]$ defined by

$$F^{-1}(B)(w):=\sup_{x\in B}F(w)(x)\,,\text{ all } w\in\Omega$$

is measurable, see [31].

Theorem 2.17. *([31]) For $F:\Omega\to\mathbb{R}_{\mathcal{F}}$,*

$$F(w)=\left\{\left(F_-^{(r)}(w)\,,F_+^{(r)}(w)\right)|0\leq r\leq 1\right\},$$

the following are equivalent

(1) F is measurable,

(2) $\forall\ r\in[0,1]$, $F_-^{(r)}$, $F_+^{(r)}$ are measurable.

Following [31], given that for each $r\in[0,1]$, $F_-^{(r)}$, $F_+^{(r)}$ are integrable we have that the parametrized representation

$$\left\{\left(\int_A F_-^{(r)}d\mu,\int_A F_+^{(r)}d\mu\right)|0\leq r\leq 1\right\}\tag{4}$$

is a fuzzy real number for each $A\in\Sigma$.

The last fact leads to

Definition 2.18. ([31]) A measurable function $F:\Omega\to\mathbb{R}_{\mathcal{F}}$,

$$F(w)=\left\{\left(F_-^{(r)}(w)\,,F_+^{(r)}(w)\right)|0\leq r\leq 1\right\}$$

is integrable if for each $r\in[0,1]$, $F_\pm^{(r)}$ is integrable, or equivalently, if $F_\pm^{(0)}$ is integrable.

In this case, the fuzzy integral of F over $A\in\Sigma$ is defined by

$$\int_A Fd\mu:=\left\{\left(\int_A F_-^{(r)}d\mu,\int_A F_+^{(r)}d\mu\right)|0\leq r\leq 1\right\}.$$

By [31], F is integrable iff $w\to\|F(w)\|_{\mathcal{F}}$ is real-valued integrable.

Here denote

$$\|u\|_{\mathcal{F}}:=D\left(u,\widetilde{0}\right),\ \ \forall\ u\in\mathbb{R}_{\mathcal{F}}.$$

We need also

Theorem 2.19. *([31]) Let $F, G : \Omega \to \mathbb{R}_{\mathcal{F}}$ be integrable. Then*

(1) Let $a, b \in \mathbb{R}$, then $aF + bG$ is integrable and for each $A \in \Sigma$,

$$\int_A (aF + bG)\, d\mu = a \int_A F d\mu + b \int_A G d\mu;$$

(2) $D(F, G)$ is a real- valued integrable function and for each $A \in \Sigma$,

$$D\left(\int_A F d\mu, \int_A G d\mu\right) \leq \int_A D(F, G)\, d\mu.$$

In particular,

$$\left\| \int_A F d\mu \right\|_{\mathcal{F}} \leq \int_A \|F\|_{\mathcal{F}}\, d\mu.$$

Above μ could be the Lebesgue measure, with all the basic properties valid here too.

Basically here we have

$$\left[\int_A F d\mu\right]^r = \left[\int_A F_-^{(r)} d\mu, \int_A F_+^{(r)} d\mu\right], \tag{5}$$

i.e.

$$\left(\int_A F d\mu\right)_{\pm}^{(r)} = \int_A F_{\pm}^{(r)} d\mu, \quad \forall\, r \in [0, 1].$$

We need

Definition 2.20. Let $\nu \geq 0$, $n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number), $f \in AC^n([a, b])$ (space of functions f with $f^{(n-1)} \in AC([a, b])$, absolutely continuous functions). We call left Caputo fractional derivative (see [24], pp. 49-52, [26], [34]) the function

$$D_{*a}^{\nu} f(x) = \frac{1}{\Gamma(n - \nu)} \int_a^x (x - t)^{n-\nu-1} f^{(n)}(t)\, dt, \tag{6}$$

$\forall\, x \in [a, b]$, where Γ is the gamma function $\Gamma(\nu) = \int_0^{\infty} e^{-t} t^{\nu-1} dt$, $\nu > 0$.

Notice $D_{*a}^{\nu} f \in L_1([a, b])$ and $D_{*a}^{\nu} f$ exists a.e. on $[a, b]$.

We set $D_{*a}^{0} f(x) = f(x)$, $\forall\, x \in [a, b]$.

Lemma 2.21. *([8]) Let $\nu > 0$, $\nu \notin \mathbb{N}$, $n = \lceil \nu \rceil$, $f \in C^{n-1}([a, b])$ and $f^{(n)} \in L_{\infty}([a, b])$. Then $D_{*a}^{\nu} f(a) = 0$.*

Definition 2.22. (see also [6], [25], [26]) Let $f \in AC^m([a, b])$, $m = \lceil \beta \rceil$, $\beta > 0$. The right Caputo fractional derivative of order $\beta > 0$ is given by

$$D_{b-}^{\beta} f(x) = \frac{(-1)^m}{\Gamma(m - \beta)} \int_x^b (\zeta - x)^{m-\beta-1} f^{(m)}(\zeta)\, d\zeta, \tag{7}$$

$\forall\, x \in [a, b]$. We set $D_{b-}^{0} f(x) = f(x)$. Notice that $D_{b-}^{\beta} f \in L_1([a, b])$ and $D_{b-}^{\beta} f$ exists a.e. on $[a, b]$.

Lemma 2.23. *([8]) Let $f \in C^{m-1}([a, b])$, $f^{(m)} \in L_{\infty}([a, b])$, $m = \lceil \beta \rceil$, $\beta > 0$, $\beta \notin \mathbb{N}$. Then $D_{b-}^{\beta} f(b) = 0$.*

Convention 2.24. We assume that

$$D_{*x_0}^{\beta} f(x) = 0, \ \text{ for } x < x_0, \tag{8}$$

and

$$D_{x_0-}^{\beta} f(x) = 0, \ \text{ for } x > x_0, \tag{9}$$

for all $x, x_0 \in [a, b]$.

We mention

Proposition 2.25. *([8]) Let $f \in C^n([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$. Then $D_{*a}^{\nu} f(x)$ is continuous in $x \in [a,b]$.*

Also we have

Proposition 2.26. *([8]) Let $f \in C^m([a,b])$, $m = \lceil \beta \rceil$, $\beta > 0$. Then $D_{b-}^{\beta} f(x)$ is continuous in $x \in [a,b]$.*

We further mention

Proposition 2.27. *([8]) Let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \beta \rceil$, $\beta > 0$ and*

$$D_{*x_0}^{\beta} f(x) = \frac{1}{\Gamma(m-\beta)} \int_{x_0}^{x} (x-t)^{m-\beta-1} f^{(m)}(t)\, dt, \tag{10}$$

for all $x, x_0 \in [a,b] : x \geq x_0$.

Then $D_{*x_0}^{\beta} f(x)$ is continuous in x_0.

Proposition 2.28. *([8]) Let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \beta \rceil$, $\beta > 0$ and*

$$D_{x_0-}^{\beta} f(x) = \frac{(-1)^m}{\Gamma(m-\beta)} \int_{x}^{x_0} (\zeta - x)^{m-\beta-1} f^{(m)}(\zeta)\, d\zeta, \tag{11}$$

for all $x, x_0 \in [a,b] : x \leq x_0$.

Then $D_{x_0-}^{\beta} f(x)$ is continuous in x_0.

We need

Proposition 2.29. *([8]) Let $g \in C([a,b])$, $0 < c < 1$, $x, x_0 \in [a,b]$. Define*

$$L(x, x_0) = \int_{x_0}^{x} (x-t)^{c-1} g(t)\, dt, \quad \text{for } x \geq x_0, \tag{12}$$

and $L(x, x_0) = 0$, for $x < x_0$.

Then L is jointly continuous in (x, x_0) on $[a,b]^2$.

We mention

Proposition 2.30. *([8]) Let $g \in C([a,b])$, $0 < c < 1$, $x, x_0 \in [a,b]$. Define*

$$K(x, x_0) = \int_{x_0}^{x} (\zeta - x)^{c-1} g(\zeta)\, d\zeta, \quad \text{for } x \leq x_0, \tag{13}$$

and $K(x, x_0) = 0$, for $x > x_0$.

Then $K(x, x_0)$ is jointly continuous from $[a,b]^2$ into $\mathbb{R}$.

Based on Propositions 2.29, 2.30 we derive

Corollary 2.31. *([8]) Let $f \in C^m([a,b])$, $m = \lceil \beta \rceil$, $\beta > 0$, $\beta \notin \mathbb{N}$, $x, x_0 \in [a,b]$. Then $D_{*x_0}^{\beta} f(x)$, $D_{x_0-}^{\beta} f(x)$ are jointly continuous functions in (x, x_0) from $[a,b]^2$ into $\mathbb{R}$.*

We need

Theorem 2.32. *([8]) Let $f : [a,b]^2 \to \mathbb{R}$ be jointly continous. Consider*

$$G(x) = \omega_1 (f(\cdot, x), \delta)_{[x,b]}, \tag{14}$$

$\delta > 0$, $x \in [a,b]$.

Then G is continuous in $x \in [a,b]$.

Also it holds

Theorem 2.33. *([8]) Let $f : [a,b]^2 \to \mathbb{R}$ be jointly continous. Then*

$$H(x) = \omega_1 (f(\cdot, x), \delta)_{[a,x]}, \tag{15}$$

$x \in [a,b]$, is continuous in $x \in [a,b]$, $\delta > 0$.

So that for $f \in C^m([a,b])$, $m = \lceil \beta \rceil$, $\beta > 0$, $\beta \notin \mathbb{N}$, $x, x_0 \in [a,b]$, we have that $\omega_1 \left(D_{*x}^{\beta} f, h \right)_{[x,b]}$, $\omega_1 \left(D_{x-}^{\beta} f, h \right)_{[a,x]}$ are continuous functions in $x \in [a,b]$, $h > 0$ is fixed.

We make

Remark 2.34. ([8]) Let $f \in C^{n-1}([a,b])$, $f^{(n)} \in L_\infty([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$, $\nu \notin \mathbb{N}$. Then we have

$$|D_{*a}^{\nu} f(x)| \leq \frac{\|f^{(n)}\|_\infty}{\Gamma(n-\nu+1)} (x-a)^{n-\nu}, \ \forall \, x \in [a,b]. \tag{16}$$

Thus we observe

$$\omega_1 (D_{*a}^{\nu} f, \delta) = \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} |D_{*a}^{\nu} f(x) - D_{*a}^{\nu} f(y)| \tag{17}$$

$$\leq \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} \left(\frac{\|f^{(n)}\|_\infty}{\Gamma(n-\nu+1)} (x-a)^{n-\nu} + \frac{\|f^{(n)}\|_\infty}{\Gamma(n-\nu+1)} (y-a)^{n-\nu} \right)$$

$$\leq \frac{2\|f^{(n)}\|_\infty}{\Gamma(n-\nu+1)} (b-a)^{n-\nu}. \tag{18}$$

Consequently

$$\omega_1 (D_{*a}^{\nu} f, \delta) \leq \frac{2\|f^{(n)}\|_\infty}{\Gamma(n-\nu+1)} (b-a)^{n-\nu}. \tag{19}$$

Similarly, let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \beta \rceil$, $\beta > 0$, $\beta \notin \mathbb{N}$, then

$$\omega_1 \left(D_{b-}^{\beta} f, \delta \right) \leq \frac{2\|f^{(m)}\|_\infty}{\Gamma(m-\beta+1)} (b-a)^{m-\beta}. \tag{20}$$

So for $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \beta \rceil$, $\beta > 0$, $\beta \notin \mathbb{N}$, we find

$$\sup_{x_0 \in [a,b]} \omega_1 \left(D_{*x_0}^{\beta} f, \delta \right)_{[x_0,b]} \leq \frac{2\|f^{(m)}\|_\infty}{\Gamma(m-\beta+1)} (b-a)^{m-\beta}, \tag{21}$$

and

$$\sup_{x_0 \in [a,b]} \omega_1 \left(D_{x_0-}^{\beta} f, \delta \right)_{[a,x_0]} \leq \frac{2\|f^{(m)}\|_\infty}{\Gamma(m-\beta+1)} (b-a)^{m-\beta}. \tag{22}$$

By Proposition 15.114, p. 388 of [7], we get here that $D_{*x_0}^{\beta} f \in C([x_0,b])$, and by [12] we obtain that $D_{x_0-}^{\beta} f \in C([a,x_0])$.

We need

Definition 2.35. ([11]) Let $f \in C_{\mathcal{F}}([a,b])$ (fuzzy continuous on $[a,b] \subset \mathbb{R}$), $\nu > 0$.

We define the Fuzzy Fractional left Riemann-Liouville operator as

$$J_a^{\nu} f(x) := \frac{1}{\Gamma(\nu)} \odot \int_a^x (x-t)^{\nu-1} \odot f(t)\, dt, \quad x \in [a,b], \tag{23}$$

$$J_a^0 f := f.$$

Also, we define the Fuzzy Fractional right Riemann-Liouville operator as

$$I_{b-}^{\nu} f(x) := \frac{1}{\Gamma(\nu)} \odot \int_x^b (t-x)^{\nu-1} \odot f(t)\, dt, \quad x \in [a,b], \tag{24}$$

$$I_{b-}^0 f := f.$$

We mention

Definition 2.36. ([11]) Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ is called fuzzy absolutely continuous iff $\forall\, \epsilon > 0$, $\exists\, \delta > 0$ for every finite, pairwise disjoint, family

$$(c_k, d_k)_{k=1}^n \subseteq (a,b) \quad \text{with} \quad \sum_{k=1}^n (d_k - c_k) < \delta$$

we get

$$\sum_{k=1}^n D(f(d_k), f(c_k)) < \epsilon. \tag{25}$$

We denote the related space of functions by $AC_{\mathcal{F}}([a,b])$.

If $f \in AC_{\mathcal{F}}([a,b])$, then $f \in C_{\mathcal{F}}([a,b])$.

It holds

Proposition 2.37. *([11])* $f \in AC_{\mathcal{F}}([a,b]) \Leftrightarrow f_{\pm}^{(r)} \in AEC([a,b])$, $\forall\, r \in [0,1]$ *(absolutely equicontinuous).*

We need

Definition 2.38. ([11]) We define the Fuzzy Fractional left Caputo derivative, $x \in [a,b]$.

Let $f \in C_{\mathcal{F}}^n([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$ ($\lceil \cdot \rceil$ denotes the ceiling). We define

$$D_{*a}^{\nu\mathcal{F}} f(x) := \frac{1}{\Gamma(n-\nu)} \odot \int_a^x (x-t)^{n-\nu-1} \odot f^{(n)}(t)\, dt \tag{26}$$

$$= \left\{ \left(\frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f^{(n)}\right)_-^{(r)}(t)\, dt, \right.\right.$$
$$\left.\left. \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f^{(n)}\right)_+^{(r)}(t)\, dt \right) | 0 \le r \le 1 \right\} =$$
$$= \left\{ \left(\frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_-^{(r)}\right)^{(n)}(t)\, dt, \right.\right.$$
$$\left.\left. \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_+^{(r)}\right)^{(n)}(t)\, dt \right) | 0 \le r \le 1 \right\}. \tag{27}$$

So, we get

$$\left[D_{*a}^{\nu\mathcal{F}} f(x)\right]^r = \left[\left(\frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_-^{(r)}\right)^{(n)}(t)\, dt,\right.\right.$$
$$\left.\left.\frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_+^{(r)}\right)^{(n)}(t)\, dt\right)\right], \quad 0 \le r \le 1. \tag{28}$$

That is

$$\left(D_{*a}^{\nu\mathcal{F}} f(x)\right)_{\pm}^{(r)} = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_{\pm}^{(r)}\right)^{(n)}(t)\, dt = \left(D_{*a}^{\nu}\left(f_{\pm}^{(r)}\right)\right)(x),$$

see [7], [24].

I.e. we get that

$$\left(D_{*a}^{\nu\mathcal{F}} f(x)\right)_{\pm}^{(r)} = \left(D_{*a}^{\nu}\left(f_{\pm}^{(r)}\right)\right)(x), \tag{29}$$

$\forall\, x \in [a,b]$, in short

$$\left(D_{*a}^{\nu\mathcal{F}} f\right)_{\pm}^{(r)} = D_{*a}^{\nu}\left(f_{\pm}^{(r)}\right), \quad \forall\, r \in [0,1]. \tag{30}$$

We need

Lemma 2.39. *([11]) $D_{*a}^{\nu\mathcal{F}} f(x)$ is fuzzy continuous in $x \in [a,b]$.*

We need

Definition 2.40. ([11]) We define the Fuzzy Fractional right Caputo derivative, $x \in [a,b]$.

Let $f \in C_{\mathcal{F}}^{n}([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$. We define

$$D_{b-}^{\nu\mathcal{F}} f(x) := \frac{(-1)^n}{\Gamma(n-\nu)} \odot \int_x^b (t-x)^{n-\nu-1} \odot f^{(n)}(t)\, dt$$
$$= \left\{\left(\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f^{(n)}\right)_-^{(r)}(t)\, dt,\right.\right.$$
$$\left.\left.\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f^{(n)}\right)_+^{(r)}(t)\, dt\right) | 0 \le r \le 1\right\} \tag{31}$$
$$= \left\{\left(\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_-^{(r)}\right)^{(n)}(t)\, dt,\right.\right.$$
$$\left.\left.\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_+^{(r)}\right)^{(n)}(t)\, dt\right) | 0 \le r \le 1\right\}.$$

We get

$$\left[D_{b-}^{\nu\mathcal{F}} f(x)\right]^r = \left[\left(\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_-^{(r)}\right)^{(n)}(t)\, dt,\right.\right.$$
$$\left.\left.\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_+^{(r)}\right)^{(n)}(t)\, dt\right)\right], \quad 0 \le r \le 1.$$

That is

$$\left(D_{b-}^{\nu\mathcal{F}} f(x)\right)_{\pm}^{(r)} = \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_{\pm}^{(r)}\right)^{(n)}(t)\, dt = \left(D_{b-}^{\nu}\left(f_{\pm}^{(r)}\right)\right)(x),$$

see [6].

I.e. we get that

$$\left(D_{b-}^{\nu\mathcal{F}} f\left(x\right)\right)_{\pm}^{(r)} = \left(D_{b-}^{\nu}\left(f_{\pm}^{(r)}\right)\right)(x), \tag{32}$$

$\forall\ x \in [a, b]$, in short

$$\left(D_{b-}^{\nu\mathcal{F}} f\right)_{\pm}^{(r)} = D_{b-}^{\nu}\left(f_{\pm}^{(r)}\right), \quad \forall\ r \in [0,1]. \tag{33}$$

Clearly,

$$D_{b-}^{\nu}\left(f_{-}^{(r)}\right) \leq D_{b-}^{\nu}\left(f_{+}^{(r)}\right), \quad \forall\ r \in [0,1].$$

We need

Lemma 2.41. *([11]) $D_{b-}^{\nu\mathcal{F}} f(x)$ is fuzzy continuous in $x \in [a, b]$.*

3 Real Neural Network Approximation

Here we follow [22].

Let $h : \mathbb{R} \to [-1, 1]$ be a general sigmoid function, such that it is strictly increasing, $h(0) = 0$, $h(-x) = -h(x)$, $h(+\infty) = 1$, $h(-\infty) = -1$. Also h is strictly convex over $(-\infty, 0]$ and strictly concave over $[0, +\infty)$, with $h^{(2)} \in C(\mathbb{R})$.

We consider the activation function

$$\psi(x) := \frac{1}{4}\left(h(x+1) - h(x-1)\right), \quad x \in \mathbb{R}, \tag{34}$$

As in [21], p. 45, we get that $\psi(-x) = \psi(x)$, thus ψ is an even function. Since $x + 1 > x - 1$, then $h(x+1) > h(x-1)$, and $\psi(x) > 0$, all $x \in \mathbb{R}$.

We see that

$$\psi(0) = \frac{h(1)}{2}. \tag{35}$$

Let $x > 1$, we have that

$$\psi'(x) = \frac{1}{4}\left(h'(x+1) - h'(x-1)\right) < 0,$$

by h' being strictly decreasing over $[0, +\infty)$.

Let now $0 < x < 1$, then $1 - x > 0$ and $0 < 1 - x < 1 + x$. It holds $h'(x-1) = h'(1-x) > h'(x+1)$, so that again $\psi'(x) < 0$. Consequently ψ is stritly decreasing on $(0, +\infty)$.

Clearly, ψ is strictly increasing on $(-\infty, 0)$, and $\psi'(0) = 0$.

See that

$$\lim_{x \to +\infty} \psi(x) = \frac{1}{4}\left(h(+\infty) - h(+\infty)\right) = 0, \tag{36}$$

and

$$\lim_{x \to -\infty} \psi(x) = \frac{1}{4}\left(h(-\infty) - h(-\infty)\right) = 0. \tag{37}$$

That is the x-axis is the horizontal asymptote on ψ.

Conclusion, ψ is a bell symmetric function with maximum

$$\psi(0) = \frac{h(1)}{2}.$$

We need

Theorem 3.1. *([22]) We have that*

$$\sum_{i=-\infty}^{\infty} \psi(x-i) = 1, \ \ \forall \ x \in \mathbb{R}. \tag{38}$$

Theorem 3.2. *([22]) It holds*

$$\int_{-\infty}^{\infty} \psi(x)\, dx = 1. \tag{39}$$

Thus $\psi(x)$ is a density function on $\mathbb{R}$.
We give

Theorem 3.3. *([22]) Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. It holds*

$$\sum_{\begin{cases} k=-\infty \\ : |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} \psi(nx-k) < \frac{\left(1 - h\left(n^{1-\alpha}-2\right)\right)}{2}. \tag{40}$$

Notice that

$$\lim_{n \to +\infty} \frac{\left(1 - h\left(n^{1-\alpha}-2\right)\right)}{2} = 0.$$

Denote by $\lfloor \cdot \rfloor$ the integral part of the number and by $\lceil \cdot \rceil$ the ceiling of the number.
We further give

Theorem 3.4. *([22]) Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)} < \frac{1}{\psi(1)}, \ \ \forall \ x \in [a,b]. \tag{41}$$

Remark 3.5. ([22]) i) We have that

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k) \neq 1, \tag{42}$$

for at least some $x \in [a,b]$.
ii) For large enough $n \in \mathbb{N}$ we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$.
In general, by Theorem 3.1, it holds

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k) \leq 1. \tag{43}$$

We give

Definition 3.6. ([22]) Let $f \in C([a,b])$ and $n \in \mathbb{N} : \lceil na \rceil \leq \lfloor nb \rfloor$. We introduce and define the linear neural network operator

$$A_n(f,x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \psi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)}, \ \ x \in [a,b]. \tag{44}$$

Clearly here $A_n(f,x) \in C([a,b])$. We present results for the pointwise and uniform convergence of $A_n(f,x)$ to $f(x)$ with rates.

We first give

Theorem 3.7. *([22]) Let $f \in C([a,b])$, $0<\alpha<1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in [a,b]$. Then*

i)

$$|A_n(f,x) - f(x)| \leq \frac{1}{\psi(1)}\left[\omega_1\left(f, \frac{1}{n^\alpha}\right) + \left(1 - h\left(n^{1-\alpha} - 2\right)\right)\|f\|_\infty\right] =: \rho, \tag{45}$$

and

ii)

$$\|A_n(f) - f\|_\infty \leq \rho. \tag{46}$$

We notice $\lim_{n\to\infty} A_n(f) = f$, pointwise and uniformly.

The speed of convergence is $\max\left(\frac{1}{n^\alpha}, \left(1 - h\left(n^{1-\alpha} - 2\right)\right)\right)$.

In the next we discuss high order neural network approximation by using the smoothness of f.

Theorem 3.8. *([22]) Let $f \in C^N([a,b])$, $n, N \in \mathbb{N}$, $0<\alpha<1$, $x \in [a,b]$ and $n^{1-\alpha} > 2$. Then*

i)

$$|A_n(f,x) - f(x)| \leq \frac{1}{\psi(1)}\left\{\sum_{j=1}^{N} \frac{\left\|f^{(j)}(x)\right\|}{j!}\left[\frac{1}{n^{\alpha j}} + \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right)}{2}(b-a)^j\right] + \right. \tag{47}$$

$$\left.\left[\omega_1\left(f^{(N)}, \frac{1}{n^\alpha}\right)\frac{1}{n^{\alpha N} N!} + \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right)\left\|f^{(N)}\right\|_\infty (b-a)^N}{N!}\right]\right\}$$

ii) assume further $f^{(j)}(x_0) = 0$, $j = 1, ..., N$, for some $x_0 \in [a,b]$, it holds

$$|A_n(f,x_0) - f(x_0)| \leq \frac{1}{\psi(1)}$$

$$\left\{\omega_1\left(f^{(N)}, \frac{1}{n^\alpha}\right)\frac{1}{n^{\alpha N} N!} + \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right)\left\|f^{(N)}\right\|_\infty (b-a)^N}{N!}\right\}, \tag{48}$$

and

iii)

$$\|A_n(f) - f\|_\infty \leq \frac{1}{\psi(1)}\left\{\sum_{j=1}^{N} \frac{\left\|f^{(j)}\right\|_\infty}{j!}\left[\frac{1}{n^{\alpha j}} + \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right)}{2}(b-a)^j\right] + \right.$$

$$\left.\left[\omega_1\left(f^{(N)}, \frac{1}{n^\alpha}\right)\frac{1}{n^{\alpha N} N!} + \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right)\left\|f^{(N)}\right\|_\infty (b-a)^N}{N!}\right]\right\}. \tag{49}$$

Again we obtain $\lim_{n\to\infty} A_n(f) = f$, pointwise and uniformly.

We present the following fractional approximation result by neural networks.

Theorem 3.9. *([22]) Let $\alpha > 0$, $N = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $f \in C^N([a,b])$, $0<\beta<1$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. Then* *i)*

$$\left| A_n(f,x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} A_n\left((\cdot - x)^j\right)(x) - f(x) \right| \leq$$

$$\frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1\left(D_{x-}^{\alpha}f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \omega_1\left(D_{*x}^{\alpha}f, \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left. \left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right) \left(\left\| D_{x-}^{\alpha}f \right\|_{\infty,[a,x]} (x-a)^{\alpha} + \left\| D_{*x}^{\alpha}f \right\|_{\infty,[x,b]} (b-x)^{\alpha} \right) \right\}, \tag{50}$$

ii) if $f^{(j)}(x) = 0$, *for* $j = 1, ..., N-1$, *we have*

$$|A_n(f,x) - f(x)| \leq \frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)}$$

$$\left\{ \frac{\left(\omega_1\left(D_{x-}^{\alpha}f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \omega_1\left(D_{*x}^{\alpha}f, \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left. \left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right) \left(\left\| D_{x-}^{\alpha}f \right\|_{\infty,[a,x]} (x-a)^{\alpha} + \left\| D_{*x}^{\alpha}f \right\|_{\infty,[x,b]} (b-x)^{\alpha} \right) \right\}, \tag{51}$$

iii)

$$|A_n(f,x) - f(x)| \leq (\psi(1))^{-1}$$

$$\left\{ \sum_{j=1}^{N-1} \frac{\left\| f^{(j)}(x) \right\|}{j!} \left\{ \frac{1}{n^{\beta j}} + (b-a)^j \left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right) \right\} + \right.$$

$$\frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1\left(D_{x-}^{\alpha}f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \omega_1\left(D_{*x}^{\alpha}f, \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left. \left. \left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right) \left(\left\| D_{x-}^{\alpha}f \right\|_{\infty,[a,x]} (x-a)^{\alpha} + \left\| D_{*x}^{\alpha}f \right\|_{\infty,[x,b]} (b-x)^{\alpha} \right) \right\} \right\}, \tag{52}$$

$\forall\ x \in [a,b]$,

and

iv)

$$\| A_n f - f \|_{\infty} \leq (\psi(1))^{-1}$$

$$\left\{ \sum_{j=1}^{N-1} \frac{\left\| f^{(j)} \right\|_{\infty}}{j!} \left\{ \frac{1}{n^{\beta j}} + (b-a)^j \left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right) \right\} + \right.$$

$$\frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\left(\sup_{x\in[a,b]} \omega_1\left(D_{x-}^{\alpha}f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \sup_{x\in[a,b]} \omega_1\left(D_{*x}^{\alpha}f, \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)(b-a)^{\alpha}\left(\sup_{x\in[a,b]}\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}+\sup_{x\in[a,b]}\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\right)\Bigg\}\Bigg\}. \tag{53}$$

Above, when $N=1$ *the sum* $\sum_{j=1}^{N-1}\cdot=0$.

As we see here we obtain fractionally type pointwise and uniform convergence with rates of $A_n\to I$ *the unit operator, as* $n\to\infty$.

4 Main Results: Approximation by General Fuzzy Neural Network Operators

Let $f\in C_{\mathcal{F}}([a,b])$ (fuzzy continuous functions on $[a,b]\subset\mathbb{R}$), $n\in\mathbb{N}$. We define the following Fuzzy Quasi-Interpolation Neural Network operator

$$A_n^{\mathcal{F}}(f,x)=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor *}f\left(\frac{k}{n}\right)\odot\frac{\psi(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)}, \tag{54}$$

$\forall\ x\in[a,b]$, see also (44).

The fuzzy sum in (54) is finite.

Let $r\in[0,1]$, we observe that

$$\left[A_n^{\mathcal{F}}(f,x)\right]^r=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left[f\left(\frac{k}{n}\right)\right]^r\left(\frac{\psi(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)}\right)=$$

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left[f_-^{(r)}\left(\frac{k}{n}\right),f_+^{(r)}\left(\frac{k}{n}\right)\right]\left(\frac{\psi(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)}\right)=$$

$$\left[\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f_-^{(r)}\left(\frac{k}{n}\right)\left(\frac{\psi(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)}\right),\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f_+^{(r)}\left(\frac{k}{n}\right)\left(\frac{\psi(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)}\right)\right] \tag{55}$$

$$=\left[A_n\left(f_-^{(r)},x\right),\ A_n\left(f_+^{(r)},x\right)\right].$$

We have proved that

$$\left(A_n^{\mathcal{F}}(f,x)\right)_{\pm}^{(r)}=\ A_n\left(f_{\pm}^{(r)},x\right), \tag{56}$$

respectively, $\forall\ r\in[0,1]$, $\forall\ x\in[a,b]$.

Therefore we get

$$D\left(A_n^{\mathcal{F}}(f,x),f(x)\right)=$$

$$\sup_{r\in[0,1]}\max\left\{\left|A_n\left(f_-^{(r)},x\right)-f_-^{(r)}(x)\right|,\left|A_n\left(f_+^{(r)},x\right)-f_+^{(r)}(x)\right|\right\}, \tag{57}$$

$\forall\ x \in [a, b]$.

We present our first fuzzy neural network approximation result.

Theorem 4.1. *Let $f \in C_{\mathcal{F}}([a, b])$, $0 < \alpha < 1$, $x \in [a, b]$, $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. Then*

1)

$$D\left(A_n^{\mathcal{F}}(f, x), f(x)\right) \leq \frac{1}{\psi(1)}\left[\omega_1^{(\mathcal{F})}\left(f, \frac{1}{n^\alpha}\right) + \left(1 - h\left(n^{1-\alpha} - 2\right)\right) D^*(f, \widetilde{o})\right] =: T_n, \tag{58}$$

and

2)

$$D^*\left(A_n^{\mathcal{F}}(f), f\right) \leq T_n. \tag{59}$$

We notice that $\lim_{n\to\infty} \left(A_n^{\mathcal{F}}(f)\right)(x) \overset{D}{\to} f(x)$, $\lim_{n\to\infty} A_n^{\mathcal{F}}(f) \overset{D^*}{\to} f$, pointwise and uniformly.

Proof. We have that $f_{\pm}^{(r)} \in C([a, b])$, $\forall\ r \in [0, 1]$. Hence by (45), we obtain

$$\left|A_n\left(f_{\pm}^{(r)}, x\right) - f_{\pm}^{(r)}(x)\right| \leq \frac{1}{\psi(1)}\left[\omega_1\left(f_{\pm}^{(r)}, \frac{1}{n^\alpha}\right) + \left(1 - h\left(n^{1-\alpha} - 2\right)\right)\left\|f_{\pm}^{(r)}\right\|_\infty\right] \tag{60}$$

(by Proposition 2.7 and $\left\|f_{\pm}^{(r)}\right\|_\infty \leq D^*(f, \widetilde{o})$)

$$\leq \frac{1}{\psi(1)}\left[\omega_1^{(\mathcal{F})}\left(f, \frac{1}{n^\alpha}\right) + \left(1 - h\left(n^{1-\alpha} - 2\right)\right) D^*(f, \widetilde{o})\right]. \tag{61}$$

Taking into account (57) the theorem is proved. □

We also give

Theorem 4.2. *Let $f \in C_{\mathcal{F}}^N([a, b])$, $N \in \mathbb{N}$, $0 < \alpha < 1$, $x \in [a, b]$, $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. Then*

1)

$$D\left(A_n^{\mathcal{F}}(f, x), f(x)\right) \leq \frac{1}{\psi(1)}$$

$$\left\{\sum_{j_*=1}^{N} \frac{D\left(f^{(j_*)}(x), \widetilde{o}\right)}{j_*!}\left[\frac{1}{n^{\alpha j_*}} + \left(\frac{1 - h\left(n^{1-\alpha} - 2\right)}{2}\right)(b-a)^{j_*}\right] +\right.$$

$$\left.\left[\omega_1^{(\mathcal{F})}\left(f^{(N)}, \frac{1}{n^\alpha}\right)\frac{1}{n^{\alpha N} N!} + \left(1 - h\left(n^{1-\alpha} - 2\right)\right) D^*\left(f^{(N)}, \widetilde{o}\right)\frac{(b-a)^N}{N!}\right]\right\}, \tag{62}$$

2) assume further that $f^{(j_*)}(x_0) = \widetilde{o}$, $j_* = 1, ..., N$, for some $x_0 \in [a, b]$, it holds

$$D\left(A_n^{\mathcal{F}}(f, x_0), f(x_0)\right) \leq$$

$$\frac{1}{\psi(1)}\left[\omega_1^{(\mathcal{F})}\left(f^{(N)}, \frac{1}{n^\alpha}\right)\frac{1}{n^{\alpha N} N!} + \left(1 - h\left(n^{1-\alpha} - 2\right)\right) D^*\left(f^{(N)}, \widetilde{o}\right)\frac{(b-a)^N}{N!}\right], \tag{63}$$

notice here the extremely high rate of convergence $n^{-(N+1)\alpha}$,

3)

$$D^*\left(A_n^{\mathcal{F}}(f), f\right) \leq \frac{1}{\psi(1)}$$

$$\left\{\sum_{j_*=1}^{N}\frac{D^*\left(f^{(j_*)},\widetilde{o}\right)}{j_*!}\left[\frac{1}{n^{\alpha j_*}}+\left(\frac{1-h\left(n^{1-\alpha}-2\right)}{2}\right)(b-a)^{j_*}\right]+\right.$$

$$\left.\left[\omega_1^{(\mathcal{F})}\left(f^{(N)},\frac{1}{n^{\alpha}}\right)\frac{1}{n^{\alpha N}N!}+\left(1-h\left(n^{1-\alpha}-2\right)\right)D^*\left(f^{(N)},\widetilde{o}\right)\frac{(b-a)^N}{N!}\right]\right\}. \tag{64}$$

Proof. Since $f\in C_{\mathcal{F}}^{N}([a,b])$, $N\geq 1$, we have that $f_{\pm}^{(r)}\in C^{N}([a,b])$, $\forall\, r\in[0,1]$. Using (47), we get

$$\left|A_n\left(f_{\pm}^{(r)},x\right)-f_{\pm}^{(r)}(x)\right|\leq\frac{1}{\psi(1)} \tag{65}$$

$$\left\{\sum_{j_*=1}^{N}\frac{\left|\left(f_{\pm}^{(r)}\right)^{(j_*)}(x)\right|}{j_*!}\left[\frac{1}{n^{\alpha j_*}}+\left(\frac{1-h\left(n^{1-\alpha}-2\right)}{2}\right)(b-a)^{j_*}\right]+\right.$$

$$\left.\left[\omega_1\left(\left(f_{\pm}^{(r)}\right)^{(N)},\frac{1}{n^{\alpha}}\right)\frac{1}{n^{\alpha N}N!}+\left(1-h\left(n^{1-\alpha}-2\right)\right)\left\|\left(f_{\pm}^{(r)}\right)^{(N)}\right\|_{\infty}\frac{(b-a)^N}{N!}\right]\right\} \tag{66}$$

(by Remark 2.12)

$$=\frac{1}{\psi(1)}\left\{\sum_{j_*=1}^{N}\frac{\left|\left(f^{(j_*)}\right)_{\pm}^{(r)}(x)\right|}{j_*!}\left[\frac{1}{n^{\alpha j_*}}+\left(\frac{1-h\left(n^{1-\alpha}-2\right)}{2}\right)(b-a)^{j_*}\right]+\right.$$

$$\left.\left[\omega_1\left(\left(f^{(N)}\right)_{\pm}^{(r)},\frac{1}{n^{\alpha}}\right)\frac{1}{n^{\alpha N}N!}+\left(1-h\left(n^{1-\alpha}-2\right)\right)\left\|\left(f^{(N)}\right)_{\pm}^{(r)}\right\|_{\infty}\frac{(b-a)^N}{N!}\right]\right\}\leq$$

$$\frac{1}{\psi(1)}\left\{\sum_{j_*=1}^{N}\frac{D\left(f^{(j_*)}(x),\widetilde{o}\right)}{j_*!}\left[\frac{1}{n^{\alpha j_*}}+\left(\frac{1-h\left(n^{1-\alpha}-2\right)}{2}\right)(b-a)^{j_*}\right]+\right.$$

$$\left.\left[\omega_1^{(\mathcal{F})}\left(f^{(N)},\frac{1}{n^{\alpha}}\right)\frac{1}{n^{\alpha N}N!}+\left(1-h\left(n^{1-\alpha}-2\right)\right)D^*\left(f^{(N)},\widetilde{o}\right)\frac{(b-a)^N}{N!}\right]\right\}, \tag{67}$$

by Proposition 2.7, $\left\|\left(f^{(N)}\right)_{\pm}^{(r)}\right\|_{\infty}\leq D^*\left(f^{(N)},\widetilde{o}\right)$ and apply (57).

The theorem is proved. □

Next we present

Theorem 4.3. *Let $\alpha>0$, $N=\lceil\alpha\rceil$, $\alpha\notin\mathbb{N}$, $f\in C_{\mathcal{F}}^{N}([a,b])$, $0<\beta<1$, $x\in[a,b]$, $n\in\mathbb{N}$, $n^{1-\beta}>2$. Then*

i)

$$D\left(A_n^{\mathcal{F}}(f,x),f(x)\right)\leq\frac{1}{\psi(1)}$$

$$\left\{\sum_{j_*=1}^{N-1}\frac{D\left(f^{(j_*)}(x),\widetilde{o}\right)}{j_*!}\left[\frac{1}{n^{\beta j_*}}+\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)(b-a)^{j_*}\right]+\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left[\omega_1^{(\mathcal{F})}\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1^{(\mathcal{F})}\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\frac{1}{n^{\beta}}\right)_{[x,b]}\right]}{n^{\alpha\beta}}+\right. \tag{68}$$

$$\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left[D^*\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[a,x]}(x-a)^{\alpha}+D^*\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[x,b]}(b-x)^{\alpha}\right]\Bigg\}\Bigg\},$$

ii) if $f^{(j)}(x_0)=0,\ j=1,...,N-1,$ *for some* $x_0\in[a,b]$*, we have*

$$D\left(A_n^{\mathcal{F}}(f,x_0),f(x_0)\right)\leq$$

$$\frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)}\left\{\frac{\left[\omega_1^{(\mathcal{F})}\left(\left(D_{x_0-}^{\alpha\mathcal{F}}f\right),\frac{1}{n^{\beta}}\right)_{[a,x_0]}+\omega_1^{(\mathcal{F})}\left(\left(D_{*x_0}^{\alpha\mathcal{F}}f\right),\frac{1}{n^{\beta}}\right)_{[x_0,b]}\right]}{n^{\alpha\beta}}+\right. \tag{69}$$

$$\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left[D^*\left(\left(D_{x_0-}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[a,x_0]}(x_0-a)^{\alpha}+D^*\left(\left(D_{*x_0}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[x_0,b]}(b-x_0)^{\alpha}\right]\right\},$$

when $\alpha>1$ *notice here the extremely high rate of convergence at* $n^{-(\alpha+1)\beta}$,

and

iii)

$$D^*\left(A_n^{\mathcal{F}}(f),f\right)\leq$$

$$\frac{1}{\psi(1)}\left\{\sum_{j_*=1}^{N-1}\frac{D^*\left(f^{(j_*)},\widetilde{o}\right)}{j_*!}\left[\frac{1}{n^{\beta j_*}}+\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)(b-a)^{j_*}\right]+\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left[\sup_{x\in[a,b]}\omega_1^{(\mathcal{F})}\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\frac{1}{n^{\beta}}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1^{(\mathcal{F})}\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\frac{1}{n^{\beta}}\right)_{[x,b]}\right]}{n^{\alpha\beta}}+\right. \tag{70}$$

$$\left.\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)(b-a)^{\alpha}\left[\sup_{x\in[a,b]}D^*\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[a,x]}+\sup_{x\in[a,b]}D^*\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[x,b]}\right]\right\}\right\},$$

above, when $N=1$ *the sum* $\sum_{j=1}^{N-1}\cdot=0.$

As we see here we obtain fractionally the fuzzy pointwise and uniform convergence with rates of $A_n^{\mathcal{F}}\to I$ *the unit operator, as* $n\to\infty$.

Proof. Here $f_{\pm}^{(r)}\in C^N([a,b])$, $\forall\ r\in[0,1]$, and $D_{x-}^{\alpha\mathcal{F}}f$, $D_{*x}^{\alpha\mathcal{F}}f$ are fuzzy continuous over $[a,b]$, $\forall\ x\in[a,b]$, so that $\left(D_{x-}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)}$, $\left(D_{*x}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)}\in C([a,b])$, $\forall\ r\in[0,1]$, $\forall\ x\in[a,b]$.

We observe by (52), $\forall\ x\in[a,b]$, that (respectively in $\pm$)

$$\left|A_n\left(f_{\pm}^{(r)},x\right)-f_{\pm}^{(r)}(x)\right|\leq\frac{1}{\psi(1)}$$

$$\left\{\sum_{j_*=1}^{N-1}\frac{\left|\left(f_{\pm}^{(r)}\right)^{(j_*)}(x)\right|}{j_*!}\left\{\frac{1}{n^{\beta j_*}}+\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)(b-a)^{j_*}\right\}+\right. \tag{71}$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{x-}^{\alpha}\left(f_{\pm}^{(r)}\right),\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D_{*x}^{\alpha}\left(f_{\pm}^{(r)}\right),\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left(\left\|D_{x-}^{\alpha}\left(f_{\pm}^{(r)}\right)\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D_{*x}^{\alpha}\left(f_{\pm}^{(r)}\right)\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\Bigg\}\Bigg\}=$$

(by Remark 2.12, (30), (33))

$$\frac{1}{\psi(1)}\left\{\sum_{j_*=1}^{N-1}\frac{\left|\left(f^{(j_*)}(x)\right)_{\pm}^{(r)}\right|}{j_*!}\left\{\frac{1}{n^{\beta j_*}}+\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)(b-a)^{j_*}\right\}+\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)},\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)},\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right. \tag{72}$$

$$\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left(\left\|\left(D_{x-}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)}\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|\left(D_{*x}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)}\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\Bigg\}\Bigg\}\leq$$

$$\frac{1}{\psi(1)}\left\{\sum_{j_*=1}^{N-1}\frac{D\left(f^{(j_*)}(x),\widetilde{o}\right)}{j_*!}\left\{\frac{1}{n^{\beta j_*}}+\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)(b-a)^{j_*}\right\}+\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left[\omega_1^{(\mathcal{F})}\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1^{(\mathcal{F})}\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\frac{1}{n^{\beta}}\right)_{[x,b]}\right]}{n^{\alpha\beta}}+\right. \tag{73}$$

$$\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left[D^{*}\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[a,x]}(x-a)^{\alpha}+D^{*}\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[x,b]}(b-x)^{\alpha}\right]\Bigg\}\Bigg\},$$

along with (57) proving all inequalities of theorem.

Here we notice that

$$\left(D_{x-}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)}(t)=\left(D_{x-}^{\alpha}\left(f_{\pm}^{(r)}\right)\right)(t)$$

$$=\frac{(-1)^{N}}{\Gamma(N-\alpha)}\int_{t}^{x}(s-t)^{N-\alpha-1}\left(f_{\pm}^{(r)}\right)^{(N)}(s)\,ds,$$

where $a\leq t\leq x$.

Hence

$$\left|\left(D_{x-}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)}(t)\right|\leq\frac{1}{\Gamma(N-\alpha)}\int_{t}^{x}(s-t)^{N-\alpha-1}\left|\left(f_{\pm}^{(r)}\right)^{(N)}(s)\right|ds$$

$$\leq\frac{\left\|\left(f^{(N)}\right)_{\pm}^{(r)}\right\|_{\infty}}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha}\leq\frac{D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha}.$$

So we have

$$\left|\left(D_{x-}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)}(t)\right|\leq\frac{D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha},$$

all $a\leq t\leq x$.

And it holds

$$\left\|\left(D_{x-}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)}\right\|_{\infty,[a,x]}\leq\frac{D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha}, \tag{74}$$

that is

$$D^{*}\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[a,x]}\leq\frac{D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma\left(N-\alpha+1\right)}\left(b-a\right)^{N-\alpha},$$

and

$$\sup_{x\in[a,b]}D^{*}\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[a,x]}\leq\frac{D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma\left(N-\alpha+1\right)}\left(b-a\right)^{N-\alpha}<\infty. \tag{75}$$

Similarly we have

$$\left(D_{*x}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)}(t)=\left(D_{*x}^{\alpha}\left(f_{\pm}^{(r)}\right)\right)(t)$$

$$=\frac{1}{\Gamma\left(N-\alpha\right)}\int_{x}^{t}\left(t-s\right)^{N-\alpha-1}\left(f_{\pm}^{(r)}\right)^{(N)}(s)\,ds,$$

where $x\leq t\leq b$.

Hence

$$\left|\left(D_{*x}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)}(t)\right|\leq\frac{1}{\Gamma\left(N-\alpha\right)}\int_{x}^{t}\left(t-s\right)^{N-\alpha-1}\left|\left(f^{(N)}\right)_{\pm}^{(r)}(s)\right|ds\leq$$

$$\frac{\left\|\left(f^{(N)}\right)_{\pm}^{(r)}\right\|_{\infty}}{\Gamma\left(N-\alpha+1\right)}\left(b-a\right)^{N-\alpha}\leq\frac{D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma\left(N-\alpha+1\right)}\left(b-a\right)^{N-\alpha},$$

$x\leq t\leq b$.

So we have

$$\left\|\left(D_{*x}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)}\right\|_{\infty,[x,b]}\leq\frac{D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma\left(N-\alpha+1\right)}\left(b-a\right)^{N-\alpha}, \tag{76}$$

that is

$$D^{*}\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[x,b]}\leq\frac{D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma\left(N-\alpha+1\right)}\left(b-a\right)^{N-\alpha},$$

and

$$\sup_{x\in[a,b]}D^{*}\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[x,b]}\leq\frac{D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma\left(N-\alpha+1\right)}\left(b-a\right)^{N-\alpha}<+\infty. \tag{77}$$

Furthermore we notice

$$\omega_{1}^{(\mathcal{F})}\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\frac{1}{n^{\beta}}\right)_{[a,x]}=\sup_{\substack{s,t\in[a,x]\\|s-t|\leq\frac{1}{n^{\beta}}}}D\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right)(s),\left(D_{x-}^{\alpha\mathcal{F}}f\right)(t)\right)\leq$$

$$\sup_{\substack{s,t\in[a,x]\\|s-t|\leq\frac{1}{n^{\beta}}}}\left\{D\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right)(s),\widetilde{o}\right)+D\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right)(t),\widetilde{o}\right)\right\}\leq 2D^{*}\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[a,x]}$$

$$\leq\frac{2D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma\left(N-\alpha+1\right)}\left(b-a\right)^{N-\alpha}.$$

Therefore it holds

$$\sup_{x\in[a,b]}\omega_{1}^{(\mathcal{F})}\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\frac{1}{n^{\beta}}\right)_{[a,x]}\leq\frac{2D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma\left(N-\alpha+1\right)}\left(b-a\right)^{N-\alpha}<+\infty. \tag{78}$$

Similarly we observe

$$\omega_1^{(\mathcal{F})}\left(\left(D_{*x}^{\alpha\mathcal{F}} f\right), \frac{1}{n^\beta}\right)_{[x,b]} = \sup_{\substack{s,t\in[x,b] \\ |s-t|\leq \frac{1}{n^\beta}}} D\left(\left(D_{*x}^{\alpha\mathcal{F}} f\right)(s), \left(D_{*x}^{\alpha\mathcal{F}} f\right)(t)\right) \leq$$

$$2D^*\left(\left(D_{*x}^{\alpha\mathcal{F}} f\right), \widetilde{o}\right)_{[x,b]} \leq \frac{2D^*\left(f^{(N)}, \widetilde{o}\right)}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha}.$$

Consequently it holds

$$\sup_{x\in[a,b]} \omega_1^{(\mathcal{F})}\left(\left(D_{*x}^{\alpha\mathcal{F}} f\right), \frac{1}{n^\beta}\right)_{[x,b]} \leq \frac{2D^*\left(f^{(N)}, \widetilde{o}\right)}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha} < +\infty. \tag{79}$$

So everything in the statements of the theorem makes sense.

The proof of the theorem is now completed. □

Corollary 4.4. *(to Theorem 4.3, $N=1$ case) Let $0<\alpha,\beta<1$, $f\in C_{\mathcal{F}}^1([a,b])$, $n\in\mathbb{N}$, $n^{1-\beta}>2$. Then*

$$D^*\left(A_n^{\mathcal{F}}(f), f\right) \leq$$

$$\frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)}\left\{\frac{\left[\sup_{x\in[a,b]} \omega_1^{(\mathcal{F})}\left(\left(D_{x-}^{\alpha\mathcal{F}} f\right), \frac{1}{n^\beta}\right)_{[a,x]} + \sup_{x\in[a,b]} \omega_1^{(\mathcal{F})}\left(\left(D_{*x}^{\alpha\mathcal{F}} f\right), \frac{1}{n^\beta}\right)_{[x,b]}\right]}{n^{\alpha\beta}} + \right. \tag{80}$$

$$\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)(b-a)^\alpha\left[\sup_{x\in[a,b]} D^*\left(\left(D_{x-}^{\alpha\mathcal{F}} f\right), \widetilde{o}\right)_{[a,x]} + \sup_{x\in[a,b]} D^*\left(\left(D_{*x}^{\alpha\mathcal{F}} f\right), \widetilde{o}\right)_{[x,b]}\right]\right\}.$$

Proof. By (70). □

Finally we specialize to $\alpha=\frac{1}{2}$.

Corollary 4.5. *(to Theorem 4.3) Let $0<\beta<1$, $f\in C_{\mathcal{F}}^1([a,b])$, $n\in\mathbb{N}$, $n^{1-\beta}>2$. Then*

$$D^*\left(A_n^{\mathcal{F}}(f), f\right) \leq$$

$$\frac{2(\psi(1))^{-1}}{\sqrt{\pi}}\left\{\frac{\left[\sup_{x\in[a,b]} \omega_1^{(\mathcal{F})}\left(\left(D_{x-}^{\frac{1}{2}\mathcal{F}} f\right), \frac{1}{n^\beta}\right)_{[a,x]} + \sup_{x\in[a,b]} \omega_1^{(\mathcal{F})}\left(\left(D_{*x}^{\frac{1}{2}\mathcal{F}} f\right), \frac{1}{n^\beta}\right)_{[x,b]}\right]}{n^{\frac{\beta}{2}}} + \right. \tag{81}$$

$$\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\sqrt{b-a}\left[\sup_{x\in[a,b]} D^*\left(\left(D_{x-}^{\frac{1}{2}\mathcal{F}} f\right), \widetilde{o}\right)_{[a,x]} + \sup_{x\in[a,b]} D^*\left(\left(D_{*x}^{\frac{1}{2}\mathcal{F}} f\right), \widetilde{o}\right)_{[x,b]}\right]\right\}.$$

Proof. By (80). □

5 Conclusion

We have extended to the fuzzy setting all the main approximation theorems of Section 3.

Conflict of Interest: The author declares no conflict of interest.

References

[1] Anastassiou GA. Rate of convergence of some neural network operators to the unit-univariate case. *Journal of Mathematical Analysis and Application*. 1997; 212: 237-262.

[2] Anastassiou GA. *Quantitative Approximation.* Chapmann and Hall/CRC, Boca Raton, New York; 2001. DOI: https://doi.org/10.1201/9781482285796

[3] Anastassiou GA. Fuzzy approximation by fuzzy convolution type operators. *Computers and Mathematics*. 2004; 48: 1369-1386. DOI: https://doi.org/10.1016/j.camwa.2004.10.027

[4] Anastassiou GA. Higher order fuzzy korovkin theory via inequalities. *Communications in Applied Analysis*. 2006; 10(2): 359-392.

[5] Anastassiou GA. Fuzzy korovkin theorems and inequalities. *Journal of Fuzzy Mathematics*. 2007; 15(1): 169-205.

[6] Anastassiou GA. On right fractional calculus. *Chaos, solitons and fractals*. 2009; 42: 365-376. DOI: https://doi.org/10.1016/j.chaos.2008.12.013

[7] Anastassiou GA. *Fractional Differentiation Inequalities.* Springer, New York; 2009. DOI: https://doi.org/10.1007/978-0-387-98128-4

[8] Anastassiou GA. Fractional korovkin theory. *Chaos, Solitons* & *Fractals*. 2009; 42(4): 2080-2094.

[9] Anastassiou GA. *Fuzzy Mathematics: Approximation Theory.* Springer, Heildelberg, New York; 2010. DOI: https://doi.org/10.1007/978-3-642-11220-1

[10] Anastassiou GA. *Intelligent Systems: Approximation by Artificial Neural Networks.* Intelligent Systems Reference Library, Springer, Heidelberg; 2011.

[11] Anastassiou GA. Fuzzy fractional calculus and ostrowski inequality. *J. Fuzzy Math.* 2011; 19(3): 577-590.

[12] Anastassiou GA. Fractional representation formulae and right fractional inequalities. *Mathematical and Computer Modelling*. 2011; 54(11-12): 3098-3115. DOI: https://doi.org/10.1016/j.mcm.2011.07.040

[13] Anastassiou GA. Univariate hyperbolic tangent neural network approximation. *Mathematics and Computer Modelling*. 2011; 53: 1111-1132. DOI: https://doi.org/10.1016/j.mcm.2010.11.072

[14] Anastassiou GA. Multivariate hyperbolic tangent neural network approximation. *Computers and Mathematics*. 2011; 61: 809-821. DOI: https://doi.org/10.1016/j.camwa.2010.12.029

[15] Anastassiou GA. Multivariate sigmoidal neural network approximation. *Neural Networks*. 2011; 24: 378-386. DOI: https://doi.org/10.1016/j.neunet.2011.01.003

[16] Anastassiou G. A. Univariate sigmoidal neural network approximation. *J. of Computational Analysis and Applications*. 2012; 14(4): 659-690.

[17] Anastassiou GA. Fractional neural network approximation. *Computers and Mathematics with Applications*. 2012; 64(6): 1655-1676. DOI: https://doi.org/10.1016/j.camwa.2012.01.019

[18] Anastassiou GA. Fuzzy fractional neural network approximation by fuzzy quasi-interpolation operators. *J. of Applied Nonlinear Dynamics*. 2013; 2(3): 235-259.

[19] Anastassiou GA. *Intelligent Systems II: Complete Approximation by Neural Network Operators.* Springer, Heidelberg, New York; 2016. DOI: https://doi.org/10.1007/978-3-319-20505-2

[20] Anastassiou GA. *Intelligent Computations: Abstract Fractional Calculus, Inequalities, Approximations.* Springer, Heidelberg, New York; 2018. DOI: https://doi.org/10.1007/978-3-319-66936-6

[21] Anastassiou GA. *Banach Space Valued Neural Network.* Springer, Heidelberg, New York; 2023. DOI: https://doi.org/10.1007/978-3-031-16400-2

[22] Anastassiou GA. General sigmoid based Banach spacevalued neural network approximation. *J. Computational Analysis and Applications.* 2023; 31(4): 520-534.

[23] Chen Z, Cao F. The approximation operators with sigmoidal functions. *Computers and Mathematics with Applications.* 2009; 58: 758-765.

[24] Diethelm K. *The Analysis of Fractional Differential Equations.* Lecture Notes in Mathematics 2004, Springer-Verlag, Berlin, Heidelberg; 2010. DOI: https://doi.org/10.1007/978-3-642-14574-2

[25] El-Sayed AMA, Gaber M. On the finite Caputo and finite Riesz derivatives. *Electronic Journal of Theoretical Physics.* 2006; 3(12): 81-95.

[26] Frederico GS, Torres DFM. Fractional optimal control in the sense of Caputo and the fractional Noether's theorem. *International Mathematical Forum.* 2008; 3(10): 479-493.

[27] Gal S. Approximation Theory in Fuzzy Setting. *Handbook of Analytic-Computational Methods in Applied Mathematics.* edited by G. Anastassiou, Chapman & Hall/CRC, Boca Raton, New York; 2000. p.617-666

[28] Goetschel JrR, Voxman W. Elementary fuzzy calculus. *Fuzzy Sets and Systems.* 1986; 18: 31-43. DOI: https://doi.org/10.1016/0165-0114(86)90026-6

[29] Haykin S. *Neural Networks: A Comprehensive Foundation.* (2 ed.) Prentice Hall, New York; 1998.

[30] Kaleva O. Fuzzy differential equations. *Fuzzy Sets and Systems.* 1987; 24: 301-317. DOI: https://doi.org/10.1016/0165-0114(87)90029-7

[31] Kim YK, Ghil BM. Integrals of fuzzy-number-valued functions. *Fuzzy Sets and Systems.* 1997; 86: 213-222.

[32] McCulloch W, Pitts W. A logical calculus of the ideas immanent in nervous activity. *Bulletin of Mathematical Biophysics.* 1943; 7: 115-133.

[33] Mitchell TM. *Machine Learning.* WCB-McGraw-Hill, New York; 1997.

[34] Samko SG, Kilbas AA, Marichev OI. *Fractional Integrals and Derivatives, Theory and Applications.* Gordon and Breach, Amsterdam; 1993. (English translation from the Russian, Integrals and Derivatives of Fractional Order and Some of Their Applications, Nauka i Tekhnika, Minsk, (1987)).

[35] Wu C, Gong Z. On Henstock integrals of interval-valued functions and fuzzy valued functions. *Fuzzy Sets and Systems.* 2000; 115(3): 377-391. DOI: https://doi.org/10.1016/S0165-0114(98)00277-2

[36] Wu C, Gong Z. On Henstock integral of fuzzy-number-valued functions (I). *Fuzzy Sets and Systems.* 2001; 120(3): 523-532. DOI: https://doi.org/10.1016/S0165-0114(99)00057-3

[37] Wu C, Ma M. On embedding problem of fuzzy numer spaces: Part 1. *Fuzzy Sets and Systems*. 1991; 44: 33-38. DOI: https://doi.org/10.1016/0165-0114(91)90030-T

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
E-mail: ganastss@memphis.edu

Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://tfss.journals.iau.ir/
Online ISSN: 2821-0131
Vol.2, No.2, (2023), 39-62
DOI: http://doi.org/10.30495/tfss.2023.1976999.1062

Article Type: Original Research Article

Intuitionistic Hesitant Fuzzy Algorithm for Multi-Objective Structural Model Using Various Membership Functions

Sanjoy Biswas*, Samir Dey

Abstract. In real life, structural problems can be described in linear and nonlinear forms. This nonlinear structural problem is very challenging to solve when its all parameters are imprecise in nature. Intuitionistic fuzzy sets were proposed to manage circumstances in which experts have some membership and non-membership value to judge an option. Hesitant fuzzy sets were used to manage scenarios in which experts pause between many possible membership values while evaluating an alternative. A new growing area of a generalized fuzzy set theory called intuitionistic hesitant fuzzy set (IHFS) provides useful tools for dealing with uncertainty in structural design problem that is observed in the actual world. In this article, we have developed a procedure to solve non-linear structural problem in an intuitionistic hesitant fuzzy (IHF) environment. The concept of an intuitionistic hesitant fuzzy set is introduced to provide a computational basis to manage the situations in which experts assess an alternative in possible membership values and non-membership values. This important feature is not available in the intuitionistic fuzzy optimization technique. Here we have discussed the solution procedure of intuitionistic hesitant fuzzy optimization technique dedicatedly for linear, exponential, and hyperbolic types of membership and non-membership functions. Some theoretical development based on these functions has been discussed. A numerical illustration is given to justify the effectiveness and efficiency of the proposed method in comparison with fuzzy multi-objective nonlinear programming method and intuitionistic fuzzy multi-objective nonlinear programming method. Finally, based on the proposed work, conclusions and future research directions are addressed.

AMS Subject Classification 2020: 49K35; 90C29; 90C70
Keywords and Phrases: Multi objective structural problem, Hesitant fuzzy set, Intuitionistic fuzzy optimization, Intuitionistic-hesitant fuzzy optimization, Pareto optimal solution.

1 Introduction

In structural and civil engineering, structural optimization is a key idea. Although the concept of structural optimization is well-established.It is frequently treated in a single objective form, with the objective being (the weight function). In addition to the minimization of the weight function, this optimization also involves satisfying one or more constants consequently. But in the real world, there are multiple competing objectives. The Multiple objective structural optimizations (MOSOs) methodology was used to address multiple competing objectives. Due to the growing technological demand for structural optimization, the MOSO is becoming a more and more important research area in the last ten years.
The development of fuzzy optimum structural design methods was required because the input data and the parameters in structural design problems are frequently/imprecise. The fuzzy set (FS) theory was first developed by Zadeh [17] to deal with erroneous and imperfect data. The decision-making problem was later addressed by Zadeh [29] and Bellman and Zadel [6] using the fuzzy set theory. Later on, Zimmermann [30]

*__Corresponding Author:__ Sanjoy Biswas, Email: sanjoy.biswas@gnit.ac.in, ORCID: 0000-0002-5438-7154
Received: 6 January 2023; **Revised**: 14 May 2023; **Accepted**: 14 May 2023; **Available Online**: 14 May 2023; **Published Online**: 7 November 2023.

How to cite: Biswas S, Dey S. Intuitionistic Hesitant Fuzzy Algorithm for Multi-Objective Structural Model Using Various Membership Functions. *Trans. Fuzzy Sets Syst.* 2023; 2(2): 39-62. DOI: http://doi.org/10.30495/tfss.2023.1976999.1062

proposed a fuzzy programming approach (FPA) for several objective optimization problems. The FS theory is also used in the structural model. Many researchers (see [27, 12, 11, 23]) have given remarkable contributions in the field of structural optimization under fuzzy environments. Also Dey et.al. [15] developed a methodology using different norm(Yager, Hamacher, Dombi) under the fuzzy environment in the context of structural design. Here they have optimized three objective functions simultaneously in three bar truss structural model. Numerous extensions of fuzzy sets emerged as a result of the growing use of FS in structural problems when the available information is ambiguous.

1.1 Literature Review

The intuitionistic fuzzy set (IFS), one of the generalizations of FS theory, was introduced by professor Atanassov [4] in 1986.IFS plays an important role when imprecise information cannot be expressed by conventional fuzzy sets. It is a more advanced version of FS. In IFS, we usually consider the degree of acceptance, degree of rejection and hesitancy such that the sum of degrees of membership should be less than or equal to one, whereas we consider the degree of acceptance only in FS. P. P. Angelov [3] introduced optimization for the first time in a widespread intuitionistic fuzzy environment(IFEv) in 1997. The field of intuitionistic fuzzy optimization (IFO) is still unexplored. There has been little research work done on IFO in terms of structural optimization. The methodology of Multi-objective linear programming(MOLP) under IFEv was developed by Jana and Roy [18] to find an optimal solution to the transportation problem. Luo et. al.[19] had discussed multi-criteria decision making (MCDM) problems based on the inclusion degree of IFS in 2008. In 2015, Dey et al. [13] used multi-objective intuitionistic fuzzy optimization approach to solve three bar truss structural model. Farther, M. Sarkar et al. [21] proposed a new computational algorithm based on t-norm and t-conorm in the intuitionistic fuzzy environment to solve a welded beam design problem. Ahmadini et al. [1] proposed intuitionistic fuzzy goal programming with preference relations to solve a multi-objective problem in 2021.M. Akram, et al.[2] introduced interval-valued Fermatean fuzzy set(IVFFS) which is the extension of Fermatean fuzzy sets(FFSs) and applied IVFFS in the fractional transportation problem. Further, M.K. Sharma, et al.[22] originally solved multi-objective transportation problem (MOTP) in Fermatean fuzzy environment. They also anticipated a new score function to convert the Fermatean fuzzy data into Crisp data to solve MOTP.In an IFS, the degree of acceptance, degree of rejection, and degree of hesitation of an element may not be a specific number in some situations. As a result, it has been extended to interval-valued intuitionistic fuzzy sets [5].

The concept of hesitant fuzzy set (HFS), which is an extension of regular FS, was first introduced by Torra [25] and Torra and Narukawa [24]. This is a useful tool because it allows for more possible degrees of an element to be in a set which is a sub-interval of [0,1]. In the literature survey, we have seen Meany researchers have implemented the concept of HFS in different fields of research. In 2016, Xu et al. [26] developed a computational programming technique based on HFS for a hybrid multi-criteria group decision making (MCGDM) model. L. Dymova [16] created a user-friendly computer application using a fuzzy multiple-criteria decision-making (MCDM) technique. In 2018, Bharati [7] developed a multi-objective hesitant fuzzy optimization technique. He also published some research articles on interval-valued intuitionistic hesitant fuzzy sets (see [8, 9]), and hesitant intuitionistic fuzzy sets [10] between 2021 and 2022. Xia et. al. [28] introduced hesitant fuzzy on decision making. In 2022, M. Ranjbar et al. [20] introduced the ranking of hesitant fuzzy numbers and new arithmetic operations based on the extension principle.

1.2 Motivation for that research

According to the literature review, numerous methods have been developed to solve multi-objective optimization problems (MOOPs) in fuzzy and intuitionistic fuzzy environments. Dey et al. [14] used fuzzy and intuitionistic fuzzy approaches to solve multi-objective three-bar truss structural model. This method can satisfy the objective(s) with a bigger degree than the analogous fuzzy optimization problem and the crisp one, but there is no space for the decision maker's point of view. In real life, the decision makers priority plays an important role in any decision-making. Therefore, it is necessary to develop a new decision-making method based on IHF decision-making set that assigns a set of potential values for each objective functions membership and non-membership in IHF environment.

1.3 Contribution of the work

Many scholars are working continuously to find the best solution to multi objective structural optimization problems (MOSOPs). A large amount of the literature is composed of fuzzy-based optimization approaches that use the generalized concept of a FS to solve MOSOPs. Many researchers optimize the MOSOPs using an intuitionistic fuzzy-based optimization technique. The study focused on IHFS under different membership and non-membership functions. After that, MOSOP can be solved by using the proposed IHF approach. However, the following are the few major aspects that guarantee a significant contribution to the field of multi objective optimization techniques.

- The intuitionistic hesitant fuzzy (IHF) is a recent extension of fuzzy sets that are explained in a structural model of three bar truss.
- In this paper, we present an IHF set theory that provides an opportunity for the decision-maker to select the best result or reject the worse result in comparison to others.
- Instead of a single fixed degree, a set of possible degrees of acceptance and degrees of rejection are defined to address the uncertainty and hesitancy of MONLPP.
- The intuitionistic hesitant Pareto optimal is also introduced in this paper.
- We have developed a multi-objective structural model under an intuitionistic hesitant fuzzy environment. A computational algorithm for intuitionistic hesitant fuzzy optimization has been developed to solve multi-objective structural models.
- The HIFS might be a useful tool to deal with any real-life situation in the context of uncertainty and hesitation.

1.4 Framework of the article

The rest of the manuscript is organized as follows: Sect. 2, we have explained the multi-objective structural optimization model. In Sect. 3 recalls some basic concepts of FS, IFS, IHFS. For the practical perspective, a computational algorithm was proposed to solve MOOP using the intuitionistic hesitant fuzzy optimization technique (IHFOT) in Sect. 4. In Sect. 5, stepwise solution procedures are described for the solution of multi-objective structural model using IHFOT. An illustrative example is examined in Sect. 6 that shows the applicability and validity of the proposed algorithm efficiently. Finally, conclusions are highlighted based on the present work in Sect. 7.

2 Mathematical form of Multi-Objective Structural Problem (MOSP)

In the structural model, the basic parameters of a bar truss structure system (such as elastic modulus, material density, height possible stress etc.) are identified, and the goal is to find the optimum cross section area of the bar truss so that we can find the lightest weight of the structure and smallest node displacement

under loading condition.
The multi-objective problem in structure model is written as follows:

$$\begin{aligned} \text{Minimize} \quad & WG(C) \\ \text{Minimize} \quad & d(C) \\ \text{Such that} \quad & T[C] \leq [T_0] \\ & C \in [C_{min}, C_{max}] \end{aligned} \tag{1}$$

Where n number design parameters $C = [C_1, C_2, C_3, ..., C_n]^T$ are considered.The design parameters are the cross section of the truss bar,the total structural weight is $WG(C) = \sum_{i=1}^{n} \delta_i C_i L_i$, $d(C)$ is the deflection of loaded joint, L_i, C_i and δ_i were the lengths of the bar, cross section area and density of the i^{th}group bars respectively. Under different conditions, the stress constraint=$T(C)$and maximum possible stress of the group bars=$[T_0]$, cross section area (minimum)= C_{min} and cross section area (maximum) = C_{max} respectively.

3 Preliminaries

In this section, we talked about several fundamental ideas related to intuitionistic fuzzy logic.

Definition 3.1. (see [4]) *(Intuitionistic Fuzzy Set(IFS)) Let $E = \{x_1, x_2, ..., x_n\}$ be the collection of finite objects then the IFS $\overline{Y}$ in E is defined as: $\overline{Y} = \{(x_j, \gamma_{\overline{Y}(x_j)}, \lambda_{\overline{Y}(x_j)}) : x_j \in E\}$, where the function $\gamma_{\overline{A}(x_j)}$: $E \to [0,1]$define the degree of membership function and $\lambda_{\overline{Y}(x_j)}$: $x_j : E \to [0,1]$ define the degree of non-membership function of an element $x_j \in E$ respectively, with the condition $0 \leq \gamma_{\overline{Y}}(x_j) + \lambda_{\overline{Y}(x_j))} \leq 1 \quad \forall x_j \in E$.For each $\overline{Y} \in E$ the amount $\overline{\pi}_{\overline{Y}}(x_j) = 1 - \gamma_{\overline{A}(x_j)} - \lambda_{\overline{Y}(x_j)}$ is called Atanassovs intuitionistic index of the element $x_j \in E$ or degree of indeterminacy (uncertainty) of x_j of the measure of hesitation.*

Definition 3.2. (see [4]) *((α, β)-cut) A subset (α, β)-cut of E generated by an IFS, where $(\alpha, \beta)$are fixed numbers such that $\alpha + \beta \leq 1$ is defined by $\overline{Y}_{\alpha,\beta}(x_j) = \{x_j \in E : \gamma_{\overline{Y}(x_j)} \geq \alpha, \lambda_{\overline{Y}(x_j))} \leq \beta\}$ .Thus (α, β) of an IFS to be denoted by $\{\overline{Y}_{\alpha,\beta}(x_j)$ as a crisp set of the element x_j which belong to $\overline{Y}_{\alpha,\beta}(x_j)$ at least to the degree α and at most to the degree β.*

Definition 3.3. (see [25, 24]) *(Hesitant Fuzzy Set(HFS)) Torra in 2009 and Torra and Narukawa in 2010, created a new tool called hesitant fuzzy sets (HFSs) and which allow the membership degree to the set of various possible values. The HFS can be stated as follows:*
*Let E be the fixed set then a HFS on E is expressed as $\overline{Y} = \{(x_j, h_{\overline{h}(x_j)}) : x_j \in E\}$, where is set of possible membership degrees of the element $x_j \in E$ in [0,1].Also, we call $h_{\overline{Y}}(x_j)$, a hesitant fuzzy element. Further, Xia and Xu [**?**] applied it in their works of research.*

Definition 3.4. [10] *(Intuitionistic Hesitant Fuzzy Set) When making a decision, a decision-maker may hesitate to determine the exact degrees of membership and non-membership between 0 and 1. In such a scenario, the IHFS, which is a generalized version of FS where the membership and nonmember ship degrees of an element to a specific set can be represented by multiple distinct values between 0 and 1. The IHFS is perfect at dealing with circumstances in which decision maker disagreement or hesitate to make a decision. Let there be a fixed setE ; a IHFS $\overline{Y}$ on E is represented as $\overline{Y} = \{(x_j, h_{\overline{h}(x_j)} : x_j \in E\}$ where $h_{\overline{Y}}(x_j)$ is set of some values of IHFSs in [0,1] , denoting the possible membership degree and non-membership degree of the element $x_j \in E$. Let I_{h_1}, I_{h_2} be two IHFSs and $h_1 \in I_{h_1}, h_2 \in I_{h_2}$.Then the complement of IHFS I_h, union and intersection of I_{h_1}, I_{h_2} are defined as follows respectively.*

- $I_h^c = \{h^c : h \in I_h\}$

- $I_{h_1} \cup I_{h_2} = \{max(h_1, h_2) : h_1 \in I_{h_1}, h_2 \in I_{h_2} \quad where \quad h_1 \cup h_2 = \{max(\gamma_{h_1}, \gamma_{h_2}), min(\lambda_{h_1}, \lambda_{h_1})\}\}$
- $I_{h_1} \cap I_{h_2} = \{min(h_1, h_2) : h_1 \in I_{h_1}, h_2 \in I_{h_2} \quad where \quad h_1 \cap h_2 = \{min(\gamma_{h_1}, \gamma_{h_2}), max(\lambda_{h_1}, \lambda_{h_1})\}\}$

Definition 3.5. [8] *(Pareto-optimal solution) An ideal solution derived from a single objective may or may not satisfy all of the conflicting objectives at the same time. However, it is difficult to find Pareto-optimal solutions, which optimize all objectives while satisfying all constraints. Mathematically, Suppose*Λ *be the collection of feasible solution for (1) of MOSOP. Then a point* x^* *is considered to be a Pareto optimal or efficient solution of (1) iff there exists no* $x \in \Lambda$ *such that* $\Theta_\sigma(x^*) \geq \Theta_\sigma(x)$ *for all* σ *and* $\Theta_\sigma(x^*) > \Theta_\sigma(x)$ *for at least one* σ*. And a point* $x^* \in \Lambda$ *is called a weak Pareto optimal solution of (1). iff there exists no* $x \in \Lambda$ *such that* $\Theta_\sigma(x^*) \geq \Theta_\sigma(x)$ *for all* σ

Definition 3.6. [10] *(Pareto-optimal solutions of IHF) The Pareto-optimal solutions for the IHF optimization can be defined as follows:*
A solution $X_0 \in \Omega$ *is said to be Pareto-optimal solution for (1) if there does not exist another* $X \in \Omega$ *such that* $\Theta_\sigma(X) \geq \Theta_\sigma(X_0)$ *with* $\gamma_\sigma^{IF}\Theta_\sigma(X) \geq \gamma_\sigma^{IF}\Theta_\sigma(X_0), \lambda_\sigma^{IF}\Theta_\sigma(X) \leq \lambda_\sigma^{IF}(\Theta_\sigma(X_0))$ *, and* $\Theta_{\sigma_0}(X) > \Theta_{\sigma_0}(X_0)$ *with* $\gamma_{\sigma_0}^{IF}(\Theta_{\sigma_0}(X)) > \gamma_{\sigma_0}^{IF}(\Theta_{\sigma_0}(X_0))$ *and* $\lambda_{\sigma_0}^{IF}(\Theta_{\sigma_0}(X)) < \lambda_{\sigma_0}^{IF}(\Theta_{\sigma_0}(X_0))$ *for at least one* $\sigma_0 = \{1, 2, ..., \Sigma\}$*.*

Definition 3.7. *(Intuitionistic Hesitant fuzzy Non Linear Programming (IHFNLP)) Most real-world problems involve the optimization of more than one objectives at the same time. The best compromise solution is the most promising solution set that efficiently satisfies each objective. Therefore, a Multi-Objective Non-Linear Programming (MONLP) with P objectives should be greater than or equal to some value* $\lesssim g_p^0(x), p = 1, 2, ..., P$ *may be taken in the following form:*

$$\begin{aligned} Minimize \quad & \Theta(x) = [\Theta_1, \Theta_2, ..., \Theta_\sigma]^T \\ subject\ to \quad & \Theta_\sigma(x) \lesssim g_\sigma^0(x), \sigma = 1, 2, ..., \Sigma \\ & g_j(x) \leq 0, x_j \geq 0 for j = 1, 2, .., m \\ & x = \{x_1, x_2, ..., x_n\} \end{aligned} \tag{2}$$

Where $g_\sigma^0(x)$ *is goal for* σ^{th} *objective and* $\lesssim$ *is uncertain form of* $\leq$*.*

4 Problem formulation and solution algorithm

4.1 Intuitionistic Hesitant Fuzzy algorithm to Solve MONLPP

A MONLP with σ objective may be taken in the following form:

$$\begin{aligned} \text{Minimize} \quad & \Theta(x) = [\Theta_1(x), \Theta_2(x), ..., \Theta_\sigma(x)]^T \\ \text{subject to} \quad & \{x \in R^n : g_j(x) \leq or = or \geq b_j \quad \text{for} \quad j = 1, 2, ..., m\} \\ & L_i \leq x_i \leq U_i \quad (i = 1, 2, ..., n) \end{aligned} \tag{3}$$

Zimmermann [30] showed that fuzzy programmin technique (FPT) can be used to solve the MOOP. To solve MONLPP, following steps are used.
Step 1 Solve the MONLP (3) as a single objective function from the set of σ objectives and solve it as a single objective subject to the given constrains and ignoring the others objective function. Determine the value of objective functions and basic feasible solutions.
Step 2 Calculate the values of the remaining $(\sigma-1)$ objectives at the basic feasible solutions that are obtained from **Step 1**.
Step 3 Repeat the **Step 1** and **Step 2** for the remaining $(\sigma - 1)$ objective functions.

Table 1: Please write your table caption here

Minimum	Θ_1	Θ_2	Θ_3	—	Θ_σ	X
Minimum Θ_1	Θ_1^*			—		X_1
Minimum Θ_2		Θ_2^*		—		X_2
Minimum Θ_3			Θ_3^*	—		X_2
—	—	—	—	—	—	—
Minimum Θ_Σ				—	Θ_σ^*	X_σ
Maximum	Θ_1'	Θ_2'	Θ_3'	—	Θ_σ'	X_σ'

Step 4 From the result of **Step 1**, **Step 2** and **Step 3**, obtained the corresponding tabulated values of objective functions from a Table 1. and these are known as positive ideal solution.
Step 5 From **Step 4**, obtain the lower bounds and upper bounds for each objective functions, where Θ_σ^* and $\Theta_\sigma^{'}$ are maximum, minimum values of Θ_Σ respectively.
Step 6 Here, we denote and define upper and lower bounds by $U_\sigma^\gamma = max\{Z_p X_p\}$ and $p = 1, 2, 3, ..., P$ for respectively for each uncertain and imprecise objective functions of MONLPPs
Step 7 Set upper bound or upper tolerance level and lower bound or lower tolerance limit for the σ^{th} objective function Θ_σ for hesitant degree of acceptance and rejection based on the set of solutions obtained in **Step 4**. For hesitant membership function: Upper and lower tolerance level for hesitant membership functions are

$$U_\sigma^\gamma = max\{\Theta_\sigma(X_p)\} \quad \text{and} \quad L_\sigma^\gamma = min\{\Theta_\sigma(X_p)\}, 1 \le p \le P, \sigma = 1, 2, ..., \Sigma$$

For hesitant non-membership function: Upper and lower tolerance level for hesitant membership functions are

$$U_\sigma^\lambda = U_\sigma^\gamma, L_\sigma^\lambda = L_\sigma^\gamma + \epsilon_\sigma$$

where $0 \le \epsilon_\sigma \le (U_\sigma - L_\sigma)$ is predetermined real numbers prescribed by decision-makers.
Step 8 In this step, we can define uncertainty and imprecise objectives of different hesitant membership functions as linear, exponential and hyperbolic more elaborately under IHF environment. Each of them is defined for the hesitant membership and a hesitant non-membership functions, which seems to be more realistic.

4.1.1 Linear-type intuitionistic hesitant membership functions approach (LTIHMFA)

The truth membership function of linear type $\gamma_\sigma^{Lf_i}(\Theta_\sigma(x))$ and a falsity membership function of linear type $\lambda_\sigma^{Lf_1}(\Theta_\sigma(x))$ functions under IHF environment can be explained in the following way:

For truth hesitant fuzzy membership functions:

$$\gamma_\sigma^{Lf_1}(\Theta_\sigma(x)) = \begin{cases} 1 & \text{if } \Theta_\sigma(x) \le L_\sigma^\gamma \\ \phi_1\Big(\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x))^\dagger}{(U_\sigma)^\dagger - (L_\sigma)^\dagger}\Big) & \text{if } L_\sigma^\gamma \le \Theta_\sigma(x)) \le U_\sigma^\gamma \\ 0 & \text{if } \Theta_p\sigma(x)) > U_\sigma^\gamma \end{cases}$$

$$\gamma_\sigma^{Lf_2}(\Theta_\sigma(x)) = \begin{cases} 1 & \text{if } \Theta_\sigma(x) \le L_\sigma^\gamma \\ \phi_2\Big(\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x))^\dagger}{(U_\sigma)^\dagger - (L_\sigma)^\dagger}\Big) & \text{if } L_\sigma^\gamma \le \Theta_\sigma(x)) \le U_\sigma^\gamma \\ 0 & \text{if } \Theta_\sigma(x)) > U_\sigma^\gamma \end{cases}$$

.....

$$\gamma_{\sigma}^{Lf_n}(\Theta_\sigma(x)) = \begin{cases} 1 & \text{if } \Theta_\sigma(x) \le L_\sigma^\gamma \\ \phi_n\left(\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x))^\dagger}{(U_\sigma)^\dagger - (L_\sigma)^\dagger}\right) & \text{if } L_\sigma^\gamma \le \Theta_\sigma(x)) \le U_\sigma^\gamma \\ 0 & \text{if } \Theta_\sigma(x)) > U_\sigma^\gamma \end{cases}$$

For Falsity hesitant fuzzy membership functions

$$\lambda_{\sigma}^{Lf_1}(\Theta_\sigma(x)) = \begin{cases} 0 & \text{if } \Theta_\sigma(x) \le L_\sigma^\lambda \\ \zeta_1\left(\frac{(\Theta_\sigma(x))^\dagger - (L_\sigma^\lambda)^\dagger}{(U_\sigma)^\dagger - (L_\sigma)^\dagger}\right) & \text{if } L_\sigma^\lambda \le \Theta_\sigma(x)) \le U_\sigma^\lambda \\ 1 & \text{if } \Theta_\sigma(x)) > U_\sigma^\lambda \end{cases}$$

$$\lambda_{\sigma}^{Lf_2}(\Theta_\sigma(x)) = \begin{cases} 0 & \text{if } \Theta_\sigma(x) \le L_\sigma^\lambda \\ \zeta_2\left(\frac{(\Theta_\sigma(x))^\dagger - (L_\sigma^\lambda)^\dagger}{(U_\sigma)^\dagger - (L_\sigma)^\dagger}\right) & \text{if } L_\sigma^\lambda \le \Theta_\sigma(x)) \le U_\sigma^\lambda \\ 1 & \text{if } \Theta_\sigma(x)) > U_\sigma^\lambda \end{cases}$$

.....

$$\lambda_{\sigma}^{Lf_n}(\Theta_\sigma(x)) = \begin{cases} 0 & \text{if } \Theta_\sigma(x) \le L_\sigma^\lambda \\ \zeta_n\left(\frac{(\Theta_\sigma(x))^\dagger - (L_\sigma^\lambda)^\dagger}{(U_\sigma)^\dagger - (L_\sigma)^\dagger}\right) & \text{if } L_\sigma^\lambda \le \Theta_\sigma(x)) \le U_\sigma^\lambda \\ 1 & \text{if } \Theta_\sigma(x)) > U_\sigma^\lambda \end{cases}$$

The mathematical expression for objective functions defined as follows

$$\begin{aligned} &\text{Max } \min_{\sigma=1,2,\ldots,\Sigma} \gamma_\sigma^{Lf_i}(\Theta_\sigma(x))^\dagger) \\ &\text{Min } \max_{\sigma=1,2,\ldots,\Sigma} \lambda_\sigma^{Lf_i}(\Theta_\sigma(x))^\dagger) \\ &i = 1, 2, ..., n \end{aligned} \tag{4}$$

Subject to all constraints of (3).
Also assume that $\gamma_\sigma^{Lf_i}(\Theta_\sigma(x))^\dagger) \ge \nu_i$ and $\lambda_r^{Lf_i}(\Theta_\sigma(x))^\dagger) \le \eta_i$ $i = 1, 2, ..., n$ for all σ. Where the parameter $\dagger > 0$
Using auxiliary parameters ν_i and η_i, the problem (4) can be transformed into the following problem (5)

$$\text{LTIHMFA} \quad Max\left(\sum_i \nu_i - \sum_i \eta_i\right)$$

Subject to

$$
\begin{aligned}
&(\Theta_\sigma(x))^\dagger + \frac{\nu_1}{\phi_1}\left((U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger\right) \leq (U_\sigma^\gamma)^\dagger,\\
&(\Theta_\sigma(x))^\dagger + \frac{\nu_2}{\phi_2}\left((U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger\right) \leq (U_\sigma^\gamma)^\dagger,\\
&.........,\\
&(\Theta_\sigma(x))^\dagger + \frac{\nu_3}{\phi_3}\left((U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger\right) \leq (U_\sigma^\gamma)^\dagger;\\
&(\Theta_\sigma(x))^\dagger - \frac{\eta_1}{\zeta_1}\left((U_\sigma^\lambda)^\dagger - (L_\sigma^\lambda)^\dagger\right) \leq (L_{\sigma^\lambda})^\dagger,\\
&(\Theta_\sigma(x))^\dagger - \frac{\eta_2}{\zeta_2}\left((U_\sigma^\lambda)^\dagger - (L_\sigma^\lambda)^\dagger\right) \leq (L_\sigma^\lambda)^\dagger,\\
&.........,\\
&(\Theta_\sigma(x))^\dagger - \frac{\eta_3}{\zeta_3}\left((U_\sigma^\lambda)^\dagger - (L_\sigma^\lambda)^\dagger\right) \leq (L_\sigma^\lambda)^\dagger;
\end{aligned}
\tag{5}
$$

$\nu_i \geq \eta_i; \nu_i + \eta_i \leq 1 \quad \text{and} \quad \eta_i, \nu_i, \phi_i, \zeta_i \in [0,1] \quad \forall i = 1, 2, ..., n$all the constraints of (3).

Theorem 4.1. *There is only one optimal solution (x^*, ν^*, η^*) of (5) that is also an efficient solution to the problem (3) where $\nu^* = (\nu_1^*, \nu_2^*, ..., \nu_n^*)$ and $\eta^* = (\eta_1^*, \eta_2^*, ..., \eta_n^*)$*

Proof. Assume that (x^*, ν^*, η^*) be the only optimal solution of (5) that it is an inefficient solution to the problem (3). Then there exist different feasible alternative $x'(x' \neq x^*)$ of problem (3), so that $\Theta_\sigma(x^*) \leq \Theta_\sigma(x') \forall \sigma = 1, 2, ..., \Sigma$ and $\Theta_\sigma(x^*) < \Theta_\sigma(x')$ for at least one σ .

$$
\begin{aligned}
\text{Therefore, we have} \quad &\phi\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x^*))^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger} \leq \phi\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x'))^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger} \quad \forall \quad \sigma = 1, 2, ..., \Sigma\\
\text{and} \quad &\phi\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x^*))^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger} < \phi\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x'))^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger} \quad \text{for atleast one} \quad \sigma,\\
\text{where} \quad &0 \leq \phi \leq 1
\end{aligned}
$$

$$
\begin{aligned}
\text{Thus,} \quad &Max_{\forall\sigma}\left(\phi\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x^*))^\dagger}{(U_\sigma^\gamma)^\dagger - ((L_\sigma^\gamma)^\dagger}\right) \leq Max_{\forall\sigma}\left(\phi\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x'))^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger}\right)\\
\text{and} \quad &Max_\sigma\left(\phi\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x^*))^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger}\right) < Max_\sigma\left(\phi\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x'))^\dagger}{(U_\sigma)^\dagger - (L_\sigma)^\dagger}\right) \quad \text{for at least one } \sigma.
\end{aligned}
$$

$$
\begin{aligned}
\text{Similarly,} \quad &Min_{\forall\sigma}\left(\zeta\frac{(\Theta_\sigma(x^*))^\dagger - (L_\sigma^\lambda)^\dagger}{(U_\sigma^\lambda)^\dagger - (L_\sigma^\lambda)^\dagger}\right) \geq Min_{\forall\sigma}\left(\zeta\frac{(\Theta_\sigma(x'))^\dagger - (L_\sigma^\lambda)^\dagger}{(U_\sigma^\lambda)^\dagger - (L_\sigma^\lambda)^\dagger}\right)\\
\text{and} \quad &Min_\sigma\left(\zeta\frac{(\Theta_\sigma(x^*))^\dagger - (L_\sigma^\lambda)^\dagger}{(U_\sigma^\lambda)^\dagger - (L_\sigma^\lambda)^\dagger}\right) > Min_\sigma\left(\zeta\frac{(\Theta_\sigma(x'))^\dagger - (L_\sigma^\lambda)^\dagger}{(U_\sigma^\lambda)^\dagger - (L_\sigma^\lambda)^\dagger}\right) \quad \text{for at least one } \sigma.\\
&\text{where} \quad 0 \leq \zeta \leq 1
\end{aligned}
$$

Now suppose that,

$$\nu' = Max_\sigma \left(\phi \frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x'))^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger} \right), \nu^* = Max_\sigma \left(\phi \frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x^*))^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger} \right),$$

$$\eta' = Min_\sigma \left(\zeta \frac{(\Theta_\sigma(x'))^\dagger - (L_\sigma^\lambda)^\dagger}{(U_\sigma^\lambda)^\dagger - (L_\sigma^\lambda)^\dagger} \right), \text{and} \quad \eta^* = Min_\sigma \left(\zeta \frac{(\Theta_\sigma(x^*))^\dagger - (L_\sigma^\lambda)^\dagger}{(U_\sigma^\lambda)^\dagger - (L_\sigma^\lambda)^\dagger} \right) \quad \text{for at least one } \sigma.$$

Then, $\nu^* \leq (<)\nu'$ and $\eta^* \geq (>)\eta'$ which gives $(\nu^* - \eta^*) < (\nu' - \eta')$ that implies the solution is not optimal which contradicts that $x'(x' \neq x^*)$ is the only one optimal solution of (5). Hence, it is an effective solution of (5). Hence the proof is now complete. □

4.1.2 Exponential-type intuitionistic hesitant membership functions approach (ETIHMFA)

The truth membership function of exponential type $\gamma_\sigma^{Ef_i}(\Theta_\sigma(x))$ and a falsity membership function of exponential type $\lambda_\sigma^{Ef_i}(\Theta_\sigma(x))$ functions under IHF environment can be explained in the following way:

For truth hesitant fuzzy membership functions:

$$\gamma_\sigma^{Ef_1}(\Theta_\sigma(x)) = \begin{cases} 1 & \text{if } \Theta_\sigma(x) \leq L_\sigma^\gamma \\ \phi_1 \left[1 - exp\left\{ -\psi \left(\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x))^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger} \right) \right\} \right] & \text{if } L_\sigma^\gamma \leq \Theta_\sigma(x)) \leq U_\sigma^\gamma \\ 0 & \text{if } \Theta_p\sigma(x)) > U_\sigma^\gamma \end{cases}$$

$$\gamma_\sigma^{Ef_2}(\Theta_\sigma(x)) = \begin{cases} 1 & \text{if } \Theta_\sigma(x) \leq L_\sigma^\gamma \\ \phi_2 \left[1 - exp\left\{ -\psi \left(\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x))^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger} \right) \right\} \right] & \text{if } L_\sigma^\gamma \leq \Theta_\sigma(x)) \leq U_\sigma^\gamma \\ 0 & \text{if } \Theta_\sigma(x)) > U_\sigma^\gamma \end{cases}$$

.....

$$\gamma_\sigma^{Ef_n}(\Theta_\sigma(x)) = \begin{cases} 1 & \text{if } \Theta_\sigma(x) \leq L_\sigma^\gamma \\ \phi_n \left[1 - exp\left\{ -\psi \left(\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x))^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger} \right) \right\} \right] & \text{if } L_\sigma^\gamma \leq \Theta_\sigma(x)) \leq U_\sigma^\gamma \\ 0 & \text{if } \Theta_\sigma(x)) > U_\sigma^\gamma \end{cases}$$

For Falsity hesitant fuzzy membership functions

$$\lambda_\sigma^{Ef_1}(\Theta_\sigma(x)) = \begin{cases} 0 & \text{if } \Theta_\sigma(x) \leq L_\sigma^\lambda \\ \zeta_1 \left[1 - exp\left\{ -\psi \left(\frac{(\Theta_\sigma(x))^\dagger - (L_\sigma^\gamma)^\dagger}{(U_\sigma^\lambda)^\dagger - (L_\sigma^\lambda)^\dagger} \right) \right\} \right] & \text{if } L_\sigma^\lambda \leq \Theta_\sigma(x)) \leq U_\sigma^\lambda \\ 1 & \text{if } \Theta_\sigma(x)) > U_\sigma^\lambda \end{cases}$$

$$\lambda_\sigma^{Ef_2}(\Theta_\sigma(x)) = \begin{cases} 0 & \text{if } \Theta_\sigma(x) \leq L_\sigma^\lambda \\ \zeta_2 \left[1 - exp\left\{ -\psi \left(\frac{(\Theta_\sigma(x))^\dagger - (L_\sigma^\gamma)^\dagger}{(U_\sigma^\lambda)^\dagger - (L_\sigma^\lambda)^\dagger} \right) \right\} \right] & \text{if } L_\sigma^\lambda \leq \Theta_\sigma(x)) \leq U_\sigma^\lambda \\ 1 & \text{if } \Theta_\sigma(x)) > U_\sigma^\lambda \end{cases}$$

.....

$$\lambda_\sigma^{Ef_n}(\Theta_\sigma(x)) = \begin{cases} 0 & \text{if } \Theta_\sigma(x) \leq L_\sigma^\lambda \\ \zeta_n \left[1 - exp\left\{ -\psi \left(\frac{(\Theta_\sigma(x))^\dagger - (L_\sigma^\gamma)^\dagger}{(U_\sigma^\lambda)^\dagger - (L_\sigma^\lambda)^\dagger} \right) \right\} \right] & \text{if } L_\sigma^\lambda \leq \Theta_\sigma(x)) \leq U_\sigma^\lambda \\ 1 & \text{if } \Theta_\sigma(x)) > U_\sigma^\lambda \end{cases}$$

Where ψ denotes the ambiguity degree or shape parameter assigned by the decision-maker.
Using by problem (4), we consider that $\gamma_\sigma^{Ef_i}(\Theta_\sigma(x)) \geq \nu_i$ and $\lambda_\sigma^{Ef_i}(\Theta_\sigma(x)) \leq \eta_i$ for $i = 1, 2, ..., n$ and $\forall\sigma$, where the parameter $\dagger > 0$.
The auxiliary parameters ν_i and η_i allow the problem (4) to be changed into (6)

$$\text{ETIHMFA} \quad Max\left(\sum_i \nu_i - \sum_i \eta_i\right)$$

Subject to

$$\begin{aligned}
&\phi_1\left[1 - exp\left\{-\psi\left(\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x))^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger}\right)\right\}\right] \geq \nu_1,\\
&\phi_2\left[1 - exp\left\{-\psi\left(\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x))^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger}\right)\right\}\right] \geq \nu_2,\\
&.........,\\
&\phi_n\left[1 - exp\left\{-\psi\left(\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x))^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger}\right)\right\}\right] \geq \nu_n;\\
&\zeta_1\left[1 - exp\left\{-\psi\left(\frac{(\Theta_\sigma(x))^\dagger - (L_\sigma^\gamma)^\dagger}{(U_\sigma^\lambda)^\dagger - (L_\sigma^\lambda)^\dagger}\right)\right\}\right] \leq \eta_1,\\
&\zeta_2\left[1 - exp\left\{-\psi\left(\frac{(\Theta_\sigma(x))^\dagger - (L_\sigma^\gamma)^\dagger}{(U_\sigma^\lambda)^\dagger - (L_\sigma^\lambda)^\dagger}\right)\right\}\right] \leq \eta_2,\\
&.........,\\
&\zeta_n\left[1 - exp\left\{-\psi\left(\frac{(\Theta_\sigma(x))^\dagger - (L_\sigma^\gamma)^\dagger}{(U_\sigma^\lambda)^\dagger - (L_\sigma^\lambda)^\dagger}\right)\right\}\right] \leq \eta_n;
\end{aligned} \tag{6}$$

$\nu_i \geq \eta_i; \nu_i + \eta_i \leq 1$ and $\eta_i, \nu_i, \phi_i, \zeta_i \in [0, 1]$ $\forall i = 1, 2, ..., n$all the constraints of (3).

