TFSS

Transactions on Fuzzy Sets and Systems

Vol. 1 No. 2 (2022)

Transactions on
Fuzzy Sets & Systems

Transactions on Fuzzy Sets and Systems

Transactions on Fuzzy Sets and Systems

About Journal

Transactions on Fuzzy Sets and Systems (TFSS) is an open access international scholarly journal. TFSS publishes new applied and pure articles related to fuzzy sets and systems as the two-quarterly journal and there is no charge for publishing an article in TFSS. All articles will be peer-review before publication. Manuscripts submitted to TFSS must be original and unpublished and not currently being considered for publication elsewhere.

The articles will be deposited immediately into the online repository, after the completion of the review processes. TFSS aims linking the ideas and techniques of fuzzy sets and systems with other disciplines to provide an international forum for refereed original research works in the theory and applications in all fields related to fuzzy science. Transactions on Fuzzy Sets and Systems (TFSS) is a two-quarterly international academic journal founded in 2022.

TFSS is an open access and free of charge and it follows the COPE publication ethics, also it follows the CC BY creative commons copyright license. TFSS aims to publish high-quality original articles that make a significant contribution to the research areas of both theoretical and applied mathematics in the field of fuzzy and all papers will be peer-review before publication.

Manuscripts submitted to the TFSS must be original and unpublished work and not currently being considered for publication elsewhere. TFSS aims to reflect the latest developments in fuzzy sets and systems and promote international academic exchanges. TFSS publishes 2 issues each year.

Aims and Scopes

TFSS aim is to present the development and new applications of fuzzy sets and systems to increase knowledge and help to publish the results of studies in the field of fuzzy logic, intelligent systems and other related topics. The scope of the journal includes fuzzy theory and its applications in every branch of science and technology. Also, TFSS publishes papers that use fuzzy in other fields, such as computer science and mathematics.

Instructions to Authors

The language of the journal is English. All accepted papers should also be prepared in LaTeX style and should be typed according to the style given in the journal homepage: https://tfss.journals.iau.ir/. Access to articles from this site is free: TFSS allows the author(s) to hold the copyright without restrictions.

We recommend that all authors review the "Guide to Authors" on the TFSS's website (https://tfss.journals.iau.ir/journal/authors.note).

Submission of Manuscripts

All papers should be prepared in LaTeX and the pdf file of the paper should be sent to the editorial office only through the electronic submission via "https://tfss.journals.iau.ir/". Manuscripts under consideration for the journal should not be published or submitted for publication elsewhere. TFSS doesn't have any submission and article processing charges.

We kindly recommend to all authors especially young ones to review publication ethics. There are many good sources for reminding publication ethics e.g., visit COPE.

Reviewers

Peer review and reviewers are at the heart of the academic publishing process. We would like to express our deepest gratitude for volunteering your time and expertise as a reviewer. Providing a voluntary service, such as peer review, under challenging circumstances is truly a dedication to your community and the advancement of research in your field. It is an honor for us to acknowledge your commitment by saying "thank you". On behalf of all our journal editorial members, we are indebted to the time you voluntarily dedicate to supporting our journal.

Cooperation with TFSS

Qualified specialists in various fields of Fuzzy Science issues who are willing to cooperate with the TFSS Journal as peer reviewers are welcomed and invited to send their CV along with their field(s) of interest to the Journal email address: tfssiauba@gmail.com.

Publication Office: Islamic Azad University, Bandar Abbas Branch, Bandar Abbas, Iran. Tel/fax: +98-76-3367-0243
Website: https://tfss.journals.iau.ir/, E-mails: tfssiauba@gmail.com; tfssiauba@yahoo.com, Online ISSN: 2821-0131

Transactions on Fuzzy Sets and Systems

(Vol.1, No.2, November 2022)

Transactions on Fuzzy Sets and Systems

URL: https://tfss.journals.iau.ir/

Online ISSN: 2821-0131

Vol.1, No.2, (2022), 1-5

DOI: http://doi.org/10.30495/TFSS.2022.1956899.1024

Article Type: Original Research Article (Invited Paper)

A Note on the Maximum Difference Between Schweizer and Wolff's σ and the Absolute Value of Spearman's ρ

Manuel Úbeda-Flores

Abstract. In this note we correct an error on the possible maximum difference between the (measure of dependence) Schweizer and Wolff's σ and the absolute value of the (measure of concordance) Spearman's ρ given in [8]. Moreover, we provide a possible value for that possible, leaving its proof as an open problem.

AMS Subject Classification 2020: 60E05; 62E10

Keywords and Phrases: Copula, Schweizer and Wolff's σ, Spearman's ρ.

1 Introduction

Aggregation functions play an important role in many applications of fuzzy set theory and fuzzy logic, among many other fields (see, e.g., [1, 3]). Copulas —multivariate probability distribution functions with uniform univariate margins on $[0,1]$— are special types of conjunctive aggregation functions, and they are used in aggregation processes because they ensure that the aggregation is stable in the sense that small error inputs correspond to small error outputs.

The importance of copulas in probability and statistics comes from *Sklar's theorem* [9], which states that the joint distribution H of a pair of random variables (X,Y) and the corresponding (univariate) marginal distributions F and G are linked by a copula C in the following manner:

$$H(x,y) = C\left(F(x),G(y)\right) \text{ for all } (x,y) \in [-\infty,\infty]^2.$$

If F and G are continuous, then the copula is unique; otherwise, C is uniquely determined on $(\mathrm{Range}\,F) \times (\mathrm{Range}\,G)$. For a review on copulas, we refer to the monographs [2, 5]

A (bivariate) *copula* is a function $C\colon [0,1]^2 \longrightarrow [0,1]$ which satisfies:

($C1$) the boundary conditions $C(t,0) = C(0,t) = 0$ and $C(t,1) = C(1,t) = t$ for all t in $[0,1]$, and

($C2$) the 2-*increasing property*, i.e., $V_C(R) := C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geq 0$, where $R = [u_1,u_2] \times [v_1,v_2]$ is a rectangle in $[0,1]^2$.

Corresponding author: Manuel Úbeda-Flores, Email: mubeda@ual.es, ORCID: 0000-0002-0675-9646
Received: 18 April 2022; Revised: 9 May 2022; Accepted: 11 May 2022; Published Online: 7 November 2022.

How to cite: M. Úbeda-Flores, A note on the maximum difference between Schweizer and Wolff's σ and the absolute value of Spearman's ρ, *Trans. Fuzzy Sets Syst.*, 1(2) (2022), 1-5.

The fundamental best-possible bounds inequality for the set of copulas is given by the Fréchet-Hoeffding bounds, i.e., for any copula C we have

$$W(u,v) := \max(0, u+v-1) \le C(u,v) \le \min(u,v) =: M(u,v) \tag{1}$$

for all $(u,v) \in [0,1]^2$. The hyperbolic paraboloid $z = uv$ —which corresponds to the copula for independence random variables, denoted by Π— sits midway between M and W.

Let $\mathcal{B}([0,1])$ and $\mathcal{B}([0,1]^2)$ denote the Borel σ-algebras in $[0,1]$ and $[0,1]^2$, respectively. A measure μ on $\mathcal{B}([0,1]^2)$ is *doubly stochastic* if $\mu(B \times [0,1]) = \mu([0,1] \times B) = \lambda(B)$ for every $B \in \mathcal{B}([0,1])$, where λ denotes the Lebesgue measure on $[0,1]$ (see [4] for details). Each copula C induces a doubly stochastic measure μ_C by setting $\mu_C(R) = V_C(R)$ for every rectangle $R \subseteq [0,1]^2$ and extending μ_C to $\mathcal{B}([0,1]^2)$. The *support* of a copula C is the complement of the union of all open subsets of $[0,1]^2$ with μ_C-measure zero, and when we refer to "mass" on a set, we mean the value of μ_C for that set.

In 1904, Charles Spearman defined the *Spearman's* ρ coefficient [10], a measure of concordance according to the set of axioms proposed by Scarsini [7]. For a pair of continuous random variables (X,Y) with associated copula C, the population version of this measure is given by

$$\rho(X,Y) = \rho_C = 12 \int_0^1 \int_0^1 [C(u,v) - uv] \, \mathrm{d}u \mathrm{d}v.$$

It represents the difference of the volume formed by the surfaces $z = C(u,v)$ and $z = uv$ on $[0,1]^2$, and where $\rho_W = -1$ and $\rho_M = 1$.

In 1959, A. Rényi proposed a set of desirable axioms for a nonparametric dependence measure for two continuously distributed random variables (X,Y) [6]. Later, those axioms were conveniently modified by Schweizer and Wolff in [8], where the authors introduced a new measure, called the *Schweizer and Wolff's* σ based upon the distance L_1 between the graphs of a copula C and Π, and which, suitably normalized, is given by

$$\sigma(X,Y) = \sigma_C = 12 \int_0^1 \int_0^1 |C(u,v) - uv| \, \mathrm{d}u \mathrm{d}v,$$

where $(X,Y) \sim C$. Note that, in this case, we have $\sigma_M = \sigma_W = 1$.

For any copula C, the quantity $|\rho_C|$ satisfies all the axioms for a measure of dependence except the fact that $\rho_C = 0$ does not necessarily imply that the random variables are independent (note that $\sigma_C = 0$ if, and only if, $C = \Pi$). If the copula C satisfies $C(u,v) \ge uv$ or $C(u,v) \le uv$ for all $(u,v) \in [0,1]^2$, then we have $\sigma_C = |\rho_C|$; but if this is not the case, σ_C is often a better measure than ρ (see [8] for several examples).

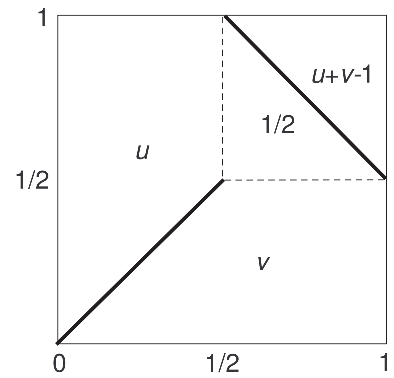
For any copula C, it is clear that $\sigma_C \ge |\rho_C|$. In [8], the authors provide an example for a possible maximum difference between the Schweizer and Wolff's σ and the absolute value of Spearman's ρ, which is ≈ 0.58 (see also [11]); however, this quantity is wrong. In the next section, we correct that error and provide a value (even greater than 0.58) for that possible difference, leaving its proof as an open problem.

2 The example, the correction and the conjecture

In [8] the authors provide the following example of the possible maximum difference between the Schweizer and Wolff's σ and the absolute value of Spearman's ρ.

Example 2.1 ([8]). Let (X,Y) be a pair of continuous random variables such that X is the identity map on $[0,1]$ and Y is defined by

$$Y(w) = \begin{cases} w, & 0 \le w \le \dfrac{1}{2} \\ \dfrac{3}{2} - w, & \dfrac{1}{2} < w \le 1. \end{cases}$$



(a) Support of C (b) $|C(u,v) - uv|$

Figure 1: Support of the copula C and the values of $|C(u,v) - uv|$ in Remark 2.2.

Then $\sigma(X,Y) - |\rho(X,Y)| = 3\ln 2 - 3/2 \approx 0.58$.

Remark 2.2. *The difference given in Example* 2.1 is not correct. Note that the copula C, associated with the pair (X,Y), is given by

$$C(u,v) = \begin{cases} \max(1/2, u+v-1), & (u,v) \in [1/2,1]^2, \\ M(u,v), & \text{otherwise.} \end{cases}$$

C is the copula whose mass is spread in two line segments, one joining the points $(0,0)$ to $(1/2,1/2)$ and the other the points $(1/2,1)$ to $(1,1/2)$. Figure 1 shows the support of the copula C and the values of $|C(u,v) - uv|$ for all $(u,v) \in [0,1]^2$.

Then, after some elementary algebra, we obtain $\sigma_C = 3\ln 2 - 5/4 \approx 0.83$ *and* $\rho_C = 0.75$ *—in* [11] *it appears* 0.25*—; whence* $\sigma_C - |\rho_C| \approx 0.08$.

In the next example, we propose a possible maximum difference $\sigma_C - |\rho_C|$ for a given copula C, even greater than the wrong value 0.58 in Example 2.1.

Example 2.3. Let $0 \le \theta \le 1$, and let C_θ be the copula given by

$$C_\theta(u,v) = \begin{cases} \max(0, u+v-\theta), & (u,v) \in [0,\theta]^2, \\ \max(\theta, u+v-1), & (u,v) \in [\theta,1]^2, \\ M(u,v), & \text{otherwise.} \end{cases}$$

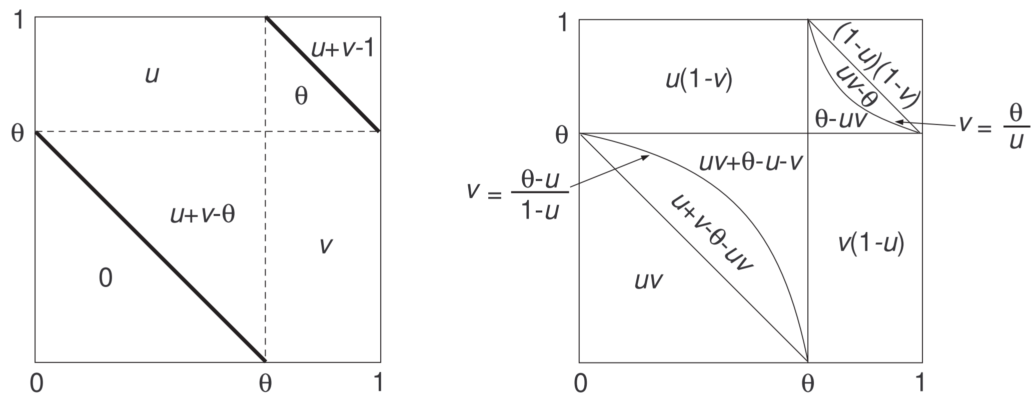
C_θ is the copula whose mass is spread in two line segments, one joining the points $(0,\theta)$ to $(\theta,0)$ and the other the points $(\theta,1)$ to $(1,\theta)$. Figure 2 shows the support of the copula C_θ and the values of $|C_\theta(u,v) - uv|$ for all $(u,v) \in [0,1]^2$. After some algebra, we obtain

$$\sigma_{C_\theta} = 1 - 18\theta(1-\theta) - 12\theta^2 \ln\theta - 12(1-\theta)^2 \ln(1-\theta)$$

and

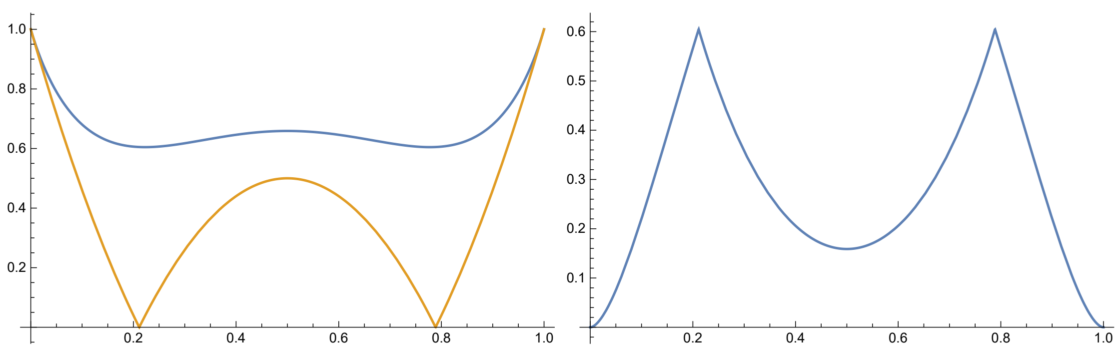
$$\rho_{C_\theta} = -1 + 6\theta(1-\theta).$$

Figure 3 shows the graphs of σ_{C_θ}, $|\rho_{C_\theta}|$ and $\sigma_{C_\theta} - |\rho_{C_\theta}|$. The maximum of the function $\sigma_{C_\theta} - |\rho_{C_\theta}|$ is reached at the points $\theta_1 = 1/2 - \sqrt{3}/6 \approx 0.21$ and $\theta_2 = 1/2 + \sqrt{3}/6 \approx 0.79$, for which $\sigma_{C_{\theta_1}} = \sigma_{C_{\theta_2}} \approx 0.60496$ and $\rho_{C_{\theta_1}} = \rho_{C_{\theta_2}} = 0$, whence $\sigma_{C_{\theta_1}} - \left|\rho_{C_{\theta_1}}\right| \approx 0.60496$.



(a) Support of C_θ (b) $|C_\theta(u,v) - uv|$

Figure 2: Support of the copula C_θ and the values of $|C_\theta(u,v) - uv|$ in Example 2.3.



(a) σ_{C_θ} (blue) and $|\rho_{C_\theta}|$ (red) (b) $\sigma_{C_\theta} - |\rho_{C_\theta}|$

Figure 3: Support of the copula C_θ and the values of $|C_\theta(u,v) - uv|$ in Example 2.3.

It remains as an open problem to check if, for any copula C, the maximum difference of $\sigma_C - |\rho_C|$ is the quantity given in Example 2.3.

Acknowledgements: The author acknowledges the support of the program FEDER-Andalucía 2014-2020 under research project UAL2020-AGR-B1783.

Conflict of Interest: The author declares that there are no conflict of interest.

References

[1] G. Beliakov, A. Pradera and T. Calvo, Aggregation Functions: A Guide for Practitioners, *Studies in Fuzziness and Soft, Springer, Berlin*, (2007).

[2] F. Durante and C. Sempi, Principles of Copula Theory, *Chapman & Hall/CRC, Boca Raton*, (2016).

[3] M. Grabisch, J. L. Marichal, R. Mesiar and E. Pap, Aggregation Functions, *Encyclopedia of Mathematics and its Applications, 127, Cambridge University Press, Cambridge*, (2009).

[4] P. R. Halmos, Measure Theory, *Springer, New York*, (1974).

[5] R. B. Nelsen, An Introduction to Copulas, Second Edition. *Springer, New York*, (2006).

[6] A. Rényi, On measures of dependence, *Acta Math. Acad. Sci. Hungar.*, 10 (1959), 441-451.

[7] M. Scarsini, On measures of concordance, *Stochastica*, 8 (1984), 201-218.

[8] B. Schweizer and E. F. Wolff, On nonparametric measures of dependence for random variables, *Ann. Statist.*, 9 (1981), 879-885.

[9] A. Sklar, Fonctions de répartition à n dimensions et leurs marges, *Publ. Inst. Statist. Univ. Paris*, 8 (1959), 229-231.

[10] C. Spearman, The proof and measurement of association between two things, *Am. J. Psychol.*, 15 (1904), 72-101.

[11] E. F. Wolff, Measures of dependence derived from copulas, *PhD Thesis, University of Massachussetts, Amherst, USA*, (1977).

Manuel Úbeda-Flores
Department of Mathematics
University of Almería
Almería, Spain
E-mail: mubeda@ual.es

Transactions on Fuzzy Sets and Systems
URL: https://tfss.journals.iau.ir/
Online ISSN: 2821-0131
Vol.1, No.2, (2022), 6-20
DOI: http://10.30495/tfss.2022.1958599.1029

Article Type: Original Research Article (Invited Paper)

Novel Characterizations of LM-Fuzzy Open Operators

Fu-Gui Shi

Abstract. In this paper, we present some characterizations of the LM-fuzzy interior operator, the LM-fuzzy closure operator, LM-fuzzy semiopen operator and LM-fuzzy preopen operator in an LM-fuzzy topological space. Based on them, we introcuced the notions of LM-fuzzy regularly open operators and LM-fuzzy regularly closed operators and show that these kinds of openness degrees are different from those defined by level L-topology.

AMS Subject Classification 2020: 54A40; 03E72
Keywords and Phrases: LM-fuzzy topology, LM-fuzzy closure operator, LM-fuzzy semiopen operator, LM-fuzzy preopen operator, LM-fuzzy regular open operator.

1 Introduction

As we all know, closure operator and interior operator are not only two important concepts in topology, but also have important applications in many other branches of mathematics. For example, it is a basic tool in functional analysis, algebra, lattice theory, matroid theory and convexity theory and so on. In [20], Shi generalized them to L-fuzzy topological spaces and called them L-fuzzy interior operators and L-fuzzy closure operators. L-fuzzy interior operators and L-fuzzy closure operators can be used to characterize L-fuzzy topology $\mathcal{T}$, but don't rely on the level L-topology $\mathcal{T}_{[r]}$.

The notions of semiopenness, preopenness and regular openness are very important in general topology [15]. They were extended to L-topological spaces by Azad, Singal and Prakash, respectively (see [1, 23]). The notions of semicontinuity and precontinuity were also extended to L-topological spaces by Azad and Nanda respectively (see [1, 17]). Moreover the notions of semiopenness and regular openness were extended to fuzzifying topological spaces by A.M. Zahran, F.M. Zeyada and A.K. Mousa respectively (see [25, 26]). Further in [12, 13, 14], S.J. Lee and E.P. Lee introduced the notions of fuzzy r-semiopen sets, fuzzy r-preopen sets and fuzzy r-regular open sets in $[0, 1]$-fuzzy topological space $(X, \mathcal{T})$ by means of the level $[0, 1]$-topology $\mathcal{T}_{[r]}$.

In 2011, Shi introduced the notions of LM-fuzzy semiopen operator and LM-fuzzy preopen operator in LM-fuzzy topological spaces by means of the idea of [20]. Further they were applied to many research fields by Ghareeb, Al-Omeri and Liang [3, 4, 5, 6, 7, 21].

In this paper, we shall present some characterizations of the LM-fuzzy interior operator, the LM-fuzzy closure operator, LM-fuzzy semiopen operator and LM-fuzzy preopen operator. We shall show that these kinds of openness degrees are different from those defined by level $[0, 1]$-topology.

Corresponding author: Fu-Gui Shi, Email: fuguishi@bit.edu.cn, ORCID: 0000-0001-8090-3872
Received: 12 May 2022; Revised: 20 May 2022; Accepted: 21 May 2022; Published Online: 7 November 2022.

How to cite: F.-G. Shi, Novel characterizations of LM-fuzzy open operator, *Trans. Fuzzy Sets Syst.*, 1(2) (2022), 6-20.

2 Preliminaries

Throughout this paper, L and M denote completely distributive lattices with order-reversing involutions, X is a nonempty set. The set of all nonzero co-prime elements of L is denoted by $J(L)$. The set of all nonzero co-prime elements of L^X is denoted by $J(L^X)$. It is easy to see that $J(L^X)$ is exactly the set of all fuzzy points x_λ ($\lambda \in J(L)$).

We say that a is a wedge below b in M, denoted by $a \prec b$, if for every subset $D \subseteq M$, $\bigvee D \geq b$ implies $d \geq a$ for some $d \in D$ [2]. A complete lattice M is completely distributive if and only if $b = \bigvee\{a \in M \mid a \prec b\}$ for each $b \in M$. $\{a \in M \mid a \prec b\}$ is called the greatest minimal family of b, denoted by $\beta(b)$. $\alpha(a) = \{b \in M \mid b' \prec a'\}$ is called the greatest maximal family of a.

In a completely distributive lattice M, α is an $\bigwedge$-$\bigcup$ map, β is a union-preserving map, and for each $a \in M$, $a = \bigvee \beta(a) = \bigwedge \alpha(a)$ (see [10, 27]).

For $A \in M^X$ and $a \in M$, we use the following symbols [18, 19].

$$A_{[a]} = \{x \in X \mid A(x) \ngeq a\}, \qquad A^{(a)} = \{x \in X \mid A(x) \nleq a\},$$
$$A_{(a)} = \{x \in X \mid a \in \beta(A(x))\}, \quad A^{[a]} = \{x \in X \mid a \notin \alpha(A(x))\}.$$

Definition 2.1. [8, 9, 11, 22, 24] *A map $\mathcal{T} : L^X \to M$ is called an LM-fuzzy pretopology on X provided that it satisfies the following conditions:*

(LFT1) $\mathcal{T}(X) = \mathcal{T}(\emptyset) = \top_M$;

(LFT2) $\mathcal{T}\left(\bigvee_{i\in\Omega} A_i\right) \geqslant \bigwedge_{i\in\Omega} \mathcal{T}(A_i),\ \forall\{A_i \mid i \in \Omega\} \subseteq L^X$.

An LM-fuzzy pretopology $\mathcal{T}$ is called an LM-fuzzy topology if it satisfies the following condition again.

(LFT3) $\mathcal{T}(U \wedge V) \geq \mathcal{T}(U) \wedge \mathcal{T}(V), \quad \forall U, V \in L^X$.

$\mathcal{T}(U)$ can be interpreted as the degree to which U is an L-open set. $\mathcal{T}^*(U) = \mathcal{T}(U')$ is called the degree of closedness of U. The pair $(X, \mathcal{T})$ is called an LM-fuzzy topological space. When $L = M$, an LM-fuzzy topology is also called an L-fuzzy topology. When $L = M = [0,1]$, an LM-fuzzy topology is called a $[0,1]$-fuzzy topology. In particular, when $M = \{0,1\}$, an LM-fuzzy topology is called an L-topology and when $L = \{0,1\}$, an LM-fuzzy topology is called an M-fuzzifying topology.

A map $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is said to be continuous with respect to LM-fuzzy topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ if $\mathcal{T}_1(f_L^{\leftarrow}(U)) \geq \mathcal{T}_2(U)$ holds for all $U \in L^Y$, where $f_L^{\leftarrow}$ is defined by $f_L^{\leftarrow}(U)(x) = U(f(x))$.

Analogous to Theorem 3.2 in [27], we have the following.

Theorem 2.2. [27] Let $\mathcal{T} : L^X \to M$ be a map. Then the following conditions are equivalent:

(T1) $\mathcal{T}$ is an LM-fuzzy topology on X.

(T2) $\forall a \in M$, $\mathcal{T}_{[a]}$ is an L-topology on X.

(T3) $\forall a \in M$, $\mathcal{T}^{[a]}$ is an L-topology on X.

Definition 2.3. [20, 22] An LM-fuzzy interior operator on X is a map $\mathrm{Int} : L^X \to M^{J(L^X)}$ satisfying the following conditions:

(FI1) $\mathrm{Int}(A)(x_\lambda) = \bigwedge\limits_{\mu\prec\lambda} \mathrm{Int}(A)(x_\mu), \quad \forall x_\lambda in J(L^X), \forall A \in L^X$;

(FI2) $\mathrm{Int}(X)(x_\lambda) = \top_M$ for any $x_\lambda \in J(L^X)$;

(FI3) $\mathrm{Int}(A)(x_\lambda) = \perp_M$ for any $x_\lambda \not\leq A$;

(FI4) $\mathrm{Int}(A \wedge B) = \mathrm{Int}(A) \wedge \mathrm{Int}(B)$;

(FI5) $\forall a \in M \backslash \{\top_M\}$, $(\mathrm{Int}(A))^{(a)} \subseteq \left(\mathrm{Int}\left(\bigvee (\mathrm{Int}(A))^{(a)}\right)\right)^{(a)}$.

Corollary 2.4. [20, 22] Let $\mathcal{T}$ be an LM-fuzzy topology on X and let $\mathrm{Int}^{\mathcal{T}}$ be the LM-fuzzy interior operator induced by $\mathcal{T}$. Then $\forall x_\lambda \in J(L^X)$, $\forall A \in L^X$,

$$\mathrm{Int}^{\mathcal{T}}(A)(x_\lambda) = \bigvee_{x_\lambda \leq V \leq A} \mathcal{T}(V) \quad \text{and} \quad \mathcal{T}(A) = \bigwedge_{x_\lambda \prec A} \mathrm{Int}^{\mathcal{T}}(A)(x_\lambda).$$

Definition 2.5. [20, 22] *An LM-fuzzy closure operator on X is a map $\mathrm{Cl} : L^X \to M^{J(L^X)}$ satisfying the following conditions:*

(FC1) $\mathrm{Cl}(A)(x_\lambda) = \bigwedge_{\mu \prec \lambda} \mathrm{Cl}(A)(x_\mu), \forall x_\lambda \in J(L^X)$;

(FC2) $\mathrm{Cl}(\emptyset)(x_\lambda) = \perp_M$ for any $x_\lambda \in J(L^X)$;

(FC3) $\mathrm{Cl}(A)(x_\lambda) = \top_M$ for any $x_\lambda \leq A$;

(FC4) $\mathrm{Cl}(A \vee B) = \mathrm{Cl}(A) \vee \mathrm{Cl}(B)$;

(FC5) $\forall a \in M \backslash \{\perp_M\}$, $\left(\mathrm{Cl}\left(\bigvee (\mathrm{Cl}(A))_{[a]}\right)\right)_{[a]} \subseteq (\mathrm{Cl}(A))_{[a]}$.

Corollary 2.6. [20, 22] Let $\mathcal{T}$ be an LM-fuzzy topology on X and let $\mathrm{Cl}^{\mathcal{T}} : L^X \to M^{J(L^X)}$ be the LM-fuzzy closure operator induced by $\mathcal{T}$. Then $\forall x_\lambda \in J(L^X)$, $\forall A \in L^X$,

$$\mathrm{Cl}^{\mathcal{T}}(A)(x_\lambda) = \bigwedge_{x_\lambda \not\leq D \geq A} \left(\mathcal{T}(D')\right)' \quad \text{and} \quad \mathcal{T}(A) = \bigwedge_{x_\lambda \not\leq A'} \mathrm{Cl}(A')(x_\lambda)'.$$

3 The characterizations of LM-fuzzy interiors and closures

In this section, our aim is to present some characterizations of LM-fuzzy interiors and LM-fuzzy closures.

Theorem 3.1. *If a map $\mathrm{Int} : L^X \to M^{J(L^X)}$ satisfies the following* (FI1)–(FI4)*:*

(FI1) $\mathrm{Int}(A)(x_\lambda) = \bigwedge_{\mu \prec \lambda} \mathrm{Int}(A)(x_\mu), \quad \forall x_\lambda \in J(L^X), \ \forall A \in L^X$;

(FI2) $\mathrm{Int}(X)(x_\lambda) = \top_M$ *for any* $x_\lambda \in J(L^X)$;

(FI3) $\mathrm{Int}(A)(x_\lambda) = \perp_M$ *for any* $x_\lambda \not\leq A$;

(FI4) $\mathrm{Int}(A \wedge B) = \mathrm{Int}(A) \wedge \mathrm{Int}(B)$,

then the following (FI5), (FI6) *and* (FI7) *are equivalent:*

(FI5) $\mathrm{Int}(A)(x_\lambda) = \bigvee_{x_\lambda \leq V \leq A} \bigwedge_{y_\mu \prec V} \mathrm{Int}(V)(y_\mu)$;

(FI6) $\forall a \in M \backslash \{\top_M\}$, $(\mathrm{Int}(A))^{(a)} \subseteq \left(\mathrm{Int}\left(\bigvee (\mathrm{Int}(A))^{(a)}\right)\right)^{(a)}$;

(FI7) $\forall a \in M\backslash\{\perp_M\}$, $(\mathrm{Int}(A))_{(a)} \subseteq \left(\mathrm{Int}\left(\bigvee(\mathrm{Int}(A))_{(a)}\right)\right)_{(a)}$.

Proof. By means of Theorem 3.3 in [22] we know that (FI5) is equivalent to (FI6). Now we prove that (FI5) is equivalent to (FI7).

In order to prove (FI5) $\Rightarrow$ (FI7), suppose $x_\lambda \in (\mathrm{Int}(A))_{(a)}$. Then $a \prec \mathrm{Int}(A)(x_\lambda)$. By (FI5) we know that there exists $V \in L^X$ such that $x_\lambda \leq V \leq A$ and

$$a \prec \bigwedge_{y_\mu \prec V} \mathrm{Int}(V)(y_\mu) \leq \mathrm{Int}(V)(y_\mu) \leq \mathrm{Int}(A)(y_\mu) \text{ for all } y_\mu \prec V\ .$$

This implies $y_\mu \in (\mathrm{Int}(V))_{(a)} \subseteq (\mathrm{Int}(A))_{(a)}$. Further we obtain $V \leq \bigvee(\mathrm{Int}(V))_{(a)} \leq \bigvee(\mathrm{Int}(A))_{(a)}$. Therefore it holds

$$a \prec \bigwedge_{y_\mu \prec V} \mathrm{Int}(V)(y_\mu) \leq \bigvee_{x_\lambda \leq V \leq \bigvee(\mathrm{Int}(A))_{(a)}} \bigwedge_{y_\mu \prec V} \mathrm{Int}(V)(y_\mu) = \mathrm{Int}\left(\bigvee(\mathrm{Int}(A))_{(a)}\right)(x_\lambda).$$

This shows $x_\lambda \in \left(\mathrm{Int}\left(\bigvee(\mathrm{Int}(A))_{(a)}\right)\right)_{(a)}$. (FI7) is proved.

(FI7) $\Rightarrow$ (FI5). It is easy to check that $\mathrm{Int}\,(A)\,(x_\lambda) \geq \bigvee_{x_\lambda \leq V \leq A} \bigwedge_{y_\mu \prec V} \mathrm{Int}\,(V)\,(y_\mu)$ holds. We only need to show that $\mathrm{Int}\,(A)\,(x_\lambda) \leq \bigvee_{x_\lambda \leq V \leq A} \bigwedge_{y_\mu \prec V} \mathrm{Int}\,(V)\,(y_\mu)$ is true.

Suppose that $a \prec \mathrm{Int}(A)(x_\lambda)$. Then by (FI7) we know $x_\lambda in (\mathrm{Int}(A))_{(a)} \subseteq \left(\mathrm{Int}\left(\bigvee(\mathrm{Int}(A))_{(a)}\right)\right)_{(a)}$. Let $V = \bigvee(\mathrm{Int}(A))_{(a)}$. Then $x_\lambda \leq V \leq A$ and $a \prec \mathrm{Int}(V)(x_\lambda)$. For all $y_\mu \prec V$, there exists $y_\gamma \in (\mathrm{Int}(A))_{(a)}$ such that $y_\mu \prec y_\gamma$. By (FI1) and (FI7) we know

$$y_\mu \in (\mathrm{Int}(A))_{(a)} \subseteq \left(\mathrm{Int}\left(\bigvee(\mathrm{Int}(A))_{(a)}\right)\right)_{(a)} = \left(\mathrm{Int}\,(V)\right)_{(a)}, i.e., a \prec \mathrm{Int}(V)(y_\mu).$$

This implies $a \leq \bigwedge_{y_\mu \prec V} \mathrm{Int}(V)(y_\mu)$. Hence we have

$$a \quad \leq \quad \bigvee_{x_\lambda \leq V \leq \bigvee(\mathrm{Int}(A))_{(a)}} \bigwedge_{y_\mu \prec V} \mathrm{Int}(V)(y_\mu) \quad \leq \quad \bigvee_{x_\lambda \leq V \leq A} \bigwedge_{y_\mu \prec V} \mathrm{Int}(V)(y_\mu).$$

This shows that $\mathrm{Int}(A)(x_\lambda) \leq \bigvee_{x_\lambda \leq V \leq A} \bigwedge_{y_\mu \prec V} \mathrm{Int}(V)(y_\mu)$ is true. The proof is completed. $\square$

Theorem 3.2. *If a map* $\mathrm{Cl} : L^X \to M^{J(L^X)}$ *satisfies the following* (FC1)–(FC4)*:*

(FC1) $\mathrm{Cl}(A)(x_\lambda) = \bigwedge_{\mu \prec \lambda} \mathrm{Cl}(A)(x_\mu), \forall x_\lambda \in J(L^X)$*;*

(FC2) $\mathrm{Cl}(\emptyset)(x_\lambda) = \perp_M$ *for any* $x_\lambda \in J(L^X)$*;*

(FC3) $\mathrm{Cl}(A)(x_\lambda) = \top_M$ *for any* $x_\lambda \leq A$*;*

(FC4) $\mathrm{Cl}(A \vee B) = \mathrm{Cl}(A) \vee \mathrm{Cl}(B)$,

then the following (FC5), (FC6) *and* (FC7) *are equivalent:*

(FC5) $\mathrm{Cl}(A)(x_\lambda) = \bigwedge_{x_\lambda \not\leq B \geq A} \bigvee_{y_\mu \not\leq B} (\mathrm{Cl}(B))(y_\mu)$;

(FC6) $\forall a \in M\backslash\{\perp_M\}$, $\left(\mathrm{Cl}\left(\bigvee(\mathrm{Cl}(A))_{[a]}\right)\right)_{[a]} \subseteq (\mathrm{Cl}(A))_{[a]}$*;*

(FC7) $\forall a \in M\backslash\{\top_M\}$, $\left(\mathrm{Cl}\left(\bigvee(\mathrm{Cl}(A))^{[a]}\right)\right)^{[a]} \subseteq (\mathrm{Cl}(A))^{[a]}$.

Proof. By means of Theorem 3.1 in [22] we know that (FC5) is equivalent to (FC6). Now we prove that (FC5) is equivalent to (FC7).

(FC5) $\Rightarrow$ (FC7). Suppose that $x_\lambda \notin \mathrm{Cl}(A)^{[a]}$. Then by (FC5) we obtain the following fact.

$$a \in \alpha\left(\mathrm{Cl}(A)(x_\lambda)\right) = \alpha\left(\bigwedge_{x_\lambda \not\leq B \geq A} \bigvee_{y_\mu \not\leq B} \mathrm{Cl}(B)(y_\mu)\right) = \bigcup_{x_\lambda \not\leq B \geq A} \alpha\left(\bigvee_{y_\mu \not\leq B} \mathrm{Cl}(B)(y_\mu)\right).$$

Hence there exists $B \in L^X$ with $x_\lambda \not\leq B \geq A$ such that $a \in \alpha\left(\bigvee_{y_\mu \not\leq B} \mathrm{Cl}(B)(y_\mu)\right)$, which implies $\forall y_\mu \not\leq B$, $a \in \alpha(\mathrm{Cl}(B)(y_\mu))$, i.e., $y_\mu \notin \mathrm{Cl}(B)^{[a]}$. Therefore it follows $\bigvee \mathrm{Cl}(B)^{[a]} \leq B$. Thus we have $x_\lambda \not\leq B \geq \bigvee \mathrm{Cl}(B)^{[a]} \geq \bigvee \mathrm{Cl}(A)^{[a]}$. Hence we obtain the following formula.

$$\begin{aligned} a &\in \bigcup_{x_\lambda \not\leq B \geq \bigvee \mathrm{Cl}(A)^{[a]}} \alpha\left(\bigvee_{y_\mu \not\leq B} \mathrm{Cl}(B)(y_\mu)\right) = \alpha\left(\bigwedge_{x_\lambda \not\leq B \geq \bigvee \mathrm{Cl}(A)^{[a]}} \bigvee_{y_\mu \not\leq B} \mathrm{Cl}(B)(y_\mu)\right) \\ &= \alpha\left(\mathrm{Cl}\left(\bigvee \mathrm{Cl}(A)^{[a]}\right)(x_\lambda)\right). \end{aligned}$$

This implies $x_\lambda \notin \mathrm{Cl}\left(\bigvee \mathrm{Int}(A)^{[a]}\right)^{[a]}$. (FC7) is proved.

(FC7) $\Rightarrow$ (FC5). It is easy to check that $\mathrm{Cl}(A)(x_\lambda) \leq \bigwedge_{x_\lambda \not\leq B \geq A} \bigvee_{y_\mu \not\leq B} (\mathrm{Cl}(B))(y_\mu)$ holds. Now we prove $\mathrm{Cl}(A)(x_\lambda) \geq \bigwedge_{x_\lambda \not\leq B \geq A} \bigvee_{y_\mu \not\leq B} (\mathrm{Cl}(B))(y_\mu)$.