Theorem 4.2. *There is only one optimal solution* (x^*, ν^*, η^*) *of (6) that is also an efficient solution to the problem (3) where* $\nu^* = (\nu_1^*, \nu_2^*, ..., \nu_n^*)$ *and* $\eta^* = (\eta_1^*, \eta_2^*, ..., \eta_n^*)$

Proof. Assume that (x^*, ν^*, η^*) be the only optimal solution of (6) that it is an inefficient solution to the problem (3). Then there exist different feasible alternative $x'(x' \neq x^*)$ of problem (3), so that $\Theta_\sigma(x^*) \leq \Theta_\sigma(x')\, \forall\sigma = 1, 2, ..., \Sigma$ and $\Theta_\sigma(x^*) < \Theta_\sigma(x')$ for at least one σ

$$\text{Therefore, we have} \quad \frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x^*))^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger} \leq \frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x'))^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger} \quad \forall \quad \sigma = 1, 2, ..., \Sigma$$

$$\text{and} \quad \frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x^*))^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger} < \frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x'))^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger} \quad \text{for atleast one} \quad \sigma,$$

$$\text{Now,} \quad 1 - exp\left\{-\psi\left(\frac{(\Theta_\sigma(x^*))^\dagger - (L_\sigma^\gamma)^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger}\right)\right\} \leq 1 - exp\left\{-\psi\left(\frac{(\Theta_\sigma(x'))^\dagger - (L_\sigma^\gamma)^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger}\right)\right\}$$

$\forall \quad \sigma = 1, 2, ..., \Sigma$

$$\text{and} \quad 1 - exp\left\{-\psi\left(\frac{(\Theta_\sigma(x^*))^\dagger - (L_\sigma^\gamma)^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger}\right)\right\} < 1 - exp\left\{-\psi\left(\frac{(\Theta_\sigma(x'))^\dagger - (L_\sigma^\gamma)^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger}\right)\right\}$$

for atleast one σ,

Thus,

$$Max_{\forall\sigma}\phi\Big[1-exp\Big\{-\psi\Big(\frac{(U_\sigma^\gamma)^\dagger-(\Theta_\sigma(x^*))^\dagger}{(U_\sigma^\gamma)^\dagger-(L_\sigma^\gamma)^\dagger}\Big)\Big\}\Big] \le Max_{\forall\sigma}\phi\Big[1-exp\Big\{-\psi\Big(\frac{(U_\sigma^\gamma)^\dagger-(\Theta_\sigma(x'))^\dagger}{(U_\sigma^\gamma)^\dagger-(L_\sigma^\gamma)^\dagger}\Big)\Big\}\Big]$$

$$\text{and}\quad Max_{\sigma}\phi\Big[1-exp\Big\{-\psi\Big(\frac{(U_\sigma^\gamma)^\dagger-(\Theta_\sigma(x^*))^\dagger}{(U_\sigma^\gamma)^\dagger-(L_\sigma^\gamma)^\dagger}\Big)\Big\}\Big] < Max_{\sigma}\phi\Big[1-exp\Big\{-\psi\Big(\frac{(U_\sigma^\gamma)^\dagger-(\Theta_\sigma(x'))^\dagger}{(U_\sigma^\gamma)^\dagger-(L_\sigma^\gamma)^\dagger}\Big)\Big\}\Big]$$

$$\text{for at least one } \sigma, \quad 0\le\phi\le 1$$

$$Min_{\forall\sigma}\zeta\Big[1-exp\Big\{-\psi\Big(\frac{(\Theta_\sigma(x^*))^\dagger-(L_\sigma^\lambda)^\dagger}{(U_\sigma^\lambda)^\dagger-(L_\sigma^\lambda)^\dagger}\Big)\Big\}\Big] \ge Min_{\forall\sigma}\zeta\Big[1-exp\Big\{-\psi\Big(\frac{(\Theta_\sigma(x'))^\dagger-(L_\sigma^\lambda)^\dagger}{(U_\sigma^\lambda)^\dagger-(L_\sigma^\lambda)^\dagger}\Big)\Big\}\Big])$$

$$\text{and}\quad Min_{\sigma}\zeta\Big[1-exp\Big\{-\psi\Big(\frac{(\Theta_\sigma(x^*))^\dagger-(L_\sigma^\gamma)^\dagger}{(U_\sigma^\lambda)^\dagger-(L_\sigma^\lambda)^\dagger}\Big)\Big\}\Big] > Min_{\sigma}\zeta\Big[1-exp\Big\{-\psi\Big(\frac{(\Theta_\sigma(x'))^\dagger-(L_\sigma^\lambda)^\dagger}{(U_\sigma^\lambda)^\dagger-(L_\sigma^\lambda)^\dagger}\Big)\Big\}\Big]$$

$$\text{for at least one } \sigma, \quad 0\le\zeta\le 1$$

Now suppose that,

$$\nu' = Max_\sigma\phi\Big[1-exp\Big\{-\psi\Big(\frac{(U_\sigma^\gamma)^\dagger-(\Theta_\sigma(x'))^\dagger}{(U_\sigma^\gamma)^\dagger-(L_\sigma^\gamma)^\dagger}\Big)\Big\}\Big]),$$

$$\nu^* = Max_\sigma\phi\Big[1-exp\Big\{-\psi\Big(\frac{(U_\sigma^\gamma)^\dagger-(\Theta_\sigma(x^*))^\dagger}{(U_\sigma^\gamma)^\dagger-(L_\sigma^\gamma)^\dagger}\Big)\Big\}\Big]),$$

$$\eta' = Min_\sigma\zeta\Big[1-exp\Big\{-\psi\Big(\frac{(\Theta_\sigma(x'))^\dagger-(L_\sigma^\lambda)^\dagger}{(U_\sigma^\lambda)^\dagger-(L_\sigma^\lambda)^\dagger}\Big)\Big\}\Big],$$

$$\text{and}\quad \eta^* = Min_\sigma\zeta\Big[1-exp\Big\{-\psi\Big(\frac{(\Theta_\sigma(x^*))^\dagger-(L_\sigma^\lambda)^\dagger}{(U_\sigma^\lambda)^\dagger-(L_\sigma^\lambda)^\dagger}\Big)\Big\}\Big] \quad \text{for at least one } \sigma.$$

Then, $\nu^* \le (<)\nu'$ and $\eta^* \ge (>)\eta'$ which gives $(\nu^*-\eta^*) < (\nu'-\eta')$ that implies the solution is not optimal which contradicts that $x'(x' \ne x^*)$ is the only one optimal solution of (6). Hence, it is an effective solution of (6). Hence the proof is now complete. □

4.1.3 Hyperbolic type intuitionistic hesitant membership functions approach (HTIHMFA)

The truth membership function of hyperbolic type $\gamma_\sigma^{Hf_i}(\Theta_\sigma(x))$ and a falsity membership function of hyperbolic type $\lambda_\sigma^{Hf_i}(\Theta_\sigma(x))$ membership functions under IHF environment can be explained in the following way: For truth hesitant fuzzy membership functions:

$$\gamma_\sigma^{Hf_1}(\Theta_\sigma(x)) = \begin{cases} 1 & \text{if } \Theta_\sigma(x) \le L_\sigma^\gamma \\ \phi_1\Big[\frac{1}{2}+\frac{1}{2}tanh\Big\{\Big(\frac{(U_\sigma^\gamma)^\dagger+(L_\sigma^\gamma)^\dagger}{2}-(\Theta_\sigma(x))^\dagger\Big)\tau_\sigma\Big\}\Big] & \text{if } L_\sigma^\gamma \le \Theta_\sigma(x)) \le U_\sigma^\gamma \\ 0 & \text{if } \Theta_p\sigma(x)) > U_\sigma^\gamma \end{cases}$$

$$\gamma_\sigma^{Hf_2}(\Theta_\sigma(x)) = \begin{cases} 1 & \text{if } \Theta_\sigma(x) \le L_\sigma^\gamma \\ \phi_2\Big[\frac{1}{2}+\frac{1}{2}tanh\Big\{\Big(\frac{(U_\sigma^\gamma)^\dagger+(L_\sigma^\gamma)^\dagger}{2}-(\Theta_\sigma(x))^\dagger\Big)\tau_\sigma\Big\}\Big] & \text{if } L_\sigma^\gamma \le \Theta_\sigma(x)) \le U_\sigma^\gamma \\ 0 & \text{if } \Theta_\sigma(x)) > U_\sigma^\gamma \end{cases}$$

.....

$$\gamma_\sigma^{Hf_n}(\Theta_\sigma(x)) = \begin{cases} 1 & \text{if } \Theta_\sigma(x) \leq L_\sigma^\gamma \\ \phi_n\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left(\frac{(U_\sigma^\gamma)^\dagger + (L_\sigma^\gamma)^\dagger}{2} - (\Theta_\sigma(x))^\dagger\right)\tau_\sigma\right\}\right] & \text{if } L_\sigma^\gamma \leq \Theta_\sigma(x)) \leq U_\sigma^\gamma \\ 0 & \text{if } \Theta_\sigma(x)) > U_\sigma^\gamma \end{cases}$$

For Falsity hesitant fuzzy membership functions

$$\lambda_\sigma^{Hf_1}(\Theta_\sigma(x)) = \begin{cases} 0 & \text{if } \Theta_\sigma(x) \leq L_\sigma^\lambda \\ \zeta_1\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left((\Theta_\sigma(x))^\dagger - \frac{(U_\sigma^\lambda)^\dagger + (L_\sigma^\lambda)^\dagger}{2}\right)\tau_\sigma\right\}\right] & \text{if } L_\sigma^\lambda \leq \Theta_\sigma(x)) \leq U_\sigma^\lambda \\ 1 & \text{if } \Theta_\sigma(x)) > U_\sigma^\lambda \end{cases}$$

$$\lambda_\sigma^{Hf_2}(\Theta_\sigma(x)) = \begin{cases} 0 & \text{if } \Theta_\sigma(x) \leq L_\sigma^\lambda \\ \zeta_2\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left((\Theta_\sigma(x))^\dagger - \frac{(U_\sigma^\lambda)^\dagger + (L_\sigma^\lambda)^\dagger}{2}\right)\tau_\sigma\right\}\right] & \text{if } L_\sigma^\lambda \leq \Theta_\sigma(x)) \leq U_\sigma^\lambda \\ 1 & \text{if } \Theta_\sigma(x)) > U_\sigma^\lambda \end{cases}$$

.....

$$\lambda_\sigma^{Hf_n}(\Theta_\sigma(x)) = \begin{cases} 0 & \text{if } \Theta_\sigma(x) \leq L_\sigma^\lambda \\ \zeta_n\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left((\Theta_\sigma(x))^\dagger - \frac{(U_\sigma^\lambda)^\dagger + (L_\sigma^\lambda)^\dagger}{2}\right)\tau_\sigma\right\}\right] & \text{if } L_\sigma^\lambda \leq \Theta_\sigma(x)) \leq U_\sigma^\lambda \\ 1 & \text{if } \Theta_\sigma(x)) > U_\sigma^\lambda \end{cases}$$

Where $\tau_\sigma = \dfrac{6}{U_\sigma - L_\sigma}$ denotes the ambiguity degree or shape parameter assigned by the decision-maker. Assume that $\gamma_\sigma^{Hf_i}(\Theta_\sigma(x)) \geq \nu_i$ and $\lambda_\sigma^{Hf_i}(\Theta_\sigma(x)) \leq \eta_i$ for $i = 1, 2, ..., n$ and $\forall\sigma$, where the parameter $\dagger > 0$. The auxiliary parameters ν_i and η_i allow the problem (4) to be changed into (7)

$$\text{HTIHMFA} \quad Max\left(\sum_i \nu_i - \sum_i \eta_i\right)$$

Subject to

$$\begin{aligned}
&\phi_1\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left(\frac{(U_\sigma^\gamma)^\dagger + (L_\sigma^\gamma)^\dagger}{2} - (\Theta_\sigma(x))^\dagger\right)\tau_\sigma\right\}\right] \geq \nu_1, \\
&\phi_2\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left(\frac{(U_\sigma^\gamma)^\dagger + (L_\sigma^\gamma)^\dagger}{2} - (\Theta_\sigma(x))^\dagger\right)\tau_\sigma\right\}\right] \geq \nu_2, \\
&........., \\
&\phi_n\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left(\frac{(U_\sigma^\gamma)^\dagger + (L_\sigma^\gamma)^\dagger}{2} - (\Theta_\sigma(x))^\dagger\right)\tau_\sigma\right\}\right] \geq \nu_n; \\
&\zeta_1\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left((\Theta_\sigma(x))^\dagger - \frac{(U_\sigma^\lambda)^\dagger + (L_\sigma^\lambda)^\dagger}{2}\right)\tau_\sigma\right\}\right] \leq \eta_1, \\
&\zeta_2\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left((\Theta_\sigma(x))^\dagger - \frac{(U_\sigma^\lambda)^\dagger + (L_\sigma^\lambda)^\dagger}{2}\right)\tau_\sigma\right\}\right] \leq \eta_2, \\
&........., \\
&\zeta_n\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left((\Theta_\sigma(x))^\dagger - \frac{(U_\sigma^\lambda)^\dagger + (L_\sigma^\lambda)^\dagger}{2}\right)\tau_\sigma\right\}\right] \leq \eta_n;
\end{aligned} \tag{7}$$

$\nu_i \geq \eta_i; \nu_i + \eta_i \leq 1$ and $\eta_i, \nu_i, \phi_i, \zeta_i \in [0,1]$ $\forall i = 1,2,...,n$ Where $\tau_\sigma = \dfrac{6}{U_\sigma - L_\sigma}$ all the constraints of (3).

Theorem 4.3. *There is only one optimal solution (x^*, ν^*, η^*) of (7) that is also an efficient solution to the problem (3) where $\nu^* = (\nu_1^*, \nu_2^*, ..., \nu_n^*)$ and $\eta^* = (\eta_1^*, \eta_2^*, ..., \eta_n^*)$*

Proof. Assume that (x^*, ν^*, η^*) be the only optimal solution of (7) that it is an inefficient solution to the problem (3). Then there exist different feasible alternative $x'(x' \neq x^*)$ of problem (3), so that $\Theta_\sigma(x^*) \leq \Theta_\sigma(x')$ $\forall \sigma = 1,2,....,\Sigma$ and $\Theta_\sigma(x^*) < \Theta_\sigma(x')$ for at least one σ .
Therefore, we have

$$tanh\left\{\left(\frac{(U_\sigma^\gamma)^\dagger + (L_\sigma^\gamma)^\dagger}{2} - (\Theta_\sigma(x^*))^\dagger\right)\tau_\sigma\right\} \leq tanh\left\{\left(\frac{(U_\sigma^\gamma)^\dagger + (L_\sigma^\gamma)^\dagger}{2} - (\Theta_\sigma(x'))^\dagger\right)\tau_\sigma\right\} \quad \forall \; \sigma = 1,2,....,\Sigma$$

$$\text{and} \quad tanh\left\{\left(\frac{(U_\sigma^\gamma)^\dagger + (L_\sigma^\gamma)^\dagger}{2} - (\Theta_\sigma(x^*))^\dagger\right)\tau_\sigma\right\} < tanh\left\{\left(\frac{(U_\sigma^\gamma)^\dagger + (L_\sigma^\gamma)^\dagger}{2} - (\Theta_\sigma(x'))^\dagger\right)\tau_\sigma\right\} \quad \text{for atleast one} \quad \sigma,$$

$$Max_{\forall\sigma}\phi\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left(\frac{(U_\sigma^\gamma)^\dagger + (L_\sigma^\gamma)^\dagger}{2} - (\Theta_\sigma(x^*))^\dagger\right)\tau_\sigma\right\}\right]$$
$$\leq Max_{\forall\sigma}\phi\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left(\frac{(U_\sigma^\gamma)^\dagger + (L_\sigma^\gamma)^\dagger}{2} - (\Theta_\sigma(x'))^\dagger\right)\tau_\sigma\right\}\right]$$
$$\text{and} \quad Max_{\sigma}\phi\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left(\frac{(U_\sigma^\gamma)^\dagger + (L_\sigma^\gamma)^\dagger}{2} - (\Theta_\sigma(x^*))^\dagger\right)\tau_\sigma\right\}\right]$$
$$< Max_{\sigma}\phi\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left(\frac{(U_\sigma^\gamma)^\dagger + (L_\sigma^\gamma)^\dagger}{2} - (\Theta_\sigma(x'))^\dagger\right)\tau_\sigma\right\}\right]$$
$$\text{for at least one } \sigma, \quad 0 \leq \phi \leq 1$$

Similarly,

$$Min_{\forall\sigma}\zeta\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left((\Theta_\sigma(x^*))^\dagger - \frac{(U_\sigma)^\dagger + (L_\sigma)^\dagger}{2}\right)\tau_\sigma\right\}\right]$$
$$\geq Min_{\forall\sigma}\zeta\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left((\Theta_\sigma(x'))^\dagger - \frac{(U_\sigma)^\dagger + (L_\sigma)^\dagger}{2}\right)\tau_\sigma\right\}\right]$$
$$\text{and} \quad Min_{\sigma}\zeta\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left((\Theta_\sigma(x^*))^\dagger - \frac{(U_\sigma)^\dagger + (L_\sigma)^\dagger}{2}\right)\tau_\sigma\right\}\right]$$
$$> Min_{\sigma}\zeta\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left((\Theta_\sigma(x'))^\dagger - \frac{(U_\sigma)^\dagger + (L_\sigma)^\dagger}{2}\right)\tau_\sigma\right\}\right]$$
$$\text{for at least one } \sigma, \quad 0 \leq \zeta \leq 1$$

Table 2: Tabulation value of objective functions

	$WG(C_1, C_2)$	$d(C_1, C_2)$
C^1	$WG(C^1)$	$d(C^1)$
C^2	$WG(C^2)$	$d(C^2)$

Now suppose that,

$$\nu' = Max_\sigma \phi\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left(\frac{(U_\sigma^\lambda)^\dagger + (L_\sigma^\lambda)^\dagger}{2} - (\Theta_\sigma(x'))^\dagger\right)\tau_\sigma\right\}\right],$$
$$\nu^* = Max_\sigma \phi\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left(\frac{(U_\sigma^\lambda)^\dagger + (L_\sigma^\lambda)^\dagger}{2} - (\Theta_\sigma(x^*))^\dagger\right)\tau_\sigma\right\}\right],$$
$$\eta' = Min_\sigma \zeta\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left((\Theta_\sigma(x'))^\dagger - \frac{(U_\sigma^\lambda)^\dagger + (L_\sigma^\lambda)^\dagger}{2}\right)\tau_\sigma\right\}\right],$$
$$\text{and} \quad \eta^* = Min_\sigma \zeta\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left((\Theta_\sigma(x^*))^\dagger - \frac{(U_\sigma^\lambda)^\dagger + (L_\sigma^\lambda)^\dagger}{2}\right)\tau_\sigma\right\}\right] \quad \text{for at least one } \sigma.$$

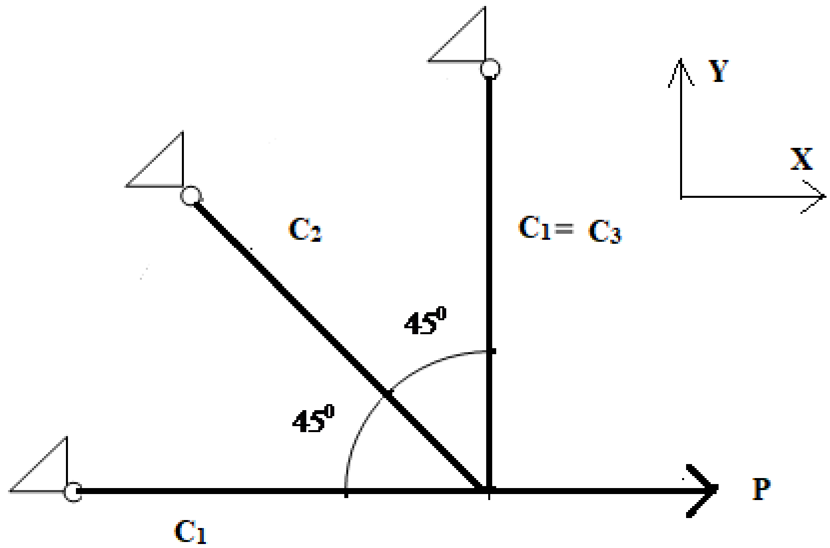
Then, $\nu^* \leq (<)\nu'$ and $\eta^* \geq (>)\eta'$ which gives $(\nu^* - \eta^*) < (\nu' - \eta')$ that implies the solution is not optimal which contradicts that $x'(x' \neq x^*)$ is the only one optimal solution of (7). Hence, it is an effective solution of (7). Hence the proof is now complete. □

5 Proposed Algorithm

5.1 Computation Algorithm for MOSP using IHF programming technique

Step 1 Solve the first goal function in the collection of objectives, (1) treating it as a single objective while taking into account the specified constraints. Evaluate the values of the objective functions and decision variables.
Step 2 Calculate the values of the remaining objectives based on the values of these decision variables.
Step 3 For the remaining objective functions, repeat **Step 1** and **Step 2**.
Step 4 As per the **Step 3**, obtained the corresponding tabulated values of objective functions from a Table 2 as follows:
Step 5 The upper and lower limits are $U_1 = max\left\{WG(C^1), WG(C^2)\right\}$,
$L_1 = min\left\{WG(C^1), WG(C^2)\right\}$ for weight function $WG(C)$, where $WG(C) \in [L_1, U_1]$ and the upper limit and lower limit of objective are $U_2 = max\left\{d(C^1), d(C^2)\right\}, L_2 = min\left\{d(C^1), d(C^2)\right\}$ for deflection function $d(C)$, where $d(C) \in [L_1, U_1]$ are identified.
Step 6 Now the IHF programming approach for MOSOP with linear (or exponential or hyperbolic) truth intuitionistic membership and falsity intuitionistic membership functions gives equivalent nonlinear programming problem as

$$\begin{aligned}
&Max\left(min\gamma_\sigma^{If_i}(WG(C))\right); Max\left(min\gamma_\sigma^{If_i}(d(C))\right);\\
&Min\left(max\lambda_\sigma^{If_i}(WG(C))\right); Min\left(max\lambda_\sigma^{If_i}(d(C))\right)\\
&\text{Subject to,} \quad [T(C)] = [T_0]\\
&C \in [C_{min}, C_{max}], \quad If_i = Lf_i, Ef_i, Hf_i; \quad i = 1, 2, .., n]\\
&\text{where} \quad x \in E = \{x \in \Re : g_j \leq or \geq b_j \quad j = 1, 2, ...m\} \quad \text{and} \quad L_i \leq x_i \leq U_i
\end{aligned} \tag{8}$$



bar truss.png bar truss.bb

Figure 1: The three-bar planar truss's design

Now, using arithmetic aggregation operator the above equation (8) can be formulated as follows:

$$\begin{aligned}
&Max\Im = \frac{\nu_1+\nu_2+.....+\nu_n}{n} - \frac{\eta_1+\eta_2+.....+\eta_n}{n} \\
\text{Subject to,}\quad &\gamma_\sigma^{If_1}(WG(C)) \geq \nu_1, \gamma_\sigma^{If_2}(WG(C)) \geq \nu_2, ..., \gamma_\sigma^{If_n}(WG(C)) \geq \nu_n; \\
&\lambda_\sigma^{If_1}(WG(C)) \leq \eta_1, \lambda_\sigma^{If_2}(WG(C)) \leq \eta_2, ..., \lambda_\sigma^{If_n}(WG(C)) \leq \eta_n; \\
&\gamma_\sigma^{If_1}(d(C)) \geq \nu_1, \gamma_\sigma^{If_2}(d(C)) \geq \nu_2, ..., \gamma_\sigma^{If_n}(d(C)) \geq \nu_n; \\
&\lambda_\sigma^{If_1}(d(C)) \leq \eta_1, \lambda_\sigma^{If_2}(d(C)) \leq \eta_2, ..., \lambda_\sigma^{If_n}(d(C)) \leq \eta_n \\
&[T(C)] = [T_0]; C \in [C_{min}, C_{max}], \quad If_i = Lf_i, Ef_i, Hf_i; C \geq 0; \\
&\nu_i, \eta_i \in [0,1]; \nu_i + \eta_i \leq 1` \quad i = 1, 2, .., n
\end{aligned} \tag{9}$$

Step 8 An appropriate mathematical programming algorithm can easily solve the above non-linear programming problem (9).

6 Numerical solution of a three-bar truss MOSOP

In Figure (1), a well-known three-bar planar truss structure is taken into consideration to minimize vertical deflection $d(C_1, C_2)$ along x and y axes at the loading point of a statistically loaded three-bar planar truss subjected to stress $T_i(C_1, C_2)$ constraints on each of the truss members $i = 1, 2, 3$ and reduce structural weight $WG(C_1, C_2)$. The MOSOP can be stated in the following manner:

Table 3: Tabulation value of objective functions

	$WG\left(C_1, C_2\right)$	$d_x\left(C_1, C_2\right)$	$d_x\left(C_1, C_2\right)$
C^1	2.187673	20	5.8578664
C^2	15	3	1
C^3	10.1	3.960784	0.03921569

$$\begin{aligned}
&\text{Minimize} \quad WG\left(C_1, C_2\right)=\delta L\left(2 \sqrt{2} C_1+C_2\right), \\
&\text{Minimize} \quad d_x\left(C_1, C_2\right)=\frac{PL\left(2 C_1+C_2\right)}{E\left(2 C_1^2+2 C_1 C_2\right)}, \\
&\text{Minimize} \quad d_y\left(C_1, C_2\right)=\frac{PLC_2}{E\left(2 C_1^2+2 C_1 C_2\right)}, \\
&\text{subject to,} \, T_1\left(C_1, C_2\right)=\frac{PL\left(2 C_1+C_2\right)}{\left(2 C_1^2+2 C_1 C_2\right)} \leq\left[T_1^T\right], \\
&T_2\left(C_1, C_2\right)=\frac{P}{\left(\sqrt{2} C_1+C_2\right)} \leq\left[T_2^T\right], \\
&T_3\left(C_1, C_2\right)=\frac{PC_2}{\left(2 C_1^2+2 C_1 C_2\right)} \leq\left[T_3^C\right], \quad C_i^{min} \leq C_i \leq C_i^{max} \quad i=1,2
\end{aligned} \tag{10}$$

Where, applied load$=P$;material density$=\delta$,$L=$ Length of each bar, maximum limit of tensile stress for bar 1 and 2 $=T_i^T$ for $i=1,2$,maximum limit of compressive stress for bar 3$=T_3^C$, Youngs modulus $=E$,$C_1=$Bar 1 and Bar 3 cross sections and $C_2=$ Bar 2 cross section. d_x and d_y are the deflection of loaded along x and $y$axes respectively.

The input data for MOSOP (10) is given as follows:

$P=20KN, \quad \delta=100KN/m^3, \quad L=1m, \left[T_1^T\right]=20KN/m^2, \quad \left[T_2^T\right]=10KN/m^2$
and $\left[T_3^C\right]=20KN/m^2, \quad E=2\times10^8 KN/m^2, \quad 0.1\times10^{-4}m^2 \leq C_1, C_2 \leq 0.5\times10^{-4}m^2$

Solution According to step 2 the corresponding tabulated values of objective functions obtained from Table 3 as follows:

Here, $WG_U^\gamma=WG_U^\lambda=15, \quad WG_L^\gamma=2.187673, WG_L^\lambda=WG_L^\gamma+\epsilon_1,$ where $0\leq\epsilon_1\leq(15-2.187673)$, $(d_x)_U^\gamma=(d_x)_U^\lambda=20, (d_x)_L^\gamma=3, (d_x)_L^\lambda+\epsilon_2,$ where $0\leq\epsilon_2\leq(20-3)$, $(d_y)_U^\gamma=(d_y)_U^\lambda=5.857864, (d_y)_L^\gamma=0.03921569, (d_y)_L^\lambda+\epsilon_3,$ where $0\leq\epsilon_3\leq(5.857864-0.03921569)$ Using the Linear type hesitant membership functions approach (LTHMFA) (5) the problem (10)
equivalent to the following (11)

$$Max\Im=\frac{1}{3}\left(\Sigma_{i=1}^3\nu_i-\Sigma_{i=1}^3\eta_i\right)$$

Subject to,
For 1st objective

$$
\begin{aligned}
&(2C_1+C_2)^{\dagger}+\Big((15)^{\dagger}-(2.187673)^{\dagger}\Big)\times\frac{\nu_1}{0.98}\le(15)^{\dagger},\\
&(2C_1+C_2)^{\dagger}+\Big((15)^{\dagger}-(2.187673)^{\dagger}\Big)\times\frac{\nu_2}{0.99}\le(15)^{\dagger},\\
&(2C_1+C_2)^{\dagger}+\Big((15)^{\dagger}-(2.187673)^{\dagger}\Big)\times\nu_3\le(15)^{\dagger}\\
&(2C_1+C_2)^{\dagger}-(2.187673)^{\dagger}-(\epsilon_1)^{\dagger}\le\Big((15)^{\dagger}-(2.187673)^{\dagger}-(\epsilon_1)^{\dagger}\Big)\times\frac{\eta_1}{0.98},\\
&(2C_1+C_2)^{\dagger}-(2.187673)^{\dagger}-(\epsilon_1)^{\dagger}\le\Big((15)^{\dagger}-(2.187673)^{\dagger}-(\epsilon_1)^{\dagger}\Big)\times\frac{\eta_2}{0.99},\\
&(2C_1+C_2)^{\dagger}-(2.187673)^{\dagger}-(\epsilon_1)^{\dagger}\le\Big((15)^{\dagger}-(2.187673)^{\dagger}-(\epsilon_1)^{\dagger}\Big)\times\eta_3
\end{aligned}
$$

For 2nd objective

$$
\begin{aligned}
&\left(\frac{20(2C_1+C_2)}{2C_1^2+2C_1C_2}\right)^{\dagger}+\Big((20)^{\dagger}-(3)^{\dagger}\Big)\times\frac{\nu_1}{0.98}\le(20)^{\dagger},\\
&\left(\frac{20(2C_1+C_2)}{2C_1^2+2C_1C_2}\right)^{\dagger}+\Big((20)^{\dagger}-(3)^{\dagger}\Big)\times\frac{\nu_2}{0.99}\le(20)^{\dagger},\\
&\left(\frac{20(2C_1+C_2)}{2C_1^2+2C_1C_2}\right)^{\dagger}+\Big((20)^{\dagger}-(3)^{\dagger}\Big)\times\nu_3\le(20)^{\dagger},\\
&\left(\frac{20(2C_1+C_2)}{2C_1^2+2C_1C_2}\right)^{\dagger}-(3)^{\dagger}-(\epsilon_2)^{\dagger}\le\Big((20)^{\dagger}-(3)^{\dagger}-(\epsilon_2)^{\dagger}\Big)\times\frac{\eta_1}{0.98},\\
&\left(\frac{20(2C_1+C_2)}{2C_1^2+2C_1C_2}\right)^{\dagger}-(3)^{\dagger}-(\epsilon_2)^{\dagger}\le\Big((20)^{\dagger}-(3)^{\dagger}-(\epsilon_2)^{\dagger}\Big)\times\frac{\eta_2}{0.99},\\
&\left(\frac{20(2C_1+C_2)}{2C_1^2+2C_1C_2}\right)^{\dagger}-(3)^{\dagger}-(\epsilon_2)^{\dagger}\le\Big((20)^{\dagger}-(3)^{\dagger}-(\epsilon_2)^{\dagger}\Big)\times\eta_3
\end{aligned}
\tag{11}
$$

For 3rd objective

$$
\begin{aligned}
&\left(\frac{20C_2}{2C_1^2+2C_1C_2}\right)^{\dagger}+\Big((5.857864)^{\dagger}-(0.03921569)^{\dagger}\Big)\times\frac{\nu_1}{0.98}\le(5.857864)^{\dagger},\\
&\left(\frac{20C_2}{2C_1^2+2C_1C_2}\right)^{\dagger}+\Big((5.857864)^{\dagger}-(0.03921569)^{\dagger}\Big)\times\frac{\nu_2}{0.99}\le(5.857864)^{\dagger},\\
&\left(\frac{20C_2}{2C_1^2+2C_1C_2}\right)^{\dagger}+\Big((5.857864)^{\dagger}-(0.03921569)^{\dagger}\Big)\times\nu_3\le(5.857864)^{\dagger},\\
&\left(\frac{20C_2}{2C_1^2+2C_1C_2}\right)^{\dagger}-(0.03921569)^{\dagger}-(\epsilon_3)^{\dagger}\le\Big((5.857864)^{\dagger}-(0.03921569)^{\dagger}-(\epsilon_3)^{\dagger}\Big)\times\frac{\eta_1}{0.98},\\
&\left(\frac{20C_2}{2C_1^2+2C_1C_2}\right)^{\dagger}-(0.03921569)^{\dagger}-(\epsilon_3)^{\dagger}\le\Big((5.857864)^{\dagger}-(0.03921569)^{\dagger}-(\epsilon_3)^{\dagger}\Big)\times\frac{\eta_2}{0.99},\\
&\left(\frac{20C_2}{2C_1^2+2C_1C_2}\right)^{\dagger}-(0.03921569)^{\dagger}-(\epsilon_3)^{\dagger}\le\Big((5.857864)^{\dagger}-(0.03921569)^{\dagger}-(\epsilon_3)^{\dagger}\Big)\times\eta_3,\\
&\nu_i\ge\eta_i,\nu_i+\eta_i\le 1,\nu_i,\eta_i\in[0,1];`\quad i=1,2,3\quad\text{and all the constraints of (10).}
\end{aligned}
$$

Using the Exponential type hesitant membership functions approach (ETHMFA) (6) the problem (10) equivalent to the following (12)

$$Max\Im = \frac{1}{3}\left(\Sigma_{i=1}^{3}\nu_i - \Sigma_{i=1}^{3}\eta_i\right)$$

Subject to,
For 1st objective

$$\begin{aligned}
&(2C_1+C_2)^{\dagger} - \left((15)^{\dagger} - (2.187673)^{\dagger}\right) \times ln\left(1-\frac{\nu_1}{0.98}\right)/\psi \leq (15)^{\dagger},\\
&(2C_1+C_2)^{\dagger} - \left((15)^{\dagger} - (2.187673)^{\dagger}\right) \times ln\left(1-\frac{\nu_2}{0.99}\right)/\psi \leq (15)^{\dagger},\\
&(2C_1+C_2)^{\dagger} - \left((15)^{\dagger} - (2.187673)^{\dagger}\right) \times ln\left(1-\nu_3\right)/\psi \leq (15)^{\dagger}\\
&(2C_1+C_2)^{\dagger} - (2.187673)^{\dagger} - (\epsilon_1)^{\dagger} \leq \left((15)^{\dagger} - (2.187673)^{\dagger} - (\epsilon_1)^{\dagger}\right) \times \left\{-ln\left(1-\frac{\eta_1}{0.98}\right)\right\}/\psi,\\
&(2C_1+C_2)^{\dagger} - (2.187673)^{\dagger} - (\epsilon_1)^{\dagger} \leq \left((15)^{\dagger} - (2.187673)^{\dagger} - (\epsilon_1)^{\dagger}\right) \times \left\{-ln\left(1-\frac{\eta_2}{0.99}\right)\right\}/\psi,\\
&(2C_1+C_2)^{\dagger} - (2.187673)^{\dagger} - (\epsilon_1)^{\dagger} \leq \left((15)^{\dagger} - (2.187673)^{\dagger} - (\epsilon_1)^{\dagger}\right) \times \{-ln\left(1-\eta_3\right)\}/\psi
\end{aligned}$$

For 2nd objective

$$\begin{aligned}
&\left(\frac{20(2C_1+C_2)}{2C_1^2+2C_1C_2}\right)^{\dagger} - \left((20)^{\dagger} - (3)^{\dagger}\right) \times ln\left(1-\frac{\nu_1}{0.98}\right)/\psi \leq (20)^{\dagger},\\
&\left(\frac{20(2C_1+C_2)}{2C_1^2+2C_1C_2}\right)^{\dagger} - \left((20)^{\dagger} - (3)^{\dagger}\right) \times ln\left(1-\frac{\nu_2}{0.99}\right)/\psi \leq (20)^{\dagger},\\
&\left(\frac{20(2C_1+C_2)}{2C_1^2+2C_1C_2}\right)^{\dagger} - \left((20)^{\dagger} - (3)^{\dagger}\right) \times ln\left(1-\nu_3\right)/\psi \leq (20)^{\dagger},\\
&\left(\frac{20(2C_1+C_2)}{2C_1^2+2C_1C_2}\right)^{\dagger} - (3)^{\dagger} - (\epsilon_2)^{\dagger} \leq \left((20)^{\dagger} - (3)^{\dagger} - (\epsilon_2)^{\dagger}\right) \times \left\{-ln\left(1-\frac{\eta_1}{0.98}\right)\right\}/\psi,\\
&\left(\frac{20(2C_1+C_2)}{2C_1^2+2C_1C_2}\right)^{\dagger} - (3)^{\dagger} - (\epsilon_2)^{\dagger} \leq \left((20)^{\dagger} - (3)^{\dagger} - (\epsilon_2)^{\dagger}\right) \times \left\{-ln\left(1-\frac{\eta_2}{0.99}\right)\right\}/\psi,\\
&\left(\frac{20(2C_1+C_2)}{2C_1^2+2C_1C_2}\right)^{\dagger} - (3)^{\dagger} - (\epsilon_2)^{\dagger} \leq \left((20)^{\dagger} - (3)^{\dagger} - (\epsilon_2)^{\dagger}\right) \times \{-ln\left(1-\eta_3\right)\}/\psi
\end{aligned} \tag{12}$$

For 3rd objective

$$\begin{aligned}
&\left(\frac{20C_2}{2C_1^2+2C_1C_2}\right)^{\dagger} - \left((5.857864)^{\dagger} - (0.03921569)^{\dagger}\right) \times ln\left(1-\frac{\nu_1}{0.98}\right)/\psi \leq (5.857864)^{\dagger},\\
&\left(\frac{20C_2}{2C_1^2+2C_1C_2}\right)^{\dagger} - \left((5.857864)^{\dagger} - (0.03921569)^{\dagger}\right) \times ln\left(1-\frac{\nu_2}{0.99}\right)/\psi \leq (5.857864)^{\dagger},\\
&\left(\frac{20C_2}{2C_1^2+2C_1C_2}\right)^{\dagger} - \left((5.857864)^{\dagger} - (0.03921569)^{\dagger}\right) \times ln\left(1-\nu_3\right)/\psi \leq (5.857864)^{\dagger},
\end{aligned}$$

$\left(\frac{20C_2}{2C_1^2+2C_1C_2}\right)^{\dagger}-(0.03921569)^{\dagger}-(\epsilon_3)^{\dagger}\le\left((5.857864)^{\dagger}-(0.03921569)^{\dagger}-(\epsilon_3)^{\dagger}\right)$
$\times\left\{-ln\left(1-\frac{\eta_1}{0.98}\right)\right\}/\psi,$
$\left(\frac{20C_2}{2C_1^2+2C_1C_2}\right)^{\dagger}-(0.03921569)^{\dagger}-(\epsilon_3)^{\dagger}\le\left((5.857864)^{\dagger}-(0.03921569)^{\dagger}-(\epsilon_3)^{\dagger}\right)$
$\times\left\{-ln\left(1-\frac{\eta_2}{0.99}\right)\right\}/\psi,$
$\left(\frac{20C_2}{2C_1^2+2C_1C_2}\right)^{\dagger}-(0.03921569)^{\dagger}-(\epsilon_3)^{\dagger}\le\left((5.857864)^{\dagger}-(0.03921569)^{\dagger}-(\epsilon_3)^{\dagger}\right)$
$\times\{-ln\,(1-\eta_3)\}/\psi,$
$\nu_i\ge\eta_i,\nu_i+\eta_i\le1,\nu_i,\eta_i\in[0,1];` \quad i=1,2,3$ and all the constraints of (10).

Using the Hyperbolic type hesitant membership functions approach (HTHMFA) (7) the problem (10) equivalent to the following (13)
For 1st objective

$$Max\Im=\frac{1}{3}\left(\Sigma_{i=1}^{3}\nu_i-\Sigma_{i=1}^{3}\eta_i\right)$$

Subject to,
For 1st objective

$$(2C_1+C_2)^{\dagger}\,\tau_{WG(C)}+tanh^{-1}\left(\frac{2\nu_1}{0.98}-1\right)\le\frac{\tau_{WG(C)}}{2}\left((15)^{\dagger}+(2.187673)^{\dagger}\right),$$
$$(2C_1+C_2)^{\dagger}\,\tau_{WG(C)}+tanh^{-1}\left(\frac{2\nu_2}{0.99}-1\right)\le\frac{\tau_{WG(C)}}{2}\left((15)^{\dagger}+(2.187673)^{\dagger}\right),$$
$$(2C_1+C_2)^{\dagger}\,\tau_{WG(C)}+tanh^{-1}(2\nu_3-1)\le\frac{\tau_{WG(C)}}{2}\left((15)^{\dagger}+(2.187673)^{\dagger}\right),$$
$$(2C_1+C_2)^{\dagger}\,\tau_{WG(C)}-tanh^{-1}\left(\frac{2\eta_1}{0.98}-1\right)\le\frac{\tau_{WG(C)}}{2}\left((15)^{\dagger}+(2.187673)^{\dagger}+(\epsilon_1)^{\dagger}\right),$$
$$(2C_1+C_2)^{\dagger}\,\tau_{WG(C)}-tanh^{-1}\left(\frac{2\eta_2}{0.99}-1\right)\le\frac{\tau_{WG(C)}}{2}\left((15)^{\dagger}+(2.187673)^{\dagger}+(\epsilon_1)^{\dagger}\right),$$
$$(2C_1+C_2)^{\dagger}\,\tau_{WG(C)}-tanh^{-1}\left(2\eta_3-1\right)\le\frac{\tau_{WG(C)}}{2}\left((15)^{\dagger}+(2.187673)^{\dagger}+(\epsilon_1)^{\dagger}\right)$$

Table 4: The input values for MOSOP (10)

P (KN)	δ (KN/m^3)	$L(m)$	$[T_1^T]$ (KN/m^2)	$[T_2^T]$ (KN/m^2)	$[T_3^C]$ (KN/m^2)	E (KN/m^2)	$C_i^{min}.C_i^{max}$ $(10^{-4}m^2)$
20	100	1	20	10	20	2×10^7	$C_1^{min}=0.1, C_1^{max}=5.0,$ $C_2^{min}=0.1, C_2^{max}=5.0$

For 2nd objective

$$\begin{aligned}
&\left(\frac{20(2C_1+C_2)}{2C_1^2+2C_1C_2}\right)^{\dagger}\tau_{d_x(C)}+tanh^{-1}\left(\frac{2\nu_1}{0.98}-1\right)\le\frac{\tau_{d_x(C)}}{2}\left((20)^{\dagger}+(3)^{\dagger}\right),\\
&\left(\frac{20(2C_1+C_2)}{2C_1^2+2C_1C_2}\right)^{\dagger}\tau_{d_x(C)}+tanh^{-1}\left(\frac{2\nu_2}{0.99}-1\right)\le\frac{\tau_{d_x(C)}}{2}\left((20)^{\dagger}+(3)^{\dagger}\right),\\
&\left(\frac{20(2C_1+C_2)}{2C_1^2+2C_1C_2}\right)^{\dagger}\tau_{d_x(C)}+tanh^{-1}\left(2\nu_3-1\right)\le\frac{\tau_{d_x(C)}}{2}\left((20)^{\dagger}+(3)^{\dagger}\right),\\
&\left(\frac{20(2C_1+C_2)}{2C_1^2+2C_1C_2}\right)^{\dagger}\tau_{d_x(C)}-tanh^{-1}\left(\frac{2\eta_1}{0.98}-1\right)\le\frac{\tau_{d_x(C)}}{2}\left((20)^{\dagger}+(3)^{\dagger}+(\epsilon_2)^{\dagger}\right),\\
&\left(\frac{20(2C_1+C_2)}{2C_1^2+2C_1C_2}\right)^{\dagger}\tau_{d_x(C)}-tanh^{-1}\left(\frac{2\eta_2}{0.99}-1\right)\le\frac{\tau_{d_x(C)}}{2}\left((20)^{\dagger}+(3)^{\dagger}+(\epsilon_2)^{\dagger}\right),\\
&\left(\frac{20(2C_1+C_2)}{2C_1^2+2C_1C_2}\right)^{\dagger}\tau_{d_x(C)}-tanh^{-1}\left(2\eta_3-1\right)\le\frac{\tau_{d_x(C)}}{2}\left((20)^{\dagger}+(3)^{\dagger}+(\epsilon_2)^{\dagger}\right)
\end{aligned}\tag{13}$$

For 3rd objective

$$\begin{aligned}
&\left(\frac{20C_2}{2C_1^2+2C_1C_2}\right)^{\dagger}\tau_{d_y(C)}+tanh^{-1}\left(\frac{2\nu_1}{0.98}-1\right)\le\frac{\tau_{d_y(C)}}{2}\left((5.857864)^{\dagger}+(0.03921569)^{\dagger}\right),\\
&\left(\frac{20C_2}{2C_1^2+2C_1C_2}\right)^{\dagger}\tau_{d_y(C)}+tanh^{-1}\left(\frac{2\nu_2}{0.99}-1\right)\le\frac{\tau_{d_y(C)}}{2}\left((5.857864)^{\dagger}+(0.03921569)^{\dagger}\right),\\
&\left(\frac{20C_2}{2C_1^2+2C_1C_2}\right)^{\dagger}\tau_{d_y(C)}+tanh^{-1}\left(2\nu_3-1\right)\le\frac{\tau_{d_y(C)}}{2}\left((5.857864)^{\dagger}+(0.03921569)^{\dagger}\right),
\end{aligned}$$

$$\begin{aligned}
&\left(\frac{20C_2}{2C_1^2+2C_1C_2}\right)^{\dagger}\tau_{d_y(C)}-tanh^{-1}\left(\frac{2\eta_1}{0.98}-1\right)\le\frac{\tau_{d_y(C)}}{2}\left((5.857864)^{\dagger}+(0.03921569)^{\dagger}+(\epsilon_3)^{\dagger}\right),\\
&\left(\frac{20C_2}{2C_1^2+2C_1C_2}\right)^{\dagger}\tau_{d_y(C)}-tanh^{-1}\left(\frac{2\eta_2}{0.99}-1\right)\le\frac{\tau_{d_y(C)}}{2}\left((5.857864)^{\dagger}+(0.03921569)^{\dagger}+(\epsilon_3)^{\dagger}\right),\\
&\left(\frac{20C_2}{2C_1^2+2C_1C_2}\right)^{\dagger}\tau_{d_y(C)}-tanh^{-1}\left(2\eta_3-1\right)\le\frac{\tau_{d_y(C)}}{2}\left((5.857864)^{\dagger}+(0.03921569)^{\dagger}+(\epsilon_3)^{\dagger}\right)
\end{aligned}$$

where $\tau_{WG(C)}=\dfrac{6}{15-2.187673}$ $\tau_{d_x(C)}=\dfrac{6}{20-3}$ and $\tau_{d_y(C)}=\dfrac{6}{5.857864-0.03921569}$,

$\nu_i\ge\eta_i,\nu_i+\eta_i\le1,\nu_i,\eta_i\in[0,1];`\quad i=1,2,3$ and all the constraints of (10).

Comparison of optimal solution of MOSOP (10) using several methods.

The Pareto optimal solution of MOSOP model (10) using fuzzy, intuitionistic fuzzy, and intuitionistic hesitant fuzzy multi-objective nonlinear programming techniques is given in Table 5. Here we get the best

Table 5: Optimal values on Structural Weight and Deflections for † = 1

Membership function	Various algorithm MONLP	$C_1 \times 10^{-4} m^2$	$C_2 \times 10^{-4} m^2$	$WG(C_1, C_2)$	$d_x(C_1, C_2)$	$d_y(C_1, C_2)$
Linear-type	Fuzzy multi-objective nonlinear programming[14]	2.677489	0.1000000	5.454979	7.335216	0.1344683
Linear-type	Intuitionistic fuzzy multi-objective nonlinear programming $\epsilon_1 = 0.76873962$, $\epsilon_2 = 1.7$, $\epsilon_3 = 0.2480392$ [14]	2.613073	0.1000000	5.326147	7.512768	0.1410545
Linear type	Proposed Method $\epsilon_1 = 0.76873962$, $\epsilon_2 = 1.7$, $\epsilon_3 = 0.2480392$	2.576483	0.1000000	5.252965	7.617507	0.1450135
Exponential -type	Proposed Method $\epsilon_1 = 0.76873962$, $\epsilon_2 = 1.7$, $\epsilon_3 = 0.2480392$	2.677490	0.1000000	5.454980	7.335215	0.1344682
Hyperbolic -type	Proposed Method $\epsilon_1 = 0.76873962$, $\epsilon_2 = 1.7$, $\epsilon_3 = 0.2480392$	2.471704	0.1000000	5.043407	7.934265	0.13573195

solution for different tolerances $\epsilon_1, \epsilon_2, \epsilon_3$ for non-membership function of objective functions. The Table 5 shows that the proposed intuitionistic hesitant fuzzy optimization technique gives a better Pareto optimal solution from the perspective of structural optimization.

7 Conclusion and Future Implication

To demonstrate the performance of the stated algorithm, a numerical example is given and compare their results with the existing studies[14]. It is concluded from this study that the proposed work gives more reasonable ways to handle the hesitant fuzzy information to solve practical problems.

In the future, we shall lengthen the methodology of intuitionistic hesitant fuzzy optimization technique to the diverse fuzzy environment as well as different fields of application such as transportation, networking, portfolio management, and emerging decision problems.

Acknowledgements: The authors are greatful to the reviewers for their useful comments.

Conflict of Interest: There is no conflict of interest among authors.

References

[1] Ahmadini AAH, Ahmad F. A novel intuitionistic fuzzy preference relations for multiobjective goal programming problems. *Journal of Intelligent & Fuzzy Systems*. 2021; 40(3): 4761-4777. DOI: http://doi.org/10.3233/JIFS-201588

[2] Akram M, Shah SMU, Al-Shamiri MMA, Edalatpanah SA. Fractional transportation problem under interval-valued Fermatean fuzzy sets. *AIMS Mathematics*. 2022; 7(9): 17327-17348. DOI: http://doi.org/10.3934/math.2022954

[3] Angelov PP. Optimization in an intuitionistic fuzzy environment. *Fuzzy sets and Systems*. 1997; 86(3): 299-306. DOI: http://doi.org/10.1016/S0165-0114(96)00009-7

[4] Atanassov KT. Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*. 1986; 20: 87-96. DOI: http://doi.org/10.1016/S0165-0114(86)80034-3

[5] Atanassov KT. Interval valued intuitionistic fuzzy sets. *Intuitionistic Fuzzy Sets: Theory and Applications*. 1999. p.139-177. DOI: http://doi.org/10.1007/978-3-7908-1870-3_2

[6] Bellman RE, Zadeh LA. Decision-making in a fuzzy environment. *Management science*. 1970; 17(4): 141-164. DOI: http://doi.org/10.1287/mnsc.17.4.B141

[7] Bharati SK. Solving optimization problems under hesitant fuzzy environment. *Life Cycle Reliability and Safety Engineering*. 2018; 7(3): 127-136. DOI: http://doi.org/10.1007/s41872-018-0052-x

[8] Bharati SK. An interval-valued intuitionistic hesitant fuzzy methodology and application. *New Generation Computing*. 2021; 39(2): 377-407. DOI: http://doi.org/10.1007/s00354-021-00132-4

[9] Bharati SK. A new interval-valued hesitant fuzzy-based optimization method. *New Mathematics and Natural Computation*. 2022; 18(02): 469-494. DOI: http://doi.org/10.1142/S1793005722500235

[10] Bharati SK. Hesitant intuitionistic fuzzy algorithm for multiobjective optimization problem. *Operational Research*. 2022; 22(4): 3521-3547. DOI: http://doi.org/10.1007/s12351-021-00685-8

[11] Changwen X. Fuzzy optimization of structures by the two-phase method. *Computers and Structures*. 1989; 31(4): 575-580. DOI: http://doi.org/10.1016/0045-7949(89)90334-9

[12] Yi-Cherng Y, Deh-Shiu H. Structural optimization with fuzzy parameters. *Computers and structures*. 1990; 37(6): 917-924. DOI: http://doi.org/10.1016/0045-7949(90)90005-M

[13] Dey S, Roy TK. Intuitionistic fuzzy goal programming technique for solving non-linear multi-objective structural problem. *Journal of Fuzzy Set Valued Analysis*. 2015; 3: 179-193. DOI: http://doi.org/10.5899/2015/jfsva-00243

[14] Dey S, Roy TK. Multi-objective Structural Optimization Using Fuzzy and Intuitionistic Fuzzy Optimization. *International Journal of Intelligent Systems Technologies and Applications*. 2015; 7(5): 57-65. DOI: http://doi.org/10.5815/ijisa.2015.05.08

[15] Dey S, Roy TK. Multi-objective structural design problem optimization using parameterized t-norm based fuzzy optimization programming Technique. *Journal of Intelligent & Fuzzy Systems*. 2016; 30(2): 971-982. DOI: http://doi.org/10.3233/IFS-151819

[16] Dymova L, Kaczmarek K, Sevastjanov P, Kulawik J. A fuzzy multiple criteria decision making approach with a complete user friendly computer implementation. *Entropy*. 2021; 23(2): 203. DOI: http://doi.org/10.3390/e23020203

[17] Goguen JA, Zadeh LA. Fuzzy sets. *Information and control*. 1965; 8: 338-353. DOI: http://doi.org/10.1016/S0019-9958(65)90241-X

[18] Jana B, Roy TK. Multi-objective intuitionistic fuzzy linear programming and its application in transportation model. *Notes on Intuitionistic Fuzzy Sets*. 2007; 13(1): 34-51. DOI: http://doi.org/nifs/13/1/34-51

[19] Luo Y, Yu C. A fuzzy optimization method for multi-criteria decision making problem based on the inclusion degrees of intuitionistic fuzzy sets. *Journal of Information & Computing Science*. 2008; 3(2): 146-152. DOI: http://doi.org/10.1007/978-3-540-85984-0_41

[20] Ranjbar M, Miri SM, Effati S. Arithmetic operations and ranking of hesitant fuzzy numbers by extension principle. *Iranian Journal of Fuzzy Systems*. 2022; 19(1): 97-114. DOI: http://doi.org/10.22111/IJFS.2022.6554

[21] Sarkar M, Roy TK. Multi-objective welded beam design optimization using T-norm and T-co-norm based intuitionistic fuzzy optimization technique. *Advances in Fuzzy Mathematics*. 2017; 12(3): 549-575. DOI: http://doi.org/10.37622/AFM/12.3.2017.549-575

[22] Sharma MK, Dhaka A, Nandal A, Rosales HG, Monteagudo FEL, Hernndez AG, Hoang VT. Fermatean Fuzzy Programming with New Score Function: A New Methodology to Multi-Objective Transportation Problems. *Electronics*. 2023; 12(2): 277. DOI: http://doi.org/10.3390/electronics12020277

[23] Shih CJ, Chi CC, Hsiao JH. Alternative α-level-cuts methods for optimum structural design with fuzzy resources. *Computers & structures*. 2003; 81(28-29): 2579-2587. DOI: http://doi.org/10.1016/S0045-7949(03)00331-6

[24] Torra V, Narukawa Y. On hesitant fuzzy sets and decision. *IEEE international conference on fuzzy systems, Jeju, Korea (South)*. 2009. p.1378-1382. DOI: http://doi.org/10.1109/FUZZY.2009.5276884

[25] Torra V. Hesitant fuzzy sets. *International journal of intelligent systems*. 2010; 25(6): 529-539. DOI: http://doi.org/10.1002/int.20418

[26] Wan SP, Qin YL, Dong JY. A hesitant fuzzy mathematical programming method for hybrid multi-criteria group decision making with hesitant fuzzy truth degrees. *Knowledge-Based Systems*. 2017; 138: 232-248. DOI: http://doi.org/10.1016/j.knosys.2017.10.002

[27] Wang GY, Wang WQ. Fuzzy optimum design of structures. *In Engineering Mechanics in Civil Engineering*. 1985; 8(4): 50-53. DOI: http://doi.org/10.1080/03052158508902494

[28] Xia M, Xu Z. Hesitant fuzzy information aggregation in decision making. *International journal of approximate reasoning*. 2011; 52(3): 395-407. DOI: http://doi.org/10.1016/j.ijar.2010.09.002

[29] Zadeh LA. Outline of a new approach to the analysis of complex systems and decision processes. *IEEE Transactions on systems, Man, and Cybernetics*. 1973; 3(1): 28-44. DOI: http://doi.org/10.1109/TSMC.1973.5408575

[30] Zimmermann HJ. Fuzzy programming and linear programming with several objective functions. *Fuzzy sets & systems*. 1978; 1(1): 45-55. DOI: http://doi.org/10.1016/0165-0114(78)90

Sanjoy Biswas
Department of Applied Science and Humanities (Mathematics)
Guru Nanak Institute of Technology
Sodepur,Kolkata 700114, West Bengal, India
E-mail: sanjoy.biswas@gnit.ac.in

Samir Dey
Department of Mathematics
JIS University
81, Nilgunj Road, Agarpara, Kolkata 700109, West Bengal, India
E-mail: samir.dey@jisuniversity.ac.in

Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://tfss.journals.iau.ir/
Online ISSN: 2821-0131
Vol.2, No.2, (2023), 63-76
DOI: http://doi.org/10.30495/TFSS.2023.1982526.1067

Article Type: Original Research Article

On the Inferential Zigzag and Its Activation Towards Clarifying What It Is Commonsense Reasoning

Marco Elio Tabacchi*, **Settimo Termini**, **Enric Trillas**

Abstract. This paper has a twofold goal: The first is to study how the inferential zigzag can be activated, even computationally, trying to analyse what kind of reasoning consists of, where its 'mechanism' is rooted, how it can be activated since without all this it can just seem a metaphysical idea. The second, not so deeply different - as it can be presumed at a first view - but complementary, is to explore the subject's link with the old thought on conjectures of the 15th Century Theologist and Philosopher Nicolaus Cusanus who was the first thinker consciously and extensively using conjectures.

AMS Subject Classification 2020: 03B65
Keywords and Phrases: Commonsense reasoning, Language at work, Inferential zigzag, 'Out of logic'.

1 Introduction

The present paper is written with a very specific target: the clarification of some aspects of the *inferential zigzag* introduced in [14, 19, 16] previously not analysed in detail. However, in order to make explicit and clear the motivations behind the technical results, the paper contains also a few general conceptual remarks as well as a final Section in which some considerations expressed by the Renaissance logician Nicolaus Cusanus are briefly surveyed in connection with what is presented in the technical side of the paper. The aim is to reach a better understanding of what the inferential zigzag seems to consist of, and of how it can be practically, specifically produced. That is, to explain how, at each statement p, a mixed inferential chain can start; to explain how the zigzag proceeds by inflexions either forward or backward and leading, finally, to another statement q, such that either $p < q$, or $q < p$ (with the symbol $<$ as a shorthand for the conditional statement If p, then q). In [14, 19, 16] attention was focused more on the concepts behind the proposal, than the practicality of the algorithmic path that could be followed. Without the clarifications in the present paper, the zigzag can be seen just as a more or less interesting, but purely theoretical idea. Let us clarify, however, that it is (and it always was) manifest from the beginning that the idea is constructive in nature, inherently lending itself to subsequent implementations. This is witnessed by the fact that a number of considerations on how to reduce the complexity of its possible implementations, which without any specific strategy appears to be exponential in time, were discussed [19]. So, the point in question has not to do with this general aspect, but with the possibility of suggesting a specific path to be followed that seems from a conceptual point of view peculiarly in synch with the theoretical aspects discussed in [14] and [19]. This opens the way for a further examination of possible strategies for implementation, based on already established mathematical and computational intelligence models that can be suitably matched to the essence

*Corresponding Author: Marco Elio Tabacchi, Email: marcoelio.tabacchi@unipa.it. ORCID: 0000-0001-7230-8192.
Received: 16 March 2023; **Revised**: 8 May 2023; **Accepted**: 21 May 2023; **Available Online**: 22 May 2023; **Published Online**: 7 November 2023.

How to cite: Tabacchi ME, Termini S, Trillas E. On the Inferential Zigzag and Its Activation Towards Clarifying What It Is Commonsense Reasoning. *Trans. Fuzzy Sets Syst.* 2023; 2(2): 63-76. DOI: http://doi.org/10.30495/TFSS.2023.1982526.1067

of the problem at hand. Detailed implementation procedures will not be discussed here, and left for a more technical paper to follow and currently under preparation. Let us, moreover, observe that our proposed path, the zigzag, in itself is quite unusual in the field of Logic, not only among the standard approaches but also in those (occasionally called deviant, but far more common than it seems at first) cases in which uncertainty and imprecision are taken into account, in the same vein of other concepts that preceded its introduction, e.g. conjecturing and linked notions. This is the true reason why some clarifying conceptual remarks appear in the paper. A few general epistemological and historical comments seem necessary to understand the way the idea has developed and how it is connected to its historical antecedents. May the zigzag be a non classic approach, it is nonetheless well routed in the history of Logic, or, perhaps better, the study of reasoning. The paper is structured in the following way. The present Introduction is followed by three Sections all devoted to purely theoretical analyses which, respectively, *present a model of commonsense reasoning*, *introduce the Zigzag*, and *develop its first properties.* Section 5 provides some preliminary considerations about the computational costs of the process considered; finally, Section 6 surveys some remarks on Conjectures done by Nicolaus Cusanus, that are connected with the inferential zigzag.

2 Around a Model of Commonsense Reasoning

Let us remark that the language or, at least, the wording used in the following, is not the usual one found in the majority of logic papers. In fact, we are here trying just to approach some specific aspects of what we called Language at Work [19] without referring to the general setting of mathematical logic, of which we are in fact out. We shall then present a very simple model of Commonsense Reasoning, another name for the language at work, where the zigzag idea was born, and that has a very soft mathematical structure in which usual laws like those of Duality are often not valid. In this setting, in fact, some laws are not universal, but have only a local validity. Few laws seem to warrant the typical flexibility shown by both natural language and commonsense reasoning. It may be worth noting that since thinking is a natural phenomenon as breathing is, reasoning should also be seen and considered as a natural phenomenon.

2.1 Reasoning from a premise

The natural phenomenon of reasoning tries, departing from a given information, linguistically compacted in a statement p called the premise, to reach a previously unknown conclusion (concerning what is described by p), also compacted in a linguistic statement q, such that either the existing knowledge on p results increased, or diminished. In the first case, symbolized by $p < q$, q is a consequence of p, in the second symbolized by $q < p$, q is a hypothesis or explanation for p. In general, reasoning is seen, and defined, as the action of refuting or conjecturing q from p, under which p and q are linked by $p < q'$ in the first case, and by $p < /q'$ in the second. That is, q is a conjecture from p whenever $not - q(q')$ is not a consequence of p; it cannot be stated $p < q'$, $not - q$ cant be deduced from p. In different words, q is not a refutation of p, q does not refute or contradict p[17].

In short, reasoning from a premise p is but finding either a refutation, or a conjecture. The only condition the premise p is supposed to verify is: $p < /p'$; p is not self-refuting, self-contradictory or, in Aristotles ancient words of wisdom, p is an inferentially impossible statement. The premise should indicate something not impossible but possible, something sensate. Notice that for these first concepts only relation $<$ and negation not $(')$ are needed.

The (inferential) binary relation $<$ between any two statements such as p and q. $p < q$, translates the conditional statement If p, then q that, as it is well known, in language is not always understood in the same form. The conditional statement $p < q$ is sometimes taken as one of the unconditional statements not p or q, or not p or (p and q), or p and q, etc.

Understanding the conditional statement $p < q$ by an unconditional one like it is, for instance, p and q, is for describing it in a form that, in principle, can be submitted to some kind of verification. Reasoning is a kingdom in which the density of conditional statements is remarkable; reasoning requires a good management of conditional statements.

It is such non-uniqueness a motive to consider $<$ as a primitive, undefined relation, only submitted to be reflexive to be sure that it is never empty and that, at each situation and/or context, should be represented by translating how the conditional If/then is there understood; for instance, and respectively, representing $p < q$, by $p' + q$, $p' + p \cdot q$, $p \cdot q$, $q + p' \cdot q'$, etc., shortening and by a point (), and or by a cross (+), like not is shortened by a comma ('). It should be noticed that such operators are not supposed to be endowed with (usual) properties like idempotence $(p+p=p)$, commutativity $(p \cdot q = q \cdot p)$,associativity$(p \cdot (q \cdot r) = (p \cdot q) \cdot r)$, etc., considered, if existing, local properties that is, not holding in all the universe of statements but only in some part, or parts, of it.

2.2 Inferential situations for a conclusion

It should come to notice that between whatsoever statements p and q, it can just exist one of the four inferential situations:

1. $p < q$; I, e., q is a consequence of p.
2. $q < p$; I, e., q is a hypothesis or explanation for p.
3. Both $p < q$ and $q < p$, written $p \sim q$ or $q \sim p$, i.e., p and q are inferentially equivalent.
4. Neither $p < q$, nor $q < p$, written $p \perp q$ or $q \perp p$, i.e., p and q are inferentially not comparable, or orthogonal.

Observe that relation $\sim$ is not necessarily an algebraic equivalence since it is just reflexive and symmetric, but its transitivity is not always warranted unless $<$ enjoys it.

A forward chain of inference like $p < u$, $u < v$, $v < w$, $w < q$, usually written $p < u < v < w < q$ is called a *deductive* process, or a deduction, and a backward chain like $p > u > v > w > q$, or $q < w < v < u < p$, is called an abductive process, or an abduction. Of course, for concluding $p < q$ in the deductive chain, and $q < p$ in the abductive chain, $<$ has to be a transitive relation at least locally for the involved terms.

Hence, and with the exception of (3) allowing the indistinguishability of p and q from the inferential point of view, and then accepting substitution of p by q, or q by p anywhere, a conclusion q of p, a conjecture q of p, only can be either a consequence, or a hypotheses, or an orthogonal element to the premise, in which case it is said that q is a speculation, or guess, from p, and, depending on how it is $p < /q'$ verified, the speculation is weak (if $q' < p$), or strong (if $q' \perp p$) [17].

Thus, reasoning just consists in refuting, deducing, abducing and speculating or guessing, i.e. obtaining orthogonal conjectures from the premise, a process that can also be understood as inducing. Thus: induction can be identified with speculation, guessing with obtaining conjectures, statements inferentially orthogonal to p.

It is interesting to observe that, under local transitivity of $<$, if r is not self-contradictory $(r < /r')$ and refutes p $(p < r')$, it is $p \perp r$. In fact, were it $p < r$, since it is $r' < p'$, then $p < r'$ and the corresponding local transitivity, forces the contradictory $p < p'$. Analogously, were $r < p$ it will follow $r < r'$ and, hence, p and r are not comparable under $<$, are othogonal.

If, under transitivity, consequences can be obtained by going forwards with $<$, and hypotheses by going backwards with $<$, how can speculations be obtained? If deduction corresponds to the first, and abduction to the second, to which inferential mechanism can speculation, induction, correspond?

3 On the Inferential Zigzag

Notice that a consequence q of p, $p < q$, is not always obtained by just an immediate first step ahead but by, at least, two different possibilities:

- The first is when q is at the end of a chain of steps such as $p < u$, $u < v$, $v < w$, and $w < q$, requiring the transitivity of $<$ to conclude $p < q$.

- The second is through the property $p < p+p^a$, with p^a brings any opposite statement of p, allowing to define $q = p + p^a$, without requiring whatsoever additional property of $<$, and if this q is not self-contradictory.

It is obvious that instead of p^a it serves any statement q, and even the same p, but combining p and one of its opposites p^a helps to cover more knowledge than that offered by only p, or by a q that is totally disconnected from p. Notice that the statement $p+p'$ has the risk of being too large (remember that $(p+p')'$ is self-contradictory and, thus, in some lattices with maximum $p + p'$ can easily be such maximum).

It analogously happens with hypotheses, $h < p$ or $p > h$, reachable either by some steps $p > u$, $u > v$, $v > w$, $w > h$, allowing to conclude $p > h$ if $>$ is transitive in the set p, u, v, w, h, or through the property $p \cdot p^a < p$, allowing to define $h = p \cdot p^a$ and without requiring additional properties for $<$, and provided $p \cdot p^a$ is not self-contradictory.

It should be noticed that the first ways correspond to what is usually done for proving that a conjecture is either a consequence, or a hypothesis; and the second to what is done for finding either a still unknown consequence or an explanation. They serve, respectively, for proving and for finding; if the first can be seen as a technical way, the second is a dialectical way.

A reason for considering p^a instead of p' lies in the fact that, under the transitivity of $<$, $p \cdot p'$ is self-contradictory; in fact:

$p \cdot p' < p$ implies $p' < (p \cdot p')'$ that $p \cdot p' < p'$ conducts to $p \cdot p' < (p \cdot p')'$, q.e.d.

This Non-contradiction theorem, obviously valid for all statement and, in particular, for those s such that $s < p'$, can be easily and directly extended to these statement s such that $s < p'$ -statements referred by p - provided $<$ is transitive where convenient, and the conjunction is monotonic that is, verifies, $p < q \Rightarrow p \cdot r < q \cdot r$ and $r \cdot p < r \cdot q$ for all r. In fact, starting from $p < q \Rightarrow q' < p'$ and from the property $s < p'$, by monotony follows $s \cdot p < p' \cdot p$ that, with $p' \cdot p < (p' \cdot p)'$ implies $s \cdot p < (p' - p)'$;but, since from the first inequality follows $(p \cdot p')' < (s \cdot p)'$, it finally results $s \cdot p < (s \cdot p)'$. Thus also $s \cdot p$ is self-contradictory.

This theorem forces to avoid as s, when the two presumed laws do hold, all statements that are refuted by p and, in particular, both the negation and whatsoever antonym of p. It suggests taking a statement $s = s(p)$ depending on the premise p but different from the negation and any opposite.

It should be noticed that given a premise p, neither refutations, nor consequences, nor hypotheses, nor speculations, are unique. Usually, there are sets of them, not reducible to a singleton. Hence, either the same person at different moments, or two different persons. will not conjecture the same from a given premise. In the same vein, refutations do not usually coincide; different people can refuse the same statement by means of different refutations. This non-uniqueness of conjectures and refutations is, of course and in fact, a matter of common experience among people, and a testament to the power of human reasoning; what can be seen of some relevance is that the current model gives a first explanation of it.

It is noteworthy that the non-uniqueness of conjectures comes directly from the non-uniqueness of those statements s such that $s < p'$; from the possible hypotheses for p'. Actually and in particular, there are a lot of words for which more than one opposite term is used in language. Analogously, it is not sure that in a mixed chain of inference the inflections are always produced in the same places and in the same sense (backwards, or forwards), and the obtained speculation at the end of a zigzag strongly depends on this.

3.1 The inferential ZigZag as a mechanism for reasoning

Basic references [14, 19, 16] have overlooked such a development, and are limited to a hinting on how the inferential zigzag and, especially speculating, guessing or also inducing can be effectively done. This paper tries to fill this gap by giving a first hint on how the zigzag can be actually developed. It can be said that reasoning is done thanks to a mechanism consisting in activating an inferential zigzag.

4 Developing the Zigzag

Before continuing, lets see how weak speculations can be effectively reached. Since they are defined by $p \perp q$ and $q' < p$, it is clear that $not - q, q'$ is a hypothesis for p. Hence, in principle q' can be reached by abduction i.e. going backward from p up to find it, and provided $<$ is locally transitive around p. Thus, two questions are posed; when to stop for finding q as the searched speculation (a question whose answer is here avoided as it corresponds to looking for the meaning of a statement), and how, once q' is given, q can be actually reached; something that depends on the character the linguistic negation can show in q:

1. If negation is weak at q, or $q < (q')' = q''$, q will be found by negating q and moving backwards from q''.
2. If negation is Intuitionistic at q, or $q'' < q$, q will be found by negating q' and moving forwards from q''.
3. If negation is strong at q, or $q'' \sim q$, it suffices to negate q' to obtain q.
4. If negation is wild at q, or $q'' \perp q$, no one of the three former situations holds, and, since it is $p \perp q$, it is not sure if one of them, previously unknown, will appear. Actually, a priori nothing can be said in general.

Thus, with the exception of (4), a weak speculation is reached at the end of a forward or backward step after negating q'.

Notice that if q is a strong speculation from p, or, it is $p \perp q$ and $p \perp q$, no similar way to the formers can be immediately inferred as we have $q' \perp p$ instead of $q' < p$. In principle, it seems that there is no inferential way of mixing deductive and abductive movements that can be foreseen to reach q. It seems that q cant be reached by enchaining statements, and one can be tempted to hope in the help of some bizarre entity, akin to the old muses, mysteriously imbuing q into the thinker. It will be seen how such suppositions are unnecessary.

4.1 Advancing and retroceding

Nevertheless, it should be noticed that from p it is possible to advance inferentially by disjunction, and to retrocede by conjunction. For instance, $p \cdot u < p < p + v$ for all statements u and v, shows a recoil by conjunction, and an advancement by disjunction, both from p. In the same way it is possible to realize alternate movements backwards/forwards or forwards/backwards, like, $p > p \cdot u < u < u + w$, etc.

In this last case, and not presuming more laws than those of the skeleton, if with $q = u + w$ it is $p \perp q$, it will depend on q' if q is a speculation from p or, simply, an element inferentially orthogonal to p. It is obvious that such inferentially mixed forms can be followed by $u = p \cdot p^a$ and $v = p + p^a$; i.e. by only using what can be known, or supposed, on p, and avoiding $p \cdot p'$ and $p + p'$ due to what was formerly stated concerning their self-contradiction if $<$ is transitive.

4.2 An example of reasoning with speculations

Lets show a very simple example starting from p and arriving at a speculation by supposing that all the statements come from the disjunction of five of them, a, b, c, d, e, expressing the available initial information on something.

- Suppose $p = a + e$, is the premise and take $q = a + b + d$. Since it is $p \cdot q = a$, lets focus our attention on a. Then: $p > a < a + b + d = q$, with $p \perp q$ and, since $q' = c + e$, it is also $q' \perp p$. In this example, and provided $p' = b + c + d$, since it is not $q < p$, is not possible to suppose the coincidence of q and p^a. Thus, what can be supposed is that, pivoting on a, $q = a + b + d$ informs on p.

Analogously,

- if with the same premise is $q = b + d$, it is $p \perp q$, and since $q' = a + e = p$ means $q' < p$, as $<$ is reflexive, q is a weak speculation from p.

Thus it seems that in all the cases in which the statements are constructed as the disjunction of some pieces of basic information on something, or atoms of knowledge, as it happens frequently, both weak and strong speculations can be obtained in ways like the former and through inferential chains mixing forwards and backwards movements. That is, through the so called inferential zigzag under which reasoning from p can be seen as a kind of Inferential Brownian Movement around the premise.

Summing up and with just the skeletons laws, deciding if the next movement in q should be either forwards, or backwards, can be done by either considering the conjunction of opposites $q \cdot q^a$, or another conjunction $p \cdot q$ if q informs on p, that is, by means of all the (available) knowledge on p. Always with care on not being $p \cdot q$ self-contradictory.

5 The Cost of Zigzagging

We have already discussed in [19] the fact that in order to render the notion of the inferential zigzag computable when using atoms of information, an exhaustive search of the problem space is necessary, to take into account all possible combinations of the morsels themselves and determine their cumulative role in achieving unlimited speculation. Such an approach would require exponential time $O(2^n)$ due to the necessity of exploring the entire power set to be performed in full, and as such would be computationally unfeasible even for a small number of atoms. This compounds with the fact that while examples are presented with atoms in the unities for sake of clarity, it is to be expected that any meaningful reasoning will require orders of magnitude more, rendering factual the worry about computational attainability.

In [19] a number of strategies that are directed toward limiting complexity by reducing the size of the searching space have been already proposed, such as reducing the number of total clauses by a plausibility selection and weighing and thresholding, where each movement in the zigzag has an associated cost, proportional to parameters inferred by the reasoning structure itself, and exploration stops when a certain threshold is passed. Here is presented a simpler strategy of reduction that preserves the polynomial complexity of search depth and is easily applicable to the specific task of speculation.

The first pass of the strategy is to add to the simple system some information about the proximity between atoms. This is necessary as without any added information there is no way of implementing a reduction of the Hasse diagram representing the power set of atoms, which is necessary to lower complexity. This can be done either by prior experience or by evaluation. In the case of prior experience, we consider a number of tuples composed of atoms (such as a, b, e, a, c and so on) that are derived from previous knowledge, e.g. instances where such atoms appeared together in a previous successful speculation or in some premise. In evaluation we have to provide such a database of tuples directly, either by directly asking a panel of humans to evaluate the proximity of atoms, or by grading single atoms and then aggregating such information. Either way, a database of atoms proximity tuples is obtained. Such a database is then used to calculate what is called a frequent itemset, a set of items appearing together, listed in order of frequency. Such structure can be stored in a DB, or structured in a Markov chain. By setting a threshold and keeping in mind the necessity for polynomiality, the atoms that are frequently found together can be clustered, and the complexity of exploring the resulting set be reduced to $O(n)$, allowing effective computation for the generation of all possible inferential zigzags.

Classical algorithms such as Apriori [1] and other members of the Apriori-like family could be used for the task, but in order to compute a frequent itemset of order l, they must produce all the subsets, bringing exponential complexity again to the table. A more suitable choice is Max-Miner [2], which obtains the same information by computing at most $l+1$ passes over the original dataset. A number of newer algorithms claim to improve on Max-Miner, but due to its simplicity and the fact that in this context a clear explanation is worth more than fractional improvements in efficiency, the choice for a better implementation is left to a more technical forthcoming paper. This approach has a number of advantages: first and foremost, it reduces complexity allowing effective computation of the inferential zigzag; second, reduction in complexity does not come at the cost of reducing the expressivity of the original idea in terms its of cognitive approach. As the zigzag is a formal version of speculation, reducing by clustering has a cognitive resonance with analogy, a process often employed in order to make effective reasoning in presence of an abundance of information. In figure 1, an example of pruning a Hasse diagram for a five atoms reasoning search is shown.

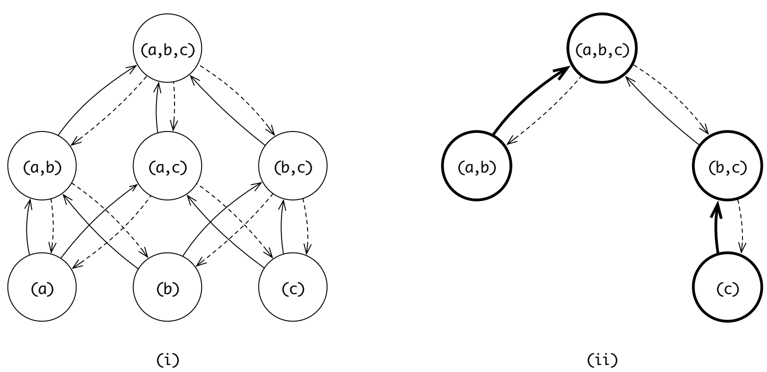


Figure 1: Example of pruning a Hasse diagram for a five atoms reasoning using Max-Miner, (i) the algorithms take as input: the starting Hasse diagram of the power set of a, b, c, d, e, which is used in [19] to explore all possible zigzag inferences when in possession of five atoms of information; a list of common atoms tuples, derived e.g. from previous reasoning on the same atoms, or by experience. (ii) the most popular couples of atoms are clumped together, and a new diagram with lower complexity is created. The threshold for clumping is chosen appropriately in order to attain effective computability.

6 Zigzagging Along the Centuries

Inferential zigzagging is something that helps to complete and extend the use of notions such as conjecturing. The general idea behind this paper without, of course, any reference to theorems was suggested to the authors by works [3, 4] of 15th Century German philosopher Nicolaus Cusanus (aka Nicholas of Cusa, 1401-1464). What specifically and concretely inspired this papers argumentation, in fact, is the continuous use made there of the methodology called unity of opposites (compositio oppositorum)[2].

The original idea behind the present paper can thus be informally traced in a few six hundred years old considerations. We already referred to old thinkers in a recent Essay [19]: what follows can be useful for outlining a general conceptual setting in which this as well as others remarks can be more adequately understood[3].

In dealing from a conceptual point of view with a few pages of Nicolaus Cusanus we shall (literally) zigzag a little around the XV Century. Since our eyes are turned to the foundational efforts done in mathematics and logic in the XX Century, we will be zigzagging through Centuries. What follows aims at drawing additional information for increasing the understanding of notions such as conjecturing[4]. These notions do not seem to play an explicit role in the bulk of present day mathematical logic, more interested in precision and accuracy. They are crucial instead in everyday language and reasoning, as well as in many facets of scientific investigations when cognition comes into play, such as AI.

In what follows we shall present:

a) a few quotations from Cusanus writings, to highlight why, in our opinion, his ideas are relevant to present day investigations

b) some remarks on the way in which the notion of *conjecture* is used by working mathematicians in the context of their daily work, and not when thinking about *foundational questions*

c) a number of reflections on the use of the term mathematical logic

6.1 Cusanus and reasoning

We feel necessary to recall some general remarks presented by Cusanus in the opening of his volume [3, 4]. Behind the veil of an old language (and notwithstanding it), they point to interesting connections with present day questions. We shall not scrutiny whether some other ideas and points of his analyses can be also and, perhaps, more incisively - useful for the same aim. The early motivations offered by Cusanus for having devoted space to a reflection precisely to the notion of conjecture are illuminating. As we shall see from the brief excerpts that follow, two points are crucial:

i. impossibility of reaching the precision of truth, and

ii. limitations of the human mind are such to imply that the conjectures of each person will be different

Cusano begins with the following statement of intents: "since a favourable opportunity to do so has now presented itself to me, I would like to illustrate my conception of conjecture". What follows is an interesting but admittedly contort presentation, due perhaps to the desire to prize the greatness of the person to which the book is sent:

> In the preceding books of the Learned Ignorance you have seen, even more profoundly and more clearly than I have done myself with all my efforts, that the accuracy of truth is unattainable. From this it follows that every positive assertion of man concerning the true is conjecture.

[2]This notion in different terms (and in a completely different context from Cusanos) has been analyzed by the third author in [18, 15].

[3]That the thought of Nicholas of Cusa can be useful for clarifying crucial aspects of problems and questions of present day relevance is also witnessed by a relatively new book [20] which collect the contributions presented at the first Congress on Cusanus to be held in Asia at the turn of the Millennium as well as a short monograph entirely devoted to the Art of Conjecture appeared in 2021 [5]. We came across both volumes when the paper was, in fact, finished and we are, then, here, simply acknowledging their existence. Their content will be very useful for further investigations along the present conceptual line, looking for stronger and less episodic connections with Cusanus suggestions as done in the present paper. Just to provide an indication a paper in the Conference volume explicitly deals with epistemology [6] and many others touch on topics of crucial present day interest.

[4]in the following we shall show how the conjunction of opposites can be connected to it

He then proceeds with a statement not easily understandable at a quick reading, but which forces a reflection in connection to present day questions of cognitive relevance: The unity of unattainable truth is therefore known through the otherness of conjecture, and the conjecture of otherness is known in the absolutely simple unity of truth. A sort of clarification follows:

> A created intelligence, which is endowed with a finite actuality, cannot but exist in one way in one individual and in another way in another individual, so that among all who formulate conjectures there is always a difference; consequently, it will always be absolutely certain that, with respect to the identity of the true which remains unattainable, the conjectures of different persons will differ in degree and yet remain without proportion to each other, so that no one will ever be able to understand perfectly what another means, although some may come closer to it than some other.

Subsequently he will explain his ways of approaching and defining conjectures, his secret being a careful and guided use of examples in a sort of maieutic or Socratic approach.

> For this reason, in order to make the secret of my conjectures clearer and easier to understand, I will first make use of a rational numerical progression, which is well known to all, and I will represent my thought by means of demonstrative examples, through which our discourse can arrive at the general art of conjecture.

Despite being useful for our general discussion, a deeper analysis is out of context here. We want instead to stress the general inspiration that can be offered by Cusanus to contemporary investigation in such new fields as information and cognitive sciences by his vision of science and logic. It is clear that, presently, we live in a very different cultural context. Not only Fregean revolution is more than one hundred and fifty years old, but also Gdel results are approaching a whole Century of life. The way in which Cusanus see the problem of truth is very different from ours, as we are acquainted with Tarksis approach. His comments about the subjectivity of conjectures would be considered, if not immediately dismissed, as opening (very interesting, maybe, but) *general* epistemological questions not something that could be of specific interest to a (traditional) working logician of our time. We shall come back to this point in 6.3.

We conclude by briefly discussing a paradigmatic, practical example, having to do with the *conjunction of opposites*, which could shed some light on this point.

With each statement p, one of its antonyms p^a (in just the former sense of being before p' respect to $<$, that is, refuted by p), by means of the linguistic conjunction and (), to obtain the statement p and p^a $(p \cdot p^a)$ jointly considering what p refers to and also what is referred to by an antonym or opposite of p, or in general by a statement refuted by p. A conjunction of opposites, in sum, with which to have a self-contradiction, an inferential impossibility, is not so immediate if it is not taken a statement refuted by p.

Before Hegel, Marx, Lenin, and all the Marxian thinkers, the conjunctio oppositorum methodology, known in English as the unity of opposites, was, formerly and systematically, managed for reasoning by a theologian and philosopher Bishop.

Notice that the disjunction $p + p^a$ represents much of what, in the universe of discourse, is specifically known on p, but without being all that is known, like with additional conditions $p + p'$ tries to give and gives effectively in Ortholattices for instance.

6.2 Cusanus as a contemporary thinker

And, however, it seems to us that from a suitable, although unusual, perspective Cusanus words are very modern, contemporary: in the sense of being able to contribute to clarify the questions (of logical nature) which are of crucial interest in topics of common sense reasoning and cognitive science and AI. More tuned, epistemologically, to them many technical papers in mathematical logic appeared in the last decades One reason for that is that they seem very direct and fresh, not burdened by the important but heavy general apparatus of mathematical logic as structured in the last century. An apparatus that, in many situations, is not destined to provide clarifications, when we are interested to investigate and scrutiny very specific and circumscribed problems. This induces us to go back to our original question. We asked if this reference to old thinkers is only casual or whether there is some deeper reason for it. We favour the second hypothesis, a position we will explain in the next subsection. Before that we shall briefly look at the question of how the notion of conjecture has been treated in math.

While the term conjecture is and always has been informally used in mathematics and considered part of the daily dialogue among mathematicians, it has very rarely been considered of crucial interest among (the very tiny tribes of) logicians. The notion has been central in Poppers reflections on the scientific method and, perhaps, this fact has contributed to consider it as important only from an epistemological point of view. Supported in this by the distinction between a logic of discovery from a logic of confirmation. Lets dwell a little bit more on this concept.

A conjecture is here understood as a proposition that is unproven (otherwise it would have been a theorem) but about which there is a sort of common consensus in the context of the already established results in the field. But there is also something else: an agreement that the conjecture could be experimentally tested and checked, in order to arrive at a proof, inside the received conceptual context.

No one would call conjecture a proposition: this wording would strongly depart from that of a traditional theorem. Many of Cantors ideas had not been considered conjectures. The same happened as well to some of his proved propositions, at least at the beginning. Similarly, at the moment of the appearance and presentation of a new conjecture, the common view is that its subsequent demonstration would not necessarily imply or, even better, require a change in the overall architecture of mathematics (at least in the specific chapter involved), especially for what regards the ontological assumptions.

It may, of course, happen (and, in fact, it does happen and did happen for the interesting ones) that proving a conjecture would force to re-discuss many general assumptions and provide also conceptual changes. In those instances, this happens along the way, not at the beginning. This is what happened with Hilberts Entscheidungsproblem or with Fermats last theorem: two crucial conjectures, although they were not, for exogenous reasons, called this way at the time of their formulation.

The former needed the creation of the completely new Theory of Computation, an ever-present notion in math that in centuries had not been in need of formalisation. The latter needed three centuries of development of new pieces of math. Both are historically akin to the inferential zigzag. The first looks like its deducting part, and the second its abductive part. Conjectures play and have played a very important role, but they have been seen as a sort of future theorems (when lucky) or statements to be refuted momentarily missing a reason for refutation. Is there a reason to be interested in the form and specificity of the logical features of conjectures? For decades starting with Frege and going on with the foundational debates at the beginning of XX Century Logic had other goals and other crucial problems to afford. It seems that no space was left for an autonomous investigation of such notions that have acquired visibility also in the development of the logical brand of AI but this is no paradox at all: such subtle results could not be achieved without sophisticated formal tools. An attitude that has only slightly modified over the decades [8], but abruptly changed when the need for studying Commonsense Reasoning, Language at Work, emerged from AI.

6.3 A fresh way of looking at Logic

Some useful suggestion on commonsense logic paradoxically comes from the general vision of sophisticated thinkers with much bigger aims, due to their theological and religious commitments. Despite that, it was clear to them that global projects as the one that in a distant future would have been envisaged by by Hilbert for math and by Lord Kelvin for physics at the end of the XIX Century were not tenable.

A general and usually tacit shared assumption in the received view of mathematical logic is that progress in the understanding of logical aspects of every facet (as well as in the nuances) of the empirical phenomenon of reasoning cannot but descend and be derived by further developments of the central bulk of logic as outlined in the Thirties of the last Century. If the subtleties and profundity of this approach successfully tackled such seminal questions, so more so this powerful edifice should be able to deal with apparently trivial matters. This can still be possible in principle, although what happened (and is happening) in AI[5] suggests some reflections.

This implicit assumption obscures and neglects the fact that many specific aspects of reasoning can be afforded by mathematical tools, in many cases of great simplicity, in a sort of Galilean approach, without referring to this magnificent but burdensome construction. We could also realize that the corpus of mathematical logic, with all its well-deserved authority, determines what is crucial and important and, in a sense, what is relevant, giving little space to what does not ontologically conform to its bases. Something that is common to all disciplines and that, usually, does not impede due to the open mindedness of the scientific way of approaching the questions that minor fields and topics are investigated and developed. Something that may lack is informal ideas and motivations for specific features of these subfields.

For many decades Logic has had other goals to look at than specific aspects, leaving them to minor applications of the big construction[6]. Due mainly to the profundity of its central results, the wonderful edifice of 20th Century Logic has tended to neglect that its main target has substantially been to put it bluntly the internal consistency of mathematics, and not a general theory of reasoning.

A more general way to express this is that classical logic has to do with specific properties of those forms of reasoning that consider clear cut situations in a static world. These instances represent but a very small percentage of human reasoning, which is dynamic par excellence, and more often than not based on incomplete and imprecise information. *Mathematical* Logic owes its name not only to the fact that it uses a mathematical *language* and mathematical *tools* but also to the fact that it is the logic of *mathematical* reasoning, but not of *Commonsense* Reasoning.

We can, perhaps, also add something more. Logic, as acutely observed by Jean van Heijenoort in Frege and Vagueness, excluded vagueness, from his horizon in its founding years (see [9]). This was a correct choice, at the beginning. One cannot consider vagaries when trying to establish a new theory: Galileo did that by forgetting friction, while constructing mechanics. But now, van Hejienoort states, some time has passed, and we must consider vagaries. Looking at vagaries and admitting vagueness into the realm of Logic imposes to look anew at many questions. Among them, the central notions of coherence and completeness. Vagueness opens the way to new motivations and the subtle analyses of old logicians provide useful inspiration, since they were thought in a period in which present day formal requirements were not required. This draws an unusual parallel with the present situation, in which we are urged to construct systems and models in which these same requirements are not strictly applicable.

When analysing questions and problems from Cognitive Science, for instance, not only the notion of conjecture is essential (perhaps with different nomenclature), but it is an everyday experience that the conjectures of each person will be different. And that is exactly what the model should consider and try to explain.

Specific aspects of reasoning can be looked at in a fresh way and not as particular cases of the big construction. In order to do so we need also to help ourselves with epistemological and conceptual reflections tuned with this approach. In this direction we found that many general remarks done by Cusanus are very stimulating and useful, which warrants the discussion of them in this paper.

[5]We refer to the well-known fact that the simpler facts affordable by humans looked the most difficult to tackle by automatic means, and also to the big steps forward obtained by brute force methods.

[6]This neither means nor implies that the sophisticated and powerful tools forged in the core of the crucial questions are not useful or cannot be applied to other conceptually very different questions [11]. The point is that they can be usefully applied and creatively used when they are specifically relevant for the problems in question, which should be looked at in their complexity and, in some cases, elusiveness.

And we can now come back again to the starting question of this Section. We think that one can affirm that the inspiration provided by ancient texts is not casual. It corresponds to a similarity (although, paradoxically, both in a very specific and vague sense) with the general conceptual framework. Once vanished the illusion of a unique, firm and stable foundation of the workings of scientific investigation along established lines, to be pursued in an automatic way [12, 13, 10], the sophisticated conceptual analyses of middle age and renaissance scholars can provide useful suggestions to be checked. Of course, through the language and methods of contemporary investigation.

6.4 The zigzag is not unique

Let us, finally, observe that, in this Section, nothing has been explicitly said about zigzagging, as formally described in Section 3. We will limit this here to a comment. Lets observe that since each person can follow a different zigzag, that could be the reason why each one can find a different conjecture as well as different proofs for either a consequence or a hypothesis. The idea of a personal, individual[7] approach to reasoning, that is so omnipresent in everyday life and often the cause of infinite discussions and diatribes and so evidently missing in ordinary logic, should (and could) finally be reconciled with implementable procedures.

7 Conclusion

In [14, 19, 16] the so called Formal Skeleton of ordinary/commonsense reasoning, was presented, using which actual reasoning can be developed through a sort of 'Brownian Movement' around a premise, called the inferential zigzag, with which refutations, consequences, hypotheses, and speculations are obtained. A process that, within conditions, is effectively realizable (i.e. programmable) [19, 17], and that can be usefully employed in a better implementation of cognitive reasoning [7]. Nevertheless a conceptual problem remained open: how such zigzag can be effectively developed. That is, if a (theoretic) automatism acting without requiring the help of any mysterious entity, but in a known and describable form, could be algorithmically implemented in at least some specific and limited cases. By acquiring total certainty on the not metaphysical character of induction through developing a mathematical theory on it, such a theoretical question is partially answered in this paper. What is here presented contributes to dissolve the old worries concerning the mystery of induction: induction, or guessing, was identified with speculation. Possibly such dissolution is not of great practical relevance, but it has, of course, a conceptual, theoretical, importance since it means but a view on how people themselves actually reason, and sometimes can quickly envisage an unexpected conjecture. In some sense at least in the context of the conceptual setting defined in [19] the problem concerning the scientific understanding of what is ordinary or commonsense reasoning has now one possible clarification. The present paper, in fact, provides an indication of how this conceptual problem can be effectively and practically solved. If reasoning is achieved by developing effectively inferential zigzags, we have shown how the forward/backward inflexions at each point in an inferential chain are produced.

Conflict of Interest: The authors declare no conflict of interest.

[7] To avoid a wrong interpretation in the direction of a sort of non-objectivity of reasoning: we are referring to the individual path that each person, in everyday reasoning, can follow and which can be very different from the one followed by other persons. This variety is irrelevant in a standard setting (complete information, no vagueness, no approximations). Everything changes in the setting of everyday life in which, moreover, also implicit (hidden) presuppositions play a role.

References

[1] Agrawal R, Mannila H, Srikant R, Toivonen H and Verkamo AI, et. al. Fast discovery of association rules. *Advances in knowledge discovery and data mining.* 1996; 12(1): 307-328.

[2] Bayardo Jr RJ. Efficiently mining long patterns from databases. *In Proceedings of the 1998 ACM SIGMOD international conference on Management of data, Seattle, Washington, USA.* 1998; 85-93. DOI: http://doi.org/10.1145/276304.276313

[3] Cusanus N. De Coniecturis/Las conjeturas (Bilingual Edition, Latin/ Spanish, translated by Sevillano C. *Las Buenas Letras.* 2011.

[4] Cusanus N, congetture Le. In Opere Filosofiche, Teologiche e Matematiche. *Bompiani, Milano.* 2017.

[5] Miller CL. The art of conjecture: Nicholas of Cusa on knowledge. *CUA Press, Washington D C.* 2021.

[6] Sato N. Cusanus epistemology in idiota de mente. *In Nicholas of Cusa, Routledge.* 2013.

[7] Tabacchi ME. Logics for Cognitive Sciences: how questions arising from Cognitive Sciences could benefit from a logic approach. *Archives for Soft Computing.* 2019; 1.

[8] Tabacchi ME. Focusing Fuzzy Logic. *UniPa Press.* 2021.

[9] Tabacchi ME. The inevitability of vagueness in fuzzy logic. *In proceedings of EuSFLaT 2023 (submitted).* 2023.

[10] Tabacchi ME, and Termini S. Varieties of vagueness, fuzziness and a few foundational (and ontological) questions. *In proceedings of the 7th conference of the European Society for Fuzzy Logic and Technology, July 18-22, 2011 Aix les Bains, France.* 2011; 578-583. DOI: http://doi.org/10.2991/eusflat.2011.57

[11] Tabacchi ME and Termini S. The fuzzy logic gambit as a paradigm of Lotfis proposals. *In Fuzzy Logic and Applications: 12th International Workshop, WILF 2018, Genoa, Italy, September 67, 2018, Revised Selected Papers.* 2019; 231-235. DOI: http://doi.org/10.1007/978-3-030-12544-8_18

[12] Termini S. Aspects of vagueness and some epistemological problems related to their formalization. *In Aspects of Vagueness, Springer.* 1984; 205-230. DOI: http://doi.org/10.1007/978-94-009-6309-2_11

[13] Termini S. On some vagaries of vagueness and information. *Annals of Mathematics and Artificial Intelligence.* 2002; 35: 343-355. DOI: http://doi.org/10.1023/A:1014512024132

[14] Trillas E. La génesis de la Lógica: reflexiones ingenuas. *Universidad de Santiago de Compostela.* 2021.

[15] Trillas E. On the Unity of Opposites, as a way to attain Hypotheses. *In Atti dellAccademia Nazionale di Scienze, Lettere e Arti di Palermo.* 2020.

[16] Trillas E. Divagaciones sobre pensar y razonar. *Universidad de Granada.* 2021.

[17] Trillas E and De Soto AR. On the search of speculations. *New Mathematics and Natural Computation.*2022; 18 : 9-18. DOI: http://doi.org/10.1142/S1793005722500028

[18] Trillas E and García-Honrado I. A reflection on the dialectic synthesis. *New Mathematics and Natural Computation.* 2019; 15: 31-46. DOI: http://doi.org/10.1142/S1793005719500029

[19] Trillas E, Termini S and Tabacchi ME. Reasoning and Language at Work: A Critical Essay. *Springer Nature.* 2022. DOI: http://doi.org/10.1007/978-3-030-86088-2

[20] Yamaki K. Nicholas of Cusa: a medieval thinker for the modern age. *Routledge*. 2013.

Marco Elio Tabacchi
Assistant Professor of Computational Logic
Dipartimento di Matematica e Informatica, Universit degli Studi di Palermo, Italy
and Istituto Nazionale di Ricerche Demopolis, Italy
E-mail: marcoelio.tabacchi@unipa.it

Settimo Termini
Dipartimento di Matematica e Informatica, Universit degli Studi di Palermo, Italy
and Accademia Nazionale di Scienze Lettere e Arti di Palermo, Italy
E-mail: settimo.termini@gmail.com

Enric Trillas
Universidad de Oviedo, Oviedo, Spain
and Accademia Nazionale di Scienze Lettere e Arti di Palermo, Italy
E-mail: etrillasetrillas@gmail.com

Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://tfss.journals.iau.ir/
Online ISSN: 2821-0131
Vol.2, No.2, (2023), 77-112
DOI: http://doi.org/10.30495/tfss.2023.1983060.1068

Article Type: Original Research Article

Gautama and Almost Gautama Algebras and their associated logics

Juan M. Cornejo, **Hanamantagouda P. Sankappanavar***

Abstract. Recently, Gautama algebras were defined and investigated as a common generalization of the variety $\mathbb{RDBLS}$t of regular double Stone algebras and the variety $\mathbb{RKLS}$t of regular Kleene Stone algebras, both of which are, in turn, generalizations of Boolean algebras. Those algebras were named in honor and memory of the two founders of Indian Logic–**Akshapada Gautama** and **Medhatithi Gautama**. The purpose of this paper is to define and investigate a generalization of Gautama algebras, called "Almost Gautama algebras ($\mathbb{AG}$, for short)." More precisely, we give an explicit description of subdirectly irreducible Almost Gautama algebras. As consequences, explicit description of the lattice of subvarieties of $\mathbb{AG}$ and the equational bases for all its subvarieties are given. It is also shown that the variety $\mathbb{AG}$ is a discriminator variety. Next, we consider logicizing $\mathbb{AG}$; but the variety $\mathbb{AG}$ lacks an implication operation. We, therefore, introduce another variety of algebras called "Almost Gautama Heyting algebras" ($\mathbb{AGH}$, for short) and show that the variety $\mathbb{AGH}$ is term-equivalent to that of $\mathbb{AG}$. Next, a propositional logic, called $\mathcal{AG}$ (or $\mathcal{AGH}$), is defined and shown to be algebraizable (in the sense of Blok and Pigozzi) with the variety $\mathbb{AG}$, via $\mathbb{AGH}$, as its equivalent algebraic semantics (up to term equivalence). All axiomatic extensions of the logic $\mathcal{AG}$, corresponding to all the subvarieties of $\mathbb{AG}$ are given. They include the axiomatic extensions $\mathcal{RDBLS}t$, $\mathcal{RKLS}t$ and $\mathcal{G}$ of the logic $\mathcal{AG}$ corresponding to the varieties $\mathbb{RDBLS}$t, $\mathbb{RKLS}$t, and $\mathbb{G}$ (of Gautama algebras), respectively. It is also deduced that none of the axiomatic extensions of $\mathcal{AG}$ has the Disjunction Property. Finally, We revisit the classical logic with strong negation $\mathcal{CN}$ and classical Nelson algebras $\mathbb{CN}$ introduced by Vakarelov in 1977 and improve his results by showing that $\mathcal{CN}$ is algebraizable with $\mathbb{CN}$ as its algebraic semantics and that the logics $\mathcal{RKLS}$t, $\mathcal{RKLS}$t$\mathcal{H}$, 3-valued Łukasivicz logic and the classical logic with strong negation are all equivalent.

AMS Subject Classification 2020: Primary: 03B50, 03G25,06D20,06D15; Secondary: 08B26, 08B15, 06D30.
Keywords and Phrases: Regular double Stone algebra, regular Kleene Stone algebra, Gautama algebra, Almost Gautama algebra, Almost Gautama Heyting algebra, subdirectly irreducible algebra, simple algebra, logic $\mathcal{AG}$, logic $\mathcal{G}$, logic $\mathcal{RDBLS}t$, logic $\mathcal{RKLS}t$.

1 Introduction

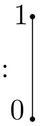
Boolean algebras have been a springboard for many new classes of algebras. Recall that an algebra $\mathbf{A} = \langle A, \vee, \wedge, ^c, 0, 1\rangle$ is a Boolean algebra if $\mathbf{A}$ is a complemented distributive lattice. The following 2-element algebra with universe $\{0, 1\}$, denoted by $\mathbf{2}$, is the smallest nontrivial Boolean algebra, up to isomorphism.

*∗***Corresponding Author:** Hanamantagouda P. Sankappanavar, Email: sankapph@hawkmail.newpaltz.edu, ORCID: 0000-0001-6878-038X
Received: 30 March 2023; **Revised**: 14 May 2023; **Accepted**: 1 June 2023; **Available Online**: 1 June 2023; **Published Online**: 7 November 2023.

How to cite: Cornejo JM and Sankappanavar HP. Gautama and Almost Gautama Algebras and their associated logics. *Trans. Fuzzy Sets Syst.* 2023; 2(2): 77-112. DOI: http://doi.org/10.30495/tfss.2023.1983060.1068

This paper is an expanded version of an invited lecture with the same title, presented at the 10th International Conference on: Non-Classical Logics: Theory and Applications, held at Łodz, Poland, during March 14-18, 2022.

2 :



$\vee$	0	1
0	0	1
1	1	1

$\wedge$	0	1
0	0	0
1	0	1

c	0	1
	1	0

Figure 1

Let $\mathbb{BA}$ denote the variety of Boolean algebras. It is well-known that $\mathbb{BA} = \mathbb{V}(\mathbf{2})$ (i.e. the variety generated by $\{\mathbf{2}\}$). In what follows, the symbol $\mathbf{2}$ denotes a two-element Boolean algebra whose signature, though varies, will be clear from the context where it appears. It is well-known that the Boolean complement has led to several weaker notions; among them are the following three:

(1) the pseudocomplement *, (2) the dual pseudocomplement $^+$, and (3) the De Morgan complement $'$.

Algebras based on the 3-element chain:

It was only natural to consider the above-mentioned operations on a 3-element chain (viewed as a bounded distributive lattice) denoted by $\mathbf{3}$. The three-element chain and the three operations mentioned above are shown below in Figure 2.

3 :

$*$:	0	a	1
	1	0	0

$+$	0	a	1
	1	1	0

$'$	0	a	1
	1	a	0

Figure 2

Let us, therefore, expand the language $\langle \vee, \wedge, 0, 1\rangle$ of bounded distributive lattices to $\langle \vee, \wedge, f, 0, 1\rangle$ by adding one unary operation symbol f and interpret f, on the chain $\mathbf{3}$, as the operation *, $^+$, or $'$, with the added restriction that $f(0) := 1$ and $f(1) := 0$. Then we get the following three algebras:

(1) $\mathbf{3_{st}} := \langle 3, \vee, \wedge, ^*, 0, 1\rangle$,

(2) $\mathbf{3_{dst}} := \langle 3, \vee, \wedge, ^+, 0, 1\rangle$,

(3) $\mathbf{3_{kl}} = \langle 3, \vee, \wedge, ', 0, 1\rangle$.

The varieties generated by $\mathbf{3_{st}}$, $\mathbf{3_{dst}}$ and $\mathbf{3_{kl}}$ are well-known, respectively, as those of Stone algebras, dual Stone algebras and Kleene algebras. We will denote these varieties by $\mathbb{St}$, $\mathbb{DSt}$ and $\mathbb{KL}$, respectively. $\mathbb{St}$ and $\mathbb{KL}$ have been researched well; as such, there is a fair amount of literature on them (see, for example, [7, 21].

In order to define Stone algebras, we need the notion of a pseudocomplemented lattice which was first introduced by Skolem [66] (see also [70]). It is clear that the usual definition of pseudocomplement (namely, $a \wedge x = 0$ iff $x \leq a^*$) is not equational. However, in 1949, Ribenboim [41] proved that the class of pseudocomplemented lattices is a variety. For our purpose here, the following axiomatization given in [49, Corollary 2.8] is more suitable.

An algebra $\mathbf{A} = \langle A, \vee, \wedge, ^*, 0, 1\rangle$ is a distributive pseudocomplemented lattice (p-algebra for short) if $\mathbf{A}$ satisfies the following:

(1) $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice,

(2) the operation * satisfies the identities:

(a) $0^* \approx 1$,

(b) $1^* \approx 0$,

(c) $(x \vee y)^* \approx x^* \wedge y^*$,
(d) $(x \wedge y)^{**} \approx x^{**} \wedge y^{**}$,
(e) $x \leq x^{**}$,
(f) $x^* \wedge x^{**} \approx 0$.

Note that the identity (f) can be replaced by the identity: $x \wedge x^* \approx 0$.

A p-algebra $\mathbf{A}$ is a Stone algebra if $\mathbf{A}$ satisfies the identity:

(3) $x^* \vee x^{**} \approx 1$ (Stone identity).

Stone algebras have an extensive literature (for example, see [7, 21, 22] and the references therein).

It is also well-known that the variety $\mathbb{S}t = \mathbb{V}(\mathbf{3_{st}})$ (the variety generated by $\mathbf{3_{st}}$). Dual Stone algebras are, of course, defined dually.

Kleene algebras are well-known too. The variety of Kleene algebras is a subvariety of that of De Morgan algebras, first introduced by Moisil [28] in 1935 (see also [29, 30]). They were further investigated later in [8, 24, 44]. They are generalized to semi-De Morgan algebras in [49], and further studied in [23, 37, 35, 36, 38, 53, 55].

An algebra $\langle A, \vee, \wedge, ', 0, 1\rangle$ is a De Morgan algebra if

(1) $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice,
(2) $0' \approx 1$ and $1' \approx 0$,
(3) $(x \wedge y)' \approx x' \vee y'$ ($\wedge$-De Morgan law),
(4) $x'' \approx x$ (Involution).

A De Morgan algebra is a Kleene algebra if it satisfies:

(5) $x \wedge x' \leq y \vee y'$ (Kleene identity).

It is also well-known that the variety $\mathbb{KL} = \mathbb{V}(\mathbf{3_{kl}})$.

Algebras on the 3-element chain with two additional unary operations:

The next natural step in this development was to consider the expansion of the language $\langle \vee, \wedge, 0, 1\rangle$ by adding two unary operation symbols corresponding to two of the above three unary operations on the 3-element chain, leading to the following three algebras on the 3-element chain:

(a) $\mathbf{3_{dblst}} = \langle 3, \vee, \wedge, ^*, ^+, 0, 1\rangle$: This is known as a "double Stone algebra." It was observed in [68] and [25] that $\mathbf{3_{dblst}}$ also satisfies an additional identity, called a "regular identity":

(R) $x \wedge x^+ \leq y \vee y^*$.

So, $\mathbf{3_{dblst}}$ is a "regular double Stone algebra."

(b) $\mathbf{3_{klst}} = \langle 3, \vee, \wedge, ^*, ', 0, 1\rangle$: This is a Kleene Stone algebra (see [43] and [48]). This algebra also satisfies an interesting identity (see [48]), also called "regular identity":

(R1) $x \wedge x'^{*'} \leq y \vee y^*$

So, $\mathbf{3_{klst}}$ is a "regular Kleene Stone algebra".

(c) $\mathbf{3_{klst}} = \langle 3, \vee, \wedge, ^+, ', 0, 1\rangle$: This, being the dual of (b), would not be of much interest to us in this paper. Thus, (a) and (b) yield the well-known varieties of regular double Stone algebras and regular Kleene Stone algebras, respectively.

An algebra $\mathbf{A} = \langle A, \vee, \wedge, ^*, ^+, 0, 1\rangle$ is a regular double Stone algebra if

(1) $\langle A, \vee, \wedge, ^*, 0, 1\rangle$ is a Stone algebra,
(2) $\langle A, \vee, \wedge, ^+, 0, 1\rangle$ is a dual Stone algebra,
(3) $\mathbf{A}$ satisfies the identity:
 (R) $x \wedge x^+ \leq y \vee y^*$.

The variety of regular double Stone algebras is denoted by $\mathbb{RDBLS}$t. For the more general variety of double p-algebras, of which $\mathbb{RDBLS}$t is a subvariety, see, for example, [68, 25, 45, 54, 58, 64, 5, 12, 17] and references therein.

We now pause briefly to recall some universal algebraic notions (see, for example, [10, 72]).

Definition 1.1. Let $\mathbf{A}$ be an algebra. An n-ary function $f : A^n \to A$ is *representable by a term* if there is a term p such that $f(a_1, \ldots, a_n) = p^{\mathbf{A}}(a_1, \ldots, a_n)$, for $a_1, \ldots, a_n \in A$. A finite algebra $\mathbf{A}$ is *primal* if every n-ary function on A, for every $n \geq 1$, is representable by a term.
The *discrimination function* on a set A is the function $t : A^3 \to A$ defined by

$$t(a, b, c) := \begin{cases} a, & \text{if } a \neq b \\ c, & \text{if } a = b. \end{cases}$$

A ternary term $t(x, y, z)$ representing the discriminator on A is called a *discriminator term* for the algebra $\mathbf{A}$. If a class $\mathbb{K}$ of algebras has a common discriminator term $t(x, y, z)$, then $\mathbb{V}(\mathbb{K})$ is called a discriminator variety. A finite algebra $\mathbf{A}$ with a discriminator term is called *quasiprimal*.

Discriminator varieties have been of great interest for a few decades now. For readers interested in this area, we recommend the books [72] and [10].

Returning to regular double Stone algebras, the following theorem is also well-known.

Theorem 1.2.

(i) $\mathbf{2}$ *and* $\mathbf{3_{dblst}}$*, up to isomorphism, are the only subdirectly irreducible (equiv. simple) algebras in* $\mathbb{RDBLS}\mathrm{t}$

(ii) *The variety* $\mathbb{RDBLS}\mathrm{t} = \mathbf{V}(\mathbf{3_{dblst}})$*,*

(iii) *The variety* $\mathbb{RDBLS}\mathrm{t}$ *is a discriminator variety* ([45])*,*

(iv) $\mathbf{3_{dblst}}$ *is quasiprimal* ([45])*,*

(v) $\mathbb{BA}$ *is the only nontrivial proper subvariety of* $\mathbb{RDBLS}\mathrm{t}$*.*

Regular Kleene Stone algebras are also well-known.
An algebra $\mathbf{A} = \langle A, \vee, \wedge, ^*, ', 0, 1\rangle$ is a regular Kleene Stone algebra if
(1) $\langle A, \vee, \wedge, ^*, 0, 1\rangle$ is a Stone algebra,
(2) $\langle A, \vee, \wedge, ', 0, 1\rangle$ is a Kleene algebra,
(3) $\mathbf{A}$ satisfies the identity:
(R1) $x \wedge x'^{*\prime} \leq y \vee y^*$ (Regularity).
The variety of regular Kleene Stone algebras is denoted by $\mathbb{RKLS}\mathrm{t}$. For the more general variety of pseudocomplemented De Morgan and Ockham algebras, of which $\mathbb{RKLS}\mathrm{t}$ is a subvariety, see [43, 48, 50, 46, 56, 58, 65, 6] and references therein.
The following theorem lists some of the known properties of the variety $\mathbb{RKLS}\mathrm{t}$.

Theorem 1.3. [58]

(i) $\mathbf{2}$ *and* $\mathbf{3_{klst}}$*, up to isomorphism, are the only subdirectly irreducible (equiv. simple) algebras in* $\mathbb{RKLS}\mathrm{t}$*.*

(ii) *The variety* $\mathbb{RKLS}\mathrm{t} = \mathbb{V}(\mathbf{3_{klst}})$*,*

(iii) *The variety* $\mathbb{RKLS}\mathrm{t}$ *is a discriminator variety,*

(iv) $\mathbf{3_{klst}}$ *is quasiprimal,*

(v) $\mathbb{BA}$ *is the only nontrivial proper subvariety of* $\mathbb{RKLS}\mathrm{t}$*.*

Remark 1.4. It is easy to verify that the algebra $\mathbf{3_{dblst}}$ also satisfies the identity (R1) and hence the variety $\mathbb{RDBLS}$t also satisfies (R1).

In view of the amazing similarities of $\mathbb{RDBLS}$t and $\mathbb{RKLS}$t, as seen in their definitions, as well as in Theorem 1.2 and in Theorem 1.3, it was only natural to ask for a common generalization of $\mathbb{RDBLS}$t and $\mathbb{RKLS}$t. To find such a common generalization, it was essential, first, to have a common generalization of dually Stone algebras and Kleene algebras, which luckily was already present since 1987, as the notion of a "dually quasi-De Morgan algebra." In 1987, the second author had introduced the variety of "upper quasi-De Morgan algebras," as a subvariety of the variety of semi-De Morgan algebras in [49]. (We drop the word "upper." here.) Actually, for our purpose here, we need the dual notion of "dually quasi-De Morgan algebra."

Definition 1.5. An algebra $\mathbf{A} = \langle A, \vee, \wedge, ', 0, 1\rangle$ is a dually quasi-De Morgan algebra if the following conditions hold:
(a) $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice,
(b) The operation $'$ is a dual quasi-De Morgan operation; that is, $'$ satisfies:

(i) $0' \approx 1$ and $1' \approx 0$,

(ii) $(x \wedge y)' \approx x' \vee y'$,

(iii) $(x \vee y)'' \approx x'' \vee y''$,

(iv) $x'' \leq x$.

The variety of dually quasi-De Morgan algebras is denoted by $\mathbb{DQD}$.

THE VARIETY OF GAUTAMA ALGEBRAS

The problem of finding a common generalization of $\mathbb{RDBLS}$t and $\mathbb{RKLS}$t, mentioned above, led the second author, to define, in [64], the variety of Gautama algebras, named in honor and memory of Medhatithi Gautama and Aksapada Gautama, the founders of Indian Logic.

Definition 1.6. An algebra $\mathbf{A} = \langle A, \vee, \wedge, ^*, ', 0, 1\rangle$ is a Gautama algebra if the following conditions hold:
(a) $\langle A, \vee, \wedge, ^*, 0, 1\rangle$ is a Stone algebra,
(b) $\langle A, \vee, \wedge, ', 0, 1\rangle$ is a dually quasi-De Morgan algebra,
(c) $\mathbf{A}$ is regular; i.e., $\mathbf{A}$ satisfies the identity:
(R1) $x \wedge x'^{*\prime} \leq y \vee y^*$,
(d) $\mathbf{A}$ is star-regular; i.e., $\mathbf{A}$ satisfies the identity:
(*) $x^{*\prime} \approx x^{**}$.

Let $\mathbb{G}$ denote the variety of Gautama algebras.

Clearly, $\mathbf{2}$, $\mathbf{3_{dblst}}$, $\mathbf{3_{klst}}$ are algebras in $\mathbb{G}$; and so, the varieties $\mathbb{BA}$, $\mathbb{RDBLS}$t, and $\mathbb{RKLS}$t are subvarieties of the variety $\mathbb{G}$.

The following theorem, proved in [64], gives a concrete description of the subdirectly irreducible algebras in the variety $\mathbb{G}$.

Theorem 1.7. [64] *Let* $\mathbf{A} \in \mathbb{G}$. *Then the following are equivalent:*
(1) $\mathbf{A}$ is simple;
(2) $\mathbf{A}$ is subdirectly irreducible;
(3) $\mathbf{A}$ is directly indecomposable;
(4) For every $x \in A$, $x \vee x^* = 1$ implies $x = 0$ or $x = 1$;
(5) $\mathbf{A} \in \{\mathbf{2}, \mathbf{3_{dblst}}, \mathbf{3_{klst}}\}$, up to isomorphism.

In this paper, we introduce and investigate a generalization of Gautam algebras, called "Almost Gautama algebras" ($\mathbb{AG}$ for short). We describe the subdirectly irreducible algebras in $\mathbb{AG}$ and then give several consequences, including the description of the lattice of subvarieties of $\mathbb{AG}$ and equational bases for all the subvarieties of $\mathbb{AG}$. It is also shown that the variety AG is a discriminator variety. Next, we consider the problem of logicizing the varety $\mathbb{AG}$, Unfortunately, $\mathbb{AG}$ lacks an implication operation. So, we introduce another variety called "Almost Gautama Heyting algebras ($\mathbb{AGH}$ for short) such that the language of $\mathbb{AGH}$ contains an implication operation symbol $\to$ and $\mathbb{AGH}$ is term-equivalent to $\mathbb{AG}$. We then consider $\mathbb{AG}$ from a logical point of view, via $\mathbb{AGH}$. More explicitly, we define a new propositional logic called $\mathcal{AG}$ (or $\mathcal{AGH}$) as an axiomatic extension of the logic $\mathcal{DHMH}$ which was introduced in [15] and show that $\mathcal{AG}$ is algebraizable with $\mathbb{AGH}$ as its equivalent algebraic semantics. Since $\mathbb{AGH}$ is term-equivalent to $\mathbb{AG}$, it can be viewed that the logic $\mathcal{AG}$ is the logic corresponding to $\mathbb{AG}$. It is also shown that the logic $\mathcal{AG}$ is decidable. Finally, all axiomatic extensions of the logic $\mathcal{AG}$, corresponding to all subvarieties of $\mathbb{AG}$ are determined. They include the axiomatic extensions $\mathcal{RDBLS}$t, $\mathcal{RKLS}$t and $\mathcal{G}$ of the logic $\mathcal{AG}$ corresponding to the varieties $\mathbb{RDBLS}$t, $\mathbb{RKLS}$t and $\mathbb{G}$, respectively. It is also deduced that none of the axiomatic extensions of $\mathcal{AG}$ has the Disjunction Property. The paper concludes with a few open problems for further research and with a fairly extensive (though not complete) bibliography.

It is assumed that the reader has had some familiarity with lattice theory and universal algebra (see [7, 21, 10], for example). As such, for notions, notations and results assumed here, the reader can refer to these or other relevant books.

2 The variety of Almost Gautama algebras

The purpose of this section is to introduce and investigate a new variety of algebras, called "Almost Gautama algebras" which, as mentioned earlier, is a generalization of Gautama algebras. The following lemma offers a hint for such a generalization.

Lemma 2.1. *Let $\mathbb{G}$ be the variety of Gautama algebras. Then*

(1) $\mathbb{G} \models x^{*\prime\prime} \approx x^*$ (Weak Star-Regular Identity),

(2) $\mathbb{G} \models (x \wedge x^{\prime *})^{\prime *} \approx x \wedge x^{\prime *}$ (L1).

Proof. Let $\mathbf{A} \in \mathbb{G}$. Let $a \in A$. Then, $a^{*\prime\prime} = a^{**\prime} = a^{***} = a^*$, proving (1), while it is routine to verify that (2) holds in $\mathbf{3_{dblst}}$ and $\mathbf{3_{klst}}$. □

We are now ready to define the variety of Almost Gautama algebras.

Definition 2.2. An algebra $\mathbf{A} = \langle A, \vee, \wedge, ^*, ^\prime, 0, 1\rangle$ is an Almost Gautama algebra if the following conditions hold:

(a) $\langle A, \vee, \wedge, ^*, 0, 1\rangle$ is a Stone algebra,

(b) $\langle A, \vee, \wedge, ^\prime, 0, 1\rangle$ is a dually quasi-De Morgan algebra,

(c) $\mathbf{A}$ is regular. That is, $\mathbf{A}$ satisfies the identity:
(R1) $x \wedge x^{\prime *\prime} \leq y \vee y^*$ (Regularity),

(d) $\mathbf{A}$ is Weak Star-Regular. That is, $\mathbf{A}$ satisfies the identity:
$(^*)_{\mathrm{w}}$ $x^{*\prime\prime} \approx x^*$, (weak star-regularity),

(e) $\mathbf{A}$ satisfies the identity:
$(x \wedge x^{\prime *})^{\prime *} \approx x \wedge x^{\prime *}$ (L1).

Let $\mathbb{AG}$ denote the variety of Almost Gautama algebras.

Clearly, in view of Lemma 2.1, every Gautama algebra is an Almost Gautama algebra. Hence, the varieties $\mathbb{BA}$ of Boolean algebras, $\mathbb{RDBLS}$t of regular double Stone algebras, and $\mathbb{RKLS}$t of regular Kleene Stone algebras and the variety $\mathbb{G}$ of Gautama algebras are all subvarieties of the variety $\mathbb{AG}$ of Almost Gautama algebras.

Consider the following 4-element algebra $\mathbf{4_{dmba}} := \langle\{0, a, b, 0, 1\}, \vee, \wedge, ^*, ', 0, 1\rangle$ (see Figure 3), where $*$ is the Boolean complement with $a^* = b$, $b^* = a$; and $0' = 1$, $1' = 0$, $a' = a$ and $b' = b$. It is easy to see that $\mathbf{4_{dmba}}$ is an Almost Gautama algebra. Observe that $\mathbf{4_{dmba}}$ is not a Gautama algebra (e.g., take $x := a$ in $(^*)_w$).

$\mathbf{4_{dmba}}$:

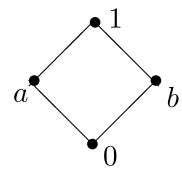


Figure 3

We close this section with a few concepts needed later in this paper.

The notion of "hemimorphic algebra" was implicit in [49] and was made explicit later in [58] (in its dual form).

Definition 2.3. An algebra $\mathbf{A} = \langle A, \vee, \wedge, ', 0, 1\rangle$ is a dually hemimorphic algebra if $\mathbf{A}$ satisfies the following conditions:

(H1) $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice,

(H2) $0' \approx 1$,

(H3) $1' \approx 0$,

(H4) $(x \wedge y)' \approx x' \vee y'$ ($\wedge$-De Morgan law).

The variety of dually hemimorphic algebras is denoted by $\mathbb{DHM}$.

We can recast the definition of a dually quasi-De Morgan algebra (see Definition 1.5) as follows:

$\mathbf{A} \in \mathbb{DHM}$ is a dually quasi-De Morgan algebra if it satisfies:

(H5) $(x \vee y)'' \approx x'' \vee y''$,

(H6) $x'' \leq x$.

The variety of dually quasi-De Morgan algebras is denoted by $\mathbb{DQD}$.

We will now introduce a far-reaching generalization of Gautama algebras.

Definition 2.4. An algebra $\mathbf{A} = \langle A, \vee, \wedge, ^*, ', 0, 1\rangle$ is a dually hemimorphic p-algebra if it satisfies:

(a) $\langle A, \vee, \wedge, ^*, 0, 1\rangle$ is a p-algebra,

(b) $\langle A, \vee, \wedge, ', 0, 1\rangle$ is a dually hemimorphic algebra.

The variety of dually hemimorphic p-algebras is denoted by $\mathbb{DHMP}$.

An algebra $\mathbf{A} = \langle A, \vee, \wedge, ^*, ', 0, 1\rangle$ is a *dually quasi-De Morgan p-algebra* if :

(a) $\langle A, \vee, \wedge, ^*, 0, 1\rangle$ is a p-algebra,

(b) $\langle A, \vee, \wedge, ', 0, 1\rangle$ is a dually quasi-De Morgan algebra.

The variety of dually quasi-De Morgan p-algebras is denoted by $\mathbb{DQDP}$. In fact, dually hemimorphic p-algebras are a common generalization of double p-algebras and pseudocomplemented Ockham algebras, which have been investigated in several papers, some of which are: [50, 51, 52, 54, 56, 65].

$\mathbf{A} \in \mathbb{DHMP}$ is regular if it satisfies (R1): $x \wedge x'^{*\prime} \leq y \vee y^*$. The variety of regular dually hemimorphic p-algebras is denoted by $\mathbb{RDHMP}$.

Clearly, $\mathbb{G} \subset \mathbb{AG} \subset \mathbb{RDBLP} \subset \mathbb{RDQDP} \subset \mathbb{DQDP} \subset \mathbb{DHMP}$. Also, $\mathbb{G} \subset \mathbb{AG} \subset \mathbb{RDMP} \subset \mathbb{RDQDP} \subset \mathbb{DQDP} \subset \mathbb{DHMP}$.

3 Subdirectly irreducible Almost Gautama Algebras

We now wish to characterize the subdirectly irreducible Almost Gautama algebras. To achieve this, we need some preliminary results. Recall the well-known fact (see [7, 21]) that if $\mathbf{A}$ is a p-algebra then $\mathbf{A} \models x \wedge (x \wedge y)^* \approx x \wedge y^*$.

Let $\mathbf{A} \in \mathbb{DQDP}$, $a \in A$ and let $(a] := \{x \in A : x \leq a\}$. Define the algebra $(\mathbf{a}]$ as follows:
$(\mathbf{a}] := \langle (a], \vee, \wedge, ^{*a}, ^{\prime a}, 0, a\rangle \in \mathbb{DQDP}$, where $x^{*a} := x^* \wedge a$ and $x^{\prime a} := x' \wedge a$, for $x \in (a]$. Similarly, the algebra $(\mathbf{a}^*]$ is defined.

Lemma 3.1. *Let $\mathbf{A} \in \mathbb{DQDP}$ satisfying* (L1): $(x \wedge x'^*)'^* \approx x \wedge x'^*$ *and let $a \in A$ such that $a \wedge a' = 0$. Then*

(i) $(\mathbf{a}] \in \mathbb{DQDP}$,

(ii) $a^* \wedge a^{*\prime} = 0$,

(iii) $(\mathbf{a}^*] \in \mathbb{DQDP}$.

Proof. Let $x, y \in (a]$. Then $x \wedge (x \wedge y)^{*a} = x \wedge (x \wedge y)^* \wedge a = x \wedge y^* \wedge a = x \wedge y^{*a}$, since * is the pseudocomplement. Also, $a^{*a} = a^* \wedge a = 0$, and $0^{*a} = 0^* \wedge a = 1 \wedge a = a$. So, x^{*a} is the pseudocomplement of $x \in (a]$. Now,

$$\begin{aligned}(x \wedge y)^{\prime a} &= (x \wedge y)' \wedge a\\ &= (x' \vee y') \wedge a\\ &= (x' \wedge a) \vee (y' \wedge a)\\ &= x^{\prime a} \vee y^{\prime a}.\end{aligned}$$

Next,

$$\begin{aligned}(x \vee y)^{\prime a \prime a} &= [(x \vee y)' \wedge a]' \wedge a\\ &= [(x \vee y)'' \vee a'] \wedge a\\ &= [x'' \vee y'' \vee a'] \wedge a\\ &= (x'' \wedge a) \vee (y'' \wedge a) \vee (a' \wedge a)\\ &= (x'' \wedge a) \vee (a' \wedge a) \vee (y'' \wedge a) \vee (a' \wedge a), \text{ as } a' \wedge a = 0\\ &= [(x'' \vee a') \wedge a)] \vee [(y'' \vee a') \wedge a]\\ &= [(x' \wedge a)' \wedge a)] \vee [(y' \wedge a)' \wedge a]\\ &= x^{\prime a \prime a} \vee y^{\prime a \prime a}.\end{aligned}$$

Also,

$$\begin{aligned}x^{\prime a \prime a} \vee x &= (x' \wedge a)' \wedge a) \vee x\\ &= (x'' \vee a') \wedge a) \vee x\\ &= [(x'' \wedge a) \vee (a' \wedge a)] \vee x\\ &= (x'' \wedge a) \vee x\\ &= (x'' \vee x) \wedge (a \vee x)\\ &= x \quad \text{as } x'' \leq x \leq a.\end{aligned}$$

Finally, $0'^{a} = 0' \wedge a = 1 \wedge a = a$, and $a'^{a} = a' \wedge a = 0$.
Thus, dual quasi-De Mogan identities hold in $(\mathbf{a}]$, proving (i).
For (ii), from $a \wedge a' = 0$, we get $a' \vee a'' = 1$ which implies $(a \wedge a') \vee (a \wedge a'') = a$. Hence $a \leq a''$ as $a \wedge a' = 0$. Thus we have

$$a'' = a. \tag{1}$$

From $a'' \wedge a''^{*} = 0$, we have $(a' \vee a'') \wedge (a' \vee a''^{*}) = a'$, whence $(a' \vee a^{*}) = a'$ as $a'' = a$ and $a \wedge a' = 0$; thus we have

$$a^{*} \leq a'. \tag{2}$$

Now, $a^{*} \wedge a^{*\prime *} = a''^{*} \wedge a''^{*\prime *} = (a' \wedge a''^{*})'^{*} = a' \wedge a''^{*} = a' \wedge a^{*} = a^{*}$ by (1), (L1) and (2). Hence,

$$a^{*} \leq a^{*\prime *}. \tag{3}$$

So, in view of (3), we get $a^{*} \wedge a^{*\prime} \leq a^{*\prime *} \wedge a^{*\prime} = 0$, implying $a^{*} \wedge a^{*\prime} = 0$, which proves (ii). (iii) follows from (i) and (ii). □

Recall the well-known result (see [7, 21]) that if $\mathbf{A}$ is a Stone algebra then $\mathbf{A} \models (x \wedge y)^{*} \approx x^{*} \vee y^{*}$.

Lemma 3.2. *Let* $\mathbf{A} \in \mathbb{AG}$ *and let* $a \in A$ *such that* $a \wedge a' = 0$ *Then*

(i) $(\mathbf{a}] = \langle (a], \vee, \wedge, ^{*a}, ^{\prime a}, 0, a \rangle \in \mathbb{AG}$.

(ii) $(\mathbf{a}^{*}] = \langle (a^{*}], \vee, \wedge, ^{*a^{*}}, ^{\prime a^{*}}, 0, a^{*} \rangle \in \mathbb{AG}$.

Proof. By Lemma 3.1, we already know that $(\mathbf{a}] \in \mathbb{DQDP}$. So, it suffices to prove (St), (R1), $(^{*})_{\mathrm{w}}$ and (L1). Toward this end, let $x, y \in A$ such that $x \leq a$ and $y \leq a$.

Since $\mathbf{A}$ is a Stone algebra, we have
$x^{*a} \vee x^{*a*a} = (x^{*} \wedge a) \vee [(x^{*} \wedge a)^{*} \wedge a] = (x^{*} \wedge a) \vee [(x^{**} \vee a^{*}) \wedge a] = (x^{*} \wedge a) \vee (x^{**} \wedge a) = (x^{*} \vee x^{**}) \wedge a = 1 \wedge a = a$.
So, Stone identity holds in $(\mathbf{a}]$.

Now,

$$\begin{aligned}
(x \wedge x^{\prime a * a \prime a}) &= x \wedge [(x' \wedge a)^{*} \wedge a]' \wedge a \\
&= x \wedge [(x' \wedge a)^{*} \wedge a]', \text{ as } x \leq a \\
&= x \wedge [(x' \wedge a)^{*\prime} \vee a'] \\
&= [x \wedge (x' \wedge a)^{*\prime}] \vee (x \wedge a'), \\
&= x \wedge (x' \wedge a)^{*\prime}, \text{ since } x \leq a \text{ and } a \wedge a' = 0.
\end{aligned}$$

Thus we have

$$x \wedge x^{\prime a * a \prime a} = x \wedge (x' \wedge a)^{*\prime}. \tag{4}$$

Since $(x' \wedge a)^{*\prime} \leq x'^{*\prime}$, we have $x \wedge (x' \wedge a)^{*\prime} \leq x \wedge x'^{*\prime} \leq y \vee y^{*}$. Also, observe that $x \wedge (x' \wedge a)^{*\prime} \leq x \leq a$. Hence, it follows that $x \wedge (x' \wedge a)^{*\prime} \leq (y \vee y^{*}) \wedge a$. Therefore, from (4) we get

$$\begin{aligned}
(x \wedge x^{\prime a * a \prime a}) \vee (y \vee y^{*a}) &= (y \vee y^{*}) \wedge a, \\
&= (y \wedge a) \vee (y^{*} \wedge a), \\
&= y \vee (y^{*} \wedge a), \\
&= y \vee y^{*a}.
\end{aligned}$$

Hence (R1) holds in $(\mathbf{a}]$.
Next,

$$\begin{aligned}
x^{*a'a'a} &= [(x^* \wedge a)' \wedge a]' \wedge a \\
&= [(x^{*\prime} \vee a') \wedge a]' \wedge a \\
&= [(x^{*\prime} \wedge a) \vee (a' \wedge a)]' \wedge a \\
&= (x^{*\prime} \wedge a)' \wedge a \\
&= (x^{*\prime\prime} \vee a') \wedge a \\
&= x^{*\prime\prime} \wedge a \\
&= x^* \wedge a, \text{ by the identity } (^*)_{\mathrm{w}} \\
&= x^{*a}.
\end{aligned}$$

Hence the weak star regular identity $(^*)_{\mathrm{w}}$ holds in $(\mathbf{a}]$.
Finally,

$$\begin{aligned}
(x \wedge x'^{a*a})'^{a*a} &= [x \wedge (x' \wedge a)^* \wedge a]'^{a*a} \\
&= [x \wedge (x' \wedge a)^*]'^{a*a} \text{ as } x \leq a \\
&= [\{x \wedge (x' \wedge a)^*\}' \wedge a]^* \wedge a \\
&= [\{x \wedge (x' \wedge a)^*\}'^* \vee a^*] \wedge a \quad \text{as } \mathbf{A} \text{ is a Stone algebra} \\
&= [x \wedge (x' \wedge a)^*]'^* \wedge a \\
&= [x \wedge (x'^* \vee a^*)]'^* \wedge a \quad \text{as } \mathbf{A} \text{ is a Stone algebra} \\
&= [(x \wedge x'^*) \vee (x \wedge a^*)]'^* \wedge a \\
&= (x \wedge x'^*)'^* \wedge a \quad \text{since } x \wedge a^* = 0 \text{ as } x \leq a \\
&= x \wedge x'^* \wedge a \quad \text{by (L1)} \\
&= x \wedge [(x'^* \wedge a) \vee (a^* \wedge a)] \\
&= x \wedge [(x'^* \vee a^*) \wedge a] \\
&= x \wedge (x' \wedge a)^* \wedge a \quad \text{as } \mathbf{A} \text{ is a Stone algebra} \\
&= x \wedge x'^{a*a}.
\end{aligned}$$

So, (L1) holds in $(\mathbf{a}]$, proving (i). The proof of (ii) is similar to (i). □

Lemma 3.3. *Let* $\mathbf{A} \in \mathbb{DQDP}$ *satisfy (L1) and let* $a \in \mathbf{A}$ *such that* $a \vee a^* = 1$ *and* $a \wedge a' = 0$. *Let* $g : \mathbf{A} \to (\mathbf{a}] \times (\mathbf{a}^*]$ *be defined by* $g(x) = \langle x \wedge a, x \wedge a^* \rangle$. *Then* g *is an isomorphism from* $\mathbf{A}$ *onto* $(\mathbf{a}] \times (\mathbf{a}^*]$.

Proof. It is easy to see that g is a lattice-homomorhism. Now,

$$\begin{aligned}
(g(x))^* &= (\langle x \wedge a, x \wedge a^* \rangle)^* \\
&= \langle (x \wedge a)^{*a}, (x \wedge a^*)^{*(a^*)} \rangle \\
&= \langle (x \wedge a)^* \wedge a, (x \wedge a^*)^* \wedge a^* \rangle \\
&= \langle x^* \wedge a,\ x^* \wedge a^* \rangle \text{ since } ^* \text{ is a pseudocomplement} \\
&= g(x^*).
\end{aligned}$$

Next,

$$\begin{aligned}
(g(x))' &= (\langle x \wedge a, x \wedge a^* \rangle)' \\
&= \langle (x \wedge a)'^a, (x \wedge a^*)'^{a^*} \rangle \\
&= \langle (x \wedge a')' \wedge a, (x \wedge a^*)' \wedge a^* \rangle \\
&= \langle (x' \vee a') \wedge a, (x' \vee a^{*\prime}) \wedge a^* \rangle \\
&= \langle x' \wedge a, (x' \wedge a^*) \vee (a^{*\prime} \wedge a^*) \rangle \text{ since } a \wedge a' = 0 \\
&= \langle x' \wedge a, x' \wedge a^* \rangle \text{ since } a^{*\prime} \wedge a^* = 0 \text{ by (ii) of Lemma 3.1} \\
&= g(x').
\end{aligned}$$

Next, suppose $g(x) = g(y)$. Then, $\langle x \wedge a, x \wedge a^* \rangle = \langle y \wedge a, y \wedge a^* \rangle$, whence $x \wedge a = y \wedge a$ and $x \wedge a^* = y \wedge a^*$. Thus, in view of the hypothesis, $x = x \wedge (a \vee a^*) = (x \wedge a) \vee (x \wedge a^*) = (y \wedge a) \vee (y \wedge a^*) = y \wedge (a \vee a^*) = y$, implying g is one-one. It is clear that g is onto. □

Lemma 3.4. *Let* $\mathbf{A} \in \mathbb{AG}$. *If* $a = a'^*$, *then* $a' = a^*$.

Proof. From $a = a'^*$, we get $a'' = a'^{*''} = a'^* = a$, in view of the axiom $(^*)_{\mathrm{w}}$. Thus we have the following:

$$a'' = a. \tag{5}$$

Next, we have $a' \leq a'^{**} = a^*$, since $a'^* = a$. Thus,

$$a' \leq a^*. \tag{6}$$

Hence, we get

$$\begin{aligned} a' &= a' \wedge a^* \text{ by (6)} \\ &= a' \wedge a''^* \text{ by (5)} \\ &= (a' \wedge a''^*)'^* \text{ by (L1)} \\ &= (a'' \vee a''^{*'})^* \\ &= a''^* \wedge a''^{*'*} \\ &= a^* \wedge a^{*'*}, \text{ by (5)}. \end{aligned}$$

Thus,

$$a' = a^* \wedge a^{*'*}. \tag{7}$$

By (6), we have $a' = a' \wedge a^*$, whence $a'' = a'' \vee a^{*'}$, implying $a = a \vee a^{*'}$ in view of (5). Hence we get $a^* = a^{*'*} \wedge a^*$. Hence, in view of (7), we conclude $a' = a^*$. □

Lemma 3.5. *Let* $\mathbf{A} \in \mathbb{DHMP}$ *satisfying* $x'' \leq x$ *and let* $a \in A$ *such that* $a' = a^*$. *Then* $a \vee a^* = 1$.

Proof. From $a \wedge a^* = 0$, we get $a' \vee a^{*'} = 1$, implying $a' \vee a'' = 1$ as $a' = a^*$. Hence $a' \vee a = 1$ since $a'' \leq a$, whence $a \vee a^* = 1$, as $a' = a^*$. □

Corollary 3.6. *Let* $\mathbf{A} \in \mathbb{AG}$ *and let* $a \in A$ *such that* $a' = a^*$. *Then* $\mathbf{A} \cong (a] \times (a^*]$.

Proof. Let $\mathbf{A}$ and a be as in the hypothesis. Then by Lemma 3.5 we have $a \vee a^* = 1$. Also, since $a' = a^*$ by hypothesis, it is clear that $a \wedge a' = 0$. Hence, in view of Lemmma 3.3 we conclude that $\mathbf{A} \cong (a] \times (a^*]$. □

Lemma 3.7. *Let* $\mathbf{A} \in \mathbb{DHMP}$ *such that*

(1) $\mathbf{A} \models x'' \leq x$ and
(2) $\mathbf{A} \models x \wedge x'^* \approx (x \wedge x'^*)'^*$ (L1).

Let $y \in A$. Then $(y \wedge y'^*) \vee (y \wedge y'^*)^* = 1$.

Proof. Observe $y \vee (y \wedge y'^*)^* \geq y'' \vee (y \wedge y'^*)^* = y'' \vee (y \wedge y'^*)'^{**} = y'' \vee (y' \vee y'^{*'})^{**} = y'' \vee (y'^* \wedge y'^{*'*})^* \geq y'' \vee y'^{*'**} \geq y'' \vee y'^{*'} = (y' \wedge y'^*)' = 0' = 1$, Thus,

$$y \vee (y \wedge y'^*)^* \approx 1, . \tag{8}$$

Now

$$\begin{aligned} (y \wedge y'^*) \vee (y \wedge y'^*)^* &= [y \vee (y \wedge y'^*)^*] \wedge [y'^* \vee (y \wedge y'^*)^*] \\ &= 1 \wedge [y'^* \vee (y \wedge y'^*)^*] \text{ by (8)} \\ &= [y'^* \vee (y \wedge y'^*)^*] \\ &= [y'^* \vee (y \wedge y'^*)'^{**}] \text{ by (L1)} \\ &= [y'^* \vee (y'^* \wedge y'^{*'*})^*] \\ &= 1, \text{ by (8)}, \end{aligned}$$

which completes the proof. □

Now we will introduce a condition for an algebra $\mathbf{A} \in \mathbb{AG}$ that will be used in the rest of the paper.

$$\text{(SC)} \qquad x \neq 1 \quad \text{then} \quad x \wedge x'^* = 0.$$

Lemma 3.8. *Let* $\mathbf{A} \in \mathbb{AG}$ *be directly indecomposable. Then* $\mathbf{A}$ *satisfies* (SC).

Proof. Suppose $\mathbf{A}$ does not satisfy (SC). Then there exists a $b \in A$ such that $b \neq 1$ and $b \wedge b'^* \neq 0$. Since $b \wedge b'^* = (b \wedge b'^*)'^*$ by (L1), we have $(b \wedge b'^*)^* = (b \wedge b'^*)'$ by Lemma 3.4, which, in view of Lemma 3.7 (b) implies $(b \wedge b'^*) \vee (b \wedge b'^*)^* = 1$. Hence by Corollary 3.6, we get $\mathbf{A} \cong (b \wedge b'^*] \times ((b \wedge b'^*)^*]$. Since $b \wedge b'^* \notin \{0, 1\}$, $\mathbf{A}$ is expressed as a nontrivial direct product, completing the proof. □

Lemma 3.9. *Let* $\mathbf{A} \in \mathbb{AG}$ *satisfying* (SC) *and* $a \in A \setminus \{0\}$*. Then* $a \vee a^{*\prime} = 1$.

Proof. Let $x \in A$. Then, since $x' \leq x \vee x'$, we get $x \geq x'' \geq (x' \vee x)'$. Hence, $x \vee (x' \vee x)' = x$, implying $x^* \wedge (x' \vee x)'^* = x^*$. Thus, we have

$$\text{For } x \in A,\ x^* \leq (x \vee x')'^*. \tag{9}$$

Replacing x by x^* in (9), we get $x \leq x^{**} \leq (x \vee x^{*\prime})'^*$. Thus,

$$\text{For } x \in A,\ x \leq (x \vee x^{*\prime})'^*. \tag{10}$$

Now, suppose the conclusion of the lemma is false. Then $a \vee a^{*\prime} \neq 1$. Therefore, $(a \vee a^{*\prime}) \wedge (a \vee a^{*\prime})'^* = 0$ by (SC), from which we get $a \wedge (a \vee a^{*\prime}) \wedge (a \vee a^{*\prime})'^* = 0$, which simplifies to $a \wedge (a \vee a^{*\prime})'^* = 0$. It follows, in view of (10), that $a = 0$, which is a contradiction to $a \neq 0.$, proving the lemma. □

Lemma 3.10. *Let* $\mathbf{A} \in \mathbb{AG}$ *and* $a, b \in A$ *such that*

(i) $\mathbf{A}$ satisfies (SC);

(ii) $0 < a < b < 1$.

Then, $a^{**} = 1$.

Proof. Assume $\mathbf{A}$ and a, b satisfy the hypothesis, and further suppose $a^{**} \neq 1$. We wish to arrive at a contradiction.

CLAIM 1: $a^{**} = a$.

For, from the supposition that $a^{**} \neq 1$, it is clear that $a^* \neq 0$ and so, $a^{*\prime} \neq 1$, in view of the axiom $(^*)_w$. Hence, $a^{*\prime} \wedge a^{*\prime\prime *} = 0$ by (SC). So,

$$\begin{aligned} a &= a \vee (a^{*\prime} \wedge a^{*\prime\prime *}) \\ &= (a \vee a^{*\prime}) \wedge (a \vee a^{*\prime\prime *}) \\ &= a \vee a^{*\prime\prime *}, \text{ since } a \vee a^{*\prime} = 1 \text{ by Lemma 3.9} \\ &= a \vee a^{**} \text{ since } a^{*\prime\prime} = a^* \text{ by the axiom } (^*)_w \\ &= a^{**}, \text{ as } a \leq a^{**}, \end{aligned}$$

proving the claim.

CLAIM 2: $a'' = a$.

For,

$$\begin{aligned} a'' &= a^{**\prime\prime} \text{ by } \textbf{CLAIM 1} \\ &= a^{**} \text{ by the axiom } (^*)_{\mathrm{w}} \\ &= a \text{ by } \textbf{CLAIM 1}. \end{aligned}$$

CLAIM 3: $b \leq a'$.

From (R1) we have $b \wedge (a' \vee a'^*) = b \wedge [(b \wedge b'^{*\prime}) \vee (a' \vee a'^*)] = (b \wedge b'^{*\prime}) \vee [b \wedge (a' \vee a'^*)] = b \wedge [b'^{*\prime} \vee (a' \vee a'^*)]$. Thus we have

$$b \wedge (a' \vee a'^*) = b \wedge [b'^{*\prime} \vee a' \vee a'^*]. \tag{11}$$

As $b \neq 1$, we have $b \wedge b'^* = 0$ by (SC). Also, as $a \leq b$, we have $a'^* \leq b'^*$ implying $b \wedge a'^* = 0$. So $a' = a' \vee (b \wedge a'^*)$, which implies $a' = (a' \vee b) \wedge (a' \vee a'^*)$, whence, $b \wedge a' = b \wedge (a' \vee a'^*)$. But we know $b \wedge (a' \vee a'^*) = b \wedge [b'^{*\prime} \vee (a' \vee a'^*)]$ from (11), whence we have the following:

$$b \wedge a' = b \wedge (b'^{*\prime} \vee a' \vee a'^*). \tag{12}$$

In view of (SC), as $b \neq 1$, we get $b \wedge b'^* = 0$. So, $b' \vee b'^{*\prime} = 1$, implying $a' \vee b'^{*\prime} = 1$, as $a \leq b$. Hence, from (12) we have $b \wedge a' = b \wedge (1 \vee a'^*)$. Thus we get $b \wedge a' = b$.

CLAIM 4: $a' = 1$.

For, suppose $a' \neq 1$. Then $a' \wedge a'^* = 0$. Hence $(a' \wedge a'^*) \vee (a' \wedge a''^{**}) = a' \wedge a''^{**}$, implying $a' \wedge (a'^* \vee a''^{**}) = a' \wedge a''^{**}$. Since $a'^* \vee a''^{**} = 1$, we have $a' = a' \wedge a''^{**}$. Hence $b \wedge a' = b \wedge a''^{**}$. But, from CLAIM 3, we have $b = b \wedge a'$. Hence, we get $b \leq a''^{**} \leq a^{**}$.

But a"=a by CLAIM 2. Hence we have $b \leq a^{**}$. Also we know $a^{**} = a$ by CLAIM 1. Thus $b \leq a$, which is a contradiction to $a < b$. Hence we conclude $a' = 1$, proving the claim.

Now, in view of CLAIM 2 and CLAIM 4, we get $a = a'' = 0$, which is a contradiction to the hypothesis that $a > 0$. This contradiction proves that our initial supposition is false. Thus the conclusion $a^{**} = 1$ holds, proving the lemma. □

Let $h(\mathbf{A})$ denote the height of $\mathbf{A} \in \mathbb{AG}$.

Lemma 3.11. *Let* $\mathbf{A} \in \mathbb{AG}$ *satisfy* (SC). *Then* $h(A) \leq 2$.

Proof. Suppose $h(A) > 2$. Hence, there exist elements $a, b \in A$ such that $0 < a < b < 1$. Since $b \neq 1$, we have $b \wedge b'^* = 0$ by (SC), implying that $a \wedge b'^* = 0$ since $a < b$. So, $a^{**} \wedge b'^{***} = 0$ which implies $b'^* = 0$, since $a^{**} = 1$ by Lemma 3.10. Thus we have

$$b'^* = 0. \tag{13}$$

From regularity it follows that $b \wedge b'^{*\prime} \leq a \vee a^*$, which implies $b \leq a$ in view of (13) and Lemma 3.10. Hence we have arrived at a contradiction, proving the lemma. □

The following theorem gives an explicit description of subdirectly irreducible Almost Gautama algebras.

Theorem 3.12. *Let* $\mathbf{A} \in \mathbb{AG}$. *Then the following are equivalent:*

(1) $\mathbf{A}$ *is simple;*
(2) $\mathbf{A}$ *is subdirectly irreducible;*
(3) $\mathbf{A}$ *is directly indecomposable;*
(4) $\mathbf{A}$ *satisfies (SC);*
(5) $\mathbf{A} \in \{\mathbf{2}, \mathbf{3_{dblst}}, \mathbf{3_{dmst}}, \mathbf{4_{dmba}}\}$, *where* $\mathbf{4_{dmba}}$ *is the algebra in Figure 3.*

Proof. It is well-known that (1) ⇒ (2) ⇒ (3). (3) ⇒ (4) is proved in Lemma 3.8. We now prove (4) ⇒ (5); so, we assume (4). We know from Lemma 3.11 that the height of the lattice reduct of $\mathbf{A}$ is ≤ 2. Now it is easy to see that the only nontrivial algebras in $\mathbb{AG}$ of height ≤ 2, up to isomorphism, are $\mathbf{2}$, $\mathbf{3_{dblst}}$, $\mathbf{3_{klst}}$, $\mathbf{4_{dmba}}$ and $\mathbf{2} \times \mathbf{2}$. But, it is easily seen that the algebra $\mathbf{2} \times \mathbf{2}$ does not satisfy (4); thus, (5) holds. Finally, it is routine to verify that $\mathbf{2}$, $\mathbf{3_{dblst}}$, $\mathbf{3_{klst}}$, and $\mathbf{4}_{dmba}$ are indeed simple, thus, (5) ⇒ (1). Hence, the proof of the theorem is complete. □

In the rest of this section we present several consequences of Theorem 3.12.

Corollary 3.13. *The variety* $\mathbb{AG}$ *is generated by* $\{\mathbf{3}_{dblst}, \mathbf{3}_{dmst}, \mathbf{4_{dmba}}\}$. *Hence, every algebra* $A \in \mathbb{AG}$ *is a subdirect product of* $\mathbf{2}$, $\mathbf{3}_{dblst}$, $\mathbf{3}_{dmst}$, *and* $\mathbf{4_{dmba}}$.

Corollary 3.13 will be improved further in Corollary 5.4.

3.1 Equational Bases for subvarieties of $\mathbb{AG}$

We now give equational bases for all subvarieties of the variety $\mathbb{AG}$. In view of Theorem 3.12, the proofs of the following theorems are easy and hence are left to the reader.

Corollary 3.14. *The variety* $\mathbb{V}(\mathbf{2})(= \mathbb{BA})$ *is defined, modulo* $\mathbb{AG}$*, by the identity:* $x^* \approx x'$.

Corollary 3.15. *The variety* $\mathbb{V}(\mathbf{3_{dblst}})$ *is defined, modulo* $\mathbb{AG}$*, by*
(i) the identity: $x \vee x' \approx 1$
or by
(ii) the identity: $x' \wedge x'' \approx 0$,
or by
(iii) the identity: $x'^{*'} \approx x'$.

Corollary 3.16. *The variety* $\mathbb{V}(\mathbf{3_{klst}})$ *is defined, modulo* $\mathbb{AG}$*, by the identities:* $x^{*'} \approx x^{**}$*, and* $x'' \approx x$.

Corollary 3.17. *The variety* $\mathbb{V}(\mathbf{4_{dmba}})$ *is defined, modulo* $\mathbb{AG}$*, by*
(i) the identity: $x \vee x^* \approx 1$.
or by
(ii) the identity: $x^{**} \approx x$,
or by
(iii) the identity: $x'^{*'} \approx x^*$.
or by
(iv) $x^{**} \approx x''$.

Corollary 3.18. *The variety* $\mathbb{V}(\{\mathbf{3_{dblst}}, \mathbf{3_{klst}}\})$ *is defined, modulo* $\mathbb{AG}$*, by:* $x^{*'} \approx x^{**}$.

Since $\mathbf{3_{dblst}}$ and $\mathbf{3_{klst}}$ are Gautama algebras and $\mathbf{4_{dmba}}$ is not, the following corollary, which was first proved in [64], is immediate.

Corollary 3.19. [64] *The variety* $\mathbb{G}$ *of Gautama algebras is generated by* $\{\mathbf{3_{dblst}}, \mathbf{3_{klst}}\}$ (i.e., $\mathbb{G} = \mathbb{V}(\mathbf{3_{dblst}}, \mathbf{3_{klst}})$).

Corollary 3.20. *The variety* $\mathbb{V}(\{\mathbf{3_{dblst}}, \mathbf{4_{dmba}}\})$ *is defined, modulo* $\mathbb{AG}$*, by the identity:*
$x' \vee (y^* \vee z) \approx (x' \vee y)^* \vee (x' \vee z)$ (A version of (JID),
or by the identity:
$(x' \vee y)^* \vee x' \approx x' \vee y^*$.

Corollary 3.21. *The variety* $\mathbb{V}(\{\mathbf{3_{klst}}, \mathbf{4_{dmba}}\})$ *is defined, modulo* $\mathbb{AG}$*, by the identity:* $x'' \approx x$.

3.2 The Lattice of Subvarieties of $\mathbb{AG}$

We give more applications of Theorem 3.12. The proof of the following corollary of Theorem 3.12 is easy.

Corollary 3.22.

(1) *The lattice of nontrivial subvarieties of* $\mathbb{AG}$ *is isomorphic to the 8-element Boolean lattice with* $\mathbb{V}(\mathbf{2})$ *(i.e., the variety of Boolean allgebras) as the least element. The Hasse diagram of this lattice is given in Figure 4.*

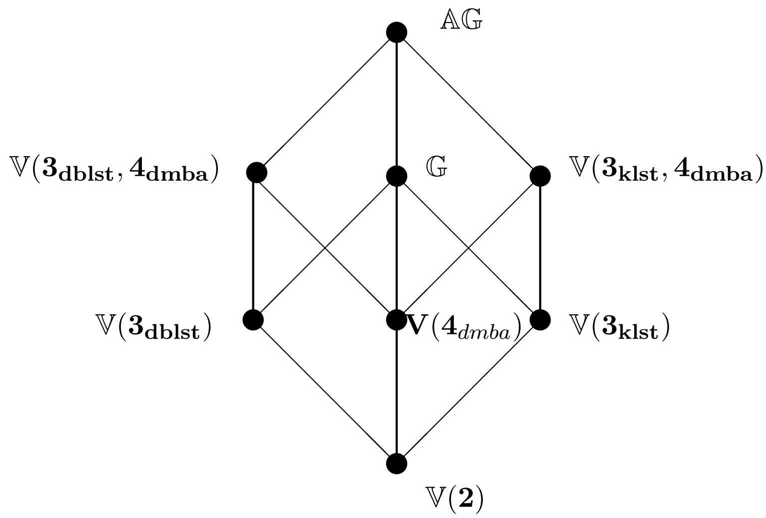


Figure 4

(2) *The lattice of nontrivial subvarieties of $\mathbb{G}$ is isomorphic to the 4-element Boolean lattice with $\mathbb{V}(\mathbf{2})$ as the least element.*

Since the variety $\mathbb{AG}$ is finitely generated, the following corollary is immediate.

Corollary 3.23. *The equational theories of $\mathbb{AG}$ and all its subvarieties are decidable.*

In fact, a much stronger result is true (see Corollary 5.5).

In passing, we mention two new axiomatizations for the variety $\mathbb{G}$, whose proofs are left to the reader. The variety of regular dually quasi-De Morgan p-algebras of level 1 (i.e., satisfying $x \wedge x'^{*} \wedge x'^{*'*} \approx x \wedge x'^{*}$) is denoted by $\mathbb{RDQDP}_1$.

Theorem 3.24. *The variety $\mathbb{G}$ is also defined, modulo $\mathbb{RDQDP}_1$, by*
$x^{*'*} = x^{*}$.

Theorem 3.25. *The variety $\mathbb{G}$ is also defined, modulo $\mathbb{RDQDP}_1$, by*
$x'^{*'*} = x'^{*}$ *and* $x^{*} \vee x^{**} \approx 1$.

4 The Variety of Almost Gautama Heyting Algebras ($\mathbb{AGH}$)

Observe that the implication connective is missing in algebras in $\mathbb{AG}$. Let us, therefore, consider the language $\mathbf{L} = \langle \vee, \wedge, \to, ', 0, 1 \rangle$. We will now define a new variety of algebras, namely the variety $\mathbb{AGH}$ of Almost

Gautama Heyting algebras of type **L** and show that it is term-equivalent to the variety $\mathbb{AG}$ of Almost Gautama algebras. This fact will play a crucial role later in "logicizing" the variety $\mathbb{AG}$.

Actually, $\mathbb{AGH}$ turns out, to our surprise, to coincide with the variety $\mathbb{RDQDS}t\mathbb{H}_1$, already introduced in [60], which is a subvariety of $\mathbb{DHMSH}$ of dually hemimorphic semi-Heyting algebras. The variety $\mathbb{DHMSH}$ and its many subvarieties have been investigated in a series of papers, some of which are: [4, 32, 45, 48, 46, 58, 59, 60, 61, 62, 63, 15, 16, 64, 17]. We, therefore, need to recall some preliminaries.

Semi-Heyting algebras were introduced in [47]; but the first results about them were published in [57]. For further results on semi-Heyting algebras, see, for example, [1, 2, 3, 14, 13].

Definition 4.1. An algebra $\mathbf{A} = \langle A, \vee, \wedge, \to, 0, 1\rangle$ is a semi-Heyting algebra if $\mathbf{A}$ satisfies the following conditions:

(i) $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice,

(ii) $x \wedge (x \to y) \approx x \wedge y$,

(iii) $x \wedge (y \to z) \approx x \wedge [(x \wedge y) \to (x \wedge z)]$,

(iv) $x \to x \approx 1$.

The variety of semi-Heyting algebras is denoted by $\mathbb{SH}$.
$\mathbf{A} \in \mathbb{SH}$ is a Heyting algebra if it satisfies:
(H) $(x \wedge y) \to x \approx 1$.
The variety of Heyting algebras is denoted by $\mathbb{H}$.

We can now define the crucial notion of a dually hemimorphic semi-Heyting algebra fundamental to the rest of this paper.

Definition 4.2. An algebra $\mathbf{A} = \langle A, \vee, \wedge, \to, ', 0, 1\rangle$ is a dually hemimorphic semi-Heyting algebra (see [58]). if $\mathbf{A}$ satisfies the following conditions:

(E1) $\langle A, \vee, \wedge, \to, 0, 1\rangle$ is a semi-Heyting algebra,

(E2) $\langle A, \vee, \wedge, ', 0, 1\rangle$ is a dually hemimorphic algebra (see Definition 2.3).

The variety of dually hemimorphic semi-Heyting algebras will be denoted by $\mathbb{DHMSH}$.
$\mathbf{A} \in \mathbb{DHMSH}$ is a dually hemimorphic Heyting algebra if it satisfies:
(H) $(x \wedge y) \to x \approx 1$.
$\mathbb{DHMH}$ denotes the variety of dually hemimorphic Heyting algebras. $\mathbf{A} \in \mathbb{DHMSH}$ is a dually quasi-De Morgan semi-Heyting algebra if its reduct $\langle A, \vee, \wedge, ', 0, 1\rangle$ is in $\mathbb{DQD}$.

The varieties of dually quasi-De Morgan semi-Heyting algebras and of dually quasi-De Morgan Heyting algebras are, respectively, denoted by $\mathbb{DQDSH}$ and $\mathbb{DQDH}$.

$\mathbf{A} \in \mathbb{DHMSH}$ *is regular* if $\mathbf{L}$ satisfies:

(R1) $x \wedge x^+ \leq y \vee y^*$, where $x^* := x \to 0$ and $x^+ := x'^{*'}$.

The variety of regular dually hemimorphic [quasi-De Morgan] Heyting algebras is denoted by $\mathbb{RDHMH}$ [$\mathbb{RDQDH}$].

$\mathbf{A} \in \mathbb{RDQDH}$ is a *regular dually quasi-De Morgan Stone Heyting algebra* if $\mathbf{A}$ satisfies:

(St) $x^* \vee x^{**} \approx 1$.

Let $\mathbb{RDQDS}\text{t}\mathbb{H}$ denote the variety of regular dually quasi De Morgan Stone Heyting algebras.

Remark 4.3. The reader is cautioned here not to confuse the notion of regularity given in the above definition with the one given in [58].

The varieties $\mathbb{DHMSH}$, $\mathbb{DHMH}$, $\mathbb{DQDH}$, $\mathbb{RDQDH}$, $\mathbb{RDQDS}\mathrm{t}\mathbb{H}$ and many of their subvarieties are examined, in [58, 59, 60, 61]. The logics associated with those subvarieties of the variety $\mathbb{DHMSH}$ are investigated in [16].

The notion of "level n" has played an important role in the classification of subvarieties of $\mathbb{DHMSH}$ in [58], although this name was not explicitly used there. We only need the definition of "level 1" here.

Definition 4.4. An algebra $\mathbf{A} \in \mathbb{DHMSH}$ is of level 1 if it satisfies the identity:
$x \wedge x'^{*} \approx x \wedge x'^{*} \wedge x'^{*'*}$ (Level 1).

Let $\mathbb{DHMSH}_1$ denote the subvariety of $\mathbb{DHMSH}$ of level 1. For a subvariety $\mathbb{V}$ of $\mathbb{DHMSH}$, we let $\mathbb{V}_1 := \mathbb{V} \cap \mathbb{DHMSH}_1$. Thus, $\mathbb{RDQDS}\mathrm{t}\mathbb{H}_1$ denotes the subvariety of $\mathbb{RDQDS}\mathrm{t}\mathbb{H}$ of level 1.

We are ready to define the variety of Almost Gautama Heyting algebras.

Definition 4.5. An algebra $\mathbf{A} = \langle A, \vee, \wedge, \to, ', 0, 1\rangle$ is an Almost Gautama Heyting algebra if $\mathbf{A} \in \mathbb{DHMSH}$ and satisfies the following additional axioms:

(1) $(x \wedge y) \to x \approx 1$ (H),

(2) $x^* \vee x^{**} \approx 1$ (St), where $x^* := x \to 0$,

(3) $(x \vee y)'' \approx x'' \vee y''$,

(4) $x'' \leq x$,

(5) $x \wedge x'^{*'} \leq y \vee y^*$ (R1),

(6) $x^{*''} \approx x^*$ $(^*)_w$,

(7) $(x \wedge x'^{*})'^{*} \approx x \wedge x'^{*}$ (L1).

The variety of Almost Gautama Heyting algebras will be denoted by $\mathbb{AGH}$.

Remark 4.6. It is clear that $\mathbb{AGH} \subseteq \mathbb{DQDH}$. Hence, it follows from Lemma 2.4 (5) of [59] that the identity (Level 1) is equivalent to the following identity in $\mathbb{GH}$:
(L1) $(x \wedge x'^{*})'^{*} \approx x \wedge x'^{*}$.

Proposition 4.7. $\mathbb{AGH} \subseteq \mathbb{RDQDS}\mathrm{t}\mathbb{H}_1 \subset \mathbb{DQDH}_1 \subset \mathbb{DHMSH}_1$.

Proof. Axioms (1) –(5) of $\mathbb{AGH}$ imply that $\mathbb{AGH} \subseteq \mathbb{RDQDS}\mathrm{t}\mathbb{H}$. Also, the variety $\mathbb{AGH}$ is of level 1 by definition and Remark 4.6, whence $\mathbb{AGH} \subseteq \mathbb{RDQDS}t\mathbb{H}_1$. The rest of the inclusions are also immediate from the relevant definitions. □

Remark 4.8.

(1) Let $\mathbf{3_{dsth}} := \langle \mathbf{3}, \vee, \wedge, \to, ^+, 0, 1\}$ be the algebra, where $\mathbf{3}$ is the 3-element chain, $0 < a < 1$ (viewed as a bounded distributive lattice), the operation $^+$ is defined as: $0^+ = 1$, $a^+ = 1$ and $1^+ = 0$, and $\to$ is defined as follows:

$\rightarrow$	0	a	1
0	1	1	1
a	0	1	1
1	0	a	1

Clearly, $\mathbf{3_{dsth}}$ is an algebra in $\mathbb{AGH}$.

(2) Let $\mathbf{3_{klh}} := \langle \mathbf{3}, \vee, \wedge, \rightarrow, ', 0, 1\rangle$ be the algebra, where $\mathbf{3}$ is the 3-element chain, $0 < a < 1$ (viewed as a bounded distributive lattice), the operation $'$ is defined as: $0' = 1$, $a' = a$ and $1' = 0$, and $\rightarrow$ is defined as in (1). Note that $\mathbf{3_{klh}}$ is also an algebra in $\mathbb{AGH}$.

(3) Let $\mathbf{4_{dmh}} := \langle \mathbf{4}, \vee, \wedge, \rightarrow, ', 0, 1\rangle$ be the De Morgan Heyting (Boolean) algebra, where $\mathbf{4}$, is the 4-element Boolean lattice shown below and the operation $'$ is defined as: $0' = 1$, $a' = a$, $b' = b$ and $1' = 0$; and $\rightarrow$ is the Boolean implication:

$\mathbf{4_{dmh}}$:

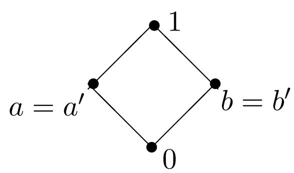


Figure 4

It is easy to see that $\mathbf{4_{dmh}} \in \mathbb{AGH}$.

We, now, wish to give an explicit description of the subdirectly irreducible algebras in $\mathbb{AGH}$. To achieve this goal, we need some definitions and results from [59].

The following lemma is a special case of Lemma 4.8 of [59], when the underlying semi-Heyting algebra is actually a Heyting algebra.

Lemma 4.9. *Let* $\mathbf{A} \in \mathbb{RDQDS}\mathrm{t}\mathbb{H}_1$ *satisfy the simplicity condition:*
(SC) For every $x \in L$, if $x \neq 1$, then $x \wedge x'^* = 0$.
Then $\mathbf{A}$ is of height at most 2.

Corollary 4.10. *Let* $\mathbf{A} \in \mathbb{AGH}$. *If* $\mathbf{A}$ *satisfies* (SC), then $\mathbf{A}$ is of height at most 2.

Proof. We know $\mathbb{AGH} \subseteq \mathbb{RDQDS}t\mathbb{H}_1$ by Proposition 4.7. Now apply Lemma 4.9. □

The following lemma is a special case of Corollary 4.1 of [59].

Lemma 4.11. *Let* $\mathbf{A} \in \mathbb{RDQDS}\mathrm{t}\mathbb{H}_1$ *with* $|A| \geq 2$. *Then TFAE:*

(1) $\mathbf{A}$ *is simple,*

(2) $\mathbf{A}$ *is subdirectly irreducible,*

(3) *For every* $x \in \mathbf{A}$, *if* $x \neq 1$, *then* $x \wedge x'^* = 0$.

We are now ready to give a concrete description of the subdirectly irreducible algebras in $\mathbb{AGH}$.

Theorem 4.12. *Let* $\mathbf{L} \in \mathbb{AGH}$ *with* $|L| \geq 2$. *Then TFAE:*

(1) $\mathbf{L}$ *is simple,*

(2) $\mathbf{L}$ *is subdirectly irreducible,*

(3) *For every* $x \in L$, *if* $x \neq 1$, *then* $x \wedge x'^{*} = 0$,

(4) $\mathbf{L} \in \{\mathbf{2},\ \mathbf{3_{dsth}},\ \mathbf{3_{klh}}, \mathbf{4_{dmh}},\}$, *up to isomorphism.*

Proof. (1) $\Rightarrow$ (2) is well-known, while (2) $\Rightarrow$ (3) by Lemma 4.11. Suppose (3) holds. Then $\mathbf{A}$ is of height at most 2 by Corollary 4.10. Then it is easy to see that the algebras of height at most 2 in $\mathbb{AGH}$ are, up to isomorphism, precisely $\mathbf{2},\ \mathbf{3_{dsth}},\ \mathbf{3_{klh}}, \mathbf{4_{dmh}}$, and $\mathbf{2} \times \mathbf{2}$. It is also clear that the algebra $\mathbf{2} \times \mathbf{2}$ does not satisfy the hypothesis (3), implying that (4) holds. Thus (3) implies (4), while it is routine to verify that (4) implies (1), proving the theorem. □

Corollary 4.13.

(i) *The smallest non-trivial subvariety of* $\mathbb{AGH}$ *is* $\mathbb{BA}$.

(ii) *The lattice of nontrivial subvarieties of* $\mathbb{AGH}$ *has exactly 3 atoms:* $\mathbb{V}(\mathbf{3_{dsth}})$, $\mathbb{V}(\mathbf{3_{klh}})$, *and* $\mathbb{V}(\mathbf{4_{dmh}})$.

(iii) *The lattice of nontrivial subvarieties of* $\mathbb{AGH}$ *is isomorphic to 8-element Boolean algebra.*

Let $\mathbb{GH}$ denote the variety $\mathbb{V}(\mathbf{3_{dsth}}), \mathbf{3_{klh}})$. We will call its elements Gautama Heyting algebras.

Corollary 4.14. *Let* $\mathbf{A} := \langle A, \vee, \wedge, \to^{\mathbf{A}},\ ', 0, 1\rangle \in \mathbb{AGH}$. *Define* $\to_k$ *on* A *by*

$$x \to_k y := (x^* \vee y^{**})^{**} \wedge [(x \vee x^*)'^{*\prime} \vee x^* \vee y \vee y^*],$$

where $x^* := x \to^{\mathbf{A}} 0$. *Then,* $\to^{\mathbf{A}}\ = \to_k$.

Proof. It suffices to show that the equality holds on the (non-trivial) subdirectly irreducible algebras in $\mathbb{AGH}$, which, in view of Theorem 4.12, are $\mathbf{2}, \mathbf{3_{dsth}}$, and $\mathbf{3_{klh}}$, and $\mathbf{4_{dmh}}$, up to isomorphism. Now it is routine to verify the equality of $\to^{\mathbf{A}}$ and $\to_k$ on these four algebras. □

Corollary 4.15. $\mathbb{AGH} = \mathbb{RDQDS}t\mathbb{H}_1$.

Proof. It is clear from Theorem 4.12 and Theorem 4.9 of [59] that both the varieties have the same subdirectly irreducible algebras. □

5 Applications

5.1 Term-Equivalence between Almost Gautama algebras and $\mathbb{AGH}$-algebras

The following theorem will play a crucial role in describing the logic associated with the variety of Almost Gautama algebras.

Theorem 5.1. *The varieties* $\mathbb{AG}$ *and* $\mathbb{AGH}$ *are term-equivalent. More explicitly,*

(a) *For* $\mathbf{A} := \langle A, \vee, \wedge, ^*, ', 0, 1\rangle \in \mathbb{AG}$, *let* $\mathbf{A}_{agh} := \langle A, \vee, \wedge, \to_k, ', 0, 1\rangle$, *where* $\to_k$ *is defined by:*

$$x \to_k y := (x^* \vee y^{**})^{**} \wedge [(x \vee x^*)'^{*\prime} \vee x^* \vee y \vee y^*].$$

Then $\mathbf{A}_{agh} \in \mathbb{AGH}$.

(b) *For* $\mathbf{A} := \langle A, \vee, \wedge, \to^{\mathbf{A}}, {}', 0, 1\rangle \in \mathbb{AGH}$, *let* $\mathbf{A}_{ag} := \langle A, \vee, \wedge, ^{\circ}, {}', 0, 1\rangle$, *where* $^{\circ}$ *is defined by* $x^{\circ} := x \to^{\mathbf{A}} 0$. *Then* $\mathbf{A}_{ag} \in \mathbb{AG}$.

(c) *If* $\mathbf{A} \in \mathbb{AG}$, *then* $(\mathbf{A}_{agh})_{ag} = \mathbf{A}$.

(d) *If* $\mathbf{A} \in \mathbb{AGH}$, *then* $(\mathbf{A}_{ag})_{agh} = \mathbf{A}$.

Proof. (a): Observe that it suffices to verify that (a) holds for subdirectly irreducible members of $\mathbb{AG}$. So, let $\mathbf{A}$ be a nontrivial subdirectly irreducible algebra in $\mathbb{AG}$. Then $\mathbf{A} \in \{\mathbf{2}, \mathbf{3_{dblst}}, \mathbf{3_{klst}}, \mathbf{4_{dmbl}}\}$ by Theorem 3.12. It is obvious that $\mathbf{A}_{agh} \in \{\mathbf{2}, \mathbf{3_{dsth}}, \mathbf{3_{klh}}, \mathbf{4_{dmh}}\}$. It is now routine to verify that $\mathbf{A}_{agh} \in \mathbb{AGH}$, whence (a) is proved.

(b): The proof of (b) is similar to that of (a), in view of Theorem 4.12.

(c): Let $\mathbf{A} := \langle A, \vee, \wedge, ^*, {}', 0, 1\rangle \in \mathbb{AG}$. Then $\mathbf{A}_{agh} \in \mathbb{AGH}$ by (a). Now, let $\mathbf{A_1} := (\mathbf{A}_{agh})_{gh} := \langle A, \vee, \wedge, ^{\circ}, {}', 0, 1\rangle$, where $x^{\circ} := x \to_k 0$. It is clear that $x \to_k 0 = x^*$. Then it follows that $x^{\circ} = x^*$, implying $\mathbf{A_1} = \mathbf{A}$.

(d): Let $\mathbf{A} := \langle A, \vee, \wedge, \to^{\mathbf{A}}, {}', 0, 1\rangle \in \mathbb{AGH}$. Then $\mathbf{A}_{ag} \in \mathbb{AG}$ by (b). Now, let $\mathbf{A_1} := (\mathbf{A}_{gh})_{agh} := \langle A, \vee, \wedge, \to_k, 0, 1\rangle$, Observe that $\to_k = \to^{\mathbf{A}}$ by Corollary 4.14. Hence, $\mathbf{A_1} = \mathbf{A}$, completing the proof. □

5.2 Discriminator Subvarieties of $\mathbb{AGH}$

Recall that the notions of a discriminator term, a discriminator variety and a quasiprimal algebra were defined in Section 1. Discriminator varieties have been a popular topic with a considerable amount of research (see for example, [10, 72]).

Theorem 5.2.

(i) *The variety* $\mathbb{AGH}$ *is a discriminator variety with the discriminator term*

$t(x, y, z) := [z \wedge d((x \vee y) \to (x \wedge y))] \vee [x \wedge (d((x \vee y) \to (x \wedge y)))^*]$, *where* $d(x) = x \wedge x'^*$.

(ii) *The algebras* $\mathbf{3_{dblst}}$, $\mathbf{3_{klst}}$ *and* $\mathbf{4_{dmbl}}$ *are quasiprimal.*

Proof. From Theorem 4.12 (3), it is clear that $x \neq 1 \Rightarrow d(x) = 0$ and $x = 1 \Rightarrow d(x) = 1$ on simple algebras. Hence, in view of Theorem 4.12 (4), if $\mathbf{L} \in \{\mathbf{2}, \mathbf{3_{dsth}}, \mathbf{3_{klh}}, \mathbf{4_{dmh}}, \}$, then it is easy to verify the following two conditions: (a) $x \neq y \Rightarrow t(x, y, z) = x$ and (b) $x = y \Rightarrow t(x, y, z) = z$. Hence $t(x, y, z)$ is a discriminator term and hence, $\mathbb{AGH}$ is a discriminator variety. □

Since $\mathbb{AG}$ and $\mathbb{AGH}$ are term-equivalent by Theorem 5.1, the following corollary is immediate.

Corollary 5.3. *The varieties* $\mathbb{AG}$, $\mathbb{G}$, $\mathbb{RDBLS}\mathrm{t}$, $\mathbb{RKLS}\mathrm{t}$ *and the remaining subvarieties of* $\mathbb{AG}$ *are discriminator varieties.*

The algebras in $\mathbb{AG}$ have a nice representation as mentioned in the next corollary which is a considerable improvement of Corollary 3.13.

Corollary 5.4. *If* $\mathbb{V} \in \{\mathbb{AG}, \mathbb{G}, \mathbb{RDBLS}\mathrm{t}, \mathbb{RKLS}\mathrm{t}\}$, *then every algebra in* $\mathbb{V}$ *is isomorphic to a Boolean Product of simple algebras in* $\mathbb{V}$.

Proof. Apply [10, Chapter IV, Theorem 9.4] and Corollary 5.3. □

The following corollary is a considerable improvement of Corollary 3.23.

Corollary 5.5.

(a) *The first-order theory of* $\mathbb{AGH}$ *is decidable,.*

(b) *The first-order theories of* $\mathbb{AG}$*,* $\mathbb{G}$*,* $\mathbb{RDBLSt}$*,* $\mathbb{RKLSt}$ *and the remaining subvarieties of* $\mathbb{AG}$ *are all decidable.*

Proof. The corollary follows from a well known result (see [11]) that a finitely generated discriminator variety of finite type has a decidable first-order theory. (b) follows from (a), in view of Theorem 5.1. □

We close this section by mentioning two new axiomatizations for the variety $\mathbb{AGH}$. The proofs of these theorems are left to the reader.

Theorem 5.6. *The variety* $\mathbb{AGH}$ *is also defined, modulo* $\mathbb{RDQDH}_1$*, by*
$x^{*\prime\prime *} = x^{**}$.

Theorem 5.7. *The variety* $\mathbb{AGH}$ *is also defined, modulo* $\mathbb{RDQDH}_1$ *by* $(x \wedge y)^{*\prime} = x^{*\prime} \wedge y^{*\prime}$.

6 Classical Nelson algebras, $\mathbb{RKLSt}$, $\mathbb{RKLStH}$ and 3-valued Lukasiewicz algebras

Nelson [34], Markov [26] and Vorobév [71] were the early contributors to the constructive logic with strong negation. Later, Rasiowa [39] introduced Nelson algebras (= quasi-pseudo-Boolean algebra) and used them to prove that the constructive logic with strong negation is implicative (see also [19]). Soon thereafter, Vakarelov [67] introduced the notion of classical Nelson algebras and proved that the variety of classical Nelson algebra is term equivalent to that of 3-valued Łukasiewicz algebras.

In this section we wish to prove this Vakarelov's result by (universal) algebraic means and then derive our main result of this section that the varieties of regular Kleene Stone algebras, of regular Kleene Stone Heyting algebras, of 3-valued Lukasiewicz algebras and of classical Nelson algebra with strong negation are all term-equivalent to one another. A logical consequence of this result will be presented in Section 8.1.

We will first recall the definition of Nelson algebras.

Definition 6.1. [33] A Nelson algebra is an algebra $\langle A, \vee, \wedge, \to, ', 1\rangle$ such that the following conditions are satisfied for all x, y, z in A:

(N1) $x \wedge (x \vee y) = x$,

(N2) $x \wedge (y \vee z) = (z \wedge x) \vee (y \wedge x)$,

(N3) $x'' = x$,

(N4) $(x \wedge y)' = x' \vee y'$,

(N5) $x \wedge x' = (x \wedge x') \wedge (y \vee y')$,

(N6) $x \to x = 1$,

(N7) $x \wedge (x \to y) = x \wedge (x' \vee y)$,

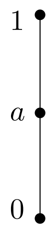
(N8) $(x \wedge y) \to z = x \to (y \to z)$.

The variety of Nelson algebras is denoted by $\mathbb{N}$. Let $1' := 0$.

A complete proof of the following Theorem, which was first proved in [31], is available in [69, Corollary 2.5].

Theorem 6.2. *A nontrivial algebra* $\mathbf{A} \in \mathbb{N}$ *is simple if and only if* $\mathbf{A} \in \{\mathbf{2}, \mathbf{3_N}\}$*, where* $\mathbf{3_N}$ *is the algebra shown in Figure 5, with* $'$ *defined as:* $0' = 1$, $a' = a$, $1' = 0$.

$\mathbf{3_N}$:



$\to$	0	a	1
0	1	1	1
a	1	1	1
1	0	a	1

The following definition is due to [67].

Definition 6.3. A Nelson algebra is a classical Nelson algebra if it satisfies:
(C) $x \vee x^+ \approx 1$, where $x^+ := x \to 0$.

We will denote by $\mathbb{CN}$ the variety of Classical Nelson algebras. Observe that $\mathbf{2}, \mathbf{3_N} \in \mathbb{CN}$. In fact, we wish to show that $\mathbb{CN}$ is generated by $\mathbf{3_N}$.

The following theorem was proved in [69, Theorem 4.13].

Theorem 6.4. $\mathbf{A} \in \mathbb{CN}$ *is semisimple if and only if* $\mathbf{A} \models x \vee x^+ \approx 1$.

Corollary 6.5. $\mathbb{CN} = \mathbb{V}(\mathbf{3_N})$.

Proof. The corollary is immediate from Theorem 6.2 and Theorem 6.4. □

Corollary 6.6. *Let* $\mathbb{V}$ *be a subvariety of* $\mathbb{N}$*. Then the following are equivalent:*
(1) $\mathbb{V}$ *is a discriminator variety,*
(2) $\mathbb{V}$ *is semisimple,*
(3) $\mathbb{V} = \mathbb{V}(\mathbf{3_N})$.
(4) $\mathbb{V} = \mathbb{CN}$.

Corollary 6.7. *Let* $\mathbf{A}$ *be a classical Nelson algebra. For* $x \in A$*, set* $x^+ := x \to 1$*. Then the reduct* $\langle A, \vee, \wedge, ^+, 0, 1\rangle$ *is a dually pseudocomplemented lattice.*

Proof. Observe that for $\mathbf{A} = \mathbf{3_N}$ the reduct in question is a dually pseudocomplemented lattice. Now apply Corollary 6.5. □

Let $\mathbf{3_L}$ denote the 3-element Łukasiewicz algebra.

Lemma 6.8. $\mathbf{3_L}$ *and* $\mathbf{3_N}$ *are term-equivalent.*

Proof. Given $\mathbf{3_N} = \langle \{0, a, 1\}, \vee, \wedge, \to, ', 1\rangle \in \mathbb{CN}$, define a new operation $\rightsquigarrow$ on $\{0, a, 1\}$ by:
$x \rightsquigarrow y := (x \to y) \wedge (y' \to x')$.
Then $\rightsquigarrow$ and $\sim$, given by:

$\rightsquigarrow$	0	a	1
0	1	1	1
a	a	1	1
1	0	a	1

and

	$\sim$
0	1
a	a
1	0

are the well known operations of Łukasiewicz's three–valued algebra. Thus $\langle\{0, a, 1\}, \vee, \wedge, \rightsquigarrow, \sim, 1\rangle$ is a 3-valued Łukasiewicz algebra isomorphic to $\mathbf{3_L}$.

On the other hand, suppose $\mathbf{3}_L = \langle\{0, a, 1\}, \vee, \wedge, \rightsquigarrow, \sim, 1\rangle$ is the three-valued Łukasiewicz algebra. Then, consider the algebra $\bar{\mathbf{3}}_\mathbf{L} := \langle\{0, a, 1\}, \vee, \wedge, \to, ', 1\rangle$, where the operations $'$ and $\to$ are defined by:

$x' := x \rightsquigarrow \sim x$,

$x \to y = x \rightsquigarrow (x \rightsquigarrow y)$, for $x \in \{0, a, 1\}$.