Suppose that $a \in \alpha\left(\mathrm{Cl}(A)(x_\lambda)\right)$. Then there exists $b \in L$ such that $a \in \alpha(b)$ and $b \in \alpha\left(\mathrm{Cl}(A)(x_\lambda)\right)$. By (FC7) we know

$$x_\lambda \notin (\mathrm{Cl}(A))^{[b]} \supseteq \left(\mathrm{Cl}\left(\bigvee (\mathrm{Cl}(A))^{[b]}\right)\right)^{[b]}.$$

Let $D = \bigvee \mathrm{Cl}(A)^{[b]}$. Then $A \leq D$ and $b \in \alpha\left(\mathrm{Cl}\left(D\right)(x_\lambda)\right)$. In this case, we must have $x_\lambda \not\leq D$. In fact, if $x_\lambda \leq D$, then $x_\mu \prec x_\lambda \leq D$ for all $\mu \prec \lambda$, hence there exists $x_\gamma \in \mathrm{Cl}(A)^{[b]}$ such that $x_\gamma \geq x_\mu$. From $\mathrm{Cl}(A)(x_\gamma) \leq \mathrm{Cl}(A)(x_\mu)$ we know $b \notin \alpha(\mathrm{Cl}(A)(x_\mu))$ for all $\mu \prec \lambda$. This implies

$$b \notin \bigcup_{\mu \prec \lambda} \alpha\left(\mathrm{Cl}(A)(x_\mu)\right) = \alpha\left(\bigwedge_{\mu \prec \lambda} \mathrm{Cl}(A)(x_\mu)\right) = \alpha\left(\mathrm{Cl}(A)(x_\lambda)\right),$$

which contradicts to $b \in \alpha\left(\mathrm{Cl}\left(D\right)(x_\lambda)\right)$. Therefore $x_\lambda \not\leq D \geq A$. For all $y_\mu \not\leq D$, by

$$(\mathrm{Cl}(A))^{[b]} \supseteq \left(\mathrm{Cl}\left(\bigvee (\mathrm{Cl}(A))^{[b]}\right)\right)^{[b]} = \left(\mathrm{Cl}\left(D\right)\right)^{[b]}$$

we know $y_\mu \not\leq \left(\mathrm{Cl}\left(D\right)\right)^{[b]}$, i.e., $b \in \alpha\left(\mathrm{Cl}\left(D\right)(y_\mu)\right)$. Further we have $b \geq \bigvee_{y_\mu \not\leq D} \mathrm{Cl}\left(D\right)(y_\mu)$. This shows

$$\begin{aligned} a \in \alpha\left(b\right) &\subseteq \alpha\left(\bigvee_{y_\mu \not\leq D} \mathrm{Cl}\left(D\right)(y_\mu)\right) \subseteq \bigcup_{x_\lambda \not\leq B \geq \bigvee \mathrm{Cl}(A)^{[b]}} \alpha\left(\bigvee_{y_\mu \not\leq B} \mathrm{Cl}(B)(y_\mu)\right) \\ &= \alpha\left(\bigwedge_{x_\lambda \not\leq B \geq \bigvee \mathrm{Cl}(A)^{[b]}} \bigvee_{y_\mu \not\leq B} \mathrm{Cl}(B)(y_\mu)\right) \subseteq \alpha\left(\bigwedge_{x_\lambda \not\leq B \geq A} \bigvee_{y_\mu \not\leq D} \mathrm{Cl}(B)(y_\mu)\right). \end{aligned}$$

Therefore it follows $\mathrm{Cl}(A)(x_\lambda) \geq \bigwedge_{x_\lambda \not\leq B \geq A} \bigvee_{y_\mu \not\leq B} (\mathrm{Cl}(B))(y_\mu)$. The proof is completed. $\square$

Theorem 3.3. *Let $\mathcal{T}$ be an LM-fuzzy topology on X, Int be the LM-fuzzy interior operator in $(X,\mathcal{T})$ and Cl be the LM-fuzzy closure operator in $(X,\mathcal{T})$. Then for any $A \in L^X$ and for any $a \in M \setminus \{\perp_M\}$, it follows*

(1) $A \in \mathcal{T}_{[a]} \;\Leftrightarrow\; \forall x_\lambda \prec A,\ \mathrm{Int}(A)(x_\lambda) \geq a \;\Leftrightarrow\; A \leq \bigvee (\mathrm{Int}(A))_{[a]}$.

(2) $A \in \mathcal{T}^*_{[a]} \;\Leftrightarrow\; \forall x_\lambda \not\leq A,\ \mathrm{Cl}(A)(x_\lambda) \leq a' \;\Leftrightarrow\; \bigvee (\mathrm{Cl}(A))^{(a')} \leq A$.

Proof. (1) From Corollary 2.4 we easily obtain

$$A \in \mathcal{T}_{[a]} \Leftrightarrow a \leq \mathcal{T}(A) \Leftrightarrow \forall x_\lambda \prec A,\ \mathrm{Int}(A)(x_\lambda) \geq a.$$

Moreover it is obvious

$$\forall x_\lambda \prec A,\ \mathrm{Int}(A)(x_\lambda) \geq a \Rightarrow A \leq \bigvee \mathrm{Int}(A)_{[a]}.$$

Now we prove

$$A \leq \bigvee \mathrm{Int}(A)_{[a]} \Rightarrow \forall x_\lambda \prec A,\ \mathrm{Int}(A)(x_\lambda) \geq a.$$

Suppose $x_\lambda \prec A$. By $A \leq \bigvee \mathrm{Int}(A)_{[a]}$, there exists $x_\mu \in \mathrm{Int}(A)_{[a]}$ such that $x_\lambda \prec x_\mu$. Hence

$$\mathrm{Int}(A)(x_\lambda) \geq \bigwedge_{\lambda \prec \mu} \mathrm{Int}(A)(x_\lambda) = \mathrm{Int}(A)(x_\mu) \geq a.$$

(2) From Corollary 2.6 we easily obtain

$$A \in \mathcal{T}^*_{[a]} \Leftrightarrow a \leq \mathcal{T}^*(A) \Leftrightarrow \forall x_\lambda \not\leq A,\ \mathrm{Cl}(A)(x_\lambda) \leq a'.$$

Moreover it is obvious

$$\bigvee \mathrm{Cl}(A)^{(a')} \leq A \Rightarrow \forall x_\lambda \not\leq A,\ \mathrm{Cl}(A)(x_\lambda) \leq a'.$$

Now we prove

$$\forall x_\lambda \not\leq A,\ \mathrm{Cl}(A)(x_\lambda) \leq a' \Rightarrow \bigvee \mathrm{Cl}(A)^{(a')} \leq A.$$

Suppose that $x_\lambda \prec \bigvee \mathrm{Cl}(A)^{(a')}$. Then there exists $x_\mu \in \mathrm{Cl}(A)^{(a')}$ (that is, $\mathrm{Cl}(A)(x_\mu) \not\leq a'$) such that $x_\lambda \prec x_\mu$. Hence by

$$\mathrm{Cl}(A)(x_\lambda) \geq \bigwedge_{\lambda \prec \mu} \mathrm{Cl}(A)(x_\lambda) = \mathrm{Cl}(A)(x_\mu) \not\leq a'$$

we obtain $\mathrm{Cl}(A)(x_\lambda) \not\leq a'$. This implies $x_\lambda \leq A$. $\bigvee \mathrm{Cl}(A)^{(a')} \leq A$ is proved. $\square$

Theorem 3.4. *Let $\mathcal{T}$ be an LM-fuzzy topology on X, Int be the LM-fuzzy interior operator in $(X,\mathcal{T})$ and Cl be the LM-fuzzy closure operator in $(X,\mathcal{T})$. Then for any $A \in L^X$ and for any $a \in M \setminus \{\perp_M\}$, it follows*

(1) $A \in \mathcal{T}^{[a]} \;\Leftrightarrow \forall x_\lambda \prec A,\ x_\lambda \in \mathrm{Int}(A)^{[a]} \Leftrightarrow\; A \leq \bigvee \mathrm{Int}(A)^{[a]}$.

(2) $A \in \mathcal{T}^{*[a]} \;\Leftrightarrow\; \forall x_\lambda \not\leq A,\ x_\lambda \notin \mathrm{Cl}(A)_{(a')} \Leftrightarrow \bigvee \mathrm{Cl}(A)_{(a')} \leq A$.

Proof. (1) From Corollary 2.4 we easily obtain

$$A \in \mathcal{T}^{[a]} \Leftrightarrow a \notin \alpha(\mathcal{T}(A)) \Leftrightarrow \forall x_\lambda \prec A,\ a \notin \alpha(\mathrm{Int}(A)(x_\lambda)) \Leftrightarrow \forall x_\lambda \prec A,\ x_\lambda \in \mathrm{Int}(A)^{[a]}.$$

Moreover it is obvious

$$\forall x_\lambda \prec A,\ x_\lambda \in \mathrm{Int}(A)^{[a]} \Rightarrow A \leq \bigvee \mathrm{Int}(A)^{[a]}.$$

Now we prove

$$A \leq \bigvee \mathrm{Int}(A)^{[a]} \Rightarrow \forall x_\lambda \prec A,\ x_\lambda \in \mathrm{Int}(A)^{[a]}.$$

Suppose $x_\lambda \prec A$. By $A \leq \bigvee \mathrm{Int}(A)^{[a]}$, there exists $x_\mu \in \mathrm{Int}(A)^{[a]}$ such that $x_\lambda \prec x_\mu$. Hence by

$$\mathrm{Int}(A)(x_\lambda) \geq \bigwedge_{\lambda \prec \mu} \mathrm{Int}(A)(x_\lambda) = \mathrm{Int}(A)(x_\mu) \text{ and } a \notin \alpha(\mathrm{Int}(A)(x_\mu))$$

we know $a \notin \alpha(\mathrm{Int}(A)(x_\lambda))$, i.e., $x_\lambda \in \mathrm{Int}(A)^{[a]}$.

(2) From Corollary 2.6 we easily obtain

$$\begin{aligned} A \in \mathcal{T}^{*[a]} &\Leftrightarrow a \notin \alpha(\mathcal{T}^*(A)) \\ &\Leftrightarrow \forall x_\lambda \not\leq A,\ a \notin \alpha(\mathrm{Cl}(A)(x_\lambda)') \\ &\Leftrightarrow \forall x_\lambda \not\leq A,\ a' \notin \beta(\mathrm{Cl}(A)(x_\lambda)) \\ &\Leftrightarrow \forall x_\lambda \not\leq A,\ x_\lambda \notin \mathrm{Cl}(A)_{(a')}. \end{aligned}$$

It is easy to check $\forall x_\lambda \not\leq A,\ x_\lambda \notin \mathrm{Cl}(A)_{(a')} \Leftrightarrow \bigvee \mathrm{Cl}(A)_{(a')} \leq A$. □

4 The Characterizations of LM-fuzzy (semiclosed, preopen) preclosed operators

In 2011, Shi presented the notions of LM-fuzzy semiopen operator and LM-fuzzy preopen operator by means of LM-fuzzy topology $\mathcal{T}$. They were applied to many research fields by Ghareeb, Al-Omeri and Liang [3, 4, 5, 6, 7, 20]. Now we give their characterizations by means of LM-fuzzy interior operator and LM-fuzzy closure operator.

Definition 4.1. [6, 16] *Let $\mathcal{T}$ be an LM-fuzzy topology on X. For any $A \in L^X$, define two mappings $\mathcal{T}_s, \mathcal{T}_p : L^X \to M$ by*

$$\mathcal{T}_s(A) = \bigvee_{B \leq A} \left\{ \mathcal{T}(B) \wedge \bigwedge_{x_\lambda \prec A} \bigwedge_{x_\lambda \not\leq D \geq B} \left(\mathcal{T}(D')\right)' \right\},$$

$$\mathcal{T}_p(A) = \bigwedge_{x_\lambda \prec A} \bigvee_{x_\lambda \prec B} \left\{ \mathcal{T}(B) \wedge \bigwedge_{y_\mu \prec B} \bigwedge_{y_\mu \not\leq D \geq A} \left(\mathcal{T}(D')\right)' \right\}.$$

Then $\mathcal{T}_s$ is called the LM-fuzzy semiopen operator induced by $\mathcal{T}$, and $\mathcal{T}_p$ is called the LM-fuzzy preopen operator induced by $\mathcal{T}$. For all $A \in L^X$, define $\mathcal{T}_s^*(A) = \mathcal{T}_s(A')$ and $\mathcal{T}_p^*(A) = \mathcal{T}_p(A')$, then $\mathcal{T}_s^*$ and $\mathcal{T}_p^*$ are respectively called the LM-fuzzy semiclosed operator and the LM-fuzzy preclosed operator induced by $\mathcal{T}$.

The next theorem presents a characterization of the LM-fuzzy semiclosed operator.

Theorem 4.2. *Let $\mathcal{T}$ be an LM-fuzzy topology on X. Then for any $A \in L^X$,*

$$\mathcal{T}_s^*(A) = \bigvee_{B \geq A} \left\{ \mathcal{T}^*(B) \wedge \bigwedge_{x_\mu \not\leq A} \left(\mathrm{Int}^{\mathcal{T}}(B)(x_\mu)\right)' \right\}. \tag{1}$$

Proof. On the one hand, we have

$$
\begin{aligned}
\mathcal{T}_s^*(A) = \mathcal{T}_s(A') &= \bigvee_{B \geq A} \left\{ \mathcal{T}(B') \wedge \bigwedge_{x_\lambda \prec A'} \mathrm{Cl}^{\mathcal{T}}(B')(x_\lambda) \right\} \\
&\geq \bigvee_{B \geq A} \left\{ \mathcal{T}(B') \wedge \bigwedge_{x_\lambda \leq A'} \mathrm{Cl}^{\mathcal{T}}(B')(x_\lambda) \right\} \\
&= \bigvee_{B \geq A} \left\{ \mathcal{T}(B') \wedge \bigwedge_{x_\lambda \leq A'} \bigwedge_{\mu \not\leq \lambda'} \left(\mathrm{Int}^{\mathcal{T}}(B)(x_\mu)\right)' \right\} \\
&\geq \bigvee_{B \geq A} \left\{ \mathcal{T}^*(B) \wedge \bigwedge_{x_\mu \not\leq A} \left(\mathrm{Int}^{\mathcal{T}}(B)(x_\mu)\right)' \right\}.
\end{aligned}
$$

On the other hand, we can prove

$$
\begin{aligned}
& \bigvee_{B \geq A} \left\{ \mathcal{T}^*(B) \wedge \bigwedge_{x_\mu \not\leq A} \left(\mathrm{Int}^{\mathcal{T}}(B)(x_\mu)\right)' \right\} \\
= & \bigvee_{B \geq A} \left\{ \mathcal{T}(B') \wedge \bigwedge_{A \not\leq (x_\lambda)'} \bigvee_{\mu \not\leq \lambda'} \mathrm{Cl}^{\mathcal{T}}(B')(x_\mu) \right\} \\
\geq & \bigvee_{B' \leq A'} \left\{ \mathcal{T}(B') \wedge \bigwedge_{x_\mu \prec A'} \mathrm{Cl}^{\mathcal{T}}(B')(x_\mu) \right\} = \mathcal{T}_s(A') = \mathcal{T}_s^*(A).
\end{aligned}
$$

The proof of (1) is completed. □

The next theorem presents a characterization of the LM-fuzzy preclosed operator.

Theorem 4.3. *Let $\mathcal{T}$ be an LM-fuzzy topology on X. Then for any $A \in L^X$,*

$$
\mathcal{T}_p^*(A) = \bigwedge_{x_\lambda \not\leq A} \bigvee_{x_\lambda \not\leq D} \left\{ \mathcal{T}^*(D) \wedge \bigwedge_{y_\gamma \not\leq D} \left(\mathrm{Int}^{\mathcal{T}}(A)(y_\gamma)\right)' \right\}. \tag{2}
$$

Proof. On the one hand, we have

$$
\begin{aligned}
\mathcal{T}_p^*(A) = \mathcal{T}_p(A') &= \bigwedge_{x_\lambda \prec A'} \bigvee_{x_\lambda \prec B} \left\{ \mathcal{T}(B) \wedge \bigwedge_{y_\mu \prec B} \mathrm{Cl}^{\mathcal{T}}(A')(y_\mu) \right\} \\
&= \bigwedge_{x_\lambda \not\leq A} \bigvee_{x_\lambda \not\leq B'} \left\{ \mathcal{T}(B) \wedge \bigwedge_{y_\mu \prec B} \mathrm{Cl}^{\mathcal{T}}(A')(y_\mu) \right\} \\
&= \bigwedge_{x_\lambda \not\leq A} \bigvee_{x_\lambda \not\leq B'} \left\{ \mathcal{T}(B) \wedge \bigwedge_{y_\mu \prec B} \bigwedge_{\gamma \not\leq \mu'} \left(\mathrm{Int}^{\mathcal{T}}(A)(y_\gamma)\right)' \right\} \\
&\geq \bigwedge_{x_\lambda \not\leq A} \bigvee_{x_\lambda \not\leq B'} \left\{ \mathcal{T}(B) \wedge \bigwedge_{y_\gamma \not\leq B'} \left(\mathrm{Int}^{\mathcal{T}}(A)(y_\gamma)\right)' \right\} \\
&= \bigwedge_{x_\lambda \not\leq A} \bigvee_{x_\lambda \not\leq D} \left\{ \mathcal{T}^*(D) \wedge \bigwedge_{y_\gamma \not\leq D} \left(\mathrm{Int}^{\mathcal{T}}(A)(y_\gamma)\right)' \right\}.
\end{aligned}
$$

On the other hand, we can prove

$$
\begin{aligned}
& \bigwedge_{x_\lambda \not\leq A} \bigvee_{x_\lambda \not\leq D} \left\{ \mathcal{T}^*(D) \wedge \bigwedge_{y_\gamma \not\leq D} \left(\mathrm{Int}^{\mathcal{T}}(A)(y_\gamma)\right)' \right\} \\
= & \bigwedge_{x_\lambda \prec A'} \bigvee_{x_\lambda \prec D'} \left\{ \mathcal{T}(D') \wedge \bigwedge_{D' \not\leq (y_\gamma)'} \left(\mathrm{Int}^{\mathcal{T}}(A)(y_\gamma)\right)' \right\} \\
\geq & \bigwedge_{x_\lambda \prec A'} \bigvee_{x_\lambda \prec D'} \left\{ \mathcal{T}(D') \wedge \bigwedge_{\exists y_\mu \prec D', y_\mu \not\leq (y_\gamma)'} \left(\mathrm{Int}^{\mathcal{T}}(A)(y_\gamma)\right)' \right\} \\
\geq & \bigwedge_{x_\lambda \prec A'} \bigvee_{x_\lambda \prec D'} \left\{ \mathcal{T}(D') \wedge \bigwedge_{\exists y_\mu \prec D', y_\mu \not\leq (y_\gamma)'} \bigvee_{\nu \not\leq \gamma'} \mathrm{Cl}^{\mathcal{T}}(A')(y_\nu) \right\} \\
\geq & \bigwedge_{x_\lambda \prec A'} \bigvee_{x_\lambda \prec D'} \left\{ \mathcal{T}(D') \wedge \bigwedge_{y_\mu \prec D'} \mathrm{Cl}^{\mathcal{T}}(A')(y_\mu) \right\} \\
\geq & \bigwedge_{x_\lambda \prec A'} \bigvee_{x_\lambda \prec B} \left\{ \mathcal{T}(B) \wedge \bigwedge_{y_\mu \prec B} \mathrm{Cl}^{\mathcal{T}}(A')(y_\mu) \right\} = \mathcal{T}_s(A') = \mathcal{T}_s^*(A).
\end{aligned}
$$

The proof of (2) is completed. $\square$

The following is a characterization of LM-fuzzy preopen operator, which is simpler than Definition 4.1.

Theorem 4.4. *Let $\mathcal{T}$ be an LM-fuzzy topology on X. Then for any $A \in L^X$,*

$$
\mathcal{T}_p(A) = \bigvee_{A \leq B} \left\{ \mathcal{T}(B) \wedge \bigwedge_{y_\mu \prec B} \bigwedge_{y_\mu \not\leq D \geq A} \left(\mathcal{T}(D')\right)' \right\}. \tag{3}
$$

Proof. First we prove

$$
\mathcal{T}_p(A) \leq \bigvee_{A \leq B} \left\{ \mathcal{T}(B) \wedge \bigwedge_{y_\mu \prec B} \bigwedge_{y_\mu \not\leq D \geq A} \left(\mathcal{T}(D')\right)' \right\}.
$$

Suppose that there exists $a \in M$ such that $a \prec \mathcal{T}_p(A)$. Then by

$$
\mathcal{T}_p(A) \quad = \quad \bigwedge_{x_\lambda \prec A} \bigvee_{x_\lambda \prec B} \left\{ \mathcal{T}(B) \wedge \bigwedge_{y_\mu \prec B} \bigwedge_{y_\mu \not\leq D \geq A} \left(\mathcal{T}(D')\right)' \right\}
$$

we know that $\forall x_\lambda \prec A$, there exists $B_{x_\lambda} \in L^X$ such that $x_\lambda \prec B_{x_\lambda}$, $\mathcal{T}(B_{x_\lambda}) \geq a$ and $\forall y_\mu \prec B_{x_\lambda}$, $a \leq \bigwedge_{y_\mu \not\leq D \geq A} (\mathcal{T}(D'))'$. Let $B = \bigvee \{B_{x_\lambda} \mid x_\lambda \prec A\}$. Then $A \leq B$, $\mathcal{T}(B) \geq a$ and $\forall y_\mu \prec B$, there exists B_{x_λ} such that $\forall y_\mu \prec B_{x_\lambda}$. This implies

$$
\bigvee_{A \leq B} \left\{ \mathcal{T}(B) \wedge \bigwedge_{y_\mu \prec B} \bigwedge_{y_\mu \not\leq D \geq A} \left(\mathcal{T}(D')\right)' \right\} \geq a.
$$

Hence

$$
\mathcal{T}_p(A) \leq \bigvee_{A \leq B} \left\{ \mathcal{T}(B) \wedge \bigwedge_{y_\mu \prec B} \bigwedge_{y_\mu \not\leq D \geq A} \left(\mathcal{T}(D')\right)' \right\}.
$$

The inverse of the above inequality is obvious. $\square$

By means of LM-fuzzy interior operator and LM-fuzzy closure operator we can give the other characterizations of LM-fuzzy preopen operator and LM-fuzzy preclosed operator.

Corollary 4.5. *In an LM-fuzzy topological space $(X, \mathcal{T})$, it holds that for any $A \in L^X$,*

$$\mathcal{T}_p(A) = \bigvee_{A \le B} \left\{ \mathcal{T}(B) \wedge \bigwedge_{y_\mu \prec B} \mathrm{Cl}(A)(y_\mu) \right\}, \tag{4}$$

$$\mathcal{T}_p^*(A) = \bigvee_{A \le D} \left\{ \mathcal{T}^*(D) \wedge \bigwedge_{y_\gamma \not\le D} \left(\mathrm{Int}^{\mathcal{T}}(A)(y_\gamma)\right)' \right\}. \tag{5}$$

Proof. (4) can be proved from Corollary 2.6 and Theorem 4.4. Based on Theorem 4.3 and analogous to the proof of Theorem 4.4 we can obtain (5) □

5 LM-fuzzy regular open operators

In this section, we shall present the notions of LM-fuzzy regularly open operators and LM-fuzzy regularly closed operators in LM-fuzzy topological spaces.

Definition 5.1. *Let $\mathcal{T}$ be an LM-fuzzy topology on X. For any $A \in L^X$, define a map $\mathcal{T}_r : L^X \to M$ by*

$$\mathcal{T}_r(A) = \mathcal{T}_s(A') \wedge \mathcal{T}(A) = \mathcal{T}_s^*(A) \wedge \mathcal{T}(A).$$

Then $\mathcal{T}_r$ is called the LM-fuzzy regularly open operator corresponding to $\mathcal{T}$, where $\mathcal{T}_r(A)$ can be regarded as the degree to which A is regular open and $\mathcal{T}_r^*(B) = \mathcal{T}_r(B')$ can be regarded as the degree to which B is regularly closed.

Theorem 5.2. *Let $\mathcal{T}_r$ be the LM-fuzzy regularly open operator in LM-fuzzy topological space $(X, \mathcal{T})$. Then*

(1) $\mathcal{T}_r(\emptyset) = \mathcal{T}_r(X) = \top_M$.

(2) $\mathcal{T}_r(A) \le \mathcal{T}(A)$ *for any* $A \in L^X$.

(3) $\mathcal{T}_r(A \wedge B) \ge \mathcal{T}_r(A) \wedge \mathcal{T}_r(B)$ *for any* $A, B \in L^X$.

Proof. (1) and (2) are obvious. In order to prove (3), we first prove the following inequality.

$$\mathcal{T}_s^*\left(\bigwedge_{i\in\Omega} A_i\right) \ge \bigwedge_{i\in\Omega} \mathcal{T}_s^*(A_i) \text{ for any subfamily } \{A_i \mid i \in \Omega\} \text{ of } L^X. \tag{6}$$

Let $a \in L$ and $a \prec \bigwedge_{i\in\Omega} \mathcal{T}_s^*(A_i)$. Then for any $i \in \Omega$, there exists $B_i \le (A_i)'$ such that

$$a \prec \mathcal{T}(B_i) \text{ and } a \prec \bigwedge_{x_\lambda \prec (A_i)'} \bigwedge_{x_\lambda \not\le D \ge B_i} \left(\mathcal{T}(D')\right)'.$$

Hence

$$a \le \bigwedge_{i\in\Omega} \mathcal{T}(B_i) \le \mathcal{T}\left(\bigvee_{i\in\Omega} B_i\right) \text{ and } a \le \bigwedge_{i\in\Omega} \bigwedge_{x_\lambda \prec (A_i)'} \bigwedge_{x_\lambda \not\le D \ge B_i} \left(\mathcal{T}(D')\right)'.$$

By

$$\left\{ x_\lambda \mid x_\lambda \prec \left(\bigwedge_{i\in\Omega} A_i\right)' \right\} = \bigcup_{i\in\Omega} \left\{ x_\lambda \mid x_\lambda \prec (A_i)' \right\},$$

we have

$$\begin{aligned}
\mathcal{T}_s^*\left(\bigwedge_{i\in\Omega} A_i\right) &= \bigvee_{B\le\left(\bigwedge_{i\in\Omega} A_i\right)'} \left\{\mathcal{T}(B)\wedge \bigwedge_{x_\lambda\prec\left(\bigwedge_{i\in\Omega} A_i\right)'}\ \bigwedge_{x_\lambda\not\le D\ge B} (\mathcal{T}(D'))'\right\}\\
&\ge \mathcal{T}\left(\bigvee_{i\in\Omega} B_i\right)\wedge \bigwedge_{i\in\Omega}\ \bigwedge_{x_\lambda\prec(A_i)'}\ \bigwedge_{x_\lambda\not\le D\ge \bigvee_{i\in\Omega} B_i} (\mathcal{T}(D'))'\\
&\ge \mathcal{T}\left(\bigvee_{i\in\Omega} B_i\right)\wedge \bigwedge_{i\in\Omega}\ \bigwedge_{x_\lambda\prec(A_i)'}\ \bigwedge_{x_\lambda\not\le D\ge B_i} (\mathcal{T}(D'))'\\
&\ge a.
\end{aligned}$$

This shows $\mathcal{T}_s^*\left(\bigwedge_{i\in\Omega} A_i\right)\ge \bigwedge_{i\in\Omega}\mathcal{T}_s^*(A_i)$.

Since $\mathcal{T}$ is an L-fuzzy topology, it follows that $\mathcal{T}(A\wedge B)\ge\mathcal{T}(A)\wedge\mathcal{T}(B)$. Hence by (6), we obtain

$$\begin{aligned}
\mathcal{T}_r(A\wedge B) &= \mathcal{T}(A\wedge B)\wedge\mathcal{T}_s^*(A\wedge B)\\
&\ge \mathcal{T}(A)\wedge\mathcal{T}(B)\wedge\mathcal{T}_s^*(A)\wedge\mathcal{T}_s^*(B)\\
&= \mathcal{T}_r(A)\wedge\mathcal{T}_r(B).
\end{aligned}$$

(3) is proved. □

Definition 5.3. *A map $f: X\to Y$ between LM-fuzzy topological spaces $(X,\mathcal{S})$ and $(Y,\mathcal{T})$ is called LM-fuzzy almost continuous if $\mathcal{T}_r(U)\le\mathcal{S}(f_L^{\leftarrow}(U))$ holds for any $U\in L^Y$.*

Obviously an LM-fuzzy continuous map is LM-fuzzy almost continuous. Moreover the following result is also obvious.

Corollary 5.4. *A map $f: X\to Y$ between LM-fuzzy topological spaces $(X,\mathcal{S})$ and $(Y,\mathcal{T})$ is almost continuous if and only if $\mathcal{T}_r^*(U)\le\mathcal{S}^*(f_L^{\leftarrow}(U))$ for any $U\in L^Y$.*

S.J. Lee and E.P. Lee presented the definitions of the fuzzy r-semiopen set, fuzzy r-preopen and fuzzy r-regularly open set, which rely on level [0,1]-topologies.

Definition 5.5. [12, 13, 14] *Let A be a [0,1]-fuzzy set of a [0,1]-fuzzy topological space $(X,\mathcal{T})$ and $r\in(0,1]$. Then A is said to be*

(1) *fuzzy r-semiopen if there is a fuzzy r-open set B such that $B\le A\le \mathrm{Cl}(B,r)$.*

(2) *fuzzy r-preopen if $A\le \imath(\mathrm{Cl}(A,r),r)$.*

(3) *fuzzy r-regularly open if $A=\imath(\mathrm{Cl}(A,r),r)$.*

Based on Definition 5.5 we can introduce the other definition of LM-fuzzy regular openness.

Definition 5.6. *Let $(X,\mathcal{T})$ be a $[0,1]$-fuzzy topological space. For any $A\in[0,1]^X$, define*

(1) $\mathcal{ST}(A)=\bigvee\{r\in(0,1]\mid A$ *is r-semiopen in* $\mathcal{T}_{[r]}\}$.

(2) $\mathcal{PT}(A)=\bigvee\{r\in(0,1]\mid A$ *is r-preopen in* $\mathcal{T}_{[r]}\}$.

(3) $\mathcal{RT}(A)=\bigvee\{r\in(0,1]\mid A$ *is r-regularopen in* $\mathcal{T}_{[r]}\}$.

In general, $\mathcal{ST} \neq \mathcal{T}_s$, $\mathcal{PT} \neq \mathcal{T}_p$, $\mathcal{RT} \neq \mathcal{T}_r$, these can be seen from the following example.

Example 5.7. Let $X = [0,1]$ and A_1, A_2, A_3 be fuzzy sets defined by

$$A_1(x) = x,\ A_2(x) = 1 - x,\ A_3(x) = 0.5,\ \forall x \in [0,1].$$

Define $\mathcal{T} : [0,1]^X \to [0,1]$ by

$$\mathcal{T}(G) = \begin{cases} 1, & \text{if } G = \emptyset, X, \\ 0.8, & \text{if } G = A_1, A_1 \vee A_2, A_1 \wedge A_2, \\ 0.6, & \text{if } G = A_3, A_1 \wedge A_3, A_2 \wedge A_3, A_1 \vee A_3, A_2 \vee A_3, \\ 0.1, & \text{if } G = A_2 \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to check that $\mathcal{T}$ is a $[0,1]$-fuzzy topology and

$$\mathcal{T}_{[1]} = \{\emptyset, X\};\quad \mathcal{T}_{[0.8]} = \{\emptyset, X, A_1, A_1 \vee A_2, A_1 \wedge A_2\};$$
$$\mathcal{T}_{[0.6]} = \{\emptyset, X, A_1, A_1 \vee A_2, A_1 \wedge A_2, A_3, A_1 \wedge A_3, A_2 \wedge A_3, A_1 \vee A_3, A_2 \vee A_3\};$$
$$\mathcal{T}_{[0.1]} = \{\emptyset, X, A_1, A_2, A_1 \vee A_2, A_1 \wedge A_2, A_3, A_1 \wedge A_3, A_2 \wedge A_3, A_1 \vee A_3, A_2 \vee A_3\}.$$

It is easy to check that A_1 is a fuzzy 0.8-open set and 0.3-closed set. This implies $\mathcal{RT}(A_1) = 0.3$. By

$$\begin{aligned} \mathcal{T}_s(A_2) &= \bigvee_{B \leq A_2} \left(\mathcal{T}(B) \wedge \bigwedge_{x_\lambda < A_2} \bigwedge_{x_\lambda \not\leq D \geq B} (\mathcal{T}(D'))' \right) \\ &= \left(\mathcal{T}(A_1 \wedge A_2) \wedge \bigwedge_{x_\lambda < A_2} \bigwedge_{x_\lambda \not\leq D \geq A_1 \wedge A_2} (\mathcal{T}(D'))' \right) \\ &\quad \vee \left(\mathcal{T}(A_2 \wedge A_3) \wedge \bigwedge_{x_\lambda < A_2} \bigwedge_{x_\lambda \not\leq D \geq A_2 \wedge A_3} (\mathcal{T}(D'))' \right) \\ &\quad \vee \left(\mathcal{T}(A_2) \wedge \bigwedge_{x_\lambda < A_2} \bigwedge_{x_\lambda \not\leq D \geq A_2} (\mathcal{T}(D'))' \right) \\ &= (0.8 \wedge 0.2) \vee (0.6 \wedge 0.4) \vee (0.1 \wedge 1) = 0.4, \end{aligned}$$

and

$$\mathcal{T}_r(A_1) = \mathcal{T}_s((A_1)') \wedge \mathcal{T}(A_1) = \mathcal{T}_s(A_2) \wedge \mathcal{T}(A_1) = 0.4$$

we know $\mathcal{RT}(A_1) \neq \mathcal{T}_r(A_1)$.

It is easy to check that A_2 is a fuzzy 0.1-open set and 0.8-closed set. This implies $\mathcal{PT}(A_2) = 0.1$. By

$$\begin{aligned} \mathcal{T}_p(A_2) &= \bigvee_{B \geq A_2} \left(\mathcal{T}(B) \wedge \bigwedge_{x_\lambda \prec B} \bigwedge_{x_\lambda \not\leq D \geq A_2} (\mathcal{T}(D'))' \right) \\ &= \left(\mathcal{T}(A_1 \vee A_2) \wedge \bigwedge_{x_\lambda \prec A_1 \vee A_2} \bigwedge_{x_\lambda \not\leq D \geq A_2} (\mathcal{T}(D'))' \right) \\ &\quad \vee \left(\mathcal{T}(A_2) \wedge \bigwedge_{x_\lambda \prec A_2} \bigwedge_{x_\lambda \not\leq D \geq A_2} (\mathcal{T}(D'))' \right) \\ &= (0.8 \wedge 0.2) \vee (0.1 \wedge 1) = 0.2. \end{aligned}$$

we know $\mathcal{PT}(A_2) \neq \mathcal{T}_p(A_2)$.

It is easy to check that $A_1 \vee A_3$ is a fuzzy 0.6-open set and 0.6-closed set. This implies $\mathcal{ST}(A_1 \vee A_3) = 0.6$. Hence by the following fact we know $\mathcal{ST}(A_1 \vee A_3) \neq \mathcal{T}_s(A_1 \vee A_3)$.

$$
\begin{aligned}
& \mathcal{T}_s(A_1 \vee A_3) \\
= & \bigvee_{B \leq A_1 \vee A_3} \left(\mathcal{T}(B) \wedge \bigwedge_{x_\lambda \prec A_1 \vee A_3} \bigwedge_{x_\lambda \not\leq D \geq B} (\mathcal{T}(D'))' \right) \\
= & \left(\mathcal{T}(A_1 \wedge A_2) \wedge \bigwedge_{x_\lambda \prec A_1 \vee A_3} \bigwedge_{x_\lambda \not\leq D \geq A_1 \wedge A_2} (\mathcal{T}(D'))' \right) \\
& \vee \left(\mathcal{T}(A_2 \wedge A_3) \wedge \bigwedge_{x_\lambda \prec A_1 \vee A_3} \bigwedge_{x_\lambda \not\leq D \geq A_2 \wedge A_3} (\mathcal{T}(D'))' \right) \\
& \vee \left(\mathcal{T}(A_1 \wedge A_3) \wedge \bigwedge_{x_\lambda \prec A_1 \vee A_3} \bigwedge_{x_\lambda \not\leq D \geq A_1 \wedge A_3} (\mathcal{T}(D'))' \right) \\
& \vee \left(\mathcal{T}(A_1) \wedge \bigwedge_{x_\lambda \prec A_1 \vee A_3} \bigwedge_{x_\lambda \not\leq D \geq A_1} (\mathcal{T}(D'))' \right) \\
& \vee \left(\mathcal{T}(A_3) \wedge \bigwedge_{x_\lambda \prec A_1 \vee A_3} \bigwedge_{x_\lambda \not\leq D \geq A_3} (\mathcal{T}(D'))' \right) \\
& \vee \left(\mathcal{T}(A_1 \vee A_3) \wedge \bigwedge_{x_\lambda \prec A_1 \vee A_3} \bigwedge_{x_\lambda \not\leq D \geq A_1 \vee A_3} (\mathcal{T}(D'))' \right) \\
= & (0.8 \wedge 0.4) \vee (0.6 \wedge 0.4) \vee (0.6 \wedge 0.4) \vee (0.8 \wedge 0.9) \vee (0.6 \wedge 0.4) \vee (0.6 \wedge 1) \\
= & 0.8.
\end{aligned}
$$

Acknowledgements: The project is supported by the National Natural Science Foundation of China (11871097).

Conflict of Interest: The author declares that there are no conflict of interest.

References

[1] K. K. Azad, On fuzzy semicontinuity, fuzzy almost continuity and fuzzy weakly continuity, *J. Math. Anal. Appl.*, 82 (1981), 14-32.

[2] P. Dwinger, Characterizations of the complete homomorphic images of a completely distributive complete lattice I, *Indag. Math.,* 85 (1982), 403-414.

[3] A. Ghareeb, A new form of F-compactness in L-fuzzy topological spaces, *Math. Comput. Modelling,* 54 (2011), 2544-2550.

[4] A. Ghareeb, Preconnectedness degree of L-fuzzy topological spaces, *Int. J. Fuzzy Log. Intell. Syst.*, 11(1) (2011), 54-58.

[5] A. Ghareeb, L-fuzzy semi-preopen operator in L-fuzzy topological spaces, *Neural Comput & Applic*, 21 (2012), 87-92.

[6] A. Ghareeb and F.-G. Shi, SP-compactness and SP-connectedness degree in L-fuzzy pretopological spaces, *J. Intell. Fuzzy Syst.*, 31 (2016), 1435-1445.

[7] A. Ghareeb, W. F. Al-Omeri, New degrees for functions in (L, M)-fuzzy topological spaces based on (L, M)-fuzzy semiopen and (L, M)-fuzzy preopen operators, *J. Intell. Fuzzy Syst.*, 36 (2019), 787-803.

[8] M. H. Ghanim and O. A. Tantawy and F. M. Selin, Gradation of supra-openness, *Fuzzy Sets Syst.*, 109 (2000), 245-250.

[9] U. Höhle and S. E. Rodabaugh (Eds), Mathematics of Fuzzy Sets: Logic, Topology, and Measure Theory, *Kluwer Academic Publishers, Boston/ Dordrecht/ London*, (1999).

[10] H. L. Huang and F.-G. Shi, L-fuzzy numbers and their properties, *Inform Sci*, 178 (2008), 1141-1151.

[11] T. Kubiak, On fuzzy topologies, *Ph.D. Thesis, Adam Mickiewicz, Poznan, Poland*, (1985).

[12] S. J. Lee and E. P. Lee, Fuzzy r-continuous and fuzzy r-semicontinuous maps, *Int. J. Math. Math. Sci.*, 27 (2001), 53-63.

[13] S. J. Lee and E. P. Lee, Fuzzy r-preopen sets and fuzzy r-precontinuous maps, *Bull. Korean Math. Soc.*, 36 (1999), 91-108.