(We could also define $\vee$ and $\wedge$ as follows: $x \vee y := (x \rightsquigarrow y) \rightsquigarrow y$, and $x \wedge y :=\sim (\sim x \vee \sim y)$.)

Then it is easy to verify that $\bar{\mathbf{3}}_\mathbf{L}$ is a classical Nelson algebra isomorphic to $\mathbf{3_N}$. The lemma follows. □

We are now ready to prove our main theorem of this section.

Theorem 6.9. *The following varieties are term equivalent to one another:*

(a) The variety $\mathbb{RKLS}$t of regular Kleene Stone algebras,

(b) The variety $\mathbb{RKLS}$t$\mathbb{H}$ of regular Kleene Stone Heyting algebras,

(c) The variety of 3-valued Lukasiewicz algebras,

(d) The variety of classical Nelson algebras with strong negation.

Proof. The equivalence of (a) and (b) follows from [64, Corollary 10]. The equivalence of (b) and (c) is proved in [15, Theorem 7.14]. It is well-known that the variety of 3-valued Lukasiewicz algebras is generated by $\mathbf{3}_L$, and we know from Theorem 6.5 that the variety of classical Nelson algebras is generated by $\mathbf{3_N}$. So, the equivalence of (c) and (d) follows from Lemma 6.8. □

We close this section by pointing out that Nelson algebras are recently generalized to semi-Nelson algebras in [18] and to quasi-Nelson algebras in [42].

7 Logical Aspects of $\mathbb{AG}$

The rest of the paper, for the most part, is concerned with defining and investigating a propositional logic, in Hilbert-style, called $\mathcal{AG}$ (also known as $\mathcal{AGH}$) from the point of view of Abstract Algebraic Logic, with the ultimate goal of showing that the logic $\mathcal{AG}$ is algebraizable (in the sense of Blok and Pigozzi [9] with the variety $\mathbb{AG}$ of Almost Gautama algebras as its equivalent algebraic semantics. Logics corresponding to the subvarieties of $\mathbb{AG}$ are also defined and studied.

7.1 Abstract Algebraic Logic

In this subsection, we present the basic definitions and results of Abstract Algebraic Logic that will play a crucial role later.

Languages, Formulas and Logics

A language $\mathbf{L}$ is a set of finitary operations (or connectives), each with a fixed arity $n \geq 0$. In this paper, we identify $\bot$ and $\top$ with 0 and 1 respectively and thus consider the languages $\langle\vee, \wedge, \to, \sim, \bot, \top\rangle$ and $\langle\vee, \wedge, \to, ', 0, 1\rangle$ as the same. For a countably infinite set *Var* of propositional variables, the *formulas* of the language $\mathbf{L}$ are inductively defined as usual. The set of formulas in the language $\mathbf{L}$ will be denoted by $Fm_\mathbf{L}$

The set of formulas $Fm_\mathbf{L}$ can be turned into an algebra of formulas, denoted by $\mathbf{Fm_L}$, in the usual way. In what follows, Γ denotes a set of formulas and lower case Greek letters denote formulas. The homomorphisms from the formula algebra $\mathbf{Fm_L}$ into an $\mathbf{L}$-algebra (i.e, an algebra of type $\mathbf{L}$) $\mathbf{A}$ are called *interpretations* (or *valuations*) in A. The set of all such interpretations is denoted by $Hom(\mathbf{Fm_L}, \mathbf{A})$. If $h \in Hom(\mathbf{Fm_L}, \mathbf{A})$ then the *interpretation of a formula* α under h is its image $h\alpha \in A$, while $h\Gamma$ denotes the set $\{h\phi \mid \phi \in \Gamma\}$.

Consequence Relations:

A *consequence relation* on $Fm_{\mathbf{L}}$ is a binary relation $\vdash$ between sets of formulas and formulas that satisfies the following conditions for all Γ, $\Delta \subseteq Fm_{\mathbf{L}}$ and $\phi \in Fm_{\mathbf{L}}$:

(i) $\phi \in \Gamma$ implies $\Gamma \vdash \phi$,

(ii) $\Gamma \vdash \phi$ and $\Gamma \subseteq \Delta$ imply $\Delta \vdash \phi$,

(iii) $\Gamma \vdash \phi$ and $\Delta \vdash \beta$ for every $\beta \in \Gamma$ imply $\Delta \vdash \phi$.

A consequence relation $\vdash$ is *finitary* if $\Gamma \vdash \phi$ implies $\Gamma' \vdash \phi$ for some finite $\Gamma' \subseteq \Gamma$.

Structural Consequence Relations:

A consequence relation $\vdash$ is *structural* if

$\Gamma \vdash \phi$ implies $\sigma(\Gamma) \vdash \sigma(\phi)$ for every substitution σ ($\in Hom(\mathbf{Fm_L}, \mathbf{Fm_L})$), where $\sigma(\Gamma) := \{\sigma\alpha : \alpha \in \Gamma\}$.

Logics:

A **logic** (or **deductive system**) is a pair $\mathcal{S} := \langle \mathbf{L}, \vdash_{\mathcal{S}} \rangle$, where $\mathbf{L}$ is a propositional language and $\vdash_{\mathcal{S}}$ is a finitary and structural consequence relation on $\mathrm{Fm}_{\mathbf{L}}$.

A *rule of inference* is a pair $\langle \Gamma, \phi \rangle$, where Γ is a finite set of formulas (the premises of the rule) and ϕ is a formula.

One way to present a logic $\mathcal{S}$ is by displaying it (syntactically) in **Hilbert-style**; that is, giving its axioms and rules of inference which induce a consequence relation $\vdash_S$ as follows:

$\Gamma \vdash_S \phi$ if there is a a **proof** (or, a **derivation**) of ϕ from Γ, where a proof is defined as a sequence of formulas $\phi_1, \ldots, \phi_n$, $n \in \mathbb{N}$, such that $\phi_n = \phi$, and for every $i \leq n$, one of the following conditions holds:

(i) $\phi_i \in \Gamma$,

(ii) there is an axiom ψ and a substitution σ such that $\phi_i = \sigma\psi$,

(iii) there is a rule $\langle \Delta, \psi \rangle$ and a substitution σ such that $\phi_i = \sigma\psi$ and $\sigma(\Delta) \subseteq \{\phi_j : j < i\}$.

Equational Consequence Relations

Let $\mathbf{L}$ denote a language. Identities in $\mathbf{L}$ are ordered pairs of $\mathbf{L}$-formulas that will be written in the form $\alpha \approx \beta$. An interpretation h in $\mathbf{A}$ satisfies an identity $\alpha \approx \beta$ if $h\alpha = h\beta$. We denote this satisfaction relation by the notation: $\mathbf{A} \models_h \alpha \approx \beta$. An algebra $\mathbf{A}$ *satisfies the equation* $\alpha \approx \beta$ if all the interpretations in $\mathbf{A}$ satisfy it; in symbols,

$$\mathbf{A} \models \alpha \approx \beta \text{ if and only if } \mathbf{A} \models_h \alpha \approx \beta, \text{ for all } h \in Hom(\mathbf{Fm_L}, \mathbf{A}).$$

A class $\mathbb{K}$ of algebras *satisfies the identity* $\alpha \approx \beta$ when all the algebras in $\mathbb{K}$ satisfy it; i.e.

$$\mathbb{K} \models \alpha \approx \beta \text{ if and only if } \mathbf{A} \models \alpha \approx \beta, \text{ for all } \mathbf{A} \in \mathbb{K}.$$

If $\bar{x}$ is a sequence of variables and h is an interpretation in $\mathbf{A}$, then we write $\bar{a}$ for $h(\bar{x})$. For a class $\mathbb{K}$ of $\mathbf{L}$-algebras, we define the relation $\models_{\mathbb{K}}$ that holds between a set Δ of identities and a single identity $\alpha \approx \beta$ as follows:

$$\Delta \models_{\mathbb{K}} \alpha \approx \beta \text{ if and only if}$$

for every $\mathbf{A} \in \mathbb{K}$ and every interpretation $\bar{a}$ of the variables of $\Delta \cup \{\alpha \approx \beta\}$ in $\mathbf{A}$,

if $\phi^{\mathbf{A}}(\bar{a}) = \psi^{\mathbf{A}}(\bar{a})$, for every $\phi \approx \psi \in \Delta$, then $\alpha^{\mathbf{A}}(\bar{a}) = \beta^{\mathbf{A}}(\bar{a})$.

In this case, we say that $\alpha \approx \beta$ is a $\mathbb{K}$-consequence of Δ. The relation $\models_K$ is called the *semantic equational consequence relation* determined by K.

Algebraic Semantics for a logic

Let $\langle \mathbf{L}, \vdash_{\mathbf{L}} \rangle$ be a finitary logic (i.e., deductive system) and $\mathbb{K}$ a class of $\mathbf{L}$-algebras. $\mathbb{K}$ is called an *algebraic semantics* for $\langle \mathbf{L}, \vdash_{\mathbf{L}} \rangle$ if $\vdash_{\mathbf{L}}$ can be interpreted in $\vdash_{\mathbb{K}}$ in the following sense:

There exists a finite set $\delta_i(p) \approx \epsilon_i(p)$, for $i \leq n$, of identities with a single variable p such that, for all

$\Gamma \cup \phi \subseteq Fm$,

$$\text{(A)} \quad \Gamma \vdash_{\mathbf{L}} \phi \Leftrightarrow \{\delta_i[\psi/p] \approx \epsilon_i[\psi/p], i \leq n, \psi \in \Gamma\} \models_K \delta_i[\phi/p] \approx \epsilon_i[\phi/p],$$

where $\delta[\psi/p]$ denotes the formula obtained by the substitution of ψ at every occurrence of p in δ.
The identities $\delta_i \approx \epsilon_i$, for $i \leq n$, are called *defining identities* for $\langle L, \vdash_L \rangle$ and $\mathbb{K}$.

Equivalent Algebraic Semantics and Algebraizable Logics

Let $\mathcal{S}$ be a logic over a language **L** and $\mathbb{K}$ an algebraic semantics of S with defining equations $\delta_i(p) \approx \epsilon_i(p)$, $i \leq n$. Then, $\mathbb{K}$ is an **equivalent algebraic semantics** of **S** if there exists a finite set $\{\Delta_j(p,q) : j \leq m\}$ of formulas in two variables satisfying the condition:

For every $\phi \approx \psi$ in the language **L**,

$$\phi \approx \psi \ \models_K \{\delta_i(\Delta_j(\phi,\psi)) \approx \epsilon_i(\Delta_j(\phi,\psi)) : i \leq n, j \leq m\}$$

and

$$\{\delta_i(\Delta_j(\phi,\psi)) \approx \epsilon_i(\Delta_j(\phi,\psi)) : i \leq n, j \leq m\} \ \models_K \ \phi \approx \psi.$$

The set $\{\Delta_j(p,q) : j \leq m\}$ is called an **equivalence system**.

A logic is **BP-algebraizable (in the sense of Blok and Pigozzi)** if it has an equivalent algebraic semantics.

Axiomatic Extensions of Algebraizable logics

A logic S' is an *axiomatic extension* of S if S' is obtained by adjoining new axioms but keeping the rules of inference the same as in S. Let $Ext(S)$ denote the lattice of axiomatic extensions of a logic S and $\mathbf{L_V}(\mathbb{K})$ denote the lattice of subvarieties of a variety $\mathbb{K}$ of algebras.

The following important theorems, due to Blok and Pigozzi, were first proved in [9].

Theorem 7.1. [9] *Let S be a BP-algebraizable logic whose equivalent algebraic semantics $\mathbb{K}$ is a variety. Then $Ext(S)$ is dually isomorphic to $\mathbf{L_V}(\mathbb{K})$.*

Theorem 7.2. [9] *Let S be a BP-algebraizable logic and S' be an axiomatic extension of S. Then $Ext(S')$ is also BP-algebraizable.*

7.2 Dually Hemimorphic Intuitionistic Logic $\mathcal{DHMH}$

The Logic $\mathcal{DHMH}$, which was first defined in [15], is slightly simplified below.
The Logic $\mathcal{DHMH}$ is defined as follows:
LANGUAGE: $\langle \vee, \wedge, \to, \sim, \bot, \top \rangle$
AXIOMS:
(a) Axioms of the Intuitionistic Logic $\mathcal{I}$ (Rasiowa-Sikorski, p.379):

(Ax1) $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$,
(Ax2) $\alpha \to (\alpha \vee \beta)$,
(Ax3) $\beta \to (\alpha \vee \beta)$,
(Ax4) $(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma))$,
(Ax5) $(\alpha \wedge \beta) \to \alpha$,
(Ax6) $(\alpha \wedge \beta) \to \beta$,
(Ax7) $(\gamma \to \alpha) \to ((\gamma \to \beta) \to (\gamma \to (\alpha \wedge \beta))$,

(Ax8) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \wedge \beta) \to \gamma)$,
(Ax9) $((\alpha \wedge \beta) \to \gamma) \to (\alpha \to (\beta \to \gamma))$,
(Ax10) $((\alpha \wedge \neg\alpha) \to \beta$, where $\neg\alpha := \alpha \to \bot$,
(Ax11) $(\alpha \to (\alpha \wedge \neg\alpha)) \to \neg\alpha$.
(b) Additional axioms:
(Ax12) $\top \to \sim \bot$,
(Ax13) $\sim \top \to \bot$,
(Ax14) $\sim (\alpha \wedge \beta) \leftrightarrow (\sim \alpha \vee \sim \beta)$.

RULES OF INFERENCE:

(MP) From ϕ and $\phi \to \gamma$, deduce γ (Modus Ponens),

(CP) From $\phi \to \gamma$, deduce $\sim \gamma \to \sim \phi$ (Contraposition).

The following lemma is crucial in proving Lemma 7.9.

Lemma 7.3.

(i) *If* $\Gamma \vdash_{\mathcal{I}} \psi$, *then* $\Gamma \vdash_{\mathcal{DHMH}} \psi$,

(ii) *If* $\Gamma \vdash_{DHMH} \psi$, *then* $\Gamma \vdash_{\mathcal{DHMH}} \alpha \to \psi$,

(iii) $\Gamma \vdash_{\mathcal{DHMH}} \bot \to \alpha$.

Proof. We only prove (iii), for which it suffices to prove that $\vdash_{\mathcal{I}} \bot \to \alpha$. Then, in view of Completeness Theorem of intuitionistic logic $\mathcal{I}$, that is equivalent to proving that the identity $0 \to x \approx 1$ holds in the variety of Heyting algebras, which immediately follows from the axiom (H): $(x \wedge y) \to x \approx 1$. ☐

7.2.1 The logic $\mathcal{DHMH}$ as an implicative logic

We first recall the definition of implicative logics that was introduced by Rasiowa [40] in 1974 (see also [20]).

Definition 7.4. [40] Let S be a logic in a language $\mathbf{L}$ that includes a binary connective $\to$, either primitive or defined by a term in exactly two variables. Then S is called an implicative logic with respect to the binary connective $\to$, if the following conditions are satisfied:

(IL1) $\vdash_S \alpha \to \alpha$,

(IL2) $\alpha \to \beta, \beta \to \gamma \vdash_S \alpha \to \gamma$,

(IL3) For each operation symbol $f \in \mathbf{L}$ of arity $n \geq 1$,
$$\left\{ \begin{array}{c} \alpha_1 \to \beta_1, \ldots, \alpha_n \to \beta_n, \\ \beta_1 \to \alpha_1, \ldots, \beta_n \to \alpha_n \end{array} \right\} \vdash_S f(\alpha_1, \ldots, \alpha_n) \to f(\beta_1, \ldots, \beta_n),$$

(IL4) $\alpha, \alpha \to \beta \vdash_S \beta$,

(IL5) $\alpha \vdash_S \beta \to \alpha$.

The following theorem is well-known.

Theorem 7.5. *The intuitionistic logic I is implicative with respect to the connective $\to$.*

Theorem 7.6. *The logic $\mathcal{DHMH}$ is implicative with respect to the connective $\to$.*

Proof. In view of Theorem 7.5, it only remains to prove (IL3) for the (unary) operation $'$ which is fulfilled by the rule **CP**. ☐

We also note here that Theorem 7.6 is a special case of [15, Theorem 3.7].

7.2.2 Algebraic Completeness of $\mathcal{DHMH}$

Definition 7.7. Rasiowa [40] Let S be an implicative logic in $\mathbf{L}$ with $\to$.
An S-algebra is an algebra $\mathbf{A}$ in the language $\mathbf{L}$ that has an element 1 with the following properties:

(LALG1) For all $\Gamma \cup \{\phi\} \subseteq Fm$ and all $h \in Hom(\mathbf{Fm_L}, \mathbf{A})$,
if $\Gamma \vdash_S \phi$ and $h\Gamma \subseteq \{1\}$ then $h\phi = 1$,

(LALG2) For all $a, b \in A$, if $a \to b = 1$ and $b \to a = 1$ then $a = b$.

The class of S-algebras is denoted by $Alg^*(S)$.

Since $\mathcal{DHMH}$ is an implicative logic we obtain the following result, in view of Rasiowa's Theorem [40, Theorem 7.1, pag 222].

Theorem 7.8. *The logic $\mathcal{DHMH}$ is complete with respect to the class $Alg^*(\mathcal{DHMH})$. In other words,*
For all $\Gamma \cup \{\phi\} \subseteq Fm$, $\Gamma \vdash_{\mathcal{DHMH}} \phi$ if and only if $\Gamma \models_{Alg^*(\mathcal{DHMH})} \phi$.

The following lemma will help us improve the above theorem. Recall the definition of the variety $\mathbb{DHMH}$ given in Definition 4.2.

Lemma 7.9. $Alg^*(\mathcal{DHMH}) = \mathbb{DHMH}$.

Proof. First of all, we note that this proof is an adaptation of the proof of [15, Lemma 4.4]. First, we wish to prove that $\mathbb{DHMH} \subseteq Alg^*(\mathcal{DHMH})$.

Let $\mathbf{A} \in \mathbb{DHMH}$, $\Gamma \cup \{\phi\} \subseteq Fm$ and $h \in Hom(Fm_{\mathbf{L}}, \mathbf{A})$ such that $\Gamma \vdash_{Alg^*(\mathcal{DHMH})} \phi$ and $h\Gamma \subseteq \{1\}$. We need to verify that $h\phi = 1$. We will proceed by induction on the length of the proof of $\Gamma \vdash_{Alg^*(\mathcal{DHMH})} \phi$.

- Assume that ϕ is an axiom.

  If ϕ is one of the axioms (Ax1) to (Ax11) then $\vdash_{\mathcal{I}} \phi$. Hence, $\models_{\mathbb{DHMH}} \phi$ and so, $h(\phi) = \top$.

  If ϕ is the axiom (Ax12) then, using (E2), we have $h(\phi) = h(\top \to \sim \bot) = 1 \to 0' = 1$.

  If ϕ is the axiom (Ax13) then, using (E3), we get that $h(\phi) = h(\sim \top \to \bot) = 0 \to 0 = 1$.

- If ϕ is the axiom (Ax14) then, using (E4), we obtain that $h(\phi) = h(\sim (\alpha \wedge \beta) \to (\sim \alpha \vee \sim \beta)) = (h(\alpha) \wedge h(\beta))' \to (h(\alpha)' \vee h(\beta)') = (h(\alpha) \wedge h(\beta))' \to (h(\alpha) \wedge h(\beta))' = 1$.

- If $\phi \in \Gamma$ then $h(\phi) = \top$ by hypothesis.

- Assume now that $\Gamma \vdash_{\mathcal{L}} \phi$ is obtained from an application of (MP). Then there exists a formula ψ such that $\Gamma \vdash_{\mathcal{L}} \psi$ and $\Gamma \vdash_{\mathcal{L}} \psi \to \phi$. By induction, $h(\psi) = 1$ and $h(\psi \to \phi) = 1$. Then $1 = h(\psi) \to h(\phi) = 1 \to h(\phi) = h(\phi)$.

- Assume that $\Gamma \vdash_{\mathcal{L}} \phi$ is the result of an application of the rule (CP). Then for $\alpha, \beta \in Fm$, $\phi = \sim \beta \to \sim \alpha$ and $\Gamma \vdash_{\mathcal{L}} \alpha \to \beta$. By induction, $1 = h(\alpha \to \beta) = h(\alpha) \to h(\beta)$ and, consequently $h(\alpha) \leq h(\beta)$. Then, using condition (E4), $h(\beta)' \leq h(\alpha)'$. Hence $h(\beta)' \to h(\alpha)' = 1$. Therefore $h(\phi) = h(\sim \beta \to \sim \alpha) = h(\beta)' \to_H h(\alpha)' = 1$.

Hence, the induction is complete and so, we conclude that $\mathbf{A}$ satisfies (LALG1). It is easy to see that the condition (LALG2) also holds, implying $\mathbf{A} \in Alg^*(\mathcal{DHMH})$.

Next, we prove the other inclusion. Let $\mathbf{A} = \langle A, \vee, \wedge, \to, ', 0, 1\rangle \in Alg^*(\mathcal{DHMH})$. Notice that $\langle A, \vee, \wedge, \to, 0, 1\rangle \in Alg^*(I)$. So, $\langle A, \vee, \wedge, \to, 0, 1\rangle \in \mathbb{H}$. Now, it only remains to show that $\mathbf{A}$ satisfies the conditions (E2) to (E4).

In view of axiom $(Ax12)$ and (LALG1), we have that $\mathbf{A} \models 1 \to 0' \approx 1$. Using (LALG1) and Lemma 7.3 (i), we get $\mathbf{A} \models 0' \to 1 \approx 1$. Then by (LALG2), $\mathbf{A} \models 1 \approx 0'$. In view of Lemma 7.3 (ii) and $(Ax13)$, together with (LALG1), we have that $\mathbf{A} \models 0 \to 1' \approx 1$ and $\mathbf{A} \models 1' \to 0 \approx 1$. Then by (LALG2), $\mathbf{A} \models 1' \approx 0$.

Using (LALG1), it can be shown that $\mathbf{A}$ satisfies the identity $(x' \vee y') \to (x \wedge y)' \approx 1$ and the identity $(x \wedge y)' \to (x' \vee y') \approx 1$. Then applying (LALG2), we see that the algebra satisfies (E4). Consequently $\mathbf{A} \in \mathbb{DHMH}$. This completes the proof. □

We are now ready to present the algebraic completeness theorem for the logic $\mathcal{DHMH}$.

Theorem 7.10. *The logic $\mathcal{DHMH}$ is complete with respect to the variety $\mathbb{DHMH}$.*

Proof. We know $Alg^*(\mathcal{DHMH}) = \mathbb{DHMH}$ by Lemma 7.9. So, the theorem follows from Theorem 7.8. □

7.2.3 The algebraizability of the logic $\mathcal{DHMH}$, $Ext(\mathcal{DHMH})$ and $\mathbf{L_V}(\mathbb{DHMH})$

The following theorem of Blok and Pigozzi shows that Rasiowa's implicative logics provide a class of examples of algebraizable logics and was proved in [9].

Theorem 7.11. [9, 20]

Every implicative logic L is algebraizable with respect to the class $Alg^*(L)$ and the algebraizability is witnessed by the set of defining identities $E = \{x \approx x \to x\}$ and the set of equivalence formulas $\Delta = \{p \to q, q \to p\}$.

Corollary 7.12. *The logic $\mathcal{DHMH}$ is algebraizable, and the variety $\mathbb{DHMH}$ is the equivalent algebraic semantics for $\mathcal{DHMH}$ with the set of defining identities $E = \{x \approx x \to x\}$ (equivlently, $x \approx 1$) and the set of equivalence formulas $\Delta = \{p \to q, q \to p\}$.*

The following theorem is immediate from Theorem 7.1 and Corollary 7.12.

Theorem 7.13. *The lattice $Ext(\mathcal{DHMH})$ of axiomatic extensions of $\mathcal{DHMH}$ is dually isomorphic to the lattice $\mathbf{L_V}(\mathbb{DHMH})$ of subvarieties of the variety $\mathbb{DHMH}$.*

8 The logic $\mathcal{AG}$

We will now present a new logic, $\mathcal{AG}$ (also known as $\mathcal{AGH}$) and its axiomatic extensions.

The logic $\mathcal{AG}$ is defined as follows:

LANGUAGE: $\langle \vee, \wedge, \to, \sim, \perp, \top \rangle$, where $\vee$, $\wedge$, and $\to$ are binary, $\sim$ is unary, and $\perp$, $\top$ are constants.

Let $\leftrightarrow$ be defined by: $\alpha \leftrightarrow \beta := (\alpha \to \beta) \wedge (\beta \to \alpha)$.

Define $\neg$ by $\neg\alpha := \alpha \to \perp$.

AXIOMS:

(1), (2), ...,(14) of the logic $\mathcal{DHMH}$, plus the following axioms:

(15) $\sim\sim (\alpha \vee \beta) \ \leftrightarrow \ (\sim\sim \alpha \vee \sim\sim \beta)$,

(16) $(\alpha \vee \sim\sim \alpha) \ \leftrightarrow \ \alpha$,

(17) $(\alpha \wedge \sim \neg \sim \alpha) \vee (\beta \vee \neg\beta) \leftrightarrow (\beta \vee \neg\beta)$ (Regularity),

(18) $\neg\alpha \vee \neg\neg\alpha$ (Stone or the Weak Law of Excluded Middle),

(19) $\sim\sim \neg\alpha \leftrightarrow \neg\alpha$ (week *-regular),

(20) $\neg \sim (\alpha \wedge \neg \sim \alpha) \approx \alpha \wedge \neg \sim \alpha$ (Level 1).

RULES OF INFERENCE:

(a) **(MP)** From ϕ and $\phi \to \gamma$, deduce γ (Modus Ponens),

(b) **(SCP)** From $\phi \to \gamma$, deduce $\sim \gamma \to \sim \phi$ (contraposition rule).

Remark 8.1. The logic $\mathcal{AG}$ is an axiomatic extension of $\mathcal{DHMH}$.

Definition 8.2.

(a) The logic $\mathcal{G}$ is the axiomatic extension of $\mathcal{AG}$ defined by the following axiom:
(G) $\sim \neg\alpha \leftrightarrow \neg\neg\alpha$.

(b) The logic $\mathcal{RDBLS}t$ is the axiomatic extension of $\mathcal{G}$ defined by the following axiom:
(DSt) $(\sim \alpha \wedge \sim\sim \alpha) \leftrightarrow \perp$.

(c) The logic $\mathcal{RKLS}t$ is the axiomatic extension of $\mathcal{G}$ defined by the following axioms:
(1) $[(\alpha \wedge \sim \alpha) \vee (\beta \vee \sim \beta)] \leftrightarrow (\beta \vee \sim \beta)$.
(2) $\sim\sim \alpha \leftrightarrow \alpha$.

Let L be an algebraizable logic with $\mathbb{K}$ as its equivalent algebraic semantics and let $\mathbb{K}'$ be a variety term-equivalent to $\mathbb{K}$. Then $\mathbb{K}'$ can be considered as an equivalent algebraic semantics for the logic L.

Corollary 8.3. *The logic $\mathcal{AG}$ is algebraizable with the variety $\mathbb{AGH}$ as its equivalent algebraic semantics, and hence with the variety $\mathbb{AG}$ of Almost Gautama algebras as its equivalent algebraic semantics.*

Corollary 8.4. *The logic $\mathcal{G}$ is algebraizable with the variety $\mathbb{GH}$ as its equivalent algebraic semantics,, and hence with the variety $\mathbb{G}$ of Gautama algebras as its equivalent algebraic semantics.*

Since the logics $\mathcal{RDBLS}t$ and $\mathcal{RKLS}t$ are axiomatic extensions of the logic $\mathcal{G}$, we have the following corollaries.

Corollary 8.5. *The logic $\mathcal{RDBLS}t$ is algebraizable with the variety $\mathbb{RDBLS}$t of regular double Stone algebras as its equivalent algebraic semantics.*

Corollary 8.6. *The logic $\mathcal{RKLS}t$ is algebraizable with the variety $\mathbb{RKLS}$t of regular Kleene algebras as the equivalent algebraic semantics.*

In a similar fashion the logics corresponding to the remaining subvarieties of $\mathbb{AG}$ can be easily axiomatized by translating the known equational bases of the corresponding subvarieties of $\mathbb{AG}$.

Corollary 8.7. *The logics $\mathcal{AG}$, $\mathcal{G}$, $\mathcal{RDBLS}t$ and $\mathcal{RKLS}t$ and the other axiomatic extensions of $\mathcal{AG}$ are decidable.*

We now consider the question as to whether $\mathcal{AG}$ or any of its axiomatic extensions have the Disjunction Property.

Definition 8.8. Let **L** be a language containing a binary connective $\vee$ and a constant 1 and let $\mathcal{L}$ be a logic in **L**. We say $\mathcal{L}$ has the Disjunction Property if the following condition holds:

For any formulas α and β, $\vdash_{\mathcal{L}} (\alpha \vee \beta)$ implies either $\vdash_{\mathcal{L}} \alpha$ or $\vdash_{\mathcal{L}} \beta$.

Since the Stone axiom holds in $\mathcal{AG}$, the following corollary is immediate.

Corollary 8.9. *The logics $\mathcal{AG}$, $\mathcal{G}$, $\mathcal{RDBLS}t$ and $\mathcal{RKLS}t$ and the other axiomatic extensions of $\mathcal{AG}$ do not have the Disjunction Property.*

Definition 8.10. Let $\mathcal{L}$ be an algebraizable logic. We say that $\mathcal{L}$ is a discriminator logic if its equivalent algebraic semantics is a discriminator variety. Furthermore, $\mathcal{L}$ is a primal logic if its equivalent algebraic semantics is a variety generated by a primal algebra. $\mathcal{L}$ is a quasiprimal logic if its equivalent algebraic semantics is a variety generated by a quasiprimal algebra.

The classical logic is the first well-known example of a primal logic (as the Boolean algebra **2** is a primal algebra).

Remark 8.11. It follows from Corollary 5.3 that $\mathcal{AG}$ and all its extensions are discriminator logics, while $\mathcal{RDBLS}t$ and $\mathcal{RKLS}t$ are quasiprimal logics.

8.1 Classical Logic with Strong Negation

Here we will give logical applications of Corollary 6.5 and Theorem 6.9.

Vakarelov introduced the notion of classical logic with strong negation. As a consequence of a completeness theorem he obtained the equivalence of this logic with the three-valued Łukasiewicz logic. In this subsection, using Corollary 6.5 and Theorem 6.9, we will show that the classical logic with strong negation is algebraizable with $\mathbb{CN}$ as its algebraic semantics and that the logics $\mathcal{RKLS}$t, $\mathcal{RKLS}$t$\mathcal{H}$, 3-valued Łukasivicz logic and the classical logic with strong negation are all equivalent, thus strengthening Vakarelov's results.

Definition 8.12. [67] The logic, in the language $\langle \vee, \wedge, \to, ', 1\rangle$, which is obtained by adding the following axioms (C1) – (C6) (for the "strong" negation) to the axioms of classical propositional calculus (also in the lanuage $\langle \vee, \wedge, \to, ', 1\rangle$), is called the *classical logic with strong negation*:

(C1) $\alpha' \to (\alpha \to \beta)$,
(C2) $(\alpha \to \beta) \leftrightarrow (\alpha \wedge \beta')$,
(C3) $(\alpha \wedge \beta)' \leftrightarrow (\alpha' \vee \beta')$,
(C4) $(\alpha \vee \beta)' \leftrightarrow (\alpha' \wedge \beta')$,
(C5) $\alpha^{*\prime} \leftrightarrow \alpha$, where $\alpha^* := \alpha \to 0$,
(C6) $\alpha'' \leftrightarrow \alpha$.

Let $\mathcal{CN}$ denote the classical logic with strong negation.

The following theorem is a strengthened version of Vakarelov's completeness theorem for $\mathcal{CN}$ (with a different proof).

Theorem 8.13. *$\mathcal{CN}$ is BP-algebraizable with $\mathbb{CN}$ as its algebraic semantics.*

Proof. It is well-known (see, for example, [20, Page 85] that the Nelson logic with strong negation is implicative and hence is BP-algebraizable with $\mathbb{N}$ as its algebraic semantics. Hence, it is easy to see, in view of Corollary 6.5, that $\mathcal{CN}$ is BP-algebraizable with $\mathbb{CN}$ as its algebraic semantics. □

The following corollary is immediate from Theorem 6.9.

Corollary 8.14. *The logics $\mathcal{RKLS}$t, $\mathcal{RKLS}$t$\mathcal{H}$, 3-valued Łukasivicz logic and the classical logic with strong negation are all equivalent.*

9 Concluding Remarks

In a forthcoming paper [17], we completely describe the subvarieties of $\mathbb{AG}$ with the Amalgamation Property and the ones without (AP).

Note that the variety $\mathbb{DHM}$ introduced in Definition 2.3 is a far-reaching–and a common– generalization of both p-algebras–more generally, semi-De Morgan algebras– and Ockham algebras. Observe also that the new variety $\mathbb{DHMP}$ that was introduced in Definition 2.4 is a sweeping generalization of the variety of Almost Gautama algebras.

We now introduce a subvariety of the variety $\mathbb{DHMP}$ whose members are called "quasi-Gautama algebras."

An algebra $\mathbf{A} = \langle A, \vee, \wedge, ^*, ', 0, 1\rangle$ is a *quasi-Gautama algebra* if $\mathbf{A}$ satisfies:

(1) $\langle A, \vee, \wedge, ^*, 0, 1\rangle$ is a p-algebra,

(2) $\langle A, \vee, \wedge, ', 0, 1\rangle$ is a dually quasi-De Morgan algebra,

(3) $\mathbf{A}$ is regular; that is, $\mathbf{A}$ satisfies the identity: (R1) $\quad x \wedge x'^{*'} \leq y \vee y^*$.

Let the variety of quasi-Gautama algebras be denoted by $\mathbb{QG}$. Notice that $\mathbb{G} \subset \mathbb{AG} \subset \mathbb{QG} \subset \mathbb{RDHMP} \subset \mathbb{DHMP}$, where $\mathbb{RDHMP}$ consists of $\mathbb{DHMP}$-algbras satisfying (R1).

Similarly, we can generalize the variety $\mathbb{AGH}$ to a new variety whose members are called quasi-Gautama semi-Heyting algebras as follows:

An algebra $\mathbf{A} = \langle A, \vee, \wedge, \rightarrow, ', 0, 1\rangle$ is a quasi-Gautama semi-Heyting algebra if $\mathbf{A}$ satisfies:

(1) $\langle A, \vee, \wedge, ^*, 0, 1\rangle$ is a semi-Heyting algebra,

(2) $\langle A, \vee, \wedge, ', 0, 1\rangle$ is a dually quasi-De Morgan algebra,

(3) $\mathbf{A}$ is regular; that is, $\mathbf{A}$ satisfies the identity: (R1) $\quad x \wedge x'^{*'} \leq y \vee y^*$.

Let the variety of quasi-Gautama semi-Heyting algebras be denoted by $\mathbb{QGSH}$, and $\mathbb{QGH}$ denotes the subvariety of $\mathbb{QGSH}$ consisting of those algebras whose semi-Heyting reduct is a Heyting algebra. Note that $\mathbb{GH} \subset \mathbb{AGH} \subset \mathbb{QGH} \subset \mathbb{QGSH} \subset \mathbb{DHMSH}$.

We know that the cardinality of the lattice of subvarieties of $\mathbb{QG}$ is 2^ω since $\mathbb{QG}$ contains each of the varieties of regular double p-algebras and of regular Kleene p-algebras, each of whose lattice of subvarieties is of cardinality 2^ω (see [5, 6]). The results presented in the present paper only describe the "bottom" of the lattices of subvarieties of $\mathbb{QG}$ and of $\mathbb{QGH}$.

We conclude the paper with some open problems for further research.

OPEN PROBLEMS:

PROBLEM 1: Find a Priestley-type duality for the varieties $\mathbb{AG}$ and $\mathbb{AGH}$. (One can ask the same question for the varieties $\mathbb{QG}$, $\mathbb{QGH}$ and $\mathbb{QGSH}$.)

PROBLEM 2: There is a representation of regular double Stone algebras in terms of rough sets. Is their a similar representation for the varieties $\mathbb{AG}$ and $\mathbb{AGH}$?

PROBLEM 3: Katriňák has given a "triple" construction for regular double Stone algebras. Is their a similar construction for the variety $\mathbb{AG}$? The same question for $\mathbb{AGH}$.

PROBLEM 4: Investigate the lattice of subvarieties of $\mathbb{QG}$, and of $\mathbb{QGSH}$ of level 1 (i.e., satisfying $x \wedge x'^* \wedge x'^{*'*} \approx x \wedge x'^*$.

PROBLEM 5: Investigate the lattice of subvarieties of the varieties $\mathbb{QG}$ and $\mathbb{QGSH}$.

Acknowledgements: We wish to acknowledge the use of [27] in the early phase of our research that led to the present paper.

Conflict of Interest: We declare that there is no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

Funding: The study was funded by Universidad Nacional del Sur and CONICET, Argentina

References

[1] Abad M, Cornejo JM and Díaz Varela JP. The variety of semi-Heyting algebras satisfying the equation $(0 \to 1)^* \vee (0 \to 1)^{**} \approx 1$. *Rep. Math. Logic*. 2011; 46: 75-90.

[2] Abad M, Cornejo JM and Diaz Varela JP. The variety generated by semi-Heyting chains. *Soft Comput.* 2011; 15:721-728. DOI: https://doi.org/10.1007/s00500-010-0604-0

[3] Abad M, Cornejo JM and Diaz Varela JP. Semi-Heyting algebras term-equivalent to Gödel algebras. *Order*. 2013; 30: 625-642. DOI: https://doi.org/10.1007/s11083-012-9266-0

[4] Abad M and Monteiro L. Free symmetric boolean algebras. *Revista de la U.M.A.*. 1976; 207-215.

[5] Adams ME, Sankappanavar HP and Vaz de Carvalho J. Regular double p-algebras. *Mathematica Slovaca*. 2019; 69(1):15-34. DOI: https://doi.org/10.1515/ms-2017-0200

[6] Adams ME, Sankappanavar HP and Vaz de Carvalho J. Varieties of regular pseudocomplemented De Morgan algebras. *Order*. 2020; 37(3): 529-557, DOI: https://doi.org/10.1007/s11083-019-09518-y

[7] Balbes R and Dwinger P. Distributive Lattices. *Missouri Press*. 1974.

[8] Bialynicki-Birula A and Rasiowa H. On the representation of quasi-Boolean algebras. *Bull. Acad. Polon. Sci. Cl. III*. 1957; 259-261.

[9] Blok WJ and Pigozzi D. Algebraizable Logics. *Mem. Amer. Math. Soc., No. 396, Providence, Rhode Island*. 1989; 396.

[10] Burris S and Sankappanavar HP. A Course in Universal Algebra. *Graduate Texts in Mathematics 78*. New York: Springer-Verlag; 1981. (The free, corrected version (2012) is available online as a PDF file at http://www.thoralf.uwaterloo.ca/htdocs/ualg.html)

[11] Burris S and Werner H. Sheaf constructions and their elementary properties. *Trans. Amer. Math Soc.* 1979; 248: 269-309.

[12] Cornejo JM, Kinyon M and Sankappanavar HP. Regular double p-algebras: A converse to a Katrinak's theorem, and applications. *Mathematica Slovaka*. (Accepted in 2022).

[13] Cornejo JM, Monteiro LF, Sankappanavar HP and Viglizzo ID. A note on chain-based semi-Heyting algebras. *Math. Log. Quart.* 2020; 66(4): 409-417. DOI: https://doi.org/10.1002/malq. 201900070

[14] Cornejo JM and Sankappanavar HP. Semi-Heyting algebras and identities of associative type. *Bulletin of the Section of Logic*. 2019; 48(2): 117-135. DOI: https://doi.org/10.18778/0138-0680.48.2.03

[15] Cornejo JM and Sankappanavar HP. A logic for dually hemimorphic semi-Heyting algebras and its axiomatic extensions. *Bulletin of the Section of Logic*. 2022; 51(4): 555-645. DOI: https://doi.org/10.18778/0138-0680.2022.2391

[16] Cornejo JM and Sankappanavar HP. *On the Disjunction Property in Semi-Heyting and Heyting Algebras.* 2022.

[17] Cornejo JM and Sankappanavar HP. *Amalgamation Property in the Varieties of Gautama and Almost Gautama Algebras.* 2023.

[18] Cornejo JM and Viglizzo I. Semi-Nelson algebras. *Order*. 2018; 35: 23-45. DOI: https://doi.org/10.1007/s11083-016-9416-x

[19] Fidel MM. An algebraic study of a propositional system of Nelson. *Mathematical Logic, Proceedings of the First Brazilian Conference, Marcel Dekker, New York.* 1978; 99-117.

[20] Font JM. Abstract Algebraic Logic: An Introductory Textbook. *Studies in Logic (London). Mathematical Logic and Foundations. College Publications, London.* 2016; 60.

[21] Grätzer G. Lattice Theory: First Concepts and Distributive Lattices. *Freeman, San Francisco.* 1971.

[22] Grtzer G and Lakser H. The structure of pseudocomplemented distributive lattices II: Congruence extension and amalgamation. *Trans. Amer. Math. Soc.* 1971; 156: 343-358. DOI: https://doi.org/10.2307/1995616

[23] Hobby D. Semi-De Morgan algebras. *Studia Logica.* 1996; 56: 150-183. DOI: https://doi.org/10.1007/BF00370145

[24] Kalman JA. Lattices with involution. *Trans. Amer. Math. Soc.* 1958; 87: 485-491.

[25] Katriňák T. The structure of distributive double p-algebras. Regularity and congruences. *Algebra Universalis.* 1973; 3: 238-246. DOI: https://doi.org/10.1007/BF02945123

[26] Markov AA. Constructive logic (in Russian). *Uspekhi Matematičeskih Nauk.* 1950; 5: 187-188.

[27] McCune W. Prover9 and Mace4, version 2009-11A, (http://www.cs.unm.edu/∼mccune/prover9/).

[28] Moisil GC. Recherches sur ĺalgèbre de la logique. *Annales scientifiques de ĺuniversité de Jassy.* 1935; 22: 1-117.

[29] Moisil GC. Logique modale. *Disquisitiones Mathematicae et Physica.* 1942; 2: 3-98. DOI: https://doi.org/10.2307/2267855

[30] Moisil GC. Essais Sur Les Logiques Non Chrysippiennes. *Editions de lÁcademie de la Republique Socialiste de Roumanie, Bucharest.* 1972.

[31] Monteiro A. Les algèbres de Nelson semi-simple. *Notas de Logica Matematica, Inst, de Mat, Universidad Nacional del Sur, Bahia Blanca.* 1996.

[32] Monteiro AA. Sur les algèbres de Heyting symétriques. *Portugal. Math.* 1980; 39(1-4): 1-237.

[33] Monteiro AA and Monteiro L. Axiomes Indépendants Pour Les Algèbres de Nelson, de Lukasiewicz Trivalentes, de De Morgan et de Kleene. *In Unpublished papers, I, Notas Lógica Matemática, Univ. Nac. del Sur, Bahía Blanca.* 1996; 40:13.

[34] Nelson D. Constructible falsity. *Journal of Symbolic Logic.* 1949; 14: 16-26. DOI: https://doi.org/10.2307/2268973

[35] Palma C. Semi-De Morgan algebras. *Ph.D. Thesis,*. 2004.

[36] Palma C. The principal join property in demi-p-lattices. *Mathematica Slovaca.* 2006; 56(2): 199-212.

[37] Palma C and Santos R. On the subdirectly irreducible semi-De Morgan algebras. *Publ. Math. Debrecen.* 1996; 49: 39-45.

[38] Palma C and Santos R. On the characterization of a subvariety of semi-De Morgan algebras. *Bol. Soc. Mat. Mexicana (3).* 2006; 12: 149-154.

[39] Rasiowa H. $\mathcal{N}$-lattices and constructive logic with strong negation. *Fundamenta Mathematicae.* 1958; 46: 61-80.

[40] Rasiowa H. An Algebraic Approach to Non-classical Logics, Studies in Logic and the Foundations of Mathematics.*North-Holland Publishing Co., Amsterdam.* 1974; 78.

[41] Ribenboim P. Characterization of the sup-complement in a distributive lattice with last element. *Summa Brasil. Math.* 1949; 2(4): 43-49.

[42] Rivieccio U and Spinks M. Quasi-Nelson algebras. *Electronic Notes in Theoretical Computer Science.* 2019; 344: 169-188.

[43] Romanowska A. Subdirectly irreducible pseudocomplemented De Morgan algebras. *Algebra Universalis.* 1981; 12(1): 70-75.

[44] Sankappanavar HP. A characterization of principal congruences of De Morgan algebras and its applications. *In: Mathematical Logic in Latin America, A.I. Arruda, R. Chuaqui and N.C.A. da Costa (editors), North- Holland, Amsterdam* . 1980; 340-349.

[45] Sankappanavar HP. Heyting algebras with dual pseudocomplementation. *Pacific J. Math.*. 1985; 117: 405-415. DOI: https://doi.org/10.2140/p jm.1985.117.405

[46] Sankappanavar HP. Pseudocomplemented Okham and De Morgan algebras. *Math. Logic Quarterly.* 1986; 32: 385-394. DOI: https://doi.org/10.1002/malq.19860322502

[47] Sankappanavar HP. Semi-Heyting Algebras. *Amer. Math. Soc. Abstracts, January 1987.* 1987; 13.

[48] Sankappanavar HP. Heyting algebras with a dual lattice endomorphism. *Math Logic Quarterly.* 1987; 33: 565-573. DOI: https://doi.org/10.1002/malq.19870330610

[49] Sankappanavar HP. Semi-De Morgan algebras. *J. Symbolic. Logic.* 1987; 52: 712-724. DOI: https://doi.org/10.2307/ 2274359

[50] Sankappanavar HP. Principal congruences of pseudocomplemented De Morgan algebras. *Math. Logic Quarterly.* 1987; 33: 3-11. DOI: https://doi.org/10.1002/malq.19870330102

[51] Sankappanavar HP. Pseudocomplemented and almost pseudocomplemented Ockham algebras: Principal congruences. *Math. Logic Quarterly.* 1989; 35: 229-236. DOI: https://doi.org/10.1002/malq.19890350306

[52] Sankappanavar HP. Linked double weak Stone algebras. *Math. Logic Quarterly.* 1989; 35: 485-494. DOI: https://doi.org/10.1002/malq.19890350604

[53] Sankappanavar HP. Demi-pseudocomplemented lattices: Principal congruences and subdirect irreducibility. *Algebra Universalis.* 1990; 27: 180-193.

[54] Sankappanavar HP. Principal congruences of double demi-p-lattices. *Algebra Universalis*. 1990; 27: 248-253.

[55] Sankappanavar HP. Varieties of demi-pseudocomplemented lattices. *Math. Logic Quarterly*. 1991; 37: 411-420. DOI: https://doi.org/10.1002/malq.19910372603

[56] Sankappanavar HP. Principal congruences of demi-pseudocomplemented Ockham algebras and applications. *Math. Logic Quarterly*. 1991; 37: 489-494.

[57] Sankappanavar HP. Semi-Heyting algebras: An abstraction from Heyting algebras. *In: Proceedings of the 9th "Dr. Antonio A. R. Monteiro" Congress (Spanish: Actas del IX Congresso Dr. Antonio A. R. Monteiro, held in BahÍa Blanca, May 30-June 1, 2007), edited by M. Abad and I. Viglizzo (Universidad Nacional del Sur)*. 2008; 33-66.

[58] Sankappanavar HP. Expansions of semi-Heyting algebras I: Discriminator varieties. *Studia Logica*. 2011; 98(1-2): 27-81. DOI: https://doi. org/10.1007/s11225-011-9322-6

[59] Sankappanavar HP. Dually quasi-De Morgan Stone semi-Heyting algebras I: Regularity. *Categories, General Algebraic Structures and Applications*. 2014; 2(1): 47-64.

[60] Sankappanavar HP. Dually quasi-De Morgan Stone semi-Heyting algebras II: Regularity. *Categories, General Algebraic Structures and Applications*. 2014; 2(1): 65-82.

[61] Sankappanavar HP. A note on regular De Morgan Stone semi-Heyting algebras. *Demonstracio mathematica*. 2016; 49(3): 252-265. DOI: https://doi.org/10.1515/dema-2016-0021

[62] Sankappanavar HP. JI-distributive dually quasi-De Morgan semi-Heyting and Heyting algebras. *Sci. Math. Jpn.* 2019; 82(3): 245-271. DOI: https://doi.org/10.32219/isms.82.3 245

[63] Sankappanavar HP. De Morgan semi-Heyting and Heyting algebras. *New Trends in Algebra and Combinatorics, Proceeding of the 3rd International Congress in Algebra and Combinatorics (ICAC2017), 25-28, August 2017, Hong Kong, China*. 2020. DOI: https://doi.org/10.1142/9789811215476 0024

[64] Sankappanavar HP. (Chapter) In: A few historical glimpses into the interplay between algebra and logic and investigations into Gautama algebras. *In: Handbook of Logical Thought in India, S. Sarukkai, M. K. Chakraborty (eds.), Springer, New Delhi*. 2022; 979-1052.

[65] Sankappanavar HP and Vaz de Carvalho J. Congruence properties of pseudocomplemented De Morgan algebras. *Mathematical logic quarterly*. 2014; 60(6): 425-436. DOI: https://doi.org/10.1002/malq.201300038

[66] Skolem T. Untersuchungen über die Axiome des Klassenkalküls und über Produktations- und Summationsprobleme. *welche gewisse Klassen von Aussagen betreffen (as reprinted in 1970)*. 1919; 67-101.

[67] Vakarelov D. Notes on $\mathcal{N}$-lattices and constructive logic with strong negation. *Stud Logica*. 1977; 36: 109-125.

[68] Varlet J. A regular variety of type (2, 2, I, 1, 0, 0). *Algebra Universalis*. 1972; 2: 218-223. DOI: https://doi.org/10.1007/bf02945029

[69] Viglizzo I. Á Lgebras De Nelson. *Instituto De Matemática De Baha Blanca, Universidad Nacional del Sur, 1999. Magister dissertation in Mathematics, Universidad Nacional del Sur, Baha Blanca*. 1999.

[70] Von Plato J. In the shadows of the Löwenheim-Skolem theorem: Early combinatorial analysis of mathematical proofs. *The Bulletin of Symbolic Logic*. 2007; 13(2): 189-225.

[71] Vorobév NN. Constructive propositional calculus with strong negation (in Russian). *Doklady Akademii Nauk SSSR*. 1952; 85: 456-468.

[72] Werner H. Discriminator Algebras. *Studien zur Algebra und ihre Anwendungen, Band 6, Academie–Verlag, Berlin*. (1978).

Juan M. Cornejo
Departamento de Matemática
Universidad Nacional del Sur
INMABB - CONICET
Bahía Blanca, Argentina
E-mail: jmcornejo@uns.edu.ar

Hanamantagouda P. Sankappanavar
Department of Mathematics
State University of New York
New Paltz, NY, USA.
E-mail: sankapph@hawkmail.newpaltz.edu

Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://tfss.journals.iau.ir/
Online ISSN: 2821-0131
Vol.2, No.2, (2023), 113-126
DOI: http://doi.org/10.30495/TFSS.2023.1972768.1055

Article Type: Original Research Article

Intuitionistic Fuzzy Modular Spaces

Tayebe Lal Shateri*

Abstract. After the introduction of the concept of fuzzy sets by Zadeh, several researches were conducted on the generalizations of the notion of fuzzy sets. There are many viewpoints on the notion of metric space in fuzzy topology. One of the most important problems in fuzzy topology is obtaining an appropriate concept of fuzzy metric space. This problem has been investigated by many authors from different points of view. Atanassov gives the concept of the intuitionistic fuzzy set as a generalization of the fuzzy set. Park introduced the notion of intuitionistic fuzzy metric space as a natural generalization of fuzzy metric spaces due to George and Veeramani. This paper introduces the concept of intuitionistic fuzzy modular space. Afterward, a Hausdorff topology induced by a δ-homogeneous intuitionistic fuzzy modular is defined and some related topological properties are also examined. After giving the fundamental definitions and the necessary examples, we introduce the definitions of intuitionistic fuzzy boundedness, intuitionistic fuzzy compactness, and intuitionistic fuzzy convergence, and obtain several preservation properties and some characterizations concerning them. Also, we investigate the relationship between an intuitionistic fuzzy modular and an intuitionistic fuzzy metric. Finally, we prove some known results of metric spaces including Baires theorem and the Uniform limit theorem for intuitionistic fuzzy modular spaces.

AMS Subject Classification 2020: 42C15; 06D22
Keywords and Phrases: Fuzzy set, Modular space, Fuzzy modular space, Intuitionistic fuzzy modular space.

1 Introduction

The notion of fuzzy sets was introduced by Zadeh [20] in 1965 and there are many viewpoints on the notion of metric space in fuzzy topology. The concept of fuzzy topology may have very important applications in quantum particle physics, [3, 4]. One of the most important problems in fuzzy topology is obtaining an appropriate concept of a fuzzy metric space. This problem has been investigated by many authors from different points of view. Kramosil and Michálek [11] introduced the concept of a fuzzy metric space, which can be regarded as a generalization of the probabilistic metric space. Afterward, Grabiec [5] defined the fuzzy metric space's completeness and extended the Banach contraction theorem to the complete fuzzy metric spaces. Next, George and Veeramani [6] modified the definition of the Cauchy sequence introduced by Grabiec. Atanassov [1] gave the concept of an intuitionistic fuzzy set as a generalization of a fuzzy set. Park [17] introduced the notion of an intuitionistic fuzzy metric space as a natural generalization of a fuzzy metric space due to George and Veeramani. He proved Baire's theorem and the Uniform limit theorem for these spaces. For more details on intuitionistic fuzzy metric space and related results, we refer the reader to [2, 18].

The concept of a modular space was founded by Nakano [14] and developed by Luxemburg [12]. Then, Musielak and Orlicz [13] redefined and generalized the notion of modular space. A real function ρ on an

***Corresponding Author:** Tayebe Lal Shateri, Email: t.shateri@hsu.ac.ir, ORCID: 0000-0002-5767-8619
Received: 15 November 2022; **Revised**: 29 May 2023; **Accepted**: 10 June 2023; **Available Online**: 10 June 2023; **Published Online**: 7 November 2023.

How to cite: Shateri TL. Intuitionistic Fuzzy Modular Spaces. *Trans. Fuzzy Sets Syst.* 2023; 2(2): 113-126. DOI: http://doi.org/10.30495/TFSS.2023.1972768.1055

arbitrary vector space $\mathcal{X}$ is called a *modular* if for arbitrary $x, y \in \mathcal{X}$ the following conditions hold:
(i) $\rho(x) = 0$ if and only if $x = 0$,
(ii) $\rho(\alpha x) = \rho(x)$ for every scaler α with $|\alpha| = 1$,
(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.
A *modular space* $\mathcal{X}_\rho$ is defined as $\mathcal{X}_\rho = \{x \in \mathcal{X} : \quad \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}$.

Based on the definition of modular space, Kozlowski [8, 9] introduced a modular function space. In the sequel, Kozlowski and Lewicki [10] considered the problem of analytic extension of measurable functions in modular function spaces and discussed some extension properties by means of polynomial approximation. Afterward, Kilmer and Kozlowski [7] studied the existence of best approximations in modular function spaces by elements of sublattices. Nourouzi [15] proposed probabilistic modular spaces based on the theory of modular spaces and in [16] he extended the well-known Baire's theorem to probabilistic modular spaces by using a special condition. Shen and Chen [19] introduced the notion of fuzzy modular space by using continuous t-norm and continuous t-conorm.

The concept of intuitionistic fuzzy modular space is first proposed in this paper. By using some ideas of [17, 19] we introduce the concept of an intuitionistic fuzzy modular space and give a Hausdorff topology in this space. We investigate some topological properties and the existence of a relationship between an intuitionistic fuzzy modular and an intuitionistic fuzzy metric. The paper is organized as follows.
First, we recall the fundamental definitions and the necessary examples of an intuitionistic fuzzy metric space. In section 2, following the idea of fuzzy modular spaces and the definition of an intuitionistic fuzzy metric space, we give a new concept named intuitionistic fuzzy modular space and give two examples to show that there does not exist a direct relationship between an intuitionistic fuzzy modular and an intuitionistic fuzzy metric. In section 3, a Hausdorff topology induced by a δ-homogeneous intuitionistic fuzzy modular is defined, and several theorems on μ-ν-completeness of the intuitionistic fuzzy modular space are given. Finally, the well-known Baire's theorem and the uniform limit theorem are extended to intuitionistic fuzzy modular spaces.

Definition 1.1. [18] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if it satisfies the following conditions:

1. $*$ is commutative and associative;
2. $*$ is continuous;
3. $a * 1 = a$ for every $a \in [0,1]$;
4. $a * b \leq c * d$ whenever $a \leq c, b \leq d$, and $a, b, c, d \in [0,1]$.

The common examples of a t-norm are as follows:
(1) $a * b = ab$ (2) $a *_M b = \min\{a, b\}$ (3) $a * b = \max\{0, a + b - 1\}$.

Definition 1.2. [18] A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-conorm if it satisfies the following conditions:

1. $\diamond$ is commutative and associative;
2. $\diamond$ is continuous;
3. $a \diamond 0 = a$ for every $a \in [0,1]$;
4. $a \diamond b \leq c \diamond d$ whenever $a \leq c, b \leq d$ and $a, b, c, d \in [0,1]$.

Some examples of a t-conorm are as follows.
(1) $a \diamond b = a + b - ab$ (2) $a \diamond_M b = \max\{a, b\}$ (3) $a \diamond b = \min\{1, a + b\}$.
By properties of t-norm and t-conorm, we get the following lemma.

Lemma 1.3. *[6] (i) If $a, b \in (0, 1)$ with $a > b$, then there exist $c, d \in (0, 1)$ such that $a * c \leq b$ and $a \geq d \diamond b$.*
(ii) If $a \in (0, 1)$, then there exist $c, d \in (0, 1)$ such that $c * c \geq a$ and $a \geq d \diamond d$.

Now, we recall the definition of an intuitionistic fuzzy metric space.

Definition 1.4. [17] A 5-tuple $(\mathcal{X}, M, N, *, \diamond)$ is said to be an intuitionistic fuzzy metric space if $\mathcal{X}$ is a real or complex vector space, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and M, N are fuzzy sets on $\mathcal{X}^2 \times (0, \infty)$ such that for all $x, y, z \in \mathcal{X}$ and $s, t > 0$ the followings hold:

1. $M(x, y, t) + N(x, y, t) \leq 1$,
2. $M(x, y, t) > 0$,
3. $M(x, , y, t) = 1$ if and only if $x = y$,
4. $M(x, y, t) = M(y, x, t)$,
5. $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
6. $M(x, y, .) : (0, \infty) \to (0, 1]$ is continuous,
7. $N(x, y, t) > 0$,
8. $N(x, y, t) = 0$ if and only if $x = y$,
9. $N(x, y, t) = N(y, x, t)$,
10. $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$,
11. $N(x, y, .) : (0, \infty) \to (0, 1]$ is continuous.

(M, N) is called an intuitionistic fuzzy metric on $\mathcal{X}$.

Example 1.5. [17, Example 2.8] Let $(\mathcal{X}, d)$ be a metric space. Denote $a * b = ab$ and $a \diamond b = \min\{1, a + b\}$ for all $a, b \in [0, 1]$ and let M_d and N_d be fuzzy sets on $\mathcal{X}^2 \times (0, \infty)$ defined as follows:

$$M_d(x, y, t) = \frac{ht^n}{ht^n + md(x, y)}, \quad N_d(x, y, t) = \frac{d(x, y)}{kt^n + md(x, y)}$$

for all $h, k, m, n \in \mathbb{R}^+$. Then $(\mathcal{X}, M_d, N_d, *, \diamond)$ is an intuitionistic fuzzy metric space.

2 Intuitionistic Fuzzy Modular Spaces

In this section, we introduce the concept of an intuitionistic fuzzy modular space by using continuous t-norm and continuous t-conorm. We investigate the relationship between an intuitionistic fuzzy modular and an intuitionistic fuzzy metric.

Definition 2.1. A 5-tuple $(\mathcal{X}, \mu, \nu, *, \diamond)$ is said to be an intuitionistic fuzzy modular space if $\mathcal{X}$ is a real or complex vector space, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and μ, ν are fuzzy sets on $\mathcal{X} \times (0, \infty)$ such that for all $x, y, z \in \mathcal{X}$, $s, t > 0$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ followings hold:

1. $\mu(x,t)+\nu(x,t)\leq 1$,
2. $\mu(x,t)>0$,
3. $\mu(x,t)=1$ if and only if $x=0$,
4. $\mu(x,t)=\mu(-x,t)$,
5. $\mu(\alpha x+\beta y, s+t)\geq \mu(x,s)*\mu(y,t)$,
6. $\mu(x,.):(0,\infty)\to(0,1]$ is continuous,
7. $\nu(x,t)>0$,
8. $\nu(x,t)=0$ if and only if $x=0$,
9. $\nu(x,t)=\nu(-x,t)$,
10. $\nu(\alpha x+\beta y, s+t)\leq \nu(x,s)\diamond\nu(y,t)$,
11. $\nu(x,.):(0,\infty)\to(0,1]$ is continuous.

Then (μ,ν) is called an intuitionistic fuzzy modular or intuitionistic $\mathfrak{F}$-modular on $\mathcal{X}$. The 5-tuple $(\mathcal{X},\mu,\nu,*,\diamond)$ is called δ-homogeneous, where $\delta\in(0,1]$, if for each $x\in\mathcal{X}$, $t>0$ and $\lambda\in\mathbb{R}-\{0\}$,

$$\mu(\lambda x,t)=\mu\Big(x,\frac{t}{|\lambda|^{\delta}}\Big),\ \nu(\lambda x,t)=\nu\Big(x,\frac{t}{|\lambda|^{\delta}}\Big).$$

Remark 2.2. (i) If $(\mathcal{X},\mu,*)$ is a $\mathfrak{F}$-modular space, then $(\mathcal{X},\mu,1-\mu,*,\diamond)$ is an intuitionistic $\mathfrak{F}$-modular space such that for any $a,b\in[0,1]$, $a\diamond b=1-((1-a)*(1-b))$.
(ii) In intuitionistic $\mathfrak{F}$-modular space $(\mathcal{X},\mu,\nu,*,\diamond)$, for all $x,y\in\mathcal{X}$, $\mu(x,y,.)$ is non-decreasing and $\nu(x,y,.)$ is non-increasing.

Example 2.3. Let $(\mathcal{X},\rho)$ be a modular space. Take $a*b=ab$ and $a\diamond b=\min\{1,a+b\}$, for all $a,b\in[0,1]$, and define fuzzy sets μ_ρ and ν_ρ on $\mathcal{X}\times(0,\infty)$ as follows:

$$\mu_\rho(x,t)=\frac{ht^n}{ht^n+m\rho(x)},\ \nu_\rho(x,t)=\frac{\rho(x)}{kt^n+m\rho(x)},$$

for all $h,k\in\mathbb{R}^+$ and $m,n\in\mathbb{N}$. Then $(\mathcal{X},\mu,\nu,*,\diamond)$ is an intuitionistic $\mathfrak{F}$-modular space. We investigate condition (5) in Definition 2.1. For this let $\alpha,\beta\geq 0$ with $\alpha+\beta=1$, since ρ is modular, we have

$$\rho(\alpha x+\beta y)\leq\rho(x)+\rho(y), \tag{1}$$

for all $x,y\in\mathcal{X}$. Hence

$$\begin{aligned}\mu(x,s)*\mu(y,t)&=\frac{hs^n}{hs^n+m\rho(x)}*\frac{ht^n}{ht^n+m\rho(y)}\\&=\frac{h^2s^nt^n}{(hs^n+m\rho(x))(ht^n+m\rho(y))}\\&\leq\frac{hs^nt^n}{hs^nt^n+m(t^n\rho(x)+s^n\rho(y))}.\end{aligned}$$

Without loss of generality, we assume that $t \leq s$. Then by using (1) we get

$$\begin{aligned}\mu(x,s) * \mu(y,t) &\leq \frac{hs^n}{hs_n + m\rho(\alpha x + \beta y)}\\ &\leq \frac{h(s+t)^n}{h(s+t)^n + m\rho(\alpha x + \beta y)}\\ &= \mu(\alpha x + \beta y, s+t).\end{aligned}$$

Remark 2.4. In Example 2.3, by taking $h = k = m = n = 1$, we get

$$\mu_\rho(x,t) = \frac{t}{t+\rho(x)},\ \nu_\rho(x,t) = \frac{\rho(x)}{t+\rho(x)}.$$

This intuitionistic $\mathfrak{F}$-modular space is called the standard intuitionistic $\mathfrak{F}$-modular space.

It should be noted that, in general, an intuitionistic fuzzy modular and an intuitionistic fuzzy metric do not necessarily induce mutually a metric when the triangular norm is the same one. In essence, the intuitionistic fuzzy modular and intuitionistic fuzzy metric can be viewed as two different characterizations for the same set. Next, we give two examples to show that there does not exist a direct relationship between an intuitionistic fuzzy modular and an intuitionistic fuzzy metric. In fact, the intuitionistic fuzzy modular and the intuitionistic fuzzy metric can be viewed as two different characterizations for the same set. The following examples are the motivation of [19, Example 8, Example 9] in the view of intuitionistic fuzzy modular spaces.

Example 2.5. Let $\mathcal{X} = \mathbb{R}$ and put $\rho(x) = |x|$, then ρ is modular on $\mathcal{X}$. Put $a * b = \min\{a,b\}$, and $a \diamond b = 1 - ((1-a)*(1-b))$ or $a \diamond b = \max\{a,b\}$. For every $t \in (0,\infty)$ and $x \in \mathcal{X}$, define $\mu(x,t) = \frac{t}{t+|x|}$. Then [19, Example 8] implies that $(\mathcal{X}, \mu, *)$ is an $\mathfrak{F}$-modular space and so by Remark 2.2, $(\mathcal{X}, \mu, 1-\mu, *, \diamond)$ is an intuitionistic $\mathfrak{F}$-modular space.
However, if we set

$$M(x,y,t) = \mu(x-y,t) = \frac{t}{t+|x-y|}, \text{ and } \quad N(x,y,t) = \frac{|x-y|}{t+|x-y|},$$

then [17, Remark 2.11] implies that (M,N) is not an intuitionistic fuzzy metric with the t-norm and t-conorm defined as $a * b = \min\{a,b\}$ and $a \diamond b = \max\{a,b\}$.

Example 2.6. Let $\mathcal{X} = \mathbb{R}$. Take t-norm $a * b = \min a, b$ and t-conorm $a \diamond b = a + b - ab$. For every $x, y \in \mathcal{X}$ and $t \in (0,\infty)$, we define

$$M(x,y,t) = \begin{cases} 1, & x = y\\ \frac{1}{2}, & x \neq y, x, y \in \mathbb{Z}\\ \frac{1}{4}, & x \in \mathbb{Z}, y \in \mathbb{R}\backslash\mathbb{Z} \ \ or \ \ x \in \mathbb{R}\backslash\mathbb{Z}, y \in \mathbb{Z}\\ \frac{1}{4}, & x \neq y, x, y \in \mathbb{R}\backslash\mathbb{Z}, \end{cases}$$

and

$$M(x,y,t) = \begin{cases} 1, & x = y\\ \frac{1}{2}, & x \neq y, x, y \in \mathbb{Z}\\ \frac{1}{4}, & x \in \mathbb{Z}, y \in \mathbb{R}\backslash\mathbb{Z} \ \ or \ \ x \in \mathbb{R}\backslash\mathbb{Z}, y \in \mathbb{Z}\\ \frac{1}{4}, & x \neq y, x, y \in \mathbb{R}\backslash\mathbb{Z} \end{cases}$$

It can easily be shown that $(M, N, *, \diamond)$ is an intuitionistic fuzzy metric on $\mathcal{X}$.
Now, set

$$\mu(x,t) = \begin{cases} 1, & x = 0\\ \frac{1}{2}, & x \in \mathbb{Z}\backslash\{0\}\\ \frac{1}{4}, & x \in \mathbb{R}\backslash\mathbb{Z} \end{cases}, \text{ and } \quad \nu(x,t) = \begin{cases} 0, & x = 0\\ \frac{1}{4}, & x \in \mathbb{Z}\backslash\{0\}\\ \frac{1}{2}, & x \in \mathbb{R}\backslash\mathbb{Z}. \end{cases}$$

If we take $\alpha = \frac{\sqrt{2}}{2}$, $\beta = 1 - \alpha$, $x \neq y$, and $x, y \in \mathbb{Z}$, then $\alpha x + \beta y \in \mathbb{R}\backslash\mathbb{Z}$. Hence for each $s, t > 0$, we have $\mu(\alpha x + \beta y, s + t) = \frac{1}{4}$, but

$$\mu(x, s) * \mu(y, t) = \frac{1}{2}.$$

Also $\nu(\alpha x + \beta y, s + t) = \frac{1}{2}$, but

$$\nu(x, s) \diamond \nu(y, t) = \frac{1}{4}.$$

Therefore (μ, ν) is not an intuitionistic fuzzy modular on $\mathcal{X}$.

3 Topology Induced by an δ-homogeneous Intuitionistic Fuzzy Modular Space

In this section, we define a topology induced by a δ-homogeneous intuitionistic $\mathfrak{F}$-modular and investigate some topological properties in δ-homogeneous intuitionistic $\mathfrak{F}$-modular space. The results obtained in this section are an extension of the results presented in [19] to intuitionistic fuzzy modular spaces.

Definition 3.1. Let $(\mathcal{X}, \mu, \nu, *, \diamond)$ be an intuitionistic $\mathfrak{F}$-modular space and let $x \in \mathcal{X},\ r \in (0, 1)$ and $t > 0$. Then the μ-ν-ball with center x and radius r with respect to t is defined as

$$B(x, r, t) = \{y \in X : \mu(x - y, t) > 1 - r,\ \nu(x - y, t) < r\}.$$

Let $E \subseteq \mathcal{X}$. An element $x \in E$ is called a μ-ν-interior point of E if there exist $r \in (0, 1)$ and $t > 0$ such that $B(x, r, t) \subseteq E$. We say that E is a μ-ν-open set in $\mathcal{X}$ if and only if every element of E is a μ-ν-interior point. Note that each open set in an intuitionistic $\mathfrak{F}$-modular space is not a μ-ν-ball in general.

Example 3.2. Let $\mathcal{X} = \mathbb{R}$ and let ρ, μ, $*$ and $\diamond$ be as in Example 2.5.
Consider $V = \{x \in \mathbb{R} : 0 < x < 1\} \cup \{x \in \mathbb{R} : 1 < x < 2\}$. Then V is an open set in $(\mathbb{R}, \mu, 1 - \mu, *, \diamond)$, but it is not a μ-$(1 - \mu)$-ball. In fact, the μ-$(1 - \mu)$-ball in $(\mathbb{R}, \mu, 1 - \mu, *, \diamond)$ with center x and radius r is as follows.

$$\begin{aligned} B(x, r, t) &= \{y \in \mathbb{R} : \frac{t}{t + |x - y|} > 1 - r,\ \frac{|x - y|}{t + |x - y|} < r\} \\ &= \{y \in \mathbb{R} : |x - y| < \frac{r}{1 - r} t\}. \end{aligned}$$

Theorem 3.3. *Each μ-ν-ball in a δ-homogeneous intuitionistic $\mathfrak{F}$-modular space is an open set.*

Proof. Let $B(x, r, t)$ be a μ-ν-ball and $y \in B(x, r, t)$. Then

$$\mu(x - y, t) > 1 - r,\ \nu(x - y, t) < r.$$

Put $t = 2t_1$. Since $\mu(x - y, .)$ and $\nu(x - y, .)$ are continuous, there exists $\varepsilon y > 0$ such that

$$\mu(x - y, \frac{t_1 - \varepsilon_y}{2^{\delta - 1}}) > 1 -,\ \nu(x - y, \frac{t_1 - \varepsilon_y}{2^{\delta - 1}}) < r.$$

For some $\varepsilon > 0$ with $\frac{t_1 - \varepsilon}{2^{\delta - 1}} > 0$ and $\frac{\varepsilon}{2^{\delta - 1}} \in (0, \varepsilon_y)$, put $r_0 = \mu(x - y, \frac{t_1 - \varepsilon}{2^{\delta - 1}})$. Since $r_0 > r - 1$, there exists $s \in (0, 1)$ such that $r_0 > 1 - s > 1 - r$, by Lemma 1.3, we can choose $r_1 \in (0, 1)$ such that

$$r_0 *\ r_0 \geq 1 - s,\ (1 - r_0) \diamond (1 - r_0) \leq s.$$

Put $r_3 = \max\{r_1, r_2\}$. We show that $B(y, 1-r_3, \frac{\varepsilon}{2^{\delta-1}}) \subseteq B(x, r, 2t_1)$. Suppose that $z \in B(y, 1-r_3, \frac{\varepsilon}{2^{\delta-1}})$ then

$$\mu(y-z, \frac{\varepsilon}{2^{\delta-1}}) > r_3,\ \nu(y-z, \frac{\varepsilon}{2^{\delta-1}}) < 1-r_3.$$

Therefore

$$\begin{aligned}\mu(x-z,t) &= \mu(x-z,2t_1) \geq \mu(2(x-y), 2(t_1-\varepsilon)) * \mu(2(y-z), 2\varepsilon)\\ &= \mu(x-y, \frac{t_1-\varepsilon}{2^{\delta-1}}) * \mu(y-z, \frac{\varepsilon}{2^{\delta-1}})\\ &\geq r_0 * r_1 \geq 1-s > 1-r,\end{aligned}$$

and

$$\begin{aligned}\nu(x-z,t) &= \nu(x-z,2t_1) \leq \nu(2(x-y), 2(t_1-\varepsilon)) \diamond \nu(2(y-z), 2\varepsilon)\\ &= \nu(x-y, \frac{t_1-\varepsilon}{2^{\delta-1}}) \diamond \nu(y-z, \frac{\varepsilon}{2^{\delta-1}})\\ &< (1-r_0) \diamond (1-r_3) \leq (1-r_0) \diamond (1-r_2) \leq s < r.\end{aligned}$$

Therefore $z \in B(x, r, t)$ and hence $B(y, 1-r_3, \frac{\varepsilon}{2^{\delta-1}}) \subseteq B(x, r, t)$. □

Now, we define a topology on a δ-homogeneous intuitionistic $\mathfrak{F}$-modular space.

Definition 3.4. Let $(\mathcal{X}, \mu, \nu, *, \diamond)$ be a δ-homogeneous intuitionistic $\mathfrak{F}$-modular space. Define

$$\tau_{(\mu,\nu)} = \{V \subseteq \mathcal{X} : \forall x \in V, \exists t > 0, r \in (0,1); B(x, r, t) \subseteq V\}.$$

Then $\tau_{(\mu,\nu)}$ is a topology on $\mathcal{X}$.

Remark 3.5. Since the family of μ-ν-balls $\{B(x, \frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$ is a local base at x, the topology $\tau_{(\mu,\nu)}$ is first countable.

Example 3.6. Let $\mathcal{X} = \mathbb{R}$ and let ρ, μ, $*$ and $\diamond$ be as in Example 2.5. Then the set of all $\{(a, b) : a, b \in \mathbb{R}\}$ induces a topology on $(\mathbb{R}, \mu, 1-\mu, *, \diamond)$.

Theorem 3.7. *Every δ-homogeneous intuitionistic $\mathfrak{F}$-modular space is Hausdorff.*

Proof. Let x, y be two distinct points in δ-homogeneous intuitionistic $\mathfrak{F}$-modular space $(\mathcal{X}, \mu, \nu, *, \diamond)$. Then for all $t > 0$, $0 < \mu(x-y, t) < 1$, $0 < \nu(x-y, t) < 1$. Put $r_1 = \mu(x-y, t)$, $r_2 = \nu(x-y, t)$ and $r = \max\{r_1, r_2\}$. For $r_0 \in (r, 1)$, there are r_3, r_4 such that

$$r_3 * r_3 \geq r_0,\ \ (1-r_4) \diamond (1-r_4) \leq 1-r_0.$$

Put $r_5 = \max\{r_3, r_4\}$. Then $B(x, 1-r_5, \frac{t}{2^{\delta+1}}) \cap B(y, 1-r_5, \frac{t}{2^{\beta+1}}) = \emptyset$. Otherwise, if there exists $z \in B(x, 1-r_5, \frac{t}{2^{\delta+1}}) \cap B(y, 1-r_5, \frac{t}{2^{\delta+1}})$, then

$$\begin{aligned}r_1 = \mu(x-y,t) &\geq \mu(2(x-z), \frac{t}{2}) * \ \mu(2(z-y), \frac{t}{2})\\ &= \mu(x-z, \frac{t}{2^{\delta+1}}) * \mu(z-y, \frac{t}{2^{\delta+1}})\\ &\geq r_5 * r_5 \geq r_3 * \ r_3 \geq r_0 > r_1,\end{aligned}$$

and

$$\begin{aligned} r_2 = \nu(x-y,t) &\leq \nu(2(x-z),\frac{t}{2}) \diamond \nu(2(z-y),\frac{t}{2}) \\ &= \nu(x-z,\frac{t}{2^{\delta+1}}) \diamond \nu(z-y,\frac{t}{2^{\delta+1}}) \\ &\leq (1-r_5)\diamond(1-r_5) \leq (1-r_4)\diamond(1-r_4) \leq 1-r_0 < r_2, \end{aligned}$$

which is a contradiction. Therefore $(\mathcal{X},\mu,\nu,*,\diamond)$ is Hausdorff. □

In the following, we give further properties of a δ-homogeneous intuitionistic $\mathfrak{F}$-modular space.

Definition 3.8. Let $(\mathcal{X},\mu,\nu,*,\diamond)$ be a δ-homogeneous intuitionistic $\mathfrak{F}$-modular space.

1. A subset $\mathcal{A}$ of $\mathcal{X}$ is said to be μ-ν-bounded if there are $t>0$ and $r\in(0,1)$ such that for all $x\in\mathcal{A}$, $\mu(x,t)>1-r$ and $\nu(x,t)<r$.

2. A subset $\mathcal{A}$ of $\mathcal{X}$ is said to be μ-ν-compact if every μ-ν-open cover of $\mathcal{A}$ has a finite subcover.

3. A sequence $\{x_n\}$ in $\mathcal{X}$ is said to be μ-ν-convergent to $x\in\mathcal{X}$ if for every $r\in(0,1)$ and $t>0$ there exists $n_0\in\mathbb{N}$ such that for each $n>n_0$, $x_n\in B(x,r,t)$.

Example 3.9. Let $\mathcal{X}=\mathbb{R}$ and let ρ, μ, $*$ and $\diamond$ be as in Example 2.5.
(i) Consider $V=\{x\in\mathbb{R}:0<x<1\}$. Then V is a bounded set in $(\mathbb{R},\mu,1-\mu,*,\diamond)$.
(ii) Each finite set in $(\mathbb{R},\mu,1-\mu,*,\diamond)$ is μ-ν-compact.
(iii) The sequence $\{\frac{1}{n}\}$ is μ-ν-convergent to 0 in $(\mathbb{R},\mu,1-\mu,*,\diamond)$ by choosing n_0 such that $1-t<\frac{1}{n_0}<r$.

Theorem 3.10. *Every μ-ν-compact subset of a δ-homogeneous intuitionistic $\mathfrak{F}$-modular space $(\mathcal{X},\mu,\nu,*,\diamond)$, is μ-ν-bounded.*

Proof. Let $\mathcal{A}$ be a μ-ν-compact subset of $(\mathcal{X},\mu,\nu,*,\diamond)$. Fix $t>0$ and $r\in(0,1)$, then the family $\{B(x,r,\frac{t}{2^{\delta+1}}): x\in A\}$ is an open cover of $\mathcal{A}$, since $\mathcal{A}$ is compact there exist $x_1,\cdots,x_n\in\mathcal{A}$ such that $\mathcal{A}\subset\cup_{i=1}^n B(x_i,r,\frac{t}{2^{\delta+1}})$. Hence for each $x\in\mathcal{A}$ there exists i such that $x\in B(x_i,r,\frac{t}{2^{\delta+1}})$. Thus

$$\mu(x-x_i,\frac{t}{2^{\delta+1}})>1-r,\ \ \nu(x-x_i,\frac{t}{2^{\delta+1}})<r.$$

Put $\alpha_1=\min\{\mu(x_i,\frac{t}{2^{\delta+1}}):1\leq i\leq n\}$ and $\alpha_2=\max\{\nu(x_i,\frac{t}{2^{\delta+1}}):1\leq i\leq n\}$, it is clear that $\alpha_1,\alpha_2>0$, hence for some $s_1,s_2\in(0,1)$ we have

$$\begin{aligned} \mu(x,t)=\mu((x-x_i)+x_i,t) &\geq \mu(2(x-x_i),\frac{t}{2})*\mu(2x_i,\frac{t}{2}) \\ &= \mu(x-x_i,\frac{t}{2^{\delta+1}})*\mu(X_I,\frac{t}{2^{\delta+1}}) \geq (1-r)*\alpha_1>1-s_1, \end{aligned}$$

and

$$\begin{aligned} \nu(x,t)=\nu((x-x_i)+x_i,t) &\leq \nu(2(x-x_i),\frac{t}{2})\diamond\nu(2x_i,\frac{t}{2}) \\ &= \nu(x-x_i,\frac{t}{2^{\delta+1}})\diamond\nu(X_i,\frac{t}{2^{\delta+1}}) \leq r\diamond\alpha_2<s_2. \end{aligned}$$

Taking $s=\max\{s_1,s_2\}$ we conclude $\mu(x,t)>1-s$ and $\nu(x,t)<s$, consequently $\mathcal{A}$ is μ-ν-bounded. □

Theorem 3.11. *Let $(\mathcal{X},\mu,\nu,*,\diamond)$ be a δ-homogeneous intuitionistic $\mathfrak{F}$-modular space and $\{x_n\}$ a sequence in $\mathcal{X}$. Then $x_n\to x$ if and only if $\mu(x_n-x,t)\to 1,\ \nu(x_n-x,t)\to 0$.*

Proof. Fix $t > 0$. Assume that $x_n \to x$, then for $r \in (0,1)$ there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$, $x_n \in B(x,r,t)$, so $\mu(x_n - x, t) > 1 - r,\ \ \nu(x_n - x, t) < r$. Hence

$$\mu(x_n - x, t) \to 1,\ \nu(x_n - x, t) \to 0.$$

Conversely, for each $t > 0$, let $\mu(x_n - x, t) \to 1,\ \nu(x_n - x, t) \to 0$. Then for $r \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$, $1 - \mu(x_n - x, t) < r,\ \ \nu(x_n - x, t) < r$. Therefore $\mu(x_n - x, t) > 1 - r$ and $\nu(x_n - x, t) < r$, for all $n \geq n_0$, that is, $x_n \in B(x,r,t)$ and so $x_n \to x$. □

In the following, we give some related results of completeness of an intuitionistic $\mathfrak{F}$-modular space.

Definition 3.12. Let $(\mathcal{X}, \mu, \nu, *, \diamond)$ be an intuitionistic $\mathfrak{F}$-modular space.

1. A sequence $\{x_n\}$ in $\mathcal{X}$ is called μ-ν-Cauchy if for every $\varepsilon > 0$ and $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $\mu(x_n - x_m, t) > 1 - r$ and $\nu(x_n - x_m, t) < r$ for all $m, n \geq n_0$.
2. $\mathcal{X}$ is called μ-ν-complete if every μ-ν-Cauchy sequence is μ-ν-convergent.

Theorem 3.13. *Let $(\mathcal{X}, \mu, \nu, *_M, \diamond_M)$ be a δ-homogeneous intuitionistic $\mathfrak{F}$-modular space. Then every μ-ν-convergent sequence in $\mathcal{X}$ is a μ-ν-Cauchy sequence.*

Proof. Let $\{x_n\}$ be μ-ν-convergent to $x \in \mathcal{X}$. Then for every $\varepsilon > 0$ and $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $\mu(x_n - x, \frac{t}{2^{\delta+1}}) > 1 - \varepsilon$ and $\nu(x_n - x, \frac{t}{2^{\delta+1}}) < \varepsilon$ for all $n \geq n_0$. For all $m, n \geq n_0$ we get

$$\begin{aligned}\mu(x_m - x_n, t) &\geq \mu(2(x_m - x), \frac{t}{2}) *\ \mu(2(x_n - x), \frac{t}{2})\\ &\geq \mu(x_m - x, \frac{t}{2^{\delta+1}}) * \mu(x_n - x, \frac{t}{2^{\delta+1}})\\ &> (1-\varepsilon) *_M (1-\varepsilon) = 1 - \varepsilon,\end{aligned}$$

and

$$\begin{aligned}\nu(x_m - x_n, t) &\leq \nu(2(x_m - x), \frac{t}{2}) \diamond \nu(2(x_n - x), \frac{t}{2})\\ &\leq \nu(x_m - x, \frac{t}{2^{\delta+1}}) \diamond\ \nu(x_n - x, \frac{t}{2^{\delta+1}}) < \varepsilon \diamond_M \varepsilon = \varepsilon.\end{aligned}$$

□

Remark 3.14. (1) Theorem 3.13 shows that in an intuitionistic $\mathfrak{F}$-modular space, a μ-ν-convergent sequence is not necessarily a μ-ν-Cauchy sequence, and the δ-homogeneity and the choice of t-norm and t-conorm are essential.
(2) From Definition 3.12, it is clear that each μ-ν-closed subspace of μ-ν-complete $\mathfrak{F}$-modular space is μ-ν-complete.

Theorem 3.15. *Let $(\mathcal{X}, \mu, \nu, *, \diamond)$ be a δ-homogeneous intuitionistic $\mathfrak{F}$-modular space and Y a subset of $\mathcal{X}$. If every μ-ν-Cauchy sequence of Y is μ-ν-convergent in $\mathcal{X}$, then every μ-ν-Cauchy sequence of $\bar{Y}$ is μ-ν-convergent in $\mathcal{X}$, where $\bar{Y}$ denotes the μ-ν-closure of Y.*

Proof. Let $\{x_n\}$ be a μ-ν-Cauchy sequence of $\bar{Y}$, then for each $n \in \mathbb{N}$ and $t > 0$, there exists $y_n \in Y$ such that $\mu(x_n - y_n, \frac{t}{4^{\delta+1}}) > 1 - \frac{1}{n+1}$ and $\nu(x_n - y_n, \frac{t}{4^{\delta+1}}) < \frac{1}{n+1}$. Since $\mu(x, .)$ is non-decreasing and $\nu(x, .)$ is non-increasing, we have $\mu(x_n - y_n, \frac{t}{2^{\delta+1}}) > 1 - \frac{1}{n+1}$ and $\nu(x_n - y_n, \frac{t}{2^{\delta+1}}) < \frac{1}{n+1}$. Moreover for each $r \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mu(x_n - x_m, \frac{t}{4^{\delta+1}}) > 1 - r$ and $\nu(x_n - x_m, \frac{t}{4^{\delta+1}}) < r$ for all $m, n \geq n_0$.

That is, $\mu(x_n - x_m, \frac{t}{4^{\delta+1}}) \to 1$ and $\nu(x_n - x_m, \frac{t}{4^{\delta+1}}) \to 0$. Now we show that $\{y_n\}$ is a μ-ν-Cauchy sequence in Y. For all $m, n \geq n_0$ we have

$$\begin{aligned}\mu(y_n - y_m, t) &\geq \mu(2(y_n - x_n), \frac{t}{2}) * \mu(2(x_n - y_n), \frac{t}{2})\\ &\geq \mu(2(y_n - x_n), \frac{t}{2}) * \mu(4(x_n - x_m), \frac{t}{4}) * \mu(4(x_m - y_m), \frac{t}{4})\\ &= \mu(y_n - x_n, \frac{t}{2^{\delta+1}}) * \mu(x_n - x_m, \frac{t}{4^{\delta+1}}) * \mu(x_m - y_m, \frac{t}{4^{\delta+1}})\\ &> (1 - \frac{1}{n+1}) * (1 - r) * (1 - \frac{1}{m+1}).\end{aligned}$$

Since $*$ is continuous $\mu(y_n - y_m, t) \to 1$, Furthermore

$$\begin{aligned}\nu(y_n - y_m, t) &\leq \nu(2(y_n - x_n), \frac{t}{2}) \diamond \nu(2(x_n - y_n), \frac{t}{2})\\ &\leq \nu(2(y_n - x_n), \frac{t}{2}) \diamond \nu(4(x_n - x_m), \frac{t}{4}) \diamond \nu(4(x_m - y_m), \frac{t}{4})\\ &= \nu(y_n - x_n, \frac{t}{2^{\delta+1}}) \diamond \nu(x_n - x_m, \frac{t}{4^{\delta+1}}) \diamond \nu(x_m - y_m, \frac{t}{4^{\delta+1}})\\ &< \frac{1}{n+1} \diamond r \diamond \frac{1}{m+1}.\end{aligned}$$

Hence $\nu(y_n - y_m, t) \to 0$, that is, $\{y_n\}$ is Cauchy in Y, so it is μ-ν-convergent to $x \in \mathcal{X}$. Thus for each $\varepsilon > 0$ and $t > 0$ there exists $n_1 \in \mathbb{N}$ such that $\mu(x - y_n, \frac{t}{2^{\delta+1}}) > 1 - \varepsilon$ and $\nu(x - y_n, \frac{t}{2^{\delta+1}}) < \varepsilon$ for all $n \geq n_1$. Therefore

$$\begin{aligned}\mu(x_n - x, t) &\geq \mu(2(x_n - y_n), \frac{t}{2}) * \ \mu(2(y_n - x_n), \frac{t}{2})\\ &= \mu(x_n - y_n, \frac{t}{2^{\delta+1}}) * \mu(x_n - y_n, \frac{t}{2^{\delta+1}})\\ &> (1 - \varepsilon) * (1 - \frac{1}{n+1}),\end{aligned}$$

consequently, $\mu(x_n - x, t) \to 1$. Similarly we have

$$\nu(x_n - x, t) \leq \nu(x_n - y_n, \frac{t}{2^{\delta+1}}) \diamond \nu(x_n - y_n, \frac{t}{2^{\delta+1}}) < \varepsilon * \frac{1}{n+1}.$$

Hence $\nu(x_n - x, t) \to 0$, and so the Cauchy sequence $\{x_n\}$ in $\bar{Y}$ converges to $x \in \mathcal{X}$. This completes the proof. □

From Theorem 3.15 we get the following result.

Corollary 3.16. *Let $(\mathcal{X}, \mu, \nu, *, \diamond)$ be a δ-homogeneous intuitionistic $\mathfrak{F}$-modular space and let Y be a dense subset of $\mathcal{X}$. If every μ-ν-Cauchy sequence of Y is μ-ν-convergent in $\mathcal{X}$, then $\mathcal{X}$ is μ-ν-complete.*

Now we extend the well-known Baire's theorem to δ-homogeneous intuitionistic $\mathfrak{F}$-modular spaces.