[14] S. J. Lee and E. P. Lee, Fuzzy r-regular open sets and fuzzy almost r-continuous maps, *Bull. Korean Math. Soc.*, 39 (2002), 441-453.

[15] N. Levine, Semi-open sets and semi-continuity in topological spaces, *Amer. Math. Monthly*, 70 (1963), 36-41.

[16] C. Y. Liang and F.-G. Shi, A new kind of measure of compactness in (L, M)-fuzzy supratopological spaces, *J. Intell. Fuzzy Syst.*, 33 (2017), 2109-2117.

[17] N. Nanda, Strongly compact fuzzy topological spaces, *Fuzzy Sets and Syst.*, 42 (1991), 259-262.

[18] F.-G. Shi, Theory of L_β-nested sets and L_α-nested sets and its applications, *Fuzzy Systems Math.*, 4 (1995), 65-72 (in Chinese).

[19] F.-G. Shi, L-fuzzy relations and L-fuzzy subgroups, *J Fuzzy Math.*, 8 (2000), 491-499.

[20] F.-G. Shi, L-fuzzy interiors and L-fuzzy closures, *Fuzzy Sets Syst.*, 160 (2009), 1218-1232.

[21] F.-G. Shi and C. Y. Liang, Measures of compactness in L-fuzzy pretopological spaces, *J. Intell. Fuzzy Syst.*, 26 (2014), 1557-1561.

[22] F.-G. Shi and B. Pang, Categories isomorphic to the category of L-fuzzy closure system spaces, *Iran. J. Fuzzy Syst.*, 10 (2013), 129-148.

[23] M. K. Singal and N. Prakash, Fuzzy preopen sets and fuzzy preseparation axioms, *Fuzzy Sets Syst.*, 44 (1991), 273-281.

[24] A. P. Šostak, On a fuzzy topological structure, *Rendiconti Circolo Matematico Palermo Suppl. Ser. II,* 11 (1985), 89-103.

[25] A. M. Zahran, Almost continuity and δ-continuity in fuzzifying topology, *Fuzzy Sets Syst.*, 116 (2000), 339-352.

[26] F. M. Zeyada and A. K. Mousa, Regular open sets in fuzzifying topology redefined, *J. Egyptian Math. Soc.*, 22 (2014), 484-488.

[27] J. Zhang, F.-G. Shi and C.-Y. Zheng, On L-fuzzy topological spaces, *Fuzzy Sets Syst.*, 149 (2005), 473-484.

Fu-Gui Shi
School of Mathematics and Statistics
Beijing Institute of Technology
Beijing, China
E-mail: fuguishi@bit.edu.cn

Transactions on Fuzzy Sets and Systems

URL: https://tfss.journals.iau.ir/

Online ISSN: 2821-0131

Vol.1, No.2, (2022), 21-31

DOI: http://doi.org/10.30495/tfss.2022.1958730.1030

Article Type: Original Research Article (Invited Paper)

Using Fuzzy C-means to Discover Concept-drift Patterns for Membership Functions

Tzung-Pei Hong*, **Chun-Hao Chen**, **Yan-Kang Li**, **Min-Thai Wu**

Abstract. People often change their minds at different times and at different places. It is important and valuable to indicate concept-drift patterns in unexpected ways for shopping behaviours for commercial applications. Research about concept drift has been growing in recent years. Many algorithms dealt with concept-drift information and detected new market trends. This paper proposes an approach based on fuzzy c-means (FCM) to mine the concept drift of fuzzy membership functions. The proposed algorithm is subdivided into two stages. In the first stage, individual fuzzy membership functions are generated from different training databases by the proposed FCM-based approach. Then, the proposed algorithm will mine the concept-drift patterns from the sets of fuzzy membership functions in the second stage. Experiments on simulated datasets were also conducted to show the effectiveness of the approach.

AMS Subject Classification 2020: 03E72; 68T30

Keywords and Phrases: Concept drift, Data mining, Fuzzy c-means, Membership function.

1 Introduction

Data processing and storage are now more readily available than ever because of the booming development of information technologies. If policy-makers can obtain information and knowledge from databases effectively and quickly, they can make better decisions. However, along with the growing number of database types, getting useful and valuable information from large databases for decision-making is difficult and important [4, 27].

Fuzzy data mining is attracting much research interest these years. In fuzzy data mining, membership functions are given and used to extract fuzzy association rules represented by linguistic terms from quantitative data [3, 8, 9, 11, 26]. Therefore, fuzzy membership functions play a crucial role in affecting the quality of the mining results. In literature, in addition to using static or manually defined membership functions, meta-heuristic methods are utilized to find appropriate membership functions [7, 24]. For example, Hong et al. proposed a genetic-fuzzy mining approach to extract fuzzy association rules with the derived membership functions from given quantitative transactions [7]. Yang et al. proposed a method to generate fuzzy membership functions using unsupervised learning of a self-organizing feature map [24].

This paper discusses the concept-drift issue of membership functions. We present a detection algorithm for concept drift of membership functions. Firstly, each item in the database was divided into several linguistic

*__Corresponding author:__ Tzung-Pei Hong, Email: tphong@nuk.edu.tw, ORCID: 0000-0001-7305-6492
Received: 14 May 2022; Revised: 12 June 2022; Accepted: 15 June 2022; Published Online: 7 November 2022.

How to cite: T. P. Hong, C. H. Chen, Y. K. Li and M. T. Wu, Using Fuzzy C-means to Discover Concept-drift Patterns for Membership Functions, *Trans. Fuzzy Sets Syst.*, 1(2) (2022), 21-31.

terms, such as low, medium, and high. The proposed FCM-based approach is then used to generate a set of relevant membership functions for each item in two given transactional databases that could be collected at different times or places. Finally, a comparison algorithm is used to obtain the concept-drift patterns for these membership functions in these two databases by a predefined threshold.

The rest of this paper is organized as follows. Section 2 reviews some related researches. Section 3 states three types of concept drift of membership functions. Section 4 describes the proposed algorithm. Section 5 shows the experimental results and analysis. At last, the conclusion and future work are given in Section 6.

2 Related Works

In this section, we review some related research in regards to concept drift, fuzzy data mining, FCM, and membership functions.

2.1 Concept Drift

The research of concept drift has become popular in recent years [4, 22, 27]. Tsymbal proposed the concept of drift as finding patterns which change over time in unexpected ways [22]. For example, assume at time t there is an association rule of "if buying milk, then buying bread" and at time $t + k$ there is another rule "if buying milk, then buying apple". For the two rules, the latter changes from the former in the consequence part along with time. The change is a type of concept-drift pattern.

The traditional methods of data mining have been used in various research areas based on the concept-drift patterns [5, 13, 20, 23]. For instance, in intrusion detection systems, Mukkavilli et al. designed an approach for detecting network attacks [13]. Hayat et al. utilized a language model to conduct concept-drift detection in junk mail filtering [5]. Lee et al. designed a rule-based model using the concept of decision trees to extract the concept-drift rules [12]. In addition, the concept of drift was often applied to classification and data stream [2, 6, 14, 16, 17, 20, 21].

Song et al. defined three types of concept-drift patterns, which could be focused in association-rule mining [18]. They are emerging patterns, unexpected changes, and added/perished patterns. Different types of concept-drift patterns indicate different meanings of concept drift for association rules. An evaluative function was then designed to calculate the degree of concept drift. If the degree between two rules is bigger than a predefined threshold, then a concept-drift pattern is generated. Hong et al. then generalized it and proposed fuzzy concept-drift patterns of association rules [10].

2.2 Fuzzy Data Mining

The fuzzy-set theory has been used in intelligent systems because of its simplicity and similarity to human reasoning [25]. The theory has been applied in fields such as manufacturing, engineering, diagnosis, and economics, among others. Many fuzzy data mining approaches have been proposed to solve real problems [8, 9, 11].

Srikant et al. proposed a mining method to handle quantitative transactions by partitioning the values of each attribute [19]. Hong et al. also proposed a fuzzy mining algorithm to mine fuzzy association rules from quantitative transaction data [8]. The fuzzy mining algorithm first used given membership functions to transform each quantitative value into a fuzzy set of linguistic terms and then used a fuzzy mining process to find fuzzy association rules. To handle the time-consuming mining task from a very large database, Fernandez-Basso et al. presented three Spark approaches to extract interesting fuzzy association rules from massive fuzzy data [3]. Based on the existing association rule mining algorithms of Apriori, Apriori-TID, and ECLAT, the map-reduced framework was utilized to speed up the data processing in mining fuzzy association rules. Besides, Zhang et al. proposed a differential evolution (DE) algorithm to extract important fuzzy association rules and reduce spurious rules [26]. They also indicated that the rules extracted were better than those by non-evolutionary and genetic algorithms. They also gave a case study to show the DE-based approach was effective in practice.

2.3 Fuzzy C-means

FCM is a popular method for clustering. It follows the fuzzy-set theory and allows one piece of data to belong to two or more clusters [1]. It is frequently used in pattern recognition. The following objective function is minimized to get good clusters:

$$\sum_{j=1}^{c}\sum_{i=1}^{n}(u_{ij})^m \left\|c_j - x_i\right\|^2, \tag{1}$$

where m is an arbitrary real number greater than 1, x_i is the i-th data, c_j is the center of the j-th cluster, u_{ij} is the membership degree of x_i in the cluster j, and $|| * ||$ is a norm expressing the similarity between a data and a center. The Euclidean distance is commonly used to calculate the norm.

FCM adopts an iterative process to minimize the above function. The process will stop when

$$\max_{i,j}\left\|u_{ij}^{k-1} - u_{ij}^{k}\right\| < \beta, \tag{2}$$

where β is a termination criterion between 0 and 1 and k is an iteration number. The algorithm of FCM is shown below.

The FCM algorithm

Step1: Initialize the U matrix, $U^{(0)}$, where U represents all the u_{ij} values.

Step2: At each k-th iteration, calculate the center of each cluster and the membership function of each data to each cluster by the following two formulas:

$$c_j = \frac{\sum_{i=1}^{n}(u_{ij})^m \times x_i}{\sum_{i=1}^{n}(u_{ij})^m}, \text{and} \tag{3}$$

$$u_{ij} = \frac{1}{\sum_{k=1}^{c}\frac{\|x_i - c_j\|}{\|x_i - c_k\|}^{(\frac{2}{m-1})}}. \tag{4}$$

Step3: If formula (2) is reached, then stop; otherwise return to Step 2.

2.4 Fuzzy Data Mining

In this paper, we use the isosceles-triangle membership functions to represent the fuzzy regions [15] for simplicity. Isosceles-triangle membership functions are shown in Figure 1. The membership function of each fuzzy region R_{jk} is represented by a (c, w) pair, where c denotes the center abscissa and w represents half the span.

3 Concept Drift of Membership Functions

In this section, we present the concept-drift process of membership functions.

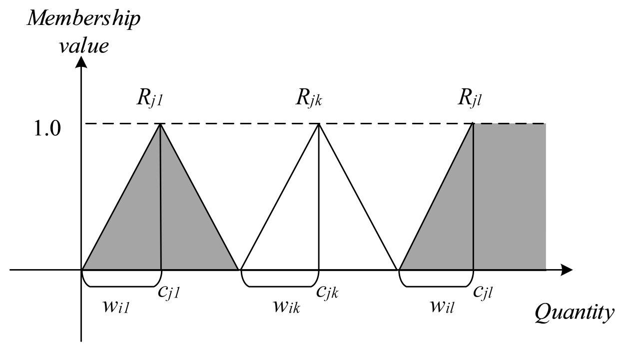


Figure 1: Representation of Membership functions.

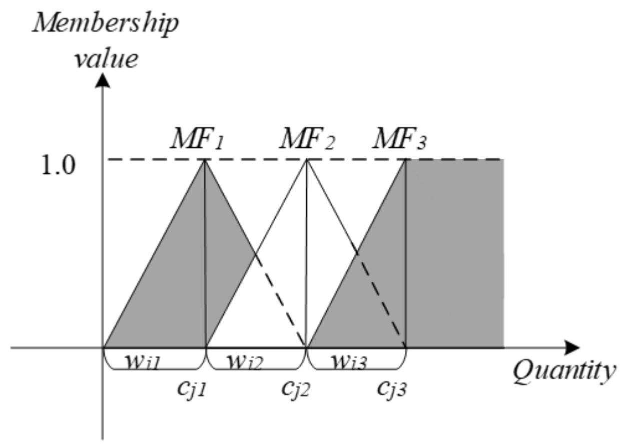


Figure 2: Membership functions of an item I_j.

3.1 Generating Membership Functions by Fuzzy C-means

In this paper, we propose a simple approach based on FCM to generate a set of membership functions for an item. As mentioned above, a membership function is designed as an isosceles triangle and encoded as a pair of (c, w). The peak of the triangle is located at c, and the distance between the peak and the left endpoint is w. If we need to generate n membership functions for an item, the proposed algorithm will obtain n cluster centers by using FCM. Each center obtained is the c value of the corresponding membership function. Then the span w is calculated as the distance between the location of the peak in this triangle with the previous one with the first one is the distance between the locations of the peak with 0. Figure 2 shows an example of three membership functions for an item I_j.

The membership functions play a critical role in converting commodity items into human semantics. Figure 3 shows a membership-function set for apples purchased in a transaction. It consists of three membership functions representing low, medium, and high, respectively, for each purchased amount. If we buy five apples, the low fuzzy value is 0.4, the medium fuzzy value is 0.6, and the high fuzzy value is 0.

Additionally, we may know the status of the concept from the membership functions. In Figure 3, the purchased amounts of three, six, and nine reach the membership value of 1 for the three membership functions, respectively. We can regard these amounts as the representative of the linguistic terms and observe the changes at different times.

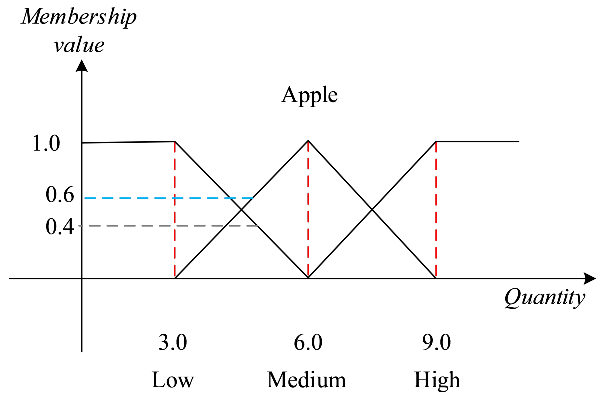


Figure 3: An example of membership functions for purchased apple amounts.

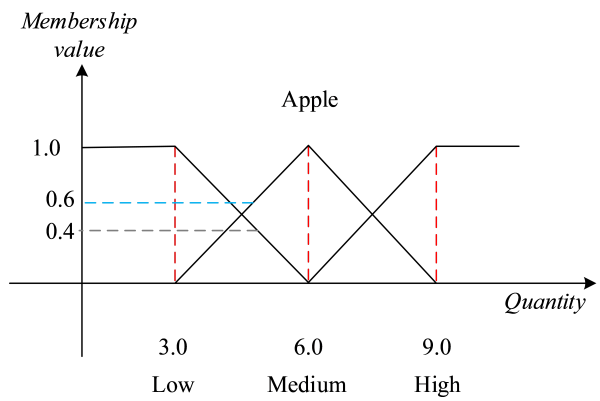


Figure 4: Changed membership functions for the purchased number of apples.

3.2 Concept-drift Patterns of Membership Functions

We proposed three types of concept-drift patterns of membership functions. The first type is the change of the representative value (the center) of a linguistic term (membership function), the second type is the change of a linguistic-term span, and the third type is the change of the fuzzy support for a linguistic term. They are described below.

(A) The concept drift of the representative value for a linguistic term

Figure 3 shows the membership functions of the purchased number of apples derived in the last year, and Figure 4 shows those derived in this year. In the membership function for the linguistic term of low, the representative value, which is the center of the membership function, reduces from three to two. On the contrary, in the membership function for the linguistic term of high, the representative value increases from nine to ten. This represents the concepts of the low and the high linguistic terms have already changed.

The concept-drift degree of the representative value of a linguistic term, denoted $cdLT$, is thus shown below:

$$cdLT = \frac{|c_{ji}^{D'} - c_{ji}^{D}|}{w_{ji}^{D}}, \tag{5}$$

where D and D' are the initial and the final transaction databases at different times or different places, c_{ji}

and w_{ji} are the center and the span values of the i-th linguistic term for the j-th commodity item. When the degree is larger than or equal to a given threshold, we may say it has a concept drift of a center.

As an alternative, we may also consider the average span as the denominator and set the formula below:

$$cdLT = \frac{|c_{ji}^{D'} - c_{ji}^{D}|}{\sum_{k=1}^{N} w_{jk}^{D}/N}, \tag{6}$$

where w_{jk} is the span value of the k-th linguistic term for the j-th commodity item, and N is the number of linguistic terms.

(B) The concept drift of the span value for a linguistic term

The meaning of the span of a membership function is the coverage of a linguistic term on data. For example, although the representative value of the medium linguistic term is not changed in Figures 3 and 4, the span of the membership function from the modified database is larger than that from the original database. The concept-drift degree of the span value of a linguistic term, denoted $cdMF$, is thus designed below:

$$cdMF = \frac{|w_{ji}^{D'} - w_{ji}^{D}|}{(c_{jN}^{D} - c_{j1}^{D})/(N-1)}, \tag{7}$$

where D and D' are the initial and the final transaction databases at different times or different places, w_{ji} is the span values of the i-th linguistic term for the j-th commodity item, c_{j1} and c_{jN} are the first and the last center values of the j-th commodity item, and N is the number of linguistic terms. When the degree is larger than or equal to a given threshold, we may say it has a concept drift of a span.

(C) The concept drift of the fuzzy support for a linguistic term

A concept drift in fuzzy support represents a group size changes for a membership function. We can use this value to measure concept drift. An example, may be the number of people that buy expensive mobile phones this year is greater than that in last year. We may use the following formula to evaluate the concept-drift degree of the fuzzy support value of a linguistic term, denoted $cdSUP$:

$$cdSUP = \frac{|sup_{ji}^{D'} - sup_{ji}^{D}|}{sup_{ji}^{D}}, \tag{8}$$

where D and D' are the initial and the final transaction databases at different times or different places, sup_{ji} is the span values of the i-th linguistic term for the j-th commodity item. When the degree is larger than or equal to a given threshold, we may say it has a concept drift of a fuzzy support. It can usually be used to represent the concept drift of customer purchase behaviour.

As an alternative, we may also consider the average support as the denominator and set the formula below:

$$cdSUP = \frac{|sup_{ji}^{D'} - sup_{ji}^{D}|}{\sum_{k=1}^{N} sup_{jk}^{D}/N}, \tag{9}$$

where N is the number of linguistic terms.

4 The Proposed Algorithm

In this section, the proposed approach that combines concept-drift and FCM is described. The algorithm is stated as follows.

The algorithm for finding the concept drift of membership functions

Input: D and D': databases; I: the number of items; S: concept-drift rule sets; M: the number of linguistic terms; α, β, γ: the thresholds for judging the concept drift of centers, spans and fuzzy supports of membership functions, respectively.

Output: The concept-drift patterns of membership functions.

Method:

Step1: Generate membership functions for each item from D and D' by the following substeps.

(a) Use the FCM algorithm to find the center values of the N clusters of each item respectively for the two databases, D and D'.

(b) Set the center points of these N clusters for each item as the centers of the membership functions.

(c) Calculate the distances of all two neighbouring centers and set them as the spans of the membership functions.

Step2: Set the initial concept-drift pattern set S as $\varnothing$.
Step3: Find the three types of concept-drift patterns of membership functions of each item between D and D' by the following substeps.

(a) Calculate the concept-drift degree ($cdLT$) of the representative value of the linguistic term and compare it with the α value. If the value of $cdLT$ is larger than or equal to α, then put the center drift pattern in S.

(b) Calculate the concept-drift degree ($cdMF$) of the span value of the linguistic term and compare it with the β value. If the value of $cdMF$ is larger than or equal to β, then put the span drift pattern in S.

(c) Calculate the concept-drift degree ($cdSUP$) of the fuzzy support value of the linguistic term and compare it with the γ value. If the value of $cdSUP$ is larger than or equal to γ, then put the support drift pattern in S.

Step4: After all the items are processed, output the concept-drift set S.

5 Experimental Results

In this section, we describe the experimental results of the concept drift of membership functions. We used a computer with an Intel Core $i5-3230M$ 2.60GHz processor with four cores, four threads, and 12 GB RAM. The operating system used was Microsoft Windows 8.1 Pro and the programming language was .NET Framework 4.5.1 C# (C# Version 5.0).

A simulated retail dataset containing $1,559$ items and $21,556$ transactions was used in the experiments. In the dataset, the number of purchased items in transactions was first randomly generated, and the purchased items and their quantities in each transaction were then generated. Each transaction was also assigned a date and a location in one year.

Table 1: The numbers of concept-drift patterns of membership functions.

	Center Drift	Span Drift	Support Drift
Case 1	1	12	52
Case 2	9	39	234
Case 3	112	236	441
Case 4	40	112	274

The cluster size was set at 3, the fuzziness index value m in FCM was set at 2, the threshold values of α, β, and γ were all set at 1. We generate the following four cases to verify the concept drift of membership functions:

Case 1. Data from two different locations are selected to form the original and drifted databases.

Case 2. Data from the first and the second half of the dataset are selected to form the original and drifted databases.

Case 3. Data from two arbitrary months are selected to form the original and drifted databases.

Case 4. The whole data set is used as the original dataset and the data from an arbitrary month is selected to form the drifted database.

Table 1 shows the experimental concept-drift results of the proposed approach. In the experimental results, we can find the number of drift patterns at different times was larger than that at different locations. The short-term databases may contain more drifted patterns, so when we compared the membership functions from two short-term databases, more concept-drifts could be found. In the contrast, since the long-term databases tended to be stable, less concept-drifts will occur. As a result, for the simulated database, the comparison between short-term databases is more preferable. The experimental results with alternative formulas for concept-drift degrees are similar.

6 Conclusion and Future Work

In this paper, we have described a simple approach to test the concept drift of membership functions. We have proposed three types of concept drifts of membership functions and designed formulas to evaluate them. We have also implemented the approach based on fuzzy c-means (FCM) and the designed formulas. Experiments on the simulated retail dataset have also been made to show the effectiveness of the proposed approach. In particular, the proposed method can help shop managers understand customer behaviour drift, analyzed from membership functions, in different times and places. In the future, we will try to design more effective ways to decrease computing time and combine the proposed concept-drift patterns with fuzzy association rules. We will also conduct more experiments to verify the approach.

Acknowledgements: This research was supported by the Ministry of Science and Technology of the Republic of China under grant MOST 109-2221-E-390-015-MY3.

Conflict of Interest: The authors declare that there are no conflict of interest.

References

[1] J. Bezdek, Pattern Recognition With Fuzzy Objective Function Algorithms, *Springer, New York*, (1981).

[2] P. B. Dongre and L. G. Malik, A review on real time data stream classification and adapting to various concept drift scenarios, *IEEE International Advance Computing Conference, ITM University, 21-22 Feb., Gurgaon, India*, (2014), 533-537.

[3] C. Fernandez-Basso, M. Ruiz and M. Martin-Bautista, Spark solutions for discovering fuzzy association rules in Big Data, *International Journal of Approximate Reasoning*, 137(07) (2021), 94-112.

[4] H. Guo, H. Lia, Q. Rena and W. Wang, Concept drift type identification based on multi-sliding windows, *Information Sciences*, (585) (2022), 1-23

[5] M. Z. Hayat, J. Basiri, L. Seyedhossein and A. Shakery, Content-based concept drift detection for Email spam filtering, *2010 5th International Symposium on Telecommunications, Iran Telecom Research Center, 4-6 Dec., Kish Island, Iran*, (2010), 531-536.

[6] M. Z. Hayat, M. R. Hashemi, A DCT based approach for detecting novelty and concept drift in data streams, *2010 International Conference of Soft Computing and Pattern Recognition, Universite de Cergy-Pontoise, 7-10 Dec., Paris, France*, (2010), 373-378.

[7] T. P. Hong, C. H. Chen, Y. L. Wu and Y. C. Lee, A GA-Based Fuzzy Mining Approach to Achieve a Trade-off Between Number of Rules and Suitability of Membership Functions, *Soft Computing*, 10(11) (2006), 10911101.

[8] T. P. Hong, C. S. Kuo and S. C. Chi, Mining association rules from quantitative data, *Intelligent Data Analysis*, 3(5) (1999), 363-376.

[9] T. P. Hong, K. Y. Lin and S. L. Wang, Fuzzy data mining for interesting generalized association rules, *Fuzzy Sets and Systems*, 138(09) (2003), 255-269.

[10] T. P. Hong, J. M. T. Wu, Y. K. Li, and C. H. Chen, Generalizing Concept-Drift Patterns for Fuzzy Association Rules, *Journal of Network Intelligence*, 3(2) (2018), 126-137.

[11] E. Hllermeier, Fuzzy sets in machine learning and data mining, *Applied Soft Computing*, 11(03) (2011), 1493-1505.

[12] C. I. Lee, C. J. Tsai, J. H. Wu and W. P. Yang, A Decision Tree-Based Approach to Mining the Rules of Concept Drift, *Fourth International Conference on Fuzzy Systems and Knowledge Discovery Hainan University*, 24-27 Aug., Haikou, China, (2007), 639-643.

[13] S. Mukkavilli and S. Shetty, Mining Concept Drifting Network Traffic in Cloud Computing Environments, *12th IEEE/ACM International Symposium on Cluster, Cloud and Grid Computing, Carleton University*, 13-16 May, Ottawa, ON, Canada, 05 (2012), 721-722.

[14] E. Padmalatha, C. R. K. Reddy and B. P. Rani, Classification of Concept Drift Data Streams, *International Conference on Information Science Applications, Kyonggi University*, 6-9 May, Seoul, Korea, (2014), 1-5.

[15] A. M. Parodi and P. Bonelli, A New Approach to Fuzzy Classifier Systems, *Proceedings of the 5th International Conference on Genetic Algorithms, University of Illinois Urbana-Champaign*, 1 June, IL, USA, (1993), 223-230.

[16] P. D. Patil and P. Kulkarni, Adaptive Supervised Learning Model for Training Set Selection under Concept Drift Data Streams, *2013 International Conference on Cloud Ubiquitous Computing Emerging Technologies, Sri Venkateshwara College of Engineering*, 15-16 Nov., Pune, India, (2013) 36-41.

[17] S. Shetty, S.K. Mukkavilli and L. H. Keel, An integrated machine learning and control theoretic model for mining concept-drifting data streams, *IEEE International Conference on Technologies for Homeland Security*, 15-17 Nov., Boston, USA, (2011), 75-80.

[18] H. S. Song, J.K. Kim and S. Kim, Mining the change of customer behaviour in an Internet shopping mall, *Expert Systems with Applications*, 21(10) (2001), 157-168.

[19] R. Srikant and R. Agrawal, Mining Quantitative Association Rules in Large Relational Tables, *ACM Special Interest Group on Management of Data*, 25(2) (1996), 1-12.

[20] J. Sun, H. Li and H. Adeli, Concept Drift-Oriented Adaptive and Dynamic Support Vector Machine Ensemble With Time Window in Corporate Financial Risk Prediction, *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 43(4) (2013), 801-813.

[21] B. Thuraisingham, Data mining for security applications: Mining concept-drifting data streams to detect peer to peer botnet traffic, *IEEE International Conference on Intelligence and Security Informatics*, 17-20 June, Taipei, Taiwan, (2008), 29-30.

[22] A. Tsymbal, The Problem of Concept Drift: Definitions and Related Work, *Technical Report*, (2004).

[23] D. H. Widyantoro and J. Yen, Relevant data expansion for learning concept drift from sparsely labeled data, *IEEE Transactions on Knowledge and Data Engineering*, 17(3) (2005), 401-412.

[24] C. C. Yang and N.K. Bose, Generating fuzzy membership function with self-organizing feature map, *Pattern Recognition Letters*, 27(5) (2006), 356-365.

[25] L. A. Zadeh, Fuzzy sets, *Information and Control*, 8(3) (1965), 338-353.

[26] A. Zhang and W. Shi, Mining significant fuzzy association rules with differential evolution algorithm, *Applied Soft Computing*, 97 (2020), 105518.

[27] X. Zheng, P. Li, X. Hu and K. Yu, Semi-supervised classification on data streams with recurring concept drift and concept evolution, *Knowledge-Based Systems*, 215(01) (2021), 106749.

Tzung-Pei Hong
Department of Computer Science and Engineering
National Sun Yat-sen University
Kaohsiung, Taiwan
E-mail: tphong@nuk.edu.tw

Chun-Hao Chen
Department of Information and Finance Management
National Taipei University of Technology
Taipei, Taiwan
E-mail: chchen@ntut.edu.tw

Yan-Kang Li
Department of Computer Science and Information Engineering
National University of Kaohsiung
Kaohsiung, Taiwan
E-mail: m1025506@mail.nuk.edu.tw

Min-Thai Wu

College of Computer Science and Engineering
Shandong University of Science and Technology
Shandong, China
E-mail: wmt@wmt35.idv.tw

Transactions on Fuzzy Sets and Systems
URL: https://tfss.journals.iau.ir/
Online ISSN: 2821-0131
Vol.1, No.2, (2022), 32-36
DOI: http://doi.org/10.30495/tfss.2022.1955757.1020

Article Type: Original Research Article

Characterization of Topological Fuzzy Sets in Hausdorff Spaces

Benard Okelo*, **Allan Onyango**

Abstract. In this paper, we have characterized big data fuzzy sets and shown that topological data points form singleton fuzzy sets which are closed. Besides, fuzzy sets of topological data points are compact and have at least one closed point. We have also shown that the fuzzy set of all condensation points of a fuzzy Hausdorff space is infinite and the cardinality of a topological data fuzzy set is also infinite and arbitrarily distributed in fuzzy Hausdorff spaces.

AMS Subject Classification 2020: 03E72; 94D05
Keywords and Phrases: Fuzzy set, Fuzzy topological space, Fuzzy Hausdorff space, Topological Data point.

1 Introduction

Studies on fuzzy sets have been carried out over a long period of time particularly in topological spaces with interesting results obtained and open problems indicated (see [1]-[16] and the references therein). From the beginning, fuzzy set theory has gained a lot of advancement in a variety of ways and in several fields [5]. Nice applications of fuzzy set theory have been seen in several disciplines like artificial intelligence, topological spaces, computer engineering, medical engineering, control and instrumentation engineering, risk theory, game theory, decision theory, expert systems, logical functions analysis, management systems science, operations research, face and pattern recognition among others [8]. With respect to mathematical developments, fuzzy set theory has led to a very high level of improvement in modern research with applications to real life problems [13]. This work describes the the pertinent logical framework of fuzzy set theory, together with very important significance of this theory to other methods and theories. Since the beginning of this area of study [15] by Lotfi Zadeh in 1965, several aspects have been considered in the study of fuzzy sets. These include: The intuitionist case [2], the empty set[3], singleton fuzzy sets among others[4]-[9]. These aspects have been utilized in several areas like logic [10]-[13], programming and decision making particularly in optimization and profit making in the business sector [14]. In this work, we consider fuzzy sets in topological spaces [8], particularly the Hausdorff space. We take advantage of the fact that for any two fuzzy sets say A, B in a Hausdorff space, $A \cap B = \emptyset$. This helps in classification of the fuzzy sets into different classes without overlaps. From early 90's, fuzzy set theory, neural circuits and programming of evolution acquired the title computational intelligence also known as soft computing [16]. There exists a very important relationship between these areas making them to be naturally equivalent in some sense. In this study, however, we particularly embark primarily on fuzzy Hausdorff spaces with applications to real life problems which are indispensable. For better under standing of this work, we give some preliminary notes which are very instrumental in the next section.

∗**Corresponding author:** Benard Okelo, Email: bnyaare@yahoo.com, ORCID: 0000-0003-3963-1910
Received: 31 March 2022; Revised: 22 May 2022; Accepted: 23 May 2022; Published Online: 7 November 2022.

How to cite: B. Okelo and A. Onyango, Characterization of Topological Fuzzy Sets in Hausdorff Spaces, *Trans. Fuzzy Sets Syst.*, 1(2) (2022), 32-36.

2 Preliminaries

We provide basic concepts which are useful in the sequel.

Definition 2.1. ([16], Definition 1) *Let X is a collection of objects denoted generically by x, then a fuzzy set $\tilde{A}$ in X is a set of ordered pairs: $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$. $\mu_{\tilde{A}}(x)$ is called the membership function (generalized characteristic function) which maps X to the membership space M. Its range is the subset of nonnegative real numbers whose supremum is finite. For $\sup \mu_{\tilde{A}}(x) = 1$ we have a normalized fuzzy set.*

Remark 2.2. *In Definition 2.1, the membership function of the fuzzy set is a crisp (real-valued) function. Zadeh [15] also defined fuzzy sets in which the membership functions themselves are fuzzy sets.*

Definition 2.3. ([17], Definition 3.2) *A type m fuzzy set is a fuzzy set whose membership values are type $m-1, m>1$, fuzzy sets on $[0,1]$.*

Remark 2.4. *For operations on fuzzy set see [16] and the references therein.*

Definition 2.5. ([6], Definition 2.3) *Let X be a fuzzy topological space and $\mathbf{H}$ be a nonempty fuzzy compact Hausdorff subspace of X. A point $\mathrm{a} \in \mathbf{H}$ is called a topological data point (TDP) if $\mathbf{H}^c \setminus \{\mathrm{a}\}$ is a compact fuzzy subspace of $\mathbf{H}$. The set of all topological data points is called a Topological Data Set (TDS). If this set is fuzzy then we call it a Fuzzy Topological Data Set (FTDS).*

Definition 2.6. ([11], Definition 1.5) *A nonempty fuzzy compact Hausdorff space $\mathbf{H}$ is called a TDP fuzzy space if every $\mathrm{a} \in \mathbf{H}$ is a TDP.*

Remark 2.7. *Let $\mathbf{H}$ be a fuzzy topological space. Then $\mathbf{H} = \mathbf{P} \dagger \mathbf{Q}$ means $\mathbf{P}$ and $\mathbf{Q}$ are nonempty fuzzy subsets of $\mathbf{H}$ such that $\mathbf{H} = \mathbf{P} \cup \mathbf{Q}$ and $\mathbf{P} \cap \bar{\mathbf{Q}} = \bar{\mathbf{P}} \cap \mathbf{Q} = \emptyset$.*

3 Topological data sets in fuzzy Hausdorff spaces

In this section, we characterize Topological Data Points in a fuzzy Hausdorff space. We begin with the following proposition.

Proposition 3.1. *Let $\mathbf{H}$ be a TDP fuzzy space and $\mathrm{a} \in \mathbf{H}$ such that $\mathbf{H}^c \setminus \{\mathrm{a}\} = \mathbf{P} \dagger \mathbf{Q}$. If $\{\mathrm{a}\}$ is open then $\mathbf{P}$ and $\mathbf{Q}$ are closed and if $\{\mathrm{a}\}$ is closed, then $\mathbf{P}$ and $\mathbf{Q}$ are open.*

Proof. Let $a \in \mathbf{H}$ be open and suppose that $\mathbf{P}$ is both open and closed in $\mathbf{H}^c \setminus \{a\}$. Without loss of generality, there exists a closed subset $\mathbf{R}$ of $\mathbf{H}$ such that $\mathbf{P} = \mathbf{R} \cap (\mathbf{H}^c \setminus \{a\}) = \mathbf{R}^c \setminus \{a\}$. Hence, $\mathbf{H}^c \setminus \{a\} = \mathbf{P} \dagger \mathbf{Q}$ meaning $\mathbf{Q}$ of $\mathbf{H}$ is closed and so is $\mathbf{P}$. Conversely, let $a \in \mathbf{H}$ be closed. Putting the same argument as the forward case, there exists an open set $\mathbf{Z}$ of $\mathbf{H}$ in which $\mathbf{P} = \mathbf{Z} \cap (\mathbf{H}^c \setminus \{a\}) = \mathbf{Z}^c \setminus \{a\}$. Therefore, $\mathbf{P} = \mathbf{R}^c \setminus \{a\} = \mathbf{Z}^c \setminus \{a\}$. Hence, $\mathbf{H}^c \setminus \{a\} = \mathbf{P} \dagger \mathbf{Q}$ meaning $\mathbf{Q}$ of $\mathbf{H}$ is open and so is $\mathbf{P}$. This completes the proof. □

This proposition leads to characterization of topological Data points in terms of compactness.

Lemma 3.2. *Let $\mathbf{H}$ be a TDP fuzzy space and $\mathrm{a} \in \mathbf{H}$. If $\mathbf{H}^c \setminus \{\mathrm{a}\} = \mathbf{P} \dagger \mathbf{Q}$ then $\mathbf{P} \cup \{\mathrm{a}\}$ is compact.*

Proof. Without loss of generality, let $\mathbf{W}$ and $\mathbf{V}$ be connected fuzzy subsets of $\mathbf{H}$ in which $\mathbf{P} \cup \{a\} = \mathbf{W} \dagger \mathbf{V}$. Let $a \in \mathbf{W}$. Then $\mathbf{V} \subseteq \mathbf{P}$. Now $(\overline{\mathbf{V}} \cup \overline{\mathbf{W}}) \cap \mathbf{V} = (\overline{\mathbf{Q}} \cap \mathbf{V}) \cup (\overline{\mathbf{W}} \cap \mathbf{V}) = \emptyset$. So $(\overline{\mathbf{V}} \cup \overline{\mathbf{W}}) \cap \mathbf{V} = \emptyset$ and consequently, $(\mathbf{Q} \cap \mathbf{W}) \cap \overline{\mathbf{V}} = \emptyset$ implying $\mathbf{H} = (\mathbf{Q} \cap \mathbf{W}) \dagger \mathbf{V}$. □

Example 3.3. Consider the points on n straight lines in the Euclidean plane with standard topology R^2. The union of these n straight lines is a compact TDP fuzzy space if and only if either all of them are concurrent or exactly $n-1$ of them are parallel.

Theorem 3.4. *Let* $\mathbf{H}$ *be a TDP fuzzy space and* $\mathrm{a} \in \mathbf{H}$. *If* $\mathbf{H}^c \setminus \{\mathrm{a}\} = \mathbf{P} \dagger \mathbf{Q}$ *and if every point of* $\mathbf{P}$ *is TDP in* $\mathbf{H}$ *then* $\mathbf{P}$ *has at least one closed point.*

Proof. Suppose that $\mathbf{P}$ is compact then by Proposition 3.1, $\mathbf{P} \cup \{a\}$ is compact. So $\{a\}$ is closed. Let $z_0 \in \mathbf{P}$. By Lemma 3.2, $\mathbf{H}^c \setminus \{z_0\} = \bigcup_{z=\mathbf{P}, z \neq z_0} \{\{a, z\} \cup (\mathbf{Q} \cup \{a\})\}$ is also compact. This contradicts the earlier hypothesis that a is TDP point of $\mathbf{H}$. □

Example 3.5. Consider the Euclidean plane with standard topology R^2. Let $X_0 = \{(x, 0) \in R^2 : x \leq 0\} \cup \{(x, 1) \in R^2 : x > 0\}$ and let for each positive integer n, $Y_n = \{(\frac{1}{n}, y) \in R^2 : 0 < y \leq 1\}$. Define $K = X_0 \cup (\cup_{n=1}^{\infty} Y_n)$. Then X is a TDP fuzzy space with at least one closed point.

Remark 3.6. *All TDP fuzzy spaces are connected spaces. However, a finite fuzzy topological space is not a TDP.*

Example 3.7. Consider the Khalimsky line given as follows. Let Z be the set of integers and let $D = \{\{2i-1, 2i, 2i+1\} : i \in Z\} \cup \{\{2i+1\} : i \in Z\}$. Then D is a base topology for Z. The set Z with this topology is a TDP fuzzy space which is connected.