Theorem 3.17. *Let $\{U_n\}_{n\in\mathbb{N}}$ be a sequence of μ-ν-dense open subsets in δ-homogeneous intuitionistic μ-ν-complete $\mathfrak{F}$-modular space $(\mathcal{X}, \mu, \nu, *_M, \diamond_M)$. Then $\bigcap_{n=1}^{\infty} U_n$ is μ-ν-dense in $\mathcal{X}$.*

Proof. Consider the μ-ν-ball $B(x, r, t)$ and let $y \in B(x, r, t)$. Then $\mu(x - y, 2t) > 1 - r$ and $\nu(x - y, 2t) < r$. Since $\mu(x - y, .)$ and $\nu(x - y, .)$ are continuous, there exists $\varepsilon_y > 0$ such that $\mu(x - y, \frac{t-\varepsilon}{2^{\delta-1}}) > 1 - r$ and

$\nu(x-y,\frac{t-\varepsilon}{2^{\delta-1}})<r$ for some $\varepsilon>0$ with $\frac{t-\varepsilon}{2^{\delta-1}}>0$ and $\frac{\varepsilon}{2^{\delta-1}}\in(0,\varepsilon_y)$. We claim that $\overline{B(y,r',\frac{\varepsilon}{4^\delta})}\subseteq B(x,r,2t)$. Choose $r'\in(0,1)$ and $z\in\overline{B(y,r',\frac{\varepsilon}{4^\delta})}$, then there exists a sequence $\{z_n\}$ in $\overline{B(y,r',\frac{\varepsilon}{4^\delta})}$ which is μ-ν-converges to z, so we have

$$\begin{aligned}\mu(z-y,\frac{\varepsilon}{2^{\delta-1}})&\geq\mu(2(z-z_n),\frac{\varepsilon}{2^\delta})*_M\mu(2(z_n-y),\frac{\varepsilon}{2^\delta})\\&=\mu(z-z_n,\frac{\varepsilon}{4^\delta})*_M\mu(z_n-y,\frac{\varepsilon}{4^\delta})>1-r,\end{aligned}$$

and

$$\begin{aligned}\nu(z-y,\frac{\varepsilon}{2^{\delta-1}})&\leq\nu(2(z-z_n),\frac{\varepsilon}{2^\delta})\diamond_M\nu(2(z_n-y),\frac{\varepsilon}{2^\delta})\\&=\nu(z-z_n,\frac{\varepsilon}{4^\delta})\diamond_M\nu(z_n-y,\frac{\varepsilon}{4^\delta})<r.\end{aligned}$$

Therefore we have

$$\begin{aligned}\mu(x-z,2t)&=\mu(2(z-y),2\varepsilon)*_M\mu(2(x-y),2(t-\varepsilon))\\&=\mu(z-y,\frac{\varepsilon}{2^{\delta-1}})*_M\mu(x-y,\frac{t-\varepsilon}{2^{\delta-1}})\\&\geq(1-r)*_M(1-r)=1-r,\end{aligned}$$

and

$$\begin{aligned}\nu(x-z,2t)&=\mu(2(z-y),2\varepsilon)\diamond_M\nu(2(x-y),2(t-\varepsilon))\\&=\nu(z-y,\frac{\varepsilon}{2^{\delta-1}})\diamond_M\nu(x-y,\frac{t-\varepsilon}{2^{\delta-1}})\\&\leq r\diamond_M r=r.\end{aligned}$$

So the claim is true and hence if V is a nonempty μ-ν-open set of $\mathcal{X}$, then $V\cap U_1$ is nonempty and μ-ν-open. Suppose $x_1\in V\cap U_1$, so there exist $r_1\in(0,1)$ and $t_1>0$ such that $B(x_1,r_1,\frac{t_1}{2^{\delta-1}})\subseteq V\cap U_1$. Choose $r_1'<r_1$ and $t_1'=\min\{t_1,1\}$ such that $B(x_1,r_1',\frac{t_1'}{2^{\delta-1}})\subseteq V\cap U_1$. Since U_2 is μ-ν-dense in $\mathcal{X}$, we have $B(x_1,r_1',\frac{t_1'}{2^{\delta-1}})\cap U_2\neq\emptyset$. Let $x_2\in B(x_1,r_1',\frac{t_1'}{2^{\delta-1}})\cap U_2$, hence there exist $r_2\in(0,\frac{1}{2})$ and $t_2>0$ such that $B(x_2,r_2,\frac{t_2}{2^{\delta-1}})\subseteq B(x_1,r_1',\frac{t_1'}{2^{\delta-1}})\cap U_2$. Choose $r_2'<r_2$ and $t_2'=\min\{t_2,\frac{1}{2}\}$ such that $\overline{B(x_2,r_2',\frac{t_2'}{2^{\delta-1}})}\subseteq V\cap U_2$. By induction, we can obtain a sequence $\{x_n\}$ in $\mathcal{X}$ and two sequences $\{r_n'\}$, $\{t_n'\}$ such that $0<r_n'<\frac{1}{n}$, $0<t_n'<\frac{1}{n}$ and $\overline{B(x_n,r_n',\frac{t_n'}{2^{\delta-1}})}\subseteq V\cap U_n$. We show that $\{x_n\}$ is μ-ν-Cauchy. Get $t>0$ and $r\in(0,1)$, then we can choose $k\in\mathbb{N}$ such that $2t_k'<t$ and $r_k'<r$. Since $x_m,x_n\in B(x_k,r_k',\frac{t_k'}{2^{\delta-1}})$, for $m,n\geq k$, we get

$$\begin{aligned}\mu(x_m-x_n,2t)&\geq\mu(x_m-x_n,4t_k')\\&\geq\mu(2(x_m-x_k),2t_k')*_M\mu(2(x_k-x_n),2t_k')\\&=\mu(x_m-x_k,\frac{t_k'}{2^{\delta-1}})*_M\mu(x_k-x_n,\frac{t_k'}{2^{\delta-1}})\\&\geq(1-r_k)*_M(1-r_k)>1-r,\end{aligned}$$

and

$$\begin{aligned}\nu(x_m-x_n,2t)&\leq\nu(x_m-x_n,4t_k')\\&\leq\nu(2(x_m-x_k),2t_k')\diamond_M\nu(2(x_k-x_n),2t_k')\\&=\nu(x_m-x_k,\frac{t_k'}{2^{\delta-1}})\diamond_M\nu(x_k-x_n,\frac{t_k'}{2^{\delta-1}})\\&\leq r_k\diamond_M r_k<r.\end{aligned}$$

Therefore $\{x_n\}$ is a μ-ν-Cauchy sequence. Since $\mathcal{X}$ is μ-ν-complete, there exists $x \in \mathcal{X}$ such that $x_n \to x$. For all $n \geq k$, $x_n \in B(x_k, r'_k, \frac{t'_k}{2^{\delta-1}})$ and hence $x \in \overline{B(x_k, r'_k, \frac{t'_k}{2^{\delta-1}})} \subseteq V \cap U_k$. This implies that $V \cap (\bigcap_{n=1}^{\infty} U_n) \neq \emptyset$. Therefore $\bigcap_{n=1}^{\infty} U_n$ is μ-ν-dense in $\mathcal{X}$. □

Finally, we give the uniform limit theorem in δ-homogeneous intuitionistic $\mathfrak{F}$-modular spaces. Let $\mathcal{X}$ be a nonempty set and let $(\mathcal{Y}, \mu, \nu, *, \diamond)$ be an intuitionistic $\mathfrak{F}$-modular space. A sequence $\{f_n\}$ of mappings from $\mathcal{X}$ to $\mathcal{Y}$ is called μ-ν-converges uniformly to a mapping $f : \mathcal{X} \to \mathcal{Y}$ if, for $t > 0$ and $r \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that $\mu(f_n(x) - f(x), t) > 1 - r$ and $\nu(f_n(x) - f(x), t) < r$, for all $n \geq n_0$ and $x \in \mathcal{X}$.

Theorem 3.18. *Let $\{f_n\}$ be a sequence of continuous mappings from a topological space $\mathcal{X}$ to a δ-homogeneous intuitionistic $\mathfrak{F}$-modular space $(\mathcal{Y}, \mu, \nu, *, \diamond)$. If $\{f_n\}$ μ-ν-convergent uniformly to $f : \mathcal{X} \to \mathcal{Y}$, then f is continuous.*

Proof. Let V be a μ-ν-open set of $\mathcal{Y}$ and $x_0 \in f^{-1}(V)$, so there exist $t > 0$ and $r \in (0,1)$ such that $B(f(x_0), r, t) \subset V$. For $r \in (0,1)$, we can choose $s \in (0,1)$ such that $*(1-s)*(1-s) > 1-r$. Since $\{f_n\}$ μ-ν-converges uniformly to f, for $s \in (0,1)$ and $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $\mu(f_n(x) - f(x), \frac{t}{4^{\delta+1}}) > 1 - s$ and $\nu(f_n(x) - f(x), \frac{t}{4^{\delta+1}}) < s$ for all $n \geq n_0$ and $x \in \mathcal{X}$. Furthermore, each f_n is continuous. Then there exists a neighborhood U of x_0 such that $f_n(U) \subset B(f_n(x_0), s, \frac{t}{4^{\delta+1}})$. Therefore $\mu(f_n(x) - f(x_0), \frac{t}{4^{\delta+1}}) > 1 - s$ and $\nu(f_n(x) - f(x_0), \frac{t}{4^{\delta+1}}) < s$ for all $n \geq n_0$ and $x \in U$ and so we have

$$\begin{aligned}
\mu(f(x) - f_n(x_0), t) &\geq \mu(2(f(x) - f_n(x)), \frac{t}{2}) * \mu(2(f_n(x) - f(x_0)), \frac{t}{2}) \\
&= \mu(f(x) - f_n(x), \frac{t}{2^{\delta+1}}) * \mu(2(f_n(x) - f(x_0)), \frac{t}{2^{\delta+1}}) \\
&\geq \mu(f(x) - f_n(x), \frac{t}{2^{\delta+1}}) * \mu(2(f_n(x) - f_n(x_0)), \frac{t}{2^{\delta+2}}) * \mu(2(f_n(x_0) - f(x_0)), \frac{t}{2^{\delta+2}}) \\
&= \mu(f(x) - f_n(x), \frac{t}{2^{\delta+1}}) * \mu(f_n(x) - f_n(x_0), \frac{t}{4^{\delta+1}}) * \mu(f_n(x_0) - f(x_0), \frac{t}{4^{\delta+1}}) \\
&\geq (1-s) * (1-s) * (1-s) > 1 - r.
\end{aligned}$$

and

$$\begin{aligned}
\nu(f(x) - f_n(x_0), t) &\leq \nu(2(f(x) - f_n(x)), \frac{t}{2}) \diamond \nu(2(f_n(x) - f(x_0)), \frac{t}{2}) \\
&= \nu(f(x) - f_n(x), \frac{t}{2^{\delta+1}}) \diamond \nu(2(f_n(x) - f(x_0)), \frac{t}{2^{\delta+1}}) \\
&\leq \nu(f(x) - f_n(x), \frac{t}{2^{\delta+1}}) \diamond \nu(2(f_n(x) - f_n(x_0)), \frac{t}{2^{\delta+2}}) \diamond \nu(2(f_n(x_0) - f(x_0)), \frac{t}{2^{\delta+2}}) \\
&= \nu(f(x) - f_n(x), \frac{t}{2^{\delta+1}}) \diamond \nu(f_n(x) - f_n(x_0), \frac{t}{4^{\delta+1}}) \diamond \nu(f_n(x_0) - f(x_0), \frac{t}{4^{\delta+1}}) \\
&\leq s \diamond s \diamond s < r.
\end{aligned}$$

This implies that $f(x) \in B(f(x_0), r, t) \subset V$, therefore $f(U) \subseteq V$, hence f is continuous. □

4 Conclusion

In this paper, we have proposed the concept of an intuitionistic fuzzy modular space based on the modular space and continuous t-norm and t-conorm, which can be regarded as a generalization of a modular space in the intuitionistic fuzzy sense. We first deal with the problem of whether there is a relationship between an intuitionistic fuzzy modular and an intuitionistic fuzzy metric. In the sequel, we have defined a Hausdorff topology induced by a δ-homogeneous fuzzy modular and examined some related topological properties.

Finally, we have extended the well-known Baire's theorem and the uniform limit theorem to δ-homogeneous intuitionistic fuzzy modular spaces.

Conflict of Interest: The author declares no conflict of interest.

References

[1] Atanassov K. Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* 1986; 20: 87-96. DOI: http://doi.org/10.1016/S0165-0114(86)80034-3

[2] Coker D. An introduction to intuitionistic fuzzy topological spaces. *Fuzzy Sets Syst.* 1997; 88: 81-89. DOI: http://doi.org/10.1016/S0165-0114(96)00076-0

[3] El Naschie MS. On the uncertainty of Cantorian geometry and two-slit experiment. *Chaos, Soliton & Fractals*. 1998; 9(3): 517-529. DOI: http://doi.org/10.1016/S0960-0779(97)00150-1

[4] El Naschie MS. On the verifications of heterotic strings theory and $\epsilon^{(\infty)}$ theory. *Chaos, Soliton & Fractals*. 2000; 11(2): 2397-2407. DOI: http://doi.org/10.1016/S0960-0779(00)00108-9

[5] Grabiec M. Fixed points in fuzzy metric spaces. *Fuzzy Sets Syst.* 1988; 27: 385-389. DOI: http://doi.org/10.1016/0165-0114(88)90064-4

[6] George A, Veeramani P. On some results in fuzzy metric spaces. *Fuzzy Sets Syst.* 1994; 64: 395-399. DOI: http://doi.org/10.1016/0165-0114(94)90162-7

[7] Kilmer SJ, Kozlowski WM, Lewicki G. Best approximants in modular function spaces. *J. Approximation Theory*. 1990; 63(3): 338-367. DOI: http://doi.org/10.1016/0021-9045(90)90126-B

[8] Kozlowski WM. Notes on modular function spaces I. *Comment. Math.* 1988; 28(1): 87-100.

[9] Kozlowski WM. Notes on modular function spaces II. *Comment. Math.* 1988; 28(1): 101-116.

[10] Kozlowski WM, Lewicki G. Analyticity and polynomial approximation in modular function spaces. *J. Approximation Theory*. 1989; 58(1): 15-35. DOI: http://doi.org/10.1016/0021-9045(89)90004-X

[11] Kramosil I, Michálek J. Fuzzy metric and statistical metric spaces. *Kybernetika*. 1975; 11(5): 336-344.

[12] Luxemburg WA. Banach function spaces. *Ph.D. thesis, Delft University of Technology, Delft, The Netherlands*. 1959. DOI: http://doi.org/10.1016/s0079-8169%2808%2960846-6

[13] Musielak J, Orlicz W. On modular Spaces. *Studia Math.* 1959; 18: 49-56. DOI: http://doi.org/10.4064/SM-18-1-49-65

[14] Nakano H. *Modulared Semi-Ordered Linear Spaces*. Tokyo Math. Book Ser. 1, Maruzen Co. Tokyo; 1950.

[15] Nourouzi K. Probabilistic modular spaces. *Proceedings of the 6th International ISAAC Congress, Ankara, Turkey*. 2007. DOI: http://doi.org/10.1142/9789812837332_0077

[16] Nourouzi K. Baire's theorem in probabilistic modular spaces. *Proceedings of the World Congress on Engineering (WCE 08)*. 2008; 2: 916-917.

[17] Park JH. Intuitionistic fuzzy metric spaces. *Chaos Solitons Fractals*. 2004; 22: 1039-1046. DOI: http://doi.org/10.1016/j.chaos.2004.02.051

[18] Schweizer B, Sklar A. Statistical metric spaces. *Pac. J. Math.* 1960; 10: 314-334. DOI: http://doi.org/10.2140/PJM.1960.10.313

[19] Shen Y, Chen W. On fuzzy modular spaces. *J. Appl. Math, Article ID 576237.* 2013; 2013: 1-8. DOI: http://doi.org/10.1155/2013/576237

[20] Zadeh LA. Fuzzy sets. *Inf. Cont.* 1965; 8: 338-353. DOI: http://doi.org/10.1016/S0019-9958(65)90241-X

Tayebe Lal Shateri
Department of Mathematics and Computer Sciences
Hakim Sabzevari University
Sabzevar, Iran
E-mail: t.shateri@hsu.ac.ir

Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://tfss.journals.iau.ir/
Online ISSN: 2821-0131
Vol.2, No.2, (2023), 127-136
DOI: http://10.30495/tfss.2023.1986166.1070

Article Type: Original Research Article

Completeness for Saturated L-Quasi-Uniform Limit Spaces

Gunther Jäger*

Abstract. We define and study two completeness notions for saturated L-quasi-uniform limit spaces. The one, that we term Lawvere completeness, is defined using the concept of promodule and lends a lax algebraic interpretation of completeness also for saturated L-quasi-uniform limit spaces. The other, termed Cauchy completeness, is defined using saturated Cauchy pair prefilters. We show that both concepts coincide with related notions in the case of saturated L-quasi-uniform spaces and that also for saturated L-quasi-uniform limit spaces, both completeness notions are equivalent.

AMS Subject Classification 2020: 54A20; 54A40; 54B30; 54E15
Keywords and Phrases: Saturated prefilter, Saturated L-quasi-uniform limit space, Completeness.

1 Introduction

Generalizing an approach in [2], completeness has recently been studied from a categorical point of view for different kinds of many-valued quasi-uniform (convergence) spaces, [12, 13, 14]. This paper adds to these investigations by considering many-valued quasi-uniform limit spaces based on saturated L-prefilters. These spaces are a slight generalization of $\top$-uniform limit spaces [6, 7, 9] and of probabilistic quasi-uniform spaces [5, 14]. We define a completeness notion using adjoint promodules, thus providing a categorical framework for completeness. Also, we define completeness with the help of saturated pair L-prefilters. The main result of the paper shows that both these approaches are equivalent.

The paper is organized as follows. In the second section we collect the necessary concepts about lattices, L-subsets, saturated L-prefilters and prorelations. The third section studies saturated L-quasi-uniform limit spaces and promodules. Sections 4 and 5 are devoted to the two concepts of completeness studied in this paper. Finally, we draw some conclusions.

2 Preliminaries

In this paper, we will consider *commutative and integral quantales* $\mathsf{L} = (L, \leq, *)$. Here, $(L, \leq)$ is a complete lattice with distinct top and bottom elements $\top \neq \bot$, $(L, *)$ is a commutative semigroup with the top element of L as the unit, that is, $\alpha * \top = \alpha$ for all $\alpha \in L$, and $*$ is distributive over arbitrary joins, that is, $(\bigvee_{i \in J} \alpha_i) * \beta = \bigvee_{i \in J}(\alpha_i * \beta)$ for all $\alpha_i, \beta \in L$, $i \in J$, see for example [4].

The *implication* in a quantale is defined by $\alpha \to \beta = \bigvee\{\delta \in L \ : \ \delta * \alpha \leq \beta\}$ and characterized by $\delta \leq \alpha \to \beta$ if and only if $\delta * \alpha \leq \beta$.

***Corresponding Author:** Gunther Jäger, Email: gunther.jaeger@hochschule-stralsund.de, ORCID: 0000-0002-1495-4564
Received: 14 May 2023; **Revised**: 9 July 2023; **Accepted**: 9 July 2023; **Available Online**: 10 July 2023; **Published Online**: 7 November 2023.

How to cite: Jäger G. Completeness for Saturated L-Quasi-Uniform Limit Spaces. *Trans. Fuzzy Sets Syst.* 2023; 2(2): 127-136. DOI: 10.30495/TFSS.2023.1986166.1070

Typical examples of commutative and integral quantales are $\mathsf{L} = ([0,1], \leq, *)$ with a left-continuous t-norm on $[0,1]$ or *Lawvere's quantale* $\mathsf{L} = ([0,\infty], \geq, +)$. Another example is given by the *quantale of distance distribution functions* $\mathsf{L} = (\Delta^+, \leq, *)$, where Δ^+ is the set of all distance distribution functions $\varphi : [0,\infty] \longrightarrow [0,1]$ which are left-continuous in the sense that $\varphi(x) = \sup_{y<x} \varphi(y)$ for all $x \in [0,\infty]$ and $*$ is a *sup-continuous triangle function*, see [3, 11].

An *L-subset of X* is a mapping $a : X \longrightarrow L$ and we denote the set of L-subsets of X by L^X. For $A \subseteq X$ we define $\top_A \in L^X$ by $\top_A(x) = \top$ if $x \in A$ and $= \bot$ otherwise. The lattice operations are extended pointwisely from L to L^X. For a mapping $\varphi : X \longrightarrow Y$ and $a \in L^X$ and $b \in L^Y$ we define $\varphi(a) \in L^Y$ by $\varphi(a)(y) = \bigvee_{\varphi(x)=y} a(x)$ for $y \in Y$ and $\varphi^{\leftarrow}(b) = b \circ \varphi \in L^X$.

For L-subsets $u \in L^{X\times Y}$ and $v \in L^{Y\times Z}$, we define $v \circ u \in L^{X\times Z}$ by $v \circ u(x,z) = \bigvee_{y\in Y} u(x,y) * v(y,z)$ for all $x \in X$ and $z \in Z$.

For $a, b \in L^X$ we denote the *fuzzy inclusion order* $[a,b] = \bigwedge_{x\in X}(a(x) \to b(x))$, [1]. The following properties are well-known.

Lemma 2.1. *Let $a, a', b, b', c \in L^X$, $d \in L^Y$, $u_1, u_2 \in L^{X\times Y}$, $, v_1, v_2 \in L^{Y\times Z}$ and let $\varphi : X \longrightarrow Y$ be a mapping. Then*

(i) $a \leq b$ *if and only if* $[a,b] = \top$*;*

(ii) $a \leq a'$ *implies* $[a',b] \leq [a,b]$ *and* $b \leq b'$ *implies* $[a,b] \leq [a,b']$*;*

(iii) $[a,c] \wedge [b,c] = [a \vee b, c]$*;*

(iv) $[\varphi(a), d] = [a, \varphi^{\leftarrow}(d)]$*;*

(v) $[u_1, v_1] * [u_2, v_2] \leq [u_2 \circ u_1, v_2 \circ v_1]$.

Definition 2.2. [5, 14] A subset $\mathbb{F} \subseteq L^X$ is called a *saturated* L*-prefilter* (on X) if

(SP1) $\top_X \in \mathbb{F}$;

(SP2) $a, b \in \mathbb{F}$ implies $a \wedge b \in \mathbb{F}$;

(SP3) $\bigvee_{b\in\mathbb{F}}[b,c] = \top$ implies $c \in \mathbb{F}$.

We denote the set of all saturated L-prefilters on X by $\mathsf{F}_{\mathsf{L}}^{\mathsf{sat}}(X)$ and we use the subsethood order on $\mathsf{F}_{\mathsf{L}}^{\mathsf{sat}}(X)$.

The condition (SP3) implies $a \leq b, a \in \mathbb{F} \Longrightarrow b \in \mathbb{F}$. If additionally $\bigvee_{x\in X} a(x) = \top$ for all $a \in \mathbb{F}$, then we speak of a $\top$*-filter* [5, 14].

Example 2.3. For $x \in X$, $[x] = \{a \in L^X : a(x) = \top\}$ is a saturated L-prefilter, the *saturated point* L*-prefilter of* x. We note that $[x]$ is a $\top$-filter. More generally, for an L-set $a \in L^X$, then $[a] = \{b \in L^X : a \leq b\}$ is a saturated L-prefilter and we have, in particular, $[x] = [\top_{\{x\}}]$.

Definition 2.4. [5, 14] A subset $\mathbb{B} \subseteq L^X$ is called a *saturated* L*-prefilter base* (on X) if

(SPB) $a, b \in \mathbb{B}$ implies $\bigvee_{c\in\mathbb{B}}[c, a \wedge b] = \top$.

For a saturated L-prefilter base $\mathbb{B}$, $[\mathbb{B}] = \{a \in L^X : \bigvee_{b\in\mathbb{B}}[b,a] = \top\}$ is the saturated L-prefilter generated by $\mathbb{B}$.

For a saturated L-prefilter $\mathbb{F} \in \mathsf{F}_{\mathsf{L}}^{\mathsf{sat}}(X)$ and a mapping $\varphi : X \longrightarrow Y$, the set $\mathbb{B} = \{\varphi(a) : a \in \mathbb{F}\}$ is a saturated L-prefilter base on Y and we denote $\varphi(\mathbb{F})$ the generated saturated L-prefilter on Y, the *image of* $\mathbb{F}$ *under* φ, see e.g. [5].

A *prorelation (from X to Y)* is a set of saturated L-prefilters $\Phi \subseteq \mathsf{F}_{\mathsf{L}}^{\mathsf{sat}}(X \times Y)$ which satisfies the axioms

(PR1) $\mathbb{F} \leq \mathbb{G}$, $\mathbb{F} \in \Phi$ implies $\mathbb{G} \in \Phi$;

(PR2) $\mathbb{F}, \mathbb{G} \in \Phi$ implies $\mathbb{F} \wedge \mathbb{G} \in \Phi$.

For $\mathbb{F} \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X \times Y)$ the set $[\mathbb{F}] = \{\mathbb{K} \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X \times Y) \,:\, \mathbb{F} \leq \mathbb{K}\}$ is a prorelation.

We consider now two prorelations $\Phi \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X \times Y)$ and $\Psi \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(Y \times Z)$ and define

$$\Psi \circ \Phi = \{\mathbb{H} \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X \times Z) \,:\, \exists \mathbb{F} \in \Phi, \mathbb{G} \in \Psi \text{ s.t. } \mathbb{G} \circ \mathbb{F} \leq \mathbb{H}\}.$$

Here, it is defined $\mathbb{G} \circ \mathbb{F} = [\{g \circ f \,:\, g \in \mathbb{G}, f \in \mathbb{F}\}]$ with $g \circ f(x,z) = \bigvee_{y \in Y} f(x,y) * g(y,z)$ for all $x \in X, z \in Z$. It is straightforward to show that $\Psi \circ \Phi$ is a prorelation from X to Z.

We denote $\Delta_X = \{(x,x) \,:\, x \in X\} \subseteq X \times X$. Then $[\top_{\Delta_X}] \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X \times X)$ and hence $[[\top_{\Delta_X}]]$ is a prorelation from X to X.

Proposition 2.5. *For a prorelation $\Phi \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X \times Y)$, we have $\Phi \circ [[\top_{\Delta_X}]] = \Phi$ and $[[\top_{\Delta_Y}]] \circ \Phi = \Phi$.*

Proof. Let $\mathbb{H} \in \Phi \circ [[\top_{\Delta_X}]]$. Then there is $\mathbb{F} \in \Phi$ such that $\mathbb{F} \circ [\top_{\Delta_X}] \leq \mathbb{H}$. For $f \in \mathbb{F}$ we have $f \circ \top_{\Delta_X}(x,y) = \bigvee_{z \in X} \top_{\Delta_X}(x,z) * f(z,y) = f(x,y)$ and hence we conclude that $g \in \mathbb{F} \circ [\top_{\Delta_X}]$ if and only if $\top = \bigvee_{f \in \mathbb{F}} [f \circ \top_{\Delta_X}, g] = \bigvee_{f \in \mathbb{F}} [f, g]$ if and only if $g \in \mathbb{F}$, as $\mathbb{F}$ is a saturated L-prefilter. Hence, $\mathbb{F} = \mathbb{F} \circ [\top_{\Delta_X}] \leq \mathbb{H}$ and we have $\mathbb{H} \in \Phi$ by (PR1). Conversely, for $\mathbb{F} \in \Phi$ we have $\mathbb{F} = \mathbb{F} \circ [\top_{\Delta_X}] \in \Phi \circ [[\top_{\Delta_X}]]$.

The second equation can be shown in a similar way. □

For $f \in L^{X \times Y}, g \in L^{Y \times Z}$ and $h \in L^{Z \times U}$ it is not difficult to show that $h \circ (g \circ f) = (h \circ g) \circ f$. From this we conclude $\mathbb{H} \circ (\mathbb{G} \circ \mathbb{F}) = (\mathbb{H} \circ \mathbb{G}) \circ \mathbb{F}$ for saturated L-prefilters $\mathbb{F} \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X \times Y), \mathbb{G} \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(Y \times Z), \mathbb{H} \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(Z \times U)$ and we obtain

Proposition 2.6. *For prorelations $\Phi \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X \times Y), \Psi \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(Y \times Z)$ and $\Theta \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(Z \times U)$ we have $(\Phi \circ \Psi) \circ \Theta = \Phi \circ (\Psi \circ \Theta)$.*

Consider now a mapping $\varphi : X \longrightarrow Y$. We define the L-relation (and denote it again by φ), $\varphi(x,y) = \top$ if $y = \varphi(x)$ and $\varphi(x,y) = \bot$ otherwise. Similarly, the opposite L-relation φ° is defined by $\varphi^\circ(y,x) = \top$ if $y = \varphi(x)$ and $\varphi^\circ(y,x) = \bot$ otherwise. Hence, $\varphi \in L^{X \times Y}$ and $\varphi^\circ \in L^{Y \times X}$ and therefore $[\varphi] \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X \times Y)$ and $[\varphi^\circ] \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(Y \times X)$ and we obtain prorelations $[[\varphi]] \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X \times Y)$ and $[[\varphi^\circ]] \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(Y \times X)$.

If $\varphi : X \longrightarrow Y$ and $\psi : Y \longrightarrow Z$, then it is not difficult to show that $[\psi \circ \varphi] = [\psi] \circ [\varphi]$. From this we immediately conclude $[[\psi]] \circ [[\varphi]] = [[\psi \circ \varphi]]$.

Proposition 2.7. *Let $\varphi : X \longrightarrow Y$. Then $[[\varphi]] \circ [[\varphi^\circ]] \subseteq [[\top_{\Delta_Y}]]$ and $[[\top_{\Delta_X}]] \subseteq [[\varphi^\circ]] \circ [[\varphi]]$.*

Proof. We have, for $y, y' \in Y$, $\varphi \circ \varphi^\circ(y,y') = \bigvee_{x \in X} \varphi^\circ(y,x) * \varphi(x,y') = \top$ if $y' = \varphi(x) = y$ for some $x \in X$ and $= \bot$ otherwise. Hence $\varphi \circ \varphi^\circ \leq \top_{\Delta_Y}$ which implies $[\top_{\Delta_Y}] \leq [\varphi \circ \varphi^\circ]$ and hence $[[\varphi]] \circ [[\varphi^\circ]] = [[\varphi \circ \varphi^\circ]] \subseteq [[\top_{\Delta_Y}]]$.

Similarly, we have, for $x, x' \in X$ that $\varphi^\circ \circ \varphi(x,x') = \bigvee_{y \in Y} \varphi(x,y) * \varphi^\circ(y,x') = \top$ if $\varphi(x') = \varphi(x)$ and $= \bot$ otherwise. Hence $\top_{\Delta_X} \leq \varphi^\circ \circ \varphi$, implying $[\top_{\Delta_X}] \geq [\varphi^\circ \circ \varphi]$. From this we conclude $[[\top_{\Delta_X}]] \subseteq [[\varphi^\circ \circ \varphi]] = [[\varphi^\circ]] \circ [[\varphi]]$. □

Lemma 2.8. *Let $\varphi : X \longrightarrow Y$ and $b \in L^{X \times X}$. Then $(\varphi \times \varphi)^{\leftarrow}(b) = \varphi^\circ \circ b \circ \varphi$.*

Proof. For all $x, x' \in X$ we have $(\varphi^\circ \circ b) \circ \varphi(x,x') = \bigvee_{y \in Y} (\varphi^\circ \circ b)(y,x') * \varphi(x,y) = \bigvee_{y \in Y} \bigvee_{x : \varphi(x) = y} \varphi^\circ \circ b(y,x') = \bigvee_{x \in X} \varphi^\circ \circ b(\varphi(x),x') = \bigvee_{y \in Y} \varphi^\circ(y,x') * b(\varphi(x),y) = b(\varphi(x),\varphi(x')) = (\varphi \times \varphi)^{\leftarrow}(b)(x,x')$. □

Lemma 2.9. *Let $\varphi : X \longrightarrow Y$ and $\mathbb{H} \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X \times X)$. Then we have, for $b \in L^{X \times Y}$, that $b \in [\varphi] \circ \mathbb{H}$ if, and only if, $\varphi^\circ \circ b \in \mathbb{H}$.*

Proof. We have with Lemma 2.1 (v), noting $[\varphi^\circ, \varphi^\circ] = \top = [\varphi, \varphi]$, for $h \in \mathbb{H}$,

$$[\varphi \circ h, b] \leq [\varphi^\circ \circ \varphi \circ h, \varphi^\circ \circ b] \leq [h, \varphi^\circ \circ b] \leq [\varphi \circ h, \varphi \circ \varphi^\circ \circ b] \leq [\varphi \circ h, b].$$

We conclude that $b \in [\varphi] \circ \mathbb{H}$ if, and only if, $\top = \bigvee_{h \in \mathbb{H}} [\varphi \circ h, b] = \bigvee_{h \in \mathbb{H}} [h, \varphi^\circ \circ b]$ if, and only if, $\varphi^\circ \circ b \in \mathbb{H}$. □

Lemma 2.10. *Let $\varphi : X \longrightarrow Y$ and $\mathbb{H} \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X \times X)$. Then we have, for $a \in L^{Y \times X}$, that $a \in \mathbb{H} \circ [\varphi^\circ]$ if, and only if, $a \circ \varphi \in \mathbb{H}$.*

Proof. Similar as in the last proof, we have, for $h \in \mathbb{H}$,

$$[h \circ \varphi^\circ, a] \leq [h \circ \varphi^\circ \circ \varphi, a \circ \varphi] \leq [h, a \circ \varphi] \leq [h \circ \varphi^\circ, a \circ \varphi \circ \varphi^\circ] \leq [h \circ \varphi^\circ, a].$$

We conclude that $a \in \mathbb{G} \circ [\varphi^\circ]$ if, and only if, $\top = \bigvee_{h \in \mathbb{H}} [h \circ \varphi^\circ, a] = \bigvee_{h \in \mathbb{H}} [h, a \circ \varphi]$ if, and only if, $a \circ \varphi \in \mathbb{H}$. □

Proposition 2.11. *For $\mathbb{H} \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X \times X)$ and $\varphi : X \longrightarrow Y$ we have $(\varphi \times \varphi)(\mathbb{H}) = [\varphi] \circ \mathbb{H} \circ [\varphi^\circ]$.*

Proof. We have $b \in [\varphi] \circ \mathbb{H} \circ [\varphi^\circ]$ if, and only if, $\varphi^\circ \circ b \in \mathbb{H} \circ [\varphi^\circ]$ if, and only if, $(\varphi \times \varphi)^{\leftarrow}(b) = \varphi^\circ \circ b \circ \varphi \in \mathbb{H}$ if, and only if, $b \in (\varphi \times \varphi)(\mathbb{H})$. □

3 Saturated L-Quasi-Uniform Limit Spaces and Promodules

Definition 3.1. Let X be a set and let $\Lambda \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X \times X)$. The pair (X, Λ) is called a *saturated* L*-quasi-uniform limit space* if

(SLUL1) $[\top_{\Delta_X}] \in \Lambda$;

(SLUL2) $\mathbb{H} \in \Lambda$, $\mathbb{H} \leq \mathbb{K}$ implies $\mathbb{K} \in \Lambda$;

(SLUL3) $\mathbb{H}, \mathbb{K} \in \Lambda$ implies $\mathbb{H} \wedge \mathbb{K} \in \Lambda$;

(SLUL4) $\mathbb{H}, \mathbb{K} \in \Lambda$ implies $\mathbb{H} \circ \mathbb{K} \in \Lambda$.

A mapping $\varphi : (X, \Lambda) \longrightarrow (X', \Lambda')$ is called *uniformly continuous* if $(\varphi \times \varphi)(\mathbb{H}) \in \Lambda'$ whenever $\mathbb{H} \in \Lambda$.

The axioms (SLUL2) and (SLUL3) show that Λ is a prorelation from X to X that satisfies, via (SLUL1) and (SLUL4), the additional axioms

$$[[\top_{\Delta_X}]] \subseteq \Lambda \quad \text{and} \quad \Lambda \circ \Lambda \subseteq \Lambda.$$

Uniform continuity of a mapping can be characterized as follows.

Proposition 3.2. *Let (X, Λ) and (X', Λ') be saturated* L*-quasi-uniform limit spaces and $\varphi : X \longrightarrow X'$ be a mapping. The following statements are equivalent.*
(1) φ is uniformly continuous.
(2) $[[\varphi]] \circ \Lambda \subseteq \Lambda' \circ [[\varphi]]$.
(3) $\Lambda \circ [[\varphi^\circ]] \subseteq [[\varphi^\circ]] \circ \Lambda'$.

Proof. We first show that (1) implies (2). Let φ be uniformly continuous and let $\mathbb{K} \in [[\varphi]] \circ \Lambda$. Then $\mathbb{K} \geq [\varphi] \circ \mathbb{H}$ for some $\mathbb{H} \in \Lambda$ and hence $\mathbb{K} \circ [\varphi^\circ] \geq [\varphi] \circ \mathbb{H} \circ [\varphi^\circ] = (\varphi \times \varphi)(\mathbb{H}) \in \Lambda'$. We conclude $\mathbb{K} = \mathbb{K} \circ [\top_{\Delta_X}] \geq \mathbb{K} \circ [\varphi^\circ] \circ [\varphi] \in \Lambda' \circ [[\varphi]]$ and we have $\mathbb{K} \in \Lambda' \circ [[\varphi]]$.

Now we show that (2) implies (3).Let $\mathbb{K} \in \Lambda \circ [[\varphi^\circ]]$. Then $\mathbb{K} \geq \mathbb{H} \circ [\varphi^\circ]$ for some $\mathbb{H} \in \Lambda$. Hence $[\varphi] \circ \mathbb{K} \geq [\varphi] \circ \mathbb{H} \circ [\varphi^\circ] \in \Lambda' \circ [[\varphi]] \circ [[\varphi^\circ]] \subseteq \Lambda' \circ [[\top_{\Delta_Y}]] = \Lambda'$ and we have that $[\varphi] \circ \mathbb{K} \in \Lambda'$. We conclude $\mathbb{K} = [\top_{\Delta_X}] \circ \mathbb{K} \geq [\varphi^\circ] \circ [\varphi] \circ \mathbb{K} \in [[\varphi^\circ]] \circ \Lambda'$ and we have $\mathbb{K} \in [[\varphi^\circ]] \circ \Lambda'$.

Finally we show that (3) implies (1). Let $\mathbb{H} \in \Lambda$. Then $(\varphi \times \varphi)(\mathbb{H}) = [\varphi] \circ \mathbb{H} \circ [\varphi^\circ] \in [[\varphi]] \circ \Lambda \circ [[\varphi^\circ]] \subseteq [[\varphi]] \circ [[\varphi^\circ]] \circ \Lambda' \subseteq [[\top_{\Delta_Y}]] \circ \Lambda' = \Lambda'$ and φ is uniformly continuous. □

Example 3.3 ([13])**.** Let X be a set. A saturated L-prefilter $\mathcal{U} \in \mathsf{F}_{\mathsf{L}}^{\mathsf{sat}}(X \times X)$ is called a *saturated* L*-quasi-uniformity* if

(U0) for all $x \in X$ and $u \in \mathcal{U}$ we have $u(x,x) = \top$;

(UC) for all $u \in \mathcal{U}$ we have $\bigvee_{v \in \mathcal{U}} [v \circ v, u] = \top$.

The pair $(X,\mathcal{U})$ is the called a *saturated* L*-quasi-uniform space*. A mapping $\varphi : (X,\mathcal{U}) \longrightarrow (X',\mathcal{U}')$ between the saturated L-quasi-uniform spaces $(X,\mathcal{U}), (X',\mathcal{U}')$ is called *uniformly continuous* if $(\varphi \times \varphi)^{\leftarrow}(v) \in \mathcal{U}$ for all $v \in \mathcal{U}'$.

We note that the conditions (U0) and (UC) are equivalent to (U0') $\mathcal{U} \leq [\top_{\Delta_X}]$ and (UC') $\mathcal{U} \leq \mathcal{U} \circ \mathcal{U}$. Uniform continuity of a mapping $\varphi : (X,\mathcal{U}) \longrightarrow (X',\mathcal{U}')$ can equivalently be expressed by $[\varphi] \circ \mathcal{U} \geq \mathcal{U}' \circ [\varphi]$.

Wang and Yue [13] call a saturated L-quasi-uniform space a fuzzy quasi-uniform space. Also, they use as order on the set of saturated L-prefilters the opposite order of the subsethood order.

For a saturated L-quasi-uniform space $(X,\mathcal{U})$ then $(X, [\mathcal{U}])$ is a saturated L-quasi-uniform limit space and a uniformly continuous mapping $\varphi : (X,\mathcal{U}) \longrightarrow (X',\mathcal{U}')$ is also uniformly continuous as a mapping $\varphi : (X, [\mathcal{U}]) \longrightarrow (X', [\mathcal{U}'])$.

Definition 3.4. Let (X, Λ) and (X', Λ') be saturated L-quasi-uniform limit spaces. A prorelation from X to X', $\Phi \subseteq \mathsf{F}_{\mathsf{L}}^{\mathsf{sat}}(X \times X')$, is called a *promodule* (from (X, Λ) to (X', Λ')) if $\Phi \circ \Lambda \subseteq \Phi$ and $\Lambda' \circ \Phi \subseteq \Phi$.

We note that for a promodule $\Phi = \Phi \circ [[\top_{\Delta_X}]] \subseteq \Phi \circ \Lambda$ and hence we even have $\Phi \circ \Lambda = \Phi$. Similarly we can see also that $\Lambda' \circ \Phi = \Phi$. Also, from (SLUL4) we see that Λ is a promodule from (X, Λ) to (X, Λ).

Example 3.5. Let $\varphi : (X, \Lambda) \longrightarrow (X', \Lambda')$ be uniformly continuous. Then $\varphi_* = \Lambda' \circ [[\varphi]]$ is a promodule from (X, Λ) to (X', Λ') and $\varphi^* = [[\varphi^\circ]] \circ \Lambda'$ is a promodule from from (X', Λ') to (X, Λ). It is easy to see that φ_* and φ^* are prorelations. Furthermore $\varphi_* \circ \Lambda = \Lambda' \circ [[\varphi]] \circ \Lambda \subseteq \Lambda' \circ \Lambda' \circ [[\varphi]] \subseteq \Lambda' \circ [[\varphi]] = \varphi_*$ and, similarly, $\Lambda' \circ [[\varphi_*]] = \Lambda' \circ \Lambda' \circ [[\varphi]] \subseteq \Lambda' \circ [[\varphi]] = \varphi_*$. The proof that φ^* is a promodule is similar and not shown.

Definition 3.6. Let (X, Λ) and (X', Λ') be saturated L-quasi-uniform limit spaces, let $\Phi \subseteq \mathsf{F}_{\mathsf{L}}^{\mathsf{sat}}(X, X')$ be a promodule from (X, Λ) to (X', Λ') and let $\Psi \subseteq \mathsf{F}_{\mathsf{L}}^{\mathsf{sat}}(X' \times X)$ be a promodule from from (X', Λ') to (X, Λ). Φ is called *left-adjoint* for Ψ (and Ψ is called *right-adjoint* for Φ) if $\Lambda \subseteq \Psi \circ \Phi$ and $\Phi \circ \Psi \subseteq \Lambda'$. In this case we write $\Phi \dashv \Psi$.

Example 3.7. For a uniformly continuous mapping $\varphi : (X, \Lambda) \longrightarrow (X', \Lambda')$ we have $\varphi_* \dashv \varphi^*$. In fact, we have $\Lambda = \Lambda \circ [[\top_{\Delta_X}]] = \Lambda \circ [[\varphi^\circ]] \circ [[\varphi]] \subseteq [[\varphi^\circ]] \circ \Lambda' \circ [[\varphi]] = [[\varphi^\circ]] \circ \Lambda' \circ \Lambda' \circ [[\varphi]] = \varphi^* \circ \varphi_*$ and also $\varphi_* \circ \varphi^* = \Lambda' \circ [[\varphi]] \circ [[\varphi^\circ]] \circ \Lambda' \subseteq \Lambda' \circ [[\top_{\Delta_Y}]] \circ \Lambda' = \Lambda' \circ \Lambda' = \Lambda'$.

We note that for a promodule $\Psi \subseteq \mathsf{F}_{\mathsf{L}}^{\mathsf{sat}}(X' \times X)$ its left-adjoint $\Phi \subseteq \mathsf{F}_{\mathsf{L}}^{\mathsf{sat}}(X, X')$ is unique. In fact, if we have $\Phi_1 \dashv \Psi$ and $\Phi_2 \dashv \Psi$, then $\Phi_1 = \Phi_1 \circ \Lambda \subseteq \Phi_1 \circ (\Psi \circ \Psi_2) = (\Phi_1 \circ \Psi) \circ \Phi_2 \subseteq \Lambda' \circ \Phi_2 = \Phi_2$. Similarly we see that $\Phi_2 \subseteq \Phi_1$ and hence $\Phi_1 = \Phi_2$. In the same way, also for a promodule $\Phi \subseteq \mathsf{F}_{\mathsf{L}}^{\mathsf{sat}}(X, X')$ its right-adjoint $\Psi \subseteq \mathsf{F}_{\mathsf{L}}^{\mathsf{sat}}(X' \times X)$ is unique.

The following lemma will come in handy later.

Lemma 3.8. *Let (X,Λ) and (X',Λ') be saturated L-quasi-uniform limit spaces, let $\Phi,\Phi' \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X,X')$ be promodules from (X,Λ) to (X',Λ') and let $\Psi,\Psi' \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X'\times X)$ be promodules from from (X',Λ') to (X,Λ). If $\Phi' \subseteq \Phi$ and $\Psi' \subseteq \Psi$, then $\Phi' = \Phi$ and $\Psi' = \Psi$.*

Proof. We have $\Phi' = \Lambda' \circ \Phi' \supseteq (\Phi \circ \Psi) \circ \Phi' \supseteq (\Phi \circ \Psi') \circ \Phi' = \Phi \circ (\Psi' \circ \Phi') \supseteq \Phi \circ \Lambda = \Phi$. Similarly we can show $\Psi \subseteq \Psi'$. □

4 Lawvere Completeness of Saturated L-Quasi-Uniform Limit Spaces

We consider a one-point set $1 = \{\bullet\}$ and the unique saturated L-quasi-uniform limit structure $\Pi = [[\top_{\{(\bullet,\bullet)\}}]]$. A mapping $\varphi : 1 \longrightarrow X$, $\varphi(\bullet) = x$ will be identified with $x \in X$ and we shall write $x : 1 \longrightarrow X$ for it. We note that $x : (1,\Pi) \longrightarrow (X,\Lambda)$ is uniformly continuous: For $\mathbb{H} \geq [\top_{\{(\bullet,\bullet)\}}]$ we find $(\varphi\times\varphi)(\mathbb{H}) \geq (\varphi\times\varphi)([\top_{\{(\bullet,\bullet)\}}]) = [\top_{\{(\varphi(\bullet),\varphi(\bullet))\}}] = [\top_{\{(x,x)\}}] \geq [\top_{\Delta_X}] \in \Lambda$ and hence $(\varphi \times \varphi)(\mathbb{H}) \in \Lambda$.

Definition 4.1. A saturated L-quasi-uniform limit space (X,Λ) is called *Lawvere complete* if for all promodules $\Phi \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(1\times X)$ from $(1,\Pi)$ to (X,Λ), $\Psi \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X\times 1)$ from (X,Λ) to $(1,\Pi)$ with $\Phi \dashv \Psi$ there is $x \in X$ such that $\Phi = x_*$ and $\Psi = x^*$.

In the sequel, we want to identify $X \times 1$ and $1 \times X$ with X. This leads to some adaptation in the concepts and definitions. For a mapping $x : 1 \longrightarrow X$ we note that $x(\bullet, y) = \top$ if and only if $x = x(\bullet) = y$ and $x(\bullet, y) = \bot$ otherwise. Hence, $x(\bullet, y) = \top_{\{x\}}(y)$ and we can write $x_* = \Lambda \circ [[x]]$ with the saturated point L-prefilter $[x]$. Similarly, $x^\circ(y,\bullet) = \top$ if $x = x(\bullet) = y$ and $x^\circ(y,\bullet) = \bot$ otherwise, so that also $x^* = [[x]] \circ \Lambda$.

More generally, for $\mathbb{F} \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X\times 1)$ (or, similarly, for $\mathbb{F} \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(1\times X)$) we identify $f \in \mathbb{F}$ with an L-subset of X (denoted again by f) via $f(x) = f(x,\bullet)$. In this sense, we define for $\mathbb{H} \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X\times X)$ and $\mathbb{F} \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X)$

$$\mathbb{H} \circ \mathbb{F} = [\{h \circ f \ : \ h \in \mathbb{H}, f \in \mathbb{F}\}]$$

with $h \circ f(x) = h \circ f(\bullet, x) = h \circ f(\bullet, x) = \bigvee_{y\in X} f(\bullet,y) * h(y,x) = \bigvee_{y\in X} f(y) * h(y,x)$ for all $x \in X$. Similarly, we define

$$\mathbb{F} \circ \mathbb{H} = [\{f \circ h \ : \ f \in \mathbb{F}, h \in \mathbb{H}\}]$$

with $f \circ h(x) = f \circ h(x,\bullet) = \bigvee_{y\in X} h(x,y) * f(y,\bullet) = \bigvee_{y\in X} h(x,y) * f(y)$.

A promodule $\Phi \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(1\times X)$ from $(1,\Pi)$ to (X,Λ) then satisfies the conditions $\Phi\circ\Pi \subseteq \Phi$ and $\Lambda\circ\Phi \subseteq \Phi$. We note that the first of these conditions is always satisfied: $\Phi \circ \Pi = \Phi \circ [[\top_{\{(\bullet,\bullet)\}}]] = \Phi \circ [[\top_{\Delta_1}]] = \Phi$. Hence it is sufficient to demand the condition $\Lambda \circ \Phi \subseteq \Phi$ in this case. Identfying $\Phi \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(1\times X)$ with $\Phi \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X)$, we call a prorelation $\Phi \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X)$ a *left-Λ-promodule* if $\Lambda \circ \Phi \subseteq \Phi$. If the saturated L-quasi-uniform limit space (X,Λ) is clear from the context, we simply speak of a *left-promodule* in this case.

Similarly, for a promodule $\Psi \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X\times 1)$ from (X,Λ) to $(1,\Pi)$ we have the conditions $\Psi \circ \Lambda \subseteq \Psi$ and $\Pi \circ \Psi \subseteq \Psi$ and again the second of these conditions will be always satisfied. We therefore call a prorelation $\Psi \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X)$ a *right-Λ-promodule* if $\Psi \circ \Lambda \subseteq \Psi$. Again, if the saturated L-quasi-uniform limit space (X,Λ) is clear from the context, we simply speak of a *right-promodule*.

For adjoint promodules, we consider prorelations $\Phi,\Psi \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X)$ as promodules (from $(1,\Pi)$ to (X,Λ) for Φ and from (X,Λ) to $(1,\Pi)$ for Ψ). Then, by definition, $\Phi \dashv \Psi$ if and only if $\Phi\circ\Psi \subseteq \Lambda$ and $\Pi \subseteq \Psi\circ\Phi$. The first condition, $\Phi\circ\Psi \subseteq \Lambda$, means that for all $\mathbb{F} \in \Phi$ and all $\mathbb{G} \in \Psi$ we have $\mathbb{F}\circ\mathbb{G} \in \Lambda$. Now we note that for $f \in \mathbb{F}$ and $g \in \mathbb{G}$ we have

$$f \circ g(x,y) = \bigvee_{z\in 1} g(x,z) * f(z,y) = f(\bullet,y) * g(x,\bullet) = f(y) * g(x) = g \otimes f(x,y)$$

and hence, $\mathbb{G} \otimes \mathbb{F} \in \Lambda$ for all $\mathbb{F} \in \Phi$ and all $\mathbb{G} \in \Psi$.

The second condition, $\Pi \subseteq \Psi \circ \Phi$, means that there are $\mathbb{F} \in \Phi$ and $\mathbb{G} \in \Psi$ such that $\mathbb{G} \circ \mathbb{F} \leq [\top_{\{(\bullet,\bullet)\}}]$, that is, that there are $\mathbb{F} \in \Phi$ and $\mathbb{G} \in \Psi$ such that $\top = g \circ f(\bullet,\bullet) = \bigvee_{x\in X} f(\bullet,x) * g(x,\bullet) = \bigvee_{x\in X} f(x) * g(x)$ for all $f \in \mathbb{F}, g \in \mathbb{G}$. So we arrive at the following characterization.

Proposition 4.2. *Let (X,Λ) be a saturated L-quasi-uniform limit space and let $\Phi \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X)$ be a left-promodule and $\Psi \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X)$ be a right-promodule. Then Φ is left-adjoint to Ψ, $\Phi \dashv \Psi$, if, and only if,*
(1) $\mathbb{G} \otimes \mathbb{F} \in \Lambda$ for all $\mathbb{F} \in \Phi$ and all $\mathbb{G} \in \Psi$; and
(2) there are $\mathbb{F} \in \Phi$ and $\mathbb{G} \in \Psi$ such that for all $f \in \mathbb{F}$ and all $g \in \mathbb{G}$ we have $\bigvee_{x\in X} f(x) * g(x) = \top$.

Proposition 4.3. *The saturated L-quasi-uniform limit space (X,Λ) is Lawvere complete if, and only if, for all left-promodules $\Phi \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X)$ and all right-promodules $\Psi \subseteq \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X)$ with $\Phi \dashv \Psi$ there is $x \in X$ such that $\Phi = \Lambda \circ [[x]]$ and $\Psi = [[x]] \circ \Lambda$.*

In [6, 13, 14], for a saturated L-quasi-uniform space $(X,\mathcal{U})$ a prorelation is defined to be a saturated prefilter $\mathbb{H} \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X)$. A prorelation $\mathbb{H}$ is a *left-$\mathcal{U}$-promodule* if $\mathbb{H} \leq \mathcal{U} \circ \mathbb{H}$ and a prorelation $\mathbb{K}$ is a *right-$\mathcal{U}$-promodule* if $\mathbb{K} \leq \mathbb{K} \circ \mathcal{U}$. (Note that in [6] the composition was defined in a different order.) A left-$\mathcal{U}$-promodule $\mathbb{H}$ is *left-adjoint* to the right-$\mathcal{U}$-promodule $\mathbb{K}$, $\mathbb{H} \dashv \mathbb{K}$, if $\mathcal{U} \leq \mathbb{K} \otimes \mathbb{H}$ and $\bigvee_{x\in X} h(x) * k(x) = \top$ for all $h \in \mathbb{H}$ and all $k \in \mathbb{K}$. Then $\mathbb{H}$ is a left-$\mathcal{U}$-promodule if and only if $[\mathbb{H}]$ is a left-$[\mathcal{U}]$-promodule. In fact, if $\mathbb{H}$ is a left-$\mathcal{U}$-promodule and $\mathbb{F} \in [\mathcal{U}] \circ [\mathbb{H}]$, then $\mathbb{H} \leq \mathcal{U} \circ \mathbb{H} \leq \mathbb{F}$ and hence, $\mathbb{F} \in [\mathbb{H}]$. Conversely, if $[\mathbb{H}]$ is a left-$[\mathcal{U}]$-promodule, then $\mathcal{U} \circ \mathbb{H} \in [\mathcal{U}] \circ [\mathbb{H}] \subseteq [\mathbb{H}]$, so that $\mathbb{H} \leq \mathcal{U} \circ \mathbb{H}$. In a similar way, we see that $\mathbb{K}$ is a right-$\mathcal{U}$-promodule if and only if $[\mathbb{K}]$ is a right-$[\mathcal{U}]$-promodule.

Furthermore, it is not difficult to show that $\mathbb{H} \dashv \mathbb{K}$ (in $(X,\mathcal{U})$) if and only if $[\mathbb{H}] \dashv [\mathbb{K}]$ (in $(X,[\mathcal{U}])$).

A saturated L-quasi-uniform space $(X,\mathcal{U})$ is called *Lawvere complete* [13] (see also [6]) if for all left-$\mathcal{U}$-promodules $\mathbb{H}$ and all right-$\mathcal{U}$-promodules $\mathbb{K}$ with $\mathbb{H} \dashv \mathbb{K}$ there is $x \in X$ such that $\mathbb{H} = \mathcal{U}(x,\cdot) = [\{u(x,\cdot) : u \in \mathcal{U}\}]$ and $\mathbb{K} = \mathcal{U}(\cdot,x) = [\{u(\cdot,x) : u \in \mathcal{U}\}]$.

Proposition 4.4. *A saturated L-quasi-uniform space $(X,\mathcal{U})$ is Lawvere complete if, and only if, $(X,[\mathcal{U}])$ is Lawvere complete.*

Proof. Let first $(X,\mathcal{U})$ be Lawvere complete and let $\Phi \dashv \Psi$. From Proposition 4.2 we see that there are $\mathbb{F} \in \Phi$ and $\mathbb{G} \in \Psi$ such that $\mathbb{F} \dashv \mathbb{G}$. By Lawvere completeness, there is $x \in X$ such that $\mathbb{F} = \mathcal{U}(x,\cdot)$ and $\mathbb{G} = \mathcal{U}(\cdot,x)$. For $u \in L^{X\times X}$ we have $u \circ \top_{\{x\}}(y) = \bigvee_{z\in X} \top_{\{x\}}(z) * u(z,y) = u(x,y)$ for all $y \in X$ and hence $\mathcal{U} \circ [x] = \mathcal{U}(x,\cdot)$. Similarly we can show $[x] \circ \mathcal{U} = \mathcal{U}(\cdot,x)$. We conclude $[\mathbb{F}] = [\mathcal{U}] \circ [[x]]$ and $[\mathbb{G}] = [[x]] \circ [\mathcal{U}]$. Clearly, we have $[\mathbb{F}] \dashv [\mathbb{G}]$ and $[\mathbb{F}] \subseteq \Phi$ and $[\mathbb{G}] \subseteq \Psi$. Lemma 3.8 implies $\Phi = [\mathbb{F}] = [\mathcal{U}] \circ [[x]] = x_*$ and $\Psi = [\mathbb{G}] = [[x]] \circ [\mathcal{U}] = x^*$ and hence $(X,[\mathcal{U}])$ is Lawvere complete.

For the converse, let $(X,[\mathcal{U}])$ be Lawvere complete and let $\mathbb{H} \dashv \mathbb{G}$. Then $[\mathbb{H}] \dashv [\mathbb{G}]$ and hence there is $x \in X$ such that $[\mathbb{H}] = [\mathcal{U}] \circ [[x]]$ and $[\mathbb{G}] = [[x]] \circ [\mathcal{U}]$. We conclude $\mathbb{H} \geq \mathcal{U} \circ [x] = \mathcal{U}(x,\cdot)$ and $\mathbb{K} \geq [x] \circ \mathcal{U} = \mathcal{U}(\cdot,x)$. As $\mathcal{U}(x,\cdot) \dashv \mathcal{U}(\cdot,x)$, see [6], we obtain $\mathbb{H} = \mathcal{U}(x,\cdot)$ and $\mathbb{K} = \mathcal{U}(\cdot,x)$ and $(X,\mathcal{U})$ is Lawvere complete. □

5 Cauchy Completeness of Saturated L-Quasi-Uniform Limit Spaces

Let (X,Λ) be a saturated L-quasi-uniform limit space and let $\mathbb{F},\mathbb{G} \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X)$. The following concepts were introduced in [13].

(1) $(\mathbb{F},\mathbb{G})$ are called a *saturated pair L-prefilter* if for all $f \in \mathbb{F}$ and all $g \in \mathbb{G}$ we have $\bigvee_{x\in X} f(x)*g(x) = \top$.

(2) A saturated pair L-prefilter $(\mathbb{F},\mathbb{G})$ is called a *Cauchy pair* if $\mathbb{G} \otimes \mathbb{F} \in \Lambda$.

(3) A saturated pair L-prefilter $(\mathbb{F},\mathbb{G})$ converges to $x \in X$, $(\mathbb{F},\mathbb{G}) \to x$, if $[x] \otimes \mathbb{F} \in \Lambda$ and $\mathbb{G} \otimes [x] \in \Lambda$.

We note that if a saturated pair L-prefilter $(\mathbb{F},\mathbb{G})$ converges to x, then $([x] \otimes \mathbb{F}) \circ (\mathbb{G} \otimes [x]) = \mathbb{G} \otimes \mathbb{F} \in \Lambda$, that is, $(\mathbb{F},\mathbb{G})$ is a Cauchy pair.

Proposition 5.1 (see also [6]). *Let (X,Λ) be a saturated* L*-quasi-uniform limit space and let $(\mathbb{F},\mathbb{G}),(\mathbb{F}',\mathbb{G}')$ be saturated pair* L*-prefilters on X.*

(SCP1) $([x],[x])$ is a Cauchy pair for all $x \in X$;

(SCP2) If $(\mathbb{F},\mathbb{G})$ is a Cauchy pair and if $\mathbb{F}' \geq \mathbb{F}$ and $\mathbb{G}' \geq \mathbb{G}$, then $(\mathbb{F}',\mathbb{G}')$ is a Cauchy pair.

(SCP3) If $(\mathbb{F},\mathbb{G}),(\mathbb{F}',\mathbb{G}')$ are Cauchy pairs and if $\bigvee_{x\in X} f(x) * g'(x) = \top$ for all $f \in \mathbb{F}$ and all $g' \in \mathbb{G}'$ and also $\bigvee_{x\in X} f'(x) * g(x) = \top$ for all $f' \in \mathbb{F}$ and all $g \in \mathbb{G}'$, then $(\mathbb{F}\wedge\mathbb{F}',\mathbb{G}\wedge\mathbb{G}')$ is a Cauchy pair.

Proof. We show only (SCP3). Obviously, $(\mathbb{F}\wedge\mathbb{F}',\mathbb{G}\wedge\mathbb{G}')$ is a pair L-prefilter. $\bigvee_{x\in X} f(x) * g'(x) = \top$ for all $f \in \mathbb{F}$ and all $g' \in \mathbb{G}'$, we conclude $(\mathbb{G}'\otimes\mathbb{F}')\circ(\mathbb{G}\otimes\mathbb{F}) = \mathbb{G}\otimes\mathbb{F}'$, see [7]. Similarly, we have $(\mathbb{G}\otimes\mathbb{F})\circ(\mathbb{G}'\otimes\mathbb{F}') = \mathbb{G}'\otimes\mathbb{F}$. By (SLUL2) then $\mathbb{G}\otimes\mathbb{F}' \in \Lambda$ and $\mathbb{G}'\otimes\mathbb{F} \in \Lambda$. Hence, using Proposition 3.10 [7], we obtain $\Lambda \ni (\mathbb{G}\otimes\mathbb{F})\wedge(\mathbb{G}\otimes\mathbb{F}')\wedge(\mathbb{G}'\otimes\mathbb{F})\wedge(\mathbb{G}'\otimes\mathbb{F}') = (\mathbb{G}\wedge\mathbb{G}')\otimes(\mathbb{F}\wedge\mathbb{F}')$. □

This proposition shows that a saturated L-quasi-uniform limit space has an underlying ⊤-quasi-Cauchy space. These spaces were introduced in [8].

Definition 5.2. A saturated L-quasi-uniform limit space (X,Λ) is called *Cauchy complete* if for all Cauchy pairs $(\mathbb{F},\mathbb{G})$ there is $x \in X$ such that $(\mathbb{F},\mathbb{G}) \to x$.

For a saturated L-quasi-uniform space $(X,\mathcal{U})$, a saturated pair L-prefilter $(\mathbb{F},\mathbb{G})$ is called a *Cauchy pair* [13] if $\mathbb{G}\otimes\mathbb{F} \geq \mathcal{U}$, that is, if $(\mathbb{F},\mathbb{G})$ is a Cauchy pair in $(X,[\mathcal{U}])$. The saturated pair L-prefilter $(\mathbb{F},\mathbb{G})$ is called *convergent* to $x \in X$ if $\mathbb{F} \geq \mathcal{U}(x,\cdot)$ and $\mathbb{G} \geq \mathcal{U}(\cdot,x)$. From $([x]\otimes\mathbb{F})\circ[x] = \mathbb{F}$ we obtain $[x]\otimes\mathbb{F} \geq \mathcal{U}$ if, and only if, $\mathbb{F} \geq \mathcal{U}\circ[x] = \mathcal{U}(x,\cdot)$ and similarly we have $\mathcal{G}\otimes[x] \geq \mathcal{U}$ if, and only if, $\mathcal{G} \geq [x]\circ\mathcal{U} = \mathcal{U}(\cdot,x)$. Hence we have $(\mathbb{F},\mathbb{G}) \to x$ in $(X,\mathcal{U})$ if, and only if, $(\mathbb{F},\mathbb{G}) \to x$ in $(X,[\mathcal{U}])$. From these observations we immediately obtain the following result.

Proposition 5.3. *A saturated* L*-quasi-uniform space $(X,\mathcal{U})$ is Cauchy complete if, and only if, $(X,[\mathcal{U}])$ is Cauchy complete.*

It is shown in [13, 14] that a saturated L-quasi-uniform space is Cauchy complete if, and only if, it is Lawvere complete. Hence, by Propositions 4.4 and 5.3, for a saturated L-quasi-uniform space $(X,\mathcal{U})$, the saturated L-quasi-uniform limit space $(X,[\mathcal{U}])$ is Cauchy complete if, and only if, it is Lawvere complete. This is also true for arbitrary saturated L-quasi-uniform limit spaces. We first show the following Lemma.

Lemma 5.4. *Let (X,Λ) be a saturated* L*-quasi-uniform limit space, $x \in X$ and $\mathbb{F},\mathbb{G} \in \mathsf{F}^{\mathsf{sat}}_{\mathsf{L}}(X)$. Then*
(1) $[x]\otimes\mathbb{F} \in \Lambda$ if, and only if, $\mathbb{F} \in \Lambda\circ[[x]]$.
(2) $\mathbb{G}\otimes[x] \in \Lambda$ if, and only if, $\mathbb{G} \in [[x]]\circ\Lambda$.

Proof. (1) Let first $[x]\otimes\mathbb{F} \in \Lambda$. Then $\mathbb{F} = ([x]\otimes\mathbb{F})\circ[x] \in \Lambda\circ[[x]]$. (We have $(\top_{\{x\}}\otimes f)\circ\top_{\{x\}}(y) = \bigvee_{z\in X}\top_{\{x\}}(z) * (\top_{\{x\}}\otimes f)(z,y) = \top_{\{x\}}\otimes f(x,y) = f(y)$.)

Let now $\mathbb{F} \in \Lambda\circ[[x]]$. Then there is $\mathbb{L} \in \Lambda$ such that $\mathbb{L}\circ[x] \leq \mathbb{F}$. We conclude $\mathbb{L} \leq [x]\otimes(\mathbb{L}\circ[x]) \leq [x]\otimes\mathbb{F}$ and hence $[x]\otimes\mathbb{F} \in \Lambda$. (We have $\top_{\{x\}}\otimes(l\circ\top_{\{x\}})(s,t) = \top_{\{x\}}(s) * \bigvee_{y\in X}\top_{\{x\}}(y) * l(y,t) = \top_{\{x\}}(s) * l(x,t) \leq l(s,t)$.)

(2) can be shown in a similar way. □

Theorem 5.5. *A saturated* L*-quasi-uniform limit space (X,Λ) is Cauchy complete if, and only if, it is Lawvere complete.*

Proof. Let first (X,Λ) be Lawvere complete and let $(\mathbb{F},\mathbb{G})$ be a Cauchy pair. We define $\Phi = \Lambda\circ[\mathbb{F}]$ and $\Psi = [\mathbb{G}]\circ\Lambda$. It is not difficult to see that Φ,Ψ are prorelations. As $\Lambda\circ\Phi = \Lambda\circ\Lambda\circ[\mathbb{F}] \subseteq \Lambda\circ\mathbb{F} = \Phi$, Φ is a left-promodule. Similarly, $\Psi\circ\Lambda = [\mathbb{G}]\circ\Lambda\circ\Lambda \subseteq [\mathbb{G}]\circ\Lambda = \Psi$, that is, Ψ is a right promodule. We show $\Phi \dashv \Psi$. Let $\mathbb{H} \in \Phi$ and $\mathbb{K} \in \Psi$. Then there are $\mathbb{L}_1,\mathbb{L}_2 \in \Lambda$ such that $\mathbb{H} \geq \mathbb{L}_1\circ\mathbb{F}$ and $\mathbb{K} \geq \mathbb{G}\circ\mathbb{L}_2$. A straightforward calculation shows that for $l_2, l_2 \in L^{X\times X}$ and $f, g \in L^X$ we have $l_1 \circ (g \otimes f) \circ l_2 = (g \circ l_2) \otimes (l_1 \circ f)$. Hence $\mathbb{K} \otimes \mathbb{H} \geq (\mathbb{G} \circ \mathbb{L}_2) \otimes (\mathbb{L}_1 \circ \mathbb{F}) = \mathbb{L}_1 \circ (\mathbb{G} \otimes \mathbb{F}) \circ \mathbb{L}_2 \in \Lambda$ by (SLUL4). Furthermore, we have $\mathbb{F} = [\top_{\Delta_X}] \circ \mathbb{F} \in \Phi$ and $\mathbb{G} = \mathbb{G} \circ [\top_{\Delta_X}] \in \Psi$ and therefore $\Phi \dashv \Psi$. As (X, Λ) is Lawvere complete, there is $x \in X$ such that $\Phi = x_*$ and $\Psi = x^*$, that is, $\Lambda \circ [\mathbb{F}] = \Lambda \circ [[x]]$ and $[\mathbb{G}] \circ \Lambda = [[x]] \circ \Lambda$. As $\mathbb{F} = [\top_{\Delta_X}] \circ \mathbb{F} \in \Lambda \circ [\mathbb{F}] = \Lambda \circ [[x]]$ we conclude with Lemma 5.4 that $[x] \otimes \mathbb{F} \in \Lambda$. In a similar way we see that $\mathbb{G} \otimes [x] \in \Lambda$ and hence $(\mathbb{F}, \mathbb{G}) \to x$ and (X, Λ) is Cauchy complete.

Let now (X, Λ) be a Cauchy complete. Let $\Phi \dashv \Psi$. From Proposition 4.2 we see that there is a Cauchy pair $(\mathbb{F}, \mathbb{G})$ with $\mathbb{F} \in \Phi$ and $\mathbb{G} \in \Psi$. By Cauchy completeness there is $x \in X$ such that $[x] \otimes \mathbb{F} \in \Lambda$ and $\mathbb{G} \otimes [x] \in \Lambda$, that is, $\mathbb{F} \in \Lambda \circ [[x]]$ and $\mathbb{G} \in [[x]] \circ \Lambda$. We define $\overline{\Phi} = \Lambda \circ [\mathbb{F}]$ and $\overline{\Psi} = [\mathbb{G}] \circ \Lambda$. Then, as in the first part of the proof, $\overline{\Phi} \dashv \overline{\Psi}$. We have $\overline{\Phi} = \Lambda \circ [\mathbb{F}] \subseteq \Lambda \circ \Phi \subseteq \Phi$. In a similar way we conclude $\overline{\Psi} \subseteq \Psi$ and hence, by Lemma 3.8, $\Phi = \Lambda \circ [\mathbb{F}]$. From $\mathbb{F} \in \Lambda \circ [[x]]$ we conclude $[\mathbb{F}] \subseteq \Lambda \circ [[x]]$ and hence $\Phi = \Lambda \circ [\mathbb{F}] \subseteq \Lambda \circ \Lambda \circ [[x]] \subseteq \Lambda \circ [[x]] = x_*$.

Let $\overline{\mathbb{F}} \in x_* = \Lambda \circ [[x]]$. Then there is $\mathbb{L} \in \Lambda$ such that $\mathbb{L} \circ [x] \leq \overline{\mathbb{F}}$. We note that for $f \in \mathbb{F}, g \in \mathbb{G}$ we have $\bigvee_{x \in X} f(x) * g(x) = \top$ and therefore $(g \otimes \top_{\{x\}}) \circ f = \top_{\{x\}}$. Hence we have $[x] = (\mathbb{G} \otimes [x]) \circ \mathbb{F} \in \Lambda \circ \Phi = \Phi$. It follows that $\overline{\mathbb{F}} \geq \mathbb{L} \circ [x] \in \Lambda \circ \Phi = \Phi$ and we have $\overline{\mathbb{F}} \in \Phi$, that is $x_* \subseteq \Phi$. Similar arguments show that $\Psi = x^*$ and (X, Λ) is Lawvere complete. □

6 Conclusion

We studied two completeness notions for saturated L-quasi-uniform limit spaces. The one is based on the concept of adjoint promodules and generalizes an approach of Clementino and Hofmann [2]. The other uses the concept of the Cauchy pair and generalizes a classical approach due to Lindgren and Fletcher [10]. We show that both approaches are equivalent.

An open problem is the construction of a completion based on either of the two completeness notions. This will still deserve more work.

Acknowledgements: Fruitful discussions which contributed to the contents of the paper and led to an improvement of the exposition with Professors Yueli Yue and Jingming Fang from Ocean University, China are gratefully acknowledged.

Conflict of Interest: The author declares no conflict of interest.

References

[1] Bělohlávek R. Fuzzy Relation Systems, Foundation and Principles. *Kluwer Academic/Plenum Publishers, New York, Boston, Dordrecht, London, Moscow.* 2002. DOI: http://doi.org/10.1007/978-1-4615-0633-1

[2] Clementino MM and Hofmann D. Lawvere completeness in topology. *Appl. Categor. Struct.* 2009; 17(2): 175-210. DOI: http://doi.org/10.1007/s10485-008-9152-5

[3] Flagg RC. Quantales and continuity spaces. *Algebra Univers.* 1997; 37(3): 257-276. DOI: http://doi.org/10.1007/s000120050018

[4] Hofmann D, Seal GJ and Tholen W. Monoidal Topology. *Cambridge University Press.* 2014. DOI: http://doi.org/10.1017/CBO9781107517288

[5] Höhle U. Probabilistic topologies induced by L-fuzzy uniformities. *Manuscripta Math.* 1982; 38(3): 289-323. DOI: http://doi.org/10.1007/BF01170928

[6] Jäger G. Sequential completeness for T-quasi-uniform spaces and a fixed point theorem. *Mathematics*. 2022; 10(13): 2285. DOI: http://doi.org/10.3390/math10132285

[7] Jäger G. Diagonal conditions and uniformly continuous extension in ⊤-uniform limit spaces. *Iranian J. Fuzzy Systems*. 2022; 19(5): 131-145. DOI: http://doi.org/10.22111/IJFS.2022.7161

[8] Jäger G. ⊤-quasi-Cauchy spaces-a non-symmetric theory of completeness and completion. *Appl. Gen. Topol*. 2023; 24(1): 205-227. DOI: http://doi.org/10.4995/agt.2023.18783

[9] Jäger G and Yue Y. T-uniform convergence spaces. *Iranian J. Fuzzy Systems*. 2022; 19(2): 133-149. DOI: http://doi.org/10.22111/IJFS.2022.6795

[10] Lindgren WF and Fletcher P. A construction of the pair completion of a quasi-uniform space. *Can. Math. Bull*. 1978; 21(1): 53-59. DOI: http://doi.org/10.4153/CMB-1978-009-2

[11] Schweizer B and Sklar A. Probabilistic Metric Spaces. *North Holland, New York*. 1983.

[12] Sun L and Yue Y. Completeness of L-quasi-uniform convergence spaces. *Iranian J. Fuzzy Systems*. 2023; 20(2): 57-67. DOI: http://doi.org/10.22111/IJFS.2023.7556

[13] Wang Y and Yue Y. Cauchy completion of Fuzzy quasi-uniform spaces. *Filomat*. 2021; 35(12): 3983-4004. DOI: http://doi.org/10.2298/FIL2112983W

[14] Yue Y and Fang J. Completeness in probabilistic quasi-uniform spaces. *Fuzzy Sets and Systems*. 2019; 370: 34-62. DOI: http://doi.org/10.1016/j.fss.2018.08.005

Gunther Jäger
School of Mechanical Engineering
University of Applied Sciences Stralsund
Stralsund, Germany
E-mail: gunther.jaeger@hochschule-stralsund.de

Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://tfss.journals.iau.ir/
Online ISSN: 2821-0131
Vol.2, No.2, (2023), 137-154
DOI: http://doi.org/10.30495/tfss.2023.1980002.1065

Article Type: Original Research Article

Lattices of (Generalized) Fuzzy Ideals in Double Boolean Algebras

Fernand Kuiebove Pefireko

Abstract. This paper develops the notion of fuzzy ideal and generalized fuzzy ideal on double Boolean algebra (dBa). According to Rudolf Wille, a double Boolean algebra $\underline{D} := (D, \sqcap, \sqcup, \neg, \lrcorner, \bot, \top)$ is an algebra of type $(2, 2, 1, 1, 0, 0)$, which satisfies a set of properties. This algebraic structure aimed to capture the equational theory of the algebra of protoconcepts. We show that collections of fuzzy ideals and generalized fuzzy ideals are endowed with lattice structures. We further prove that (by isomorphism) lattice structures obtained from fuzzy ideals and generalized fuzzy ideals of a double Boolean algebra D can entirely be determined by sets of fuzzy ideals and generalized fuzzy ideals of the Boolean algebra $D_\sqcup$.

AMS Subject Classification 2020: 03B15, 03B16, 03B35, 03B38.
Keywords and Phrases: Double Boolean algebras, Fuzzy ideals, Fuzzy primary ideal.

1 Introduction

Nowadays, fuzzy logic is used in numerous applications such as facial pattern recognition, air conditioners, washing machines, vacuum cleaners, anti-skid braking systems, transmission systems and unmanned helicopters knowledge-based systems for multi-objective optimization of power systems. In Machine Learning, fuzzy logic can be applied in some models such as MLP (Multi Layers Perceptron) model which is a fully connected class of feed-forward artificial neural network (ANN). In forecasting, fuzzification is incorporated at the input layers by considering the degree of participation of each of the features in the prediction model [2]. Fuzzy layers can also be seen as a circuit design as it is an application of Boolean algebra and therefore, we strongly believe that the way of connecting layers can be related to a lattice structure. Lattices can also appear in analysis of cellular traffic for finding anomalies in the performance and provisioning of demand resources [3]. Another application of double Boolean algebra is in multi-layer neural networks, in fact considering multilayer neural network design. Different blocks made between layers represent ordered structures of dBas. So with a specific dBa, we can easily design a multilayer neural network based on connection between layers. This task can therefore be added to artificial intelligence purpose on designing circuits that are used in digital computers.

So far, fuzzification of ideals has been studied on bounded lattices [1, 7, 10]. Mezzomo et *al*, based on Chon's approach [4], has defined the notion of fuzzy ideals and fuzzy filters on the product operators of bounded lattices. They have also proved some properties that are analogous to the classical theory of fuzzy ideals and fuzzy filters such as, the class of fuzzy ideals being closed under fuzzy union and fuzzy intersection. Thus this leads to the study of fuzzy topology on bounded lattices. Attalah has studied complete fuzzy prime ideals on distributive lattices. Fuzzification of ideals has been tackled in other algebraic structures such as, IL

∗**Corresponding Author:** Fernand Kuiebove Pefireko, Email: fkuiebove@yahoo.fr, ORCID: 0000-0002-9908-9708
Received: 12 February 2023; **Revised**: 29 June 2023; **Accepted**: 16 July 2023; **Available Online**: 16 July 2023; **Published Online**: 7 November 2023.

How to cite: Kuiebove F. Lattices of (Generalized) Fuzzy Ideals in Double Boolean algebras. *Trans. Fuzzy Sets Syst.* 2023; 2(2): 137-154. DOI: http://doi.org/10.30495/tfss.2023.1980002.1065

algebras [6], in that structure the concept of fuzzy ideal generalizes the notion of fuzzy ideals in BL-algebra and MTL-algebra. Kuanyum et *al* [12] have tackled the question of fuzzy ideals in residuated lattices. They first defined the generalized fuzzy ideals and they showed that the set of generalized fuzzy ideals is endowed with a lattice structure. As it is known that a double Boolean algebra is a more general structure than a residuated lattice and does not necessarily have a subjacent structure of lattice and Tatuéné [5] has shown that transfer of structure from that algebra to the fuzzy structure does not holds. The question that captures our interest is whether for the case of a double Boolean algebra, we still have a lattice structure by endowing the set of fuzzy ideals with some operators. From the best of our knowledge, this direction has not yet been tackled. So our goal in this paper is mainly focused on the study of fuzzy ideals of a double Boolean algebra. We fuzzify the notion of ideals on double Boolean algebra. Moreover we prove that the collection of fuzzy ideals of a double Boolean algebra $\underline{D}$ is endowed with a lattice structure which is an extension of the work done by Kuanyun et *al* [12].

The paper is organized as follows: in section 2, we present a background which contains definitions and related properties of ideals and filters in the double Boolean algebra for a better understanding of the structure. In section 3, we introduce the concept of fuzzy ideals and fuzzy filters on double Boolean algebras and then we characterize them. In section 4, we study the lattice structure of the set of all fuzzy ideals of a double Boolean algebra $\underline{D}$. In section 5, we draw a generalization of the concept of fuzzy ideal and then we study the bounded lattice structure of the set of generalized fuzzy ideals of a double Boolean algebra.

2 Background

In this section, we present double Boolean algebras, ideals of double Boolean algebras and related properties. We then give some results obtained by Tenkeu et al [11] for this structure. These notions will be useful for the rest of the paper.

2.1 Double Boolean algebras and related properties

Definition 2.1. [9] A double Boolean algebra is an algebra $\underline{D} = (D, \sqcap, \sqcup, \neg, \lrcorner, \bot, \top)$ of type $(2,2,1,1,0,0)$ that satisfies (1a) to (11a) and (1b) to (11b).

(1a) $(x \sqcap x) \sqcap y = x \sqcap y$

(2a) $x \sqcap y = y \sqcap x$

(3a) $x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$

(4a) $\neg(x \sqcap x) = \neg x$

(5a) $x \sqcap (x \sqcup y) = x \sqcap x$

(6a) $x \sqcap (y \vee z) = (x \sqcap y) \vee (x \sqcap z)$

(7a) $x \sqcap (x \vee y) = x \sqcap x$

(8a) $\neg\neg(x \sqcap y) = x \sqcap y$

(9a) $x \sqcap \neg x = \bot$

(10a) $\neg\bot = \top \sqcap \top$

(11a) $\neg\top = \bot$

(1b) $(x \sqcup x) \sqcup y = x \sqcup y$

(2b) $x \sqcup y = y \sqcup x$

(3b) $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$

(4b) $\lrcorner(x \sqcup x) = \lrcorner x$

(5b) $x \sqcup (x \sqcap y) = x \sqcup x$

(6b) $x \sqcup (y \wedge z) = (x \sqcup y) \wedge (x \sqcup z)$

(7b) $x \sqcup (x \wedge y) = x \sqcup x$

(8b) $\lrcorner\lrcorner(x \sqcup y) = x \sqcup y$

(9b) $x \sqcup \lrcorner x = \top$

(10b) $\lrcorner\top = \bot \sqcup \bot$

(11b) $\lrcorner\bot = \top$

$$(12)\ (x \sqcap x) \sqcup (x \sqcap x) = (x \sqcup x) \sqcap (x \sqcup x)$$

Where the supremum (join) is defined by $x \vee y := \neg(\neg x \sqcap \neg y)$, and the infimum (meet) is defined by: $x \wedge y := \lrcorner(\lrcorner x \sqcup \lrcorner y)$, $1 := \neg\bot$ and $0 := \lrcorner\top$. The relation defined by $x \sqsubseteq y \Longleftrightarrow x \sqcap y = x \sqcap x$ *and* $x \sqcup y = y \sqcup y$ is a quasi-order.

A double Boolean algebra is called pure if it satisfies: $x \sqcap x = x$ *or* $x \sqcup x = x$. This relation also holds in algebra of semiconcepts.

The following notations are adopted $x_\sqcap := x \sqcap x$, $D_\sqcap = \{x_\sqcap : x \in D\}$ and $x_\sqcup = x \sqcup x$, $D_\sqcup = \{x_\sqcup : x \in D\}$. The algebras $(D_\sqcap, \sqcap, \vee, \neg, \bot, 1)$ and $(D_\sqcup, \wedge, \sqcup, \lrcorner, 0, \top)$ are Boolean algebras.

$$\begin{aligned} x_\sqcap \leq y_\sqcap &\Longleftrightarrow x \sqsubseteq y \\ &\Longleftrightarrow x_\sqcap \sqcap y_\sqcap = (x \sqcap x) \sqcap (y \sqcap y) = x \sqcap x = x_\sqcap \end{aligned}$$

and in the same way, we have: $x_\sqcup \sqcup y_\sqcup = y_\sqcup$. As $\sqcap$ is the meet and $\sqcup$ is the join operator in the Boolean algebra $(D_\sqcap, \sqcap, \vee, \neg, \bot, 1)$ and $(D_\sqcup, \wedge, \sqcup, \lrcorner, 0, \top)$ respectively. We get $x \sqsubseteq y \Longleftrightarrow x_\sqcap \leq y_\sqcap$ *and* $x_\sqcup \leq y_\sqcup$ where $\leq$ is the induced order in the corresponding Boolean algebra.

A double Boolean algebra is called regular if the relation $\sqsubseteq$ is an order relation.

Example 2.2. Let $D = \{a, b, c, d, e = a \sqcap b = c \sqcup d, \bot, \top\}$ with the diagram given by the following:

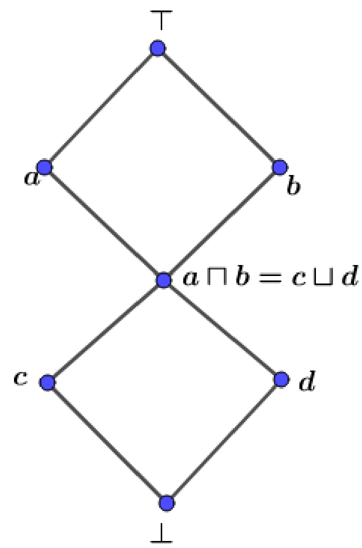


Figure 1: dBa pure

With the following table:

$\sqcup$	a	b	c	d	$\bot$	$\top$	e
a	a	$\top$	a	a	a	$\top$	a
b	$\top$	b	b	b	b	$\top$	b
c	a	b	e	e	e	$\top$	e
d	a	b	e	e	e	$\top$	e
e	a	b	e	e	e	$\top$	e
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
$\bot$	e	b	e	e	$\bot$	$\top$	e

x	a	b	c	d	$\bot$	$\top$	e
$\neg x$	$\bot$	$\bot$	d	c	e	$\bot$	$\bot$

x	a	b	c	d	$\bot$	$\top$	e
$\lrcorner x$	b	a	$\top$	$\top$	$\top$	e	$\top$

$\sqcap$	a	b	c	d	$\bot$	$\top$	e
a	e	e	c	d	$\bot$	e	e
b	e	b	c	d	$\bot$	b	b
c	c	c	c	$\bot$	$\bot$	c	e
d	d	d	$\bot$	d	$\bot$	d	e
$\top$	a	b	c	d	$\bot$	$\top$	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	e
e	a	b	e	e	e	$\top$	e

$D = D_{\sqcup} \cup D_{\sqcap}$, *with* $D_{\sqcup} = \{a, b, a \sqcap b, \top\}$and $D_{\sqcap} = \{c, d, c \sqcup d, \bot\}$ $a \sqcup a = a$, $b \sqcup b = b$, $, \top \sqcup \top = \top$. Hence D is a pure double Boolean algebra.

Example 2.3. Let $D = \{\alpha, \beta, \gamma, \lambda, \bot, \top\}$ with the diagram and tables given by the following:

$\sqcap$	$\bot$	γ	λ	β	α	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
γ	$\bot$	γ	$\bot$	γ	γ	γ
λ	$\bot$	$\bot$	λ	λ	$\bot$	λ
β	$\bot$	λ	β	γ	γ	β
α	$\bot$	γ	$\bot$	γ	γ	γ
$\top$	$\bot$	γ	λ	β	γ	β

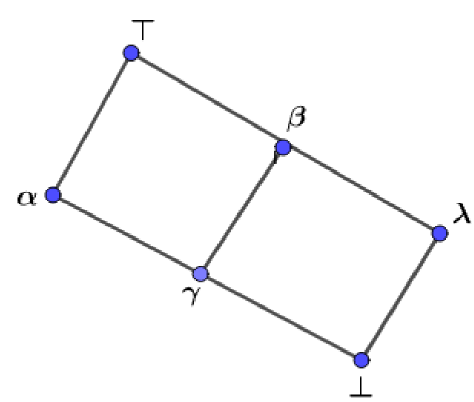


Figure 2: Hasse diagram of D

$\sqcup$	$\bot$	γ	λ	β	α	$\top$
$\bot$	γ	γ	β	β	α	$\top$
γ	γ	γ	β	β	α	$\top$
λ	β	β	β	β	$\top$	$\top$
β	β	β	β	β	$\top$	$\top$
α	α	α	$\top$	$\top$	α	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

x	$\bot$	γ	λ	β	α	$\top$
$\neg x$	β	λ	γ	$\bot$	λ	$\bot$
$\lrcorner x$	$\top$	$\top$	α	α	β	γ

The dBa D is pure and we have $D_{\sqcap} = \{\bot, \gamma, \lambda, \beta\}$ and $D_{\sqcup} = \{\alpha, \beta, \gamma, \top\}$

2.2 Ideals on double Boolean algebras and their properties

Definition 2.4. (see [8]) Let $\underline{D}$ be a double Boolean algebra. A non empty subset F of $\underline{D}$ is called a filter if it satisfies:

(i) $x, y \in F \Longrightarrow x \sqcap y \in F$;
(ii) $x \in F, y \in D, x \sqsubseteq y \Longrightarrow y \in F$.

Dually, ideals of double Boolean algebras is defined.

Definition 2.5. (see [8]) Let $\underline{D}$ be a double Boolean algebra. A filter F is called proper if $F \neq D$, and primary if it is proper and satisfies $x \in F$ or $\neg x \in F$, for all $x \in D$.
Dually are defined primary ideals.

Tenkeu et *al.* [11] showed that primary ideals are exactly maximal ideals in the framework of double Boolean algebras.

Proposition 2.6. *(see [8]) Let I an ideal of D, then $I_{\sqcup} = \{x_{\sqcup} : x \in I\}$ is an ideal of $D_{\sqcup}$.*

We call double Boolean algebra **trivial** iff $\top \sqcap \top = \bot \sqcup \bot$.

Proposition 2.7. *(see [11]) Let D be a dBa and $X \subseteq D$ a non empty subset of D. F_1, F_2 two filters of $\underline{D}$ and I_1, I_2 two ideals of $\underline{D}$.*

1. *$I(a) = \{x \in D : x \sqsubseteq a \sqcup a\}$, where $I(a)$ stand for the ideal generated by a and $F(a) = \{x \in D : a \sqcap a \sqsubseteq x\}$, where $F(a)$ stand for the filter generated by a.*
2. *$Ideal(\emptyset) = I(\bot) = \{x \in D : x \subseteq \bot \sqcup \bot\}$ and $Filter(\emptyset) = F(\top) = \{x \in D : \top \sqcap \top \sqsubseteq x\}$*
3. *$Ideal(X) = \{x \in D : x \sqsubseteq b_1 \sqcup b_2 \sqcup \cdots \sqcup b_n,$ for some $b_1, b_2, \cdots, b_n \in X, n \geq 1\}$*
4. *$Filter(X) = \{x \in D : x \sqsupseteq b_1 \sqcap b_2 \sqcap \cdots \sqcap b_n,$ for some $b_1, b_2, \cdots, b_n \in X, n \geq 1\}$*
5. *$Ideal\,(I_1 \cup I_2) = \{x \in D : x \sqsubseteq i_1 \sqcup i_2, i_1 \in I_1, i_2 \in I_2\} = I_1 \vee I_2$*
6. *$Filter\,(F_1 \cup F_2) = \{x \in D : x \sqsubseteq f_1 \sqcap f_2, f_1 \in F_1, f_2 \in F_2\} = F_1 \vee F_2$*

The following proposition gives the distributivity-like property of dBa.

Proposition 2.8. *(see [11]) Let $\underline{D} = (D, \sqcap, \sqcup, \bot, \top)$ be a dBa and $a, b, c \in D$. We have:*

(i) $a \vee (b \sqcap c) = (a \vee b) \sqcap (a \vee c)$
(ii) $a \wedge (b \sqcup c) = (a \wedge b) \sqcup (a \wedge c)$
(iii) $a \vee (a \sqcap b) = a \sqcap a$
(iv) $a \wedge (a \sqcup b) = a \sqcup a$
(v) $(a \sqcap a) \vee (b \sqcap b) = a \vee b$
(vi) $(a \sqcup a) \wedge (b \sqcup b) = a \wedge b.$
(vii) $a \sqcup b,\ c \sqcup d \Longrightarrow a \sqcap c \sqsubseteq b \sqcap d,\ a \sqcup c \sqsubseteq b \sqcup d,\ a \vee c \sqsubseteq b \vee d$ *and* $a \wedge c \sqsubseteq b \wedge d$.

In the next section we are going to introduce fuzzy ideals and fuzzy filters on double Boolean algebras and give some related properties.

3 Fuzzy ideals and fuzzy filters on double Boolean algebras

In this section we introduce the notion of fuzzy ideals and fuzzy filters in the context of double Boolean algebras. Namely, we characterized these concepts. A fuzzy set on $\underline{D}$ is a function $\mu : D \longrightarrow [0,1]$. Let $\alpha \in [0,1]$, the α-cut of μ is defined by $\mu_\alpha = \{x \in D : \mu(x) \geq \alpha\}$.

Definition 3.1. Let $\underline{D} = (D, \sqcap, \sqcup, \neg, \lrcorner, \bot, \top)$ be a double Boolean algebra and μ, ν two fuzzy subsets of $\underline{D}$. The fuzzy subset μ of $\underline{D}$ is a fuzzy filter if for all $x, y \in D$

(i) $\mu(x \sqcap y) \geq \mu(x) \wedge \mu(y)$;

(ii) $x \sqsubseteq y \Longrightarrow \mu(x) \leq \mu(y)$.

Dually the fuzzy subset ν of $\underline{D}$ is a fuzzy ideal if for all $x, y \in D$, the following inequalities hold:

(i) $\nu(x \sqcup y) \geq \nu(x) \wedge \nu(y)$;

(ii) $x \sqsubseteq y \Longrightarrow \nu(x) \geq \nu(y)$.

The above two relations on Definition 3.1 are equivalent to say that : μ is fuzzy ideal of $\underline{D}$ iff $\mu(x \sqcup y) = \mu(x) \wedge \mu(y)$, for all $x, y \in D$. And μ is fuzzy filter of $\underline{D}$ iff $\mu(x \sqcap y) = \mu(x) \wedge \mu(y)$, for all $x, y \in D$.

Proposition 3.2. *Let μ be a fuzzy ideal of $\underline{D}$, and ν the fuzzy filter of $\underline{D}$, then we have the following: $\mu(x \sqcup x) = \mu(x)$ and $\nu(x \sqcap x) = \nu(x)$, for all $x \in D$.*

Proof. Since μ is a fuzzy ideal then we have: $\mu(x \sqcup x) \geq \mu(x) \wedge \mu(x) = \mu(x)$ this implies that $\mu(x \sqcup x) \geq \mu(x)$. In addition, we have $x \sqsubseteq x \sqcup x$, then $\mu(x) \geq \mu(x \sqcup x)$. Hence $\mu(x \sqcup x) = \mu(x)$. Similarly, the cases of the fuzzy filter can be shown. □

The previous proposition shows that the fuzzy ideals of the Boolean algebra $D_\sqcup$ can be extended as fuzzy ideals of the double Boolean $\underline{D}$.

Let denote by $FI(D_\sqcup)$ and $FI(D)$ the collection of fuzzy ideals of $D_\sqcup$ and D respectively. We have the following proposition which is a characterization of fuzzy ideals and fuzzy filters with their α-cuts.

Proposition 3.3. *Let μ, ν be two fuzzy subsets of $\underline{D}$.*

(i) μ is a fuzzy ideal iff for all $\alpha \in [0,1]$, $\mu_\alpha = \emptyset$ or μ_α is an ideal of $\underline{D}$.

(ii) ν is a fuzzy filter iff for all $\alpha \in [0,1]$, $\nu_\alpha = \emptyset$ or ν_α is an filter of $\underline{D}$.

Proof.

Case of (i)

Let μ be a fuzzy subset of $\underline{D}$.

$\Longrightarrow$) Let us assume that μ is a fuzzy ideal of $\underline{D}$. Let $\alpha \in [0,1]$, such that $\mu_\alpha \neq \emptyset$. we need to show that μ_α is an ideal of $\underline{D}$.

Suppose $x, y \in \mu_\alpha$ then we have $\mu(x) \geq \alpha$ and $\mu(y) \geq \alpha$, hence $\mu(x) \wedge \mu(y) \geq \alpha$. But by hypothesis, μ is a fuzzy ideal of $\underline{D}$. So we have $\mu(x \sqcup y) \geq \mu(x) \wedge \mu(y)$. Thus $\mu(x \sqcup y) \geq \alpha$. Hence $x \sqcup y \in \mu_\alpha$.

Now suppose that $x \sqsubseteq y$ then $\mu(y) \leq \mu(x)$ since μ is a fuzzy ideal of $\underline{D}$. And since $y \in \mu_\alpha$, this means that $\mu(y) \geq \alpha$, so $\mu(x) \geq \alpha$, by transitivity of $\leq$, therefore $x \in \mu_\alpha$. Hence μ_α is an ideal of $\underline{D}$.

$\Longleftarrow$) Let us assume that for any $\alpha \in [0,1]$, μ_α is an ideal of $\underline{D}$.