At this juncture, we locate Topological Data points of Big Data fuzzy Sets in a fuzzy Hausdorff space. We state the following proposition.

Proposition 3.8. *Let* $\mathbf{H}$ *be a TDP fuzzy space. The set* $\mathbf{A}_0$ *of all condensation points of* $\mathbf{H}$ *is a fuzzy TDS which is infinite.*

Proof. Let $a_1, a_2, ...$ be a sequence of distinct condensation points in $\mathbf{H}$. By induction, we have a condensation point a_0 in $\mathbf{A}_0 \subseteq \mathbf{H}$. But a_0 is a TDP of $\mathbf{H}$. So we have open TDS $\mathbf{W}_1$ and $\mathbf{V}_1$ of $\mathbf{H}$ such that $\mathbf{H}^c \setminus \{a_1\} = \mathbf{W}_1 \dagger \mathbf{V}_1$. Suppose that $a_1, a_2, ..., a_n$ are in $\mathbf{H}$ and open subsets $\mathbf{W}_i$ and $\mathbf{V}_i (i \in \mathbf{N})$ are picked such that $\mathbf{H}^c \setminus \{a_1\} = \mathbf{W}_i \dagger \mathbf{V}_i$, where $i = 1, ..., n$. Clearly, by induction and considering $\mathbf{W}_{i+1}$ and $\mathbf{V}_{i+1}$, the set $\mathbf{A}_0$ of all condensation points of $\mathbf{H}$ is infinite. □

The above Proposition 3.8 takes us to characterization of the size of the fuzzy sets. We give the size of the fuzzy data set in the next lemma.

Corollary 3.9. *Let* X *be a fuzzy topological space and* $\mathbf{H}$ *be a TDP fuzzy subspace of* X. *Then* $Card\mathbf{H} = \infty$.

Proof. By Hausdorff Maximal Principle (HMP) and by Proposition 3.8, the proof is complete. □

Next, we establish the distribution patterns of the topological Data Points within a fuzzy Hausdorff space in the following theorem.

Theorem 3.10. *All TDS in TDP fuzzy space are arbitrarily distributed if they are* $\boldsymbol{T}_2$. *Moreover, each FTDS has at least two TDPs with closed subsets of FTDS which are singletons.*

Proof. Let $\mathbf{H}$ be a TDP fuzzy space with two subsets $\mathbf{H}_1$ and $\mathbf{H}_2$. Let $\mathbf{H}_1$ and $\mathbf{H}_2$ be TDP fuzzy subspaces of $\mathbf{H}$. Then it implies that if $\mathbf{H}_1$ and $\mathbf{H}_2$ are both empty then trivially we are done. Let $\mathbf{H}_1$ and $\mathbf{H}_2$ be non-empty. It remains to show that $\mathbf{H}_1 \cap \mathbf{H}_2 = \emptyset$ and hence it is $\mathbf{T}_2$. To see this, consider $a_1 \in \mathbf{H}_1$ and $a_2 \in \mathbf{H}_2$ such that $a_2 \notin \mathbf{H}_1$ and $a_1 \notin \mathbf{H}_2$. Clearly, $\mathbf{H}_1 \cap \mathbf{H}_2 = \emptyset$, hence it is Hausdorff. Now we show that $\mathbf{H}$ has at least two TDPs. Let $\mathbf{H}$ be such that it has at most one TDP. Let $a_1 \in \mathbf{H}$ and $\mathbf{H} \setminus \{a_1\} = \mathbf{P}_0 \dagger \mathbf{Q}_0$ for some $\mathbf{P}_0, \mathbf{Q}_0$, which are subsets of $\mathbf{H}$. Since $\mathbf{H}$ has only one TDP then either $\mathbf{P}_0$ or $\mathbf{Q}_0$ has TDPs. By proposition 4.8, $\mathbf{P}_0$ has some condensation point of $\mathbf{P}$ say a. Let $\mathbf{H} \setminus \{a\} = \mathbf{P} \dagger \mathbf{Q}$. Without loss of generality, let $a \in \mathbf{Q}$. By Hausdorff Maximal Principle, there is an optimal chain $\mathbf{C}$ in $\mathbf{S}$ of $\mathbf{H}$ such that for some $\mathbf{U}_\alpha$ of $\mathbf{S}$, $\bigcup_{\alpha \in \Lambda} \mathbf{U}_\alpha \in \mathbf{H}$. Hence, $\mathbf{H}$ is compact. Let $\mathbf{V} = \bigcup_{\alpha \in \Lambda} \mathbf{U}_\alpha$, then by Lemma 4.9, we can get at least two points of $\mathbf{H}$ which give a subcover for $\mathbf{H}$. Since the subcovers are open by Heine-Borel Property, each set forms a singleton set. □

Example 3.11. Consider the Euclidean plane with standard topology R^2. Let $X_1 = \{(x, y) \in R^2 : x \leq 0 \text{ and } |y| = 1\}$ and let $X_2 = \{(x, y) \in R^2 : x > 0 \text{ and } y = \sin \frac{1}{x}\}$. Define $X = X_1 \cup X_2$. Then X is a FTDS with at least two TDPs with closed subsets of FTDS which are singletons.

4 Conclusion

In this work, we have characterized big data fuzzy sets and shown that topological data points form singleton fuzzy sets which are closed. Besides, fuzzy sets of topological data points are compact and have at least one closed point. We have also shown that the fuzzy set of all condensation points of a fuzzy Hausdorff space is infinite and the cardinality of a fuzzy topological data set is also infinite and arbitrarily distributed in fuzzy Hausdorff spaces. For further research, this work can be extended by characterizing topological data point and sets in soft fuzzy Haursdorff spaces.

Acknowledgements: The authors are grateful to the reviewers for their useful comments.

Conflict of Interest: The authors declare that there are no conflict of interest.

References

[1] K. P. Adlassnig, A survey on medical diagnosis and fuzzy subsets, *Springer Verlag, New York*, (1982).

[2] R. C. Berkan and S. L. Trubatch, Fuzzy Systems Design Principles: Building Fuzzy IF-THEN Rule Bases, *New York, NY: IEEE Press*, (1997).

[3] G. Carlsson, Topology and data, *Bull. Amer. Math. Soc.*, 46 (2009), 255-308.

[4] R. W. David, Topological Spaces, *Springer Verlag, New York*, (2008).

[5] D. Dubois and H. Prade, Towards fuzzy differential calculus: Part 1 integration of fuzzy mappings; Part 2: integration of fuzzy intervals; Part 3: Differentiation, *Fuzzy Set Syst.*, 8 (1982), 1-233.

[6] S. Gottwald, Foundations of a theory for fuzzy sets: 40 years of development, *Int. J. Gen. Syst.*, 37 (2008), 69-82.

[7] M. Inuiguchi, H. Ichihashi and Y. Kume, A solution algorithm for fuzzy linear programming with piecewise linear membership functions, *Fuzzy Set Syst.*, 34 (1990), 5-31.

[8] J. Kacprzyk, A. Wilbik and S. Zadrzny, Linguistic summarization of time series using quantifier driven aggregation, *Fuzzy Set Syst.*, 159 (2008), 1485-1499.

[9] P. E. Kloeden, Chaotic iterations of fuzzy sets, *Fuzzy Sets and Systems*, 42 (1991), 37-42.

[10] W. A. Lodwick and K. D. Jamison, Interval-valued probability in the ananlysis of problems containing a mixture of possibilistic, probabilistic and interval uncertainty, *Fuzzy Set Syst.*, 159 (2008), 2845-2858.

[11] T. Maeda, On characterization of fuzzy vectors and its application to fuzzy mathematical programming problems, *Fuzzy Set Syst.*, 159 (2008), 3333-3346.

[12] E. H. Mamdani and S. Assilian, An experiment in linguistic synthesis with a fuzzy logic controller, *Int. J. Man. Mach. Stud.*, 7 (1975), 1-13.

[13] W. Pedrycz and P. Rai, Collaborative clustering with the use of Fuzzy C-Means and its quantification. *Fuzzy Set Syst.*, 159 (2008), 2399-2427.

[14] D. Ruan, A critical study of widely used fuzzy implication operators and their inference on the inference rules in fuzzy expert systems. *Ph.D. Thesis, Gent*, (1990).

[15] L. A. Zadeh, Fuzzy sets, *Information and Control*, 8 (1965), 338-353.

[16] H. J. Zimmermann, Fuzzy set theory, *Comp. Stat.*, 2 (2010), 317-332.

[17] H. J. Zimmermann, Fuzzy set theory- and its applications, 4th Ed., *Springer Dordrechtg, New York*, (2001).

Benard Okelo
Department of Pure and Applied Mathematics
Jaramogi Oginga Odinga University of Science and Technology
Bondo, Kenya
E-mail: bnyaare@yahoo.com

Allan Onyango
Department of Pure and Applied Mathematics
Jaramogi Oginga Odinga University of Science and Technology
Bondo, Kenya
E-mail: onyango.onyangoa@gmail.com

Transactions on Fuzzy Sets and Systems
URL: https://tfss.journals.iau.ir/
Online ISSN: 2821-0131
Vol.1, No.2, (2022), 37-45
DOI: http://doi.org/10.30495/tfss.2022.1960391.1037

Article Type: Original Research Article (Invited Paper)

Łukasiewicz Anti Fuzzy Set and Its Application in BE-algebras

Young Bae Jun

Abstract. The idea of Łukasiewicz t-conorm is used to construct the concept of Łukasiewicz anti fuzzy sets based on a given anti fuzzy set, and it is applied to BE-algebras. The notion of Łukasiewicz anti fuzzy BE-ideal is introduced, and its properties are investigated. The conditions under which Łukasiewicz anti fuzzy set will be Łukasiewicz anti fuzzy BE-ideal are explored, and the relationship between anti fuzzy BE-ideal and Łukasiewicz anti fuzzy BE-ideal are discussed. Three types of subsets so called $\lessdot$-subset, Υ-subset, and anti subset are constructed, and the conditions under which they can be BE-ideals are explored.

AMS Subject Classification 2020: 03G25; 06F35; 08A72.
Keywords and Phrases: Anti fuzzy BE-ideal, Łukasiewicz anti fuzzy set, Łukasiewicz anti fuzzy BE-ideal, $\lessdot$-subset, Υ-subset, anti subset.

1 Introduction

BCK-algebra and BCI-algebra, introduced by Y. Imai, K. Iski and S. Tanaka in 1966, are algebraic structures of universal algebra which describe fragments of propositional calculus related to implications known as BCK and BCI-logic. After that, various generalizations were attempted, and BCC-algebras, BCH-algebras, BH-algebras, d-algebras etc. appeared. In 2007, H. S. Kim and Y. H. Kim [7] introduced the notion of a BE-algebra as a dualization of a generalization of a BCK-algebra. Since then, the fuzzy set theory in BE-algebras has been studied (see [2, 5, 8]). S. S. Ahn and K. S. So [3] introduced the notion of ideals in BE-algebras, and S. Abdullah et al. [1] studied anti fuzzy ideals in BE-algebras. In mathematics, a triangular norm (briefly, t-norm) is a kind of binary operation used in the framework of probabilistic metric spaces and in multi-valued logic, specifically in fuzzy logic. The Łukasiewicz t-norm is a nice example of t-norm. A t-conorm is dual to a t-norm under the order-reversing operation that assigns $1x$ to x on $[0, 1]$, and the Łukasiewicz t-conorm is dual to the Łukasiewicz t-norm. It is the standard semantics for strong disjunction in Łukasiewicz fuzzy logic.

In this paper, we establish the concept of the Łukasiewicz anti-fuzzy set using the idea of the Łukasiewicz t-conorm and anti-fuzzy set, and apply it to BE-algebra. We introduce the notion of Łukasiewicz anti fuzzy BE-ideal and investigate its properties. We explore the conditions under which Łukasiewicz anti fuzzy set will be Łukasiewicz anti fuzzy BE-ideal. We discuss the relationship between anti fuzzy BE-ideal and Łukasiewicz anti fuzzy BE-ideal. We look for conditions under which the $\lessdot$-subset, Υ-subset, and anti subset can be BE-ideal.

Corresponding author: Young Bae Jun, Email: skywine@gmail.com, ORCID: 0000-0002-0181-8969
Received: 6 June 2022; Revised: 5 July 2022; Accepted: 14 July 2022; Published Online: 7 November 2022.

How to cite: Y. B. Jun, Łukasiewicz anti fuzzy set and its application in BE-algebras, *Trans. Fuzzy Sets Syst.*, 1(2), (2022), 37-45.

2 Preliminaries

This section lists the known default content that will be used later.

A *BE-algebra* (see [7]) is defined to be a set X together with a binary operation "$*$" and a special element "1" satisfying the conditions:

(BE1) $(\forall a \in X)\ (a * a = 1)$,

(BE2) $(\forall a \in X)\ (a * 1 = 1)$,

(BE3) $(\forall a \in X)\ (1 * a = a)$,

(BE4) $(\forall a, b, c \in X)\ (a * (b * c) = b * (a * c))$.

In the following, the BE-algebra is expressed as $(X, 1)_*$.

A relation "$\leq$" in $(X, 1)_*$ is defined as follows:

$$(\forall a, b \in X)(a \leq b \ \Leftrightarrow \ a * b = 1). \tag{1}$$

In $(X, 1)_*$, the following conditions are valid.

$$(\forall a, b \in X)\,(a * (b * a) = 1)\,. \tag{2}$$

$$(\forall a, b \in X)\,(a * ((a * b) * b) = 1)\,. \tag{3}$$

A BE-algebra $(X, 1)_*$ is said to be *transitive* (see [3]) if it satisfies:

$$(\forall a, b, c \in X)\,(b * c \leq (a * b) * (a * c))\,. \tag{4}$$

A BE-algebra $(X, 1)_*$ is said to be *self-distributive* (see [7]) if it satisfies:

$$(\forall a, b, c \in X)\,(a * (b * c) = (a * b) * (a * c))\,. \tag{5}$$

Note that if a BE-algebra $(X, 1)_*$ is self-distributive, then it is transitive, but the converse is not valid (see [3]).

A subset K of X is called a *BE-ideal* of $(X, 1)_*$ (see [3]) if it satisfies:

$$(\forall a, b \in X)\,(b \in K \ \Rightarrow \ a * b \in K)\,, \tag{6}$$

$$(\forall a, b, c \in X)\,(b, c \in K \ \Rightarrow \ (b * (c * a)) * a \in K)\,. \tag{7}$$

Lemma 2.1 ([6]). *A subset K of X is a BE-ideal of $(X, 1)_*$ if and only if it satisfies:*

$$1 \in K, \tag{8}$$

$$(\forall x, y, z \in X)(x * (y * z) \in K,\ y \in K \ \Rightarrow \ x * z \in K). \tag{9}$$

Given two fuzzy sets f and g in a set X, their union $f \cup g$ and intersection $f \cap g$ are defined as follows:

$$f \cup g : X \to [0, 1],\ b \mapsto \max\{f(b), g(b)\},$$

$$f \cap g : X \to [0, 1],\ b \mapsto \min\{f(b), g(b)\}.$$

A fuzzy set g in X is called an *anti fuzzy BE-ideal* of $(X, 1)_*$ (see [1]) if it satisfies:

$$(\forall a, b \in X)\,(g(a * b) \leq g(b))\,, \tag{10}$$

$$(\forall a, b, c \in X)\,(g((b * (c * a)) * a) \leq \max\{g(b), g(c)\})\,. \tag{11}$$

3 Łukasiewicz anti fuzzy sets

A fuzzy set g in a set X of the form

$$g(b) := \begin{cases} s \in [0,1) & \text{if } b = a, \\ 1 & \text{if } b \neq a, \end{cases} \tag{12}$$

is called an *anti fuzzy point* with support a and value s, and is denoted by $\frac{a}{s}$. A fuzzy set g in a set X is said to be *non-unit* if there exists $a \in X$ such that $g(a) \neq 1$.

For a fuzzy set g in a set X, we say that an anti fuzzy point $\frac{a}{s}$ is said to

(i) *beside* in g, denoted by $\frac{a}{s} \lessdot g$, (see [4]) if $g(a) \leq s$.

(ii) *be non-quasi coincident* with g, denoted by $\frac{a}{s}\,\Upsilon\, g$, (see [4]) if $g(a) + s < 1$.

If $\frac{a}{s} \lessdot g$ or $\frac{a}{s}\,\Upsilon\, g$ (resp., $\frac{a}{s} \lessdot g$ and $\frac{a}{s}\,\Upsilon\, g$), we say that $\frac{a}{s} \lessdot\!\vee\Upsilon\, g$ (resp., $\frac{a}{s} \lessdot\!\wedge\Upsilon\, g$). Given $\beta \in \{\lessdot, \Upsilon\}$, to indicate $\frac{a}{s}\,\overline{\beta}\, g$ means that $\frac{a}{s}\,\beta\, g$ is not established.

Based on the Łukasiewicz t-conorm, we define Łukasiewicz anti fuzzy set.

Definition 3.1. Let ε be an element of the unit interval $[0,1]$ and let g be a fuzzy set in a set X. A function

$$Ł_g^{\varepsilon} : X \to [0,1],\ x \mapsto \min\{1, g(x) + \varepsilon\}$$

is called a *Łukasiewicz anti fuzzy set* of g in X.

Let $Ł_g^{\varepsilon}$ be a Łukasiewicz anti fuzzy set of a fuzzy set g in X. If $\varepsilon = 0$, then $Ł_g^{\varepsilon}(x) = \min\{1, g(x) + \varepsilon\} = \min\{1, g(x)\} = g(x)$ for all $x \in X$. This shows that if $\varepsilon = 0$, then the Łukasiewicz anti fuzzy set of a fuzzy set g in X is the classisical fuzzy set g itself in X. If $\varepsilon = 1$, then $Ł_g^{\varepsilon}(x) = \min\{1, g(x)+\varepsilon\} = \min\{1, g(x)+1\} = 1$ for all $x \in X$, that is, if $\varepsilon = 1$, then the Łukasiewicz anti fuzzy set is the constant function with value 1. Therefore, in handling the Łukasiewicz anti fuzzy set, the value of ε can always be considered to be in $(0,1)$.

Let g be a fuzzy set in a set X and $\varepsilon \in (0,1)$. If $g(x) + \varepsilon \geq 1$ for all $x \in X$, then the Łukasiewicz anti fuzzy set $Ł_g^{\varepsilon}$ of g in X is the constant function with value 1, that is, $Ł_g^{\varepsilon}(x) = 1$ for all $x \in X$. Therefore, in order for the Łukasiewicz anti fuzzy set to have a meaningful shape, a fuzzy set g in X and $\varepsilon \in (0,1)$ shall be set to satisfy condition "$g(x) + \varepsilon < 1$ for some $x \in X$".

Proposition 3.2. *If g is a fuzzy set in a set X and $\varepsilon \in (0,1)$, then its Łukasiewicz anti fuzzy set $Ł_g^{\varepsilon}$ satisfies:*

$$(\forall x, y \in X)(g(x) \geq g(y) \ \Rightarrow\ Ł_g^{\varepsilon}(x) \geq Ł_g^{\varepsilon}(y)), \tag{13}$$

$$(\forall x \in X)\left(\tfrac{x}{\varepsilon}\,\Upsilon\, g \ \Rightarrow\ Ł_g^{\varepsilon}(x) = g(x) + \varepsilon\right). \tag{14}$$

$$(\forall x \in X)(\forall \varepsilon, \gamma \in (0,1))(\varepsilon \geq \gamma \ \Rightarrow\ Ł_g^{\varepsilon}(x) \geq Ł_g^{\gamma}(x)). \tag{15}$$

Proof. Straightforward. □

Proposition 3.3. *If f and g are fuzzy sets in a set X, then*

$$(\forall \varepsilon \in (0,1))\left(Ł_{f\cap g}^{\varepsilon} = Ł_f^{\varepsilon} \cap Ł_g^{\varepsilon},\ Ł_{f\cup g}^{\varepsilon} = Ł_f^{\varepsilon} \cup Ł_g^{\varepsilon}\right). \tag{16}$$

Proof. For every $y \in X$, we have

$$\begin{aligned} Ł_{f\cap g}^{\varepsilon}(y) &= \min\{1, (f \cap g)(y) + \varepsilon\} = \min\{1, \min\{f(y), g(y)\} + \varepsilon\} \\ &= \min\{1, \min\{f(y) + \varepsilon, g(y) + \varepsilon\}\} \\ &= \min\{\min\{1, f(y) + \varepsilon\}, \min\{1, g(y) + \varepsilon\}\} \\ &= \min\{Ł_f^{\varepsilon}(y), Ł_g^{\varepsilon}(y)\} = (Ł_f^{\varepsilon} \cap Ł_g^{\varepsilon})(y), \end{aligned}$$

and

$$\begin{aligned} Ł^{\varepsilon}_{f\cup g}(y) &= \min\{1, (f\cup g)(y) + \varepsilon\} = \min\{1, \max\{f(y), g(y)\} + \varepsilon\} \\ &= \min\{1, \max\{f(y) + \varepsilon, g(y) + \varepsilon\}\} \\ &= \max\{\min\{1, f(y) + \varepsilon\}, \min\{1, g(y) + \varepsilon\}\} \\ &= \max\{Ł^{\varepsilon}_{f}(y), Ł^{\varepsilon}_{g}(y)\} = (Ł^{\varepsilon}_{f} \cup Ł^{\varepsilon}_{g})(y). \end{aligned}$$

Hence (16) is valid. □

Given a Łukasiewicz anti fuzzy set $Ł^{\varepsilon}_{g}$ of a fuzzy set g in X and $s \in [0,1)$, consider the sets:

$$(Ł^{\varepsilon}_{g}, s)_{\lessdot} := \{y \in X \mid \tfrac{y}{s} \lessdot Ł^{\varepsilon}_{g}\} \text{ and } (Ł^{\varepsilon}_{g}, s)_{\Upsilon} := \{y \in X \mid \tfrac{y}{s}\, \Upsilon\, Ł^{\varepsilon}_{g}\}$$

which are called the $\lessdot$-*subset* and Υ-*subset* of $Ł^{\varepsilon}_{g}$ in X. Also, we consider the following set

$$Anti(Ł^{\varepsilon}_{g}) := \{y \in X \mid Ł^{\varepsilon}_{g}(y) < 1\}$$

and it is called the *anti subset* of $Ł^{\varepsilon}_{g}$ in X. It is observed that

$$Anti(Ł^{\varepsilon}_{g}) = \{y \in X \mid g(y) + \varepsilon < 1\}.$$

It is clear that if $s < \varepsilon$, then $(Ł^{\varepsilon}_{g}, s)_{\lessdot} = \emptyset$, and if $s + \varepsilon \geq 1$, then $(Ł^{\varepsilon}_{g}, s)_{q} = \emptyset$.

Example 3.4. Consider a set $X := \{x \in \mathbb{N} \mid x \leq 10\}$ and define a fuzzy set g in X as follows:

$$g : X \to [0,1],\ x \mapsto \begin{cases} 0.5 & \text{if } x = 5, \\ 0.3 & \text{if } x \in \{1,2\}, \\ 0.6 & \text{if } x \in \{3,4\}, \\ 0.8 & \text{if } x \in \{5,6,7\}, \\ 0.1 & \text{if } x \in \{8,9\}, \\ 1.0 & \text{if } x = 10. \end{cases}$$

If we take $\varepsilon := 0.28$ and $s := 0.59$, then $(Ł^{\varepsilon}_{g}, s)_{\lessdot} = \{1,2,8,9\}$, $(Ł^{\varepsilon}_{g}, \Upsilon)_{\lessdot} = \{8,9\}$, and $Anti(Ł^{\varepsilon}_{g}) = \{1,2,3,4,8,9\}$.

4 Łukasiewicz anti fuzzy BE-ideals

In this section, let g and ε be a fuzzy set in X and an element of $(0,1)$, respectively, unless otherwise specified.

Definition 4.1. A Łukasiewicz anti fuzzy set $Ł^{\varepsilon}_{g}$ in X is called a *Łukasiewicz anti fuzzy BE-ideal* of $(X,1)_{*}$ if it satisfies:

$$(\forall x, y \in X)(\forall s \in [0,1))\left(\tfrac{y}{s} \lessdot Ł^{\varepsilon}_{g} \;\Rightarrow\; \tfrac{x*y}{s} \lessdot Ł^{\varepsilon}_{g}\right), \tag{17}$$

$$(\forall x, y, z \in X)(\forall s_a, s_b \in [0,1))\left(\tfrac{x}{s_a} \lessdot Ł^{\varepsilon}_{g},\ \tfrac{y}{s_b} \lessdot Ł^{\varepsilon}_{g} \;\Rightarrow\; \tfrac{(x*(y*z))*z}{\max\{s_a, s_b\}} \lessdot Ł^{\varepsilon}_{g}\right). \tag{18}$$

Example 4.2. Let $X = \{1, a, b, c, d, 0\}$ and $*$ be given by the following Cayley table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Then $(X,1)_*$ is a BE-algebra (see [7]). Let g be a fuzzy set in X defined as follows:

$$g : X \to [0,1],\ x \mapsto \begin{cases} 0.43 & \text{if } x \in \{1,a,b\}, \\ 0.86 & \text{if } x = c, \\ 0.67 & \text{if } x = d, \\ 0.79 & \text{if } x = 0. \end{cases}$$

Given $\varepsilon := 0.35$, the Łukasiewicz anti fuzzy set $Ł_g^\varepsilon$ of g in X is given as follows:

$$Ł_g^\varepsilon : X \to [0,1],\ y \mapsto \begin{cases} 0.78 & \text{if } y \in \{1,a,b\}, \\ 1.00 & \text{if } y \in \{c,d,0\}. \end{cases}$$

It is routine to verify that $Ł_g^\varepsilon$ is a Łukasiewicz anti fuzzy BE-ideal of $(X,1)_*$.

Theorem 4.3. *A Łukasiewicz anti fuzzy set $Ł_g^\varepsilon$ in X is a Łukasiewicz anti fuzzy BE-ideal of $(X,1)_*$ if and only if it satisfies:*

$$(\forall x,y \in X)\left(Ł_g^\varepsilon(x*y) \le Ł_g^\varepsilon(y)\right). \tag{19}$$

$$(\forall x,y,z \in X)\left(Ł_g^\varepsilon((x*(y*z))*z) \le \max\{Ł_g^\varepsilon(x), Ł_g^\varepsilon(y)\}\right). \tag{20}$$

Proof. Assume that $Ł_g^\varepsilon$ is a Łukasiewicz anti fuzzy BE-ideal of $(X,1)_*$. Let $x,y \in X$. Since $\frac{y}{Ł_g^\varepsilon(y)} \lessdot Ł_g^\varepsilon$, we have $\frac{x*y}{Ł_g^\varepsilon(y)} \lessdot Ł_g^\varepsilon$ by (17), and so $Ł_g^\varepsilon(x*y) \le Ł_g^\varepsilon(y)$. Note that $\frac{x}{Ł_g^\varepsilon(x)} \lessdot Ł_g^\varepsilon$ and $\frac{y}{Ł_g^\varepsilon(y)} \lessdot Ł_g^\varepsilon$ for all $x,y \in X$. It follows from (18) that $\frac{(x*(y*z))*z}{\max\{Ł_g^\varepsilon(x),Ł_g^\varepsilon(y)\}} \lessdot Ł_g^\varepsilon$, that is, $Ł_g^\varepsilon((x*(y*z))*z) \le \max\left\{Ł_g^\varepsilon(x), Ł_g^\varepsilon(y)\right\}$ for all $x,y,z \in X$.

Conversely, let $Ł_g^\varepsilon$ be a Łukasiewicz anti fuzzy set satisfying (19) and (20). If $\frac{y}{s} \lessdot Ł_g^\varepsilon$ for all $y \in X$ and $s \in [0,1)$, then $Ł_g^\varepsilon(x*y) \le Ł_g^\varepsilon(y) \le s$ for all $x \in X$ by (19). Hence $\frac{x*y}{s} \lessdot Ł_g^\varepsilon$. Let $x,y,z \in X$ and $s_a, s_b \in [0,1)$ be such that $\frac{x}{s_a} \lessdot Ł_g^\varepsilon$ and $\frac{y}{s_b} \lessdot Ł_g^\varepsilon$. Then $Ł_g^\varepsilon(x) \le s_a$ and $Ł_g^\varepsilon(y) \le s_b$. It follows from (20) that

$$Ł_g^\varepsilon((x*(y*z))*z) \le \max\{Ł_g^\varepsilon(x), Ł_g^\varepsilon(y)\} \le \max\{s_a, s_b\}.$$

Hence $\frac{(x*(y*z))*z}{\max\{s_a,s_b\}} \lessdot Ł_g^\varepsilon$, and therefore $Ł_g^\varepsilon$ is a Łukasiewicz anti fuzzy BE-ideal of $(X,1)_*$. □

Proposition 4.4. *Every Łukasiewicz anti fuzzy BE-ideal $Ł_g^\varepsilon$ of $(X,1)_*$ satisfies:*

$$(\forall x \in X)(\forall s \in [0,1))\left(\tfrac{x}{s} \lessdot Ł_g^\varepsilon \ \Rightarrow\ \tfrac{1}{s} \lessdot Ł_g^\varepsilon\right). \tag{21}$$

$$(\forall x,y \in X)(\forall s \in [0,1))\left(\tfrac{x}{s} \lessdot Ł_g^\varepsilon \ \Rightarrow\ \tfrac{(x*y)*y}{s} \lessdot Ł_g^\varepsilon\right). \tag{22}$$

$$(\forall x,y \in X)(\forall s \in [0,1))\left(x \le y,\ \tfrac{x}{s} \lessdot Ł_g^\varepsilon \ \Rightarrow\ \tfrac{y}{s} \lessdot Ł_g^\varepsilon\right). \tag{23}$$

$$(\forall x,y \in X)(\forall s_a,s_b \in [0,1))\left(\tfrac{x*y}{s_b} \lessdot Ł_g^\varepsilon,\ \tfrac{x}{s_a} \lessdot Ł_g^\varepsilon \ \Rightarrow\ \tfrac{y}{\max\{s_a,s_b\}} \lessdot Ł_g^\varepsilon\right). \tag{24}$$

$$(\forall x,y,z \in X)(\forall s_a,s_b \in [0,1))\left(\tfrac{x*(y*z)}{s_a} \lessdot Ł_g^\varepsilon,\ \tfrac{y}{s_b} \lessdot Ł_g^\varepsilon \ \Rightarrow\ \tfrac{x*z}{\max\{s_a,s_b\}} \lessdot Ł_g^\varepsilon\right). \tag{25}$$

Proof. The combination of (BE1) and (17) induces the condition (21). Let $x \in X$ and $s \in [0,1)$ be such that $\frac{x}{s} \lessdot Ł_g^\varepsilon$. Then $\frac{(x*y)*y}{s} = \frac{(x*(1*y))*y}{s} = \frac{(x*(1*y))*y}{\max\{s,s\}} \lessdot Ł_g^\varepsilon$ by (BE3), (18) and (21). The combination of (BE3), (1) and (22) induces (23). Let $x,y \in X$ and $s_a, s_b \in [0,1)$ be such that $\frac{x*y}{s_b} \lessdot Ł_g^\varepsilon$ and $\frac{x}{s_a} \lessdot Ł_g^\varepsilon$. Then

$$\frac{y}{\max\{s_a,s_b\}} = \frac{1*y}{\max\{s_a,s_b\}} = \frac{((x*y)*(x*y))*y}{\max\{s_a,s_b\}} \lessdot Ł_g^\varepsilon$$

by (BE1), (BE3) and (18), which proves (24). The condition (25) is derived from the combination of (BE4) and (24). □

Lemma 4.5. *If a Łukasiewicz anti fuzzy set $Ł_g^\varepsilon$ in X satisfies (21) and (25), then it satisfies the conditions (22) and (23).*

Proof. Let $x, y \in X$ and $s \in [0,1)$ be such that $x \leq y$ and $\frac{x}{s} \lessdot Ł_g^\varepsilon$. Then $x * y = 1$ and $\frac{1*(x*y)}{s} = \frac{1*1}{s} = \frac{1}{s} \lessdot Ł_g^\varepsilon$ by (BE1) and (21). It follows from (BE3) and (25) that $\frac{y}{s} = \frac{1*y}{s} \lessdot Ł_g^\varepsilon$. Hence (23) is valid. Since $x * ((x * y) * y) = (x * y) * (x * y) = 1$, i.e., $x \leq (x * y) * y$, for all $x, y \in X$, it follows from (23) that $\frac{(x*y)*y}{s} \lessdot Ł_g^\varepsilon$ which proves (22). □

Theorem 4.6. *Let $(X, 1)_*$ be a transitive BE-algebra. If a Łukasiewicz anti fuzzy set $Ł_g^\varepsilon$ in X satisfies conditions (21) and (25), then it is a Łukasiewicz anti fuzzy BE-ideal of $(X, 1)_*$.*

Proof. Assume that $Ł_g^\varepsilon$ satisfies conditions (21) and (25). Since $(X, 1)_*$ is transitive, we have

$$(\forall x, y, z \in X)\,(((y * z) * z) * ((x * (y * z)) * (x * z)) = 1)\,. \tag{26}$$

Let $y \in X$ and $s \in [0,1)$ be such that $\frac{y}{s} \lessdot Ł_g^\varepsilon$. Then $\frac{x*(y*y)}{s} = \frac{1}{s} \lessdot Ł_g^\varepsilon$ by (BE1), (BE2) and (21). It follows from (25) that $\frac{x*y}{s} \lessdot Ł_g^\varepsilon$. Let $x, y, z \in X$ and $s_a, s_b \in [0,1)$ be such that $\frac{x}{s_a} \lessdot Ł_g^\varepsilon$ and $\frac{y}{s_b} \lessdot Ł_g^\varepsilon$. Then $\frac{(y*z)*z}{s_b} \lessdot Ł_g^\varepsilon$ by Lemma 4.5, and so $\frac{(x*(y*z))*(x*z)}{s_b} \lessdot Ł_g^\varepsilon$ by the combination of Lemma 4.5 and (26). It follows from (25) that $\frac{(x*(y*z))*z}{\max\{s_a, s_b\}} \lessdot Ł_g^\varepsilon$. Therefore $Ł_g^\varepsilon$ is a Łukasiewicz anti fuzzy BE-ideal of $(X, 1)_*$. □

Since every self-distributive BE-algebra is transitive, we have the following corollary.

Corollary 4.7. *Let $(X, 1)_*$ be a self-distributive BE-algebra. Then every Łukasiewicz anti fuzzy set $Ł_g^\varepsilon$ in X is a Łukasiewicz anti fuzzy BE-ideal of $(X, 1)_*$ if and only if it satisfies conditions (21) and (25).*

Theorem 4.8. *If g is an anti fuzzy BE-ideal of $(X, 1)_*$, then $Ł_g^\varepsilon$ is a Łukasiewicz anti fuzzy BE-ideal of $(X, 1)_*$.*

Proof. Let $x, y, z \in X$. Then $Ł_g^\varepsilon(x * y) = \min\{1, g(x * y) + \varepsilon\} \leq \min\{1, g(y) + \varepsilon\} = Ł_g^\varepsilon(y)$ and

$$\begin{aligned}
Ł_g^\varepsilon((x * (y * z)) * z) &= \min\{1, g((x * (y * z)) * z) + \varepsilon\} \\
&\leq \min\{1, \max\{g(x), g(y)\} + \varepsilon\} \\
&= \min\{1, \max\{g(x) + \varepsilon, g(y) + \varepsilon\}\} \\
&= \max\{\min\{1, g(x) + \varepsilon\}, \min\{1, g(y) + \varepsilon\}\} \\
&= \max\{Ł_g^\varepsilon(x), Ł_g^\varepsilon(y)\}
\end{aligned}$$

Hence $Ł_g^\varepsilon$ is a Łukasiewicz anti fuzzy BE-ideal of $(X, 1)_*$ by Theorem 4.3. □

In Example 4.2, $Ł_g^\varepsilon$ is a Łukasiewicz anti fuzzy BE-ideal of $(X, 1)_*$. But g is not an anti fuzzy BE-ideal of $(X, 1)_*$ since $g(b * 0) = g(c) = 0.86 \nleq 0.79 = g(0)$. Therefore, the converse of Theorem 4.8 may not be true. In the sense of Theorem 4.8, we can say that Łukasiewicz anti fuzzy BE-ideal is a generalization of anti fuzzy BE-ideal.

We explore the conditions under which $\lessdot$-subset and Υ-subset of the Łukasiewicz anti fuzzy set can be BE-ideal.

Theorem 4.9. *Let $Ł_g^\varepsilon$ be a Łukasiewicz anti fuzzy set in X. Then $\lessdot$-subset $(Ł_g^\varepsilon, s)_\lessdot$ of $Ł_g^\varepsilon$ with value $s \in [0, 0.5)$ is a BE-ideal of $(X, 1)_*$ if and only if $Ł_g^\varepsilon$ satisfies:*

$$(\forall x \in X)\left(Ł_g^\varepsilon(x) \geq \min\{Ł_g^\varepsilon(1), 0.5\}\right), \tag{27}$$

$$(\forall x, y, z \in X)\left(\min\{Ł_g^\varepsilon(x * z), 0.5\} \leq \max\{Ł_g^\varepsilon(x * (y * z)), Ł_g^\varepsilon(y)\}\right). \tag{28}$$

Proof. Assume that $(Ł_g^\varepsilon, s)_\lessdot$ is a BE-ideal of $(X,1)_*$ for $s \in [0, 0.5)$. If $Ł_g^\varepsilon(a) < \min\{Ł_g^\varepsilon(1), 0.5\}$ for some $a \in X$, then $Ł_g^\varepsilon(a) \in [0, 0.5)$ and $Ł_g^\varepsilon(a) < Ł_g^\varepsilon(1)$. Hence $\frac{a}{Ł_g^\varepsilon(a)} \lessdot Ł_g^\varepsilon$, and so $a \in (Ł_g^\varepsilon, Ł_g^\varepsilon(a))_\lessdot$, but $1 \notin (Ł_g^\varepsilon, Ł_g^\varepsilon(a))_\lessdot$. This is a contradiction, and thus $Ł_g^\varepsilon(x) \geq \min\{Ł_g^\varepsilon(1), 0.5\}$ for all $x \in X$. If the condition (28) is not valid, then there exist $a, b, c \in X$ such that $\min\{Ł_g^\varepsilon(a * c), 0.5\} > \max\{Ł_g^\varepsilon(a * (b * c)), Ł_g^\varepsilon(b)\}$. If we take $s := \max\{Ł_g^\varepsilon(a * (b * c)), Ł_g^\varepsilon(b)\}$, then $s \in [0, 0.5)$ and $\frac{a*(b*c)}{s} \lessdot Ł_g^\varepsilon$ and $\frac{b}{s} \lessdot Ł_g^\varepsilon$, but $\frac{a*c}{s} \overline{\lessdot} Ł_g^\varepsilon$, that is, $a * (b * c) \in (Ł_g^\varepsilon, s)_\lessdot$ and $b \in (Ł_g^\varepsilon, s)_\lessdot$, but $a * c \notin (Ł_g^\varepsilon, s)_\lessdot$. This is a contradiction, and thus (28) is valid.

Conversely, suppose that $Ł_g^\varepsilon$ satisfies (27) and (28), and let $s \in [0, 0.5)$. For every $x \in (Ł_g^\varepsilon, s)_\lessdot$, we have $\min\{Ł_g^\varepsilon(1), 0.5\} \leq Ł_g^\varepsilon(x) \leq s < 0.5$ by (27). Hence $1 \in (Ł_g^\varepsilon, s)_\lessdot$. Let $x, y, z \in X$ be such that $x * (y * z) \in (Ł_g^\varepsilon, s)_\lessdot$ and $y \in (Ł_g^\varepsilon, s)_\lessdot$. Then $Ł_g^\varepsilon(x * (y * z)) \leq s$ and $Ł_g^\varepsilon(y) \leq s$, which imply from (28) that

$$\min\{Ł_g^\varepsilon(x * z), 0.5\} \leq \max\{Ł_g^\varepsilon(x * (y * z)), Ł_g^\varepsilon(y)\} \leq s < 0.5.$$

Hence $\frac{x*z}{s} \lessdot Ł_g^\varepsilon$, that is, $x * z \in (Ł_g^\varepsilon, s)_\lessdot$. Therefore $(Ł_g^\varepsilon, s)_\lessdot$ is a BE-ideal of $(X,1)_*$ for $s \in [0, 0.5)$ by Lemma 2.1. □

Theorem 4.10. *The Υ-subset of the Łukasiewicz anti fuzzy BE-ideal is a BE-ideal.*

Proof. Let $Ł_g^\varepsilon$ be a Łukasiewicz anti fuzzy BE-ideal of $(X,1)_*$ and let $s \in [0,1)$. If $1 \notin (Ł_g^\varepsilon, s)_\Upsilon$, then $\frac{1}{s} \overline{\Upsilon} Ł_g^\varepsilon$, i.e., $Ł_g^\varepsilon(1) + s \geq 1$. Since $\frac{x}{Ł_g^\varepsilon(x)} \lessdot Ł_g^\varepsilon$ for all $x \in X$, we get $\frac{1}{Ł_g^\varepsilon(x)} \lessdot Ł_g^\varepsilon$ for all $x \in X$ by (21). Hence $Ł_g^\varepsilon(1) \leq Ł_g^\varepsilon(x)$ for $x \in (Ł_g^\varepsilon, s)_\Upsilon$, and so $1 - s \leq Ł_g^\varepsilon(1) \leq Ł_g^\varepsilon(x)$. This shows that $\frac{x}{s} \overline{\Upsilon} Ł_g^\varepsilon$, that is, $x \notin (Ł_g^\varepsilon, s)_\Upsilon$, a contradiction. Thus $1 \in (Ł_g^\varepsilon, s)_\Upsilon$. Let $x, y, z \in X$ be such that $x * (y * z) \in (Ł_g^\varepsilon, s)_\Upsilon$ and $y \in (Ł_g^\varepsilon, s)_\Upsilon$. Then $\frac{x*(y*z)}{s} \Upsilon Ł_g^\varepsilon$ and $\frac{y}{s} \Upsilon Ł_g^\varepsilon$, that is, $Ł_g^\varepsilon(x * (y * z)) < 1 - s$ and $Ł_g^\varepsilon(y) < 1 - s$. It follows from (25) that $Ł_g^\varepsilon(x * z) \leq \max\left\{Ł_g^\varepsilon(x * (y * z)), Ł_g^\varepsilon(y)\right\} < 1 - s$. Hence $\frac{x*z}{s} \Upsilon Ł_g^\varepsilon$, and so $x * z \in (Ł_g^\varepsilon, s)_\Upsilon$. Therefore $(Ł_g^\varepsilon, s)_\Upsilon$ is a BE-ideal of $(X,1)_*$ by Lemma 2.1. □

Corollary 4.11. *If g is an anti fuzzy BE-ideal of $(X,1)_*$, then the Υ-subset of $Ł_g^\varepsilon$ is a BE-ideal of $(X,1)_*$.*

Theorem 4.12. *For the Łukasiewicz anti fuzzy set $Ł_g^\varepsilon$ in X, if the Υ-subset of $Ł_g^\varepsilon$ is a BE-ideal of $(X,1)_*$, then the following arguments are satisfied.*

$$1 \in (Ł_g^\varepsilon, s)_\lessdot, \tag{29}$$

$$\tfrac{x}{s_a} \Upsilon Ł_g^\varepsilon,\ \tfrac{y}{s_b} \Upsilon Ł_g^\varepsilon \ \Rightarrow\ (x * (y * z)) * z \in (Ł_g^\varepsilon, \min\{s_a, s_b\})_\lessdot \tag{30}$$

for all $x, y, z \in X$ and $s, s_a, s_b \in [0.5, 1)$.

Proof. Assume that the Υ-subset of $Ł_g^\varepsilon$ is a BE-ideal of $(X,1)_*$. If $1 \notin (Ł_g^\varepsilon, s)_\lessdot$ for some $s \in [0.5, 1)$, then $\frac{1}{s} \overline{\lessdot} Ł_g^\varepsilon$. Hence $Ł_g^\varepsilon(1) > s \geq 1 - s$ since $s \in [0.5, 1)$, and so $\frac{1}{s} \overline{\Upsilon} Ł_g^\varepsilon$, i.e., $1 \notin (Ł_g^\varepsilon, s)_\Upsilon$. This is a conradiction, and thus $1 \in (Ł_g^\varepsilon, s)_\lessdot$. Let $x, y, z \in X$ and $s_a, s_b \in [0.5, 1)$ be such that $\frac{x}{s_a} \Upsilon Ł_g^\varepsilon$ and $\frac{y}{s_b} \Upsilon Ł_g^\varepsilon$. Then $x \in (Ł_g^\varepsilon, s_a)_\Upsilon \subseteq (Ł_g^\varepsilon, \min\{s_a, s_b\})_\Upsilon$ and $y \in (Ł_g^\varepsilon, s_b)_\Upsilon \subseteq (Ł_g^\varepsilon, \min\{s_a, s_b\})_\Upsilon$, from which $(x * (y * z)) * z \in (Ł_g^\varepsilon, \min\{s_a, s_b\})_\Upsilon$ is derived. Hence

$$Ł_g^\varepsilon((x * (y * z)) * z) < 1 - \min\{s_a, s_b\} \leq \min\{s_a, s_b\},$$

that is, $\frac{(x*(y*z))*z}{\min\{s_a, s_b\}} \lessdot Ł_g^\varepsilon$. Therefore $(x * (y * z)) * z \in (Ł_g^\varepsilon, \min\{s_a, s_b\})_\lessdot$. □

Theorem 4.13. *If g is an anti fuzzy BE-ideal of $(X,1)_*$, then the non-empty anti subset of $Ł_g^\varepsilon$ is a BE-ideal of $(X,1)_*$.*

Proof. If g is an anti fuzzy BE-ideal of $(X,1)_*$, then Ł_g^{ε} is a Łukasiewicz anti fuzzy BE-ideal of $(X,1)_*$ (see Theorem 4.8). It is clear that $1 \in Anti(\text{Ł}_g^{\varepsilon})$. Let $x, y, z \in X$ be such that $x*(y*z) \in Anti(\text{Ł}_g^{\varepsilon})$ and $y \in Anti(\text{Ł}_g^{\varepsilon})$. Then $\text{Ł}_g^{\varepsilon}(x*(y*z)) < 1$ and $\text{Ł}_g^{\varepsilon}(y) < 1$. Since $\frac{x*(y*z)}{\text{Ł}_g^{\varepsilon}(x*(y*z))} \lessdot \text{Ł}_g^{\varepsilon}$ and $\frac{y}{\text{Ł}_g^{\varepsilon}(y)} \lessdot \text{Ł}_g^{\varepsilon}$, we have $\frac{x*z}{\max\left\{\text{Ł}_g^{\varepsilon}(x*(y*z)),\text{Ł}_g^{\varepsilon}(y)\right\}} \lessdot \text{Ł}_g^{\varepsilon}$ by (25). It follows that

$$\text{Ł}_g^{\varepsilon}(x*z) \leq \max\left\{\text{Ł}_g^{\varepsilon}(x*(y*z)), \text{Ł}_g^{\varepsilon}(y)\right\} < 1.$$

Hence $x*z \in Anti(\text{Ł}_g^{\varepsilon})$, and therefore $Anti(\text{Ł}_g^{\varepsilon})$ is a BE-ideal of $(X,1)_*$ by Lemma 2.1. □

Theorem 4.14. *If a Łukasiewicz anti fuzzy set Ł_g^{ε} in X satisfies (21) and*

$$(\forall x, y, z \in X)(\forall s_a, s_b \in [0,1)) \left(\begin{array}{c} \frac{x*(y*z)}{s_a} \lessdot \text{Ł}_g^{\varepsilon},\ \frac{y}{s_b} \lessdot \text{Ł}_g^{\varepsilon} \\ \Rightarrow \frac{x*z}{\min\{s_a,s_b\}} \,\Upsilon\, \text{Ł}_g^{\varepsilon} \end{array} \right). \tag{31}$$

then the non-empty anti subset of Ł_g^{ε} is a BE-ideal of $(X,1)_*$.

Proof. Let $Anti(\text{Ł}_g^{\varepsilon})$ be a non-empty anti subset of Ł_g^{ε}. Then there exists $x \in Anti(\text{Ł}_g^{\varepsilon})$, and so $s := \text{Ł}_g^{\varepsilon}(x) < 1$, i.e., $\frac{x}{s} \lessdot \text{Ł}_g^{\varepsilon}$ for $s < 1$. Hence $\frac{1}{s} \lessdot \text{Ł}_g^{\varepsilon}$ by (21), and thus $\text{Ł}_g^{\varepsilon}(1) \leq s < 1$. Thus $1 \in Anti(\text{Ł}_g^{\varepsilon})$. Let $x, y, z \in X$ be such that $x*(y*z) \in Anti(\text{Ł}_g^{\varepsilon})$ and $y \in Anti(\text{Ł}_g^{\varepsilon})$. Then $g(x*(y*z)) + \varepsilon < 1$ and $g(y) + \varepsilon < 1$. Since $\frac{x*(y*z)}{\text{Ł}_g^{\varepsilon}(x*(y*z))} \lessdot \text{Ł}_g^{\varepsilon}$ and $\frac{y}{\text{Ł}_g^{\varepsilon}(y)} \lessdot \text{Ł}_g^{\varepsilon}$, it follows from (31) that $\frac{x*z}{\min\left\{\text{Ł}_g^{\varepsilon}(x*(y*z)),\, \text{Ł}_g^{\varepsilon}(y)\right\}} \,\Upsilon\, \text{Ł}_g^{\varepsilon}$. If $x*z \notin Anti(\text{Ł}_g^{\varepsilon})$, then $\text{Ł}_g^{\varepsilon}(x*z) = 1$, and so

$$\begin{aligned}
&\text{Ł}_g^{\varepsilon}(x*z) + \min\left\{\text{Ł}_g^{\varepsilon}(x*(y*z)),\, \text{Ł}_g^{\varepsilon}(y)\right\} = 1 + \min\left\{\text{Ł}_g^{\varepsilon}(x*(y*z)),\, \text{Ł}_g^{\varepsilon}(y)\right\} \\
&= 1 + \min\left\{\min\{1, g(x*(y*z)) + \varepsilon\},\, \min\{1, g(y) + \varepsilon\}\right\} \\
&= 1 + \min\left\{g(x*(y*z)) + \varepsilon,\, g(y) + \varepsilon\right\} \\
&= 1 + \min\left\{g(x*(y*z)),\, g(y)\right\} + \varepsilon \\
&\geq 1 + \varepsilon > 1.
\end{aligned}$$

Hence $\frac{x*z}{\min\left\{\text{Ł}_g^{\varepsilon}(x*(y*z)),\, \text{Ł}_g^{\varepsilon}(y)\right\}} \,\overline{\Upsilon}\, \text{Ł}_g^{\varepsilon}$, a contradiction. Thus $x*z \in Anti(\text{Ł}_g^{\varepsilon})$, and therefore $Anti(\text{Ł}_g^{\varepsilon})$ is a BE-ideal of $(X,1)_*$ by Lemma 2.1. □

Theorem 4.15. *Let Ł_g^{ε} be a Łukasiewicz anti fuzzy set in X that satisfies $\frac{1}{\varepsilon} \,\Upsilon\, g$ and the condition (30) for all $x, y, z \in X$ and $s_a, s_b \in [0,1)$. Then the anti subset of Ł_g^{ε} is a BE-ideal of $(X,1)_*$.*

Proof. Let $Anti(\text{Ł}_g^{\varepsilon})$ be the anti subset of Ł_g^{ε}. If $\frac{1}{\varepsilon} \,\Upsilon\, g$, then $g(1) + \varepsilon < 1$ and so $\text{Ł}_g^{\varepsilon}(1) = \min\{1, g(1) + \varepsilon\} = g(1) + \varepsilon < 1$. Hence $1 \in Anti(\text{Ł}_g^{\varepsilon})$. Let $x, y, z \in X$ be such that $x, y \in Anti(\text{Ł}_g^{\varepsilon})$. Then $\text{Ł}_g^{\varepsilon}(x) < 1$ and $\text{Ł}_g^{\varepsilon}(y) < 1$, which imply that $\frac{x}{0} \,\Upsilon\, \text{Ł}_g^{\varepsilon}$ and $\frac{y}{0} \,\Upsilon\, \text{Ł}_g^{\varepsilon}$. It follows from (30) that $(x*(y*z))*z \in (\text{Ł}_g^{\varepsilon}, \min\{0,0\})_{\lessdot} = (\text{Ł}_g^{\varepsilon}, 0)_{\lessdot}$. Hence $\text{Ł}_g^{\varepsilon}((x*(y*z))*z) = 0 < 1$, and so $(x*(y*z))*z \in Anti(\text{Ł}_g^{\varepsilon})$. Therefore $Anti(\text{Ł}_g^{\varepsilon})$ is a BE-ideal of $(X,1)_*$. □

Theorem 4.16. *Let Ł_g^{ε} be a Łukasiewicz anti fuzzy set in X that satisfies $\frac{1}{\varepsilon} \,\Upsilon\, g$ and*

$$(\forall x, y, z \in X)(\forall s_a, s_b \in [0,1)) \left(\begin{array}{l} \frac{x*(y*z)}{s_a} \,\Upsilon\, \text{Ł}_g^{\varepsilon},\ \frac{y}{s_b} \,\Upsilon\, \text{Ł}_g^{\varepsilon} \\ \ \Rightarrow x*z \in (\text{Ł}_g^{\varepsilon}, \min\{s_a, s_b\})_{\lessdot} \end{array} \right). \tag{32}$$

Then the anti subset of Ł_g^{ε} is a BE-ideal of $(X,1)_*$.

Proof. Let $Anti(Ł_g^\varepsilon)$ be an anti subset of $Ł_g^\varepsilon$. Then $1 \in Anti(Ł_g^\varepsilon)$ in the proof of Theorem 4.15. Let $x, y, z \in X$ be such that $x * (y * z) \in Anti(Ł_g^\varepsilon)$ and $y \in Anti(Ł_g^\varepsilon)$. Then $Ł_g^\varepsilon(x * (y * z)) < 1$ and $Ł_g^\varepsilon(y) < 1$. Thus $\frac{x*(y*z)}{0} \Upsilon Ł_g^\varepsilon$ and $\frac{y}{0} \Upsilon Ł_g^\varepsilon$. Using (32) leads to $x * z \in (Ł_g^\varepsilon, \min\{0, 0\})_\lessdot = (Ł_g^\varepsilon, 0)_\lessdot$ Hence $Ł_g^\varepsilon(x * z) = 0 < 1$, and so $x * z \in Anti(Ł_g^\varepsilon)$. It follows from Lemma 2.1 that $Anti(Ł_g^\varepsilon)$ is a BE-ideal of $(X, 1)_*$. □

Conflict of Interest: The author declares that there are no conflict of interest.

References

[1] S. Abdullah, T. Anwar, N. Amin and M. Taimur, anti fuzzy BE-ideals in BE-algebra, *Ann. Fuzzy Math. Inform.*, 6(3) (2013), 487–494.

[2] S. S. Ahn, Y. H. Kim and K. S. So, Fuzzy BE-algebras, *J. Appl. Math. Inform.*, 29 (2011), 1049–1057.

[3] S. S. Ahn and K. S. So, On ideals and upper sets in BE-algebras, *Sci. Math. Jpn.*, 68(2) (2008), 279–285.

[4] A. Borumand Saeid and Y. B. Jun, Redefined fuzzy subalgebras of BCK/BCI-algebras, *Iran. J. Fuzzy Syst.*, 5(2) (2008), 63–70.

[5] G. Dymek and A. Walendziak, Fuzzy filters of BE-algebras, *Math. Slovaca*, 63 (2013), 935–946. DOI: 10.2478/s12175-013-0145-y

[6] Y. B. Jun, K. J. Lee and S. Z. Song, Fuzzy ideals in BE-algebra, *Bull. Malays. Math. Sci. Soc.*, 33 (2010), 147–153.

[7] H. S. Kim and Y. H. Kim, On BE-algebras, *Sci. Math. Jpn.*, 66 (2007), 113–116.

[8] A. Rezaei and A. Borumand Saeid, On fuzzy subalgebras of BE-algebras, *Afr. Mat.*, 22 (2011), 115–127. DOI 10.1007/s13370-011-0011-4

Young Bae Jun
Department of Mathematics Education
Gyeongsang National University
Jinju 52828, Korea
E-mail: skywine@gmail.com

Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://tfss.journals.iau.ir/
Online ISSN: 2821-0131
Vol.1, No.2, (2022), 46-58
DOI: http://10.30495/tfss.2022.1960147.1036

Article Type: Original Research Article (Invited Paper)

Managing the Uncertainty: From Probability to Fuzziness, Neutrosophy and Soft Sets

Michael Gr. Voskoglou

Abstract. The present paper reviews and compares the main theories reported in the literature for managing the existing real life uncertainty by listing their advantages and disadvantages. Starting with a comparison of the bivalent logic (including probability) and fuzzy logic, proceeds to a brief description of the primary generalizations of fuzzy sets (FSs) including interval valued FSs, type-2 FSs, intuitionistic FSs, neutrosophic sets, etc. Alternative theories related to fuzziness are also examined including grey system theory, rough sets and soft sets. The conclusion obtained at the end of this discussion is that there is no ideal model for managing the uncertainty; it all depends upon the form, the available data and the existing knowledge about the problem under solution. The combination of all the existing models, however, provides a sufficient framework for efficiently tackling several types of uncertainty appearing in real life.

AMS Subject Classification 2020: Primary 03E72; Secondary; 03B52; 03B53
Keywords and Phrases: Uncertainty, Fuzzy set (FS), Interval valued FS (IVFS), Type-2 FS, Intuitionistic FS (IFS), Neutrosophic set (NS), Rough set, Soft set, Grey system (GS).

1 Introduction

The frequently appearing in the real world, in science and in everyday life *uncertainty* is due to a shortage of knowledge regarding some situations. Roughly speaking, the amount of the existing uncertainty is equal to the difference in the amount of the necessary knowledge needed for interpreting or determining the evolution of a situation, minus the already existing knowledge about this situation. In other words, uncertainty represents the total amount of potential information in the situation, which implies that a reduction of uncertainty due to new evidence (e.g. receipt of a message) indicates a gain of an equal amount of information. This is the reason for which the classical measures of uncertainty under crisp or fuzzy conditions (Hartley's formula, Shannon's entropy, etc. [15, Chapter 5]) have also been adopted as measures of information, comprising a powerful tool for dealing with problems such as systems modeling analysis and design, decision making, etc. Different kinds of uncertain environments exist in real life [15]. A typical taxonomy of the uncertainties that can arise includes *vagueness, imprecision, ambiguity* and *inconsistency.* The uncertainty due to vagueness is created when one is unable to clearly differentiate between two classes, such as "a person of average height" and "a tall person". In case of imprecision the available information has not an exact value; e.g. "the temperature tomorrow will be between 27° and 32° C". In ambiguity then existing information leads to several interpretations by different observers. For example, the statement "Boy no girl" written as "Boy, no

Corresponding author: M. Gr. Voskoglou, Email: voskoglou@teiwest.gr, ORCID: 0000-0002-4727-0089
Received: 1 June 2022; Revised: 11 June 2022; Accepted: 11 June 2022; Published Online: 7 November 2022.

How to cite: M. Gr. Voskoglou, Managing the uncertainty: from probability to fuzziness, neutrosophy and soft sets, *Trans. Fuzzy Sets Syst.*, 1(2) (2022), 46-58.

girl" means boy, but written as "Boy no, girl" means girl. Finally, inconsistency appears when two or more pieces of information cannot be true at the same time. As a result the obtainable in this case information is conflicted or undetermined. For example, "the probability for raining tomorrow is 80%, but this does not mean that the probability of not raining is 20%, because they might be hidden weather factors".
Note that several other taxonomies of the uncertainty exist. One such taxonomy, for example, includes the *epistemic (or subjective)* uncertainty and the *linguistic* uncertainty. The former is due to a lack of knowledge, whether the latter is produced by statements expressed in natural language. Another taxonomy includes the uncertainty due to *randomness* and the uncertainty due to *imprecision.* The uncertainty due to randomness is related to well-defined events whose outcomes cannot be predicted in advance, like the turning of a coin, the throwing of a die, etc. On the other hand, uncertainty due to imprecision occurs when the events are well defined, but the possible outcomes cannot be expressed in a crisp form.
The uncertain problems need imprecise methods that could deal with different types of uncertainties to increase the understanding of the outcomes. Several theories have been proposed for tackling such kinds of problems. The target of the present work is to review and compare the primary among those theories and list their advantages and disadvantages. The rest of the paper is formulated as follows: Section 2 compares the bivalent logic (including probability) with the fuzzy logic. Section 2 describes the headlines of the primary generalizations of fuzzy sets (FSs), such as interval valued FSs, type-2 FSs, intuitionistic FSs, neutrosophic sets, etc. Alternative theories related to fuzziness are examined in section 3, including grey system theory, rough sets and soft sets. The paper closes with a discussion including some hints for future research and the final conclusion, which are contained in section 5.

2 Fuzzy Vs Bivalent Logic

Logic is the study of correct reasoning, involving the drawing of inferences. There is no doubt that the enormous progress of science and technology owes a lot to Aristotle's (384-322 BC) *bivalent logic,* which dominated for centuries the human way of thinking.
Bivalent logic is based on Aristotle's law of the *excluded middle,* according to which, for all propositions p, either p or not p must be true and there is no middle (third) true proposition between them; all its other principles are mere elaborations of this law [16].
From the time of Buddha Siddhartha Gautama, however, who lived in India around 500 BC, Heraclitus (535-475 BC) and Plato (427-377 BC) views have appeared to discuss the existence of a third area between "true" and "false", where those two opposites can exist together. More recent philosophers like Hegel, Marx, Engels, Russel and others supported and cultivated further those ideas, but the first integrated propositions of multivalued logics appeared only during the 20^{th} century by Jan Lukasiewicz (1858-1956) and Alfred Tarski (1901-1983) [37, Section 2]. Max Black [2] introduced in 1937 the concept of the *vague set* being a premonition of the Zadeh's *fuzzy* set (FS) introduced in 1965 [44].
Let U be the universal set of the discourse. It is recalled that a fuzzy set F on U is defined with the help of its *membership function* $m : U \rightarrow [0, 1]$ as the set of the ordered pairs

$$F = \{(x, m(x)) \ : \ x \in U\}. \tag{1}$$

The real number $m(x)$ is called the *membership degree* of x in F. The greater is $m(x)$, the more x satisfies the characteristic property of F.
A crisp subset A of U is a fuzzy set on U with a membership function taking the values $m(x) = 1$ if x belongs to A and 0 otherwise. Most notions and operations concerning the crisp sets, e.g. subset, complement, union, intersection, Cartesian product, binary and other relations, etc., can be extended to FS. For general facts about FSs and the connected to them uncertainty we refer to the chapters 4-7 of the book [36].
The infinite-valued on the interval $[0, 1]$ *fuzzy logic* (FL) is defined with the help of the concept of FS [45].

Through FL, the fuzzy terminology is translated by algorithmic procedures into numerical values, operations are performed upon those values and the outcomes are returned into natural language statements in a reliable manner [17]. FL is useful for handling real-life situations that are inherently fuzzy, calculating the existing in such situations fuzzy data and describing the operation of the corresponding fuzzy systems. An important advantage of FL is that its rules are set in natural language with the help of linguistic, and therefore fuzzy, variables [46].

The process of reasoning with fuzzy rules involves:

• *Fuzzification* of the problem's data by utilizing the suitable membership functions to define the required FSs.

• Application of FL operators on the defined FSs and combination of them to obtain the final result in the form of a unique FS.

• *Defuzzification* of the final FS to return to a crisp output value, in order to apply it to the real world situation for resolving the corresponding problem.

Among the more than 30 defuzzification methods in use, the most popular is probably the *Centre of Gravity* (*COG*) technique. According to it, a problem's fuzzy solution is represented by the coordinates of the COG of the level's section contained between the graph of the MF involved and the OX axis [35].

But, while Zadeh was trying to spread out the message of fuzziness, he received many tough critiques for his radical ideas from three different directions [10].

The first direction of critique came from a great number of scientists who asked for some practical applications. In fact, such applications started to appear in the industry during the 1970's, the first one being in the area of cement kiln control [42]. This is an operation demanding the control of a highly complex set of chemical interactions by dynamically managing 40-50 "rules of thumb". This was followed by E. H. Mamdani's [21] work in the Queen Mary College of London, who designed the first fuzzy system for controlling a steam engine and later the operation of traffic lights. Another type of fuzzy inference system was developed later in Japan by Takagi-Sugeno-Kang [32]. Nowadays FSs and FL have found many important applications in almost all sectors of human activity. It must be mentioned that fuzzy mathematics has also been significantly developed on a theoretical level, providing important contributions even in branches of classical mathematics, such as algebra, analysis, geometry, etc. (e.g. see [3]).

The second direction was related to a great part of the *probability* theorists, who claimed that FL could not do any more than probability does. Membership degrees, taking values in the same with probabilities interval $[0, 1]$, are actually hidden probabilities, fuzziness is a kind of disguised randomness, and the multi-valued logic is not a new idea. It took a long time to become universally understood that fuzziness does not oppose probability, but actually supports and completes it by successfully treating the cases of the existing the real world uncertainty which is caused by reasons different from randomness [9].

The expressions "John's membership degree in the FS of clever people is 0.7" and "the probability of John to be clever is 0.7", although they look similar, they actually have essentially different meanings. The former means that John is a rather clever person, whereas the latter means that John, according to the principle of the excluded middle, is either clever or not, but his outlines (heredity, academic studies, etc.) suggest that the probability to be clever is high (70%).

There are also other differences between the two theories mainly arising from the way of defining the corresponding notions and operations. For instance, whereas the sum of the probabilities of all the single events (singleton subsets) of the universal set is always equal to 1 (probability of the certain event), this is not necessarily true for the membership degrees. Consequently a probability distribution could be used to define membership degrees, but the converse does not hold in general.

Note that Edwin T. Jaynes, Professor of Physics at the University of Washington, argued that Probability theory could be considered as a generalization of the bivalent logic reducing to it in the special case where our hypothesis is either absolutely true or absolutely false [12]. Many eminent scientists have been inspired

by the ideas of Janes, like the expert in Algebraic Geometry David Mumford, who believes that Probability and Statistics are emerging as a better way of building scientific models [26].
Probability and Statistics are related mathematical topics having, however, fundamental differences. In fact, Probability is a branch of theoretical mathematics dealing with the estimation of the likelihood of future events, whereas Statistics is an applied branch , which tries to make sense by analyzing the frequencies of past events. Nevertheless, both Probability and Statistics have been developed on the basis of the principles of the bivalent logic. As a result, they are tackling effectively only the cases of the existing in the real world uncertainty which are due to randomness [18]. In other words, Janes' probabilistic logic "covers" only the cases of uncertainty wich are due to randomness.
One could argue, however, that *Bayesian Reasoning* constitutes an interface between bivalent and FL [38]. In fact, the *Bayes' rule* (see equation 2 below) calculates the *conditional probability* $P(A/B)$ with the help of the *inverse in time* conditional probability $P(B/A)$, the *prior probability* $P(A)$ and the *posterior probability* $P(B)$:

$$P(A/B) = \frac{P(B/A)P(A)}{P(B)}. \tag{2}$$

In other words, the Bayes' rule calculates the probability of an event based on prior knowledge of conditions related to that event. The value of the prior probability $P(A)$ is fixed before the experiment, whereas the value of the posterior probability is derived from the experiment's data. Usually, however, there exists an uncertainty (not necessarily due to randomness) about the value of $P(A)$. In such cases, considering all the possible values of $P(A)$, we obtain different values for the conditional probability $P(A/B)$. Therefore, the Bayes' rule introduces a kind of multi-valued logic tackling the existing, due to the different values of the prior probability, uncertainty in a way analogous to FL.
The third direction of the critiques against FL comes from bivalent logic. Many of its traditional supporters, based on a culture of centuries, argue that, since this logic works effectively in science, functions the computers and explains satisfactorily the phenomena of the real world, except perhaps those that happen in the boundaries, there is no reason to make things more complicated by introducing the unstable principles of a multi-valued logic.
FL, however, aims exactly at smoothing the situation in the boundaries! Look, for example, at the graph in Figure 1 corresponding to the FS T of "tall people". People with heights less than 1.50 m are considered to have a membership degree 0 in T. The membership degree is continuously increasing for heights greater than 1.50 m, taking its maximal value 1 for heights equal or greater than 1.80 m. Therefore, the "fuzzy part" of the graph - which is conventionally represented in Figure 1 by the straight line segment AC, but its exact form depends upon the way in which the membership function has been defined - lies in the area of the rectangle ABCD defined by the OX axis, its parallel through the point E and the two perpendiculars to it lines at the points A and B.

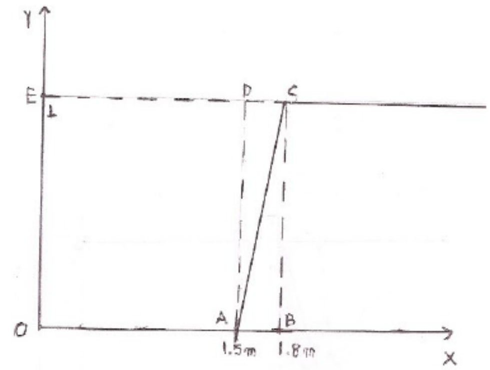


Figure 1: The fuzzy set of "tall people"

In fact, the way of perceiving a concept (e.g. "tall") is different from person to person, depending on the "signals" that each one receives from the real world about it. Mathematically speaking, this means that the definition of the membership function of a FS is not unique, depending on the observer's personal criteria. The only restriction is that this definition must be compatible to the common logic, because otherwise the corresponding FS does not give a reliable description of the corresponding real situation.
On the contrary, bivalent logic defines a bound, e.g. 1.80 m, above which people are considered to be tall and under which are considered to be short. Consequently, one with a height 1.79 m is considered to be short, whether another with a height 1.81 m is tall!
Bivalent logic is able to verify the validity/consistency of an argument only and not its truth. A deductive argument is always valid, even if its inference is false. A characteristic example can be found in the function of computers. A computer is unable to judge, if the input data inserted into it is correct, and therefore if the result obtained by elaborating this data is correct and useful for the user. The only thing that it guarantees is that, if the input is correct, then the output will be correct too. On the contrary, always under the bivalent logic approach, an inductive argument is never valid, even if its inference is true. To put it in a different way, if a property p is true for a sufficient large number of cases, the expression "the property p is possibly true in general" is not acceptable, since it does not satisfy the principle of the excluded middle.
People, however, always want to know the truth in order to organize better, or even to protect, their lives. Consequently, under this option, the significance of an argument has greater importance than its validity/precision. In Figure 2 [4], for example, the extra precision on the left makes things worse for the poor man in danger, who has to spend too much time trying to understand the data and misses the opportunity to take the much needed action of getting out of the way. On the contrary, the rough / fuzzy warning on the right could save his life.

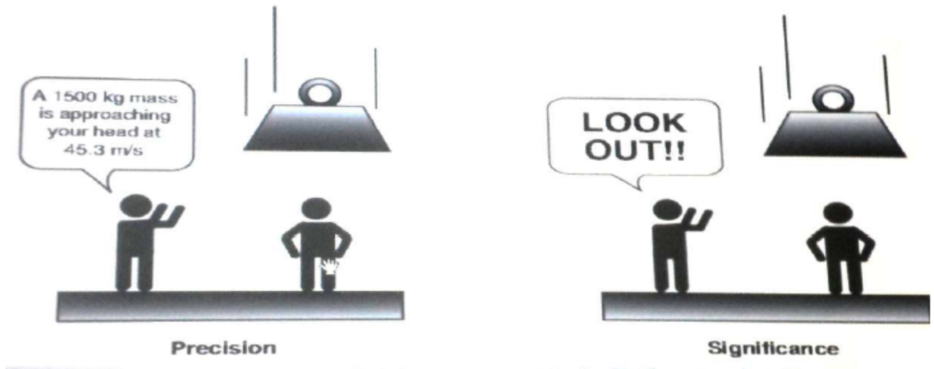


Figure 2: Validity/precision vs significance

Figure 2 illustrates very successfully the importance of FL for the real life situations. Real-world knowledge generally has a different structure and requires different formalization than the existing formal systems. FL, which according to Zadeh is "a precise logic of imprecision and approximate reasoning" [45], serves as a link between classical logic and human reasoning/experience, which is two incommensurable approaches. Having a much higher generality than bivalent logic, FL is capable of generalizing any bivalent logic-based theory.
They have appeared also whit strong voices of anger against FL, without bothering to present any logical arguments about it. Those voices, characterize FL as the tool for making the science unstable, or more emphatically as the "cocaine of science"! Such voices, however, frequently appear in analogous cases of the history of science and must be simply ignored.
Zadeh introduced further *fuzzy numbers (FNs)* [46] as a special form of FSs on the set of the real numbers. He defined the basic arithmetic operations on them in terms of his *extension principle*, which provides the means for any function mapping the crisp set X to the crisp set Y to be generalized so that to map fuzzy subsets of X to fuzzy subsets of Y [15]. FNs play an important role in fuzzy mathematics, analogous to the role of ordinary numbers in traditional mathematics. For general facts on FNs we refer to the book [13]. The present author has used in earlier works *triangular FNs (TFNs)*, the simplest form of FNs, as tools in assessment processes; e.g. [37, section 5].
Zadeh realized that FSs are connected to words (adjectives and adverbs) of the natural language; e.g. the adjective "tall" indicates the FS of the tall people, since "how tall is everyone" is a matter of degree. A grammatical sentence may contain many adjectives and/or adverbs, therefore it correlates a number of FSs. A synthesis of grammatical sentences, i.e. a group of FSs related to each other, forms what we call a *fuzzy system*. A fuzzy system provides empirical advice, mnemonic rules and common logic in general. It is not only able to use its own knowledge to represent and explain the real world, but can also increase it with the help of the new data; in other words, it learns from the experience. This is actually the way in which humans think. Nowadays, for example, a fuzzy system can control the function of an electric washing-machine or send signals for purchasing shares from the stock exchange, etc. [39]. Fuzzy systems are considered to be a part of the wider class of *Soft Computing*, which also includes probabilistic reasoning and *neural networks* (Figure 3) [27].

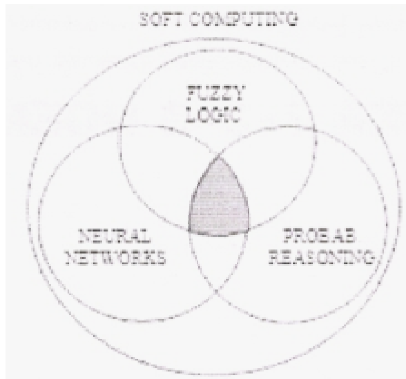


Figure 3: A graphical approach to the contents of Soft Computing

One may say that fuzzy systems and neural networks try to emulate the operation of the human brain. Neural networks have the ability to learn and also have a parallel structure that can rapidly process the information. In other words they concentrate on the structure of the human brain, i.e. on the "hardware", emulating its basic functions. On the other hand, fuzzy systems concentrate on the "software", emulating fuzzy and symbolic reasoning. Fuzzy systems make decisions based on the raw and ambiguous data given to them, whereas neural networks try to learn from the data, incorporating the same way involved in the biological neural networks.
Intersections in Figure 2 include *neuro-fuzzy systems* and techniques, probabilistic approaches to neural networks and Bayesian Reasoning. A neuro-fuzzy system is a fuzzy system that uses a learning algorithm derived from or inspired by neural network theory to determine its parameters (FSs and fuzzy rules) by processing data samples. Characteristic examples of such kinds of systems are the *Adaptive Neuro- Fuzzy Inference Systems (ANFIS)* [11].

3 Generalizations of Fuzzy Sets

As has been already mentioned in the previous section, the probability is suitable for managing the cases of uncertainty due to randomness. Fuzziness, on the other hand, treats as well the cases of vagueness. for the purpose of managing the existing real world uncertainty in a better way, a lot of research has been carried out during the last 60 years to improve/generalize the FS theory.
Zadeh, Sambuc, Jahn and Grattan Guiness introduced in 1975, independently from each other, the concept of the *interval-valued FS (IVFS)* [8]. The idea behind IVFs is that the membership degrees of the traditional FSs, as has been already explained in the previous section, can hardly be precise Thus, an IVFS, defined by a mapping from the universe U to the set of closed intervals in $[0, 1]$, replaces the membership degrees with closed sub-intervals of $[0, 1]$.
Similar to the concept of IVFS is the *hesitant FS (HFS)* introduced by Torra and Narukawa in 2009 [34]. The difference in the definition of a HFS with respect to an IVFS is that the *hesitant degree* $h(x)$ of an element x of U is not a single value like its membership degree, but a set of some values in $[0, 1]$. For example, if $U = \{a, b, c\}$, we could have $h(a) = \{0.2, 0.3\}$, $h(b) = \{0.75, 0.8, 0.82\}$ and $h(c) = \{0.9\}$.
Zadeh also introduced in 1975 the concept of *type-2 FS* [46], so that more uncertainty could be handled connected to the membership function. The membership function of a type-2 FS is three - dimensional, its third dimension is the value of the membership function at each point of its two-dimensional domain, which is called *Footprint of Uncertainty (FOU)*. The FOU is completely determined by its two bounding functions,

a *lower* membership function and an *upper* membership function, both of which are ordinary FSs (otherwise called *type-1 FSs*). When no uncertainty exists about the membership function, then a type-2 FS reduces to a type-1 FS, in a way analogous to probability reducing to determinism when unpredictability vanishes. Zadeh in the same paper [46] generalized the type-2 FS to the *type-n FS* $n = 1, 2, 3, \ldots$. When Zadeh proposed the type-2 FS, however, the time was not right for researchers to drop what they were doing with type-1 FS and focus on type-2 FS. This changed in the late 1990s as a result of Prof. Jerry Mendel and his students' works on type-2 FS [22]. Since then, more and more researchers around the world have been writing articles about type-2 FSs and systems.

Another application of FS, inspired by Zadeh, is the process of *Computing with Words (CWW)*, a methodology in which the objects of computation are words and propositions drawn from a natural language [47]. The idea was that computers would be activated by words, which would be converted into a mathematical representation using FSs and that these FSs would be mapped by a CWW engine into some other FS, after which the latter would be converted back into a word. Much research is under way about CWW. As Mendel has argued [23], a type-2 fuzzy set should be used as a model for a word.

Ramot et al. [29] introduced in 2002 the notion of *Complex FS (CFS)* characterized by a complex-valued MF, whose range is extended from the traditional fuzzy range of $[0, 1]$ to the unit circle in the complex plane. More explicitly, the membership function of a CFS is of the form $m(x) = r(x)e^{i\theta(x)} = r(x)[\cos[\theta(x)] + i\sin[\theta(x)]$. In the above formula $r(x)$ is the *amplitude term* and $\theta(x)$ is the *phase term* of the membership function. The terms $r(x)$ and $\theta(x)$ are both real-valued and $r(x)$ is in $[0, 1]$ for all x in the universal set U. Since $m(x)$ is a periodic function, one may only consider $\theta(x)$ in $[0, 2\pi]$. When $\theta(x) = 0$ for all x in U, then $m(x)$ reduces to the membership function of an ordinary FS.