Let $x, y \in D$, for $\alpha = \mu(x) \wedge \mu(y)$, $x, y \in \mu_\alpha$, since μ_α is an ideal of $\underline{D}$, we have $x \sqcup y \in \mu_\alpha$. Thus $\mu(x \sqcup y) \geq \mu(x) \wedge \mu(y)$. Also $x \sqsubseteq y \Longrightarrow x \sqcup y = y \sqcup y$ (Prop.3). For $\alpha = \mu(y)$, $y \in \mu_\alpha$, but $x \sqsubseteq y$ and μ_α ideal implies $x \in \mu_\alpha$. Thus $\mu(x) \geq \alpha = \mu(y)$.

The proof of (ii) is similar.
□

Example 3.4. Let us consider the double Boolean algebra defined in the example 2.2 with it diagram in Figure 1. Then we have the following:

$$\mu(x) = \begin{cases} \frac{1}{2} & if \quad x \in \{a, b, c, d, e, \top\} \\ 1 & if \quad x = \bot \end{cases}$$

is a fuzzy ideal of the double Boolean algebra D, defined in example 1 of the paper. In fact, let $\alpha \in [0, 1]$, if $\alpha \leq \frac{1}{2}$, $\mu_\alpha = \{a, b, c, d, e, \top\}$ which is an ideal of D. If $\alpha > \frac{1}{2}$, then $\mu_\alpha = \{\bot\}$, which is an ideal of D.

$$\nu(x) = \begin{cases} \frac{3}{4} & if \quad x \in \{a, b, e\} \\ 1 & if \quad x \in \{c, d, \top\} \\ 0 & if \quad x = \bot \end{cases}$$

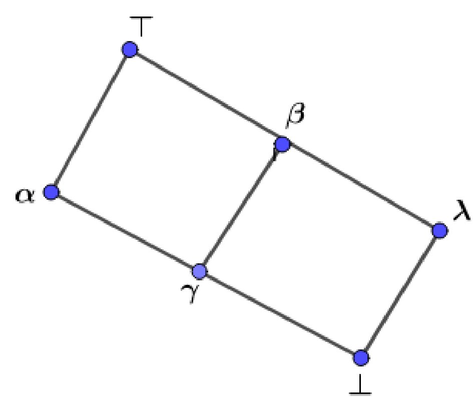
is a fuzzy filter of the double Boolean algebra D, defined in example 2.2

Definition 3.5. Let $\underline{D}$ a double Boolean algebra. A fuzzy filter μ is called proper if μ is a non constant function.

Definition 3.6. Let μ be a fuzzy subset of $\underline{D}$, μ is a fuzzy primary filter of $\underline{D}$ if it is proper and satisfies $\mu(x) \vee \mu(\neg x) = 1$, for all $x \in D$. Dually, a fuzzy subset ν is a fuzzy primary ideal of $\underline{D}$ if $\nu(x) \vee \nu(\lrcorner x) = 1$, for all $x \in D$.

Example 3.7. Let us consider the double Boolean algebra of the example 2.3, where $D = \{\bot, \alpha, \beta, \gamma, \lambda, \top\}$ with it diagram in Figure 2.

$$\mu(x) = \begin{cases} 1 & \text{if } x \in \{\bot, \alpha, \beta, \gamma\} \\ \frac{1}{3} & \text{if } x = \lambda \\ \frac{1}{10} & \text{if } x = \top \end{cases} \quad (1)$$



Then μ is a fuzzy primary ideal of D. In fact, μ is a fuzzy ideal of D, since it is a decreasing function. And it is primary: let $x \in D = \{\bot, \alpha, \beta, \gamma, \lambda, \top\}$

If $x = \bot$, then since $\mu(\bot) = 1$, we directly have $\mu(x) \vee \mu(\lrcorner x) = 1$;

If $x = \alpha$, then since $\mu(\alpha) = 1$, we directly have $\mu(\alpha) \vee \mu(\lrcorner \alpha) = 1$;

If $x = \beta$, then since $\mu(\beta) = 1$, we directly have $\mu(\beta) \vee \mu(\lrcorner \beta) = 1$;

If $x = \gamma$, then since $\mu(\gamma) = 1$, we directly have $\mu(\gamma) \vee \mu(\lrcorner \gamma) = 1$;

If $x = \lambda$, then since $\mu(\lrcorner \lambda) = 1$, we directly have $\mu(\lambda) \vee \mu(\lrcorner \lambda) = 1$;

If $x = \top$, then since $\mu(\lrcorner \top) = 1$, we directly have $\mu(\top) \vee \mu(\lrcorner \top) = 1$.

Proposition 3.8. *Let ν and μ be two fuzzy subsets of D,*

(i) ν is a fuzzy primary ideal of D if and only if for all $\alpha \in [0,1]$, such that $\emptyset \neq \nu_\alpha \neq D$, ν_α is a primary ideal of D;

(ii) μ is a fuzzy primary filter of D if and only if for all $\alpha \in [0,1]$, $\emptyset \neq \mu_\alpha \neq D$, μ_α is a primary filter of D.

Proof. Case of (i)
$\Longrightarrow$) Suppose ν is a fuzzy primary ideal of D, let $\alpha \in [0,1]$ and $x \in D$. We have $\nu(x) \vee \nu(\lrcorner x) = 1$ this implies $\nu(x) \vee \nu(\lrcorner x) \geq \alpha$. Thus we have: $\nu(x) \geq \alpha$ or $\nu(\lrcorner x) \geq \alpha$. Hence $x \in \nu_\alpha$ or $\lrcorner x \in \nu_\alpha$.
$\Longleftarrow$) Conversely, let's assume that for any $\alpha \in [0,1]$, ν_α is a primary ideal of D. let us show that ν is a fuzzy primary ideal of D. Let $x \in D$, since ν_α is a primary ideal of D, either $x \in \nu_\alpha$ or $\lrcorner x \in \nu_\alpha$ this implies $\nu(x) \geq \alpha$ or $\nu(\lrcorner x) \geq \alpha$, thus $\nu(x) \vee \nu(\lrcorner x) \geq \alpha$ by taking on both side the sup on α, we get $\nu(x) \vee \nu(\lrcorner x) \geq 1$. Hence $\nu(x) \vee \nu(\lrcorner x) = 1$. The proof of (ii) is similar to the one of (i). □

Definition 3.9. Let μ be a fuzzy subset of $\underline{D}$. Then μ is a fuzzy maximal ideal if $\mu \neq \underline{1}$ and for any fuzzy ideal ν, $\mu \leq \nu \Longrightarrow \nu = \underline{1}$ or $\mu = \nu$.

Dually, fuzzy maximal filter of $\underline{D}$ is defined.

Example 3.10. Let us consider the double Boolean algebra illustrated in example 2.2, with it diagram in Figure 1. Then we have the following:

$$\mu(x) = \begin{cases} \frac{1}{2} & if \quad x \in \{a,b,c,d,e,\top\} \\ 1 & if \quad x = \bot \end{cases}$$

is a maximal fuzzy ideal. In fact by taking $\alpha \in [0,1]$, we have $\mu_\alpha = D$, if $\alpha > \frac{3}{5}$ and $\mu_\alpha = \{a,b,c,d,e,\top\}$ if $\alpha \leq \frac{3}{5}$. So μ_α is a maximal ideal of D.

Proposition 3.11. *Let μ, ν be two fuzzy subsets of $\underline{D}$.*

(i) If μ is fuzzy primary ideal iff μ is fuzzy maximal ideal.

(ii) If ν is fuzzy primary filter iff ν is fuzzy maximal filter.

Proof. Case of (i)
Let μ a fuzzy primary ideal of $\underline{D}$, let us show that μ is a fuzzy maximal ideal of $\underline{D}$. Since μ is a fuzzy primary ideal, μ is proper, then $\mu \neq \underline{1}$. Let ν be a fuzzy ideal of D such that $\mu \leq \nu$ and let us assume that $\nu \neq \underline{1}$ and show that necessarily we have $\mu = \nu$. If $\mu \neq \nu$ then there exists $a \in D$, such that $\mu(a) < \nu(a)$. According to Proposition 3.8, for $\alpha = 1$, $\mu_1 = \{x \in D : \ \mu(x) = 1\}$, since $\mu \neq 1$, and $\bot \in \mu_1$ then $\emptyset \neq \mu_1 \neq D$ is a primary ideal of D. But $\mu \leq \nu$ implies that $\mu_1 \subseteq \nu_1$ and using the fact that any primary ideal of D is maximal, we therefore have $\mu_1 = \nu_1$ or $\nu_1 = D$. Having $\mu < \nu$ implies that $\mu_1 \neq \nu_1$. Hence by maximality of μ_1, we have $\nu_1 = D$ and this implies that $\nu = \underline{1}$ which is absurd ($\nu \neq \underline{1}$ by hypothesis). Conversely, let assume that μ is a maximal fuzzy ideal and let us show that μ is a fuzzy primary ideal.

Suppose that there exists $x \in D$ such that $\mu(x) \vee \mu(\lrcorner x) < 1$, then we have $\mu(x) < 1$ and $\mu(\lrcorner x) < 1$. Then there exists $x \in D$, $x \notin \mu_1$ and $\lrcorner x \notin \mu_1$. Since μ is maximal fuzzy ideal, then μ_1 is a maximal ideal. But $x \notin \mu_1$ implies that $Ideal < x > \cup \mu_1$ strictly contains μ_1 and by maximality of μ_1, we have $Ideal < x > \cup \mu_1 = D$. Thus $\lrcorner x \in \mu_1$, which is absurd.

The case of (ii) is the dual version. □

Proposition 3.12. *Let μ be a fuzzy subset of $\underline{D}$.*

(i) μ is a fuzzy ideal of D iff $\mu(\bot) \geq \mu(x)$ and $\mu(y) \geq \mu(x) \wedge \mu(\lrcorner x \wedge y), \forall x, y \in D$

(ii) μ *is a fuzzy filter of* D *iff* $\mu(\top) \geq \mu(x)$ *and* $\mu(x) \geq \mu(y) \wedge \mu(x \vee \neg y)$.

Proof. Case (i)

$\Longrightarrow$) Let us assume that μ is a fuzzy ideal of $\underline{D}$. Then it is obvious that $\mu(\bot) \geq \mu(x)$. Since μ is a fuzzy ideal of D, we have: $\mu(x \sqcup (\lrcorner x \wedge y)) = \mu(x) \wedge \mu(\lrcorner x \wedge y)$ but $y \sqsubseteq x \sqcup (\lrcorner x \wedge y)$, since $x \sqcup (\lrcorner x \wedge y) = (x \sqcup \lrcorner x) \wedge (x \sqcup y) = \top \wedge (x \sqcup y) = x \sqcup y$. Thus $\mu(y) \geq \mu(x) \wedge \mu(\lrcorner x \wedge y)$.

$\Longleftarrow$) Now let us assume that $\mu(\bot) \geq \mu(x)$ and $\mu(y) \geq \mu(x) \wedge \mu(\lrcorner x \wedge y), \forall x, y \in D$. Then we have $\mu(x \sqcup y) \leq \mu(x)$ and $\mu(x \sqcup y) \leq \mu(y)$. Thus $\mu(x \sqcup y) \leq \mu(x) \wedge \mu(y)$ and $\mu(x \sqcup y) \geq \mu(x) \wedge \mu(\lrcorner x \sqcup (x \sqcup y)) \geq \mu(y) \wedge \mu(y)$. Thus $\mu(x \sqcup y) = \mu(x) \wedge \mu(y)$.

Let x, y such that $x \sqsubseteq y$, then we have

$$\begin{aligned} \mu(x) &\geq \mu(y) \wedge \mu(\lrcorner y \wedge x) \\ &= \mu(y \sqcup (\lrcorner y \wedge x)) \quad \textit{By definition} \\ &\geq \mu(y \sqcup y) = \mu(y) \quad \textit{Proposition } 3.3 \end{aligned}$$

Finally, we have the equivalence. The case (ii) is similar. □

4 Lattices of fuzzy ideals in double Boolean algebras

Kaunyun et *al.* in [12] have introduced the concept of tip-extended. In the context of double Boolean algebra, having μ and ν to be two fuzzy ideals, it is not always true that $\mu \vee \nu$ is a fuzzy ideal too. So to solve this problem we need to introduce the concept of tip-extended in double Boolean algebra.

Definition 4.1. Let μ and ν be two fuzzy sets of $\underline{D}$. Then the tip-extended pair of μ and ν of $\underline{D}$ can be defined as follows:

$$\mu^{\nu}(x) = \begin{cases} \mu(x) & \text{if } x \neq \bot \\ \mu(\bot) \vee \nu(\bot) & \text{if } x = \bot \end{cases}$$

and

$$\nu^{\mu}(x) = \begin{cases} \nu(x) & \text{if } x \neq \bot \\ \nu(\bot) \vee \mu(\bot) & \text{if } x = \bot \end{cases}.$$

Lemma 4.2. *Let* μ *be a fuzzy ideal of* $\underline{D}$ *and* $t \in [0, 1]$. *Then*

$$\mu^{t}(x) = \begin{cases} \mu(x) & \textit{if } x \neq \bot \\ \mu(\bot) \vee t & \textit{if } x = \bot \end{cases}$$

is a fuzzy ideal of $\underline{D}$.

Proof. Let $x, y \in D$, and $t \in [0, 1]$

$$\mu^{t}(x \sqcup y) = \begin{cases} \mu(x \sqcup y) & \text{if } x \sqcup y \neq \bot \\ \mu(\bot) \vee t & \text{if } x \sqcup y = \bot \end{cases}$$

- If $x \sqcup y \neq \bot$, then we have $\mu^{t}(x \sqcup y) = \mu(x \sqcup y) = \mu(x) \wedge \mu(y)$

- If $x = \bot$ and $y \neq \bot$

$$\begin{aligned}\mu^t(x \sqcup y) &= \mu(x \sqcup y)\\ &= \mu(x) \wedge \mu(y)\\ &= (\mu(\bot) \vee t) \wedge \mu^t(y)\\ &= \mu^t(x) \wedge \mu^t(y)\end{aligned}$$

- If $x \neq \bot$ and $y \neq \bot$, then

$$\begin{aligned}\mu^t(x \sqcup y) &= \mu(x \sqcup y)\\ &= \mu(x) \wedge \mu(y)\\ &= \mu^t(x) \wedge \mu^t(y)\end{aligned}$$

Let $x, y \in D$ such that $x \sqsubseteq y$ we need to show that $\mu^t(x) \geq \mu^t(y)$.
If $y = \bot$, then $\mu^t(x) = \mu^t(y)$.
If $y \neq \bot$ and $x = \bot$ then we have $\mu^t(y) = \mu(y) \leq \mu(\bot) = \mu(x) \leq \mu(\bot) \vee t = \mu^t(x)$. If $y \neq \bot$ and $x \neq \bot$ we have $\mu^t(y) = \mu(y) \geq \mu(x) = \mu^t(x)$. Thus in any case, $\mu^t(y) \geq \mu^t(x)$. Thus for all $x, y \in D$, $\mu^t(x \sqcup y) = \mu^t(x) \wedge \mu^t(y)$.

□ In general, when ν is a non constant function, the tip-extended pair μ^ν is a fuzzy ideal. Here we defined the join of two fuzzy ideals.

Definition 4.3. Let μ and ν be two fuzzy sets of $\underline{D}$. Then the operation $\sqcup^*$ is defined as follows:

$$(\mu \sqcup^* \nu)(x) = \bigvee_{x \sqsubseteq y \sqcup z} (\mu(y) \vee \nu(z)), \quad \forall x \in D. \tag{2}$$

The following theorem characterizes the fuzzy ideal of $\underline{D}$ generated by a fuzzy subset.

Lemma 4.4. *Let μ be a fuzzy set of D. Define a fuzzy set ν of D as follows:*

$$\nu(x) = \bigvee_{x \sqsubseteq x_1 \sqcup x_2 \sqcup \cdots \sqcup x_n} (\mu(x_1) \wedge \mu(x_2) \wedge \cdots \wedge \mu(x_n)) \tag{3}$$

for some $x_1, x_2 \cdots, x_n \in D$. Then ν is the smallest fuzzy ideal of D that contains μ.

Proof. Let us first show that ν is a fuzzy ideal of D. Let $x, y \in D$. By definition of $\nu(x \sqcup y)$, we have

$$\nu(x \sqcup y) = \bigvee_{x \sqcup y \sqsubseteq x_1 \sqcup x_2 \sqcup \cdots \sqcup x_n} (\mu(x_1) \wedge \mu(x_2) \wedge \cdots \wedge \mu(x_n)).$$

By definition of $\nu(x)$ and $\nu(y)$, we have

$$\nu(x) = \bigvee_{x \sqsubseteq a_1 \sqcup a_2 \sqcup \cdots \sqcup a_n} (\mu(a_1) \wedge \mu(a_2) \wedge \cdots \wedge \mu(a_n))$$

$$\nu(y) = \bigvee_{y \sqsubseteq b_1 \sqcup b_2 \sqcup \cdots \sqcup b_m} (\mu(b_1) \wedge \mu(b_2) \wedge \cdots \wedge \mu(b_m)).$$

We know from [11] that if $x \sqsubseteq a_1 \sqcup a_2 \sqcup \cdots \sqcup a_n$ and $y \sqsubseteq b_1 \sqcup b_2 \sqcup \cdots \sqcup b_m$, then $x \sqcup y \sqsubseteq (a_1 \sqcup a_2 \sqcup \cdots \sqcup a_n) \sqcup (b_1 \sqcup b_2 \sqcup \cdots \sqcup b_m)$ thus we have the following:

$$\begin{aligned}
\nu(x) \wedge \nu(y) &= \left(\bigvee_{x \sqsubseteq a_1 \sqcup a_2 \sqcup \cdots \sqcup a_n} \mu(a_1) \wedge \mu(a_2) \wedge \cdots \wedge \mu(a_n) \right) \wedge \\
&\quad \left(\bigvee_{y \sqsubseteq b_1 \sqcup b_2 \sqcup \cdots \sqcup b_m} \mu(b_1) \wedge \mu(b_2) \wedge \cdots \wedge \mu(b_m) \right) \\
&= \bigvee (\mu(a_1) \wedge \mu(a_2) \wedge \cdots \wedge \mu(a_n) \wedge \mu(b_1) \wedge \mu(b_2) \wedge \cdots \wedge \mu(b_m)) \\
&= \nu(x \sqcup y).
\end{aligned}$$

Thus ν is a fuzzy ideal of $\underline{D}$.

Let $x \in D$, then we have that $\nu(x) \geq \mu(x)$, this shows that ν contains μ.

Let η be a fuzzy ideal of L that contains μ $(\eta(x) \geq \mu(x))$ and let $x \in D$,

$$\begin{aligned}
\nu(x) &= \bigvee_{x \sqsubseteq x_1 \sqcup x_2 \sqcup \cdots \sqcup x_n} \mu(x_1) \wedge \mu(x_2) \wedge \cdots \wedge \mu(x_n) \\
&\leq \bigvee_{x \sqcup y \sqsubseteq x_1 \sqcup x_2 \sqcup \cdots \sqcup x_n} \eta(x_1) \wedge \eta(x_2) \wedge \cdots \wedge \eta(x_n) \\
&\leq \eta(x).
\end{aligned}$$

□

Notation:

Let μ be a fuzzy set of $\underline{D}$. We denote by $< \mu >$, the fuzzy ideal generated by μ. That is the smallest fuzzy ideal containing μ.

Lemma 4.5. *Let $\underline{D}$ be a double Boolean algebra, μ and ν two fuzzy ideals of $\underline{D}$. Then $\mu^\nu \sqcup^* \nu^\mu =< \mu \vee \nu >$. That is: the fuzzy ideal generated by μ and ν is exactly the supremum between the tip-extended pair of μ and ν.*

Proof. We need first to show that $\mu^\nu \sqcup^* \nu^\mu$ is a fuzzy ideal.

Let $x, y \in D$,

$$\begin{aligned}
(\mu^\nu \sqcup^* \nu^\mu)(x \sqcup y) &= \bigvee_{x \sqcup y \sqsubseteq a \sqcup b} \mu^\nu(a) \wedge \nu^\mu(b) \\
&\geq \bigvee_{x \sqsubseteq q \sqcup p, y \sqsubseteq r \sqcup s} \mu^\nu(p \sqcup r) \wedge \nu^\mu(q \sqcup s) \\
&\geq \bigvee_{x \sqsubseteq q \sqcup p, y \sqsubseteq r \sqcup s} \mu^\nu(p) \wedge \mu^\nu(r) \wedge \nu^\mu(q) \wedge \nu^\mu(s) \\
&= \left(\bigvee_{x \sqsubseteq p \sqcup q} \mu^\nu(p) \wedge \nu^\mu(q) \right) \wedge \left(\bigvee_{y \sqsubseteq r \sqcup s} \mu^\nu(r) \wedge \nu^\mu(s) \right) \\
&= (\mu^\nu \sqcup^* \nu^\mu)(x) \wedge (\mu^\nu \sqcup^* \nu^\mu)(y).
\end{aligned}$$

Let $x, y \in D$ such that $x \sqsubseteq y$, we need to show that $(\mu^\nu \sqcup^* \nu^\mu)(x) \geq (\mu^\nu \sqcup^* \nu^\mu)(y)$

If $y = \bot$ then it is obvious that $x = \bot$ and the result holds.
If $y \neq \bot$

$$(\mu^\nu \sqcup^* \nu^\mu)(x) = \bigvee_{x \sqsubseteq t \sqcup z} \mu^\nu(t) \wedge \nu^\mu(z) \tag{4}$$

$$(\mu^\nu \sqcup^* \nu^\mu)(y) = \bigvee_{y \sqsubseteq r \sqcup s} \mu^\nu(r) \wedge \nu^\mu(s) \tag{5}$$

since $x \sqsubseteq y$ so by transitivity of $\sqsubseteq$, $x \sqsubseteq r \sqcup s$.
Thus $\bigvee_{y \sqsubseteq r \sqcup s} \mu^\nu(r) \wedge \nu^\mu(s) \leq \bigvee_{x \sqsubseteq t \sqcup z} \mu^\nu(t) \wedge \nu^\mu(z)$. That is $(\mu^\nu \sqcup^* \nu^\mu)(x) \leq (\mu^\nu \sqcup^* \nu^\mu)(x)$
Thus $\mu^\nu \sqcup^* \nu^\mu$ is a fuzzy ideal of D. Now let us show that we have $\mu^\nu \sqcup^* \nu^\mu \geq \mu \vee \nu$ to conclude with Lemma 4.2 and Lemma 4.4.
Let $x \in D$,

$$\begin{aligned}(\mu^\nu \sqcup^* \nu^\mu)(x) &= \bigvee_{x \sqsubseteq y \sqcup z} \mu^\nu(y) \wedge \nu^\mu(z) \\ &\geq \mu^\nu(x) \wedge \nu^\mu(\bot) \\ &\geq \mu(x) \wedge \mu(\bot) \\ &= \mu(x).\end{aligned}$$

Thus $\mu^\nu \sqcup^* \nu^\mu$ contains μ. Similarly,

$$\begin{aligned}(\mu^\nu \sqcup^* \nu^\mu)(x) &= \bigvee_{x \sqsubseteq y \sqcup z} \mu^\nu(y) \wedge \nu^\mu(z) \\ &\geq \nu^\mu(x) \wedge \mu^\nu(\bot) \\ &\geq \nu(x) \wedge \nu(\bot) \\ &= \nu(x).\end{aligned}$$

Let η be a fuzzy ideal of D containing $\mu \vee \nu$. We need to show that $\eta(x) \geq (\mu^\nu \sqcup^* \nu^\mu)(x)$
Let $x \in D$, if $x = \bot$, then $(\mu^\nu \sqcup^* \nu^\mu)(\bot) = \mu(\bot) \vee \nu(\bot) \leq \eta(\bot)$.
If $x \neq \bot$

$$\begin{aligned}(\mu^\nu \sqcup^* \nu^\mu)(x) &= \bigvee_{x \sqsubseteq y \sqcup z} \mu^\nu(y) \wedge \nu^\mu(z) \\ &= \left(\bigvee_{x \sqsubseteq y \sqcup z, y \neq \bot, z \neq \bot} \mu(y) \wedge \nu(z)\right) \vee \left(\bigvee_{x \sqsubseteq y} \mu(y)\right) \vee \left(\bigvee_{x \sqsubseteq z} \nu(z)\right) \\ &= \left(\bigvee_{x \sqsubseteq y \sqcup z, y \neq \bot, z \neq \bot} \mu(y) \wedge \nu(z)\right) \vee \left(\bigvee_{x \sqsubseteq y} \mu(y)\right) \vee \left(\bigvee_{x \sqsubseteq z} \nu(z)\right) \\ &\leq \left(\bigvee_{x \sqsubseteq y \sqcup z, y \neq \bot, z \neq \bot} \eta(y) \wedge \eta(z)\right) \vee \left(\bigvee_{x \sqsubseteq y} \eta(y)\right) \vee \left(\bigvee_{x \sqsubseteq z} \eta(z)\right) \\ &= \bigvee_{x \sqsubseteq y \sqcup z} \eta(y) \wedge \eta(z) \\ &\leq \eta(x).\end{aligned}$$

Thus $\mu^\nu \sqcup^* \nu^\mu = < \mu \vee \nu >$. □

Theorem 4.6. *If we consider $FI(D)$ to be the collection of all fuzzy ideals of $\underline{D}$, then $(FI(D), \sqcup^*, \sqcap^*, \underline{0}, \underline{1})$ is a bounded lattice, where $\sqcap^*$ is defined as follows: $\mu \sqcap^* \nu = \mu \wedge \nu$, $\underline{0} : D \mapsto [0,1]$, $\underline{0}(x) = 0$, for all $x \in D$. $\underline{1} : D \mapsto [0,1]$, and $\underline{1}(x) = 1$, for all $x \in D$.*

Proof. From the previous proposition, any pair of elements of $FI(D)$ has a supremum (Lemma 4.5) and the upper bound is $\underline{1}$, the lower bound is $\underline{0}$. □

Theorem 4.7. *The map*

$$\overline{\varphi}\colon FI(D_\sqcup) \longrightarrow FI(D)$$
$$\mu \longmapsto \tilde{\mu}$$

Where

$$\tilde{\mu}\colon D \longrightarrow [0,1]$$
$$x \longmapsto \mu(x \sqcup x)$$

is an isomorphism of lattices.

Proof. The map $\overline{\varphi}$ is well defined, in fact for $\mu \in FI(D_\sqcup)$, $\tilde{\mu} \in FI(D)$ according to Proposition 3.2. $\overline{\varphi}$ preserve the order $\leq$. Suppose $\mu_1 \leq \mu_2$ then let $x \in D$, $\tilde{\mu_1}(x) = \mu_1(x \sqcup x) \leq \mu_2(x \sqcup x)$. Thus $\tilde{\mu_1}(x) \leq \tilde{\mu_2}(x)$. Hence $\tilde{\mu_1} \leq \tilde{\mu_2}$. Conversely, if $\tilde{\mu_1} \leq \tilde{\mu_2}$, then $\tilde{\mu}_{1/D_\sqcup} = \tilde{\mu}_{2/D_\sqcup}$ i.e $\mu_1 \leq \mu_2$. Thus $\mu_1 \leq \mu_2 \Longleftrightarrow \overline{\varphi}(\mu_1) \leq \overline{\varphi}(\mu_2)$. $\overline{\varphi}$ is surjective since for any $\mu \in FI(D)$, $\mu_{/D_\sqcup} \in FI(D_\sqcup)$ and $\overline{\varphi}\left(\mu_{/D_\sqcup}\right) = \mu$. $\overline{\varphi}$ is injective, in fact let $\mu_1, \mu_2 \in FI(D_\sqcup)$ such that $\tilde{\mu_1} = \tilde{\mu_2}$ Let us show that $\mu_1 = \mu_2$. Let $x \in D_\sqcup$,

$$\begin{aligned} \mu_1(x) &= \mu_1(x \sqcup x) \;\; by\ (i)\ of\ Proposition\ 3.2 \\ &= \tilde{\mu_1}(x) \;\; by\ definition\ of\ \overline{\varphi} \\ &= \tilde{\mu_2}(x) \;\; by\ hypothesis\ (\tilde{\mu_1} = \tilde{\mu_2}) \\ &= \mu_2(x \sqcup x) \\ &= \mu_2(x). \end{aligned}$$

Thus $\mu_1 = \mu_2$. □

5 Lattices of generalized fuzzy ideals in dbas

In this section, we introduce the notion of generalized fuzzy ideal which is a more general definition of fuzzy ideal on dBas. Namely we show that the collection of generalized fuzzy ideals of a dBa is endowed with a lattice structure.

Let us first define the concept of generalized fuzzy ideal.

Let $m, n \in [0,1]$ and $m < n$, then a fuzzy set μ of D is called a generalized fuzzy ideal of D if:

(I_1) $\mu(x \sqcup y) \vee m \geq \mu(x) \wedge \mu(y) \wedge n$

(I_2) $x \sqsubseteq y \Longrightarrow \mu(x) \vee m \geq \mu(y) \wedge n$.

Here generalized fuzzy ideals are defined with respect to a fixed pair (m, n). We denote by $GFI(D)$ the set of all generalized fuzzy ideals.

Let $m, n \in [0,1]$ and $m < n$. Then the fuzzy set μ of $\underline{D}$ is called generalized fuzzy ideal if :

(I_3) $\mu(\perp) \vee m \geq \mu(x) \wedge n$;

(I_4) $\mu(y) \vee m \geq \mu(x) \wedge \mu(\lrcorner x \sqcup y) \wedge n$.

Proof. Let μ be a generalized fuzzy ideal of $\underline{D}$. Since $\perp \sqsubseteq x$, for any $x \in D$, it follows that $\mu(\perp) \vee m \geq \mu(x) \wedge n$. On other hand, since $y \sqsubseteq x \sqcup (\lrcorner x \sqcup y)$, we have $\mu(y) \vee m \geq \mu\left(x \sqcup (\neg x \sqcup y)\right) \wedge n$.

Thus one can write:

$$\begin{aligned}
\mu(y) \vee m \vee m &= \mu(y) \vee m \\
&\geq \left(\mu\left(x \sqcup (\lrcorner x \sqcup y)\right) \wedge n\right) \vee m \\
&= (\mu(x \sqcup (\lrcorner x \sqcup y)) \vee m) \wedge (n \vee m) \\
&\geq \mu(x) \wedge \mu(\lrcorner x \sqcup y) \wedge n \wedge n \\
&= \mu(x) \wedge \mu(\lrcorner x \sqcup y) \wedge n.
\end{aligned}$$

Conversely, let us now assume that I_3 and I_4 hold. Let $x, y \in L$ and $x \sqsubseteq y$. Then $\lrcorner y \sqsubseteq \lrcorner x$ and $x \wedge \lrcorner y \sqsubseteq x \wedge \lrcorner y$ by compatibility. So $\mu(\perp) = \mu(\neg y \wedge x)$ and from (I_2) we have that

$$\begin{aligned}
\mu(x) \vee m &\geq \mu(y) \wedge \mu(\neg y \wedge x) \wedge n \\
&= \mu(y) \wedge \mu(\perp) \wedge n.
\end{aligned}$$

Thus

$$\begin{aligned}
\mu(x) \vee m \vee m &\geq ((\mu(y) \wedge n) \vee m) \wedge (\mu(\perp) \vee m) \\
&\geq (\mu(y) \wedge n) \vee m) \wedge (\mu(y) \wedge n) \\
&= \mu(y) \wedge n.
\end{aligned}$$

This implies that I_2 holds. On the other hand, since $\lrcorner x \wedge (x \sqcup y) \sqsubseteq y$,
we have $\mu\left(\lrcorner x \wedge (x \sqcup y)\right) \vee m \geq \mu(y) \wedge n$ and from I_4 we have that $\mu(x \sqcup y) \vee m \geq \mu(x) \wedge \mu(\lrcorner x \wedge (x \sqcup y)) \wedge n$.
Thus

$$\begin{aligned}
\mu(x \sqcup y) \vee m \vee m &\geq ((\mu(x) \wedge n) \vee m) \wedge (\mu(y) \wedge n) \vee m \\
&= ((\mu(x) \wedge n) \vee m) \wedge \left(\mu\left(\lrcorner x \wedge (x \sqcup y)\right) \vee m\right) \\
&= ((\mu(x) \wedge n) \vee m) \wedge (\mu(y) \wedge n) \\
&= (\mu(x) \wedge n) \wedge (\mu(y) \wedge n) \\
&= \mu(x) \wedge \mu(y) \wedge n.
\end{aligned}$$

Therefore I_1 is satisfied and this shows that μ is a generalized fuzzy ideal of D. □

Theorem 5.1. *Let $m, n \in [0, 1]$ and $m < n$, Then a fuzzy set μ of D is a generalized fuzzy ideal if and only if for all $x, y, z \in D$,*

$$x \sqsubseteq y \sqcup z \Longrightarrow \mu(x) \vee m \geq \mu(y) \wedge \mu(z) \wedge n.$$

Proof. Let μ be a generalized fuzzy ideal of $\underline{D}$ and $x \sqsubseteq y \sqcup z$, then $\mu(x) \vee m \geq \mu(y \sqcup z) \wedge n$.

Hence one can write

$$\begin{aligned}
\mu(x) \vee m &\geq (\mu(y \sqcup z) \vee m) \wedge (n \vee m) \\
&\geq \mu(y) \wedge \mu(z) \wedge n \wedge n \\
&= \mu(y) \wedge \mu(z) \wedge n.
\end{aligned}$$

Thus $\mu(x) \vee m \geq \mu(y) \wedge \mu(z) \wedge n$, conversely, since we know that we have $\mu(\perp) \vee m \geq \mu(x) \wedge \mu(x) \wedge n$. Thus $\mu(\perp) \vee m \geq \mu(x) \wedge n$.

On the other side, since we know that $y \sqsubseteq x \sqcup (\lrcorner x \wedge y)$ we have:

$$\mu(y) \vee m \geq \mu(x) \wedge \mu(\lrcorner x \wedge y) \wedge n$$

Hence μ is a generalized fuzzy ideal of D.

□

Example 5.2. By considering the dBa of example 2.2, we can easily show that the following is a generalized fuzzy dBa:

$$\mu(x) = \begin{cases} \frac{1}{3} & if \quad x \in \{a, b, c, d, e, \top\} \\ 1 & if \quad x = \perp \end{cases}$$

In fact, let $m, n \in [0, 1]$ such that $m < n$. Then the following property is verify:

$$x \sqsubseteq y \sqcup z \Longrightarrow \mu(x) \vee m \geq \mu(z) \wedge n.$$

Corollary 5.3. *Let $m, n \in [0, 1]$ and $m < n$. Then a fuzzy set μ of $\underline{D}$ is called a generalized fuzzy ideal if and only if for all $x, y_1, \cdots, y_n \in D$, $x \sqsubseteq y_1 \sqcup y_2 \sqcup \cdots \sqcup y_n$ implies that $\mu(x) \vee m \geq \mu(y_1) \wedge \cdots \wedge \mu(y_n) \wedge n$.*

Let μ be a fuzzy set of $\underline{D}$, $m, n \in [0, 1]$ and $m < n$. Then the intersection of all generalized fuzzy ideals containing μ is called the generated generalized fuzzy ideal by μ, denoted $< \mu >^{(m,n)}$.

Theorem 5.4. *Let $\underline{D}$ be a double Boolean algebra μ be a fuzzy set of L, $m, n \in [0, 1]$ and $m < n$ then*

$$\begin{aligned} \langle \mu \rangle^{(m,n)}(x) &= m \vee \left(\bigvee_{x \sqsubseteq a_1 \sqcup a_2 \sqcup \cdots \sqcup a_n} \mu(a_1) \wedge \cdots \wedge \mu(a_n) \wedge n \right) \\ &= \bigvee_{x \sqsubseteq a_1 \sqcup a_2 \sqcup \cdots \sqcup a_n} (\mu(a_1) \vee m) \wedge \cdots \wedge (\mu(a_n) \vee m) \wedge n, \quad for\ all\ x \in D. \end{aligned}$$

Proof. Let

$$\theta(x) = m \vee \left(\bigvee_{x \sqsubseteq a_1 \sqcup a_2 \sqcup \cdots \sqcup a_n} \mu(a_1) \wedge \cdots \wedge \mu(a_n) \wedge n \right).$$

Let us show that θ is a generalized fuzzy ideal which contains μ.

For $n = 1$ and $a_1 = x$, $m \vee (n \wedge \mu(x)) \leq \theta(x)$. Hence $n \wedge \mu(x) \leq \theta(x) \vee m$. We can easy check that θ is a generalized fuzzy ideal. Now let us focus on how to show that θ is the smallest generalized fuzzy ideal which contains μ.

Let us assume that there is a generalized fuzzy ideal η such that $\forall x \in D$, $\eta(x) \vee m \geq n \wedge \mu(x)$.

Then we need to show that η contains θ too.

$$\begin{aligned} n \wedge \theta(x) &= n \wedge \left(m \vee \left(\bigvee_{x \sqsubseteq a_1 \sqcup a_2 \sqcup \cdots \sqcup a_n} \mu(a_1) \wedge \mu(a_2) \wedge \cdots \wedge \mu(a_n) \wedge n \right) \right) \\ &\leq m \vee \left(\bigvee_{x \sqsubseteq a_1 \sqcup a_2 \sqcup \cdots \sqcup a_n} \eta(x) \wedge \cdots \wedge \eta(x) \wedge n \right) \\ &\leq m \vee \eta(x). \end{aligned}$$

Thus θ is the smallest generalized fuzzy ideal which contains μ. □

Definition 5.5. Let μ and ν be two fuzzy sets of L, $m, n \in [0,1]$ and $m < n$ the operation $\tilde{\sqcup}^{(m,n)}$ is defined by:

$$\mu\tilde{\sqcup}^{(m,n)}\nu = \bigvee_{x \sqsubseteq y \sqcup z} \left((\mu(y) \vee m) \wedge (\nu(z) \vee m) \wedge n\right).$$

Let $GFI(D)$ the set of generalized fuzzy ideals of $\underline{D}$.

Remark 5.6. Let μ be a generalized fuzzy ideal of D and $t \in [0,1]$. Then μ^t is also a generalized fuzzy ideal of $\underline{D}$.

Theorem 5.7. *Let $\underline{D}$ be a double Boolean algebra, $m, n \in [0,1]$ such that $m < n$. Then $\mu^\nu\tilde{\sqcup}^{(m,n)}\nu^\mu = \langle\mu \vee \nu\rangle^{(m,n)}$. More over, $(GFI(D), \tilde{\sqcap}, \tilde{\sqcup}, \underline{0}, \underline{1})$ is a bounded lattice, where $GFI(D)$ the set of generalized fuzzy ideals of $\underline{D}$.*

Proof.

$$\begin{aligned}
m \vee \left(\mu^\nu\tilde{\sqcup}^{(m,n)}\nu^\mu\right)(x \wedge y) &= \left(\mu^\nu\tilde{\sqcup}^{(m,n)}\nu^\mu\right)(x \wedge y)\\
&= \bigvee_{x \sqcup y \sqsubseteq u \sqcup v} \left((\mu^\nu(u) \vee m) \wedge (\nu^\mu(v) \vee m) \wedge n\right)\\
&\geq \bigvee_{x \sqsubseteq p \sqcup q, y \sqsubseteq r \sqcup s} \left(\mu^\nu(p \sqcup r) \vee m\right) \wedge \left(\nu^\mu(q) \wedge \nu^\mu(p) \wedge n\right)\\
&= \left(\mu^\nu\tilde{\sqcup}^{(m,n)}\nu^\mu\right)(x) \wedge \left(\mu^\nu\tilde{\sqcup}^{(m,n)}\nu^\mu\right)(y)\\
&= \left(\mu^\nu\tilde{\sqcup}^{(m,n)}\nu^\mu\right)(x) \wedge \left(\mu^\nu\tilde{\sqcup}^{(m,n)}\nu^\mu\right)(y) \wedge n.
\end{aligned}$$

Let $x, y \in D$ and $x \sqsubseteq y$. Then it is easy to see that $m \vee (\mu^\nu\tilde{\sqcup}\nu^\mu)(x) \geq n \wedge (\mu^\nu\tilde{\sqcup}\nu^\mu)(y)$.
So $\mu^\nu\tilde{\sqcup}^{(m,n)}\nu^\mu$ is a generalized fuzzy ideal of $\underline{D}$.
Let $x \in D$,

$$\begin{aligned}
m \vee \left(\mu^\nu\tilde{\sqcup}^{(m,n)}\nu^\mu\right)(x) &= \left(\mu^\nu\tilde{\sqcup}^{(m,n)}\nu^\mu\right)(x)\\
&= \bigvee_{x \sqsubseteq y \sqcup z} \left((\mu^\nu(y) \vee m) \wedge (\nu^\mu(z) \vee m) \wedge n\right)\\
&\geq \left((\mu^\nu(x) \vee m) \wedge (\nu^\mu(\bot) \vee m) \wedge n\right)\\
&\geq \left((\mu^\nu(x) \vee m) \wedge (\mu^\nu(\bot) \vee m) \wedge n\right)\\
&= \left((\mu^\nu(x) \wedge n) \vee m\right)\\
&\geq \mu^\nu(x) \wedge n\\
&\geq \mu(x) \wedge n.
\end{aligned}$$

Hence $m \vee \mu^\nu\tilde{\sqcup}\nu^\mu \geq \mu \wedge n$. In a similar way, we have $m \vee \mu^\nu\tilde{\sqcup}\nu^\mu \geq \nu \wedge n$.
Hence $\mu^\nu\tilde{\sqcup}\nu^\mu \geq \mu \vee \nu^{(m,n)}$.
Last, let us verify that is the smallest. Let $\lambda \in GFI(D)$ such that $n \wedge (\mu(x) \vee \nu(x)) \leq \lambda(x) \wedge n$.
Consider the following cases:
If $x = 0$ then $n \wedge \left(\mu^\nu\tilde{\sqcup}^{(m,n)}\nu^\mu\right)(\bot) = \left(\mu^\nu\tilde{\sqcup}^{(m,n)}\nu^\mu\right)(0) = (\mu(\bot) \vee \nu(\bot) \vee m) \wedge n \leq \lambda(\bot) \vee m$.

Else:

$$\begin{aligned} n \wedge \left(\mu^{\nu} \tilde{\sqcup}^{(m,n)} \nu^{\mu}\right)(x) &= \left(\mu^{\nu} \tilde{\sqcup}^{(m,n)} \nu^{\mu}\right)(x) \\ &= \bigvee_{x \sqsubseteq y \sqcup z} (((\mu^{\nu}(y) \vee m)) \wedge (\nu^{\mu}(y) \vee m) \wedge n) \\ &\leq m \vee \bigvee_{x \sqsubseteq y \sqcup z} (\lambda(y) \wedge \lambda(z) \wedge n) \\ &\leq m \vee \lambda(x). \end{aligned}$$

Thus

$$n \wedge \left(\mu^{\nu} \tilde{\sqcup}^{(m,n)} \nu^{\mu}\right)(x) \leq m \vee \lambda(x)$$

We can conclude that $\mu^{\nu} \tilde{\sqcup}^{(m,n)} \nu^{\mu} = \langle \mu \vee \nu \rangle^{(m,n)}$. Thus we have just proved that any pair of elements of $FI(D)$ has a supremum. The upper bound is $\underline{1}$, the lower bound is $\underline{0}$.

□

6 Conclusion

In this paper fuzzy ideals on double Boolean algebras have been studied. Various properties and characterizations of fuzzy ideals have been proved. Based on the notion of tip-extended pair inspired by Kuanyun [12], it was proved that the set of fuzzy ideals on double Boolean algebra is endowed with a structure of the bounded lattice. This is a new type of lattice structure constructed from fuzzy ideals. This lattice structure has the particularity that it is isomorphic to the collection of fuzzy ideals of the Boolean algebra $D_{\sqcup}$. Thus we can conclude that the collection of fuzzy ideals of the dBa D can fully be determined by knowing just fuzzy ideals of the Boolean algebra $D_{\sqcup}$. We have also introduced the concept of fuzzy primary ideals and fuzzy primary filters and have established that the set of generalized fuzzy ideals of a double Boolean algebra has the structure of a bounded lattice. Generalized fuzzy ideals of D can also be entirely determined by generalized fuzzy ideals of $D_{\sqcup}$. In the future, it will be interesting to investigate L-fuzzy ideals (where L is a bounded lattice) and generalized L-fuzzy ideals in the framework of double Boolean algebras.

Conflict of Interest: The author declares no conflict of interest.

References

[1] Attallah M. Complete Fuzzy Prime Ideals of Distributives Lattices. *The journal of fuzzy Mathematics*, 2000; 8(1): 151-156.

[2] Awoke T, Rout M, Mohandi L, and Satapathy S. Bitcoin Price Prediction and Analysis Using Deep Learning Models. *Networks and Systems,*. 2020; 1-5. DOI: https://doi.org/10.1007/978-981-15-5397-4-63

[3] Azari A, Papapetrou P, Denic S and Peters G. Cellular Traffic Prediction and Classification: a comparative evaluation of LSTM and ARIMA. *arXiv.* 2019; 1-8. DOI: https://doi.org/10.48550/arXiv.1906.00939

[4] Chon I. Fuzzy Partial Order Relations and Fuzzy Lattices. *Korean Journal Mathematics.* 2009; 17(4): 361-374.

[5] Fotso Tatuéné A, Tonga M. *Transfer of structure between a universal algebra and its fuzzy algebra.* 2016.

[6] Islam S, Sanyal A and Sen J. Fuzzy filters of IL-algebras. *Foundation, algebraic, and analytical methods in soft computing, Springer*. 2022; 26: 7017-7027. DOI: https://doi.org/10.1007/s00500-022-06985-1

[7] Koguep BBN, Nkuimi C and Lele C. On Fuzzy Primes Ideals of Lattices. *SAMSA Journal of Pure and Applied Mathematics, (SJPAM)*. 2008; (3): 1-11.

[8] Kwuida L. Prime ideal theorem for double Boolean algebra. *Discussiones Mathematicae. General algebra and Applications*. 2007; 27: 263-275. DOI: 10.7151/dmgaa.1130.

[9] Howlader P and Banerjee M. Remarks on Prime Ideal and Representation Theorems for Double Boolean Algebras. *Proceedings of the 15th International conference on Concept Lattices and their Applications, CLA*. 2020; 83-94.

[10] Mezzomo I, Bredegal BC, Santiago RHN and Reiser RHS. Fuzzy ideals of product operators on bounded fuzzy lattices. *Second workshop-school of theoretical computer sciences*. 2014; 1-7.

[11] Tenkeu Jeufack Y, Temgoua Alomo E and Kwuida L. Filters Ideals and Congruences on Double Boolean Algebras. *International conference on Formal Concept analysis*. 2021; 270-280. DOI: https://doi.org/10.1007/978-3-030-77867-5-18

[12] Zhu K, Wang J and Yang Y. Lattices of (Generalized) Fuzzy Ideals in Residuated Lattices. *International Journal of Applied Mathematics*. 2020; 50(3).

Fernand Kuiebove Pefireko
Department of Mathematics
University of Yaoundé 1
Yaoundé, Cameroon
E-mail: fkuiebove@yahoo.fr

Transactions on Fuzzy Sets and Systems (TFSS)

URL: https://tfss.journals.iau.ir/

Online ISSN: 2821-0131

Vol.2, No.2, (2023), 155-183

DOI: http://doi.org/10.30495/tfss.2023.1990572.1079

Article Type: Original Research Article

An Artificial Intelligence Framework for Supporting Coarse-Grained Workload Classification in Complex Virtual Environments

Alfredo Cuzzocrea*, **Enzo Mumolo**, **Islam Belmerabet**, **Abderraouf Hafsaoui**

Abstract. We propose *Cloud-based machine learning tools for enhanced Big Data applications*, where the main idea is that of predicting the "*next*" *workload* occurring against the target Cloud infrastructure via an innovative *ensemble-based approach* that combines the effectiveness of different well-known *classifiers* in order to enhance the whole accuracy of the final classification, which is very relevant at now in the specific context of *Big Data*. The so-called *workload categorization problem* plays a critical role in improving the efficiency and reliability of Cloud-based big data applications. Implementation-wise, our method proposes deploying Cloud entities that participate in the distributed classification approach on top of *virtual machines*, which represent classical "commodity" settings for Cloud-based big data applications. Given a number of known reference workloads, and an unknown workload, in this paper we deal with the problem of finding the reference workload which is most similar to the unknown one. The depicted scenario turns out to be useful in a plethora of modern information system applications. We name this problem as *coarse-grained workload classification*, because, instead of characterizing the unknown workload in terms of finer behaviors, such as CPU, memory, disk, or network intensive patterns, we classify the whole unknown workload as one of the (possible) reference workloads. Reference workloads represent a category of workloads that are relevant in a given applicative environment. In particular, we focus our attention on the classification problem described above in the special case represented by *virtualized environments*. Today, *Virtual Machines* (VMs) have become very popular because they offer important advantages to modern computing environments such as cloud computing or server farms. In virtualization frameworks, workload classification is very useful for accounting, security reasons, or user profiling. Hence, our research makes more sense in such environments, and it turns out to be very useful in a special context like Cloud Computing, which is emerging now. In this respect, our approach consists of running several machine learning-based classifiers of different workload models, and then deriving the best classifier produced by the *Dempster-Shafer Fusion*, in order to magnify the accuracy of the final classification. Experimental assessment and analysis clearly confirm the benefits derived from our classification framework. The running programs which produce unknown workloads to be classified are treated in a similar way. A fundamental aspect of this paper concerns the successful use of data fusion in workload classification. Different types of metrics are in fact fused together using the Dempster-Shafer theory of evidence combination, giving a classification accuracy of slightly less than 80%. The acquisition of data from the running process, the pre-processing algorithms, and the workload classification are described in detail. Various classical algorithms have been used for classification to classify the workloads, and the results are compared.

AMS Subject Classification 2020: 62H30; 68T10

Keywords and Phrases: Virtual machines, Workload, Dempster-Shafer theory, Classification.

∗**Corresponding Author:** Alfredo Cuzzocrea, Email: alfredo.cuzzocrea@unical.it, ORCID: 0000-0002-7104-6415

Received: 4 July 2023; **Revised**: 5 September 2023; **Accepted**: 6 September 2023; **Available Online**: 7 September 2023; **Published Online**: 7 November 2023.

How to cite: Cuzzocrea A, Mumolo E, Belmerabet I, Hafsaoui A. An artificial intelligence framework for supporting coarse-grained workload classification in complex virtual environments. *Trans. Fuzzy Sets Syst.* 2023; 2(2): 155-183. DOI: http://doi.org/10.30495/tfss.2023.1990572.1079

1 Introduction

In this paper, we propose *Cloud-based machine learning tools for enhanced big data applications* (e.g., [34, 7, 21]), where the main idea is that of predicting the "*next*" *workload* occurring against the target Cloud infrastructure via an innovative *ensemble-based* (e.g., [47]) *approach* combining the effectiveness of different well-known *classifiers* in order to enhance the whole accuracy of the final classification, which is very relevant at now in the specific context of *Big Data* (e.g., [17]). So-called *workload categorization problem* plays a critical role in improving the efficiency and the reliability of Cloud-based big data applications (e.g., [60, 62]). Implementation-wise, our method proposes deploying Cloud entities that participate to the distributed classification approach on top of *virtual machines* (e.g., [28]), which represent classical "commodity" settings for Cloud-based big data applications (e.g., [37]).

Virtualization technology has become fundamental in modern computing environments such as cloud computing [9, 13, 22, 8] and server farms [57, 23]. By running multiple virtual machines on the same hardware, virtualization allows us to achieve a high utilization of the available hardware resources. Moreover, virtualization brings advantages in security, reliability, scalability, and resource management (e.g., [10, 58, 12]). Resource management in the virtualized context can be performed by *classifying the workload of the virtualized application* (e.g., [66]). As a consequence, workload characterization and prediction have been widely studied during past research efforts (e.g., [14, 3]). More recently, some work has been done on workload characterization in data center environments [26]. On the other hand, workload modeling and prediction in virtualization environments have been addressed in [24, 27, 2], while a virtualized workload balancing approach is described in [29] which uses virtual machine migration, and another approach that focuses on server farms is presented in [61]. From a methodological point of view, *workload classification* is a critical task that integrates the previously-mentioned ones, is performed by collecting suitable metrics during the execution of reference applications, and running a pattern classifier on the collected data, which allows us to discriminate among the different classes. At a base level, the workload can be classified as CPU-intensive or I/O-intensive. In [32], Hu *et al.* perform asymmetric virtual machine scheduling based on this base classification level. At a finer level, the workload can be classified as CPU-intensive, memory-intensive, disk read/write-intensive, and network I/O-intensive. Zhao *et al.* [66] describe a workload classification model based on such a finer classification level. In [65], Zhang *et al.* address the problem of automatically selecting the metrics which provide the best accuracy in the classification task. Also, it has been studied that workloads can be classified by considering memory references as signals, which can be analyzed using spectral parameters (e.g., [53, 42]). Results on instrumented machines and in simulation show that *Hidden Markov Model* (HMM) *classifiers* [4] can be used to model memory references created and managed by processes under execution.

In our proposed research, the classification phase works as follows. First, in a virtualized environment, we run some programs we take as reference (in this work, we make use of the well-known *SPEC CINT2006 benchmarks* [54]) and, then, from their execution, we extract some features using the APIs of the *Virtual Machine Monitor*. With these so-collected features, we train a model of the workload of each benchmark program according to various and well-understood machine learning algorithms. Unknown programs are executed in the same environment, and their features are fed to models of the reference workloads, in order to find the belief that the unknown workload could be associated with each model. Finally, beliefs obtained by means of different classification algorithms are fused using the *Dempster-Shafer rule of evidence combination* [48] in order to derive a higher-quality classifier (e.g., [6]).

In particular, we discriminate among application workloads. In fact, we run the SPEC2006 benchmarks under a virtualized operating system, and we collect some features through the Virtual Machine Monitor. Using machine learning algorithms, we develop a model for each workload. Unknown workloads are then classified among the different models. The classification among application workloads running in virtualization gives interesting potential applications. For example, if the benchmarks are chosen appropriately, it may be determined what the main characteristics of the processes running in the virtual machine are. Another possibility might be to know what are the processes that a given customer typically executes. Other possible applications are in the area of *malware detection* [31]. In this respect, running processes can be monitored to see if their workload is the same or if it changes over time (e.g., [35]). Preliminary experimental assessment and analysis clearly confirm the benefits derived from our classification framework.

The remaining part of this paper is organized as follows. Section 2 considers related work relevant for our research. In Section 3, we highlight the process of workload categorization using common classification algorithms. Section 4 presents the SPEC 2006 Benchmarks used in this work. In Section 5, we provide a description of the virtual environment settings used to elaborate our experiments. Section 6 introduces the aspects on which we based our data analysis (i.e., memory reference and resource demand). In Section 7, we present a detailed description of our methodology along with the used classification algorithms, i.e., Neural Networks, Hidden Markov Models, k-NN, and ARMA. Section 8 demonstrates the fundamentals of the Dempster-Shafer theory of evidence adopted in our approach. Section 9 shows an innovative case study where we describe workload categorization in the context of anomaly detection. In Section 10, we report our extensive experimental assessment and the obtained results. Finally, Section 11 contains conclusions and future work of our research.

2 Related Work

The problem of workload categorization has gained a great deal of attention from researchers, and as a result, several works have appeared in the active literature. In this Section, we will discuss some of the most relevant to our work.

In [36], the authors provide a solution to reduce the risk of incidents and injury in *hazardous work conditions*, especially in the forestry industry, which is one of the most dangerous industries in New Zealand, by proposing a *semantic paradigm for workload classification*. The model takes a collection of *multi-modal physiological measures* as input and categorizes a sequence of workloads (resting, cognitive, and physical workloads). The proposed model was subjected to a series of experimental assessments with participants ranging in age from 22 to 39 based on three different scenarios: (*i*) relaxing and refraining from any physically or intellectually demanding tasks; (*ii*) performing a cognitively intense activity; (*iii*) walking, jogging, and running. The obtained results in these experiments achieved an average accuracy of 89% for resting workload, 76% for cognitive workload, and 97% for physical workload. Finally, the contribution reported in this work, by proposing the model to forecast fatigue in hazardous sectors, opens the doors to a wider research initiative focused on technological applications in hazardous work situations.

[55] presents a *workload categorization-based resource allocation framework* for balancing the load between active physical machines and leveraging their resource capacities. The *CloudSim simulator* is used to run simulation-based experiments using three separate sets of tasks having 10000, 20000, and 30000 tasks. During the experiments, the imbalance in workload among active physical machines and the disparity in resource utilization, specifically CPU and RAM, are observed and measured. According to the simulation results, the proposed framework outperforms similar methods in the literature in terms of balancing the load among active physical machines and using their various resource capabilities.

In [40], they focus the attention on performance testing in new application developments and propose a *performance engineering strategy* that extracts the workload of an existing legacy *Enterprise Resource Planning* (ERP) application with over 1 million users and produces workload for a new version of this application. The proposed method demonstrates that (*i*) workload for new application testing and architecture validation can be generated from legacy application behavior; (*ii*) end user organizations have significantly different usage patterns; (*iii*) high-level operations provide a useful method for analyzing and generating workload for ERP applications as opposed to low-level page views. The experimental tests of the proposed method performed on a Dutch software firm show that leveraging this approach gives better results in performance engineering.

In [41], the authors investigate and classify *Infrastructure as a Service* (IaaS) cloud workloads into patterns based on their behavioral features as effective characterization of workloads plays a crucial role in driving *Capacity Planning and Performance Management* in IaaS Cloud environments. Various workload metrics, including CPU utilization, memory usage, throughput, and response time, can be leveraged and modeled to understand their interrelationships. Furthermore, different types of behavioral patterns that can be observed within workloads and an outline of statistical techniques to be employed in identifying and determining these patterns are presented in this work. To support their research, they present initial results obtained from the analysis of development workload data collected in a controlled lab environment. These results highlight the potential of the proposed approach in uncovering meaningful workload patterns and highlight the importance of effective workload characterization for efficient Capacity Planning and Performance Management in IaaS Clouds.

[44] introduces the analysis of Cloud workloads and evaluation of the effectiveness of two commonly used prediction techniques, namely *Markov Modeling* and *Bayesian Modeling*, using a dataset comprising 7 hours of Google cluster data. The primary objective is to assess the performance of these methodologies in accurately forecasting user demand. Moreover, a key aspect of this study involves the categorization and characterization of Cloud workloads, which enables the modeling of essential parameters for user demand forecasting. By understanding the patterns and characteristics of workloads, the authors aim to enhance the accuracy of demand prediction, thereby facilitating efficient resource allocation and energy consumption management, they present an optimal solution to minimize idle resources and reduce unnecessary energy consumption while ensuring *Quality of Service* (QoS) maintenance. Through the experimental analysis and assessments, the research provides insights into the effectiveness of different prediction methodologies for Cloud workload forecasting. This research contributes to the development of energy-efficient Cloud environments while maintaining optimal QoS levels.

In [51], the authors manage the *dynamic scalability of resources* in IaaS environments by studying different workloads and classifying them based on their features and limits. Additionally, metrics aligned with QoS requirements are defined and analyzed for each task, enabling the creation of improved application architectures, as efficiently managing these workloads is essential for the *optimal utilization of dynamic natural resources*. This research contributes to enhancing the efficiency of Cloud resource utilization by considering workload as a core capacity. Therefore, by effectively classifying and characterizing workloads, organizations can optimize resource allocation and ensure that QoS demands are met. This research emphasizes the importance of aligning application architecture with workload characteristics and QoS metrics to achieve optimal performance in IaaS Cloud environments.

[49] discusses the problem of *Instruction Set Architecture* (ISA)-independent workload characterization for significant program features related to compute, memory, and control flow by employing a *Just-In-Time* (JIT) compiler that generates ISA-independent instructions. Through a comparative analysis with an x86 trace, they evaluate the impact of different ISAs on the workload characterization results. revealing that certain aspects of the study exhibit significant sensitivity to the ISA employed. This highlights the importance of adopting ISA-independent workload characterization methodologies for designers of specialized architectures. Based on these results, one can notice that specialized architecture designers must utilize ISA-independent workload characterization methodologies to ensure accurate and reliable assessments of program features. By decoupling workload characterization from specific ISAs, designers can effectively optimize specialized architectures for energy efficiency while considering the unique demands and characteristics of the workload, providing insights into the design and development of energy-efficient computing solutions in the industry.

3 Operational Principles

The training and testing phases of the classification algorithm are described in Figure 1. The idea behind training is to use the different execution sequences produced by a program when fed by different inputs to train the workload model of that program. On the other hand, when an unknown execution sequence is given to a workload model, the probability that the workload of the unknown sequence is similar to that of the model is produced.

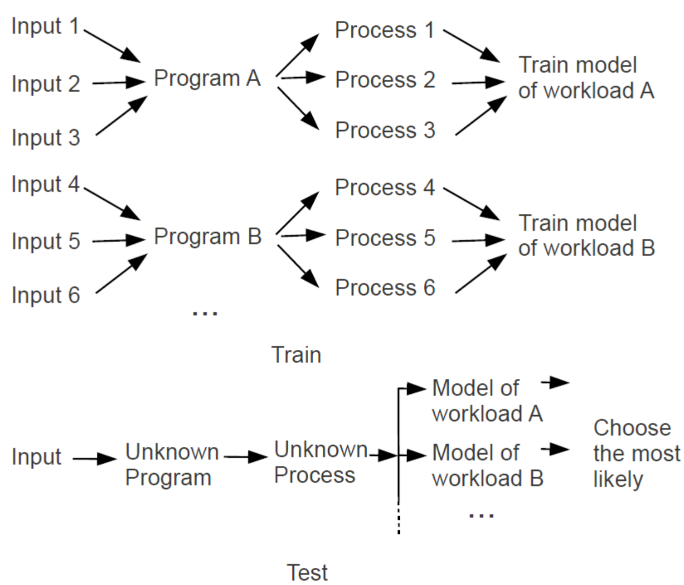


Figure 1: Training and testing of workload models

It is also worth remarking that a number of inputs are given to the benchmarks. In other words, we generate different executions from a given benchmark using different input data. The executions generated from the same benchmark are different because they are obtained with different inputs. Nevertheless, the executions have in common the fact that they come from the same benchmark. The correct classification of the workload of one process means that the classifier is able to understand that different executions come from a single benchmark. Furthermore, we perform the workload classification using four classifiers, namely *Neural Networks* (e.g., [46]), *Hidden Markov Models*, *k-Nearest Neighbors* [1] and *ARMA* [33].

After that, we show that the Dempster-Shafer data fusion algorithm can be successfully used with two different and independent types of metrics. The final classification rate is slightly less than 80% over six benchmarks. In this work, we derived workload models from six benchmarks.

We performed two classification experiments: first, we tested the workload models with the same six benchmarks used to derive the models. However, the input data is different from that used in training, and therefore the processes are always different. Secondly, the other six benchmarks are used for evaluating the similarity with the workload models.

Different classifiers and features can be considered in a data fusion framework to improve classification accuracy, as reported in Figure 2. As it will be shown at the end of the paper, these higher-quality classifiers can be used to find the category of an unknown workload.

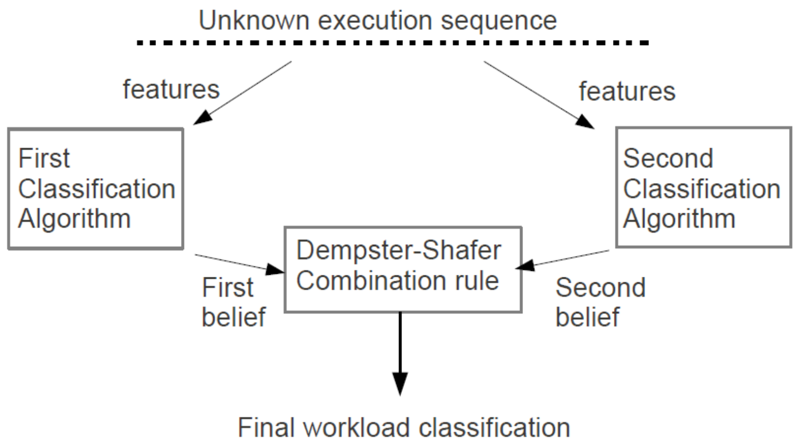


Figure 2: Data fusion of two classifiers

4 SPEC 2006 Benchmarks

CINT2006 [54] is SPEC's CPU-intensive benchmark suite, stressing a system's processor, memory subsystem, and compiler. SPEC designed CPU2006 to provide a comparative measure of compute-intensive performance across the widest practical range of hardware using workloads developed from real user applications. All the benchmarks are provided as source code. The twelve programs included in the benchmark suite can be grouped into the following classes according to their functionality: compiler class, game class, compression class, scientific computing class, and optimization class.

In this work, we derived workload models from six benchmarks, namely *401.bzip2*, *403.gcc*, *458.sjeng*, *471.omnetpp*, *400.perlbench* and *462.libquantum*. In the first experiment, we tested the derived models with the same six benchmarks. It is important to note that the input data is different from that used in training, and therefore the execution sequences are always different. In the second experiment, the other six benchmarks are used to evaluate the similarity between the workload models.

It is worth observing that the used benchmarks represent only a fraction of what applications look like because I/O and memory activity are missing. Thus, the reported results have to be considered as preliminary from a general point of view, being valid only within computer-intensive workloads.

It is important to describe how the input data for the benchmarks are organized in order to make the reported results repeatable. SPEC gives six different inputs for *bzip*, nine for *gcc* and three for *perlbench*. The *sjeng* benchmark has only one input; two other inputs for *sjeng* have been obtained from the first chess positions of *chess.html* downloaded from WWW.DOWNSCRIPTS.COM/CHESS-DATABASE.

Similarly, *omnetpp* has only one input furnished by SPEC; other inputs have been obtained from the first example networks reported in HTTP://INET.OMNETPP.ORG/DOC/INET/NEDDOC/.

Finally, *libquantum* was given the following two additional pairs of numbers: $(159, 15)$ and $(1413, 17)$. In this way, all the benchmarks have at least three inputs that are used for training. Additional input for the test has been obtained in a similar way.

5 Virtual Machine Setting

The virtualization infrastructure used in this work is provided by VirtualBox, which is an open source full virtualization Virtual Machine Monitor (VMM) that runs on both Linux and Windows operating systems running on x86 and x64-based architectures [43]. The useful thing is that VirtualBox offers a rich set of APIs that easily allow to collect metrics on the virtualized process, and the complete set of available APIs is described in [56]. The SDK provided with Virtual Box allows third parties to develop applications that can directly interact with it. It is designed in levels, and at the bottom level, we find the VMM (hypervisor) which is the heart of the virtualization engine that allows for monitoring the performance of virtual machines, providing security, and ensuring the absence of conflicts between virtual machines and the host. Above the hypervisor, there are modules that provide additional functionality, for example, the RDP server (*Remote Desktop Protocol*). Finally, there is the API level, which is implemented above these functional blocks.

VirtualBox comes with a web service that, once running, acts as an HTTP server, accepts SOAP connections [52] (*Simple Object Access Protocol*) and processes them. And the interface of this service is described in a *Web Services Description Language* (WSDL) file [59]. In this way, it is possible to write client programs in any programming language that has provided the tools to process WSDL files, such as Java, C++, NET PHP, Python, and Perl. In addition to Java and Python, the SDK contains many libraries that are ready for use. Internally, the API is implemented using *Component Object Model* (COM) as a reference model. In Windows, it is natural to use Microsoft COM, however, in other hosts, where COM is not present, XPCOM, which is a free implementation of COM, can be used.

Despite the numerous advantages of Web service API, we used the COM method because it allows a lower overhead and thus a higher data rate. We conducted a data exchange experiment, and it turns out that the web service is able, on average, to collect data every 5.96 ms, while using COM, data can be collected every 0.49 ms. As other interesting features regarding the collection of statistical data about resources usage, the API provides functions for:

- specifying which groups of indicators we are interested in (CPU, RAM, Network, and Disk);
- setting the measuring range (minimum interval of 1 s);
- setting the frame size for statistics (Min, Max, and Average);
- making queries on the usage of a single resource.

6 Metrics

As regards metrics, we developed the acquisition system described in the block diagram reported in Figure 3. The acquisition is driven by the host, and all the commands and the acquired data use the COM interface. There is also a web interface to the VirtualBox VMM, but it is much slower, as specified above.

The measured quantities used for workload characterization are of two types, namely memory references and resource demand. Both quantities are gathered using the VirtualBox VMM's API classes. The memory references are the instruction addresses generated by the virtualized process. The VMM's IMachineDebugger API can collect the value of the Program Counter related to instructions every 0.5 ms. In Figure 4 (a), we report, as an example, a chunk of memory references collected during the virtualized execution of one SPEC benchmark (gobmk). On the other hand, Figure 4 (b) presents a chunk of memory reference for another SPEC benchmark (perlbench).

The data is collected in a <*time stamp*><*address*> format. Using the IPerformanceCollector API, we collect resource demand features generated by the virtualized process. The resource demand features we acquired are the following:

- the CPU used in user mode;
- the CPU used in system mode;

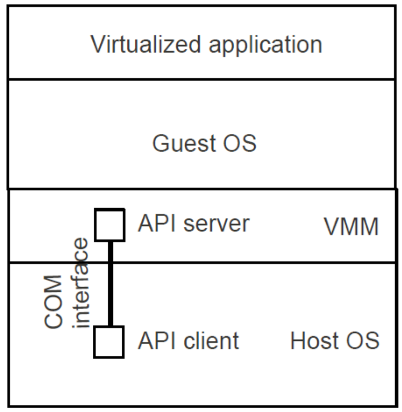
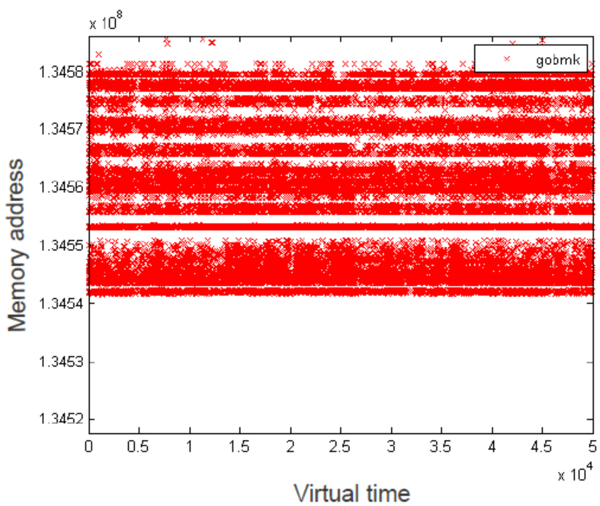
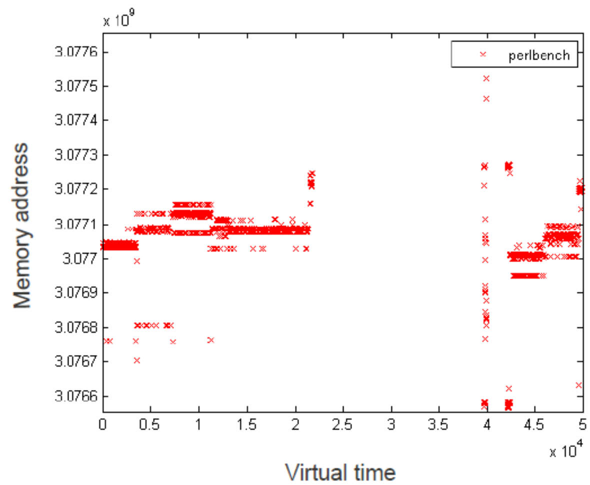


Figure 3: Acquisition system



(a)



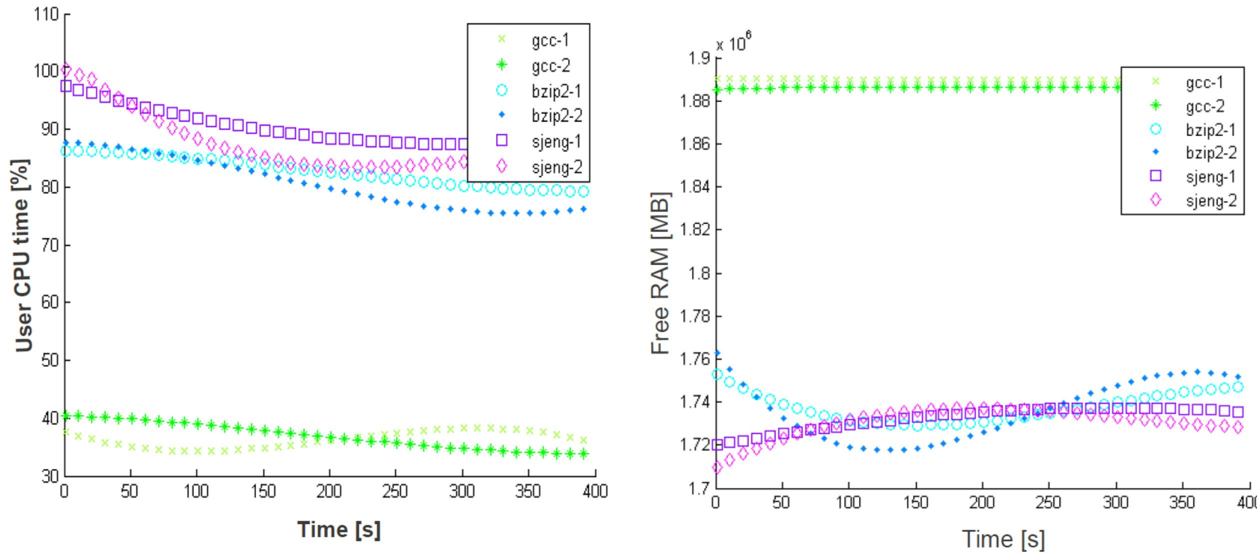
(b)

Figure 4: Memory references generated by (a) the *gobmk* benchmark - (b) the *perlbench* benchmark

- memory fragmentation (free memory / total memory).

In Figure 5 (a), we report, as an example, a portion of 400 s of the CPU used when in user mode collected during the execution of the virtualized process. In Figure 5 (b), we report, instead, the amount of free RAM memory available during the execution of the virtualized process.

Each curve is related to the execution of a different process in the virtual machine, and resource demand feature data is also acquired in a time-stamp format as shown in Figure 6, which reports a piece of resource demand metrics acquired during execution.



(a) **(b)**

Figure 5: User CPU time feature (a) - free RAM feature (b) for different programs in two different executions

Timestamp	CPU user	CPU system	Free Ram
701893535744118	90.0	10.0	1016784.0
701893637910209	90.0	10.0	1016784.0
701893741889815	90.0	10.0	1016784.0
701893845900845	90.0	10.0	1016784.0
701893949900057	90.0	10.0	1016784.0
701894051790540	90.0	10.0	1016784.0
701894153906958	90.0	10.0	1016784.0
701894257888280	90.0	10.0	1016784.0
701894361898999	90.0	10.0	1016784.0

Figure 6: Resource demand format

7 Data Analysis Methodology

7.1 Pre-Processing Algorithms

It is well known that the initial instructions of a running code are highly non-representative of the steady-state behavior of the program. In fact, the first billion instructions do very little except for file I/O and memory allocation as data structures are set up and populated before getting to the real computation to be performed by the program. In this work, we do not use techniques for discovering program phases such as those described in [50] to find the beginning of the steady-state phase of the programs. Instead, we simply blindly fast forward for 1 billion instructions before starting data analysis.

We consider the memory reference sequence as a one-dimensional signal and the resource demand sequence as a multi-dimensional signal. Similarly to what happens in signal processing, we use a parametric description of the sequences. We remark that events in the process, such as for examples loops or sequential program behaviors, produce important events in the metrics sequences and then in the signal spectrum. For instance, loops introduce peaks in the spectrum, while a sequential address sequence produces a DC spectral component. Moreover, the sequences dynamically change their properties. For these reasons, we used a short-time spectral description of the memory reference sequence. Thus, the sequences are divided into overlapped analysis frames of a given size, as reported in Figure 7. The frames are further divided into blocks. Hence, for instance, a frame of memory references is represented by a set of blocks. Also, the multi-dimensional sequence of resource demand features is divided into overlapped frames and blocks.

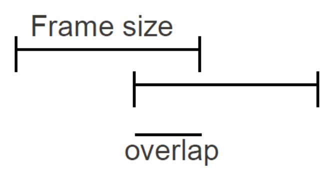


Figure 7: Analysis frames

The next step is to perform a spectral analysis of the blocks. Among the possible spectral-related parameters, we chose the *Discrete Cosine Transform* (DCT) representation. DCT is a well-known signal processing operation with important properties [30]. For example, it is useful for reducing signal redundancy since it places as much energy as possible in as few coefficients as possible (energy compaction). The first DCT coefficients are given as input to the classification algorithm. The effects of retaining the first DCT coefficients are shown in Figure 8. A frame of 1024 memory references is plotted in this Figure. On this frame, we perform a DCT transform; the first sixteen coefficients are used to obtain a 1024-coefficient vector with zero-padding. By inversely transforming this sequence, we obtain the curve plotted in the same Figure 8. It is evident that the effect of retaining the first coefficients is to smooth the peaks of the original sequence while still representing the overall sequence behavior, reflecting the signal redundancy reduction property.

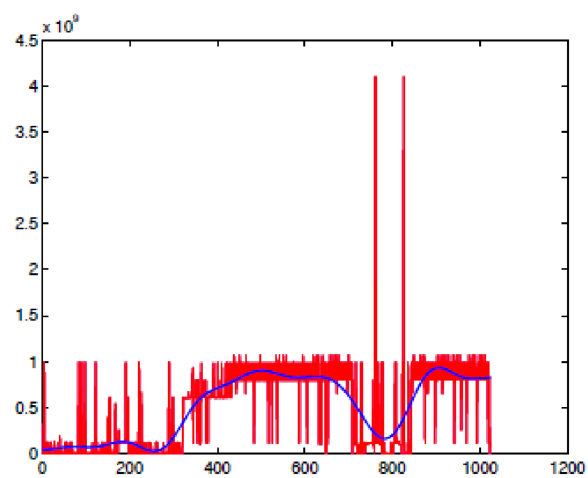


Figure 8: Signal reconstruction by inverse transforming the first DCT coefficients

Concerning the resource demand, since there are three types of features in a frame, the DCT is applied separately to each feature. In every case, we take a small number of DCT coefficients per feature to represent the frame. In this case, a frame is spectrally described by a three-component vector, each formed by a DCT coefficient for a single feature. Eventually, the mono or multi-dimension DCT representations are vector quantized with a 128 entries codebook. The result is that each block of the acquired sequences, both in terms of memory references and resource demand, is described by an integer number.

7.2 Process Selection

The SPEC CPU2006 is formed by two sets of benchmarks: CINT2006 benchmarks, integer benchmarks, and CFP2006, floating point benchmarks. In our experiments, we consider all the CINT2006 benchmarks, formed by twelve programs. Early classification experiments gave the impression that some benchmarks were classified as the same workload. To explore this impression, we performed the following experiment. Using the analysis algorithm of Section 7.1, we train a three-hidden-layer Neural Network for each benchmark, using three different input sets per benchmark. Hence, we have a Neural Network trained for each workload. Then, the vector-quantized parametric sequence obtained from each benchmark in execution was given as input to the Neural Network. The output is very close to one if a given workload is given as input to the Neural Network trained for the same type of workload, and between one and zero for all the others. A distance matrix among all the twelve workloads is obtained by computing $y = 1 - out$, where out is the Neural Network output, and averaging y for the same type of workloads. The size of the distance matrix is clearly twelve by twelve.

By k-means clustering of the distance matrix, we have a reduction to six processes, which confirms our first impression. To get a graphical view of this, in Figure 9, we report the 3D graphical plot, obtained with multidimensional scaling, of the distance matrix. From this Figure, we can see that the distance among astar, h264, gobmk, and perl is very small, so they can be represented by only one program. Similarly, the distance between hmm and sjeng is very small and the distance among libquantum, mcf, and xalancbmk is also very small. In conclusion, the workloads resulting from the clustering reduce to the six benchmarks: bzip, gcc, sjecg, omnetpp, perlbench, and libquantum. The workload classifications reported in the following are performed using these six benchmarks.

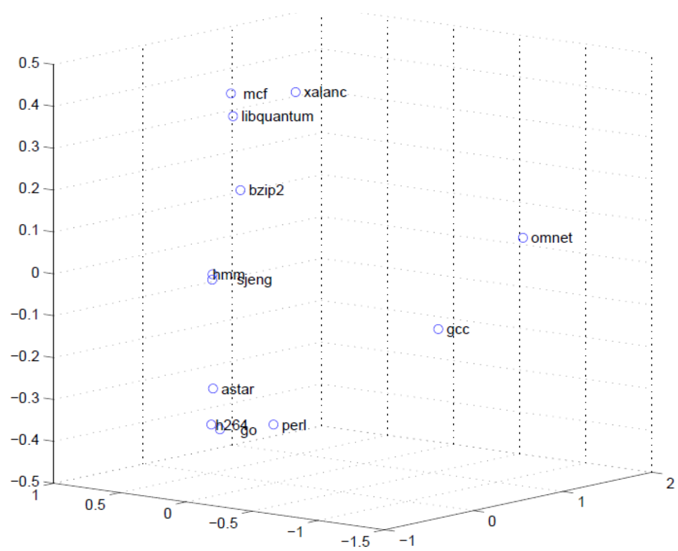


Figure 9: 3D visualization of the distances among programs

7.3 The Input Data to the Benchmarks

Each benchmark has three sets of input data supplied by SPEC in a text file. The sets of input data provided by SPEC are used for training the classifiers. Each benchmark, when run, gives rise to three different processes, one for each set of input data. We can say that each process represents the workload of the benchmark that generated that process. The processes are used to train the classifiers. Each classifier is trained with the different processes so that you can classify the workload of the benchmark.

For each benchmark, we constructed three other sets of input data, other than the above, for the classification stage. Each process generated during benchmark execution with the new sets of input data is classified by the classifiers trained earlier. This generates a number of classification experiments equal to $3 \cdot N$, where N is the number of benchmarks.

7.4 Sampling Rate

A memory reference value is acquired every 0.5 ms. It is important for computational complexity reasons to ascertain how much this sample rate can be reduced. We therefore performed classification experiments with memory reference features at various sampling rates.

For each benchmark, with three different inputs, a Neural Network was trained with the parameters reported in Table 1.

Table 1: Initial analysis parameters

Frame size	50s
Overlap	50%
Number of DCT coeffs	10
Number of blocks per frame	50
Vector quantization	128 levels

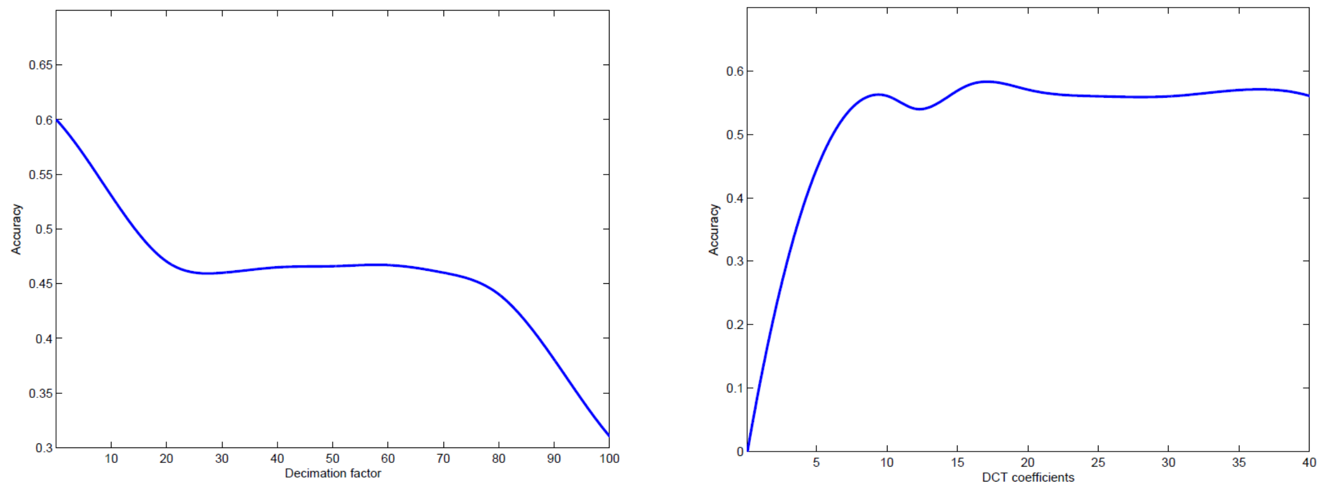
In Figure 10 (a), we show the accuracy obtained at different values of the decimation of the original sampling rates. We note that by acquiring the memory references and decimating the original sampling rates by ten, the accuracy drops from 60% to 53%. In view of the fact that this accuracy is improved if the parameters are tuned, and for reducing the algorithm complexity, we acquired the memory references are acquired at a 5 ms sampling rate, which corresponds to a decimation factor of ten. The resource demand features acquisition was decimated accordingly.

Starting from the initial analysis parameters, we conducted some experiments of *Neural Network Classification*, using three hidden layers and 50 neurons. First, we obtained the accuracy with different values of frame size. We find that by slightly decreasing the frame duration from 50 s to 48 s we get a slight accuracy improvement. We also find that the frame overlap we used initially lead to the best accuracy. The next set of experiments is directed to the number of DCT coefficients. We found that the best accuracy is obtained by setting the number of DCT coefficients to sixteen, which is shown in Figure 10 (b).

Other classification experiments indicate that the best number of blocks per frame is forty. Thus, the final optimal analysis parameters are reported in Table 2.

Table 2: Final parameters setup

Frame size	48s
Overlap	50%
Number of DCT coeffs	16
Number of blocks per frame	40
Vector quantization	128 levels



(a) **(b)**

Figure 10: Accuracy versus decimation factor (a) - DCT coefficients (b)

7.5 Classification Algorithms

For the sake of completeness, in this Section we present a brief description of the machine learning algorithms that we used as classifiers.

7.5.1 k-Nearest Neighbor Algorithm

The *k-Nearest Neighbor algorithm* (k-NN) is a simple machine learning algorithm that does not use any underlined model acquired during the training phase, as other machine learning algorithms do. Instead, k-NN is based on the principle that instances within a dataset will generally exist in close proximity to other instances that have similar properties. If the objects are tagged with a classification label, they are classified by a majority vote of their neighbors and are assigned to the class most common amongst their k-nearest neighbors.

k is usually a small odd positive number, and the correct classification of the neighbors is known a priori. The objects can be considered n-dimensional points within an n-dimensional instance space, where each point corresponds to one of the n features describing the objects. The distance or closeness to the neighbors of an unclassified object is determined by using a distance metric (also called the similarity function), for example, the Euclidean distance or the Manhattan distance.

Our k-NN uses the Euclidean distance to represent the closeness to the neighbors of an unclassified object, as the high degree of local sensitivity makes k-NN highly susceptible to noise in the training data. In other words, the value of k strongly influences the performance of the k-NN algorithm.

7.5.2 Neural Network Algorithm

The well-known neural networks are composed of a set of simple processing units which communicate with each other through a large number of weighted connections. In most cases, it is assumed that each neuron makes an additive contribution to the neuron to which it is connected. The total input s_k is simply the weighted sum of the different outputs of the connected neurons plus a noise factor:

$$s_k(t) = \sum_j w_{jk}(t)u_j(t) + \theta_k(t) \tag{1}$$

where positive w_{jk} are said to excite the neuron input, and negative w_{jk} are said to inhibit the neuron.

We need to have rules to determine the effect of the total input on the activation of the neuron. It is well defined a function F_k that, based on the total input $s_k(t)$ and the current activation of the neuron $y_k(t)$ produces the new value of the activation:

$$y_k(t+1) = F_k(y_k(t), s_k(t)) \tag{2}$$

Very often, F_k depends only on the total input at that moment, and then the last Equation can be written as follows:

$$y_k(t+1) = \sum_j w_{jk}(t)y_j(t) + \theta(t) \tag{3}$$

Normally F_k has values in the range $[-1, \ldots, +1]$. The most commonly used functions are the *sign function*, the *hyperbolic tangent*, or the *sigmoid function*.

We used a neural network with a Feed-forward topology with three hidden layers, where the flow of information between the input and the output travels one-way to the exit.

- feed-forward networks: here, the flow of information between the input and the output travels one way to the exit. Processing can be extended to many levels of neurons;
- recursive networks: contrary to feed-forward networks, they have feedback connections. The network is bound to evolve towards a stable state in which the functions of activation do not change.

As a classic example of the first type is the perception, while the second type is the *Hopfield network*, a neural network must be configured in such a way that the application of inputs produce the desired output. It is therefore necessary to modify the weight of the connections during a training phase.

7.5.3 Hidden Markov Model Algorithm

Markov models are stochastic interpretations of time series. The basic Markov model is the Markov chain, which is represented by a graph composed by a set of N states, the graph describes the fact that the probability of the next event depends on the previous event. A Markov chain is described by the transition matrix A whose elements are:

$$a_{i,j} = Prob(S_{t+1} = j | S_t = i) \tag{4}$$

and the initial probability vector π_i:

$$\pi_i = Prob(S_1 = i) \tag{5}$$

where:

$$\sum_{i=1}^{N} \pi_i = 1 \tag{6}$$

In homogeneous Markov chains, the transition probability depends only on the previous state; in such cases, the transition probabilities can be represented by a transition matrix. However, in many cases, Markov models are too simple to describe complex real-life systems and signals.

In the case of Hidden Markov Models (HMMs), the output of each state corresponds to an output probability distribution instead of a deterministic event. That is, if the observations are sequences of discrete symbols chosen from a finite alphabet, then for each state, there is a corresponding discrete probability distribution that describes the stochastic process to be modeled. In HMMs, the state sequence is hidden and can only be observed through another set of observable stochastic processes. Thus, the state sequence is recovered with a suitable algorithm on the basis of optimization criteria. It is important to note that the observation probabilities can be discrete or continuous feature vectors.

7.5.4 ARMA

Considering the memory reference sequence as a time series of data M_t, the ARMA model is a tool for understanding and, perhaps, predicting future values in this series. The model consists of two parts, an *Auto Regressive* (AR) part and a *Moving Average* (MA) part. Thus, the model is referred to as the $ARMA(p,q)$ model, where p is the order of the auto-regressive part and q is the order of the moving average part:

$$M_t = c + \epsilon_t + \sum_{i=1}^{p} \varphi_i M_{t-i} + \sum_{i=1}^{q} \psi \epsilon_{t-i} \tag{7}$$

where φ_i and ψ_i are the parameters of the model, ϵ_t is white noise, and c is a constant. Classification with ARMA model is performed using the generalized linear model.

8 The Dempster-Shafer Fusion

The goal of the Dempster-Shafer (DS) theory of evidence [48], is to combine different measures of evidence. At the base of the theory is a finite set of possible hypotheses, say $\theta = \{\theta_1, \ldots, \theta_K\}$.

8.1 Basic Belief Assignment

The Basic Belief Assignment (BBA) can be viewed as a generalization of a probability density function. More precisely, a basic belief assignment m is a function that assigns a value in $[0,1]$ to every subset $\mathcal{A}$ of θ that satisfies the following conditions:

$$\sum_{\mathcal{A} \subseteq \Theta} m(\mathcal{A}) = 1, \quad m(\varnothing) = 0 \tag{8}$$

It is worth noting that $m(\mathcal{A})$ is the belief that supports the subset $\mathcal{A}$ of θ, not the elements of $\mathcal{A}$. This reflects some ignorance because it means that we can assign belief only to subsets of θ, not to the individual hypothesis.

8.2 Belief Function

The belief function, $bel(.)$, associated with the Basic belief assignment $m(.)$, assigns a value in $[0,1]$ to every nonempty subset $\mathcal{B}$ of θ. It is defined by:

$$bel(\mathcal{B}) = \sum_{\mathcal{A} \subseteq \mathcal{B}} m(\mathcal{A}) \tag{9}$$

where the belief function can be viewed as a generalization of a probability function.

8.3 Combination of Evidence

Considering two Basic belief assignments, $m_1(.)$ and $m_2(.)$ and the corresponding belief functions, $bel_1(.)$ and $bel_2(.)$. Let $\mathcal{A}_j$ and $\mathcal{B}_k$ be subsets of θ. Then, $m_1(.)$ and $m_2(.)$ can be combined to obtain the belief mass assigned to $\mathcal{C} \subset \theta$ according to the following formula [48]:

$$m(\mathcal{C}) = m_1 \bigoplus m_2 = \frac{\sum_{j,k,\mathcal{A}_j \cap \mathcal{B}_k = \mathcal{C}} m_1(\mathcal{A}_j) m_2(\mathcal{B}_k)}{1 - \sum_{j,k,\mathcal{A}_j \cap \mathcal{B}_k = 0} m_1(A_j) m_2(B_k)} \tag{10}$$

where the denominator is a normalizing factor, which measures how much $m_1(.)$ and $m_2(.)$ are conflicting.

8.4 Belief Functions Combination

The combination rule can be easily extended to several belief functions by repeating the rule for new belief functions. Thus the sum of n belief functions, $bel_1, bel_2, \ldots, bel_n$, can be formed as:

$$((bel_1 \bigoplus bel_2) \bigoplus bel_3) \ldots bel_n = \bigoplus_{i=1}^{n} bel_i \tag{11}$$

It is important to note that the basic belief combination formula given above assumes that the belief functions to be combined are independent.

8.5 BBA Based on Single Class Classifiers

In our case, a hypothesis set is defined for each texel in which the image is divided. Within each texel, the hypothesis concerns the possibility that the pixel (i, j) corresponds to an object or not. In other words, we have eight hundred hypotheses for each texel, namely:

$$\theta = \{\theta_1(0,0), \ldots, \theta_1(19,19), \theta_2(0,0), \ldots, \theta_2(19,19)\} \tag{12}$$

where $\theta_1(i, j)$ is the belief that the pixel (i, j) of that texel belongs to an object in the environment and $\theta_2(i, j)$ is the belief that the pixel (i, j) doesn't belong to an object.

If we have K benchmarks, we can use K classifiers, each trained using the processes generated by each of the K benchmarks. Each classifier is used as an expert in DS fusion. The goal of the classifiers is to infer from the benchmark where an unknown process comes from.