Kassimir Atanassov, Professor of Mathematics at the Bulgarian Academy of Sciences, introduced in 1986, as a complement of Zadehs membership degree $m(x)$, $x \in U$, the *degree of non-membership* $n(x)$. In a FS is always $m(x) + n(x) = 1$, but this need not be always true in real applications; e.g. see the example of section 2 with the rainy weather. Atanassov proposed the notion of *intuitionistic FS (IFS)* for more accurate quantification of the uncertainty [1].

An IFS A is formally defined as the set of the ordered triples

$$A = \{(x, m(x), n(x)) \ : \ x \in U, \quad 0 \leq m(x) + n(x) \leq 1\}. \tag{3}$$

One can write $m(x) + n(x) + h(x) = 1$, where $h(x)$ is called the *hesitation* or *uncertainty degree* of x. If $h(x) = 0$, then the corresponding IFS reduces to an ordinary FS. The characterization of intuitionistic is due to the fact that an IFS contains the intuitionistic idea, as it incorporates the degree of hesitation.

Most notions and operations concerning the crisp sets can be extended to IFS [1]. A *Pythagorean FS (PFS)*, introduced by Yager in 2013 [43], considers the membership degree $m(x)$ and non-membership degree $n(x)$ satisfying the condition $m^2(x) + n^2(x) \leq 1$. PFSs have a stronger ability than IFS to manage uncertainty in real-world decision-making problems [48].

The Romanian-American writer and mathematician Florentin Smarandache, Professor at the branch of Gallup of the New Mexico University, introduced in 1995 the degree of *indeterminancy/neutrality* membership of the elements of the universal set U in a subset of U and defined the concept of *neutrosophic set (NS)*, which generalizes the notions of FS and IFS [30].

A *single valued NS (SVNS)* A on U is of the form

$$A = \{(x, T(x), I(x), F(x)) \ : \ x \in U, \quad T(x), I(x), F(x) \in [0, 1], \quad 0 \leq T(x) + I(x) + F(x) \leq 3\}. \tag{4}$$

In (4) $T(x)$, $I(x)$, $F(x)$ are the degrees of truth, indeterminancy and falsity membership of x in A respectively, called the *neutrosophic components* of x. The etymology of the term "neutrosophy" comes from the adjective "neutral" and the Greek word "sophia" (wisdom) and means, according to Smarandanche who introduced it, "the knowledge of neutral thought".

For example, let U be the set of the players of a football team and let A be the SVNS of the good players of U. Then each player x of U is characterized by a *neutrosophic triplet* (t, i, f) with respect to A, with t, i, f in $[0, 1]$. For instance, $(0.6, 0.2, 0.4) \in A$ means that there is a 60% probability for x to be in A, a 20% probability to be unknown if x is in A and a 40% probability for x to not be in A. In particular, $x(0, 1, 0) \in A$ means that we do not know absolutely nothing about x's affiliation with A.
Indeterminancy is understood to be in general everything which is between the opposites of truth and falsity [31]. One can find plenty of real examples of neutrosophic triplets, like (friend, neutral, enemy), (positive, zero, negative), (small, medium, high), (male, transgender, female), (win, draw, defeat), etc. This means that the previously given definition of SVNS is well placed.
In an IFS the inderterminancy is equal by default with the hesitancy, i.e. we have $I(x) = 1 - T(x) - F(x)$. Also, in a FS is $I(x) = 0$ and $F(x) = 1 - T(x)$, whereas in a crisp set is $T(x) = 1$ (or 0) and $F(x) = 0$ (or 1). In other words, crisp sets, FSs and IFSs are special cases of SVNSs.
When the sum $T(x) + I(x) + F(x)$ of the neutrosophic components of $x \in U$ in a SVNS A on U is < 1, then it leaves room for incomplete information about x, when is equal to 1 for complete information and when is greater than 1 for *parasconsistent* (i.e. contradiction tolerant) information about x. A SVNS may contain simultaneously elements leaving room for all the previous types of information.
When $T(x) + I(x) + F(x) < 1$, $\forall\ x \in U$, then the corresponding SVNS is usually referred as *picture FS (PiFS)* [39]. In this case $1 - T(x) - I(x) - F(x)$ is called the degree of *refusal membership* of x in A. The PiFSs based models are adequate in situations where we face human opinions involving answers of types yes, abstain, no and refusal Voting is a good example of such a situation.
The difference between the *general definition of a NS* and the previously given definition of a SVNS is that in the general definition $T(x)$, $I(x)$ and $F(x)$ may take values in the non-standard unit interval $]-0, 1+[$ (including values < 0 or > 1) [36]. This could happen in real world applications. For example, in a company with full-time work for its employees 40 hours per week an employee, upon his work, could belong by $\frac{40}{40} = 1$ to the company (full-time job) or by $\frac{30}{40} < 1$ (part-time job) or by $\frac{45}{40} > 1$ (over-time job). Assume further that a full-time employee caused a damage to his job's equipment, the cost of which must be taken from his salary. Then, if the cost is equal to $\frac{50}{40}$ of his weekly salary, the employee belongs this week to the company by $-\frac{10}{40} < 0$.
Most notions and operations concerning the crisp sets can be extended to NSs [31].

4 Alternative Theories Related to Fuzziness

In 1982 Julong Deng, Professor of the Huazhong University of Science and Technology, Wuhan, China, introduced the theory of *Grey System (GS)* [6] for handling the approximate data that are frequently appear in the study of large and complex systems, like the socio-economic, the biological ones, etc. The systems which lack information, such as structure message, operation mechanism and behaviour document, are referred to as GSs. Usually, on the grounds of existing grey relations and elements one can identify where "grey" means poor, incomplete, uncertain, etc. The GS theory was mainly developed in China and it has found many applications in agriculture, economy, management, industry, ecology and in many other fields of the human activity [7].
An effective tool of the GS theory is the use of *Grey Numbers (GNs)* that are indeterminate numbers defined in terms of the closed real intervals. More explicitly, a GN, say A, is of the form $A \in [a, b]$, where a, b are real numbers with $a \leq b$. In other words, the range in which A lies is known, but not its exact value. A GN may enrich its uncertainty representation with respect to the interval $[a, b]$ by a *whitenization function* $g : [a, b] \longrightarrow [0, 1]$, which defines the *degree of greyness* $g(x)$ for each x in $[a, b]$. The closer is $g(x)$ to 1, the

better x approximates the real value of A. The real number which is used as the crisp representative of the GN $A \in [a, b]$ is denoted by $W(A)$. When the distribution of A is unknown, i.e. no whitenization function has been defined for it, one usually takes $W(A) = \frac{a+b}{2}$ [5].
The arithmetic of the real intervals introduced by Moore et al. [25] has been used to define the basic arithmetic operations among the GNs. For general facts on GNs we refer to the book [19]. The present author has utilized in earlier works GNs as tools in assessment processes; e.g. see section 6 of [37])
A *rough set*, first described by the Polish computer scientist Zdzislaw Pawlak in 1991 [28] is a formal approximation of a crisp set in terms of a pair of sets which give the *lower* and the *upper approximation* of the original set. In the standard version of rough set theory the lower and upper-approximation sets are crisp sets, but in other variations, the approximating sets may be FSs. The theory of rough sets has found important applications to many scientific fields and in particular in Informatics.
In 1999 Dmtri Molodstov, Professor of the Computing Center of the Russian Academy of Sciences in Moscow, in order to overpass the existing difficulty for defining properly the membership functions of FSs, IFSs, NSs, etc. proposed the notion of *soft set* as a new mathematical tool for dealing with the uncertainty in a parametric manner [24].
Let E be a set of parameters, let A be a subset of E and let f be a mapping of A into the set $P(U)$ of all subsets of U. Then the soft set on U connected to A, denoted by (f, A), is defined as the set of the ordered pairs

$$(f, A) = \{(e, f(e)) \ : \ e \in A\}. \tag{5}$$

In other words, a soft set is a paramametrized family of subsets of U. Intuitively, it is "soft" because the boundary of the set depends on the parameters.
For example, let $V = \{C_1, C_2, C_3, C_4, C_5, C_6\}$ be a set of cars and let $E = \{e_1, e_2, e_3, e_4, e_5\}$ be the set of the parameters e_1=high-speed, e_2 =automatic (gear-box), e_3 =hybrid (petrol and electric power), $e_4 = 4x4$ and e_5 =cheap. Consider the subset $A = \{e_1, e_2, e_3, e_5\}$ of E and assume that C_1, C_2, C_6 are the high-speed, C_2, C_3, C_5, C_6 are the automatic, C_3, C_5 are the hybrid cars and C_4 is the unique cheap car. Then a map $g : A \longrightarrow P(V)$ is defined by $g(e_1) = \{C_1, C_2, C_6\}$, $g(e_2) = \{C_2, C_3, C_5, C_6\}$, $g(e_3) = \{C_3, C_5\}$, $g(e_5) = \{C_4\}$ and the soft set

$$(g, A) = \{(e_1, \{C_1, C_2, C_6\}), (e_2, \{C_2, C_3, C_5, C_6\}), (e_3, \{C_3, C_5\}), (e_5, \{C_4\})\}. \tag{6}$$

A FS on U with membership function $y = m(x)$ is a soft set on U of the form $(f, [0, 1])$, where $f(\alpha) = \{x \in U : m(x) \geq \alpha\}$ is the corresponding $\alpha - cut$ of the FS, for each α in $[0, 1]$. Most notions and operations concerning the crisp sets can be extended to soft sets [20].
Soft sets have found important applications to several sectors of human activity [14, 33]. The present author has used recently soft sets as tools for assessment processes [40], as well as a combination of soft sets and GNs for developing a hybrid decision making method under fuzzy conditions [41].
The catalogue of the extensions of FS and of the related to the fuzziness theories does not end here. Several other alternatives have been proposed, many of them being hybrid constructions of the above mentioned primary approaches. For example, if in the definition of the soft set the set of all subsets of U is replaced by the set of all fuzzy subsets of U, one gets the notion of the *fuzzy soft set*. Also, the notion of neutrosophic set has been combined with that of the IVFS to form a new hybrid set called *interval valued neutrosophic set*, etc.

5 Discussion and Conclusion

The frequently existing in real life situations uncertainty is connected to the available information about the corresponding situation and appears in several types, like randomness, vagueness, imprecision, etc. The study performed in this work leads to the conclusion that there is no ideal model for managing the uncertainty; it all depends upon the form, the available data and the existing knowledge about the problem under solution. Probability treats efficiently the cases of randomness, FS describes vagueness, type-2 FS describes vagueness and imprecision by a 3-dimensional range of membership values, IFS is suitable for simulating imprecision in human thinking, NS can deal with vagueness, imprecision, ambiguity and inconsistency, etc. The combination of all these models, however, provides a sufficient framework for tackling the several types of uncertainty, but further research is needed to improving the existing methods, probably by using hybrid approaches.

Conflict of Interest: The author declares that there are no conflict of interest.

References

[1] K. T. Atanassov, Intuitionistic Fuzzy sets, *Fuzzy Sets and Systems*, 20(1) (1986), 87-96.

[2] Black, M., Vagueness, Phil. of Science, 4 (1937) 427-455. Reprinted in Int. J. of General Systems, 17 (1990), 107-128.

[3] S. L. Chang, Fuzzy topological spaces, *Journal of Mathematical Analysis and Applications*, 24(1) (1968), 182-190.

[4] B. C. Cuong, Picture Fuzzy sets, *Journal of Computer Science and Cybernetics*, 30(4) (2014), 409-420.

[5] F. Dernoncourt, Fuzzy logic: Between human reasoning and Artificial Intelligence, *Report, Ecole Normale Supperieure, Paris*, (2011). Retrieved from: https://www.researchgate.net/publication/235333084-Fuzzy-logic-between-human-reasoning-and-artificial-intelligence

[6] J. Deng, Control problems of grey systems, *Systems and Control Letters*, (1982), 288-294.

[7] J. Deng, Introduction to grey system theory, *The Journal of Grey System*, 1 (1989), 1-24.

[8] D. Dubois and H. Prade, Interval-Valued Fuzzy Sets, Possibility Theory and Imprecise Probability, *Proceedings EUSFLAT-LFA*, (2005), 314-319. Retrieved from: http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.207.932rep=rep1type=pdf

[9] J. Fox, Towards a reconciliation of fuzzy logic and standard logic, *Int. J. of Man-Machine Studies*, 15 (1981), 213-220.

[10] S. Haack, Do we need fuzzy logic? *Int. J. of Man-Machine Studies*, 11 (1979), 437-445.

[11] J.-S. R. Jang, ANFIS: adaptive network-based fuzzy inference system, *IEEE Transactions on Systems, Man and Cybernetics*, 23(3) (1993), 665-685.

[12] E. T. Jaynes, Probability Theory: The Logic of Science, *Cambridge University Press, UK, 8th Printing*, (2011), (first published, 2003).

[13] A. Kaufmann and M. Gupta, Introduction to Fuzzy Arithmetic, *Van Nostrand Reinhold Company, New York*, (1991).

[14] A. Kharal and B. Ahmad, Mappings on Soft Classes, *New Mathematics and Natural Computation* , 7(3) (2011), 471-481.

[15] G. J. Klir and T. A. Folger, Fuzzy Sets, *Uncertainty and Information, Prentice-Hall, London*, (1988).

[16] S. Korner, Laws of Thought, *In Encyclopedia of Philosophy; Mac Millan: New York, NY, USA.*, 4 (1967), 414-417.

[17] B. Kosko, Fuzzy Thinking: The New Science of Fuzzy Logic, *Hyperion: New York*, (1993).

[18] B. Kosko, Fuzziness Vs Probability, *Int. J. of General Systems*, 17(2-3) (1990), 211-240.

[19] S. F. Liu and Y. Lin. (Eds.), Advances in Grey System Research, *Berlin-Heidelberg: Springer*, (2010).

[20] P. K. Maji, R. Biswas, and A. R. Roy, Soft Set Theory, *Computers and Mathematics with Applications*, 45 (2003), 555-562.

[21] E. H. Mamdani and S. Assilian, An experiment in linguistic synthesis with a fuzzy logic controller, *Int. J. of Man-Machine Studies*, 7(1) (1975), 1-13.

[22] J. M. Mendel, Uncertain Rule-Based Fuzzy Logic Systems: Introduction and New Directions, *Prentice-Hall, Upper-Saddle River, NJ*, (2001).

[23] J. M. Mendel, Fuzzy Sets for Words: a New Beginning, *Proc. IEEE FUZZ Conference, St. Louis, MO*, May 26-28, (2003), 37-42.

[24] D. Molodtsov, Soft Set Theory-First Results, *Computers and Mathematics with Applications*, 37(4-5) (1999), 19-31.

[25] R. A. Moore, R. B. Kearfort and M. J. Clood, Introduction to Interval Analysis, *2nd Printing, Philadelphia, SIAM*, (1995).

[26] D. Mumford, The Dawing of the Age of Stochasticity, *in V. Amoid, M. Atiyah, P. Laxand* & *B. Mazur (Eds.), Mathematics: Frontiers and Perspectives, AMS*, (2000), 197-218.

[27] A. P. Paplinski, Neuro-Fuzzy Computing, *Lecture Notes, Monash University, Australia*, (2005).

[28] Z. Pawlak, Rough Sets: Aspects of Reasoning about Data, *Kluer Academic Publishers, Dordrecht,* (1991).

[29] D. Ramot, R. Milo, M. Friedman and A. Kandel, Complex fuzzy set, *IEEE Transactions on Fuzzy Systems*, 10 (2002), 171-186.

[30] F. Smarandache, Neutrosophy/Neutrosophic probability, *set, and logic, Proquest, Michigan, USA*, (1998).

[31] F. Smarandache, Indeterminancy in Neutrosophic Theories and their Applications, *International Journal of Neutrosophic Science*, 15(2) (2021), 89-97.

[32] M. Sugeno, Industrial applications of fuzzy control, *Elsevier Science Pub. Co.*, (1985).

[33] B. K. Tripathy and K. R. Arun, Soft Sets and Its Applications, *J. S. Jacob (Ed.), Handbook of Research on Generalized and Hybrid Set Structures and Applications for Soft Computing, IGI Global, Hersey, PA*, (2016), 65-85.

[34] V. Torra and Y. Narukawa, On hesitant fuzzy sets and decision, *18th IEEE Int. Conference on Fuzzy Systems, Jeju island, Korea*, 544 (2009), 1378-1382.

[35] E. Van Broekhoven, and B. De Baets, Fast and accurate centre of gravity defuzzification of fuzzy systems outputs defined on trapezoidal fuzzy partitions, *Fuzzy Sets Syst.*, 157 (2006), 904-918.

[36] M. Gr. Voskoglou, Finite Markov Chain and Fuzzy Logic Assessment Models: Emerging Research and Opportunities, *Create Space Independent Publishing Platform, Amazon, Columbia, SC., USA.*, (2017).

[37] M. Gr. Voskoglou, Methods for Assessing Human-Machine Performance under Fuzzy Conditions, *Mathematics*, 7(3), article 230, (2019). DOI:10.3390/math7030230

[38] M. Gr. Voskoglou and E. Athanassopoulos, The Importance of Bayesian Reasoning in Everyday Life and Science, *Int. J. of Education, Development, Society and Technology*, 8(2) (2020), 24-33.

[39] M. Gr. Voskoglou, Fuzzy Control Systems, *WSEAS Transactions on Systems*, 19 (2020), 295-300. DOI: 10.37394/23202.2020.19.33

[40] M. Gr. Voskoglou, Use of Soft Sets and the Bloom's Taxonomy for Assessing Learning Skills, *Transactions on Fuzzy Sets and Systems*, 1(1) (2022), 106-113.

[41] M. Gr. Voskoglou, A Combined Use of Soft Sets and Grey Numbers in Decision Making, *Journal of Computational and Cognitive Engineering https://doi.org/10.47852/bonviewjcce2202237*, (2022).

[42] I. G. Umbers and P. J. King, An analysis of human decision-making in cement kiln control and the implications for automation, *Int. J. of Man-Mach. Stud.*, 12 (1980), 11-23.

[43] R. R. Yager, Pythagorean fuzzy subsets, *in Proceedings of Joint IFSA World Congress and NAFIPS Annual Meeting, Edmonton, Canada*, (2013), 57-61.

[44] L. A. Zadeh, Fuzzy Sets, *Information and Control*, 8 (1965), 338-353.

[45] L. A. Zadeh, Outline of a new approach to the analysis of complex systems and decision processes, *IEEE Trans. Syst. Man Cybern.*, 3 (1980), 28-44.

[46] L. A. Zadeh, The Concept of a Linguistic Variable and its Application to Approximate Reasoning, *Information Science*, 8 (1975), 199-249.

[47] L. A. Zadeh, Fuzzy logic = computing with words, *IEEE Trans. on Fuzzy Systems*, 4 (1996), 103-111.

[48] Z. Zhang and Z. Hu, Extension of TOPSIS to Multiple Criteria Decision Making with Pythagorean Fuzzy Sets, *Int. J. of Information Systems*, 29(12) (2014), 1061-1078.

Michael Gr. Voskoglou
Department of Mathematics
School of Technological Applications
Patras, Greece
E-mail: voskoglou@teiwest.gr, mvoskoglou@gmail.com

Transactions on Fuzzy Sets and Systems
URL: https://tfss.journals.iau.ir/
Online ISSN: 2821-0131
Vol.1, No.2, (2022), 59-71
DOI: http://doi.org/10.30495/tfss.2022.1959658.1035

Article Type: Original Research Article (Invited Paper)

Forensic Dynamic Łukasiewicz Logic

Antonio Di Nola , **Revaz Grigolia***

Abstract. A forensic dynamic n-valued Łukasiewicz logic $FD\text{Ł}_n$ is introduced on the base of n-valued Łukasiewicz logic Ł_n and corresponding to it forensic dynamic MV_n-algebra (FDL_n-algebra), $1 < n < \omega$, which are algebraic counterparts of the logic, that in turn represent two-sorted algebras $(\mathcal{M}, \mathcal{R}, \Diamond)$ that combine the varieties of MV_n-algebras $\mathcal{M} = (M, \oplus, \odot, \sim, 0, 1)$ and regular algebras $\mathcal{R} = (R, \cup, ;, ^*)$ into a single finitely axiomatized variety resemblig R-module with "scalar" multiplication $\Diamond$. Kripke semantics is developed for forensic dynamic Łukasiewicz logic $FD\text{Ł}_n$ with application to Digital Forensics.

AMS Subject Classification 2020: 03B421; 03B50; More
Keywords and Phrases: Lukasiewiz Logic, Dynamic Logic, Epistemic Logic, MV-algebra.

1 Introduction

Digital forensics involve securing and analyzing digital information stored on a computer for use as evidence in civil, criminal, or administrative cases. Forensics, network forensics, video forensics, and plenty of others are defined as the application of computer science and investigative procedures for a legal purpose involving the analysis of digital evidence (information of probative value that is stored or transmitted in binary form) after proper search authority, chain of keeping, validation with mathematics, use of validated tools, repeatability, reporting and possible expert presentation. The field of digital forensics can also encompass items such as research and incident response.

We introduce the notion of forensic dynamic n-valued Łukasiewicz logic $FD\text{Ł}_n$ $(1 < n < \omega)$ which permits compound investigation built up from given initial investigations and facts as well. Given investigations a and b, the compound investigations $a \cup b$, choice, is performed by performing one of a or b. The compound investigation $a;b$, sequence, is performed by performing first a and then b. The compound investigation a^*, iteration, is performed by performing a one or more times, sequentially. The constant investigation 0 does nothing and does not terminate, whereas the constant action 1, definable as 0^*, does nothing but does terminate.

Dynamic logic [14, 8] (see also [11] and cited their literature) is a classical formal system for reasoning about programs. Dynamic logic is a classical modal logic for reasoning about dynamic behavior taking into account a discrete time. Dynamic logic is an extension of modal logic originally intended for reasoning about computer programs.

Modal logic is characterized by the modal operators $\Box p$ asserting that p is necessarily the case, and $\Diamond p$ asserting that p is possibly the case. Dynamic logic extends this by associating every action (execution of the program) a the modal operators $[a]$ and $\langle a \rangle$, thereby making it a multimodal logic.

*Corresponding author: R. Grigolia, Email: revaz.grigolia@tsu.ge, ORCID: 0000-0003-0842-138X
Received: 30 May 2022; Revised: 12 June 2022; Accepted: 12 June 2022; Published Online: 7 November 2022.

How to cite: A. Di Nola, R. Grigolia, Forensic Dynamic Łukasiewicz Logic, *Trans. Fuzzy Sets Syst.*, 1(2) (2022), 59-71.

We adapt the dynamic logic, which is presented on the base of classical logic and R-module, to non-classical finitely valued Łukasiewicz logic $Ł_n$ and R-module, and the investigating group consisting of a set of investigators with communications between them represented as a Kripke frame, i.e. relational system - a non-empty set with binary relation on it. The meaning of $[a]p$ is that after performing fact-finding (investigation) a, i.e. to examine the validity of a hypothesis (proposition), it is necessarily the case that p holds, that is, a must bring about p. The meaning of $\langle a\rangle p$ is that after performing a it is possible that p, that is, a might bring about p. These operators are related by $[a]p \equiv \neg\langle a\rangle\neg p$ and $\langle a\rangle p \equiv \neg[a]\neg p$, analogously to the relationship between the universal $\forall$ and existential $\exists$ quantifiers.

Following D. Kozen [8] and V. Pratt [11], who have been introduced dynamic algebra, we propose the notion of a forensic dynamic MV_n-algebra[1] (FDL_n-algebra) ($1 < n < \omega$), which integrates an abstract notion of proposition with an equally abstract notion of investigation. Just as propositions tend to band together to form MV_n-algebras with operations $x \oplus y$, and $\sim x$, so do experiments organize themselves into regular algebras, with operations $a \cup b$, $a;b$, and a^*. Analogously to the proposition $p\underline{\vee}q$ being the strong disjunction (the algebraic counterpart of which is $x \oplus y$), $p \vee q$ being the disjunction of propositions p and q, and $\neg p$ the negation of p, the investigation $a \cup b$ is the choice of investigations a or b, $a;b$, or just ab, is the sequence a followed by b, and a^* is the iteration of a indefinitely often.

Just as $p \vee q$ has natural set theoretic interpretation, namely union, so do $a \cup b$, $a;b$ and a^* have natural interpretations on such concrete kinds of investigations as additive functions, binary relations, trajectory sets and languages over regular algebras, to name those regular algebras that are suited to foresinc dynamic MV_n-algebra.

It is natural to think of fact-finding as being able to bring about a proposition (hypothesis about the fact-findings). We write $\langle a\rangle p$ pronounced "$fact-finding\ a\ enables\ p$", as the proposition that fact-finding a can bring about proposition p. Then a forensic dynamic MV_n-algebra is a MV_n-algebra $(A, \oplus, \odot, \sim, 0, 1)$, a regular algebra $(R, \cup, ;, ^*)$, and the *enables operation* $\Diamond : R \times A \to A$.

Suppose now that either p holds, or a can bring about a situation from which a can eventually (by being iterated) bring about p. Then a can eventually bring about p. That is, $p \vee aa^*p \leq a^*p$. (We write $p \leq q$ to indicate that p implies q, defined as $p \vee q = q$). In turn, if a can eventually bring about p, then either p is already the case or a can eventually bring about a situation in which p is not the case but one further iteration of a will bring about p. That is, $a^*p \leq p \vee a^*(\neg p \wedge ap)$. $[a]$ is the dual of $\langle a\rangle$, and $[a]p$ asserts that whatever a does, p will hold.

We axiomatically define the Forensic Dynamic Lukasiewicz logic, its algebraic counterpart and the corresponding Kripke model which are suitable for digital forensics.

2 Forensic dynamic n-valued Łukasiewicz logic $FDŁ_n$

Forensic dynamic n-valued Łukasiewicz logic $FDŁ_n$ is designed for representing and reasoning about propositional Łukasiewicz logic expected results (hypothesis) of investigations. Its syntax is based upon two sets of symbols: a countable set **Var** ($= \{p, p_1, p_2, \ldots, q, q_1, q_2, \ldots\}$) of propositional variables, that encompass hypotheses, and a countable set **Inv** ($= \{a, b, c, ...\}$) of atomic investigations, that encompass the initial facts and investigations. So the language $\mathcal{L}$ of $FDŁ_n$ is given by a countable set **Var** of propositional variables and a countable set **Inv** of atomic investigations. We suppose that investigations are performed by some computer programs. Formulas and investigations $FI(\mathcal{L})$, which we name formulas, over this base are defined as follows:

- Every propositional variable is a formula;

[1] MV_n-algebras, which are algebraic models of n-valued Łukasiewicz logic $Ł_n$, where introduced by Grigolia in [6]. The variety $\mathbf{MV_n}$ of MV_n-algebras is a subvariety of the variety $\mathbf{MV}$ [2].

- $\perp$ ($false$) is a formula;
- If φ is a formula then $\neg\varphi$ ($not\varphi$) is a formula;
- If φ and ψ are formulas then $(\phi\underline{\vee}\psi)$ ($\underline{\vee}$ is a strong disjunction) is a formula;
- If φ and ψ are formulas then $(\varphi\&\psi)$ ($\&$ is a strong conjunction) is a formula;
- If φ and ψ are formulas then $(\varphi \vee \psi)$ (φ *or* ψ) is a formula;
- If a is an investigation and φ is a formula then $[a]\varphi$ (every made investigation a from the present state leads to a state where φ is *true*) is a formula;
- Every atomic investigation is an investigation;
- If a and b are investigations then $(a;b)$ (do a followed by b) is a investigation;
- If a and b are investigations then $(a \cup b)$ (do a or b, *non-deterministically*) is an investigation;
- If a is an investigation then a^* (repeat a a finite, but non-deterministically determined, number of times) is an investigation.

The other Łukasiewicz connectives $1, \to$ and $\leftrightarrow$ are used as abbreviations in the standard way ($1 \equiv \perp\underline{\vee}\neg\perp,\ p \to q \equiv \neg p\underline{\vee}q,\ p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$). In addition, we abbreviate $\neg[a]\neg\varphi$ to $\langle a\rangle\varphi$ (performing some investigation a from the present state leads to a state where φ is true) as in modal logic. We write a^n for $a;\ldots;a$ with n occurrences of a. More formally:

- $a^1 =_{df} a$
- $a^{n+1} =_{df} a;a^n$

The axioms of $FD\text{Ł}_n$ are the axioms of Łukasiewicz logic Ł:

(L1) $\varphi \to (\psi \to \varphi)$,
(L2) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$,
(L3) $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$,
(L4) $((\varphi \to \psi) \to \psi) \to (\psi \to \varphi) \to \varphi)$,

plus the axioms of the logic Ł_n, that was given by R. Grigolia [6]:

(L_n5) $\varphi^n \leftrightarrow \varphi^{n-1}$,
(L_n6) $n(\varphi^k) \leftrightarrow (k(\varphi^{k-1}))^n$,

for every integer $2 \leq k \leq n-2$ that does not divide $n-1$ and for any formulas $\varphi,\ \psi$ and any investigation:

Ax0 $[a](\varphi \to \psi) \to ([a]\varphi \to [a]\psi)$,
Ax1 $[a]1 \leftrightarrow 1$,
Ax2 $[a;b]\varphi \leftrightarrow [a][b]\varphi$,
Ax3 $[a \cup b]\varphi \leftrightarrow [a]\varphi \wedge [b]\varphi$,
Ax4 $[a](\varphi \wedge \psi) \leftrightarrow ([a]\varphi \wedge [a]\psi)$.
Ax5 $[a^*]\varphi \leftrightarrow \varphi \wedge [a][a^*]\varphi$,
Ax6 $\varphi \wedge [a^*](\varphi \to [a]\varphi) \to [a^*]\varphi$,
Ax7 $[a](\varphi\&\varphi) \leftrightarrow [a]\varphi\&[a]\varphi$,
Ax8 $[a](\varphi \veebar \varphi) \leftrightarrow [a]\varphi \veebar [a]\varphi$.

and closed under the following rules of inference:

(MP) from φ and $\varphi \to \psi$ infer ψ,
(N) from φ infer $[a]\varphi$,
(I) from $\varphi \to [a]\varphi$ infer $\varphi \to [a^*]\varphi$.

3 Forensic dynamic MV_n-algebras

An algebra $A = (A, 0, \neg, \oplus)$ with one binary and one unary and one nullary operations is a MV-*algebras* if it satisfies: MV1. $(A, 0, \oplus)$ is an abelian monoid

MV2. $\neg\neg x = x$
MV2. $x \oplus \neg 0 = \neg 0$
MV3. $y \oplus \neg(y \oplus \neg x) = x \oplus \neg(x \oplus y)$.

We set $1 = \neg 0$ and $x \odot y = \neg(\neg x \oplus \neg y)$. We shall write ab for $a \odot b$ and a^n for $\underbrace{a \odot \cdots \odot a}_{n \text{ times}}$, for given $a, b \in A$. Every MV-algebra has an underlying ordered structure defined by

$$x \leq y \text{ iff } \neg x \oplus y = 1.$$

Then $(A; \leq, 0, 1)$ is a bounded distributive lattice. Moreover, the following property holds in any MV-algebra:

$$xy \leq x \wedge y \leq x \vee y \leq x \oplus y.$$

An MV-algebra $A = (A, 0, \neg, \oplus)$ is MV_n-algebra if it satisfies the identities: $x^n = x^{n-1}$, $n(x^k) = (k(x^{k-1}))^n$ for every integer $2 \leq k \leq n-2$ that does not divide $n-1$ [6].

Recall that MV_n-algebras are algebraic models of n-valued Łukasiewicz logic $Ł_n$.

The unit interval of real numbers $[0, 1]$ endowed with the following operations:

$$xx \oplus y = \min(1, x+y), x \odot y = \max(0, x+y-1), \sim x = 1-x,$$

becomes an MV-algebra [2]. From these operations are defined the lattice operations

$$x \vee y = \max(x, y) = (x \odot \sim y) \oplus y \quad and \quad x \wedge y = \min(x, y) = (\sim x \oplus y) \odot x.$$

It is well known that the MV-algebra $S = ([0, 1], \oplus, \odot, \sim, 0, 1)$ generate the variety $\mathbf{MV}$ of all MV-algebras, i.e. $\mathcal{V}(S) = \mathbf{MV}$. The algebra $S_n = (\{0, 1/n-1, ..., n-2/n-1, 1\}, \oplus, \odot, \sim, 0, 1)$ generates the subvariety $\mathbf{MV}_n$ $(1 < n < \omega)$, the algebras of which is called MV_n-algebras [6], i.e. $\mathcal{V}(S_n) = \mathbf{MV}_n$. Notice that $\mathbf{MV} = \mathcal{V}(\bigcup_{i=1}^{\infty} \mathbf{MV}_n)$.

The algebra $\mathcal{S} = ([0, 1], \odot, \Rightarrow, 0)$ (which is functionally equivalent to the MV-algebra defined above), where a binary operation $\odot$ called Łukasiewicz t-norm and defined as $x \odot y = max\{0, x+y-1\}$, for all $x, y \in [0, 1]$; a binary operation $\Rightarrow$ called the residuum (of the t-norm $\odot$) and defined as $x \Rightarrow y = min\{1, 1-x+y\}$, and $\sim x = x \Rightarrow 0 = 1-x$, $x \oplus y = \sim(\sim x \odot \sim y) = min(1, x+y)$, for all $x, y \in [0, 1]$.

Firstly define regular algebras that are also named Kleene algebras. There exist several definitions of regular algebras. We use J.H. Conway's definition of regular algebras [3] to whom Kozen follows [8]. A *Kleene algebra* is a structure $(K, +, \cdot, ^*, 0, 1)$ such that $(K, +, 0)$ is a commutative monoid, $(K, \cdot, 1)$ is a monoid, and the following laws hold:

$$\begin{array}{ll} a + a = a, & a \cdot (a+b) = a \cdot a + a \cdot b, \\ a \cdot 0 = 0 \cdot a = 0, & (a+b) \cdot c = a \cdot c + b \cdot c, \\ 1 + a \cdot a^* = a^*, & b + a \cdot c \leq c \Rightarrow a^* \cdot b \leq c, \\ 1 + a^* \cdot a = a^*, & b + c \cdot a \leq c \Rightarrow b \cdot a^* \leq c, \end{array}$$

where $\leq$ is the partial order induced by $+$, that is, $a \leq b \Leftrightarrow a + b = b$.

For us it is interesting regular algebras represented by algebras of binary relations. Algebras of relations over a set X: $(2^{X \times X}, \cup, ;, ^*, \emptyset, Id)$, where $\cup$ is set-theoretic union, ; is relational composition, * is reflexive-transitive closure and Id is the identity relation. Notice that this algebra is a complete lattice with respect to $\cup$. In the sequel, following Pratt [11], we represent regular algebras as $(R, \cup, ;, ^*)$.

Forensic dynamic MV_n-algebra, $n \in Z^+$, combine MV_n-algebra $\mathcal{M} = (M, \oplus, \odot, \sim, 0, 1)$ and regular algebra $\mathcal{R} = (R, \cup, ;, ^*)$ into a single finitely axiomatized class $(\mathcal{M}, \mathcal{R}, \Diamond)$ resembling an R-module with scalar multiplication $\Diamond : R \times M \to M$. A forensic dynamic MV_n-algebra $\mathcal{D} = (\mathcal{M}, \mathcal{R}, \Diamond)$ satisfies the following axioms: for any $x, y \in M$ and $a, b \in R$

1. $\mathcal{M}$ is MV_n-algebra.
2. $a0 = 0$.
3. $a(x \vee y) = ax \vee ay$.
4. $(a \cup b)x = ax \vee bx$.
5. $(ab)x = a(bx)$.
6. $a(x \oplus x) = ax \oplus ax$.
7. $a(x \odot x) = ax \odot ax$.
8. $x \vee aa^*x \leq a^*x \leq x \vee a^*(\sim x \wedge ax)$.

If in addition a dynamic MV_n-algebra satisfies the following condition

9. $x?y = x \wedge y$,

then it is called *test algebra*.

Notice that we may think $\langle a \rangle x$ as a function on M. The alternative notation ax is to suggest that we may think of a itself as a function, in spite of the fact that we may have $ax = bx$ for all $x \in M$ yet not have $a = b$.

In the following instead of a variable x sometimes we will use a propositional variable p. If $ap = bp$ for all p we call a and b *inseparable* and write $a \equiv b$, an equivalence relation which we shall later show to be a congruence relation on (forensic) dynamic algebras. We call *separable* any forensic dynamic algebra in which inseparability is the identity relation [8]. More precisely, forensic dynamic MV_n-algebra $\mathcal{D} = (\mathcal{M}, \mathcal{R}, \Diamond)$, $n \in Z^+$, is called *separable* iff $(\forall a_1, a_2 \in R)(\exists x \in M)(a_1 \neq a_2 \Rightarrow a_1x \neq a_2x)$. We let $\mathbf{SFD_nA}$ denote the class of separable forensic dynamic MV_n-algebras.

On R we define a quasiorder $\leq$: $a \leq b$ means that $ap \leq bp$ for all p. It follows that $\leq$ on R is reflexive and transitive but not antisymmetric, and so is a quasiorder. In a separable forensic dynamic MV_n-algebra it becomes a partial order.

Using the axioms 2, 3, 4 and 8, Pratt have proven in [11] that if $a \equiv b$ then $a^* \equiv b^*$ and hence $\equiv$ is a congruence relation on R. Moreover (a) if $a \leq b$ then $a^* \leq b^*$, (b) $a \leq a^*$, (c) $a^* = a^{**}$ [11].

Let us consider M as a lattice, and write aS for $\{as : s \in S\}$ for any $S \subset M$ and $a \in R$. We call a finitely additive (completely additive) if $a(\bigvee S) = \bigvee a(S)$ for any finite subset $S \subset M$ (for any subset $S \subset M$ for which $\bigvee S$ exists). Notice that the regular algebra operations $\cup, ;, ^*$ preserve finitely additivity (completely additivity), i.e. if a and b are finite (completely) additive, so are $a \cup b$, $a; b$, a^* [11].

Example 3.1. Full forensic dynamic MV_n-algebras. Given a complete MV_n-algebra $\mathcal{M} = (M, \oplus, \odot, \sim, 0, 1)$, let R be the set of all finitely (resp. completely) additive functions on M, with conditions $f(0) = 0$, $f(x \oplus x) = f(x) \oplus f(x)$ and $f(x \odot x) = f(x) \odot f(x)$, and let $\Diamond : R \times M \to M$ be application of elements of R to elements of M. We call it the full (completely full) forensic dynamic MV_n-algebra on M.

Example 3.2. Functional MV_n-algebra. Let W be non-empty set (of states) and

$$M_W = \{f : f \text{ is a function from } W \textit{ to } S_n\},$$

$n \in Z^+$, - the set of all functions, which is complete MV_n-algebra. More precisely, we have MV_n-algebra $(M_W, \oplus, \otimes, \sim, 0, 1)$, where $(f \circ g)(x) = f(x) \circ g(x)$, $\sim f(x) = f(\sim x)$ with conditions $f(x \circ x) = f(x) \circ f(x)$ where $\circ \in \{\oplus, \otimes\}$, and

$$R = \{r | r : M_W \to M_W \text{ is additive (completelly additive) functions}\}$$

with $r(f) = r \circ f$. Then the full functional MV_n-algebra on W is the completely full forensic dynamic MV_n-algebra on M_W.

Remark 3.3. Notice, that the full functional MV_n-algebra on W is separable. Indeed, recall that forensic dynamic MV_n-algebra $\mathcal{D} = (\mathcal{M}, \mathcal{R}, \lozenge)$, $n \in Z^+$, is called *separable* iff

$$(\forall a_1, a_2 \in R)(\exists x \in M)(a_1 \neq a_2 \Rightarrow a_1 x \neq a_2 x).$$

Then, if we take as element $x \in M$ the constant function 1, then if $a_1 \neq a_2$, then

$$a_1 1 = a_1 \circ 1 = a_1 \neq a_2 = a_2 \circ 1 = a_2 1.$$

4 Completeness Theorem

Recall that dynamic MV_n-algebra $\mathcal{D} = (\mathcal{M}, \mathcal{R}, \lozenge)$ is called separable iff

$$(\forall a_1, a_2 \in R)(\exists x \in M)(a_1 \neq a_2 \Rightarrow \lozenge(a_1, x) \neq \lozenge(a_2, x)).$$

In this case x is called a separator for the actions a_1 and a_2. $\mathbf{SFD_nA}$ denotes the class of all separable dynamic MV_n-algebras, and $\mathbf{V}_n$ denotes the variety generated by $\mathbf{SFD_nA}$, i.e. $\mathbf{V_n} = \mathcal{V}(\mathbf{SFD_nA})$.

The notion of heterogeneous algebra and products, subalgebras and homomorphisms of heterogeneous algebras can be found in [1]. A *subalgebra* $\mathcal{D}' = (\mathcal{M}', \mathcal{R}', \lozenge)$ of an algebra $\mathcal{D} = (\mathcal{M}, \mathcal{R}, \lozenge)$ is a set of subsets $M' \subset M, R' \subset R$ closed under the corresponding operations, and $\lozenge(a', x') \in M'$ for any $a' \in R'$ and $x' \in M'$. A *homomorphism* $h : \mathcal{D} \to \mathcal{D}'$ is a pair (h_1, h_2) homomorphisms $h_1 : M \to M',\ h_2 : R \to R'$, and $h(\lozenge(a, x)) = \lozenge(h_1(a), h_2(x))$. A *congruence* E on an algebra $\mathcal{D}$ is a pair of congruences (E_1, E_2) on M and R respectively, and if aE_1b and xE_2y, then $\lozenge(a, x)E_1\lozenge(b, y)$.

Let $\mathbf{D_n}$ be the variety of all forensic dynamic MV_n-algebra.

Let $\mathcal{F}(\mathbf{Var}, \mathbf{Inv})$ denote the absolutely free algebra (or term-algebra) with similarity $(2, 2, 1, 0, 0; 2, 2, 1)$ and generate by the set of variables and set of investigations. We can restrict the cardinality of the set of variables (say finite set of variable) and the cardinality of the set of ivestigations (say finite set of investigations). Then we will have finitely generated absolutely free algebra. Denote by $\mathcal{F}(\mathbf{Var}_f, \mathbf{Inv}_f)$ finitely generated absolutely free algebra.

Let $x, y..., a, b, ..., \alpha, \beta, ...$ range over the set of generators in $M, R, M \cup R$ respectively, and write M_0, R_0, D_0 for the respective generator sets. Let $\mathcal{F}_{\mathbf{V}_n}(M_0, R_0)$ denotes the free V_n-algebra (free algebra over $\mathbf{V}_n$) freely generated by the sets R_0 and M_0 as free generators of sorts MV_n-algebra and actions respectively [5]. We can represent $\mathcal{F}_{\mathbf{V}_n}(M_0, R_0)$ as $(\mathcal{F}_{\mathbf{MV}_n}(M_0), \mathcal{F}_{\mathbf{R}}(R_0), \lozenge)$.
Notice that $(\mathcal{F}_{\mathbf{MV}_n}(M_0), \mathcal{F}_{\mathbf{R}}(R_0), \lozenge)$ is a homomorphic image of the absolutely free term forensic dynamic MV_n-algebra. In other words $(\mathcal{F}_{\mathbf{MV}_n}(M_0), \mathcal{F}_{\mathbf{R}}(R_0), \lozenge)$ is a Lindenbaum algebra of the forensic dynamic Lukasiewicz logic on a finitely many generating sets.

According to well known Birkhoff's theorem we have

Theorem 4.1. *D_n-algebra $\mathcal{D} = (\mathcal{M}, \mathcal{R}, \lozenge)$ is isomorphic to a subdirect product of subdirectly irreducible D_n-algebras.*

According to this theorem $(\mathcal{F}_{\mathbf{MV}_n}(M_0), \mathcal{F}_{\mathbf{R}}(R_0), \lozenge)$ is represented as a subdirect product of subdirectly irreducible D_n-algebras where $\mathcal{F}_{\mathbf{MV}_n}(M_0)$ is a subdirect product of finite chain MV_n-algebras and $\mathcal{F}_{\mathbf{R}}(R_0)$ is a separable regular algebras. Notice that when M_0 is finite then $\mathcal{F}_{\mathbf{MV}_n}(M_0)$ is finite.

Taking into account that the variety of MV_n-algebras is locally finite and adapting Segerberg's technique of filtration (for modal logic) [14] for dynamic MV_n-algebras it holds

Theorem 4.2. *For a free forensic dynamic MV_n-algebra $\mathcal{F}_{\boldsymbol{D}_n}(M_0, R_0)$ and a finite subset M_g of $\mathcal{F}_{\boldsymbol{MV}_n}(M_0)$, there exists a forensic dynamic MV_n-algebra $\mathcal{D} = (\mathcal{M}, \mathcal{R}, \Diamond)$ and a homomorphism $f : \mathcal{F}_{\boldsymbol{D}_n}(M_0, R_0) \to \mathcal{D}$ injective on M_g, with $f(\mathcal{F}_{\boldsymbol{D}_n}(M_0, R_0))$ finite and separable.*

Theorem 4.3. *Every finite separable forensic dynamic MV_n-algebra $\mathcal{D} = (\mathcal{M}, \mathcal{R}, \Diamond)$ is isomorphic to a (finite) functional MV_n-algebra.*

Proof. Let $\mathcal{D} = (\mathcal{M}, \mathcal{R}, \Diamond)$ be a finite separable forensic dynamic MV_n-algebra. Let (W, R, V) be the Kripke model such that:

i) W is the set of all additive functions $f : M \to S_n$;
ii) the binary relation R is defined on W by

$(u, v) \in R$ if for every formula $\varphi \in \mathcal{F}_{\mathbf{MV}_n}(M_0)$ and $a \in \mathbf{Inv}$

$$u([a]\varphi) = 1 \Rightarrow v(\varphi) = 1;$$

iii) the valuation map $V : W \times \mathbf{Var} \to S_n$ is defined by

$$V(u, p) = u(p).$$

By the fact that every finite MV_n-algebra is isomorphic to ta direct product $\prod_{i \in I} S_i$, where i divides n, and by separability, $\mathcal{D}$ is isomorphic to a subalgebra of the full (hence completely full by the finiteness of M) forensic dynamic MV_n-algebra, which is a functional MV_n-algebra by definition. $\square$

From the theorems 1 - 3 we can conclude that the variety $\mathbf{V_n}$ coincides with $\mathbf{D_n}$.

Let $\theta(n) = (\theta(n)_1, \theta(n)_2$ be an equivalence relation on $\mathcal{F}(\mathbf{Var}_f, \mathbf{Inv}_f)$ defined as follows: $\alpha\theta_1\beta$ iff $\alpha \to \beta$ and $\beta \to \alpha$ are theorems of $FD\text{Ł}_n$ and $a\theta_2 b$ iff $ax = bx$ for all $x \in M$.

It holds

Theorem 4.4. $(\mathcal{F}(Var_f, \Pi_f)/\theta(n)$ *is forensic dynamic MV_n-algebra.*

Theorem 4.5. *(Completeness theorem) A formula φ of forensic dynamic logic $FD\text{Ł}_n$ is a tautology iff it is a theorem of the logic.*

Proof. It is obvious that if φ is a theorem, then φ is a tautology. Let us suppose that φ is not a theorem. Then $\varphi/\theta(n) \neq 1$ in the Lindenbaum algebra $\mathcal{F}(\mathbf{Var}_f, \mathbf{Inv}_f)/\theta(n)$ $(n \in \omega)$. $\mathcal{F}(\mathbf{Var}_f, \mathbf{Inv}_f)/\theta(n)$ is isomorphic to $\mathcal{F}_{\mathbf{V}_n}(M_0, R_0)$ for some finite M_0 and R_0. Then there exists a homomorphism $h : \mathcal{F}_{\mathbf{V}_n}(M_0, R_0) \to \mathcal{D}$ with injection on M_0, R_0 where $\mathcal{D}$ is finite and separable with $h(\varphi/\theta(n)) \neq 1$. So, φ is not a tautology. $\square$

5 Kripke semantics

Formulas can be used to describe the properties that hold after the successful investigation. For example, the formula $[a \cup b]\varphi$ means that whenever investigations a or b is successfully finalized, a state is reached where φ holds, whereas the formula $\langle (a; b)^* \rangle \varphi$ means that there is a sequence of alternating investigations of a and b such that a state is reached where φ holds. Semantically speaking, formulas are interpreted by sets of states and investigations are interpreted by binary relations over states in a Kripke model. More precisely, the meaning of $FD\text{Ł}_n$ formulas and investigations are interpreted over Kripke models (KM) $\mathcal{K} = (W, R, V)$ where W is a nonempty set of worlds or states, R is a mapping from the set $\mathbf{Inv}$ of atomic investigations into binary relations on W (i.e. $R : \mathbf{Inv} \to r : W^2 \to \{0, 1\}$) and V is a mapping from the set $\mathbf{Var}$ of atomic formulas into S_n. Informally, the mapping R assigns to each atomic investigation $a \in \mathbf{Inv}$ some binary relation $R(a)$ on W with intended meaning $xR(a)y$ iff there exists an execution of a from x that leads to

y, whereas the mapping V assigns to each pair $(p, x) \in Var \times W$, where $p \in Var$ is an atomic formula and $x \in W$, some element $V(p, x) \in S_n$ with intended meaning $V(p, x) = 1$ iff p is true in x. Given our readings of $0, \neg\varphi, \varphi \veebar \psi$, $[a]\varphi, a; b, a \cup b, a^*$ and $\varphi?$, it is clear that R and V must be extended inductively as follows to supply the intended meanings for the complex investigations and formulas:

- $xR(a; b)y$ iff there exists a world z such that $xR(a)z$ and $zR(b)y$,
- $xR(a \cup b)y$ iff $xR(a)y$ or $xR(b)y$,
- $xR(a^*)y$ iff there exists a non-negative integer n and there exist worlds $z_0, \ldots, z_n$ such that $z_0 = x$, $z_n = y$ and for all $k = 1, \ldots, n$, $z_{k-1}R(a)z_k$,
- $xR(\varphi?)y$ iff $x = y$ and $V(\varphi, y) = 1$,
- $V(\perp) = 0$.
- $V(\neg\varphi, x) = 1 - V(\varphi, x)$,
- $V(\varphi \veebar \psi, x) = V(\varphi, x) \oplus V(\psi, x)$,
- $V(\varphi \vee \psi, x) = V(\varphi, x) \vee V(\psi, x)$,
- $V([a]\varphi, x) = \bigwedge\{V(\varphi, y) : xR(a)y\}$,
- $V(\langle a\rangle\varphi, x) = \bigvee\{V(\varphi, y) : xR(a)y\}$.

If $V(\varphi, x) = 1$ then we say that φ is satisfied at state x in $\mathcal{K}$, or "$\mathcal{K}, x$ *sat* φ".

Now consider a formula φ. We say that φ is *valid* in $\mathcal{K}$ or that $\mathcal{K}$ is a model of φ, or "$\mathcal{K} \models \varphi$", iff for all worlds x, $V(\varphi, x) = 1$. φ is said to be *valid*, or " $\models \varphi$", iff for all models $\mathcal{K}$, $\mathcal{K} \models \varphi$. We say that φ is *satisfiable* in $\mathcal{K}$ or that $\mathcal{K}$ *satisfies* φ, or "$\mathcal{K}$ *sat* φ", iff there exists a world x such that $V(\varphi, x) = 1$. φ is said to be *satisfiable*, or " *sat* φ", iff there exists a model $\mathcal{K}$ such that $\mathcal{K}$ sat φ. Interestingly, sat φ iff not $\models \neg\varphi$, $\models \varphi$ iff not sat $\neg\varphi$.

Some remarkable formulas of $FD\text{Ł}_n$ are valid.

$\models [a; b]\varphi \leftrightarrow [a][b]\varphi$
$\models [a \cup b]\varphi \leftrightarrow [a]\varphi \vee [b]\varphi$
$\models [a^*]\varphi \leftrightarrow \varphi \wedge [a][a^*]\varphi$
$\models [\varphi?]\psi \leftrightarrow (\varphi \to \psi)$

Equivalently, we can write them under their dual form.

$\models \langle a; b\rangle\varphi \leftrightarrow \langle a\rangle\langle b\rangle\varphi$
$\models \langle a \cup b\rangle\varphi \leftrightarrow \langle a\rangle \wedge \langle b\rangle\varphi$
$\models \langle a^*\rangle\varphi \leftrightarrow \varphi \vee \langle a\rangle\langle a^*\rangle\varphi$
$\models \langle\varphi?\rangle\psi \leftrightarrow (\varphi \wedge \psi)$.

We define propositional forensic dynamic Łukasiewicz logic $FD\text{Ł}_n$ as the set of all formulas that are valid in all Kripke models, i.e.

$$FD\text{Ł}_n = \{\varphi : \models_{FD\text{Ł}_n} \varphi\}.$$

Completeness theorem for classical and non-classical case with respect to Kripke models was proven by many authors. Adapting the existing methods for $FD\text{Ł}_n$ it is easy to prove the following

Theorem 5.1. *(Completeness theorem) The following assertions are equivalent: for any formula φ*
i) φ is a theorem of FDL_n ($n \in Z^+$),
ii) φ is valid.

Proof. We give a sketch of the proof.

i) $\Rightarrow$ ii). It follows from the immediate inspection, i.e. showing that every axiom Ax0 - Ax8 are valid and inference rules preserve validity. It is routine to check every axiom and inference rules. But we show validity of one of them, namely the axiom Ax0. Firstly, notice that the identity

$$\bigwedge_{i\in I}(x_i \oplus y_i) = \bigwedge_{i\in I}(x_i) \oplus \bigwedge_{i\in I}(y_i) \qquad (\#)$$

holds in the MV-algebra S and, hence, in the MV_n-algebra S_n. Let $\mathcal{K}_n = (W, R, V)$ be any Kripke model. Then $V([a](\varphi \to \psi) \to ([a]\varphi \to [a]\psi), x) =$
$V(\neg[a](\neg\varphi\underline{\vee}\psi)\underline{\vee}(\neg[a]\varphi\underline{\vee}[a]\psi), x) =$
$\sim V([a](\neg\varphi\underline{\vee}\psi), x) \oplus (\sim V([a]\varphi, x) \oplus V([a]\psi), x)) =$
$\sim (\bigwedge_{y\in W}\{\sim V(\varphi, y) \oplus V(\psi, y) : xR_ay\}) \oplus (\sim \bigwedge_{y\in W}\{V(\varphi, y) : xR_ay\} \oplus \bigwedge_{y\in W}\{V(\psi, y) : xR_ay\})$. Using (#) we have
$\sim (\bigwedge_{y\in W}\{\sim V(\varphi, y) \oplus V(\psi, y) : xR_ay\}) = (\sim \bigwedge_{y\in W}\{V(\varphi, y) : xR_ay\} \oplus \bigwedge_{y\in W}\{V(\psi, y) : xR_ay\})$. So, $V([a](\varphi \to \psi) \to ([a]\varphi \to [a]\psi), x) = 1$.

ii) $\Rightarrow$ i). This part is the completeness theorem concerning Kripke models. The completeness theorem for the classical case was given by Segerberg [13], Parikh [10], Kozen and Parikh [9]. In the proof of the theorem, they mainly use the fact that the set of the subformulas of the formula is finite and by the Boolean combination on the given subformulas, we also get finite set (because of locally finiteness of Boolean algebras), and then use filtration method. Since we have locally finiteness of MV_n-algebras (which is an algebraic counterpart of n-valued Łukasiewicz logic) G. Hansoul and B. Teheux in [7] adapted the Segerberg's proof for (mono)modal n-valued Łukasiewicz logic where they have proved Kripke completeness of (mono)modal n-valued Łukasiewicz logic.

Using an abstract version of the modal logic technique of filtration, which is a Kripke structure setting is the process of dividing a Kripke model of a given formula φ by an equivalence relation on its worlds to yield a finite Kripke model of φ. Fischer and Ladner [4] showed that filtration could be made to work for propositional dynamic logic just as well as for modal logic. Prat [12] has extended their result that filtration does not depend on any special properties of Kripke structures but works for all dynamic logic. Adapting G. Hansoul and B. Teheux technique of filtration (for Łukasiewicz modal logic) [7] for (multimodal) dynamic propositional Łukasiewicz logic $FDŁ_n$ we arrive to the assertion $(ii) \Rightarrow (i)$. $\square$

6 Application

We study logical system and their Kripke semantics (Kripke frames) for an application to the forensic system. In turn, the forensic system consists of special kind of investigations interacting between themselves, depending on the state of an environment, which afterward is predetermined by investigators behavior. So, their behavior depends on so far as finding facts (evidence) possess full information about the environment and presented facts.

Our basic aim is to give to the investigators some useful tools for diagnosis about a state of a forensic system having some initial data. These data represent some properties, which may estimate, that possess some parts of a forensic system, in particular some evidence being fundamental elements of the forensic system.

6.1 A fragment of a forensic system as a Kripke Frame

In this section, we try to represent some simple fragment of a forensic system by n-valued Kripke frame with the following interpretation in forensic models that is different, but similar.

Now we give a naive definition of forensic system FS. A forensic system FS is a set of investigations with some actions between them. Identifying some investigation with a possible world and an action between investigations with the relation between corresponding words we can represent a forensic system FS as a n-valued descriptive Kripke frame.

$FragmFS = (S, Q)$, where $S = \{Fact_1, ..., Fact_n, Inv_1, ..., Inv_m\}$, forms a fragment of a forensic system with communication between its members which is expressed by some reflexive and transitive binary relation Q pointed out in Fig. 1. In the sequel, we assume that the binary relation Q is reflexive and transitive.

Now we will give some representation of a fragment of a forensic system by Kripke frame. Let $\mathfrak{J} = (W, R)$ be n-valued Kripke frame, where $R \subset W \times W$ is a binary reflexive and transitive relation on finite set W (called *the accessibility relation between possible words* from W). By the representation of a forensic system $FragmImS = (S, Q)$ by Kripke frame $\mathfrak{J} = (W, R)$ we mean a bijective function $\varphi : S \to W$ such that $(t_1, t_2) \in Q \Rightarrow (\varphi(t_1), \varphi(t_2)) \in R$.

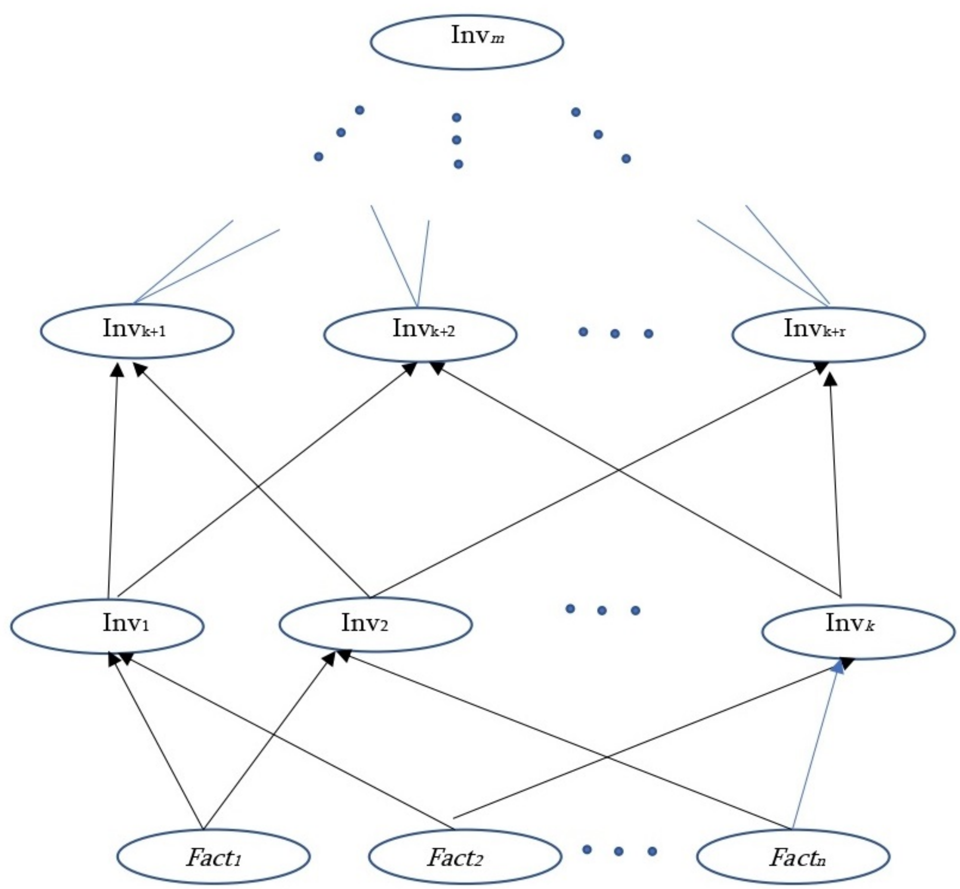


Fig. 1 Kripke frame

6.2 The modal aspects of forensic system

Given a n-valued Kripke frame $\mathfrak{J}$ which is a representation of a fragment of the forensic system, then we consider some a forensic system, represented by the Kripke frame $\mathfrak{J} = (W, R)$,where W is a finite set of forensic investigations, and let $\mathfrak{M} = (\mathfrak{J}, e)$ be n-valued Kripke model and $e : Var \times W \to S_n$. Representing a Kripke frame as a set of forensic investigations, in addition we can interpret a propositional variable $p \in Var$ as a sentence about the investigation $w \in W$. The value $e(p, w)$ expresses how much p fits a certain property of w.

We say that $w \in W$, where W is a finite set of investigations, is p-activated if $e(p, w) = 1$, it is not p-activated if $e(p, w) = 0$, it is p-activated in some degree $s \in S_n$ if $e(p, w) \in S_n - \{0, 1\}$. Note that for $w \in W$ there are finitely many ways to be p-activated for an investigation w. So, for evaluation e we have the set of points of W (i.e. the set of investigations) such that part of them are activated, part of them is not activated and part of them is activated to some degree.

A function $S : W \to S_n$ is named *a state function* (or simply *state*) if for every $w, w' \in W$ it holds

$$(w, w') \in R \Rightarrow (S(w) = 1 \Rightarrow S(w') = 1).$$

Let $e : Var \times W \to S_n$ be an evaluation. A formula φ defines a function $S^e_\varphi : W \to S_n$, such that $S^e_\varphi(w) = e(w, \varphi)$. We say that a formula φ is *labelled by the evaluation* e if S^e_φ is an state function and denote such kind of function by S^e_φ. The process of transformation of one state function S_1 $(= S^{e_1}_\varphi)$ to an another function S_2 $(= S^{e_2}_\varphi)$ will be named "$\varphi - activation$". So, for a formula φ a transferring of the state function $S^{e_1}_\varphi$ to the state function $S^{e_2}_\varphi$ is a φ-activation of points of W.

We described a forensic system as a Kripke Frame. It means that by Kripke frame we capture just the relational structure of a forensic system.

This representation of the forensic system neglects some information about the forensic system, that is some knowledge on the points w are not represented. So to recover such information we give the notion of *forensic system state function* (or simply *forensic state function* of a forensic system). This is done by a function S defined on all possible worlds to S_n. Of course S satisfies some suitable conditions, which are essentially compatibility conditions with respect to the relational structure of the forensic system. In this way we have a more faithful representation of the knowledge about the given forensic system. It is reasonable to think that to get the value $S(w)$ it is needed some intellectual work (maybe an experiment). We plan mathematically to study the set of all forensic states. Our aim is to help the investigators to have a formal and canonical way to explore the possible forensic state (function) of a forensic system. We have a variety of forensic state functions. Roughly speaking we have any allocation of the elements of S_n with any elements of W. But we need the allocations which are compatible with n-valued Kripke frame. So we single out such kinds of forensic state functions which are defined by some logical formulas, say φ, and an evaluation e is denoted as S^e_φ.

Since a forensic system, as defined in the paper, can be associated whit a logic which is complete with respect to certain Kripke frames, and since forensic system representation gives us a Kripke Frame, we use formulas of the logic of our Kripke Frame forensic system, to define some forensic states of the forensic system. Actually we use a formula φ and an evaluation e of φ, in the following way: $S^e_\varphi(w) = e(\varphi, w)$.

It is worth to note that a single formula φ essentially represents a set of forensic states (investigation), actually all such states are defined by S^e_φ when e varies in the set of all evaluations. In this way a given formula represents a collection of forensic states of the forensic system. It could be of interest to explore the possibility of checking whether given a collection of forensic states we can find a formula representing such a collection.

We defined the Activation function as a function defined on the set of all the forensic System States with value in the same set. This is a way to represent how changes the forensic information after, say an experiment, that produces new information about the forensic state values of all points w. To know facts about the function means to know facts about possible variations of the forensic state of the system, and to check whether these variations can be described by formulas.

Conflict of Interest: The authors declare that there are no conflict of interest.

References

[1] G. Birkhoff and J. D. Lipson, *Heterogeneous Algebras*, Journal of Combinatorial Theory, 8 (1970), 115-133.

[2] C. C. Chang, *Algebraic Analysis of Many-Valued Logics*, Trans. Amer. Math. Soc., 88 (1958), 467-490.

[3] J. H. Conway, *Regular Algebra and Finite Machines*, Chapman and Hall, London, (1971).

[4] M. J Fischer and R. E. Ladner, *Propositional dynamic logic of regular programs*, JCSS, 18(2) (1979).

[5] G. Grätzer, *Univarsal algebra*, Springer, New York, (1979).

[6] R. Grigolia, *Algebraic analysis of Lukasiewicz-Tarski n-valued logical systems*, Selected papers on Lukasiewicz Sentential Calculi, Wroclaw, (1977), 81-91.

[7] G. Hansoul and B. Teheux, *Completeness results for many-valued Lukasiewicz modal systems and relational semantics*, (2006).

[8] D. Kozen, *A representation theorem for models of ∗-free PDL*, Technical Report RC7864, IBM, September, (1979).

[9] D. Kozen and R. Parikh, *An Elementary Proof of the completeness of PDL*, Theoretical Computer Science, North-Holland Publishing Company, 14 (1981), 113-118.

[10] R. Parikh, *A completeness result for PDL*, in Symposium on Mathematical Foundations of Computer Science, Zakopane, Warsaw, Sept., (1978).

[11] V. R. Pratt, *Dynamic algebras and the nature of induction*, 12th ACM Symposium on Theory of Computation, Los Angeles, April, (1980).

[12] V. Pratt, *Dynamic algebras: Examples, constructions, applications*, Stud Logica 50, (1991), 571-605.

[13] K. Segerberg, *A completeness theorem in the modal logic of programs*, Notices AMS, 24(6) (1977), A-522.

[14] K. Segerberg, *An Essay in Classical Modal Logic*, Ph.D. Thesis, Stanford University, May, (1971).

Antonio Di Nola
Department of Mathematics
University of Salerno
Italy.
E-mail: adinola@unisa.it

Revaz Grigolia
Department of Mathematics
Tbilisi State University

Department of Mathematical Cybernetics
Georgian Technical University
Georgia
E-mail: revaz.grigolia@tsu.ge

Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://tfss.journals.iau.ir/
Online ISSN: 2821-0131
Vol.1, No.2, (2022), 72-89
DOI: http://10.30495/tfss.2022.1961482.1039

Article Type: Original Research Article (Invited Paper)

Markov Characteristics for IFSP and IIFSP

Nan Jiang*, **Wei Li**, **Fei Li**, **Jun-tao Wang**

Abstract. As the research object of modern nonlinear science, a fractal theory has been an important research content for scholars since it comes into the world. Moreover, iterated function system (IFS) is a significant research object of fractal theory. On the other hand, the Markov process plays an important role in the stochastic process. In this paper, the iterated function system with probability(IFSP) and the infinite function system with probability(IIFSP) are investigated by using interlink, period, recurrence and some related concepts. Then, some important properties are obtained, such as: 1. The sequence of stochastic variable $\{\zeta_n, (n \geq 0)\}$ is a homogenous Markov chain. 2. The sequence of stochastic variable $\{\zeta_n, (n \geq 0)\}$ is an irreducible ergodic chain. 3. The distribution of transition probability $p_{ij}^{(n)}$ based on $n \to \infty$ is a stationary probability distribution. 4. The state space can be decomposed of the union of the finite(or countable) mutually disjoint subsets, which are composed of non-recurrence states and recurrence states respectively.

AMS Subject Classification 2020: MSC 60J05; MSC 60J10; MSC 60J20
Keywords and Phrases: Fractal, Markov process, Iterated function system, Probability.

1 Introduction

As the research object of modern nonlinear science, a fractal theory has been an important research content for scholars since it comes into the world. One of its most important features is that fractal can truly describe natural phenomena. It was founded by an American mathematician Mandelbrot in the 1970s. With the vigorous promotion of Mandelbrot, fractal makes people's understanding of object shape change from regular to irregular gradually, and provides a new mathematical tool for the research of nonlinear characteristics and irregular phenomena. Fractal is being applied and explored in many fields with a new concept and theory, and its research has also greatly expanded the human cognitive domain [8, 10].

There are a lot of both this and that phenomenon, which have no clear boundary in the real world. This makes it necessary to go through a continuous repetition and accumulation change process from complete coincidence to complete non coincidence. In other words, the fractal set in nature cannot be described by the characteristic function with only two values of 0 or 1 in classical set theory. In 1965, American cybernetic expert Zadeh extended the value range of characteristic function in classical set theory from $\{0, 1\}$ to the closed interval $[0, 1]$, which is the core idea of fuzzy set. In order to apply fractal set to this fuzzy phenomenon, Xie Heping et al gave the concept of fuzzy fractal in 1990.

Iterated function system (IFS) is a significant research object of fractal theory founded by Hutchinson [6]. As the framework of fractal theory, the affine transformation is a critical mathematical tool. According

∗**Corresponding author:** Nan Jiang, Email: jiangnan@xsyu.edu.cn, ORCID: 0000-0002-8116-620X
Received: 21 June 2022; Revised: 8 July 2022; Accepted: 9 July 2022; Published Online: 7 November 2022.

How to cite: Nan Jiang, Wei Li, Fei Li and Jun-tao Wang, Markov Characteristics for IFSP and IIFSP, *Trans. Fuzzy Sets Syst.*, 1(2) (2022), 72-89.

to the self similar formation of the part and whole about the objective object, the overall shape is iterated on the basis of some affine transformations with a given probability until emerging a pretty fractal figure. With the help of the powerful iterative computing ability of a computer, IFS applies the essence of fractal theory such as self similarity, a multiplicity of levels and unity of rules at different levels to the field of computer graphics, and can produce many graphics with infinite detail and exquisite texture.

Markov process plays an important role in the stochastic process. The mathematician Markov proposed an interesting chain in 1907 called Markov chain nowadays. So far, it has formed a branch of mathematics with rich content, complete theory and wide application. The Markov process is a significant method for researching the discrete state space based on the theory of stochastic process [18].

The random fractal is another important field of the fractal theory. Because it is closely related to nature, it has become hot research in recent years. Moreover, the random iterative function system is also an important research object of the random fractal, and many scholars have done a lot of research work in this field. Such as Joanna used two methods to construct a gradually stable IFSP [11]. Weihrauch et al studied the IFSP from the perspective of measure [3, 9, 21, 22]. Andrzej et al researched the fractal dimension through IFSP [1, 4, 5, 16]. There is also some literature have researched the related problems between IFSP and Markov process [7, 13, 15, 19]. In particular, John et al used the Markov process to study the linear properties of IFS as early as 1990 [12]. However, some specific Markov Characteristics for IFSP and IIFSP almost no one has studied yet. Therefore, Our main purpose is to find and prove specific Markov Characteristics for IFSP and IIFSP in this paper.

In this article, we first review some important concepts and properties of the stochastic process and fractal theory in section 2. Then, we will introduce the homogeneous property for IFSP in section 3. In section 4, we will introduce the ergolic property for IFSP. In section 5, we will introduce the distribution property for IFSP. In section 6, we will introduce the decomposition of state space for IFSP. Then, we will extend the Markov Characteristics to IIFSP in section 7. The last section is conclusion and future work.

2 Preliminaries

Before studying the Markov Characteristics for IFSP and IIFSP, we introduce some important concepts that will be useful later in the subsections. First, we begin with the concept of the fractal theory.

Definition 2.1. [14] Let $y_n \in Y$ be a point sequence, if there exists a positive integer number $N(\varepsilon)$, such that $d(y_m, y_n) < \varepsilon$ for all $m, n > N(\varepsilon)$ and $\forall \varepsilon > 0$, then y_n is said to be a Cauchy sequence. Further, (Y, d) is known as a complete metric space, if each Cauchy sequence in Y converges to a point y in Y.

According to the complete metric space, we give the definitions of iterated function system(IFS) and hyperbolic iterated function system (HIFS).

Definition 2.2. [2, 14, 17] Suppose (Y, d) be a complete metric space, if there exists a family of continuous functions $f_k(k \in \{1, 2, \cdots, N\})$: Y → Y. Then

$$\{Y; f_k, k \in \{1, 2, \cdots, N\}\}$$

is called an iterated function system(IFS). Further,

$$\{Y; H; f_k, k \in \{1, 2, \cdots, N\}\}$$

is called a hyperbolic iterated function system (HIFS), if $f_k(k \in \{1, 2, \cdots, N\})$ is the contraction mapping (for $\forall x, y \in X$, there exists $0 \leq \alpha < 1$, such that $d(f_k(x), f_k(y)) \leq \alpha d(x, y))$ based on (Y, d).

With the help of IFS, HIFS and probability vector, we will introduce the definition of IFSP and HIFSP which are the most important concepts in this paper.

Definition 2.3. [2, 14, 17] Let $\{Y; f_k, k \in \{1, 2, \cdots, N\}\}$ be an IFS, and $P = \{p_1, p_2, \cdots, p_N\}$ be a probability vector, where $\sum_{i=1}^{N} p_i = 1$ and $p_i \geq 0$ for all $i \in \{1, 2, \cdots, N\}$. We have $P(\zeta_n = i) = p_i$ for the given independent random variables sequence $\zeta_1, \zeta_2, \cdots, \zeta_n, \cdots$, where $i \in \{1, 2, \cdots, N\}; n \in \{1, 2, 3, \cdots\}$. Then we call it is an iterated function system with probability(IFSP), denoted by

$$\{Y; P; f_k, k \in \{1, 2, \cdots, N\}\}.$$

Further,

$$\{Y; H; P; f_k, k \in \{1, 2, \cdots, N\}\}$$

is known as a hyperbolic iterated function system (HIFSP), if $f_k(k \in \{1, 2, \cdots, N\})$ is the contraction mapping based on (Y, d).

Theorem 2.4. *[2, 14, 17] Let $\{Y; H; f_k, k \in \{1, 2, \cdots, N\}\}$ be a HIFS based on (Y, d). Then $A \subset Y$ is a unique non-empty compact subset, such that*

$$A = f(A) = \bigcup_{k=1}^{N} f_k(A), \tag{1}$$

and $f^n(A_0) \longrightarrow A$, where A_0 is any element of the all nonempty compact subsets of Y. A is the attractor (or invariant set) of IFS.

The next theorem will give the relationship of probability between the random iterative sequence and the attractor of IFS through hausdorff distance h. $(h(A, B) = max\{d(A, B), d(B, A)\}, d(A, B) = max_{a \in A}\{min_{b \in B} d(a, b)\})$.

Theorem 2.5. *[2, 14, 17] Let $\{Y; H; P; f_k, k \in \{1, 2, \cdots, N\}\}$ be a HIFSP. For any $y_0 \in Y$, let $y_{n+1} = f_{\zeta_n}(y_n), n \in \{0, 1, 2, \cdots\}.$, where $P(\zeta_n = i) = p_i, i \in \{1, 2, \cdots, N\}$. Then there exists $n_0 = n_0(\varepsilon)$ and $k_0 = k_0(\varepsilon)$, for any $\varepsilon > 0$, we have*

$$P\{h(\{y_n, y_{n+1}, \cdots, y_{n+k}\}, A) < \varepsilon\} > 1 - \varepsilon, \tag{2}$$

if $n \geq n_0$ and $k \geq k_0$, where A is the attractor (or invariant set) of IFS.

The above theorem tells us such a fact that if we remove n_0 items in front of the random iterative sequence $\{y_n\}$. So the possibility of the hausdorff distance between sufficiently long sequence $\{y_n, y_{n+1}, \cdots, y_{n+k}\}$ and the attractor is less than ε will exceed $1 - \varepsilon$. We have introduced the related concepts and properties of IFS in the previous. And we will introduce some significant concepts about Markov process in the sequel, which play an important role in this paper.

Definition 2.6. [18, 20, 23] Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $\{X(n), (n \geq 0)\}$ be a random sequence. Then we have

$$\begin{aligned} &P\{X(t_{m+1}) = i_{m+1} | X(t_1) = i_1, X(t_2) = i_2, \cdots, X(t_m) = i_m\} \\ &\qquad = P\{X(t_{m+1}) = i_{m+1} | X(t_m) = i_m\}, \end{aligned} \tag{3}$$

if for any $m \geq 1$, and nonnegative integer $t_1 < t_2 < \cdots < t_m < t_{m+1}$, where $i_1, i_2, \cdots, i_{m+1} \in E$, E is the state space of $\{X(n), (n \geq 0)\}$. If the conditional probabilities at both ends of the equation are meaningful, then $\{X(n), (n \geq 0)\}$ is a Markov chain.

The above equation is often called the markov attribute(or the attribute of no aftereffect) in the Markov process. We find the random variables at each time have a certain dependence(i.e.,non independence) in the above Definition. More specifically, the past only affects the present, not the future.

Definition 2.7. [18, 20, 23] The equation

$$p_{ij}^{\mu}(m) = P\{X(m+\mu) = j|X(m) = i\}, i, j \in E, \mu \geq 1, \tag{4}$$

is said to be the $\mu - step$ transition probability of transferring to state j after μ steps, if the system is in state i at m.

It is said to be one-step transition probability apparently if $\mu = 1$, and short for transition probability. As we know, $p_{ij}^{\mu}(m)$ has the following important properties since it is a probability.

$$\begin{cases} p_{ij}^{(\mu)}(m) \geq 0, \quad j \in E, \\ \sum_{j\in I} p_{ij}^{(\mu)}(m) = \sum_{j\in I} P\{X(m+\mu) = j|X(m) = i\} = 1. \end{cases} \tag{5}$$

Then the matrix

$$P^{(\mu)}(m) = (p_{ij}^{(\mu)}(m))_{i,j\in E}, \quad m \in T = \{0, 1, 2, \cdots\},$$

is said to be the k steps transition matrix of $\{X(m)\}$.

It is not difficult to see $\{p_{ij}^{(\mu)}(m), j \in E\}$ is a probability distribution for any given $i \in E$ and $m \geq 0, \mu \geq 1$. We will introduce other important concepts called absolute probability and initial probability in the next Definition.

Definition 2.8. [18, 20, 23] $p_j(\mu) = P\{X(\mu) = j, j \in E\}$ is known as absolute probability, if μ is a nonnegative integer. Particularly, $p_j = p_j(0) = P\{X(0) = j, j \in E\}$ is known as initial probability.

Similarly, $p_j(\mu)$ and p_j also has the same properties as below:

$$\begin{cases} p_j(\mu) \geq 0, \quad j \in E, \\ \sum_{j\in I} p_j(\mu) = 1, \end{cases} \tag{6}$$

$$\begin{cases} p_j \geq 0, \quad j \in E, \\ \sum_{j\in I} p_j = 1, \end{cases} \tag{7}$$

Therefore, $\{p_j(\mu), (\mu \geq 0)\}$ and $\{p_j\}$ are both probability distributions. Particularly, $\{p_j\}$ is also called initial distribution. And $\{p_j(\mu), (\mu \geq 0)\}$ is the one dimension distribution in Markov chain known as absolute distribution commonly.