The classifier C_i provides a probability p_i as output, $i = 1, \ldots, K$, where p_i is the probability that the process analyzed by the classifier has been generated by the i-th benchmark. The hypothesis set is given by:

$$\Theta = \{\theta_1, \ldots, \theta_K\} \tag{13}$$

where θ_i is the event that the process comes from the benchmark i, $i = 1, \ldots, K$. Under this assumption, the expert i provides the probability of the subset $\{\theta_i\}$:

$$m_i(\{\theta_i\}) = p_i \tag{14}$$

If the classifier has been trained with the processes generated by a given benchmark, the probability of the event θ_i shall be distributed to all the other subsets $\mathcal{C}$ of Θ:

$$m_i(\mathcal{C}) = \frac{1 - p_i}{2^K - 1} \tag{15}$$

Then, we consider K single-class classifiers; each classifier is specialized to recognize the workload of a particular benchmark. Under this BBA, the i-th classifier is trained using a set of data produced by the benchmark i and a set of data not produced by the benchmark i, and it provides as output a real value p_i in the range [0,1], that represents the probability that the current input comes from the benchmark i. Each expert shall assign a belief to the subsets of Θ. Under this BBA, the i-th expert assigns $m_i(\{\theta_i\}) = p_i$ and $m_i(\mathcal{C}) = \frac{1-p_i}{2^K-1}$, $\mathcal{C} \subseteq \Theta$, $\mathcal{C} \neq \{\theta_i\}$.

8.6 Pseudocode

The application of the rule follows the following pseudocode.

```
Foreach classifier
   Foreach classifier
      Translates a set of hypotheses
         to the internally used representation.
      Add a set of hypotheses to the
         evidence and assign a mass to it.
      Combine two evidences.
```

9 Case Study: Workload Categorization in the Context of Anomaly Detection Caused by Android Malware

Android mobile devices have become significantly more popular recently, and at the same time, the number of malicious programs operating on them has also grown significantly. As a result, business and academics have given a lot of attention to the security and privacy concerns of Android applications and systems, as Anomaly Detection is crucial due to the increasing dependability of these systems and applications. To this end, behavior-based anomaly detection systems have been developed to identify irregularities brought on by Android malware. These systems analyze these anomalies in data on network traffic, battery temperature, and power usage using machine learning classification algorithms. Hence, we provide in this Section a detailed case study where we show how our proposed approach can be employed within the context of Anomaly Detection caused by Android Malware.

According to a new security report, 4.9 million malware attacks were prevented in the first quarter of 2023 by *Kaspersky mobile security solutions*. Each malware sample must be thoroughly analyzed, which takes time. Malware analysis systems are therefore overloaded by the sheer volume of malware samples. Most newly discovered malware samples are polymorphic versions of already existing malware. By grouping malware samples into different families and then choosing representative samples from each family, we can speed up the analysis of malware. The familial classification of Android malware is difficult, though, for two reasons:

- it is difficult to distinguish between dangerous and legitimate components in the bulk of Android malware samples, which are repackaged versions of popular apps. In fact, 86% of Android malware samples are malicious component-infected repackaged apps [67]. In most cases, just a tiny part of the repackaged apps have dangerous components that have been introduced, which are concealed inside the features of popular programs. System calls [64] and sensitive paths [63] are examples of existing features that make it difficult to distinguish between malware's legitimate and harmful parts;

- the same destructive actions are carried out with multiple implementations by polymorphic Android malware versions that belong to the same family. As a result, malware of this type is readily able to avoid existing classification techniques that look for a precise match to a given specification [11]. As an illustration, two malware samples have various ways of carrying out the same task (i.e., obtaining the device id, phone number, and voice mail number).

Monitoring data is collected and analyzed to determine system health, workload patterns, and metric spaces, which are then utilized to discover anomalies. Furthermore, to test the efficacy of anomaly detection, the detection can be evaluated using different faults to analyze: sensitive API calls, CPU heavy loops, memory leaks, disk I/O errors, and network anomalies:

- sensitive API calls: Android malware typically uses sensitive Application Programming Interface (API) methods, such as *getDeviceId()*, *getLine1Number()*, and *getVoiceMailNumber()*, that operate on sensitive data to carry out harmful actions. For example, in order to gather the consumers' phone numbers, the malware may invoke *getLine1Number()*;

- CPU-intensive loops: Malware causes circular waits and never-ending loops in applications, such as spin lock faults, where CPU resource-intensive operations result in server requests timing out in Android systems and application failures. These injections of fault components result in more computing processes, which need more CPU resources;

- memory leak: this is brought on by assigning heap memory to objects without releasing them, which steadily depletes the system's memory resource and finally causes a system crash. It might take a system with a memory leak a long time before any significant issues arise, making it challenging to identify the issue right away;

- disk I/O error: disk I/O access has a predictable pattern, however, disk access can also be impacted by application-level or hard disk failures in a particular workload pattern. By adding additional disk reading and writing operations to application components, malware trigger this type of these errors;

- network anomaly: Android-based systems and programs are vulnerable to network assaults because rogue scripts may be placed into programs to broadcast schematics that suck up network capacity, saturating servers and leading to service denial. When the injected servlet components are invoked, the malicious code in the application sends UDP packets to any host on the local network.

To this end, the reference architecture shown in Figure 11 for Workload Analysis in Familial Classification of Android Malware, where the architecture is depicted, can be integrated with our proposed Workload Categorization approach using the Dempster-Shafer theory of evidence [48] that we previously described in detail in Section 8. And the hierarchical levels include the following:

- Detection of malicious API calls and FASTA files generation: To identify the APIs, malware applications from different families in reference datasets and benign apps retrieved from Google Play Store are disassembled and analyzed. Furthermore, the frequency of each API call mentioned in both groups is computed independently. API calls are found and documented that are regularly used in malware apps but not in benign apps. Then, based on the suspected Android API types, the API classes are categorized, and these groupings constitute the API class sequence in FASTA format [45]. This structure is required to feed the sequence file into machine learning tools for training and assessment;

- *Multiple Sequence Alignment* (MSA) generation: MSA is created for each app in the family and is used to build a *Profile Hidden Markov Model* (PHMM) using machine learning techniques, which is then trained and utilized as one of the classifiers of unfamiliar Android applications based on a derived score;

- Training and classification: in this layer, the PHMM is trained along with other classifiers such as a Neural Network and k-NN using some reference datasets which consist of malware samples from several families, where these samples are selected from the datasets such that they contain multiple apps from every family. From the selected malware samples, a big percentage of apps is used for training the classifiers and the rest for testing, and the training is conducted by decompiling the apps and creating the API call list. Then, from the API call list, the suspicious APIs are identified, the final APIs are represented in FASTA format, and the corresponding MSA files are generated. Finally, MSA for all the families are given to the classifiers to create the profile files corresponding to malware families for classification. And the scores generated by each classifier will be combined using the Dempster-Shafer theory of evidence for a final classification;

- Dempster-Shafer fusion: in this layer, the Dempster-Shafer theory of evidence is employed, which combines different measures of evidence on the basis of a finite set of possible hypotheses, which are in our case the scores provided by the classifiers. These scores are also, as a result, the belief values of the assignments represented by the malware families or benign apps that the unknown workload should be classified to. All follow the particular formula presented in Equation 10.

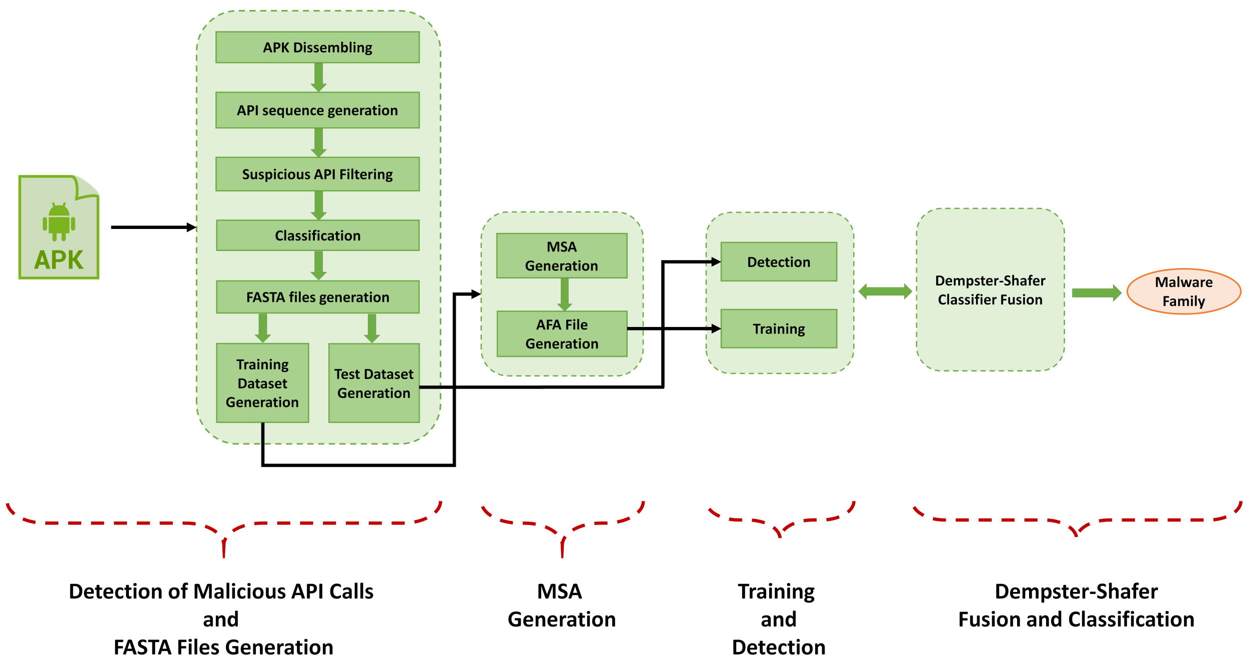


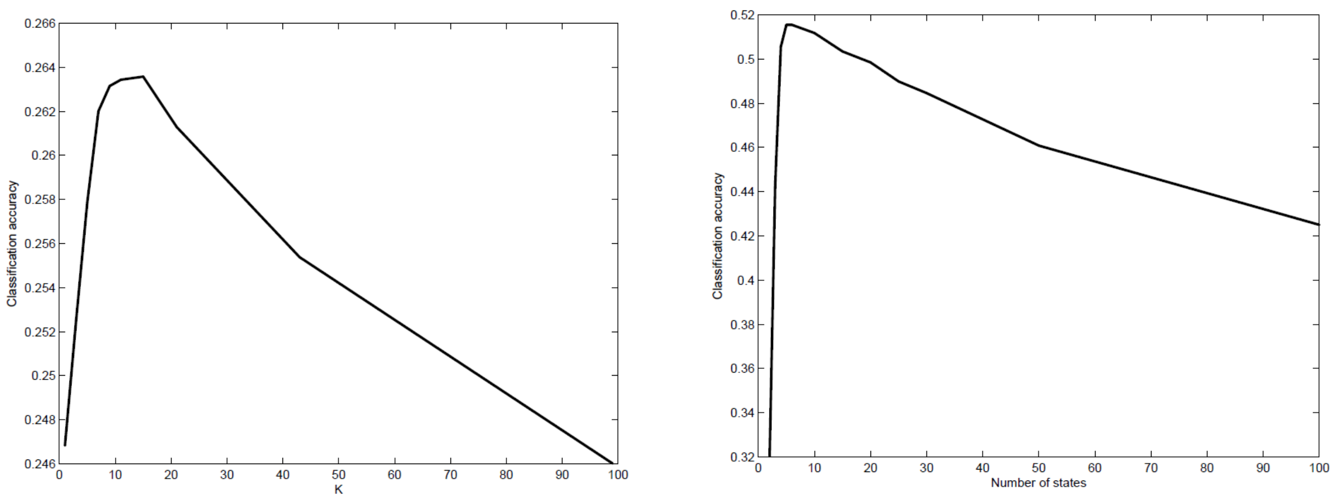
Figure 11: Familial classification of android malware using Dempster-Shafer Fusion

10 Experimental Results and Analysis

First, we specify how the classification experiments that we report in this Section were made. The input data to the classifiers are the metrics acquired during the execution of the benchmarks, namely the memory reference and the resource demand features. The six selected benchmarks are executed three times, each with a different set of input data. We thus obtained the execution sequences that describe the workloads of the different benchmarks. These execution sequences are preprocessed as described in Section 7 and are used to train the classifiers that are so ready to perform the classification. For example, a neural network trained with three different sets of input data but with the same benchmark will become a model of that workload.

The same six benchmarks are then executed with three different sets of input values. Of course, as mentioned before, these executions are completely different from the first ones, but they share the workload for the same benchmark. These executions are classified according to the models derived above.

Let us first consider the result obtained with the memory reference metric. Figure 12 (a) represents the accuracy obtained with the k-NN classifier (e.g., [1]) versus K. From this Figure, we also note that the best accuracy (26.3%) is obtained for $k = 15$. Figure 12 (b) represents the accuracy of the classification with HMM. We used a continuous HMM in which the output distribution is represented by 20 Gaussian mixtures. The used topology is ergodic. This graph shows the classification accuracy as a function of the number of states. The best accuracy is 52.8% and is obtained with six states.



(a) **(b)**

Figure 12: Average classification accuracy of k-NN versus k (a) and HMM versus the number of states (b) using memory reference metrics

Figure 13 shows the accuracy obtained with the neural network. The graph displays the accuracy versus the number of neurons in each hidden layer. Each curve is related to a different number of hidden layers. In particular, curves marked with circles, squares and diamonds are obtained with three, four, and five hidden layers, respectively, and all the other curves are obtained with six up to nine layers. The best accuracy (65.32%) is obtained with four hidden layers and 120 neurons per layer. However, with 50 neurons per hidden layer, the accuracy is around 60% for every number of layers.

Finally, Figure 14 shows the accuracy obtained with the *ARMA model* (e.g., [33]) with $p = 8$ and $q = 4$. The average classification accuracy is 44.15%. The benchmarks are respectively from 1 to 6: *401.bzip2*, *403.gcc*, *458.sjeng*, *471.omnetpp*, *400.perlbench*, and *462.libquantum*.

The Dempster-Shafer data fusion combines the output values from the classifiers. We used different workload metrics and the best classification algorithms according to the results reported in Table 2. The results shown in Figure 15 are obtained by fusing Neural Networks with memory reference features, indicated by NNmr, Neural Networks with resource demand features, indicated by NNrd and HMM with memory references, denoted by HMMmr. In this Figure, the bars show the classification accuracies obtained with, respectively from left to right, the fusion between NNmr and HMMmr, between NNmr and NNrd, between HMMmr and NNrd, and the fusion of HMMmr, NNrd, and NNmr. As shown, the best results are obtained with data fusion between NN with memory reference features and NN with resource demand features, and the obtained accuracy is 79.35%.

The first bar is the data fusion of different classifiers with the same feature (mr), while the other bars are related to the usage of two different features (mr and rd). As shown in Figure 15, the best results are obtained with data fusion between NN with memory reference features and NN with resource demand features, and the

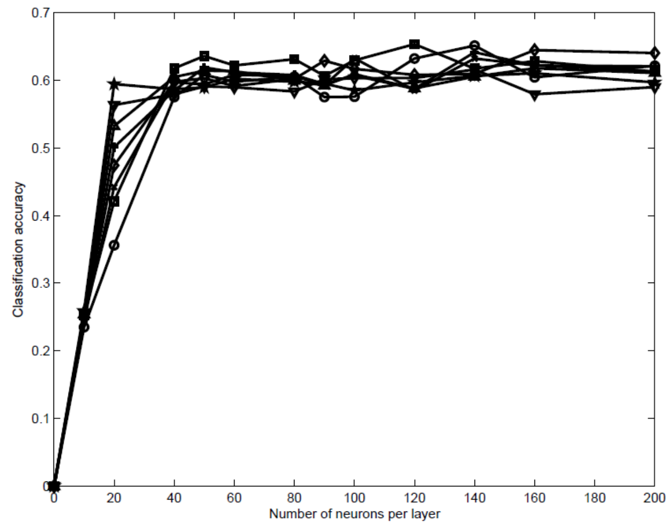


Figure 13: Average classification accuracy of neural networks versus the number of neurons per Layer with memory reference features

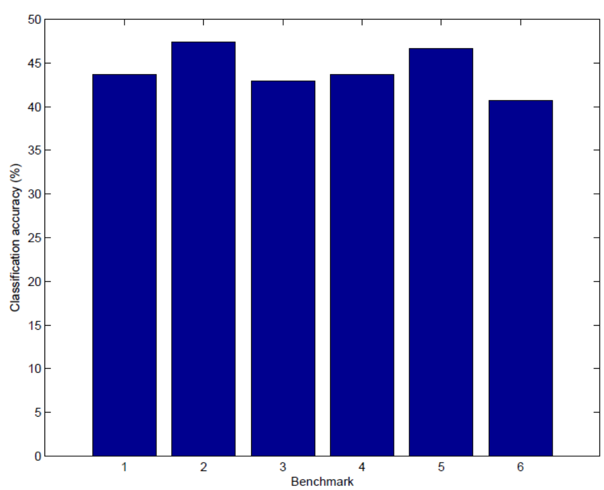


Figure 14: Average classification accuracy obtained with the ARMA model with MA order of 8 and AR order of 4

obtained accuracy is 79.3%. This is an important result that shows that data fusion is effective in boosting classification accuracy with features having different time constants and different computational complexity.

As highlighted so far, using Dempster-Shafer data fusion, we can build a high-quality classifier using lower-quality classifiers. Now, we will show that these high-quality classifiers can be used to find the category of an unknown workload. By testing an unknown execution sequence with the classifier trained on a given workload W, we get an indication of how much the unknown execution sequence can be assigned to the category of the workload W. In fact, if an execution sequence with the workload W is tested with the high-quality classifier trained with a given workload W, the output will be close to one. If an execution sequence with a workload similar to W is tested with the same classifier, the output will be close to 1, and

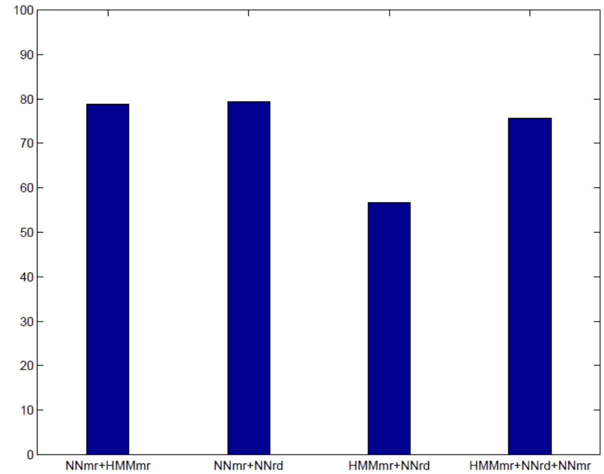


Figure 15: Classification accuracies of the fusion between (from left to right) NNmr and HMMmr, NNmr and NNrd, HMMmr and NNrd, and HMMmr, NNrd and NNmr, respectively

so forth. This property can be used to assign a workload category to an unknown execution sequence. In the experiment described in this Section, we used this property to evaluate distances among the benchmarks from the point of view of the workload they represent. Figure 16 shows a graphical view of the distances among workloads.

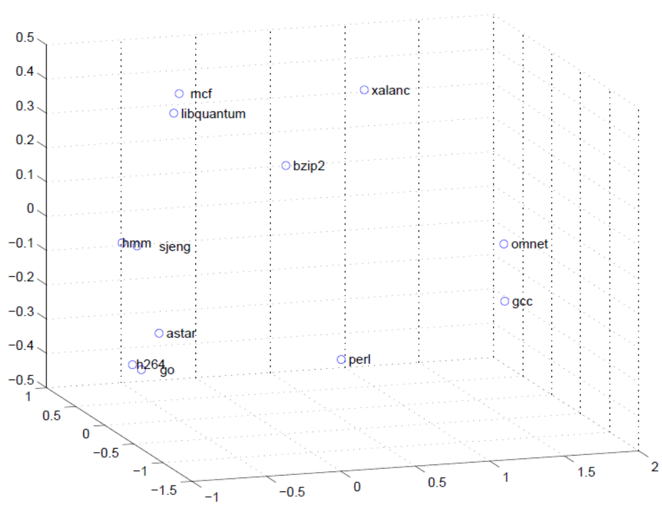


Figure 16: 3D visualization of distances among programs

This Figure shows that *429.mcf* can be given the category of *462.libquantum*, and that *456.hmmer* can be given the category of *458.sjeng* and the workloads of *445.gobmk* and *464.h264ref* are very close.

11 Concluding Remarks and Future Work

In this paper, we deal with the classification of application workloads in a virtualized environment as a means of improving the efficiency and reliability of Cloud-based big data applications. We show that, using lower-quality classifiers, we can build a higher-quality classifier using data fusion algorithms. There may be several applications for this type of classification, from user profiling to malware detection. To this end, we used the SPEC benchmarks that were run in a virtual environment. Different sets of input data were used. The Neural Network classifier gives better results than Hidden Markov Models and K-NN. Final results are obtained by Dempster-Shafer fusion of the Neural Network classification with memory reference and resource demand features, which are workload metrics with completely different time constants. The best classification rate is about 80%.

An obvious extension of the work described in this paper is to use other benchmarks in order to include other workload activities. Also, we plan to further improve the characteristics of our framework by integrating solutions for dealing with novel aspects of massive big data set processing, on top of which workloads may still be defined, such as *data compression techniques* (e.g., [20]), *fragmentation approaches* (e.g., [19]), *privacy-preservation approaches* (e.g., [18]) that, particularly, may be extremely useful when combined with malware detection issues. In addition, we are planning to further enrich our proposed framework by means of emerging big data trends (e.g., [5, 16, 39, 15, 38, 25])

Conflict of Interest: "The authors declare no conflict of interest."

Funding: "This work was partially supported by project SERICS (PE00000014) under the MUR National Recovery and Resilience Plan funded by the European Union - NextGenerationEU."

References

[1] Altman NS. An introduction to kernel and nearest-neighbor nonparametric regression. *The American Statistician.* 1992; 46(3): 175-185. DOI: https://doi.org/10.1080/00031305.1992.10475879

[2] Azmandian F, Moffie M, Dy JG, Aslam JA, Kaeli DR. Workload characterization at the virtualization layer. In: *19th Annual IEEE International Symposium on Modelling, Analysis, and Simulation of Computer and Telecommunication Systems, MASCOTS 2011, 25-27 July 2011, Singapore.* 2011. p.63-72. DOI: https://doi.org/10.1109/MASCOTS.2011.63

[3] Barford P, Crovella M. Generating representative web workload for network and server performance evaluation. In: *Proceedings of the 1998 ACM SIGMETRICS Joint International Conference on Measurement and Modeling of Computer Systems, SIGMETRICS 1998 / PERFORMANCE 1998, 22-26 June 1998, Madison, Wisconsin, USA.* 1998. p.151-160. DOI: https://doi.org/10.1145/277851.277897

[4] Baum LE, Petrie T. Statistical inference for probabilistic functions of finite state Markov chains. *The Annals of Mathematical Statistics.* 1966; 37(6): 1554-1563. DOI: https://doi.org/10.1214/aoms/1177699147

[5] Bellatreche L, Cuzzocrea A, Benkrid S. *F&A*: A Methodology for effectively and efficiently designing parallel relational data warehouses on heterogenous database clusters. In: *Proceedings of the Springer 12th International Conference on Data Warehousing and Knowledge Discovery, DAWAK 2010, 30 June*

2010 - 3 September 2010, Bilbao, Spain. 2010. p.89-104. DOI: https://doi.org/10.1007/978-3-642-15105-7_8

[6] Bhowal P, Sen S, Yoon JH, Geem ZW, Sarkar R. Evaluation of Fuzzy measures using Dempster-Shafer belief structure: A classifier Fusion framework. *IEEE Transacations on Fuzzy Systems*. 2023; 31(5): 1593-1603. DOI: https://doi.org/10.1109/TFUZZ.2022.3206504

[7] Biswas A, Majumdar S, Nandy B, El-Haraki A. Automatic resource provisioning: A machine learning based proactive approach. In: *IEEE 6th International Conference on Cloud Computing Technology and Science, CloudCom 2014, 15-18 December 2014, Singapore*. 2014. p.169-173. DOI: https://doi.org/10.1109/CloudCom.2014.147

[8] Bleikertz S, Vogel C, Groß T. Cloud radar: Near real-time detection of security failures in dynamic virtualized infrastructures. In: *Proceedings of the 30th Annual Computer Security Applications Conference, ACSAC 2014, 8-12 December 2014, New Orleans, USA*. 2014. p.26-35. DOI: https://doi.org/10.1145/2664243.2664274

[9] Bruder G, Steinicke F, Nüchter A. Poster: Immersive point cloud virtual environments. In: *IEEE Symposium on 3D User Interfaces, 3DUI 2014, 29-30 March 2014, Minneapolis, USA*. 2014. p.161-162. DOI: https://doi.org/10.1109/3DUI.2014.6798870

[10] Carlson M. Systems and virtualization management: Standards and the cloud. A report on SVM 2013. *Journal of Network Systems Management*. 2014; 22(4): 709-715. DOI: https://doi.org/10.1007/s10922-014-9315-7

[11] Chen KZ, Johnson NM, D'Silva V, Dai S, MacNamara K, Magrino TR, Wu EX, Rinard MC, Song DX. Contextual policy enforcement in Android applications with permission event graphs. In: *20th Network and Distributed System Security Symposium, NDSS 2013, 24-27 February 2013, San Diego, USA*. 2013.

[12] Cho Y, Choi J, Choi J. An integrated management system of virtual resources based on virtualization API and data distribution service. In: *ACM Cloud and Autonomic Computing Conference, CAC 2013, 5-9 August 2013, Miami FL, USA*. 2013. p.1-7. DOI: https://doi.org/10.1145/2494621.2494648

[13] Chuang IH, Tsai YT, Horng MF, Kuo YH, Hsu JP. A GA-based approach for resource consolidation of virtual machines in clouds. In: *Springer 6th Asian Conference on Intelligent Information and Database Systems, ACIIDS 2014, 7-9 April 2014, Bangkok, Thailand*. 2014. p.342-351. DOI: https://doi.org/10.1007/978-3-319-05476-6_35

[14] Cirne W, Berman F. A comprehensive model of the supercomputer workload. In: *Proceedings of the 4th Annual IEEE International Workshop on Workload Characterization, WWC-4 (Cat. No. 01EX538), 2 December 2001, Austin, TX, USA*. 2001. p.140-148. DOI: https://doi.org/10.1109/WWC.2001.990753

[15] Coronato A, Cuzzocrea A. An innovative risk assessment methodology for medical information systems. *IEEE Transactions on Knowledge and Data Engineering*. 2022; 34(7): 3095-3110. DOI: https://doi.org/10.1109/TKDE.2020.3023553

[16] Cuzzocrea A, Martinelli F, Mercaldo F, Vercelli GV. Tor traffic analysis and detection via machine learning techniques. In: *2017 IEEE International Conference on Big Data, IEEE BigData 2017, 11-14 December 2017, Boston, MA, USA*. 2017. p.4474-4480. DOI: https://doi.org/10.1109/BigData.2017.8258487

[17] Cuzzocrea A, Saccà D, Ullman JD. Big data: A research agenda. In: *ACM 17th International Database Engineering & Applications Symposium, IDEAS 2013, 09-11 October 2013, Barcelona, Spain*. 2013. p.198-203. DOI: https://doi.org/10.1145/2513591.2527071

[18] Cuzzocrea A, Saccà D. Balancing accuracy and privacy of OLAP aggregations on data cubes. In: *ACM 13th International Workshop on Data Warehousing and OLAP, DOLAP 2010, 30 October 2010, Toronto, Ontario, Canada*. 2010. p.93-98. DOI: https://doi.org/10.1145/1871940.1871960

[19] Cuzzocrea A, Darmont J, Mahboubi H. Fragmenting very large XML data warehouses via K-means clustering algorithm. *International Journal of Business Intelligence and Data Mining*. 2009; 4(3-4): 301-328. DOI: https://doi.org/10.1504/IJBIDM.2009.029076

[20] Cuzzocrea A, Saccà D, Serafino P. A hierarchy-driven compression technique for advanced OLAP visualization of multidimensional data cubes. In: *Proceedings of the Springer 8th International Conference on Data Warehousing and Knowledge Discovery, DaWaK 2006, 4-8 September 2006, Krakow, Poland*. 2006. p.106-119. DOI: https://doi.org/10.1007/11823728_11

[21] Da Mota B, Tudoran R, Costan A, Varoquaux G, Brasche G, Conrod PJ, Lemaître H, Paus T, Rietschel M, Frouin V, Poline JB, Antoniu G, Thirion B. Generic machine learning pattern for neuroimaging-genetic studies in the cloud. *Frontiers in Neuroinformatics*. 2014; 8: art.31. DOI: https://doi.org/10.3389/fninf.2014.00031

[22] Deng Y, Shen S, Huang Z, Iosup A, Lau RWH. Dynamic resource management in cloud-based distributed virtual environments. In: *Proceedings of the ACM 22nd International Conference on Multimedia, MM 2014, 3-7 November 2014, Orlando, FL, USA*. 2014. p.1209-1212. DOI: https://doi.org/10.1145/2647868.2655051

[23] DiFranzo D, Graves A. A farm in every window: A study into the incentives for participation in the Windowfarm virtual community. In: *ACM 3rd International Web Science Conference, WebSci 2011, 15-17 June 2011, Koblenz, Germany*. 2011. p.1-8. DOI: https://doi.org/10.1145/2527031.2527042

[24] El-Refaey MA, Rizkaa MA. Virtual systems workload characterization: An overview. In: *18th IEEE International Workshops on Enabling Technologies: Infrastructures for Collaborative Enterprises, WETICE 2009, 29 June 2009 - 1 July 2009, Groningen, The Netherlands*. 2009. p.72-77. DOI: https://doi.org/10.1109/WETICE.2009.13

[25] Genkin M, Dehne F. Autonomic workload change classification and prediction for big data workloads. In: *2019 IEEE International Conference on Big Data, IEEE BigData 2019, 9-12 December 2019, Los Angeles, CA, USA*. 2019. p.2835-2844. DOI: https://doi.org/10.1109/BigData47090.2019.9006149

[26] Gmach D, Rolia J, Cherkasova L, Kemper A. Workload analysis and demand prediction of enterprise data center applications. In: *IEEE 10th International Symposium on Workload Characterization, IISWC 2007, September 27-29, Boston, MA, USA*. 2007. p.171-180. DOI: https://doi.org/10.1109/IISWC.2007.4362193

[27] Goel G, Ganesan R, Sarkar S, Kaup K. Workload analysis for virtual machine placement. In: *IEEE 18th International Conference on Parallel and Distributed Systems, ICPADS 2012, 17-19 December 2012, Singapore*. 2012. p.732-737. DOI: https://doi.org/10.1109/ICPADS.2012.118

[28] Goldberg RP. Survey of virtual machine research. *Computer*. 1974; 7(6): 34-45. DOI: https://doi.org/10.1109/MC.1974.6323581

[29] Gutiérrez-García JO, Ramirez-Nafarrate A. A policy-based agents for virtual machine migration in cloud data centers. In: *IEEE International Conference on Services Computing, 28 June 2013 - 3 July 2013, Santa Clara, CA, USA*. 2013. p.603-610. DOI: https://doi.org/10.1109/SCC.2013.55

[30] Hou HS, Tretter DR, Vogel MJ. Interesting properties of the discrete cosine transform. *Journal of Visual Communication and Image Representation.* 1992; 3: 73-83. DOI: https://doi.org/10.1016/1047-3203(92)90031-N

[31] Hsiao SW, Chen YN, Sun YS, Chen MC. Combining dynamic passive analysis and active fingerprinting for effective bot malware detection in virtualized environments. In: *Proceedings of the Springer 7th International Conference on Network and System Security, NSS 2013, 3-4 June 2013, Madrid, Spain.* 2013. p.699-706. DOI: https://doi.org/10.1007/978-3-642-38631-2_59

[32] Hu Y, Long X, Wen C. Asymmetric virtual machine scheduling model based on workload classification. In: *IEEE International Conference on Computer Science and Service System, 11-13 August 2012, Nanjing, China*. 2012. p.2231-2234. DOI: https://doi.org/10.1109/CSSS.2012.554

[33] Joachims T. Text categorization with support vector machines: Learning with many relevant features. In: Nedellec C, Rouveirol C. (eds.) *Lecture Notes in Computer Science: Proceedings of the Springer 10th European Conference on Machine Learning, 21-23 April 1998, Chemnitz, Germany.* 1998. p.137-142. DOI: https://doi.org/10.1007/BFb0026683

[34] Kejela G, Esteves RM, Rong C. Predictive analytics of sensor data using distributed machine learning techniques. In: *IEEE 6th International Conference on Cloud Computing Technology and Science, CloudCom 2014, 15-18 December 2014, Singapore.* 2014. p.626-631. DOI: https://doi.org/10.1109/CloudCom.2014.44

[35] Kim BK, Jang JH, Hur KW, Lee JG ,Woong Ko Y. Monitoring and feedback tools for realtime workloads for Xen virtual machine. In: *Proceedings of the International Conference on IT Convergence and Security, ICITCS 2011, 14-16 December 2011, Suwon, Korea.* 2011. p.151-161. DOI: https://doi.org/10.1007/978-94-007-2911-7_13

[36] König JL, Hinze A, Bowen J. Workload categorization for hazardous industries: The semantic modelling of multi-modal physiological data. *Future Generation Computer Systems.* 2023; 141(4): 369-381. DOI: https://doi.org/10.1016/j.future.2022.11.019

[37] Krishna DS, Srinivasi G, Reddy PVGDP. Novel private cloud architecture: A three tier approach to deploy private cloud using virtual machine manager. *Intelligent Decision Technologies.* 2023; 17(2): 275-285. DOI: https://doi.org/10.3233/IDT-229035

[38] Leung CK, Braun P, Hoi CSH, Souza J,Cuzzocrea A. Urban analytics of big transportation data for supporting smart cities. In: *Springer 21st International Conference on Big Data Analytics and Knowledge Discovery, DaWaK 2019, 26-29 August 2019, Linz, Austria.* 2019. p.24-33. DOI: https://doi.org/10.1007/978-3-030-27520-4_3

[39] Leung CK, Cuzzocrea A, Mai JJ, Deng D, Jiang F. Personalized DeepInf: Enhanced social influence prediction with deep learning and transfer learning. In: *2019 IEEE International Conference on Big Data, BigData 2019, 9-12 December 2019, Los Angeles, CA, USA.* 2019. p.2871-2880. DOI: https://doi.org/10.1109/BigData47090.2019.9005969

[40] Maddodi G, Jansen S, de Jong R. Generating workload for ERP applications through end-user organization categorization using high level business operation data. In: *Proceedings of the 2018 ACM/SPEC International Conference on Performance Engineering, ICPE 2018, 09-13 April 2018, Berlin, Germany.* 2018. p.200-210. DOI: https://doi.org/10.1145/3184407.3184432

[41] Mahambre S, Kulkarni P, Bellur U, Chafle G, Deshpande D. Workload characterization for capacity planning and performance management in IaaS cloud. In: *2012 IEEE International Conference on Cloud Computing in Emerging Markets, CCEM 2012, October 11-12, Bangalore, India.* 2012. p.1-7. DOI: https://doi.org/10.1109/CCEM.2012.6354624

[42] Manté C. Application of resampling and linear Spline methods to spectral and dispersional analyses of long-memory processes. *Computational Statistics & Data Analysis.* 2007; 51(9): 4308-4323. DOI: https://doi.org/10.1016/j.csda.2006.05.015

[43] Oracle VM VirtualBox. *User Manual.* https://www.virtualbox.org/manual/ [Accessed 15th January 2023].

[44] Panneerselvam J, Liu L, Antonopoulos N, Yuan B. Workload analysis for the scope of user demand prediction model evaluations in cloud environments. In: *Proceedings of the 7th IEEE/ACM International Conference on Utility and Cloud Computing, UCC 2014, 8-11 December 2014, London, United Kingdom.* 2014. p.883-889. DOI: https://doi.org/10.1109/UCC.2014.144

[45] Pearson WR, Lipman DJ. Improved tools for biological sequence comparison. *Proceedings of the National Academy of Sciences.* 1988; 85(8): 2444-2448. DOI: https://doi.org/10.1073/pnas.85.8.2444

[46] Ripley BD. Neural networks and related methods for classification. *Journal of the Royal Statistical Society: Series B (Methodological).* 1994; 56(3): 409-437. DOI: https://doi.org/10.1111/j.2517-6161.1994.tb01990.x

[47] Rokach L. Ensemble-based classifiers. *Artificial Intelligence Review.* 2010; 33(1-2): 1-39. DOI: https://doi.org/10.1007/s10462-009-9124-7

[48] Shafer G. *A Mathematical Theory of Evidence.* Princeton University Press; 1976.

[49] Shao YS, Brooks DM. ISA-independent workload characterization and its implications for specialized architectures. In: *2012 IEEE International Symposium on Performance Analysis of Systems & Software, 21-23 April 2013, Austin, TX, USA*. 2013. p.245-255. DOI: https://doi.org/10.1109/ISPASS.2013.6557175

[50] Sherwood T, Perelman E, Hamerly G, Sair S, Calder B. Discovering and exploiting program phases. *IEEE Micro.* 2003; 23(6): 84-93. DOI: https://doi.org/10.1109/MM.2003.1261391

[51] Singh S, Chana I. Metrics based workload analysis technique for IaaS Cloud. *arXiv* [Preprint] 2014. Doi: https://arxiv.org/abs/1411.6753

[52] SOAP. *Simple object access protocol.* https://www.w3.org/TR/soap/ [Accessed 15th January 2023].

[53] Sousa Vieira ME, Suaŕez-Gonzaĺez A, Fernańdez-Veiga M, Lopéz-Ardao JC, Lopéz-Garca C. Model selection for long-memory processes in the spectral domain. *Computer Communications.* 2013; 36(13): 1436-1449. DOI: https://doi.org/10.1016/j.comcom.2013.06.002

[54] SPEC. *The standard performance evaluation corporation.* http://www.spec.org/ [Accessed 15th January 2023].

[55] Thakur A, Goraya MS. A workload and machine categorization-based resource allocation framework for load balancing and balanced resource utilization in the cloud. *International Journal of Grid and High Performance Computing.* 2022; 14(1): 1-16. DOI: https://doi.org/10.4018/IJGHPC.301594

[56] VirtualBox Main API. *VirtualBox main API documentation.* https://www.virtualbox.org/sdkref/ [Accessed 15th January 2023].

[57] Van Do T. Comparison of allocation schemes for virtual machines in energy-aware server farms. *The Computer Journal.* 2011; 54(11): 1790-1797. DOI: https://doi.org/10.1093/comjnl/bxr007

[58] Vandromme N, Dandres T, Maurice E, Samson R, Khazri S, Moghaddam RF, Nguyen KK, Lemieux Y, Cheriet M. Life cycle assessment of videoconferencing with call management servers relying on virtualization. In: *Proceedings of the 2014 conference ICT for Sustainability, ICT4S-14, 25 August 2014, Stockholm, Sweden.* 2014. p.281-289. DOI: https://doi.org/10.2991/ict4s-14.2014.34

[59] WSDL. *Web services description language.* https://www.w3.org/TR/wsdl20/ [Accessed 15th January 2023].

[60] Xiao P, Hu Z, Liu D, Zhang X, Qu X. Energy-efficiency enhanced virtual machine scheduling policy for mixed workloads in cloud environments. *Computers & Electrical Engineering.* 2014; 40(5): 1650-1665. DOI: https://doi.org/10.1016/j.compeleceng.2014.03.002

[61] Xiao L, Chen S, Zhang X. Dynamic cluster resource allocation for jobs with known and unknown memory demand. *IEEE Transactions on Parallel and Distributed Systems.* 2002; 13(3): 223-240. DOI: https://doi.org/10.1109/71.993204

[62] Xu Y, Musgrave Z, Noble B, Bailey M. Workload-aware provisioning in public clouds. *IEEE Internet Computing.* 2014; 18(4): 15-21. DOI: https://doi.org/10.1109/MIC.2014.38

[63] Yang C, Xu Z, Gu G, Yegneswaran V, Porras PA. DroidMiner: automated mining and characterization of fine-grained malicious behaviors in android applications. In: Kutylowski M, Vaidya J. (eds.) *Lecture Notes in Computer Science: Proccedings of the 19th European Symposium on Research in Computer Security, ESORICS 2014, 7-11 September 2014, Wroclaw, Poland.* 2014. p.163-182. DOI: https://doi.org/10.1007/978-3-319-11203-9_10

[64] Ying-Dar L, Yuan-Cheng L, Chien-Hung C, Hao-Chuan T. Identifying Android malicious repackaged applications by thread-grained system call sequences. *Computers & Security.* 2013; 39: 340-350. DOI: https://doi.org/10.1016/j.cose.2013.08.010

[65] Zhang J, Figueiredo RJ. Autonomic feature selection for application classification. In: *Proceedings of the IEEE 3rd International Conference on Autonomic Computing, ICAC 2006, 13-16 June 2006, Dublin, Ireland.* 2006. p.43-52. DOI: https://doi.org/10.1109/ICAC.2006.1662380

[66] Zhao X, Yin J, Chen Z, He S. Workload classification model for specializing virtual machine operating system. In: *2013 IEEE 6th International Conference on Cloud Computing,* June 28 - July 3, Santa Clara, CA, USA. 2013. p.343-350. DOI: https://doi.org/10.1109/CLOUD.2013.144

[67] Zhou Y, Jiang X. Dissecting Android malware: Characterization and evolution. *33rd IEEE Symposium on Security and Privacy, SP 2012, 21-23 May 2012, San Francisco, CA, USA.* 2012. p.95-109. DOI: https://doi.org/10.1109/SP.2012.16

Alfredo Cuzzocrea
iDEA Lab
University of Calabria
Rende, Italy
&
Department of Computer Science
University of Paris City
Paris, France
E-mail: alfredo.cuzzocrea@unical.it

Enzo Mumolo
Department of Engineering
University of Trieste
Trieste, Italy
E-mail: mumolo@units.it

Islam Belmerabet
iDEA Lab
University of Calabria
Rende, Italy
E-mail: ibelmerabet.idealab.unical@gmail.com

Abderraouf Hafsaoui
iDEA Lab
University of Calabria
Rende, Italy
E-mail: ahafsaoui.idealab.unical@gmail.com

Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://tfss.journals.iau.ir/
Online ISSN: 2821-0131
Vol.2, No.2, (2023), 184-193
DOI: http://doi.org/10.30495/tfss.2023.1991906.1082

Article Type: Original Research Article

On The Spectrum of Countable MV-algebras

Giacomo Lenzi*

Abstract. In this paper we consider MV-algebras and their prime spectrum. We show that there is an uncountable MV-algebra that has the same spectrum as the free MV-algebra over one element, that is, the MV-algebra $Free_1$ of McNaughton functions from $[0,1]$ to $[0,1]$, the continuous, piecewise linear functions with integer coefficients. The construction is heavily based on Mundici equivalence between MV-algebras and lattice ordered abelian groups with the strong unit. Also, we heavily use the fact that two MV-algebras have the same spectrum if and only if their lattice of principal ideals is isomorphic. As an intermediate step we consider the MV-algebra A_1 of continuous, piecewise linear functions with rational coefficients. It is known that A_1 contains $Free_1$, and that A_1 and $Free_1$ are equispectral. However, A_1 is in some sense easy to work with than $Free_1$. Now, A_1 is still countable. To build an equispectral uncountable MV-algebra A_2, we consider certain "almost rational" functions on $[0,1]$, which are rational in every initial segment of $[0,1]$, but which can have an irrational limit in 1.

We exploit heavily, via Mundici equivalence, the properties of divisible lattice ordered abelian groups, which have an additional structure of vector spaces over the rational field.

AMS Subject Classification 2020: MSC 06D35; MSC 06F20
Keywords and Phrases: MV-algebras, Prime spectrum, Lattice ordered abelian groups.

1 Introduction

MV-algebras are the algebraic counterpart of Łukasiewicz fuzzy logic in the same sense as Boolean algebras are the counterpart of classical logic. An important topological invariant of MV-algebras is the prime spectrum. However, unlike the particular case of Boolean algebras, whose prime spectrum is a complete invariant by Stone duality, see [11], there are different MV-algebras with the same spectrum. A simple example is given by the Boolean algebra $\{0,1\}$ and the real interval $[0,1]$. These two MV-algebras are not isomorphic (one has two elements, and the other has the power of the continuum) but their prime spectrum is the one point topological space.

As usual we denote the set of natural numbers by $\mathbb{N} = \{0,1,2,\ldots\}$, the set of positive integers $\mathbb{N}^+ = \{1,2,3,\ldots,\}$, the integer ring by $\mathbb{Z}$, the rational field by $\mathbb{Q}$, and the real field by $\mathbb{R}$.

MV-algebras are axiomatized in the following way (see [5] for a basic treatment and [9] for a more advanced text). They are algebraic structures of the form $(A,\oplus,0,\neg,1)$ where

- $(A,\oplus,0)$ is a commutative monoid;
- $\neg\neg x = x$;
- $\neg 0 = 1$;

*Corresponding Author: Giacomo Lenzi, Email: gilenzi@unisa.it, ORCID: 0000-0002-5599-2604
Received: 19 July 2023; **Revised**: 8 August 2023; **Accepted**: 21 September 2023; **Available Online**: 28 September 2023; **Published Online**: 7 November 2023.

How to cite: Lenzi G., On the spectrum of countable MV-algebras *Trans. Fuzzy Sets Syst.* 2023; 2(2): 184-193. DOI: http://doi.org/10.30495/tfss.2023.1991906.1082

- $x \oplus 1 = 1$;
- $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$ (Mangani axiom).

We will denote an MV-algebra by A.

The most important (and motivating) example of MV-algebra is $[0, 1]$, where $x \oplus y = min(x + y, 1)$ and $\neg x = 1 - x$. We have:

Lemma 1.1. *(see [6]) The MV-algebra* $[0, 1]$ *generates the variety of MV-algebras.*

For $n \in \mathbb{N}^+$ and $x \in A$ we denote $nx = x \oplus x \ldots \oplus x$ where $\oplus$ occurs $n-1$ times. We let also $0x = 0$.

Recall that a lattice is a partially ordered set $(L, \leq)$ where for every $x, y \in L$ there is the supremum (least upper bound) $x \vee y$ and the infimum (greatest lower bound) $x \wedge y$. We will denote a lattice by L.

In any MV-algebra we have the partial order such that $x \leq y$ if and only if there is z such that $y = x \oplus z$. This order is a (distributive) lattice where $x \vee y = \neg(\neg x \oplus y) \oplus y$ and $x \wedge y = \neg(\neg x \vee \neg y)$.

2 Abelian ℓ-groups

A kind of algebraic structure close to MV-algebras are abelian ℓ-groups, see [5]. An abelian ℓ group is an abelian group with a lattice structure such that $x \leq y$ implies $x + z \leq y + z$. A strong unit of an ℓ-group G is an element $u \geq 0$ such that for every $x \in G$ there is a positive integer $n \in \mathbb{N}$ such that $x \leq nu$.

We will denote an abelian ℓ-group by G.

For $x \in G$ and $m \in \mathbb{Z}$ we denote by mx the usual multiplication by m.

We collect some useful ℓ-group rules of commutation (or anticommutation):

Lemma 2.1. *For every* a, b, c *in an abelian* ℓ*-group* G *and for every* $m, m' \in \mathbb{Z}$ *we have*

$$\begin{aligned}
a + (b \wedge c) &= (a + b) \wedge (a + c) \\
a + (b \vee c) &= (a + b) \vee (a + c) \\
0a &= 0 \\
m(a \wedge b) &= ma \wedge mb, m > 0 \\
m(a \vee b) &= ma \vee mb, m > 0 \\
m(a \wedge b) &= ma \vee mb, m < 0 \\
m(a \vee b) &= ma \wedge mb, m < 0 \\
m(a + b) &= ma + mb \\
m(m'a) &= (mm')a
\end{aligned} \tag{1}$$

Proof. It is known that the variety of abelian ℓ-groups is generated by $\mathbb{Z}$, see [1]. So it is enough to prove these assertions in $\mathbb{Z}$, which is easy. □

We find it useful to recall divisible abelian ℓ-groups. An abelian ℓ-group G is called divisible if for every $x \in G$ and for every $n \in \mathbb{N}^+$ there is $y \in G$ such that $ny = x$. Abelian ℓ-groups are torsion free, see [4], so when y exists, it is unique, and it will be denoted by x/n.

A divisible abelian ℓ-group has a natural structure of a vector space over $\mathbb{Q}$, that is, if $m \in \mathbb{Z}$ and $n \in \mathbb{N}^+$, we denote $(m/n)x = (mx)/n$.

Now for divisible abelian ℓ-groups we not only have all properties of ℓ-groups, but some properties concerning multiplication by elements of $\mathbb{Q}$:

Lemma 2.2. *For every a, b, c in a divisible abelian ℓ-group G and for every $q, q' \in \mathbb{Q}$ we have*

$$\begin{aligned} q(a \wedge b) &= qa \wedge qb, q > 0 \\ q(a \vee b) &= qa \vee qb, q > 0 \\ q(a \wedge b) &= qa \vee qb, q < 0 \\ q(a \vee b) &= qa \wedge qb, q < 0 \\ q(a + b) &= qa + qb \\ q(q'a) &= (qq')a \end{aligned} \tag{2}$$

Proof. The idea is to write $q = m/n$ with $m \in \mathbb{Z}$ and $n \in \mathbb{N}^+$ and reduce to the integer case, which is treated in the Lemma 2.1. □

We derive from Lemmas 2.1 and 2.2 a lemma on subgroups:

Lemma 2.3. *Let G be a divisible abelian ℓ-group and S a subset of G. The divisible abelian ℓ-group generated by S is given by the lattice combinations of rational linear combinations of elements of S.*

Proof. Let T be the set of the lattice combinations of rational linear combinations of elements of S. It is enough to show that T is a divisible ℓ-group.

First, $0 \in T$ trivially, and by definition, T is closed under the lattice operations $\wedge$ and $\vee$.

To prove that T is closed under sum, we have to show that if $t, t' \in T$ then $t + t' \in T$. To this aim, if u is a ℓ-group term, let us denote by $n(u)$ the number of lattice operations in u.

Closure under sum can be proved by induction on $n(t + t')$. In fact, if $n(t + t') = 0$ then the thesis is clear. If $n(t + t') > 0$ then at least one of t, t' begins with $\circ$, where $\circ$ is one of $\wedge, \vee$. By symmetry we can suppose $t' = u \circ v$. Now $t + t' = t + (u \circ v) = (t + u) \circ (t + v)$ by Lemma 2.1. But $t + u$ and $t + v$ have less lattice operators than $t + t'$, so we can apply the inductive hypothesis and find $t + u \in T$ and $t + v \in T$, so $t + t' = (t + u) \circ (t + v) \in T$. This completes the inductive proof.

Finally, T is closed under multiplication by any $q \in \mathbb{Q}$ since the latter commutes (or anticommutes) with lattice operations and rational linear combinations, in the sense of Lemma 2.2. In particular taking $q = -1$, T is closed under inverse, so is a group, and taking $q = 1/n$, $n \in \mathbb{N}^+$, T is divisible. □

Mundici in [10] discovered an equivalence Γ between the category of MV-algebras and the category of abelian ℓ-groups with strong unit. Namely, $\Gamma(G, u)$ is the MV-algebra with universe $\{x \in G \mid 0 \leq x \leq u\}$ and operations $x \oplus y = (x + y) \wedge u$ and $\neg x = u - x$. The motivation of Mundici was the study of AF C^*-algebras in quantum mechanics. The equivalence Γ will be very useful in this work.

3 Ideals in MV-algebras and lattices

An ideal of an MV-algebra A is a subset I of A which is a monoid and is closed downwards. Ideals in ℓ-groups also exist, but they will not be used in this work.

An ideal of a lattice L is a subset I of L closed under $\vee$ and closed downwards.

An ideal I is called principal if it is generated by one element. In MV-algebras or lattices, every finitely generated ideal is principal.

If $a \in A$, we denote by $id_A(a)$ the principal ideal generated by a in A. So, $id_A(a)$ is the set of all $b \in A$ such that $b \leq na$ for some $n \in \mathbb{N}$.

An ideal P of an MV-algebra A is called prime if $P \neq A$ and $x \wedge y \in P$ implies $x \in P$ or $y \in P$. The same holds for prime ideals of a lattice L.

Following [13], we denote by $Id_c(A)$ the lattice of principal ideals of an MV-algebra A, where c stands for compact, since principal ideals are exactly the compact elements of the lattice of all ideals of A.

The lattice $Id_c(A)$ when A is an MV-algebra can also be characterized in the following way:

Lemma 3.1. *(see [3] for the introduction of the Belluce lattice) Let A be an MV-algebra. The lattice $Id_c(A)$ is isomorphic to the Belluce lattice $\beta(A)$, which is A modulo the equivalence of lying in the same prime ideals, where $[x] \wedge [y] = [x \wedge y]$ and $[x] \vee [y] = [x \vee y]$, and $[x]$ denotes the equivalence class of x.*

Proof. It is enough to show that, for every $x, y \in A$, we have $x \in id_A(y)$ if and only if every prime ideal containing y contains x.

Suppose $x \in id_A(y)$. Then $x \leq ny$ for some $n \in \mathbb{N}$. Let P prime with $y \in P$. Since P is an ideal, $ny \in P$ and $x \in P$.

Conversely, suppose $x \notin id_A(y)$. Let S be the set of ideals I of A such that $x \notin I$ and $y \in I$. Then S is nonempty since $id_A(y) \in S$. Every chain in S has an upper bound since S is closed under union. So, by Zorn's Lemma, S has a maximal element P. It is enough to show that P is prime.

First, P is a proper ideal since $x \notin P$.

Moreover, suppose $a \wedge b \in P$ but $a, b \notin P$. Then $P \cup \{a\}$ and $P \cup \{b\}$ generate an ideal containing x. So, $x \leq p_1 \vee n_1 a$ and $x \leq p_2 \vee n_2 b$ with $p_1, p_2 \in P$ and $n_1, n_2 \in \mathbb{N}$. We can write $n_1 + n_2 = n$ and $p_1 \vee p_2 = p$. Then $x \leq p \vee na$ and $x \leq p \vee nb$. Taking the infimum we have

$$x \leq (p \vee na) \wedge (p \vee nb) = p \vee (na \wedge nb) = p \vee n(a \wedge b) \qquad (3)$$

(this can be proved by Lemma 1.1).

But $p \in P$ and $a \wedge b \in P$, so $x \in P$, contrary to the fact that $x \notin P$. □

The prime spectrum $Spec(A)$ is the topological space of the prime ideals of A where the basic opens are $O(a) = \{P \in Spec(A) | a \notin P\}$, where $a \in A$. This topology is called the Zariski topology. The prime spectrum of a lattice L, $Spec(L)$, is defined in the same way.

Spectra of MV-algebras have been characterized in [7] in terms of their compact open sets. They are particular spectral spaces. So, we can take advantage of Stone duality between spectral spaces and bounded distributive lattices, see [12]. If X is a spectral space, its Stone dual is the lattice $\overset{\circ}{K}(X)$ of compact open sets of X. Conversely, if L is a bounded distributive lattice, then $Spec(L)$ is the prime spectrum of L with the Zariski topology.

Moreover, the following are known:

Proposition 3.2. *For every MV-algebra A, $Spec(Id_c(A))$ is homeomorphic to $Spec(A)$.*

Proof. This follows from [3] and Lemma 3.1. □

Proposition 3.3. *For every MV-algebra A, the lattice $\overset{\circ}{K}(Spec(A))$ is isomorphic to $Id_c(A)$.*

Proof. By the previous proposition $\overset{\circ}{K}(Spec(A)) = \overset{\circ}{K}(Spec(Id_c(A)))$ and this is $Id_c(A)$ by the Stone duality of [12]. □

Summing up, it follows:

Lemma 3.4. *Let A, B be two MV-algebras. Then the topological spaces $Spec(A)$ and $Spec(B)$ are homeomorphic if and only if the lattices $Id_c(A)$ and $Id_c(B)$ are isomorphic.*

Proof. Suppose $Id_c(A) = Id_c(B)$. Then $Spec(Id_c(A)) = Spec(Id_c(B))$. So by Proposition 3.2, $Spec(A) = Spec(B)$.

Conversely, suppose $Spec(A) = Spec(B)$. Then $\overset{\circ}{K}(Spec(A)) = \overset{\circ}{K}(Spec(B))$ and by Proposition 3.3 we conclude $Id_c(A) = Id_c(B)$. □

4 Piecewise-F functions and free MV-algebras

In the next section we will build an MV-algebra consisting of one-dimensional functions with real values. So, in this section we introduce some notation which is ad hoc for the study of this kind of functions.

Given $a, b \in \mathbb{R}$ we use the standard notations for intervals like $[a, b] = \{x|a \leq x \leq b\}$ and $]a, b[= \{x|a < x < b\}$ and $[a, b[= \{x|a \leq x < b\}$.

Given a real valued function f, let $zeros(f)$ denote the set of zeros of f.

Let F be a class of functions. Let $a, b \in \mathbb{Q}$.

A continuous function $f : [a, b] \to \mathbb{R}$ is piecewise-F if there is $n \in \mathbb{N}$ and there are n rationals $c_1 = a < c_2 < \ldots < c_n = b$ such that every restriction $f|_{[c_i, c_{i+1}]}$ is in F.

More generally, we say that a continuous function $f : [a, b] \to \mathbb{R}$ is almost (piecewise) F if for every closed rational interval $J = [a, c]$ with $a < c < b$, the restriction $f|_J$ is piecewise F. Note that J is an initial segment of $[a, b]$.

Since MV-algebras are axiomatized by equations, they form a variety, and there are free MV-algebras $Free_k$ over any cardinal k. For $k = 1$ we have:

Theorem 4.1. *(see [5]) The MV-algebra $Free_1$ is given by one variable McNaughton functions, that is, the piecewise-$AFFINT$ functions from $[0, 1]$ to $[0, 1]$, where $AFFINT$ is the set of affine linear functions with integer coefficients.*

More generally one can consider the family $AFFRAT$ of affine linear functions with *rational* coefficients, or $AFFREAL$ with real coefficients (we do not need real coefficients in this work, but they are important in the context of Riesz MV-algebras, see [8]).

It is convenient to say that a segment in the Cartesian plane is rational if it has two rational extremes (this implies that the slope is rational unless the segment is vertical).

Lemma 4.2. *The graph of a piecewise $AFFRAT$ function from $[a, b]$ to $\mathbb{R}$ is a finite union of rational segments.*

Corollary 4.3. *Let f be a piecewise $AFFRAT$ function from $[a, b]$ to $\mathbb{R}$. The intersection of the graph of f with a rational segment is a finite union of rational points and rational segments.*

Proof. The intersection of two rational segments, if nonempty, is either a rational point or a rational segment (by analytic geometry). Now the thesis follows from the previous lemma by subdividing the graph of f into finitely many rational segments. □

5 On the spectrum of $Free_1$

We have seen that there are different MV-algebras with homeomorphic spectrum.

We call equispectral two MV-algebras with homeomorphic spectrum.

We recall from a submitted work:

Proposition 5.1. *(see [2]) Let A be an MV-algebra. Then the equispectrality class of A is a set (in Zermelo-Fraenkel set theory) and it has at least cardinality $2^{\aleph_0}$.*

We conjecture that every equispectrality class has at least cardinality $2^{2^{\aleph_0}}$. This happens, for instance, for the class of $A = \{0, 1\}$.

The conjecture implies that every equispectrality class contains an uncountable MV-algebra. This question is relevant in view of the results of [10], where countable MV-algebras play a major role, since they are put in correspondence with certain AF-C^*-algebras in view of applications to quantum mechanics.

In this paper we focus on the equispectrality class of the MV-algebra $Free_1$. We will prove:

Theorem 5.2. *There is an uncountable MV-algebra equispectral with $Free_1$.*

Proof.

It is enough to find an uncountable MV-algebra with the same principal ideal lattice as $Free_1$. We will build two MV-algebras A_1 and A_2; A_1 is already known, whereas A_2 (to our knowledge) is new and is the witness of the theorem. We have $Free_1 \subseteq A_1 \subseteq A_2$.

$Free_1$, A_1 and A_2 are MV-algebras of continuous functions from $[0,1]$ to itself.

Let G_1 be the divisible ℓ-group of piecewise $AFFRAT$ functions from $[0,1]$ to $\mathbb{R}$, where $AFFRAT$ is the set of affine linear functions with *rational* coefficients.

Let $A_1 = \Gamma(G_1, 1)$. Note that A_1 is an MV-algebra consisting of all functions of the form $trunc(g)$ for $g \in G_1$, where $trunc$ is the truncation operator:

$$trunc(g) = (g \vee 0) \wedge 1. \tag{4}$$

Equivalently, A_1 is the MV-algebra of piecewise $AFFRAT$ functions from $[0,1]$ to $[0,1]$.

We can say that the elements of A_1 are generalized McNaughton functions, where integer coefficients are replaced by rational coefficients. In particular $Free_1$ is a MV-subalgebra of A_1.

It is known:

Lemma 5.3. *Two elements of A_1 generate the same ideal if and only if they have the same zeros.*

Proof. Let $f, g \in A_1$. If $id_{A_1}(f) = id_{A_1}(g)$ then $f \leq ng$ and $g \leq nf$ for some $n \in \mathbb{N}$, so they have the same zeros.

Conversely, suppose f, g have the same zeros.

Let us call maximal nonzero interval any maximal open interval $]a, b[\subseteq [0,1]$ where f, g have no zero. Note that a and b can be zeros of f, or 0, or 1.

Let $]a, b[$ be a maximal nonzero interval.

If c is sufficiently close to a, then f and g are linear in $]a, c[$. Likewise if d is sufficiently close to b, then f and g are linear in $]d, b[$. So, f/g is bounded in $]a, c[\cup]d, b[$. Moreover, $g \neq 0$ in $[c, d]$; so, by continuity, g is bounded from below in $[c, d]$, and since f is bounded in $[c, d]$, f/g is bounded in $[c, d]$. Summing up, f/g is bounded in $]a, b[$, and taking the supremum over all maximal nonzero intervals $]a, b[$, f/g is bounded in $[0,1]$ whenever $g \neq 0$. So, there is $n \in \mathbb{N}$ such that $f \leq ng$ whenever $g \neq 0$, hence $f \leq ng$ everywhere in $[0,1]$ since $zeros(f) = zeros(g)$.

By symmetry, there is also $n' \in \mathbb{N}$ such that $g \leq n'f$. So $id_{A_1}(f) = id_{A_1}(g)$. □

It is also known:

Lemma 5.4. *The MV-algebras $Free_1$ and A_1 are equispectral.*

Proof. By Lemma 3.4 it is enough to show that $Free_1$ and A_1 have the same lattice of principal ideals.

Given a function $f \in A_1$, let us consider the MV-algebraic sum nf, by a sufficiently large integer $n \in \mathbb{N}$. Then rational slopes of f become integer in nf, and the intersection of every piece of the function with the y axis is an integer point. So in nf, every piece is affine linear with integer coefficients (or constantly 1, which is trivially an affine function with integer coefficients), so nf is piecewise $AFFINT$, that is, $nf \in Free_1$.

Moreover, clearly the zeros of f and nf are the same. So, by Lemma 5.3, the ideals generated in A_1 by f and nf are the same. □

The previous lemma allows us to use A_1 rather than $Free_1$, which is more convenient for our purposes.

Let us begin our construction of A_2.

Let B be a vector space basis of $\mathbb{R}$ over $\mathbb{Q}$ such that $1 \in B$. Note that B has size $2^{\aleph_0}$.

For every $b \in B, b \neq 1$, let $(q_n(b))_{n \in \mathbb{N}}$ be a sequence of rationals converging to b and we define $u_b : [0,1] \to \mathbb{R}$ as the function such that

- u_b is continuous
- $u_b(1 - 1/n) = q_n(b)$
- u_b is linear between $1 - 1/n$ and $1 - 1/n + 1$.

Note that u_b is almost $AFFRAT$ in $[0, 1]$ and $u_b(1) = b$.

Let G_2 be the divisible ℓ-group generated by $G_1 \cup \{u_b, b \in B, b \neq 1\}$ in the function space $\mathbb{R}^{[0,1]}$ of all functions from $[0, 1]$ to $\mathbb{R}$. Note that every element of G_2 is a continuous, almost $AFFRAT$ function from $[0, 1]$ to $\mathbb{R}$.

Let $A_2 = \Gamma(G_2, 1)$ be the corresponding MV-algebra.

Note that A_2 is an MV-algebra consisting of the functions $trunc(g)$ for $g \in G_2$.

Since $G_1 \subseteq G_2$ we have $A_1 \subseteq A_2$. Moreover:

Corollary 5.5. *Two elements f, g of A_1 generate the same ideal in A_2 if and only if they have the same zeros.*

Proof. By Lemma 5.3, $zeros(f) = zeros(g)$ if and only if $f \leq ng$ and $g \leq nf$ for some $n \in \mathbb{N}$, that is, f an g generate the same ideal in A_2. □

Note that the proof of the previous corollary works only for $f, g \in A_1$.

The elements of G_2 have the following local representation as a sum:

Lemma 5.6. *Locally in 1, every element g_2 of G_2 has the form*

$$g_2 = g_1 + \Sigma_{b \neq 1} q_b u_b, \tag{5}$$

where $g_1 \in G_1$, Σ is a finite sum, $b \in B$, and $q_b \in \mathbb{Q}$. (Note that no lattice operations are necessary).

Proof. Let f, g be continuous functions from $[0, 1]$ to $\mathbb{R}$. If $f(1) < g(1)$, then $f \wedge g = f$ locally in 1. Likewise if $f(1) > g(1)$, then $f \vee g = f$ locally in 1.

So, let g_2 be any element of G_2. By Lemma 2.3, g_2 is a lattice combination of rational linear combinations of $G_1 \cup \{u_b, b \in B, b \neq 1\}$.

So, g_2 is locally in 1 a lattice combination of a family $(f_i)_{i \in I}$ of rational linear combinations of $G_1 \cup \{u_b, b \in B, b \neq 1\}$ such that $f_i(1) = g_2(1)$ for every $i \in I$.

Since B is a basis of $\mathbb{R}$ over $\mathbb{Q}$, and $f_i(1) = g_2(1)$ is independent of i, every f_i is a sum of a rational linear combination of the u_b's independent of i, say $u_B = \Sigma_{b \neq 1} q_b u_b$, and a part $g_i \in G_1$ possibly dependent on i. So $f_i = u_B + g_i$ for every $i \in I$.

Hence, by Lemma 2.2, g_2 is, locally in 1, the sum of u_B and an element g_1 which is a lattice combination of rational linear combinations of elements of G_1, so $g_1 \in G_1$ since G_1 is a divisible ℓ-group. □

Lemma 5.7. *Let $g_2 \in G_2$, with*

$$g_2 = g_1 + \Sigma_{b \neq 1} q_b u_b \tag{6}$$

locally in 1 as in the previous lemma.

If $q_b \neq 0$ for some b, then $g_2(1)$ is irrational.

Proof. Evaluating the equation (6) in 1 we derive

$$g_2(1) = g_1(1) + \Sigma_{b \neq 1} q_b u_b(1) = g_1(1) + \Sigma_{b \neq 1} q_b b.$$

Since each b is in the base B of $\mathbb{R}$ and $q_b \in \mathbb{Q}$, it follows $\Sigma_{b \neq 1} q_b b$ is irrational; and since $g_1(1)$ is rational, $g_2(1)$ is irrational. □

Let $f \in A_2$. We want to show that there is $f_1 \in A_1$ such that $id_{A_2}(f) = id_{A_2}(f_1)$. The key idea is to "sandwich" f between two elements of A_1 whenever possible.

Suppose $f(1) \in \{0, 1\}$. Then $f \in A_1$. In fact, locally in 1,

$$f = trunc(g_1 + \Sigma), \tag{7}$$

where $g_1 \in G_1$ and $\Sigma = \Sigma_{b \neq 1} q_b u_b$. Let us distinguish the possible values of $(g_1 + \Sigma)(1)$.

If $(g_1 + \Sigma)(1) > 1$ then f is an element of A_1 up to some rational abscissa less then 1, followed by a segment constant 1, so $f \in A_1$.

If $(g_1 + \Sigma)(1) = 1$ then $\Sigma = 0$ otherwise $\Sigma(1)$ should be irrational by Lemma 5.7, so $f = trunc(g_1) \in A_1$.

The case $0 < (g_1 + \Sigma)(1) < 1$ is impossible, otherwise by truncation we have $0 < f(1) < 1$, which is false by hypothesis.

If $(g_1 + \Sigma)(1) = 0$ then $\Sigma = 0$ otherwise $\Sigma(1)$ should be irrational by Lemma 5.7, so $f = trunc(g_1) \in A_1$.

Finally if $(g_1 + \Sigma)(1) < 0$ then f is an element of A_1 up to some rational abscissa less than 1, followed by a segment constant 0, so $f \in A_1$.

Otherwise suppose $f(1) \notin \{0, 1\}$. We build two functions $h_1, h_2 \in A_1$ which "sandwich" f.

Let $p_f < 1$ be a rational such that f is nonzero in $[p_f, 1]$ (p_f exists since f is continuous and $f(1) \neq 0$).

Let q_f be a rational with $0 < q_f < f(1)$. Consider the segment s_1 joining $(1, q_f)$ and $(p_f, f(p_f))$. Note that the slope of s_1 is finite and negative, and that the extremes of s_1 are rational, so s_1 is a rational segment. Let us write $y = s_1(x)$ when $(x, y) \in s_1$.

Let $(y_1, f(y_1))$ be the rightmost intersection of s_1 and the graph of f. This intersection point exists since $(p_f, f(p_f))$ is in s_1 and is also in the graph of f. Moreover the point $(y_1, f(y_1))$ has rational coordinates, by Corollary 4.3. The function $h_1 : [0, 1] \to [0, 1]$ is defined by

$$h_1(x) = \begin{cases} f(x), x < y_1 \\ s_1(x), x \geq y_1. \end{cases} \tag{8}$$

Note that $h_1 \in A_1$, $h_1 \leq f$, and f and h_1 have the same zeros.

Likewise consider the segment s_2 joining $(1, 1)$ and $(p_f, 0)$. The segment s_2 has finite positive slope and has two rational extremes, so it is rational. Let us write $y = s_2(x)$ when $(x, y) \in s_2$.

Let $(y_2, f(y_2))$ be the leftmost intersection of s_2 and the graph of f. This intersection exists by continuity since $(1, 1)$ is above the graph of f and $(p_f, 0)$ is below the graph of f. Moreover the point $(y_2, f(y_2))$ has rational coordinates, by Corollary 4.3.

The function $h_2 : [0, 1] \to [0, 1]$ is defined by

$$h_2(x) = \begin{cases} f(x), x < y_2 \\ s_2(x), x \geq y_2. \end{cases} \tag{9}$$

Note that $h_2 \in A_1$, $f \leq h_2$, and f and h_2 have the same zeros.

Summing up, we have $h_1 \leq f \leq h_2$ and $zeros(h_1) = zeros(f) = zeros(h_2)$; so by Corollary 5.5, we have $id_{A_2}(h_1) = id_{A_2}(h_2)$; and since $h_1 \leq f \leq h_2$ we have $id_{A_2}(h_1) = id_{A_2}(h_2) = id_{A_2}(f)$. $\square$

6 Conclusion

We hope to extend theorem 5.2 to every countable MV-algebra. Also we hope to solve the problem whether all equispectrality classes have size exactly $2^{2^{\aleph_0}}$.

Acknowledgements: The author would like to thank professor Antonio Di Nola for discussions about the problem and professor Arsham Borumand Saeid for inviting me to publish this paper.

Conflict of Interest: The author declares no conflict of interest.

References

[1] Anderson M. and Feil T. *Lattice-ordered groups. An introduction.* Reidel Texts in the Mathematical Sciences. Dordrecht: D. Reidel Publishing Co.; 1988. x+190. DOI: https://doi.org/10.1007/978-94-009-2871-8

[2] Barbieri G, Di Nola A and Lenzi G. *Topological classes of MV-algebras.* Submitted.

[3] Belluce L P. Semisimple algebras of infinite-valued logic and bold fuzzy set theory. *Canad. J. Math.* 1986; 38(6): 1356-1379. DOI: https://doi.org/10.4153/cjm-1986-069-0

[4] Bigard A, Keimel K and Wolfenstein S. *Groupes et anneaux réticulés.* (French) Lecture Notes in Mathematics, Vol. 608. Berlin-New York: Springer-Verlag; 1977. DOI: https://doi.org/10.1007/bfb0067004

[5] Cignoli R, D'Ottaviano I and Mundici D. *Algebraic foundations of many-valued reasoning.* Trends in Logic-Studia Logica Library, 7. Dordrecht: Kluwer Academic Publishers; 2000. DOI: https://doi.org/10.1007/978-94-015-9480-6

[6] Di Nola A. Representation and reticulation by quotients of MV-algebras. *Ric. Mat.* 1991; 40(2): 291-297. DOI: https://doi.org/10.1007/11587.1827-3491

[7] Di Nola A and Lenzi G. The spectrum problem for Abelian ℓ-groups and MV-algebras. *Algebra Universalis* 2020; 81(39), Paper No. 39, 43 pp. DOI: https://doi.org/10.1007/s00012-020-00668-4

[8] Di Nola A and Leustean I. Łukasiewicz logic and Riesz spaces. *Soft Comput.* 2014; 18(12): 2349-2363. DOI: https://doi.org/10.1007/s00500-014-1348-z

[9] Mundici D. *Advanced Łukasiewicz calculus and MV-algebras.* Trends in Logic-Studia Logica Library, 35. Dordrecht: Springer; 2011. https://doi.org/10.1007/978-94-007-0840-2

[10] Mundici D. Interpretation of AF C^*-algebras in Łukasiewicz sentential calculus. *J. Funct. Anal.* 1986; 65(1): 15-63. DOI: https://doi.org/10.1016/0022-1236(86)90015-7

[11] Stone M. The theory of representations of Boolean algebras. *Trans. Amer. Math. Soc.* 1936; 40(1): 37-111. DOI: https://doi.org/10.2307/1989664

[12] Stone M. Topological representation of distributive lattices and Brouwerian logics. *Casopis Pest. Mat. Fys.* 1938; 67(1): 1-25. DOI: https://doi.org/10.21136/cpmf.1938.124080

[13] Wehrung F. Spectral spaces of countable Abelian lattice-ordered groups. *Trans. Amer. Math. Soc.* 2019; 371(3): 2133-2158. DOI: https://doi.org/10.1090/tran/7596

Giacomo Lenzi
Department of Mathematics
University of Salerno
Fisciano (SA), Italy
E-mail: gilenzi@unisa.it

Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://tfss.journals.iau.ir/
Online ISSN: 2821-0131
Vol.2, No.2, (2023), 194-218
DOI: http://doi.org/10.30495/tfss.2023.1987633.1074

Article Type: Original Research Article

Provenance Based Trust Boosted Recommender System Using Boosted Vector Similarity Measure

Dhanalakshmi Teekaraman*, **Sendhilkumar Selvaraju**, **Mahalakshmi Guruvayur Suryanarayanan**

Abstract. As users in an online social network are overwhelmed by the abundant amount of information, it is very hard to retrieve the preferred or required content. In this context, an online recommender system helps to filter and recommend content such as people,items or services. But, in a real scenario, people rely more on recommendations from trusted sources than distrusting sources. Though, there are many trust based recommender systems that exist, it lag in prediction error. In order to improve the accuracy of the prediction, this paper proposes a Trust-Boosted Recommender System (TBRS). Since, the provenance derives the trust in a better way than other approaches, TBRS is built from the provenance concept. The proposed recommender system takes the provenance based fuzzy rules which were derived from the Fuzzy Decision Tree. TBRS then computes the multi-attribute vector similarity score and boosts the score with trust weight. This system is tested on the book-review dataset to recommend the top-k trustworthy reviewers.The performance of the proposed method is evaluated in terms of MAE and RMSE. The result shows that the error value of boosted similarity is lesser than without boost. The reduced error rates of the Jaccard, Dice and Cosine similarity measures are 18%, 15% and 7% respectively. Also, when the model is subjected to failure analysis, it gives better performance for unskewed data than slewed data. The models fbest, average and worst case predictions are 90%, 50% and <23% respectively.

AMS Subject Classification 2020: 91D30; 03B52; 03E72; 15A03
Keywords and Phrases: Social network, Provenance, Trust, Fuzzy rule, Fuzzy vector space, Multi-attribute.

1 Introduction

The social network is overloaded with a huge number of posts such as blogs, reviews, opinions, images, videos, etc. People use such web information, to make decisions about what to buy, how to spend free time, what to study, etc. An online recommender system helps to retrieve the desired information from this crowded network. For example, to recommend an item in Amazon's recommender system [19], an item-to-item collaborative filtering approach is used. Similarly, Facebook, LinkedIn and other social networking sites examine the network of connections between a user and their friends to suggest a new group, based on interest. Such a recommendation does not guarantee an accurate recommendation, since it is received from an anonymous person. Therefore, people tend to rely more on a trusted person's recommendation than an untrusted online recommendation [27]. Consequently, the quality of recommendations is ensured by exploiting the trust

∗**Corresponding Author:** Dhanalakshmi Teekaraman, Email: dhanalakshmi.t@jerusalemengg.ac.in, ORCID: 0000-0001-8039-3120
Received: 31 May 2023; **Revised**: 29 August 2023; **Accepted**: 8 October 2023; **Available Online**: 27 October 2023; **Published Online**: 7 November 2023.

How to cite: Teekaraman D, Sendhilkumar S, Mahalakshmi G. S. Provenance based trust boosted recommender system using boosted vector similarity measure. *Trans. Fuzzy Sets Syst.* 2023; 2(2): 194-218. DOI: http://doi.org/10.30495/tfss.2023.1987633.1074

values among recommenders. For example, some of the single rating or single preference based trust models [1, 10, 13, 15, 21], represent the trust value '2' as low trust, while trust value '5' represents very high trust in the five rating scale. With a single rating or preference, the multiple features of the user or item cannot be stated which will either directly or indirectly reduce the recommendation quality. Therefore, if the trust rating is derived using multiple features, for example 'Food Quality', 'Food Service', and 'Cleanliness' rated as (4, 3, 2) to recommend a hotel, then evidently the quality of recommendation is improved. Therefore, the proposed recommender system considers multiple attributes or preferences (here 5).

Although many researchers have been successfully working on the integration of trust networks in recommender systems, some more directions are yet to be explored. The issues that are addressed in this work are stated below.

- The derivation of the trust score plays a vital role in any trust based recommender system. Only, countable number of approaches exist that derive the trust score using provenance, but fail to prove the reduced prediction rate.
- Next is, many trust based recommender systems handle only the crisp input (trust score) i.e. 5 (highly trusted) and 1 (meagerly trusted) but unable to handle the vague trust score effectively.
- The final issue that is addressed here is the recommendation of top-k trustworthy reviewers with reduced prediction error.

The first issue is handled by adopting the W7 provenance model to compute the trust score. The second issue is handled by generating fuzzy rules from the Fuzzy Decision Tree based classifier. The first two issues are solved in [31]. This work is an extension of the above two works and solves the last issue. The TBRS works by first extracting the conditional and decision attributes from fuzzy rules and forms a Fuzzy Vector Space (FVSP). Then it finds the similarity between trusted users using the vector based similarity measures, namely Jaccard, Dice and Cosine [8]. Then computes the weighted similarity score by taking the attribute gain as a weight component. Finally, this similarity is boosted by the user's respective trust degree and Top-k similar users are recommended to the target user. The three major contributions of this paper are as follows:

- User profile Modeling
- Formation of Fuzzy Vector Space
- Prediction and Recommendation

This paper is organized as follows. Section 2 briefs about the existing trust based recommender systems. A detailed discussion of the proposed recommender system is given in section 3. Performance evaluation and results are discussed in section 4. Finally, the conclusion and future works are stated in section 5.

2 Related Research

This section discusses the various related articles in the field of trust based recommender systems. The taxonomy of background work is graphically represented in Figure 1. Based on the information collected from an online trust network, recommendation is generated in trust enhanced recommender systems. The most common trust enhanced recommender strategy is, asking the users to explicitly mention the trust statements about other users. For instance, the Moleskiing recommender system [4] uses FOAF files that contain trusted information scales ranging from 1 to 9. The Trust model proposed by A. Abdul Rahman and S. Hailes [1] for virtual communities is grounded in real-world social trust characteristics, reputation or word-of-mouth. Falcone et.al. proposed a fuzzy cognitive map model [10] to derive the trust based on the belief value of an agent. This model shows how different components (belief) may change and how their impact can change depending on the specific situation and from the agent's personality. The aim of a Golbeck's trust model [13] is, to determine how much one person in the network should trust another person to whom they are not directly connected. This algorithm accurately analyses the opinions of the people in the system. TidalTrust algorithm works based on trust-based weighted mean which uses the trust value of users as a weight for the ratings of other users. Hang et al. [14] used a graph-based approach to recommend a node in a social network using similarity in trust networks. Massa and Aversani [21] proposed a trust-based recommendation system where it is possible to search for trustable users by exploiting trust propagation over the trust network. Andersen et. al. [2] explored an axiomatic approach for trust-based recommendation and proposed several recommendation models, some of which are incentive compatible. In the MoleTrust method the similarity weight is attributed to ratings by users.

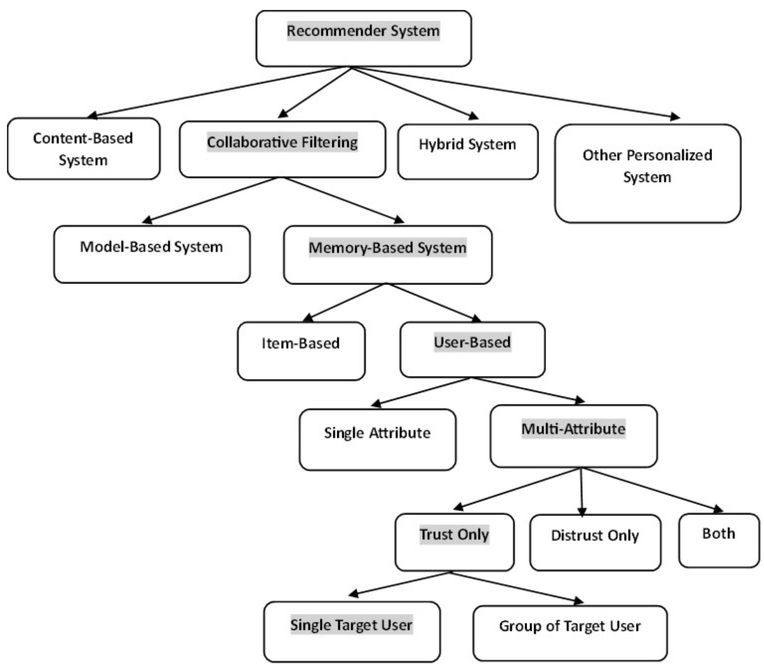


Figure 1: Taxonomy of related work

A trust-filtered collaborative filtering technique is used by O'Donovan and Smith in [29]. Here, the trust value is used as a filtering mechanism, to choose only, the item raters who are trusted above a certain threshold. An Ensemble trust technique proposed by Victor et al. [32] aims to take into account all possible ways to obtain a positive weight for a rater of an item while favoring trust over similarity. Tomislav Duricic et al. [9] proposed a solution to solve the cold start problem in a collaborative-filtering method using regular equivalence. This regular equivalence is applied to a trusted network to generate a similarity matrix using the Katz similarity measure. Abdelghani Bellaachia & Deema [6] proposed a recommendation algorithm called Averaged Localized Trust-Based Ant Recommender (ALT-BAR) to increase the prediction accuracy. The base for this algorithm is Ant Colony Optimization (ACO). To overcome the cold start problem (lack of ratings), ALT-BAR emphasizes the significance of trust between users by modifying the initial pheromone levels of edges. Vahid Faridani et al. [11] proposed a method called effective trust to solve the data sparsity and cold start issues by combining the ratings of trusted neighbors to complement and represent the preferences of active users. Liaoliang et al. [16] proposed a slope one algorithm based on the fusion of trusted data and user similarity. The procedure for the above recommendation algorithm consists of selecting trusted data, calculating the similarity between users and adding this similarity to the weight factor. To address the data sparsity and the cold start problem, Bo Yang et al. [35] proposed a social-collaborative filtering by utilizing the trust data to give high quality recommendations. The author used the matrix factorization technique which maps users into small dimensional latent feature spaces a trust relationship (trustee and truster model). This mapped model is combined into TrustMF model.

Usually e-commerce sites face a large amount of information which leads to sparsity in the data. This causes low accuracy during recommendation. To solve this issue Li Ye et al. [36] proposed a collaborative filtering recommendation which is based on a trust model with fused similar factors. This is nothing but combining the trust model with the user similarity. Modified cosine similarity is introduced in this fused similar factor. One of the key challenges in a recommender system is an accurate prediction of unknown ratings of the target user. During prediction, selecting an appropriate set of users is the major issue in Collaborative Filtering (CF). Hashem Parvin et al. [23] proposed a novel CF method called Trust-aware CF by Ant Colony Optimization (TCF-ACO) to predict missing ratings. First using available ratings and social trust relationship, the users are ranked. Next, proper weight values are assigned to users using ACO. Finally, a set of top-k similar users are filtered out and are used for predicting unknown ratings of the target user. To solve the sparsity and low recommendation accuracy in CF, Kejia & Junyi [34] proposed an improved CF algorithm. This algorithm calculates the user's attribute preference, trust relationships and weight of interest based on time and recommends the items with the highest prediction score. A Graph Convolutional Network via a Reliable and Informative Motif-based Attention Model (CNRIM) [20] is developed to investigate user-user heterogeneous trust relationships and user-item heterogeneous interactivity. Varying reliability and informative motifs introduce heterogeneity. The experiments on publicly available real-world datasets, and empirical analyses present the superiority of our model over popular baselines.

Rad D et al. [24] focus on the study of how socioeconomic status affects trust in recommender systems. It shapes users' perceptions of accuracy, fairness, and transparency in recommender systems. This study is done by exploring the curvilinear effects of the predictor variables on the outcome variable using quadratic regression analysis. The positive and negative aspect of traditional recommendation approaches namely collaborative, content-based and Demographic filtering as discussed in [18]. Also, the potential biases, theoretical insights, design implications and practical solutions for the cold start problem are discussed. Richa and Punam developed a Cross Domain Recommender System (CDRS) [25], which employs data from multiple domains to reduce the problem of sparsity. This model uses a combination of trust as well as distrust which helps in improving trustworthiness of generated recommendation. By incorporating knowledge about the malicious users, the distrust measure shows higher accuracy. This CDRS is developed using JADE and Java technology for the tourism domain. Knowledge graph based trustworthy recommendation system was developed by Nidhi and Richa [7]. Pu Li et al. proposed a scholarly recommendation method by high-order propagation of knowledge graph (HoPKG) [17]. This HoPKG analyzes the high-order semantic information in the KG and generates richer entity representations to obtain users' potential interest by distinguishing the importance of different entities. In current scenario, the demand for senior care services is high. From the crowded data, it has become more difficult to get matching services. This paper proposed a service recommendation framework PCE-CF [33] based on an embedded user portrait model. An automated and personalized meal plan generation was introduced by George and Tekli [26]. This method adapted to the transportation optimization problem. This is a simulation of the human thought process in generating a daily meal plan. The relation between nodes in online social network is filtered out with the help of an ontology. This paper proposed a recommender system using ontology [3]. Choosing a best fit elective course for a student is a challenging task especially at the higher education level. This issue is solved in this paper by utilizing the versatile ontology and sequence prediction algorithm and compact prediction tree [12].

Many works on trust-based collaborative filtering have been carried out to solve the cold start and data sparsity problem. There exist only a few works that attempt to improve accuracy and error minimization.

Most of the work simply uses the similarity score for recommendation without enhancing it. The proposed trust-boosted recommender system recommends the top-k users with minimized error.

3 Proposed Trust Boosted Recommendation System

An architecture of the proposed TBRS is shown in Figure 2. It consists of two major modules. The first module is the provenance based user classification using Fuzzy Decision Tree (FDT) and second module is the recommendation of Top-k trustworthy users. Here users refer to book reviewers. This article depicts an overview of the first module which is given in the following subsection and subsequent section discuss about the proposed trust boosted recommender system.

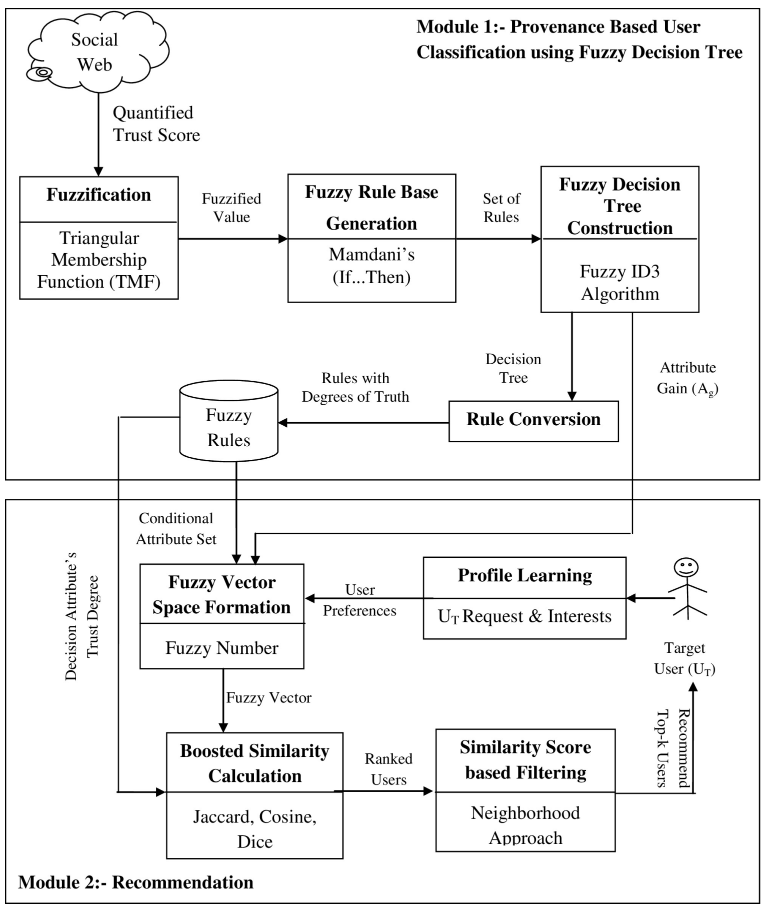


Figure 2: Architecture of trust boosted recommender system

3.1 Overview of Provenance Based User Classification Using Fuzzy Decision Tree (FDT)

This section gives an overview of a classifier built using FDT [31] which is able to solve the first two issues mentioned in the introduction section. The process flow is shown in the top portion of Figure 2.

3.1.1 Provenance Based Trust Quantification

The user's trust in social networks is guaranteed by provenance of web resources [5] The W7 is a provenance model which examines the data in semantic perspective. The plus point of W7 model is that, domain specific provenance requirements are easily adapted. The seven provenance elements are ($WHAT, WHEN, WHERE, WHO, HOW, WHICH, WHY$) [30]. Therefore the trust value of the content creator is assessed using this model. From Goodreads book domain, the reviewer's data are collected by invoking ad-hoc APIs (Application Programming Interface) and HTML scrapping. The Table 1 shows the sample data collected.

Table 1: An example provenance relevant fields

S.No	Field_Name	Example
1	Review Text(RT)	Ever hear people talk about wanting to write the "'Great American Novel"'? Well, it's already done and this is it.This novel is one of my longest standing favorites. It's a profound meditation on the nature of freedom,full of clever Southern folk wisdom, deeply sensitive and insightful.
2	Year Month of Joining(YMJ)	August 2006
3	Review Postdate(RPDT)	September 27, 2013
4	Rate of Review (RVR)	5
5	Likes Received(NLK)	14
6	Post Count(PCNT)	1
7	Comments Received(NCMT)	0
8	Reply Received(NRPY)	90
9	Average Review Rate(ARR)	3.79
10	Key Terms(KT) (Since the terms are exhaustive, only partial terms are given here)	slouch, blame, bowie, buckle, bullyragged, cairo, doxolojer, erysipelas, fan, tod, rod, fox, fire, gabble, gingham, barlow, knife, bars, bilgewater, black, galoot, gar, habob, allycumpain, harrow, teeth, toned, hived, irish, potato, jackstaff, jimpson, weed, juice, harp, langudoc, ambuscade, liberty, pole, melodeum, mesmerism, methusalem, mud, cat, muddy, mug, mulatter, mullen, stalk, mushmelon,calico,camp, capet, bag ,congress, water, corn, consumption, pone, curry, comb,dauphin, delirium, tremens, dog, fennel, doggery, irons, bills, nation, navarre, ...
11	Matched Reviews(MR)	I was surprised by how much I liked this book.There were a couple of parts that dragged for me a bit, but all in all I though it was a very clever, entertaining read. I'm glad I read it as an adult, because I think I liked a lot more now than I would have in high school, especially being the mother of a boy. And I can only hope that my son never is, or has a friend, like Tom Sawyer.
12	Matched Review Postdate(MRPDT)	March 26,2011
13	Time of System Initialization(TSI)	January 2007

The actual description of provenance element as per Bunge's Ontology is given in Table 2. The Table 3 shows the description in the context of trust and its relevant fields are given in Table 4.

P_{WHAT}**:** The trust score of the reviewer is assessed based on the review(s) that are relevant to the title of the book.

Table 2: Description of provenance elements as per Bunge's ontology

Provenance Elements	Description
WHAT	An event that occurred to the data during its lifetime.
WHEN	The time of the event.
WHERE	Location of the event.
WHO	An organization or agent involved in the event.
HOW	The one or more actions that lead to the event.
WHICH	The software or instruments used in the event.
WHY	The reason behind the occurrence of an event.

Table 3: Description of provenance elements in the context of trust

Provenance Elements	Description
P_{WHAT}	Describes the review content that is relevant to the topic.
P_{WHEN}	Represents the effective time spent by the reviewer.
P_{WHERE}	Refers to the location (IP_Address, Domain_name) from where review is posted.
P_{WHO}	Refers to the reviewer who is an author (creator) of the review (originator).
P_{HOW}	Describes how review content is deviated from the rating given by the reviewer.
P_{WHICH}	Refers to the application or device used to post the review.
P_{WHY}	Describes the intention behind the post of review content.

P_{HOW}**:** The reviewer's the trust score of is judged based on how much the RT is deviated from the RVR.

P_{WHO}**:** Here, based on the originality of the review, the trust score of the reviewer is evaluated.

P_{WHY}**:** The trust score of the reviewer is assessed based on the truthfulness of the review.

P_{WHEN}**:** Here, trust scores of the reviewer is assessed based on following three factors.These are

- **Activity_Factor** ($P_{WHEN_{AF}}$):- Measures the active participation or involvement of the reviewer
- **Presence_Factor** ($P_{WHEN_{PF}}$):- Measures how long the reviewer is present in the domain.
- **Frequency_Factor** ($P_{WHEN_{FF}}$):- Calculates how frequently reviewer makes an interaction at awaited frequency constant (π). The π can take the value as one week, two week, three week and upto seven week.
- **Final trust score** $(P_{WHEN_{TF}}) : -(Wt_1)P_{WHEN_{AF}} + (Wt_2)P_{WHEN_{PF}} + (Wt_3)P_{WHEN_{FF}}$.
  Here, Wt_1, Wt_2 and Wt_3 are weight values of $P_{WHEN_{AF}}$, $P_{WHEN_{PF}}$, and $P_{WHEN_{FF}}$ respectively. The weight values can be from 0 to 1 and sum of weight should be 1. For eaxmple, $Wt_1 = 0.6$, $Wt_2 = 0.25$ and $Wt_3 = 0.15$.

A sample quantified value (trust score) of these five provenance elements is shown in Table 5. The score P_{WHAT}=0.222 means RT is highly relevant to the title or concept. The trust score P_{HOW}=0.827 shows that there is not much deviation between RT and his/her RVR whereas P_{HOW}=2.836 shows the more deviation. The $P_{WHEN} = 0.3296$ and P_{WHEN} =0.1222 means that the effective time spent by the reviewer is more in former case and less in latter case.

Table 4: Required fields of provenance elements

Provenance Elements	Required Fields
P_{WHAT}	RT, KT
P_{WHEN}	RPDT, YMJ, NLK, NCMT, NRPY, TSI, PCNT
P_{WHO}	RT, RPDT
P_{HOW}	RT, RVR
P_{WHY}	RT, RVR

Finally, the trust score of each reviewer is given as input to the learning model to classify reviewers with gradual trust levels.

Table 5: Sample quantified value

Reviewer_ID	P_{WHO}	P_{HOW}	P_{WHY}	P_{WHAT}	P_{WHEN}
1	0.0403	2.761	1.351	0.094	0.3296
2	0.0125	2.831	1.421	0.11	0.1222
3	0.0896	0.827	1.417	0.182	0.1344
4	0.0062	2.836	1.426	0.222	0.2812
5	0.0023	1.791	1.799	0.066	0.1278

3.1.2 Fuzzy Decision Tree Based Classification

The classification process comprises of four major steps.

- Fuzzification of Trust Score
- Fuzzy Rule Base Generation
- Fuzzy Decision Tree Construction and
- Rule Conversion

(a) Fuzzification of Trust Score The quantified trust value derived above is taken as a training data for fuzzification process which converts it into linguistic terms. The proposed model uses the Triangular Membership Function(TMF) for fuzzification process, since it allows a maximum number of instances to fall into this class than any other MF. Each attribute (P_{WHAT}, P_{WHAT}, P_{WHAT}, P_{WHAT}, P_{WHAT}) is partitioned into 5 regions as R_1 to R_5 and the corresponding linguistic space is given in Equation 1.

$$LinguisticSpace = \begin{pmatrix} P_{WHAT} = [HIR, MIR, NR, MR, HR] \\ P_{HOW} = [HSM, MSM, NSM, MD, HD] \\ P_{WHEN} = [HITM, MITM, NETM, METM, HETM] \\ P_{WHY} = [HTR, MTR, NTR, MUTR, HUTR] \\ P_{WHO} = [HDSML, MDSML, NDSML, MSML, HSML] \end{pmatrix} \tag{1}$$

(b) Fuzzy Rule Base Generation Fuzzy sets and fuzzy logic are used as tools for representing the knowledge in Fuzzy Rule Based System (FRBS). The fuzzy knowledge base comprises vague facts and vague rules. Each rule contains an antecedent ('IF' part) and a consequent ('THEN' part). Now, this fuzzy input is then transformed into a set of fuzzy rules (rule base) using Mamdani's 'If...Then' interpretation. The two major steps for deriving a rule base are (i) T-norm to evaluate the firing strength of a rule and (ii) S-norm to compute the qualified membership value. The sample fuzzy rule base is as follows.

$$P_{WHO}(\text{HSML}) \wedge P_{WHEN}(\text{HITM}) \wedge P_{HOW}(\text{HD}) \wedge P_{WHY}(\text{MUTR}) \wedge P_{WHAT}(\text{HR}) \implies U_{TRUST}(\text{LT})$$

(c) Fuzzy Decision Tree Construction FDT takes the rule base and generates decision trees using a fuzzy ID3 [22] algorithm. In FDT, provenance element having highest information gain is chosen as a root node and trust decisions are denoted in a leaf node. Each distinctive path from root to a leaf gives distinct rule. The predecessor part ('IF') of the rule contains node(s) and edge(s) of a path excluding leaf. If more than one node exists in 'IF' part, then they are joined by AND/OR operator or both. The consequent part ('THEN') of the rule contains a leaf node alone. A Degree of Truth (DoT) [28] is assigned to each generated rule to state that how much truth value it holds. DoT is computed using (i) Certainty Factor and (ii) Subsethood based approaches. The range of DoT from 0 to 0.5 represents the false degree and 0.6 to 0.9 denotes the truth degree. If DoT is 1, it means the rule is absolutely true which (i) takes a minimum number of nodes and hence reduced rule generation time and (ii) acquire the knowledge with the least number of feature itself. The sample decision tree is shown in Figure 3.

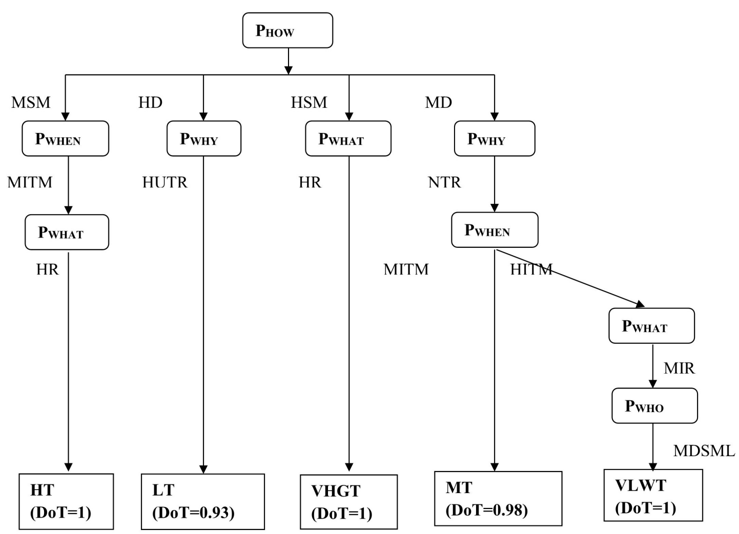


Figure 3: Sample fuzzy decision tree

(d) Rule Conversion The rule is an easy and comprehensive form of knowledge representation than any other representation The corresponding fuzzy rule is shown in Figure 4. Each distinct path from a root to a leaf is called a rule.

3.2 Recommendation of Top-k Trustworthy Users

The three major steps of the recommendation process are (i) User profile modeling, (ii) Formation of FVSP and (iii) Prediction and recommendation. Let the user U_T is a target user who sends a request for recommendations. If U_T is an existing user then the details such as name, number of ratings given, number of

Rule 1: If P_{HOW} is MSM and P_{WHEN} is MITM and P_{WHAT} is HR Then Trust is HT (DoT=1)

Rule 2: If P_{HOW} is HD and P_{WHY} is HUTR Then Trust is LT (DoT=0.93)

Rule 3: If P_{HOW} is HSM and P_{WHAT} is HR Then Trust is VHGT (DoT=1)

ule4: If P_{HOW} is MD and P_{WHY} is NTR and P_{WHEN} is MITM Then Trust is MT (DoT=0.98)

Rule 5: If P_{HOW} is MD and P_{WHY} is NTR and P_{WHEN} is HITM and P_{WHAT} is MIR and P_{WHO} is MDSML Then Trust is VLWT (DoT=1)

Figure 4: Corresponding Fuzzy rules

reviews given, average rating, the interest and trust score of the user are known and can directly access the trust network. If U_T is new user then profile of the user needs to be learned prior to network access. The contents of the profile learned are name, location, join date, favorite books. Initially U_T's area of interest and training example, or already labeled items are collected and sent to the profile learner. Then the set of feedback and request are merged with the output of profile learner. This forms the U_T's file database and sets as user preference.

3.2.1 Formation of FVSP

The fuzzy rules extracted from the trust network as discussed in section 3.1 are partitioned into conditional attribute sets and decision attributes set. The conditional attributes consist of all the trust attributes $P_{HOW}, P_{WHY}, P_{WHEN}, P_{WHAT}, \textit{and}\ P_{WHO}$. The decision attributes consists of trust decision VLWT, LT, MT, HT and VHGT. The following steps explain how to form FVSP using conditional attribute set.

Step 1: For each trust attributes in the conditional attribute set, assign attribute grade. This is based on the position of the TMF. For example, in P_{WHAT} attribute the position of 'HIR' has low grade, i.e. 1 and 'HR' has high grades, i.e. 5. Similarly, for other trust attributes.

Step 2: Now, assign the fuzzy number for each linguistic term based on the grade. Since it follows the triangular fuzzy logic, the fuzzy number assigned for each grade is shown in Table 6.

Table 6: Fuzzy number for each grade

Grade	Fuzzy Number
1	(0.0, 0.0, 0.25)
2	(0.0, 0.25, 0.50)
3	(0.25, 0.50, 0.75)
4	(0.50, 0.75, 1.0)
5	(0.75, 1.0, 1.0)

For example, the fuzzy number of the linguistic term for the attribute P_{HOW} is shown in Table 7. Similarly for other attributes, fuzzy number is same as that shown in Table 7. The corresponding fuzzy number line is given in Figure 5.

Step 3: The fuzzy number for each attribute is now represented as a vector in FVSP. The FVSP for each rule is represented as $< A_K, FN_{AK} >$.
where,

Table 7: Linguistic values of P_{HOW} Fuzzy number

Linguistic Term	Fuzzy Number
HSM (Highly Same)	(0.0, 0.0, 0.25)
MSM (Moderately Same)	(0.0, 0.25, 0.50)
NSM (Neutrally Same)	(0.25, 0.50, 0.75)
MD (Moderately Deviated)	(0.50, 0.75, 1.0)
HD (Highly Deviated)	(0.75, 1.0, 1.0)

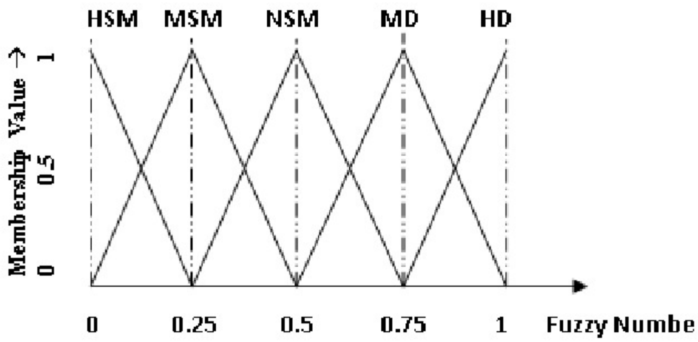


Figure 5: Fuzzy line of P_{HOW} attribute.

- K is the number of attributes (Here, 5)
- A_K is the current attribute and
- FN_{AK} is the fuzzy number for the specified attribute A_K

That is FVSP $= < A_1, (a_{11}, a_{12}, a_{13}) >, < A_2, (a_{21}, a_{22}, a_{23}) > \cdots < A_5, (a_{51}, a_{52}, a_{53}) >$. The (a_{11}, a_{12}, a_{13}) is a triplet used in TMF to define the fuzzy number and the range of value is $0 \leq a_{11} \leq a_{12} \leq a_{13} \leq 1$.

For example, consider the following fuzzy rule.

If P_{HOW} is MD $\wedge$ P_{WHY} is NTR $\wedge$ P_{WHEN} is HITM $\wedge$ P_{WHAT} is MIR $\wedge$ P_{WHO} is MDSML $\rightarrow T_{VLWT}$.

The FVSP for the above fuzzy rule is $< P_{HOW}, (0.5, 0.7, 1) >, < P_{WHY}, (0.25, 0.5, 0.7) >, < P_{WHEN}, (0, 0, 0.25) >, < P_{WHAT}, (0, 0.25, 0.5) >$ and $< P_{WHO}, (0, 0.25, 0.5) >$. Here, MD (Moderately Deviated), NTR (Neutrally Truthful), HITM (Highly Ineffective Time Spent), MIR (Moderately Irrelevant) and MDSML (Moderately Dissimilar) are the linguistic terms of the P_{HOW}, P_{WHY}, P_{WHEN}, P_{WHAT}, and P_{WHO} attributes respectively.

This FVSP is taken as input to calculate the vector similarity and to suggest the top-k trustworthy users.

3.2.2 Prediction and Recommendation

Some of the similarity measures in vector space have been positively applied in fields such as pattern recognition, decision making problems and classification complex objects. The familiar vector similarity measures are Jaccard, Dice and Cosine. The proposed recommender system uses these three measures separately to compute the similarity score between two vectors as shown in Equations 2, 3 and 4. The weighted similarity is obtained by taking the gain value of each attribute (A_G) as weight.

Let $X = U_T = (a_1, a_2, a_3)$ and $Y = U_N = (b_1, b_2, b_3)$ be the fuzzy number of the target user U_T and the other user U_N from the trust network respectively, then

$$S = Jaccard(U_T, U_N) = \sum_{k=1}^{5} A_{G_k} * \frac{\sum_{f=1}^{3}(FN_{A_{Tkf}}.FN_{A_{Nkf}})}{\sum_{f=1}^{3}(FN^2_{A_{Tkf}})+\sum_{f=1}^{3}(FN^2_{A_{Nkf}})-\sum_{f=1}^{3}(FN_{A_{Tkf}}.FN_{A_{Nkf}})} \tag{2}$$

$$S = Dice(U_T, U_N) = \sum_{k=1}^{5} A_{G_k} \frac{2\sum_{f=1}^{3}(FN_{A_{Tkf}}.FN_{A_{Nkf}})}{\sum_{f=1}^{3}(FN^2_{A_{Tkf}}) + \sum_{f=1}^{3}(FN^2_{A_{Nkf}})} \tag{3}$$

$$S = Cosine(U_T, U_N) = \sum_{k=1}^{5} A_{G_k} \frac{\sum_{f=1}^{3}(FN_{A_{Tkf}}.FN_{A_{Nkf}})}{\sqrt{\sum_{f=1}^{3}(FN^2_{A_{Tkf}})}.\sqrt{\sum_{f=1}^{3}(FN^2_{A_{Nkf}})}} \tag{4}$$

where,

- A_G- Represents the attribute gain
- f - Represents the fuzzy number of values in each fuzzy number
- a_1, a_3, b_2, b_3 are the endpoints and a_2, b_2 are the peak point of fuzzy numbers

After finding the similarity (S), boost this value by corresponding trust score (T_{wt}) of the user U_N as shown in Equation 5.

$$S_b = S * S^{T_{wt}-1} \tag{5}$$

Using this boosted similarity (S_b), prediction of the target user's trust score is carried out. The prediction formula is given in Equation 6.

$$Pred(U_T, I_j) = \begin{cases} tr_{U_T}, \quad if\ S_b = 0\ or\ if\ tr_{U_N,I_j} = \bar{tr_{U_N}} \\ tr_{U_T} + \frac{\sum U_N \in NB S_b(U_N,U_T) x (tr_{U_N,I_j} - \bar{tr_{U_N}}}{\sum U_N \in NB |S_b(U_N,U_T)|},\ Else \end{cases} \tag{6}$$

where,

- t_r - Represents the trust value
- I_j – Represents items (books) which are not given any review
- NB – Represents the number of neighbors chosen

Consider the randomly chosen reviewer say reviewer 631 (R_{631}) requesting for the recommendation of k users (Let k=15) as shown in Table 8. The similarity (S) between the requester and the rest of the users is calculated. Then it is boosted using Equation 5. The Table 8 shows the similarity and boosted similarity (S_b) score of the top-15 reviewer where the reviewers are sorted based on similarity scores from highest to lowest. Though both similarities show the highest score for the top reviewers, the trust level differs. The trust level of highly matched reviewer with R_{631} is 'LT'. The top 2 reviewers for both the case, i.e. with and without boost are same. In the case without boosting, top 3rd to 11th and 13th reviewers have other trust level ('MT') instead of 'LT'. But, in case of boosting 3rd to 6th and 13th reviewer has 'VLWT' trust level. Also, 8th and 12th reviewer has 'MT' trust level. This shows that there is an error while carrying out the prediction of trust levels.

Though both without boost and with boost method shows some kind of prediction error, the percentage of the prediction error is less in a later case (46.66%) than the former case (66.66%).

Table 8: Similarity score with and without boost

Without Boost			With Boost			Top-k Reviewer
Reviewer Number	S	Trust Level	Reviewer Number	S_b	Trust Level	
531	0.86012	LT	50	0.94150	LT	1
50	0.86012	LT	531	0.94150	LT	2
630	0.83144	MT	837	0.92353	VLWT	3
26	0.80813	MT	988	0.92353	VLWT	4
1123	0.78135	MT	618	0.90491	VLWT	5
973	0.77931	MT	947	0.90491	VLWT	6
942	0.77931	MT	453	0.90411	LT	7
842	0.77931	MT	630	0.89515	MT	8
356	0.77931	MT	500	0.89250	LT	9
257	0.77931	MT	650	0.89250	LT	10
236	0.77931	MT	678	0.89250	LT	11
453	0.77725	LT	26	0.88001	MT	12
119	0.76748	MT	662	0.87366	VLWT	13
678	0.75253	LT	637	0.87018	LT	14
650	0.75253	LT	679	0.87018	LT	15

3.2.3 Illustrative Example

Let us take random users for whom the recommendation need to be done. The fuzzy rule for Target User (U_T) and user from the trust network (U_N)is given below.

Rule of U_N If P_{HOW} is MD $\wedge$ P_{WHY} is HUTR $\wedge$ P_{WHEN} is NETM $\wedge$ P_{WHAT} is MIR $\wedge$ P_{WHO} is HDSML $\rightarrow T_{LT}$.
Rule of U_T If P_{HOW} is HD $\wedge$ P_{WHY} is HUTR $\wedge$ P_{WHEN} is NETM $\wedge$ P_{WHAT} is NR $\wedge$ P_{WHO} is HDSML.

The Table 9 shows the fuzzy number of U_N and U_T. The similarity calculations are given in Table 10. The attribute gain value of P_{HOW}, P_{WHY}, P_{WHEN}, P_{WHAT} and P_{WHO} are 0.3393, 0.2363, 0.1825, 0.1696 and 0.0723 respectively. The GainWtSim is calculated using these values. FinalSim is the sum of GainWtSim of all the attributes. Finally, BoostedSim is calculated using Equation 5.

Table 9: Fuzzy number for sample input

LingValue (U_N)	Fuzzy Number			LingValue (U_T)	Fuzzy Number		
MD	0.5	0.75	1	HD	0.75	1	1
HUTR	0.75	1	1	HUTR	0.75	1	1
NETM	0.25	0.5	0.75	NETM	0.25	0.5	0.75
MIR	0	0.25	0.5	NR	0.25	0.25	0.25
HDSML	0	0	0.25	HDSML	0	0	0.25

Table 10: Vector similarity score calculation

Similarity Measure	Similarity	GainWtSim	FinalSim	BoostedSim
Cosine	0.986	0.335	0.988	0.995
	1.00	0.236		
	1.00	0.186		
	0.956	0.162		
	1.00	0.072		
Dice	0.971	0.329	0.963	0.985
	1.00	0.236		
	1.00	0.1825		
	0.842	0.143		
	1.00	0.072		
Jaccard	0.944	0.320	0.935	0.973
	1.00	0.236		
	1.00	0.183		
	0.727	0.123		
	1.00	0.072		

4 Performance Evaluation and Result Discussion

To evaluate the performance of the proposed TBRS, experiments are conducted on the popular book based social network called Goodreads.com. The performance measures such as Mean Absolute Error (MAE), Root Mean Squared Error (RMSE) and Average Precision (AP) are defined in the following subsection. Then, an evaluation is carried out with other weight strategies and results are discussed. The proposed TBRS is compared with other trust based recommender systems and the outcomes are described. Finally, the failure scenarios of the proposed recommender system are discussed.

4.1 Performance Measures

The performance of the proposed recommendation strategy is measured with respect to quality of predictions and quality of recommendations. The quality of prediction is done by measuring MAE and RMSE given in Equation 7 and 8 respectively. Similarly the quality of recommendation is done by measuring AP as shown in Equation 9. The TBRS uses the Leave-one-out method to evaluate recommendation systems. This technique involves withholding one rating and trying to predict it with remaining ratings. Then the predicted rating can be compared with the actual rating and the difference will be considered as the prediction error.

$$MAE = \frac{1}{NB}\sum_{i=1}^{NB} \mid Y_i - \hat{Y}_i \mid \tag{7}$$

$$RMSE = \sqrt{\frac{1}{NB}\sum_{i=1}^{NB}(Y_i - \hat{Y}_i)^2} \tag{8}$$

where,

- y_i – Represents the actual value and $\widehat{y_i}$ - Represents predicted value

$$AP@N = \frac{1}{m}\sum_{k=1}^{N} P(k).rel(k) \tag{9}$$

AP is an average of the precision value obtained after each relevant document is retrieved and corresponds to the area under the precision-recall curve. Here, N be the number of items to be recommended, m be the number of relevant items and P(k) refers to precision at k^{th} item.

4.2 Evaluation of Different Weight Approaches

The different weight approaches considered for evaluation are expected weight, preference based weight and proposed gain weight. The three vector similarity measures, namely Cosine, Dice and Jaccard are carried out on the above mentioned weight approaches. The Figures 6, 7 and 8 shows the MAE value obtained from the above three similarity measures. The RMSE value obtained for the above three similarity methods is shown in figures 9, 10 and 11. From the Figures 6, 7 and 8, it is observed that the proposed gain based method shows the less MAE than the other two methods in all the three similarity cases. Also, the RMSE value of the proposed method is less when compared with the expected weight method in all the three cases. The preference based method shows more error rate than the other two methods.

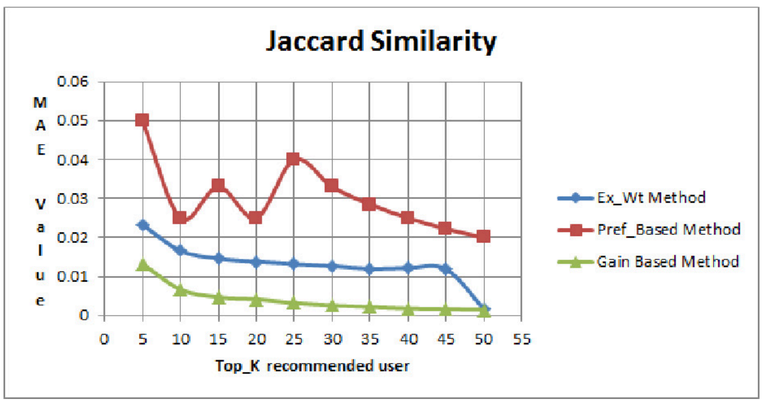


Figure 6: Jaccard MAE measure

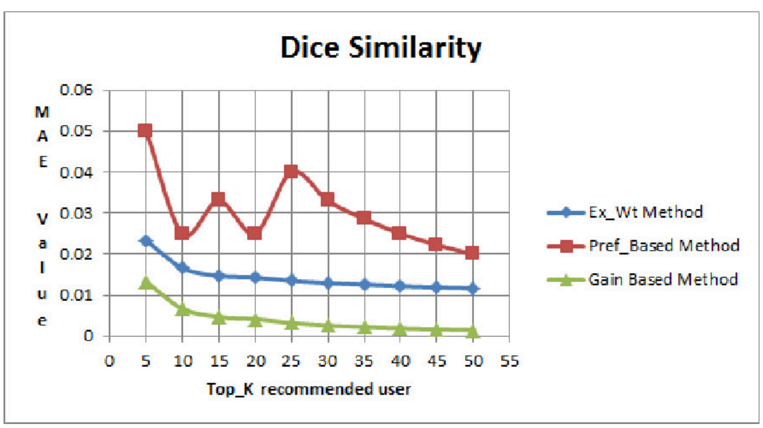


Figure 7: Dice MAE measure

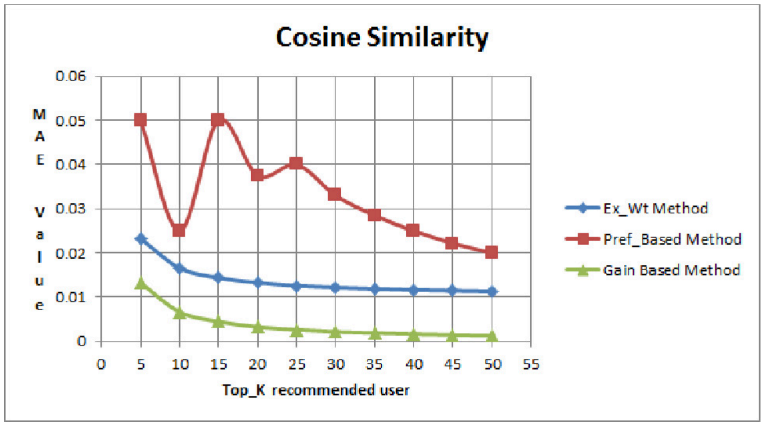


Figure 8: Cosine MAE measure

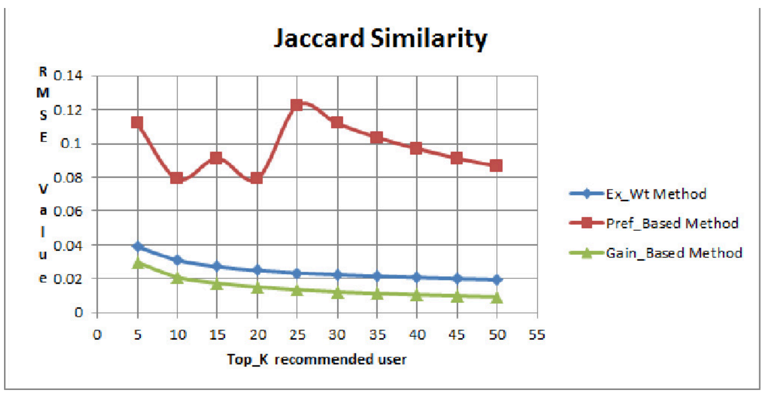


Figure 9: Jaccard RMSE measure

The average precision value is shown in Figures 12, 13 and 14 for all the three similarity methods. The precision value for the proposed method is higher than the other two methods. In all the three cases the average precision is almost same for top-5 and top-10 users. Up to top 20 users precision value is greater than or equal to 0.90. After that the precision value is started decreasing gradually. For top-50$^{\text{th}}$ user, the precision value is very less in preference based method.

The reason for low MAE, lowest RMSE and high AP in proposed gain based method is as follows.

- In preference based method, fuzzy numbers are ranked based on surface area measurement method. The magnitude of the surface area depends on the location of each fuzzy number on the real line. Possible surface values are $0, 0.25, 1, 2, \cdots n-2$. An overall evaluation of edge is calculated by arithmetic average of the fuzzy weight of all the values of involved edges. The value of an edge is adjusted, i.e. rounded-up or rounded-down to one of the above possible surface values. This result in almost equal weight for all the five attributes.

- The expected weight method assigns a weight or grade for each linguistic term of an attribute on the real line. For example, HSM, NTR, HITM, NR, HDSML assigned the weight of 5, 3, 1, 3, 5 as per position on the real line. This method also results in almost equal weight for all five attributes.

- The proposed gain based method uses the information gain value as a weight. The information gain is derived while constructing a fuzzy decision tree. This results in maximum gain for more significant

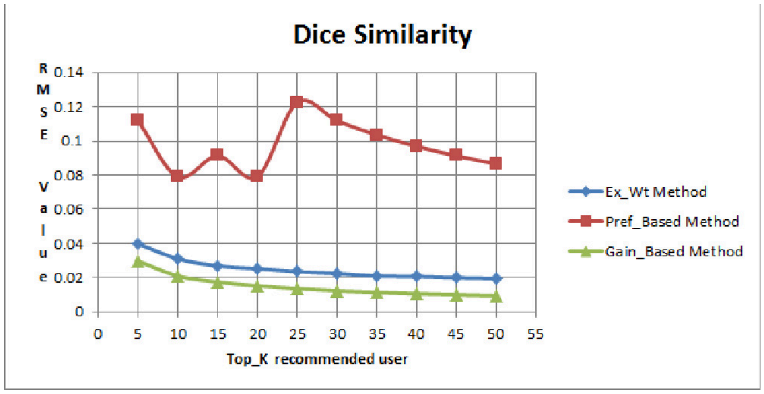


Figure 10: Dice RMSE measure

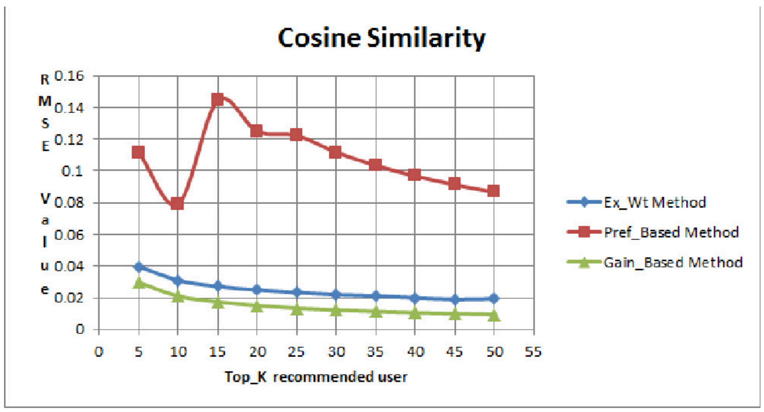


Figure 11: Cosine RMSE measure

attribute P_{HOW} and minimum gain value for least significant attribute P_{WHO}.

Equal weights treats all the attributes as either most significant or least significant. But this is not the case in fuzzy real time data. Therefore, the preference based and expected weight method unable to predict the highly matched reviewer accurately. This compromises the quality of the prediction and hence it leads to prediction error.

4.3 Comparing with Other Trust-Based Recommender System

The proposed recommender system is compared with other trust based recommender system. The evaluation is done on MAE and RMSE measures. First, the proposed method (Boost) is compared against without boosting the similarity. The MAE value of this comparison is shown in Figure 15. In case of NoBst the MAE value for all the three measures are larger than a Boost (Proposed). In the Boost (proposed) the MAE value is very less in Jaccard , slightly higher MAE in Dice followed by Cosine. The MAE and RMSE values of the proposed approach when compared with other trust-based classifier is shown Figure 16 and 17 respectively. The compared methods are Tidal Trust, Mole Trust, Fuzzy Trust Filtering (FTF), Ensemble and Hybrid.

The MAE value of the proposed method is minimized when compared to other methods. When compared to mole trust the error value of the proposed method is slightly lesser. Similarly, the proposed approach results in minimum RMSE value when compared with other approaches. The reason for lesser MAE and RMSE of the proposed method is as follows.

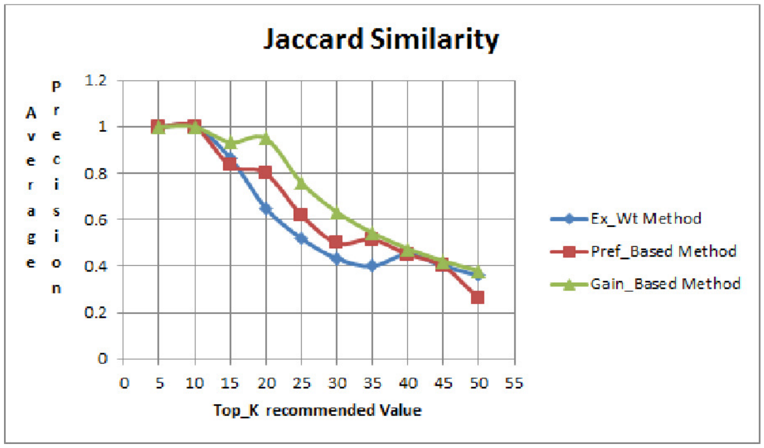


Figure 12: Jaccard AP measure

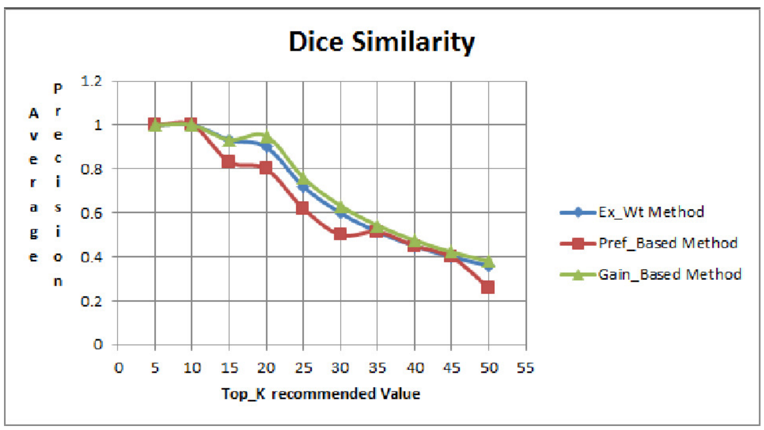


Figure 13: Dice AP measure

- The Tidal Trust method assigns greater weight for more trustworthy users in the prediction process. For example, the trust score of Average Trust, High Trust, Very High Trust and Completely Trust are 0.50, 0.67, 0.83 and 1.00 respectively. This weight is based on the core of the corresponding triangular fuzzy set. The proposed method also assigns more weight to higher trust users and less weight to the low trust user, but the weights are uniformly assigned.

- Mole Trust works by aggregating all the trust statements to produce a trust network. The trust metric is computed based on the maximum propagation distance (MPD). If MPD is 4, then trust metric is 1 (High Trust), 0.75, 0.5, 0.25 (Low Trust). If MPD is 5, then trust metric is 1 (High Trust), 0.833, 0.666, 0.5, 0.33, 0.16 (Low Trust). Since the trust weights are more or less same as proposed, the MAE and RMSE values are slightly closer. But, if MPD is greater than 5, then certainly Mole Trust shows higher MAE and RMSE.

- FTF chose only item rater who are above a certain threshold. That is, it filters neighbors prior to recommendation so that, only the High Trust, Very High Trust and Completely Trust users can participate in the recommendation process. Because of the threshold restriction, the MAE and RMSE values are very large.

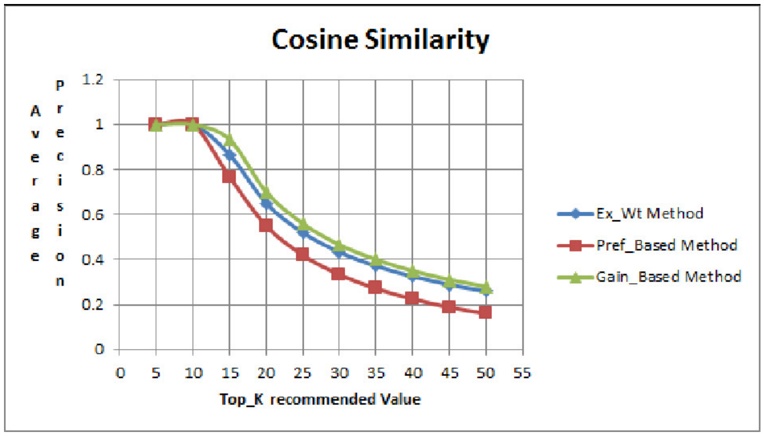


Figure 14: Cosine AP measure

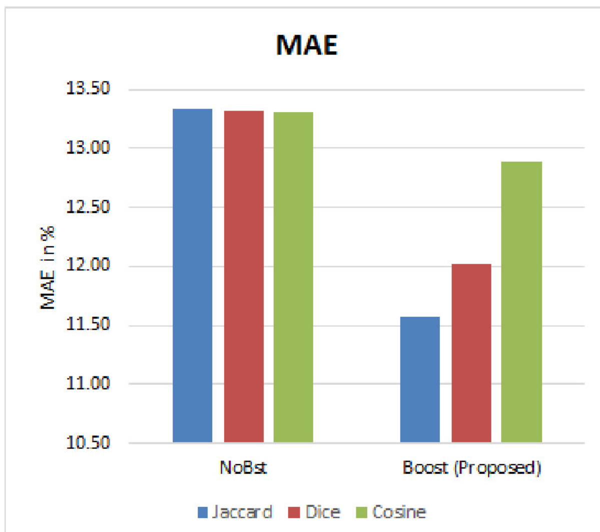


Figure 15: MAE with and without boost

- The ensemble method takes all possible ways to obtain a positive weight for the rater. Thus, it aims to increase the percentage of predictions made by the RS called coverage.

- The hybrid method combines explicit and implicit ratings. Explicit ratings are derived using the Mole Trust method. An implicit rating is computed based on similarity and knowledge factors. It finds the rating difference between two users and assigns weights. If the difference is 0 to 0.5 then weights of 5 is assigned. If the difference is 2 to 3 then weights of 2 is assigned. If it is >3 then weights of 1 is assigned.

To conclude, each method applies a different trust metric for different level of trust. Since the trust metric is not uniformly distributed in the compared trust based recommender systems, it shows the higher MAE and RMSE than proposed.

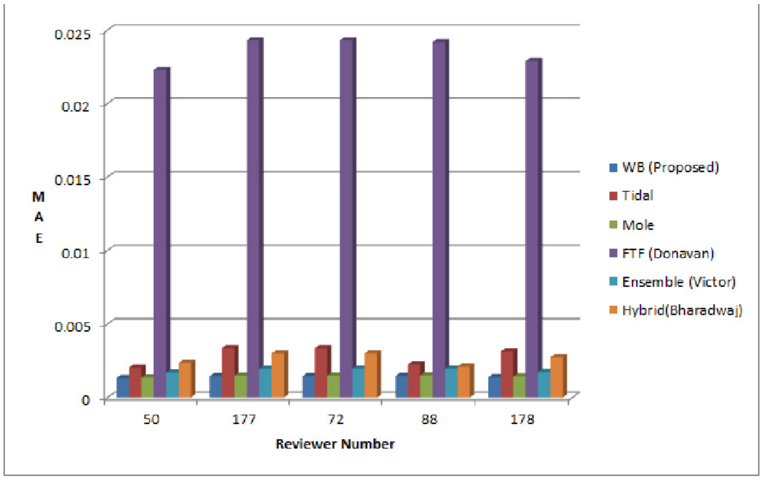


Figure 16: MAE compare

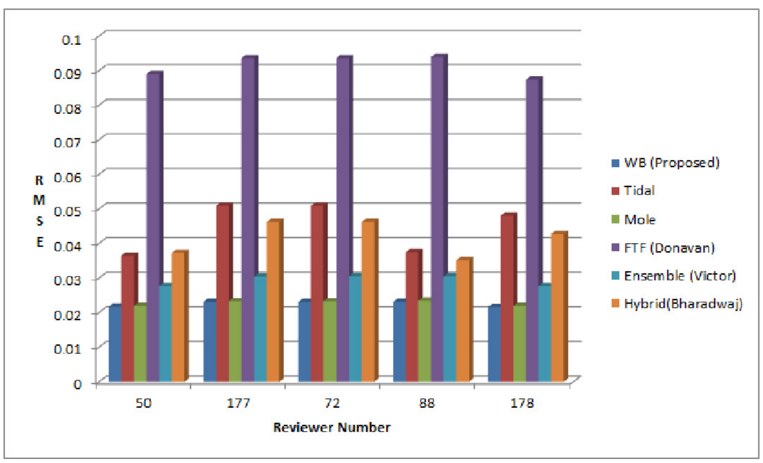


Figure 17: RMSE compare

4.4 Failure Scenarios of Recommender System

To analyze the failure of prediction, let us take a sample from the population. With the margin of error 5% and confidence interval of 95%, the required sample size is obtained. The Figures 18, 19 and 20 shows the prediction score of Best Case, Average Case and Worst Case scenario respectively. The x-axis represents the active users (here, reviewers), y-axis refer to percentage of correct prediction, BJ refers to Boosted Jaccard, BD refers to Boosted Dice and BC refers to Boosted Cosine.

The best prediction score is obtained when the recommendation is made for the Moderately Trusted Users. Here more than 90% of score is achieved in all the three similarity measures. The reason for high prediction score is the highest number of reviewers are classified into this category. An average prediction score is attained when the recommendation is made from Low Trusted and High Trusted users. In this scenario maximum of 50% data is correctly predicted. Because, the number of reviewers classified into LT and HT is lesser when compared with MT.

The recommendation made from the Very Low Trust and few cases of Low Trust users gives the worst prediction score. The maximum of 23% and a minimum of 0% of data can be predicted here. For the user

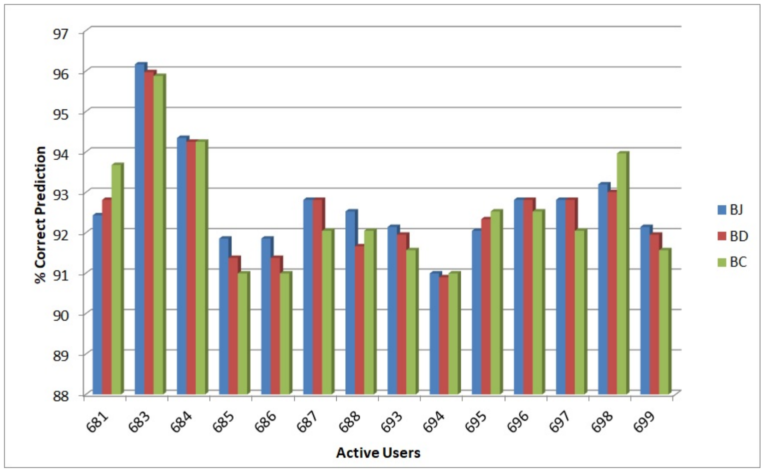


Figure 18: Best case scenario

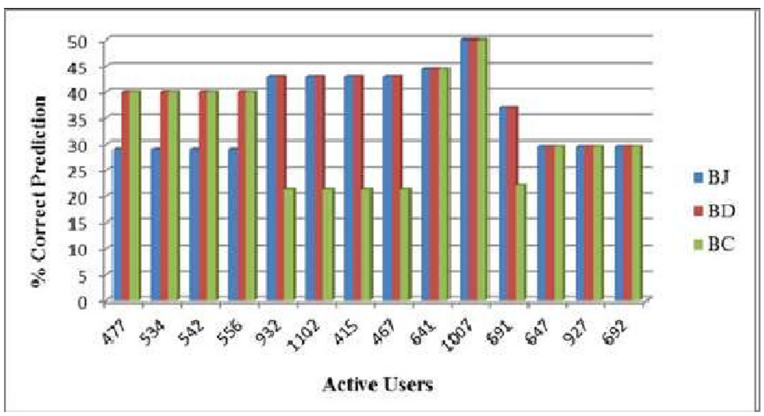


Figure 19: Average case scenario

120 given in Figure 20, the recommender system is unable to recommend a none of the user. As the number of VLWT users is very, very minimum the class becomes skewed and hence worst prediction performance.

To conclude, when the skewness of the data is normal, Jaccard similarity gives the excellent output (Best Case). Similarly, when the data has less skewness, Dice similarity measure performs better than Jaccard and Cosine (Average Case). All the three similarity measures perform poorly (Worst Case) when the skewness of the data is more.

5 Conclusion and Future Work

The proposed TBRS aimed to recommend top-k trustworthy users using a vector similarity measure. To model the user, the contents of the user profile are extracted and formed into a profile database. To find the similarity the fuzzy rules are converted into fuzzy vector space by assigning a fuzzy number for each linguistic term in the rule. To compute the similarity score, the proposed model uses the Jaccard, Dice and Cosine vector similarity measures with information gain as weight. This weighted similarity score is boosted by the trust level of the decision attribute. The performance of the proposed recommender system shows better results in terms of MAE, RMSE and AP when compared with preference based method and expected weight

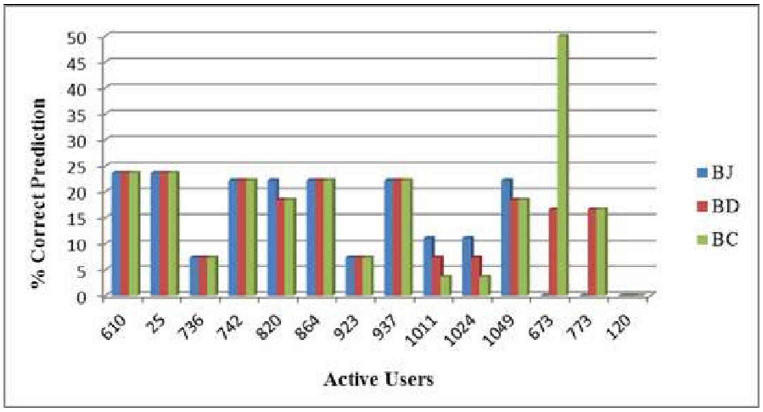


Figure 20: Worst case scenario

methods. Also, the proposed recommender system shows less MAE and less RMSE when are compared with other trust based recommender system. When the data is highly skewed the proposed system fails to give better results.

The limitation of the proposed TBRS is that it needs to be improved to handle the highly skewed data. For example, by applying the log transformation of the skewed data. Also, the TBRS can be extended to recommend a group of users than a single user. For example, recommending the top-5 or top-10 restaurant to the family members.

Acknowledgements: This Publication is an outcome of the R&D work undertaken in the project under the Visvesvaraya PhD Scheme (Unique Awardee Number: VISPHD-MEITY-2959) of Ministry of Electronics & Information Technology, Government of India, being implemented by Digital India Corporation (formerly Media Lab Asia).

Conflict of Interest: The authors declare no potential conflict of interest regarding the publication of this work. In addition, the ethical issues including plagiarism, informed consent, misconduct, data fabrication and, or falsification, double publication and, or submission, and redundancy have been completely witnessed by the authors.

References

[1] Abdul-Rahman A, Hailes S. Supporting trust in virtual communities. In: *Proceedings of the 33rd Hawaii International Conference on System Sciences. January 7, Maui, HI, USA*. 2000. p.1769-1777. DOI: 10.1109/HICSS.2000.926814

[2] Andersen R, Borgs C, Chayes J, Feige U, Flaxman A, Kalai A, Mirrokni V, Tennenholtz M. Trust-based recommendation systems: An axiomatic approach. In: *Proceedings of the 17th ACM International Conference on World Wide Web, April 21–25, Beijing, China.* 2008. p.199-208. DOI: https://doi.org/10.1145/1367497.1367525

[3] Arafeh M, Ceravolo P, Mourad A, Damiani E, Bellini E. Ontology based recommender system using social network data. *Future Generation Computer Systems.* 2021; 115: 769–779. DOI: https://doi.org/10.1016/j.future.2020.09.030

[4] Avesani P, Massa P, Tiella R. Moleskiing.it: A trust-aware recommender system for ski mountainteering. *International Journal for Infonomics*. 2005; 1-10. DOI: https://api.semanticscholar.org/CorpusID:10873049

[5] Barbier GB, Huan Liu. *Finding Provenance Data in Social Media*. Arizona State University; 2011.

[6] Bellaachia A, Alathel D. Improving the recommendation accuracy for cold start users in trust-based recommender systems. *International Journal of Computer and Communication Engineering*. 2016; 5(3): 206-214. DOI: 10.17706/ijcce.2016.5.3.206-214

[7] Channappagoudar NB, Singh R. Trust based recommendation system using knowledge graph (KGTRS). In: *ICIDSSD 2022: Proceedings of the 3rd International Conference on ICT for Digital, Smart, and Sustainable Development, ICIDSSD 2022, 24-25 March 2022, New Delhi, India*. 2023. p. 25-36. DOI: https://eudl.eu/pdf/10.4108/eai.24-3-2022.2318767

[8] Choi SS, Cha SH, Tappert CC. A survey of binary similarity and distance measures. *Journal of Systemics, Cybernetics and Informatics*. 2010; 8(1): 43-48. DOI: https://www.iiisci.org/Journal/pdv/sci/pdfs/GS315JG.pdf

[9] Duricic T, Lacic E, Kowald D, Lex E. Trust based collaborative filtering: Tackling the cold start problem using regular equivalence. In: *Proceedings of the 12th ACM Conference on Recommender Systems*. 2018. p.446-450. DOI: https://dl.acm.org/doi/10.1145/3240323.3240404

[10] Falcone R, Pezzulo G, Castelfranchi C. A fuzzy approach to a belief based trust computation. In: *Lecture Notes in Artificial Intelligence*. 2003. p.73-86. DOI: https://doi.org/10.1007/3-540-36609-1_7

[11] Faridani V, Jalali M, Jahan MV. Collaborative filtering-based recommender systems by effective trust. *International Journal of Data Science and Analytics*. 2017; 3(4): 297-307. DOI: https://link.springer.com/article/10.1007/s41060-017-0049-y

[12] George G, Lal AM. A personalized approach to course recommendation in higher education. *International Journal on Semantic Web and Information Systems*. 2021; 17(2): 100-114. DOI: 10.4018/IJSWIS.2021040106

[13] Golbeck JA. *Computing and Applying Trust in Web-Based Social Networks*. United States: University of Maryland at College Park; 2005.

[14] Hang CW, Singh MP. Trust based recommendation based on graph similarities. 2010. DOI: http://www.csc.ncsu.edu/faculty/mpsingh/papers/mas/aamas-trust-10-graph.pdf

[15] Janowicz K. Trust and provenance you can't have one without the other. *Muenster, Germany*. 2009. DOI: https://api.semanticscholar.org/CorpusID:13276942

[16] Jiang L, Cheng Y, Yang L, Li J, Yan H, Wang X. A trust-based collaborative filtering algorithm for E-commerce recommendation system. *Journal of Ambient Intelligence and Humanized Computing*. 2019; 10(8): 3023–3034. DOI: https://doi.org/10.1007/s12652-018-0928-7

[17] Li P, Li T, Wang X, Zhang S, Jiang Y, Tang Y. Scholar recommendation based on high-order propagation of knowledge graphs. *International Journal on Semantic Web and Information Systems*. 2022; 18(1): 1-19. DOI: https://doi.org/10.4018/IJSWIS.297146

[18] Liao M, Sundar SS, Walther JB. User trust in recommendation systems: A comparison of content-based, collaborative and demographic filtering. In: *Proceedings of the 2022 CHI Conference on Human Factors in Computing Systems (CHI '22), Association for Computing Machinery, April 29 - May 5, New York, USA*. 2022. p.1–14. DOI: https://doi.org/10.1145/3491102.3501936

[19] Linden G, Smith B, York J. Amazon.com recommendations: Item-to-item collaborative filtering. *Journal of IEEE Internet Computing*. 2003; 7(1): 76-80. DOI: 10.1109/MIC.2003.1167344

[20] Mandal S, Maiti A. Heterogeneous trust-based social recommendation via reliable and informative motif-based attention. In: 2022 *International Joint Conference on Neural Networks (IJCNN), Padua, Italy*. 2022 p.1-8. DOI: 10.1109/IJCNN55064.2022.9892977

[21] Massa P, Avesani P. Trust-aware recommender systems. In: *Proceedings of the first ACM Conference on Recommender Systems, October 19-20, Minneapolis MN USA*. 2007. p.17-24. DOI: https://doi.org/10.1145/1297231.1297235

[22] Olaru C, Wehenkel L. A complete fuzzy decision tree technique. *Journal of Fuzzy Sets and Systems*. 2003; 138(2): 221-254. DOI: https://doi.org/10.1016/S0165-0114(03)00089-7

[23] Parvin H, Moradi P, Esmaeili Sh. TCFACO: Trustaware collaborative filtering method based on ant colony optimization. *Journal of Expert Systems with Application*. 2019; 118: 152-168. DOI: https://doi.org/10.1016/j.eswa.2018.09.045

[24] Rad D, Cuc LD, Feher A, Joldes CSR, Batca-Dumitru GC, Sendroiu C, Almasi RC, Chis S, Popescu MG. The influence of social stratification on trust in recommender systems. *Electronics*. 2023; 12(10): 2160. DOI: 10.3390/electronics12102160

[25] Richa, Bedi P. Trust and distrust based cross-domain recommender system. *Applied Artificial Intelligence*. 2021; 35(4): 326-351. DOI: https://doi.org/10.1080/08839514.2021.1881297

[26] Salloum G, Tekli J. Automated and personalized meal plan generation and relevance scoring using a multi-factor adaptation of the transportation problem. *Soft Computing*. 2022; 26: 2561–2585. DOI: https://doi.org/10.1007/s00500-021-06400-1

[27] Sinha RR, Swearingen K. Comparing recommendations made by online systems and friends. In: *Proceedings of the DELOS-NSF Workshop on Personalisation and Recommender Systems in Digital Libraries*. 2001. DOI: https://api.semanticscholar.org/CorpusID:15526356

[28] Smith NJJ. *Vagueness and Degrees of Truth*. Oxford, 2008; online edn, Oxford Academic, 1 Jan. 2009. DOI: https://doi.org/10.1093/acprof:oso/9780199233007.001.0001

[29] Smyth B, O'Donovan J. Trust in recommender systems. In: *Proceedings of 10^{th} international conference on intelligent user interfaces, January 10-13, San Diego, California, USA*. 2005. p.167-174. DOI: https://doi.org/10.1145/1040830.1040870

[30] Sudha Ram, Jun Liu. *Book on Active conceptual modeling of learning*. Peter P. Chen, Leah Y. Wong (eds.).Lecture Notes in Computer Science. Springer-Verlag Berlin, Heidelberg; 2007; 17-29. DOI: https://link.springer.com/book/10.1007/978-3-540-77503-4

[31] Teekaraman D, Sendhilkumar S, Mahalakshmi GS. Semantic provenance based trustworthy user classification on book-based social network using Fuzzy decision tree. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*. 2020; 28(1): 47-77. DOI: https://doi.org/10.1142/S0218488520500038

[32] Victor P, Cornellis C , Cock MD, Teredesai AM. Trust and distrust based recommendations for controversial reviews. *IEEE Intelligent Systems*. 2011; 26: 48–55. DOI: 10.1109/MIS.2011.22

[33] Xiao J, Liu X, Zeng J, Cao Y, Feng Z. Recommendation of healthcare services based on an embedded user profile model, *International Journal on Semantic Web and Information Systems*. 2022; 18(1): 1-21. DOI: 10.4018/IJSWIS.313198

[34] Xue K, Wang J. Collaborative filtering recommendation algorithm for user interest and relationship based on score matrix. In: 2018 *International Conference on Mathematics, Modelling, Simulation and Algorithms*. 2018. p.217-221. DOI: 10.2991/mmsa-18.2018.49

[35] Yang B, Lei Y, Liu J, Li W. Social collaborative filtering by trust. *IEEE Transactions on Pattern Analysis and Machine Intelligence*. 2017; 39(8): 1633-1647. DOI: 10.1109/TPAMI.2016.2605085

[36] Ye L, Wu C, Li M. Collaborative filtering recommendation based on trust model with fused similar factor. In: *MATEC Web of Conferences*. 2017. p.1-7. DOI: https://doi.org/10.1051/matecconf/201713900010

Dhanalakshmi Teekaraman
Department of Computer Science and Engineering
Associate Professor
Jerusalem College of Engineering
Chennai, India
E-mail: dhanalakshmi.t@jerusalemengg.ac.in

Sendhilkumar Selvaraju
Department of Information Science and Technology
Professor
Anna University, CEG Campus
Chennai, India
E-mail: ssk_pdy@yahoo.ac.in

Mahalakshmi Guruvayur Suryanarayanan
Department of Computer Science and Engineering
Associate Professor
Anna University, CEG Campus
Chennai, India
E-mail: smaha@annauniv.edu

Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://tfss.journals.iau.ir/
Online ISSN: 2821-0131
Vol.2, No.2, (2023), 219-228
DOI: http://doi.org/10.30495/tfss.2023.1980759.1066

Article Type: Original Research Article

Similarity Measure: An Intuitionistic Fuzzy Rough Set Approach

Jaydip Bhattacharya *

Abstract. In fuzzy set theory, the concept of a non-membership function and the hesitation margin were not considered while these two concepts have been included along with the membership function for intuitionistic fuzzy sets. It is also to be noted that the intuitionistic fuzzy set is reflected as an extension of the fuzzy set accommodating both membership and non-membership functions together with a hesitation margin. In the intuitionistic fuzzy set theory, the sum of the membership function and the non-membership function is a value between 0 and 1. In recent times, intuitionistic fuzzy rough set theory has emerged as a powerful tool for dealing with imprecision and uncertain information in relational database theory. Measures of similarity between fuzzy rough sets as well as intuitionistic fuzzy rough sets provide wide applications in real-life problems and that is why many researchers paid more attention to this concept. Intuitionistic fuzzy rough set theory behaves like an excellent tool to tackle impreciseness or uncertainties. In this paper, we propose a new approach of similarity measure on an intuitionistic fuzzy rough set based on a set-theoretic approach. The proposed measure is able to give an exact result. In the application part, we consider a real-life problem for selecting a fair play award-winning team in a cricket tournament and describe the algorithm.

AMS Subject Classification 2020: 90C70; 03F55
Keywords and Phrases: Similarity measure, Intuitionistic fuzzy set, Rough set.

1 Introduction

The Rough set theory introduced by Pawlak [13, 14] is an excellent and elegant mathematical tool for the analysis of uncertainty, inconsistency and vague descriptions of objects. The basic idea of a rough set is based upon the approximation of sets by a pair of sets known as lower approximation and upper approximation. Here, the lower and upper approximation operators are based on equivalence relation. However, in many real-life problems, a rough set model cannot be applied due to the restrictions of the requirement of equivalence relation. For this reason, the rough set is generalized to fuzzy sets such as fuzzy rough set and rough fuzzy set [9].

In 1965, L.A.Zadeh [17] first introduced the concept of a Fuzzy set. Atanassov [1] generalized this concept into an intuitionistic fuzzy set in 1983. Since then many authors [3, 4] have been concentrating as well as developing the concepts like algebraic laws of IFSs, basic operations on IFSs, modal operators and normalization of IFSs etc. In fuzzy set theory it is taken into consideration that there exists a membership value for all the elements of the set and we do not consider non-membership values of the elements of the set. But in real-life problems, we feel the existence of hesitation. In fuzzy set theory, if $\mu(x)$ is the degree of membership of an element x, then the degree of non-membership of x is calculated by $1-\mu(x)$. But this concept is not always applicable to all real-life problems and that is why the notion of an intuitionistic fuzzy set is introduced. It may be

*Corresponding Author: Jaydip Bhattacharya, Email: jay73bhatta@gmail.com, ORCID: 0000-0003-3026-9441
Received: 2 February 2023; **Revised**: 3 October 2023; **Accepted**: 24 October 2023; **Available Online**: 24 October 2023; **Published Online**: 7 November 2023.

How to cite: Bhattacharya J. Similarity measure: An intuitionistic fuzzy rough set approach. *Trans. Fuzzy Sets Syst.* 2023; 2(2): 219-228. DOI: http://doi.org/10.30495/tfss.2023.1980759.1066

mentioned that the intuitionistic fuzzy set theory reduces to fuzzy set theory if the in-deterministic part is zero. Combining the fuzzy sets with the rough sets, Nanda and Majumdar [11] proposed the concept of fuzzy rough sets in 1992. Subsequently, Coker [7] pointed out fuzzy rough sets are the intuitionistic L-fuzzy sets. In this paper, we utilize the concept of the intuitionistic fuzzy rough set [16, 19] model to determine the similarity measure between two given intuitionistic fuzzy rough sets. Furthermore, using this concept we illustrate an example for selecting the procedure of fair play award in a cricket tournament.

2 Preliminaries

Definition 2.1. [2] *Let X be a nonempty set. An intuitionistic fuzzy set A in X is an object having the form $A=\{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$, where the functions $\mu_A, \nu_A : x \to [0,1]$ define respectively, the degree of membership and degree of non-membership of the element $x \in X$ to the set A, which is a subset of X, and for every element $x \in X$, it holds that $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.*
Furthermore, we have $\pi_A(x)= 1- \mu_A(x) - \nu_A(x)$ called the intuitionistic fuzzy set index or hesitation margin of x in A. $\pi_A(x)$ is the degree of indeterminacy of $x \in X$ to the IFS A and $\pi_A(x) \in [0,1]$ that is $\pi_A : X \to [0,1]$ and $0 \leq \pi_A(x) \leq 1$ for every $x \in X$. $\pi_A(x)$ expresses the lack of knowledge of whether x belongs to IFS A or not.

The definition of rough sets is based upon the approximation of a set by a pair of sets known as a lower and an upper approximation. Let U be the universe of a finite non-empty set of objects. Let $R \subseteq U \times U$ be an equivalence relation on U. The equivalence relation R partitions the set U into disjoint classes and it is denoted as U/R. Let X be a subset of U. Therefore the target set X can be described by a lower and an upper approximation as below, where $\underline{R}X$ and $\overline{R}X$ are $R-lower$ and $R-upper$ approximations of X respectively.
$\underline{R}X = \cup\{X' \in U/R : X' \subseteq X\}$ and $\overline{R}X = \cup\{X' \in U/R : X' \cap X \neq \emptyset\}$
Boundary region of the set X, $BN_R(X)$, is the objects in X that can be distinguished neither as a member nor as a non-member of x employing the relation R. It is denoted as $BN_R(X) = \underline{R}X - \overline{R}X$.
A set X is said to be definable if $\underline{R}X = \overline{R}X$ and the target set is a crisp set i.e., there is no boundary line objects. Similarly, it is said to be rough if $\underline{R}X \neq \overline{R}X$ or equivalently $BN_R(X) \neq \emptyset$.

Definition 2.2. [4] Let U be a non-empty and finite universe of discourse and IFR be an intuitionistic fuzzy relation defined on $U \times U$. The pair (U, IFR) is called an intuitionistic fuzzy rough approximation space. For any $A \in IF(U)$, where IF(U) denotes the intuitionistic fuzzy power set of U, the lower and upper approximations of A with respect to (U, IFR) denoted by $IF\underline{R}(A)$ and $IF\overline{R}(A)$ are defined as follows:
$IF\underline{R}(A) = \{\langle x, \mu_{IF\underline{R}(A)}(x), \nu_{IF\underline{R}(A)}(x)\rangle : x \in U\}$
$IF\overline{R}(A) = \{\langle x, \mu_{IF\overline{R}(A)}(x), \nu_{IF\overline{R}(A)}(x)\rangle : x \in U\}$
Where
$\mu_{IF\underline{R}(A)}(x) = \wedge_{y \in U}[\nu_{IFR}(x,y) \vee \mu_A(y)]$
$\nu_{IF\underline{R}(A)}(x) = \vee_{y \in U}[\mu_{IFR}(x,y) \wedge \nu_A(y)]$
$\mu_{IF\overline{R}(A)}(x) = \vee_{y \in U}[\mu_{IFR}(x,y) \wedge \mu_A(y)]$
$\nu_{IF\overline{R}(A)}(x) = \wedge_{y \in U}[\nu_{IFR}(x,y) \vee \nu_A(y)]$
The pair $(IF\underline{R}(A), IF\overline{R}(A))$ is called the intuitionistic fuzzy rough set associated with A denoted by IFR(A). Then, an IF rough set $A \in IF(U)$ could be denoted by $A = \{\langle x, \mu_{\underline{A}}(x), \mu_{\overline{A}}(x), \nu_{\underline{A}}(x), \nu_{\overline{A}}(x)\rangle : \forall x \in U\}$.

Definition 2.3. [3] Let U be a non-empty and finite universe of discourse and $A, B \in IF(U)$.
Then (i) The complement of $A = \langle \mu_{\underline{A}}(x), \mu_{\overline{A}}(x), \nu_{\underline{A}}(x), \nu_{\overline{A}}(x)\rangle$ is defined as $A^c = \langle \nu_{\underline{A}}(x), \nu_{\overline{A}}(x), \mu_{\underline{A}}(x), \mu_{\overline{A}}(x)\rangle$, for any $x \in U$.
(ii) $A \subseteq B$ if for any $x \in U$, $\mu_{\underline{A}}(x) \leq \mu_{\underline{B}}(x), \mu_{\overline{A}}(x) \leq \mu_{\overline{B}}(x)$ and $\nu_{\underline{A}}(x) \geq \nu_{\underline{B}}(x), \nu_{\overline{A}}(x) \geq \nu_{\overline{B}}(x)$.

Definition 2.4. [10, 15] Let U be a non-empty and finite universe of discourse and $A \in IFR(U)$.
Then $M : A \times A \to [0,1]$ is called the similarity measure on A and $M(x,y)$ is called the similarity degree between the intuitionistic fuzzy rough values $x = (\mu_{\underline{A}}(x), \mu_{\overline{A}}(x), \nu_{\underline{A}}(x), \nu_{\overline{A}}(x), \pi_{\underline{A}}(x), \pi_{\overline{A}}(x))$,
$y = (\mu_{\underline{A}}(y), \mu_{\overline{A}}(y), \nu_{\underline{A}}(y), \nu_{\overline{A}}, \pi_{\underline{A}}(y), \pi_{\overline{A}}(y))$, if M satisfies the following conditions:

1. $0 \leq M(x,y) \leq 1$.
2. $M(x,y) = M(y,x)$.
3. $\forall x \in A, M(x,y) = M(x,z) \Rightarrow M(y,z) = 1$.
4. $M(x,y) = M(x^c, y^c)$, where x^c and y^c are complements of x and y respectively.
5. If $x \leq y \leq z$, then $M(x,z) \leq min\{M(x,y), M(y,z)\}$, $\forall x, y, z \in A$.

3 Similarity Measures

Many researchers [[6], [8], [10], [19], [18], [20]] have paid their concentration to develop the concept of similarity measure between fuzzy sets, intuitionistic fuzzy sets and intuitionistic fuzzy rough sets. On the basis of the set-theoretic approach, Pappis and Karacapilidis [12] defined the similarity measure between fuzzy sets A and B with fuzzy values $a_i \in A$ and $b_i \in B$ as follows.

$$M_p(A,B) = \frac{(|A \cap B|)}{(|A \cup B|)} = \frac{\sum_{i=1}^{n}(a_i \wedge b_i)}{\sum_{i=1}^{n}(a_i \vee b_i)} \tag{1}$$

In [5] Chen defined a similarity measure between two IF sets with IF values x and y (from the set-theoretic point of view) as follows:

$$M_C(x,y) = \frac{(min(\mu(x),\mu(y)) + min(\nu(x),\nu(y)) + min(\pi(x),\pi(y)))}{(max(\mu(x),\mu(y)) + max(\nu(x),\nu(y)) + max(\pi(x),\pi(y)))} \tag{2}$$

Atanassov [3] also gives a similarity measure between fuzzy rough values as follows: Let A be a fuzzy rough set in X, $x = \langle \mu_{\underline{A}}(x), \mu_{\overline{A}}(x) \rangle, y = \langle \mu_{\underline{A}}(y), \mu_{\overline{A}}(y) \rangle$be the fuzzy rough values in A. The degree of similarity between the fuzzy rough values x and y can be evaluated by the function $M_Z(x,y)$.

$$M_Z(x,y) = 1 - \frac{1}{2}(\left|\mu_{\underline{A}}(x) - \mu_{\underline{A}}(y)\right| + \left|\mu_{\overline{A}}(x) - \mu_{\overline{A}}(y)\right|) \tag{3}$$

Gangwal et.al. [10] also mentioned a similarity measure between IF rough values based on a set-theoretic approach as mentioned below.
Let A be an IF rough set in X, $x = \langle \mu_{\underline{A}}(x), \mu_{\overline{A}}(x), \nu_{\underline{A}}(x), \nu_{\overline{A}}(x) \rangle, y = \langle \mu_{\underline{A}}(y), \mu_{\overline{A}}(y), \nu_{\underline{A}}(y), \nu_{\overline{A}}(y) \rangle$ be two IF rough values in A. The degree of similarity between the IF rough values x and y can be defined by the function $M(x,y)$ as follows:

$$M(x,y) = \frac{(\left|\mu_{\underline{A}}(x) \wedge \mu_{\underline{A}}(y) + \mu_{\overline{A}}(x) \wedge \mu_{\overline{A}}(y) + \nu_{\underline{A}}(x) \wedge \nu_{\underline{A}}(y) + \nu_{\overline{A}}(x) \wedge \nu_{\overline{A}}(y)\right|)}{(\left|\mu_{\underline{A}}(x) \vee \mu_{\underline{A}}(y) + \mu_{\overline{A}}(x) \vee \mu_{\overline{A}}(y) + \nu_{\underline{A}}(x) \vee \nu_{\underline{A}}(y) + \nu_{\overline{A}}(x) \vee \nu_{\overline{A}}(y)\right|)} \tag{4}$$

In the above definition, the IF rough values of 4-tuples are used. Instead of IF rough values of 4-tuples, we consider IF rough values of 6-tuples in the rest of the paper.

Definition 3.1. Let A be an IF rough set in X,

$$x = \langle \mu_{\underline{A}}(x), \mu_{\overline{A}}(x), \nu_{\underline{A}}(x), \nu_{\overline{A}}(x), \pi_{\underline{A}}(x), \pi_{\overline{A}}(x) \rangle,$$

$$y = \langle \mu_{\underline{A}}(y), \mu_{\overline{A}}(y), \nu_{\underline{A}}(y), \nu_{\overline{A}}(y), \pi_{\underline{A}}(y), \pi_{\overline{A}}(y) \rangle$$

be two IF rough values in A. The degree of similarity between the IF rough values x and y can be defined by the function $M_J(x, y)$ as follows:

$$M_J(x,y) = \frac{(\mu_{\underline{A}}(x) \wedge \mu_{\underline{A}}(y) + \mu_{\overline{A}}(x) \wedge \mu_{\overline{A}}(y) + \nu_{\underline{A}}(x) \wedge \nu_{\underline{A}}(y) + \nu_{\overline{A}}(x) \wedge \nu_{\overline{A}}(y) + \pi_{\underline{A}}(x) \wedge \pi_{\underline{A}}(y) + \pi_{\overline{A}}(x) \wedge \pi_{\overline{A}}(y))}{(\mu_{\underline{A}}(x) \vee \mu_{\underline{A}}(y) + \mu_{\overline{A}}(x) \vee \mu_{\overline{A}}(y) + \nu_{\underline{A}}(x) \vee \nu_{\underline{A}}(y) + \nu_{\overline{A}}(x) \vee \nu_{\overline{A}}(y) + \pi_{\underline{A}}(x) \vee \pi_{\underline{A}}(y) + \pi_{\overline{A}}(x) \vee \pi_{\overline{A}}(y))} \tag{5}$$

The larger the value of $M_J(x, y)$, the more the similarity between the IF rough values x and y.

Example 3.2. Let x and y be two IF rough values, where $x = \langle 0.6, 0.5, 0.3, 0.4, 0.1, 0.1 \rangle$ and $y = \langle 0.7, 0.65, 0.25, 0.3, 0.05, 0.05 \rangle$. Then the degree of similarity between x and y can be evaluated as

$$M_J(x,y) = \frac{min(0.6,0.7)+min(0.5,0.65)+min(0.3,0.25)+min(0.4,0.3)+min(0.1,0.05)+min(0.1,0.05)}{max(0.6,0.7)+max(0.5,0.65)+max(0.3,0.25)+max(0.4,0.3)+max(0.1,0.05)+max(0.1,0.05)}$$

$$= \frac{0.6+0.5+0.25+0.3+0.05+0.05}{0.7+0.65+0.3+0.4+0.1+0.1} = \frac{1.75}{2.25} \approx 0.7778.$$

Example 3.3. Let x and y be two IF rough values, where $x = \langle 0.6, 0.5, 0.3, 0.4, 0.1, 0.1 \rangle$ and $y = \langle 0.7, 0.65, 0.25, 0.3, 0.05, 0.05 \rangle$.
Then the complementary of x and y can be given by $x^c = \langle 0.3, 0.4, 0.6, 0.5, 0.1, 0.1 \rangle$ and $y^c = \langle 0.25, 0.3, 0.7, 0.65, 0.05, 0.05 \rangle$.
Hence the degree of similarity between x^c and y^c can be evaluated as

$$M_J(x^c,y^c) = \frac{min(0.3,0.25)+min(0.4,0.3)+min(0.6,0.7)+min(0.5,0.65)+min(0.1,0.05)+min(0.1,0.05)}{max(0.3,0.25)+max(0.4,0.3)+max(0.6,0.7)+max(0.5,0.65)+max(0.1,0.05)+max(0.1,0.05)}$$

$$= \frac{0.25+0.3+0.6+0.5+0.05+0.05}{0.3+0.4+0.7+0.65+0.1+0.1} = \frac{1.75}{2.25} \approx 0.7778.$$

From examples 3.2 and 3.3, it is observed that $M_J(x, y) = M_J(x^c, y^c)$.

Example 3.4. Let x and y be two IF rough values, where $x = \langle 0.6, 0.5, 0.3, 0.4, 0.1, 0.1 \rangle$ and $y = \langle 0, 0, 0, 0, 0, 0 \rangle$.

Hence the degree of similarity between x and y can be evaluated as

$$M_J(x,y) = \frac{min(0.6,0)+min(0.5,0)+min(0.3,0)+min(0.4,0)+min(0.1,0)+min(0.1,0)}{max(0.6,0)+max(0.5,0)+max(0.3,0)+max(0.4,0)+max(0.1,0)+max(0.1,0)} = 0.$$

Example 3.5. Let x and y be two IF rough values, where $x = y = \langle 0.6, 0.5, 0.3, 0.4, 0.1, 0.1 \rangle$

Hence the degree of similarity between x and y can be evaluated as

$$M_J(x,y) = \frac{min(0.6,0.6)+min(0.5,0.5)+min(0.3,0.3)+min(0.4,0.4)+min(0.1,0.1)+min(0.1,0.1)}{max(0.6,0.6)+max(0.5,0.5)+max(0.3,0.3)+max(0.4,0.4)+max(0.1,0.1)+max(0.1,0.1)} = 1.$$

Example 3.6. Let x and y be two IF rough values, where $x = \langle 0.6, 0.5, 0.3, 0.4, 0.1, 0.1 \rangle$ and $y = \langle 1, 1, 0, 0, 0, 0 \rangle$.

Hence the degree of similarity between x and y can be evaluated as

$$M_J(x,y) = \frac{min(0.6,1)+min(0.5,1)+min(0.3,0)+min(0.4,0)+min(0.1,0)+min(0.1,0)}{max(0.6,1)+max(0.5,1)+max(0.3,0)+max(0.4,0)+max(0.1,0)+max(0.1,0)} \approx 0.3793.$$

Theorem 3.7. *Let A be an IF rough set in X where x, y, z be the IF rough values in A. Then the following statements are true:*

1. *$M_J(x,y)$ is bounded i.e.; $0 \leq M_J(x,y) \leq 1$.*

2. *$M_J(x,y) = M_J(y,x)$.*

3. *$\forall x \in X, M_J(x,y) = M_J(x,z) \Rightarrow M_J(y,z) = 1$.*

4. *$M_J(x,y) = M_J(x^c, y^c)$.*

5. *If $x \leq y \leq z$, then $M_J(x,z) \leq min\{M_J(x,y), M_J(y,z)\}$ for $x, y, z \in X$.*

Proof. Let A be an IF rough set in X where $x = \langle \mu_{\underline{A}}(x), \mu_{\overline{A}}(x), \nu_{\underline{A}}(x), \nu_{\overline{A}}(x), \pi_{\underline{A}}(x), \pi_{\overline{A}}(x) \rangle$, $y = \langle \mu_{\underline{A}}(y), \mu_{\overline{A}}(y), \nu_{\underline{A}}(y), \nu_{\overline{A}}(y), \pi_{\underline{A}}(y), \pi_{\overline{A}}(y) \rangle$ and $z = \langle \mu_{\underline{A}}(z), \mu_{\overline{A}}(z), \nu_{\underline{A}}(z), \nu_{\overline{A}}(z), \pi_{\underline{A}}(z), \pi_{\overline{A}}(z) \rangle$ be the IF rough values in A. We may define the order relation of the intuitionistic fuzzy rough values as
$x \leq y \Longleftrightarrow (\mu_{\underline{A}}(x) \leq \mu_{\underline{A}}(y),\ \mu_{\overline{A}}(x) \leq \mu_{\overline{A}}(y))$ and $(\nu_{\underline{A}}(x) \geq \nu_{\underline{A}}(y),\ \nu_{\overline{A}}(x) \geq \nu_{\overline{A}}(y))$.

1. The minimum value of (5) is 0 and the maximum is 1. In other cases, the value of the expression (5) must be positive and lesser than one as the value of the numerator is less than the value of denominator. Thus $0 \leq M_J(x,y) \leq 1$.

2. M_J is symmetric as min and max operations are both symmetric.

3. Since $M_J(x,y) = M_J(x,z)$, $\forall x \in X$ then for $x = y$ we get that $1 = M_J(y,y) = M_J(y,z)$.
   Similarly, for $x = z$, we get $1 = M_J(z,z) = M_J(z,y) = M_J(y,z)$.

4. In this case, $x^c = \langle \nu_{\underline{A}}(x), \nu_{\overline{A}}(x), \mu_{\underline{A}}(x), \mu_{\overline{A}}(x), \pi_{\underline{A}}(x), \pi_{\overline{A}}(x) \rangle$
   and $y^c = \langle \nu_{\underline{A}}(y), \nu_{\overline{A}}(y), \mu_{\underline{A}}(y), \mu_{\overline{A}}(y), \pi_{\underline{A}}(y), \pi_{\overline{A}}(y) \rangle$ and hence point 4 holds.

5. Given, $x \leq y \leq z$.
   Substituting $\pi_{\underline{A}}(x), \pi_{\overline{A}}(x), \pi_{\underline{A}}(z), \pi_{\overline{A}}(z)$ we get $M_J(x,z) = \frac{1-\underline{U}_{xz}+1-\overline{U}_{xz}}{1+\underline{U}_{xz}+1+\overline{U}_{xz}}$,
   where $\underline{U}_{xz} = (\mu_{\underline{A}}(z) - \mu_{\underline{A}}(x)) \vee (\nu_{\underline{A}}(x) - \nu_{\underline{A}}(z))$ and $\overline{U}_{xz} = (\mu_{\overline{A}}(z) - \mu_{\overline{A}}(x)) \vee (\nu_{\overline{A}}(x) - \nu_{\overline{A}}(z))$.
   Similarly, $M_J(x,y) = \frac{1-\underline{U}_{xy}+1-\overline{U}_{xy}}{1+\underline{U}_{xy}+1+\overline{U}_{xy}}$, where $\underline{U}_{xy} = (\mu_{\underline{A}}(y) - \mu_{\underline{A}}(x)) \vee (\nu_{\underline{A}}(x) - \nu_{\underline{A}}(y))$ and $\overline{U}_{xy} = (\mu_{\overline{A}}(y) - \mu_{\overline{A}}(x)) \vee (\nu_{\overline{A}}(x) - \nu_{\overline{A}}(y))$.
   Clearly, $(1 - \underline{U}_{xz} + 1 - \overline{U}_{xz}) \leq (1 - \underline{U}_{xy} + 1 - \overline{U}_{xy})$ as $\underline{U}_{xz} \geq \underline{U}_{xy}$ and $\overline{U}_{xz} \geq \overline{U}_{xy}$
   And $(1 + \underline{U}_{xz} + 1 + \overline{U}_{xz}) \geq (1 + \underline{U}_{xy} + 1 + \overline{U}_{xy})$ as $\underline{U}_{xz} \geq \underline{U}_{xy}$ and $\overline{U}_{xz} \geq \overline{U}_{xy}$
   Hence $M_J(x,z) \leq M_J(x,y)$.
   Similarly, it can be shown that $M_J(x,z) \leq M_J(y,z)$.

□

Now, the similarity measure between two given IF rough sets is generalized. Let A and B be two IF rough sets in the universe of discourse $U = \{u_1, u_2, u_3, , u_n\}$, where
$A = \langle \mu_{\underline{A}}(u_1), \mu_{\overline{A}}(u_1), \nu_{\underline{A}}(u_1), \nu_{\overline{A}}(u_1), \pi_{\underline{A}}(u_1), \pi_{\overline{A}}(u_1)\rangle/u_1 +$
$+ \langle \mu_{\underline{A}}(u_n), \mu_{\overline{A}}(u_n), \nu_{\underline{A}}(u_n), \nu_{\overline{A}}(u_n), \pi_{\underline{A}}(u_n), \pi_{\overline{A}}(u_n)\rangle/u_n$
and $B = \langle \mu_{\underline{B}}(u_1), \mu_{\overline{B}}(u_1), \nu_{\underline{B}}(u_1), \nu_{\overline{B}}(u_1), \pi_{\underline{B}}(u_1), \pi_{\overline{B}}(u_1)\rangle/u_1 +$
$+ \langle \mu_{\underline{B}}(u_n), \mu_{\overline{B}}(u_n), \nu_{\underline{B}}(u_n), \nu_{\overline{B}}(u_n), \pi_{\underline{B}}(u_n), \pi_{\overline{B}}(u_n)\rangle/u_n$

Then based on definition 3.1, the degree of similarity between the IF rough sets A and B can be defined as follows:

$$T_J(A,B) = \frac{1}{n}\sum_{i=1}^{n} M_J(\langle \mu_{\underline{A}}(u_i), \mu_{\overline{A}}(u_i), \nu_{\underline{A}}(u_i), \nu_{\overline{A}}(u_i), \pi_{\underline{A}}(u_i), \pi_{\overline{A}}(u_i)\rangle, \langle \mu_{\underline{B}}(u_i), \mu_{\overline{B}}(u_i), \nu_{\underline{B}}(u_i), \nu_{\overline{B}}(u_i), \pi_{\underline{B}}(u_i), \pi_{\overline{B}}(u_i)\rangle)$$

So, $T_J(A,B) =$

$$\frac{1}{n}\sum_{i=1}^{n} \frac{(min(\mu_{\underline{A}}(u_i),\mu_{\underline{B}}(u_i))+min(\nu_{\underline{A}}(u_i),\nu_{\underline{B}}(u_i))+min(\pi_{\underline{A}}(u_i),\pi_{\underline{B}}(u_i))+min(\mu_{\overline{A}}(u_i),\mu_{\overline{B}}(u_i))+min(\nu_{\overline{A}}(u_i),\nu_{\overline{B}}(u_i))+min(\pi_{\overline{A}}(u_i),\pi_{\overline{B}}(u_i))}{(max(\mu_{\underline{A}}(u_i),\mu_{\underline{B}}(u_i))+max(\nu_{\underline{A}}(u_i),\nu_{\underline{B}}(u_i))+max(\pi_{\underline{A}}(u_i),\pi_{\underline{B}}(u_i))+max(\mu_{\overline{A}}(u_i),\mu_{\overline{B}}(u_i))+max(\nu_{\overline{A}}(u_i),\nu_{\overline{B}}(u_i))+max(\pi_{\overline{A}}(u_i),\pi_{\overline{B}}(u_i))}$$

Here $T_J(A,B) \in [0,1]$. The larger the value of $T_J(A,B)$, the more similarity between the IF rough sets A and B.

Theorem 3.8. *Let X be the set of all IF rough sets on the fixed finite universe of discourse U and A, B, C ∈ X. Then the following statements are true:*

1. T_J *is bounded, i.e.,* $0 \leq T_J(A,B) \leq 1$.

2. $T_J(A,B) = T_J(B,A)$.

3. $\forall A \in X,\ T_J(A,B) = T_J(A,C) \Rightarrow T_J(B,C) = 1$.

4. $T_J(A,B) = T_J(A^c, B^c)$.

5. *If* $A \subseteq B \subseteq C$, *then* $T_J(A,C) \leq min\{T_J(A,B), T_J(B,C)\}$ *for* $A, B, C \in X$.

Proof. Similar to the Theorem 3.7. □

4 Application

In this section, we are considering the selection procedure for a Fair Play award in a cricket tournament. The fair play award is to make sure that the teams show the best behavior and sporting spirit while also being competitive. The award motivates the teams to play the game fairly.
The main factors on which the fair play award for a team depends are described below.

- Teams that uphold the spirit of the game$\rightarrow e_1$
- Teams that respect the opposition team$\rightarrow e_2$
- Teams that show respect towards the laws and rules of cricket$\rightarrow e_3$
- Teams that respect the umpires and officials$\rightarrow e_4$

4.1 Algorithm

The steps of the algorithm of this method are as follows:
First step: Construct an intuitionistic fuzzy rough set for a standard alternative.
Second step: Construct an intuitionistic fuzzy rough set for the available alternatives.
Third step: Calculate the similarity measure.
Fourth step: Arrange alternatives in order to their ranking.
Fifth step: Choose the best alternative.

4.2 Computation

Let U be the universal set where $U = \{TeamA, TeamB, TeamC, TeamD, TeamE, TeamF\}$ and $S = \{e_1, e_2, e_3, e_4\}$ be the parameters.
Also let S be the standard alternative and A, B, C, D, E, and F are the available alternatives.
S $=\langle 1, 1, 0, 0, 0, 0\rangle/e_1+\langle 0.9, 0.9, 0.05, 0.05, 0.05, 0.05\rangle/e_2+\langle 1, 0.9, 0, 0.1, 0, 0\rangle/e_3+\langle 0.95, 0.95, 0.05, 0.05, 0, 0\rangle/e_4$
A $= \langle 0.6, 0.5, 0.2, 0.3, 0.2, 0.2\rangle/e_1 + \langle 0.45, 0.5, 0.25, 0.4, 0.3, 0.1\rangle/e_2 + \langle 0.5, 0.6, 0.4, 0.4, 0.1, 0\rangle/e_3$
$+ \langle 0.75, 0.6, 0.2, 0.2, 0.05, 0.2\rangle/e_4$
B $=\langle 0.8, 0.7, 0.2, 0.2, 0, 0.1\rangle/e_1 + \langle 0.7, 0.5, 0.2, 0.3, 0.1, 0.2\rangle/e_2 + \langle 0.55, 0.65, 0.35, 0.25, 0.1, 0.1\rangle/e_3$
$+ \langle 0.6, 0.4, 0.3, 0.5, 0.1, 0.1\rangle/e_4$
C $= \langle 0.8, 0.9, 0.1, 0.1, 0.1, 0\rangle/e_1 + \langle 0.75, 0.85, 0.15, 0.1, 0.1, 0.05\rangle/e_2 + \langle 0.85, 0.9, 0.1, 0.05, 0.05, 0.05\rangle/e_3$
$+ \langle 0.5, 0.6, 0.3, 0.3, 0.2, 0.1\rangle/e_4$
D $=\langle 0.85, 0.6, 0.1, 0.2, 0.05, 0.2\rangle/e_1 + \langle 0.7, 0.65, 0.2, 0.3, 0.1, 0.05\rangle/e_2 + \langle 0.5, 0.5, 0.4, 0.4, 0.1, 0.1\rangle/e_3$
$+ \langle 0.6, 0.6, 0.3, 0.3, 0.1, 0.1\rangle/e_4$
E $= \langle 0.6, 0.8, 0.2, 0.2, 0.2, 0\rangle/e_1 + \langle 0.65, 0.75, 0.25, 0.2, 0.1, 0.05\rangle/e_2 + \langle 0.7, 0.7, 0.3, 0.3, 0, 0\rangle/e_3$
$+ \langle 0.8, 0.9, 0.1, 0.1, 0.1, 0\rangle/e_4$
F $=\langle 0.9, 0.8, 0.05, 0.15, 0.05, 0.05\rangle/e_1 + \langle 0.7, 0.5, 0.2, 0.3, 0.1, 0.2\rangle/e_2 + \langle 0.8, 0.6, 0.1, 0.3, 0.1, 0.1\rangle/e_3$
$+ \langle 0.7, 0.6, 0.25, 0.35, 0.05, 0.05\rangle/e_4$

Now using the formula of $T_J(A, B)$, we can evaluate

$$T_J(S, A) = \frac{1}{4}[\frac{min(1,0.6)+min(1,0.5)+min(0,0.2)+min(0,0.3)+min(0,0.2)+min(0,0.2)}{max(1,0.6)+max(1,0.5)+max(0,0.2)+max(0,0.3)+max(0,0.2)+max(0,0.2)} +$$

$$\frac{min(0.9,0.45)+min(0.9,0.5)+min(0.05,0.25)+min(0.05,0.4)+min(0.05,0.3)+min(0.05,0.1)}{max(0.9,0.45)+max(0.9,0.5)+max(0.05,0.25)+max(0.05,0.4)+max(0.05,0.3)+max(0.05,0.1)} +$$

$$\frac{min(1,0.5)+min(0.9,0.6)+min(0,0.4)+min(0.1,0.4)+min(0,0.1)+min(0,0)}{max(1,0.5)+max(0.9,0.6)+max(0,0.4)+max(0.1,0.4)+max(0,0.1)+max(0,0)} +$$

$$\frac{min(0.95,0.75)+min(0.95,0.6)+min(0.05,0.2)+min(0.05,0.2)+min(0,0.05)+min(0,0.2)}{max(0.95,0.75)+max(0.95,0.6)+max(0.05,0.2)+max(0.05,0.2)+max(0,0.05)+max(0,0.2)}]$$

$= 0.4450.$

Thus we get,

Similarity Measure	Value
$T_J(S,A)$	0.4450
$T_J(S,B)$	0.4998
$T_J(S,C)$	0.7010
$T_J(S,D)$	0.5155
$T_J(S,E)$	0.6558
$T_J(S,F)$	0.6040

This indicates that team C will receive the fair play trophy.

5 Conclusion

In this paper, we describe an intuitionistic fuzzy rough set model or approach to find the similarity measure between intuitionistic fuzzy rough sets. The main feature of this model is that we have considered and calculated the hesitation margin. We also establish some rules for measuring the degree of similarity between elements and between intuitionistic fuzzy rough sets. Based on this concept we solve a problem related to fair play winner in a cricket tournament. Many such problems also can be solved by applying this method. As the proposed similarity measures have some good properties, they can provide a useful way for measuring the similarity between intuitionistic fuzzy rough sets.

Acknowledgements: The author is most grateful to the referees for carefully reading the article and for their constructive comments and suggestions, which have helped to develop the paper.

Conflict of Interest: The author declares that there is no conflict of interest regarding the publication of this paper.

References

[1] Atanassov KT. (1983). Intuitionistic fuzzy sets. *VII ITKRs Session, Sofia*, 20-23 June 1983 (Deposed in Centr. Sci.-Techn. Library of the Bulg. Acad. of Sci., 1697/84) (in Bulgarian). Reprinted: International Journal of Bioautomation, 2016, 20(S1), S1-S6.

[2] Atanassov KT. Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*. 1986; 20(1): 87-96. DOI: https://doi.org/10.1016/S0165-0114(86)80034-3

[3] Atanassov KT. Intuitionistic fuzzy sets. In: Theory and Applications, *Physica-Verlag, Heidelberg*; 1999. p.1-137. DOI: https://doi.org/10.1007/978-3-7908-1870-3-1

[4] Atanasova L. On intuitionistic fuzzy versions of L. Zadeh's extension principle. *Notes on Intuitionistic Fuzzy Sets*. 2007; 13(3): 33-36. http://ifigenia.org/wiki/issue:nifs/13/3/33-36

[5] Chen SM, Yeh MS, Hsiao PY. A comparison of similarity measures of fuzzy values. *Fuzzy Sets and Systems*. 1995; 72(1): 79-89. DOI: https://doi.org/10.1016/0165-0114(94)00284-E

[6] Chengyi Z, Pingan D, Haiyan F. On measures of similarity between fuzzy rough sets. *International Journal of Pure and Applied Mathematics*. 2004; 10(4), 451-458.

[7] Coker D. Fuzzy rough sets are intuitionistic L-fuzzy sets. *Fuzzy Sets and System*. 1998; 96(3): 381-383. DOI: https://doi.org/10.1016/S0165-0114(97)00249-2

[8] Das TK, Acharjya DP, Patra MR. Multi-criteria decision making using intuitionistic Fuzzy rough set on two universal sets. *International Journal of Intelligent Systems and Applications*. 2015; 7(4): 26-33. DOI: https://doi.org/10.5815/ijisa.2015.04.04

[9] Dubois D, Prade H. Rough fuzzy sets and fuzzy rough sets. *International Journal of General System*. 1990; 17(2-3): 191-209. DOI: https://doi.org/10.1080/03081079008935107

[10] Gangwal C, Bhaumik RN. Functional dependences in intuitionistic fuzzy rough relational databases, In: S.Bhattacharya Halder (ed.), *Rough sets, Fuzzy sets and Soft Computing*. 2015; 111-120.

[11] Nanda S, Majumdar S. Fuzzy rough sets. *Fuzzy sets and System*. 1992; 45(2): 157-160. DOI: https://doi.org/10.1016/0165-0114(92)90114-J

[12] Pappis CP, Karacapilidis NI. A comparative assessment of measures of similarity of fuzzy values. *Fuzzy Sets and Systems*. 1993; 56(2): 171-174. DOI: https://doi.org/10.1016/0165-0114(93)90141-4

[13] Pawlak Z. Rough sets. *International Journal of Computer and Information Sciences*. 1982; 11: 341-356. DOI: https://doi.org/10.1007/BF01001956

[14] Pawlak Z. *Rough Sets: Theoretical Aspects of Reasoning about Data*. Netherlands: Kluwer Academic Publishers, Dordrecht; 1991. DOI: https://doi.org/10.1007/978-94-011-3534-4

[15] Sharma O, Rani A, Gupta P. Some similarity and distance measures on fuzzy rough sets and it's applications. *International Journal of Engineering, Science and Mathematics*. 2017; 6(5): 85-105.

[16] Yun SM, Lee SJ. New approach to intuitionistic fuzzy rough sets. *International Journal of Fuzzy Logic and Intelligent Systems*. 2020; 20(2): 129-137. DOI: https://doi.org/10.5391/IJFIS.2020.20.2.129

[17] Zadeh LA. Fuzzy sets. *Information and Control*. 1965; 8(3): 338-353. DOI: https://doi.org/10.1016/S0019-9958(65)90241-X

[18] Zhou L, Wu WZ. On generalized intuitionistic fuzzy rough approximation operators. *Information Sciences*. 2008; 178(11): 2448-2465. DOI: https://doi.org/10.1080/03081070802187723

[19] Zhou L, Wu WZ, Zhang WX. On intuitionistic fuzzy rough sets and their topological structures. *International Journal General Systems*. 2009; 38 (6): 589-616.

[20] Zwick R, Carlstein E, Budescu DV. Measures of similarity among fuzzy concepts: A comparative analysis. *International Journal of Approximate Reasoning*. 1987; 1(2): 221-242. DOI: https://doi.org/10.1016/0888-613X(87)90015-6

Jaydip Bhattacharya
Department of Mathematics
Bir Bikram Memorial College
Agartala, Tripura, India
E-mail: jay73bhatta@gmail.com