Theorem 2.9. *[18, 20, 23] Let $\{X(n), n \geq 0\}$ be a Markov chain. Then the following formula holds for any nonnegative integer μ, ν, m.*

$$p_{ij}^{(\mu+\nu)}(m) = \sum_{s\in E} p_{is}^{(\mu)}(m) p_{sj}^{(\nu)}(m+\mu), i, j \in E. \tag{8}$$

The above equation is known as Chapman–Kolmogorov equation, abbreviated as C-K equation. The C-K equation is an important result of transition probability.

3 The homogeneous property for IFSP

Homogeneity is a very important mathematical property, which describes the change characteristics of transition probability. In this section, we will discuss the homogeneity for IFSP through transition probability in Markov process.

Definition 3.1. [18, 20, 23] The Markov chain $\{X(n), n \geq 0\}$ is called homogeneous, if its one-step transition probability $\{p_{ij}(m), i, j \in E\}$ is independent of m, where E is the state space.

We will consider stochastic dynamical system which is determined by IFSP in Definition 2.3. In the stochastic dynamical system:

$$y_{n+1} = f_{\zeta_n}(y_n), n \in \{0, 1, 2, \cdots\}.$$

As we know, the steps of this iterative process are: first, we take the origin point $y_0 \in Y$, then we take the value of probability p_{j_0}

$$y_1 = f_{j_0}(y_0),$$

further, we take the value of probability p_{j_1}

$$y_2 = f_{j_1}(y_1).$$

In the same way, we obtain a random iterative sequence $\{y_n, (n \geq 0)\}$ by iterating one by one. What interests us is to select a random variable sequence $\{\zeta_n, (n \geq 0)\}$ that is determined by the random iterative sequence $\{y_n, (n \geq 0)\}$. $(\zeta_n = g(y_n) = j_n, j_n \in \{1, 2, \cdots, N\})$

Proposition 3.2. *Let the random variable sequence $\{\zeta_n, (n \geq 0)\}$ be determined by the random iterative sequence $\{y_n, (n \geq 0)\}$ on the IFSP, then $\{\zeta_n, (n \geq 0)\}$ is a finite homogeneous Markov chain.*

Proof. Due to the characteristic of random variable sequence $\{\zeta_n, (n \geq 0)\}$ in IFSP, it's not hard for us to find out y_{n+1} is determined by $y_0, y_1, \cdots, y_n$, then the following conditional probability equation holds:

$$P\{y_{n+1} = f_{j_n}(y_n) | y_1 = f_{j_0}(y_0), y_2 = f_{j_1}(y_1), \cdots, y_n = f_{j_{n-1}}(y_{n-1})\}$$
$$= P\{y_{n+1} = f_{j_n}(y_n) | y_n = f_{j_{n-1}}(y_{n-1})\}. \tag{9}$$

Where $j_0, j_1, \cdots, j_n \in \{1, 2, \cdots, N\}$, since $\zeta_n = j_n; n \in \{0, 1, 2, \cdots\}$. Thus $\{1, 2, \cdots, N\}$ is the state space of random variable sequence $\{\zeta_n, (n \geq 0)\}$, ζ_n is determined by x_n. So we obtain:

$$P\{\zeta_n = j_n | \zeta_0 = j_0, \zeta_1 = j_1, \cdots, \zeta_{n-1} = j_{n-1}\}$$
$$P\{\zeta_n = j_n | \zeta_{n-1} = j_{n-1}\}, \tag{10}$$

where $j_0, j_1, \cdots, j_n \in \{1, 2, \cdots, N\}$, Obviously, the probabilities at both ends of the equation make sense. So $\{\zeta_n, (n \geq 0)\}$ is a Markov chain. We will proof it is finite and homogeneous in the next.

In the iterative process of IFSP, we let

$$p_{ij}(m) = P\{\zeta_{m+1} = j | \zeta_m = i\}, i, j \in \{1, 2, \cdots, N\}$$

is the transition probability from state i to state j after m-th iteration of the stochastic system. Obviously, $p_{ij}(m)$ is the one-step transition probability of $\{\zeta_n, (n \geq 0)\}$. The iterative process is independent of m, and N is a finite positive integer in the state space. Thus $\{\zeta_n, (n \geq 0)\}$ is a finite homogeneous Markov chain. This completes the proof. □

Example 3.3. (General random walk) There is a particle in the line segment $[1, 3]$. It can only stay at the three points $1, 2, 3$, one movement per second. The move rule is: the particle is at any one of the points $1, 2, 3$ before moving, it either stays where it is or moves to any of the remaining three points in the next second, the probability are both $\frac{1}{3}$.

Firstly, according to the above example, we can construct a model of IFSP. i.e.,let

$$\{Y; P; f_k, k \in \{1, 2, 3\}\}$$

be an IFSP, where $\{x_n, n \geq 0\}$ is a random iterative sequence,

$$P = \{p_1, p_2, p_3\} = \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$$

is the probability vector,and

$$f_{\zeta_n}(y) = \zeta_n, \quad \zeta_n \in \{1, 2, 3\}, \quad y \in \{1, 2, 3\}$$

is the iterated function.

Secondly, we can let $X(n) = \zeta_n = i$ be the particle is at point i at $t = n$ $(i = 1, 2, 3, n = 0, 1, 2, \cdots)$ through the above analysis. Then $\{y_n, (n \geq 0)\}$ is a random sequence, and the state space is $E = \{1, 2, 3\}$. Thus, ζ_n is a finite homogeneous Markov chain based on IFSP, owing to

$$P\{X(m+1) = j|A, X(m) = i\} = P\{X(m+1) = j|X(m) = i\}$$
$$= p_{ij}(m) = \frac{1}{3},$$

where A is known as any one event, which is determined by $X(0), \cdots, X(m-1)$.

Finally, we will give the transition probability matrix (one step) of $\{X(m)\}$ in the following.

$$P = (p_{ij}) = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

It not difficult to obtain the two step (or n step) transition probability matrix of $\{X(m)\}$ in the following.

$$P^{(2)} = (p_{ij}^{(2)}) = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$
$$= (p_{ij}^{(n)}) = P^{(n)}.$$

Example 3.4. (The random walk that cannot cross the wall) There is a particle in the line segment $[1, 5]$. It can only stay at the five points $1, 2, 3, 4, 5$, one movement per second. The move rule is: the particle is at any one of the points $2, 3, 4$ before moving, it either stays where it is, or move one space to the left, or move one space to the right in the next second, the probability are both $\frac{1}{3}$. If the particle is at 5 before moving, then it will move to point 4 with a probability of 1 in the next second. If the particle is at 1 before moving, then it will move to point 2 with a probability of 1 in the next second. Because 1 and 5 are "insurmountable walls" of the particle.

Firstly, similar to Example 3.3, we can construct a model of IFSP. i.e., let

$$\{Y; P; f_k, k \in \{1, 2, 3\}\}$$

be an IFSP, where $\{x_n, n \geq 0\}$ is a random iterative sequence.

$$f_1(y) = y, f_2(y) = y - 1, f_3(y) = y + 1$$

are the iterated functions,

$$P = \{p_1, p_2, p_3\} = \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$$

is the probability vector, if the particle is at any one of the points $2, 3, 4$ before moving.

$$P = \{p_1, p_2, p_3\} = \{0, 0, 1\}$$

is the probability vector, if the particle is at the point 1 before moving.

$$P = \{p_1, p_2, p_3\} = \{0, 1, 0\}$$

is the probability vector, if the particle is at the point 5 before moving. It's quite easy to know, $\zeta_n \epsilon \{1, 2, 3\}$ which is determined by $X(n) = i$, so it is a finite homogeneous Markov chain based on IFSP due to Proposition 3.2.

Secondly, we can let $X(n) = i$ be the particle is at point i at time $t = n$ $(i = 1, 2, 3, 4, 5, n = 0, 1, 2, \cdots)$. Then $\{y_n, (n \geq 0)\}$ is a random sequence, and the state space is $E = \{1, 2, 3, 4, 5\}$. Thus, $X(n) = i$ is also a finite homogeneous Markov chain, owing to

$$P\{X(m+1) = j|A, X(m) = i\} = P\{X(m+1) = j|X(m) = i\}$$

$$= p_{ij}(m) = \begin{cases} 1, & if|j-i| = 1, i = 1, 5, \\ \frac{1}{3}, & if|j-i| \leq 1, i = 2, 3, 4, \\ 0, & otherwise, \end{cases}$$

where A is known as any one event, which is determined by $X(0), \cdots, X(m-1)$.

Finally, we will give the transition probability matrix (one step) of $\{X(m)\}$ in the following.

$$= P = (p_{ij}) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

It not difficult to obtain the transition probability matrix (two step) of $\{X(m)\}$ in the following.

$$P^{(2)} = (p_{ij}^{(2)}) = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{9} & \frac{5}{9} & \frac{2}{9} & \frac{1}{9} & 0 \\ \frac{1}{9} & \frac{2}{9} & \frac{1}{3} & \frac{2}{9} & \frac{1}{9} \\ 0 & \frac{1}{9} & \frac{2}{9} & \frac{5}{9} & \frac{1}{9} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

4 The ergodic property for IFSP

The development of the physical system can be regarded as a random process from the viewpoint of a quantitative relationships. The physical system always reaches equilibrium after a period of time, if there is no significant change in the reasons affecting the development of the system. It is of great significance to expose the internal law of this phenomenon with mathematical theory. This law is called ergodicity in a random process. More specifically, The ergodic property is to study the limit case of transition probability $p_{ij}^{(m)}$, for $m \to \infty$.

"Recurrent" is an important concept in the Markov chain. We can use it to further reveal many characteristics of the state. For the state j, we can pull-in random variables

$$F_j = min\{m : X(m) = j, (m \geq 1)\},$$

It indicates the time when the system enters the state j for the first time. If the set on the right of the above formula is empty (i.e. for any $m \geq 1, Y(m) \doteq j$), we stipulate $min\phi = +\infty$, and let

$$g_{ij}^{(m)} = P\{F_j = m|X(0) = i\}, m \geq 1,$$

be the probability that the system first reaches state j after m steps from state i. Now, we let

$$g_{ij} = \sum_{m=1}^{\infty} g_{ij}^{(m)} = \sum_{m=1}^{\infty} P\{F_j = m|(X0) = i\}$$

$$= P\{F_j < +\infty|X(0) = i\},$$

be the probability that the system will arrive sooner or later reaches state j from state i. In particular, g_{jj} means the probability that the system starting from state j and returning to state j sooner or later if $i = j$.

Definition 4.1. [18, 20, 23] The state j is said to be recurrence, if $g_{jj} = 1$; The state j is known as non-recurrence (or transience), if $g_{jj} < 1$.

Definition 4.2. [18, 20, 23] The greatest common divisor T of the positive integer set $\{m : m \geq 1, p_{jj}^{(m)} > 0\}$ is said to be the period of state j for state j, if the set is non empty. The state j is called periodic, if $T > 1$. The state j is known as aperiodic, if $T = 1$. The period of the state j cannot be defined, if the positive integer set $\{m : m \geq 1, p_{jj}^{(m)} > 0\}$ is an empty set.

Remark 4.3. Given recurrence state j, owing to

$$g_{jj} = P\{F_j < +\infty\} = 1,$$

this shows that starting from state i must return to itself. We can further subdivide the recurrence state, because of

$$g_{jj} = \sum_{m=1}^{\infty} g_{jj}^{(m)} = 1,$$

thus $g_{jj}^{(m)}$ is a probability distribution. We can describe this phenomenon with mathematical expectation, i.e

$$\nu_j = \sum_{m=1}^{\infty} m g_{jj}^{(m)} = \sum_{m=1}^{\infty} mP\{F_j = m|X(0) = j\}$$

$$= E\{F_j|X(0) = j\}.$$

It not hard to get $\nu_j \geq 1$, which signifies the mean of times (or steps) that the system starting from state j and also returning to state j. The state j is said to be positive recurrence, if $\nu_j < +\infty$. And the state j is known as null recurrence, if $\nu_j = +\infty$. Then the aperiodic and positive recurrence state is called ergodic state.

Lemma 4.4. *[18, 20, 23] Let j be a recurrence state and its period is T, then*

$$\lim_{m\to\infty} p_{jj}^{(mT)} = \frac{T}{\nu_j}.$$

The right end of the equation is equal to zero when $\nu_j = +\infty$. Most notably, the necessary and sufficient condition of the positive recurrence state j is

$$\overline{\lim_{m\to\infty}} p_{jj}^{(m)} > 0.$$

Definition 4.5. [18, 20, 23] For the state i and j, if there exists $m \geq 1$ satisfy $p_{ij}^{(m)} > 0$, i.e., Starting from the state i, after certain m steps, it can reach the state j. Then it is called the state i can reach state j, denoted by $i \to j$. Then, the state i and j are said to be interlinked, if $j \to i$ hold simultaneously. A chain is called irreducible, if any two states are interlinked in this chain.

Based on the above definitions and lemma, we will investigate the ergodic property for Markov chain $\{\zeta_n, (n \geq 0)\}$ in next.

Theorem 4.6. *Let $\{\zeta_n, (n \geq 0)\}$ be the random variable sequence which is determined by the random iterative sequence $\{y_n, (n \geq 0)\}$ in the IFSP. Then $\{\zeta_n, (n \geq 0)\}$ is an irreducible ergodic chain.*

Proof. Let $\{1, 2, \cdots, N\}$ be the state space of random variable sequence $\{\zeta_n, (n \geq 0)\}$, j is an any state in the state space$\{1, 2, \cdots, N\}$. According to the iterative process in IFSP, it is obvious that the positive integer set $\{n : n \geq 1, p_{jj}^{(n)} > 0\}$ is an empty set, and its greatest common divisor is $T = 1$. Therefore, the state j is called aperiodic owing to Definition 4.2.

Next, we will show the state j of the random variable sequence $\{\zeta_n, (n \geq 0)\}$ is a positive recurrence. It is not difficult to verify that the upper limit of transition probability of the state j returns to itself after n-step iteration is always greater than zero. i.e.

$$\overline{\lim_{n\to\infty}} p_{jj}^{(n)} > 0.$$

This completes the proof due to Lemma 4.4. Thus the state i of the random variable sequence $\{\zeta_n, (n \geq 0)\}$ is ergodic.

Finally, we obtain the state i and j are interlinked, for $\forall i, j \in \{1, 2, \cdots, N\}$, by the arbitrariness of iteration in IFSP. i.e.,

$$i \longleftrightarrow j.$$

It shows this chain is irreducible. Therefore, $\{\zeta_n, (n \geq 0)\}$ is an irreducible ergodic chain. $\square$

5 The distribution property for IFSP

The law of probability distribution is used to describe the random value of probability variables. The stationary distribution is an important type of probability distribution, which has a certain kind of invariable property. It is often used to describe some characteristics of Markov process. In this section, we will consider the relationship between IFSP and Markov chain through the property of stationary distribution. The definition of stationary distribution in Markov chain and some properties will be given in the sequel.

Definition 5.1. [18, 20, 23] A probability distribution $\{u_j, j \in E\}$ is called stationary in the homogeneous Markov chain, if it satisfies

$$u_j = \sum_{i\in I} u_i p_{ij}, j \in E.$$

Remark 5.2. For the stationary distribution $\{u_j\}$, if $n \geq 1$ is integer number. It not difficult to verify the following equation hold

$$u_j = \sum_{i \in I} u_i p_{ij}^{(n)}, j \in E$$

Therefore, the initial distribution of the homogeneous Markov chain is stationary.

In the light of the ergodic in the previous section, the Markov chain $\{y_n, (n \geq 0)\}$ is known as ergodic, if there exists a constant π_j which be independent of i such that the following equation, for all state i and j of the homogeneous Markov chain $\{y_n, (n \geq 0)\}$.

$$\lim_{n \to \infty} p_{ij}^{(n)} = \pi_j.$$

It means the probability of transferring to the state j is approach to a constant π_j no matter what state the system starts, if the "transition step" n is large enough. We will give an important property related to the constant π_j in the next lemma.

Lemma 5.3. *[18, 20, 23] Let $\{y_n, (n \geq 0)\}$ be a finite homogeneous Markov chain (without losing generality, we can set the state space $E = \{1, 2, \cdots, N\}$). The Markov chain is ergodic, if there exists positive integer t for all state i and j satisfy*

$$p_{ij}^{(n)} > 0.$$

Therefore $\lim_{n \to \infty} p_{ij}^{(n)} = \pi_j$, where the constant π_j is independent of i. Moreover, $\pi_j (j \in \{1, 2, \cdots, N\})$ is the unique solution in the following formulas

$$\pi_j = \sum_{i=1}^{N} \pi_i p_{ij}, \quad j \in \{1, 2, \cdots, N\},$$

if it satisfies the conditions

$$\pi_j > 0, \quad j \in \{1, 2, \cdots, N\}, \quad \sum_{j=1}^{N} \pi_j = 1.$$

In the next theorem, we will discuss the distribution property of random iterative sequences based on IFSP.

Theorem 5.4. *Let $\{\zeta_n, (n \geq 0)\}$ be the random variable sequence which is determined by the random iterative sequence $\{y_n, (n \geq 0)\}$ on the IFSP. The state $i, j \in \{1, 2, \cdots, N\}$ are any two states of the state space. Then the limit distribution of transition probability $p_{ij}^{(n)}$ is a stationary probability distribution, if the state i transfers to the state j after n-step iteration.*

Proof. $\{\zeta_n, (n \geq 0)\}$ is an irreducible ergodic (aperiodic and positive recurrence) chain owing to Theorem 4.6. Then we will obtain the following equation through Lemma 5.3.

$$\lim_{n \to \infty} p_{ij}^{(n)} = \frac{1}{\psi_j}, \quad j \in \{1, 2, \cdots, N\}.$$

where $\frac{1}{\psi_j}$ is similar to π_j in Lemma 5.3. On the other hand, according to Definition 2.7 we get

$$\sum_{j=1}^{N} p_{ij}^{(n)} = 1.$$

Through the C-K equation

$$p_{ij}^{(\mu+\nu)} = \sum_{k=1}^{N} p_{ik}^{(\mu)} p_{kj}^{(\nu)}.$$

Without loss of generality, let $\mu = m$, and $\nu = n$. Then if $m \to \infty$, we gain

$$\frac{1}{\psi_j} = \sum_{k=1}^{N} (\frac{1}{\psi_k}) p_{kj}^{(n)}, \quad j \in \{1, 2, \cdots, N\}, \quad n \geq 1. \tag{11}$$

Thus $\frac{1}{\psi_j}$ is a stationary distribution. We will proof it is also a probability distribution. Also let $n \to \infty$, we have

$$\frac{1}{\psi_j} = \sum_{k=1}^{N} (\frac{1}{\psi_k})(\lim_{n\to\infty} p_{kj}^{(n)})$$

$$= \sum_{k=1}^{N} (\frac{1}{\psi_k})(\frac{1}{\psi_j}), \quad j \in \{1, 2, \cdots, N\}. \tag{12}$$

From the above equation, we immediately get

$$\sum_{k=1}^{N} \frac{1}{\psi_k} = 1.$$

Therefore, the limit of $p_{ij}^{(n)}$, i.e. $\{\frac{1}{\psi_j}, j \in \{1, 2, \cdots, N\}\}$ is a probability distribution. This completes the proof. □

6 The decomposition of state space for IFSP

The state space is an important concept in Markov process. The system state and the minimum number of variables in the system can be determined by the ordered set of variables known as the state. Therefore, the set of all possible states in the system constitutes a state space. It can considered to be the space with state variables as the coordinate axis. In this section, we will make a new state space that has a little different from the above, then extend the related properties of the Markov process to the IFSP.

Definition 6.1. $\{Y; P; f_1, f_2, \cdots, f_N\}$ is an IFSP, then

$$E = \{y : f_{\zeta_n}(y_n) = y, n \in \{0, 1, 2, \cdots\}\}$$

is the state space based on IFSP, where $\{y_n, (n \geq 0)\}$ is an random iterative sequence related to IFSP.

Definition 6.2. The subset D of the state space E based on IFSP is called a closed set, if the state inside D cannot reach the state outside D. i.e., $p_{ij} = 0$, for any $i \in D$, and $j \in E - D$.

As can be seen from the above definition, once the particle enters a closed set, it will always move in it and cannot reach the outside, denoted by

$$p_{ij}^{n} = 0, n \geq 1.$$

It not difficult to find E is the maximum closed set, and the minimum closed set is compose by all absorption state j, i.e. $p_{jj} = 1$.

Proposition 6.3. *A is the minimum closed set, if A is the attractor of a HIFS.*

According to Theorem 2.4, if A is the attractor of a HIFS, it can be deduced that once the random iterative sequences $y(n)$ enter to the attractor A, it can not get out. i.e.,

$$P_{ij} = 0, i \in A, j \in E - A.$$

Proposition 6.4. *Let $\{Y; H; P; f_k, k \in \{1, 2, \cdots, N\}\}$ be a HIFSP, if A is the minimum closed set based on it, then for any $y_0 \in Y$, let $y_{n+1} = f_{\zeta_n}(y_n), n \in \{0, 1, 2, \cdots\}.$, where $P(\zeta_n = i) = p_i, i \in \{1, 2, \cdots, N\}$, there exists $n^* = n^*(\varepsilon)$ and $k^* = k^*(\varepsilon)$, for any $\varepsilon > 0$, such that if $n \geq n^*$ and $k \geq k^*$, we have*

$$P\{h(\{y_n, y_{n+1}, \cdots, y_{n+k}\}, A) < \varepsilon\} > 1 - \varepsilon.$$

Proof. A is the minimum closed set based on a HIFSP in the above proposition, it not difficult to verify A is an attractor based on the HIFS in nature. Therefore, according to Theorem 2.5, the relationship between the minimum closed set A and the random iterative sequence $Y(n)$ can be obtained as follows.

$$P\{h(\{y_n, y_{n+1}, \cdots, y_{n+k}\}, A) < \varepsilon\} > 1 - \varepsilon.$$

□

Definition 6.5. [18] The closed set D is said to be irreducible, if D does not contain non empty true closed sets. The Markov chain $\{Y(n), (n \geq 0)\}$ is an irreducible chain, if its state space E is an irreducible set. i.e.there are no non empty sets except E. Otherwise, it is a reducible chain.

Proposition 6.6. *Let the random variable sequence $\{\zeta_n, (n \geq 0)\}$ be determined by the random iterative sequence $\{y_n, (n \geq 0)\}$ based on the HIFS, then $\{\zeta_n, (n \geq 0)\}$ is an reducible chain. Otherwise, $\{\zeta_n, (n \geq 0)\}$ is an irreducible chain.*

In virtue of Definition 6.1, $E = \{y : f_{\zeta_n}(y_n) = y, n \in \{0, 1, 2, \cdots\}\}$ is the state space based on IFS. If the IFS is hyperbolic, then $A \subset E$ is the attractor of IFS. Therefore E has non empty true closed sets. so $\{\zeta_n, (n \geq 0)\}$ is an reducible chain. On the other hand, if the IFS is not hyperbolic, $\{\zeta_n, (n \geq 0)\}$ is an irreducible chain.

Proposition 6.7. *Let the random variable sequence $\{\zeta_n, (n \geq 0)\}$ be determined by the random iterative sequence $\{y_n, (n \geq 0)\}$ based on the HIFSP. If $\{\zeta_n, (n \geq 0)\}$ is an reducible chain, then there exists $n^* = n^*(\varepsilon)$ and $k^* = k^*(\varepsilon)$, for any $\varepsilon > 0$, such that if $n \geq n^*$ and $k \geq k^*$, we have*

$$P\{h(\{y_n, y_{n+1}, \cdots, y_{n+k}\}, A) < \varepsilon\} > 1 - \varepsilon,$$

where A is the attractor of the IFS. Otherwise, $\{\zeta_n, (n \geq 0)\}$ is an irreducible chain.

Proof. The key to solving proposition is to ascertain the relationship between the attractor of the IFS and the random iterative sequence $Y(n)$. Similar to the above, we get the following formula.

$$P\{h(\{y_n, y_{n+1}, \cdots, y_{n+k}\}, A) < \varepsilon\} > 1 - \varepsilon.$$

Therefore, the conclusion of the proposition is tenable. □

Lemma 6.8. *[18] The homogeneous Markov chain is called irreducible, if and only if any two states in its state space are interlinked.*

Proposition 6.9. *Let the random variable sequence $\{\zeta_n, (n \geq 0)\}$ be determined by the random iterative sequence $\{y_n, (n \geq 0)\}$ based on the IFSP. $\{\zeta_n, (n \geq 0)\}$ is an reducible chain, if the IFSP is hyperbolic.*

Proof. If IFSP is hyperbolic, we can deduce that the IFSP must have a attractor A. Then there exists an n^*, such that if $n > n^*$, we have $x_n \in A$. That is to say, the random iterative sequence $\{y_n\}$ can not get out of A, if $n > n^*$.

Now we suppose E is the state space of the IFSP, if $j \in A, i \in E - A$, it not difficult to get $i \to j$ is true, and $j \to i$ is not true. i.e. the states A and $E - A$ are not interlinked. Therefore, $\{\zeta_n, (n \geq 0)\}$ is an reducible chain due to Lemma 6.8. □

Remark 6.10. The random variable sequence $\{\zeta_n, (n \geq 0)\}$ are determined by the random iterative sequence $\{y_n, (n \geq 0)\}$ based on the IFSP. It is not difficult to verify that $\{\zeta_n, (n \geq 0)\}$ is an irreducible chain, if the IFSP is not hyperbolic.

Lemma 6.11. *[18] The equivalence class $E(i)$ is irreducible, if it is a closed set.*

Theorem 6.12. *Let E be the state space of an IFSP. Then it can be decompose of mutually disjoint subsets which are the union of the finite(or countable) of states $G, D_1, D_2, \cdots$. i.e.,*

$$E = G \bigcup D_1 \bigcup D_2 \bigcup \cdots , \tag{13}$$

where G is the set that compose of the all non-recurrence states, and every $D_n(n = 1, 2, \cdots)$ is the closed set that compose of the recurrence states.

Proof. Let F be the set that compose of all the recurrence states based on the IFSP, and $G = E - F$ be the set that compose of all the non-recurrence states based on the IFSP. Take $i_1 \in F$ arbitrarily, denoted by $D_1 = E(i_1)$.

Now, let any $j \in E(i_1), k \in E$, if $j \leftrightarrow i$, then j and i are interlinked, thus $k \in E(i_1)$, $D_1 = E(i_1)$ is a closed set. Therefore, D_1 is an irreducible closed set owing to Lemma 6.11.

Finally, take any $i_2 \in F - D_1$, denoted by $D_2 = E(i_2)$. D_2 can be verified is an irreducible closed set as above. Go on like this, we get $D_1, D_2, D_3, \cdots$, and all of the recurrence state closed sets $\{D_n\}$ are mutually disjoint.Moreover,

$$F = D1 \bigcup D_2 \bigcup D_3 \bigcup \cdots .$$

This completes the proof. □

Example 6.13. Let $Y \times Z = [0, 1] \times [0, 1]$,

$$f_1 \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix},$$

$$f_2 \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix},$$

$$f_3 \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}.$$

and $P_1 = P_2 = P_3 = 1/3$, we can construct a HIFSP, denoted by $\{Y; P; f_k, k \in \{1, 2, 3\}\}$. The attractor A of $\{Y; P; f_k, k \in \{1, 2, 3\}\}$ is called Sierpinski right triangle in fractal theory.

Through the above analysis, Sierpinski right triangle is the minimum closed set of $\{Y; P; f_k, k \in \{1, 2, 3\}\}$, and it is irreducible. Let E be the state space of $\{Y; P; f_k, k \in \{1, 2, 3\}\}$, then it can be decomposed of

$$E = A \bigcup E - A,$$

where A is the minimum closed set that compose of the all recurrence states, and $E - A$ is the set that compose of the non-recurrence states.

7 The Markov characteristics for IIFSP

We have investigated the Markov characteristics for IFSP in the above sections, and obtained many interesting results. However, these results are based on finite state space. That is to say, the iterative functions in the iterated function system (IFS) must be finite. We will study further what Markov characteristics will emerge if the iterative functions in the iterated function system are changed to denumerably infinite in the sequel, which also leads to denumerably infinite state space. The definition of IIFSP based on IFSP will be given first.

Definition 7.1. Let $\{Y; f_k, k \in \{1, 2, \cdots, N, \cdots\}\}$ be an IIFS, $P = \{p_1, p_2, \cdots, p_N, \cdots\}$ is a probability vector, where $p_i \geq 0$ for all $i \in 1, 2, \cdots, N, \cdots$, and$\sum_{i=1}^{\infty} p_i = 1$. We have $P(\zeta_n = i) = p_i$ for the independent random variables sequence $\zeta_1, \zeta_2, \cdots, \zeta_n, \cdots$, where $i \in \{1, 2, \cdots, N, \cdots\}, n \in \{1, 2, 3, \cdots\}$. Then it is called infinite iterated function system with probability(IIFSP),denoted by $\{Y; P; f_k, k \in \{1, 2, \cdots, N, \cdots\}\}$.

Corollary 7.2. *Let the random variable sequence $\{\zeta_n, (n \geq 0)\}$ be determined by the random iterative sequence $\{y_n, (n \geq 0)\}$ based on the IIFSP, then $\{\zeta_n, (n \geq 0)\}$ is an infinite homogeneous Markov chain.*

Corollary 7.3. *Let $\{\zeta_n, (n \geq 0)\}$ be the random variable sequence which is determined by the random iterative sequence $\{y_n, (n \geq 0)\}$ in the IIFSP. Then $\{\zeta_n, (n \geq 0)\}$ is an irreducible ergodic chain.*

Corollary 7.2 and Corollary 7.3 are the important extension of Proposition 3.2 and Theorem 4.6. The proof thought and process are also similar to the previous two theorems. Therefore, we omit the proof of Corollary 7.2 and Corollary 7.3 here.

Theorem 7.4. *Let $\{\zeta_n, (n \geq 0)\}$ be the random variable sequence which is determined by the random iterative sequence $\{y_n, (n \geq 0)\}$ in the IIFSP. $i, j \in Z^+$ are any two states in the state (positive integer) space of $\{\zeta_n, (n \geq 0)\}$, where $Z^+ = \{1, 2, \cdots, N, \cdots\}$. Then the limit distribution of transition probability $p_{ij}^{(n)}$ is said to be a stationary probability distribution, if the state i transfers to j after n-step iteration.*

Theorem 7.4 are an important extension of Theorem 5.4, but the proof thought and process is different to Theorem 5.4. We will give the detailed proof process in the following.

Proof. Similar to Theorem 5.4, $\{\zeta_n, (n \geq 0)\}$ is an irreducible ergodic (aperiodic and positive recurrence) chain owing to Theorem 4.6. Then we will obtain the following equation through Lemma 5.3.

$$\lim_{n \to \infty} p_{ij}^{(n)} = \frac{1}{\psi_j}, j \in Z^+.$$

On the other hand,

$$\sum_{j=1}^{N} p_{ij}^{(n)} \leq \sum_{j=1}^{\infty} p_{ij}^{(n)} = \sum_{j \in Z^+} p_{ij}^{(n)} = 1.$$

Let $n \to \infty$, and $N \to \infty$, then we get

$$\sum_{j=1}^{\infty} \frac{1}{\psi_j} \leq 1.$$

According to the C-K equation again, we have

$$\sum_{k=1}^{N} p_{ik}^{(\mu)} p_{kj}^{(\nu)} \leq \sum_{k=1}^{\infty} p_{ik}^{(\mu)} p_{kj}^{(\nu)} = p_{ij}^{(\mu+\nu)}.$$

Without loss of generality, suppose $\mu = m$, and $\nu = n$, and let $m \to \infty$, and $N \to \infty$, then we get

$$\sum_{k=1}^{\infty}(\frac{1}{\psi_k})p_{kj}^{(n)} \leq \frac{1}{\psi_j}, \quad j,n \in Z^+. \tag{14}$$

We will prove the equal sign of the above formula is also true by means of counter evidence for $j,n \in Z^+$ in the sequel. Suppose for some j or n, the equal sign of the above formula does not hold, then we get

$$\sum_{k=1}^{\infty}\frac{1}{\psi_k} = \sum_{k=1}^{\infty}(\frac{1}{\psi_k})[\sum_{j=1}^{\infty}p_{kj}^{(n)}]$$

$$= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(\frac{1}{\psi_k})p_{kj}^{(n)} < \sum_{j=1}^{\infty}\frac{1}{\psi_j} \leq 1.$$

This is a contradiction. Thus, for all $j,n \in Z^+$, the equal sign hold, i.e.,

$$\frac{1}{\psi_j} = \sum_{k=1}^{\infty}(\frac{1}{\psi_k})p_{kj}^{(n)}, \quad j,n \in Z^+. \tag{15}$$

Thus $\frac{1}{\psi_j}$ is a stationary distribution. We will proof it is also a probability distribution. The following inequality hold due to the above equation.

$$\sum_{k=1}^{N}(\frac{1}{\psi_k})p_{kj}^{(n)} \leq \frac{1}{\psi_j} \leq \sum_{k=1}^{N}(\frac{1}{\psi_k})p_{kj}^{(n)} + \sum_{k=N+1}^{\infty}(\frac{1}{\psi_k}).$$

Then let $k \to \infty$,$N \to \infty$ again, we get

$$\frac{1}{\psi_j} = \sum_{k=1}^{\infty}(\frac{1}{\psi_k})\lim_{n\to\infty}(p_{kj}^{(n)})$$

$$= \sum_{k=1}^{\infty}(\frac{1}{\psi_k})\frac{1}{\psi_j}, \quad j \in Z^+.$$

Through the above formula, we can get the following immediately.

$$\sum_{k=1}^{\infty}\frac{1}{\psi_k} = 1.$$

Therefore, the limit of $p_{ij}^{(n)}$, i.e. $\{\frac{1}{\psi_j}, j \in Z^+\}$ is a probability distribution. This completes the proof. □

Corollary 7.5. *Let E be the state space of an IIFSP. Then it can be decompose of mutually disjoint subsets which are the union of the finite (or countable) of states $G, D_1, D_2, \cdots$. i.e.,*

$$E = G\bigcup D_1\bigcup D_2\bigcup\cdots, \tag{16}$$

where G is the set that compose of the all non-recurrence states. And every $D_n(n = 1,2,\cdots)$ is the closed set that compose of the recurrence states.

The proof process is similar to Theorem 6.12, we omit the proof here.

8 Conclusions

In this article, we research the Markov Characteristics for IFSP and IIFSP through interlink, period, recurrence and some related concepts and properties on the basis of predecessors's work. Then, there are four important results are obtained as follows:

1. The sequence of stochastic variable $\{\zeta_n, (n \geq 0)\}$ is a homogenous Markov chain.
2. The sequence of stochastic variable $\{\zeta_n, (n \geq 0)\}$ is an irreducible ergodic chain.
3. The distribution of transition probability $p_{ij}^{(n)}$ based on $n \to \infty$ is a stationary probability distribution.
4. The state space can be decomposed of the union of the finite(or countable) mutually disjoint subsets, which are composed of non-recurrence states and recurrence states respectively.

In the future, we can further study IFSP by some important theories in stochastic processes and fuzzy fractal, such as martingale theory, Poisson process, renewal process et al. These studies will not only enrich the fractal theory, but also enhance the relationship between random fractal and real life.

Acknowledgements: This research thanks to the support of National Natural Science Foundation of China (11926503, 11901444, 12001423, 12001413) and China Postdoctoral Science Foundation (2020M683693XB, 2019T120938, 2018M643716).

Conflict of Interest: The authors declare that there are no conflict of interest.

References

[1] L. Andrzej, A variational principle for fractal dimensions, Nonlinear Analysis, 64 (2006) 618-628.

[2] M. F. Barnsley, Fractal everywhere, Academic press, New York, (1988).

[3] C. Escribano, M. A. Sastre, E. Torrano, A fixed point theorem for moment matrices of self-similar measures, Journal of Computational and Applied Mathematics, 207 (2007) 352-359.

[4] J. M. Fraser, Fractal geometry of Bedford-McPsillen carpets, Thermodynamic Formalism, (2021) 495-516.

[5] E. Guariglia, Harmonic sierpinski gasket and applications, Entropy, 20 (2018) 714.

[6] J. E. Hutchinson, Fractal and self-similarity, Indiana Univ Math., 30 (1981) 713-747.

[7] M. S. Islam, M. J. Islam, Dynamics of fractal in Euclidean and measure spaces: a recent study, Newest Updates in Physical Science Research, 13 (2021) 43-60.

[8] N. Jiang, Brief history of fractal, Publishing House of Electronics Industry, Bei Jing, (2020).

[9] N. Jiang, N. N. Ma, Probability measure and dirac measure based on hyperbolic IFSP, Journal of Jilin University (Science Edition), 55 (2017) 581-586.

[10] N. Jiang, A. J. Qu, The generation of the concept of fractional dimension, Studies in Philosophy of Science and Technology, 37 (2020) 87-93.

[11] J. Joanna, How to construct asymptotically stable iterated function system, Statistics and Probability Letters, (2008) 1-7.

[12] W. L. John, E. W. Tad, Linear Markov iterated function system, Computers and Graphics, 14 (1990) 343-353.

[13] A. M. Jurgens, P. C. James, Shannon entropy rate of hidden Markov processes, Journal of Statistical Physics, 183 (2021) 1-18.

[14] F. Kenneth, Fractal Geometry Mathematical Foundations And Applications, Thomson Press (India) Ltd, London, (1990).

[15] K. Leniak, S. Nina, Iterated function system enriched with symmetry, Constructive Approzetamation, (2021) 1-21.

[16] I. D. Morris, S. Cagri, A strongly irreducible affine iterated function system with two invariant measures of mazetamal dimension, Ergodic Theory and Dynamical System, 41 (2021) 3417-3438.

[17] Z. Sha, H. J. Ruan, Fractal and fitting, Zhejiang University Press, Hang zhou, (2005).

[18] D. W. Stroock, An introduction to Markov processes, Springer Science & Business Media, Berlin, (2013).

[19] T. Szarek, Z. Anna, The central limit theorem for iterated function system on the circle, Mosc. Math. J., 21 (2021) 175-190.

[20] Z. Tian, C. Y. Qing, et al, Stochastic process and its application, Science Press, Bei Jing, (2007).

[21] D. L. Torre, et al, Iterated function system with place-dependent probabilities and the inverse problem of measure approzetamation using moments, Fractals, 26 (2018) 1850076.

[22] K. Weihrauch, Computability on the probability measures on the Bore1 sets of the unit interval, Theoretical Computer Science, 219 (1999) 421-437.

[23] E. H. Ye, D. P. Zhang, Probability theory and stochastic process, Science Press, Bei Jing, (2005).

Nan Jiang
School of Science
Xi'an Shiyou University
Xi'an, China
E-mail: jiangnan@xsyu.edu.cn

Wei Li
Institute for Advanced Studies in The History of Science
Northwest University
Xi'an, China
E-mail: lw5607@163.com

Fei Li
School of Science
Xi'an Shiyou University
Xi'an, China
E-mail: nancong1987@163.com

Juntao Wang
School of Science
Xi'an Shiyou University

Xi'an, China
E-mail: wjt@xsyu.edu.cn

Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://tfss.journals.iau.ir/
Online ISSN: 2821-0131
Vol.1, No.2, (2022), 90-98
DOI: http://doi.org/10.30495/tfss.2022.1954304.1016

Article Type: Original Research Article

Safety Risk Assessment; Using Fuzzy Failure Mode and Effect Analysis and Fault Tree Analysis

Mazdak Khodadadi-Karimvand*, **Sara TaheriFar**

Abstract. The failure mode and effects analysis (FMEA) is a qualitative, Inductive and effective method for detecting errors, faults, and failures in a system and fuzzy logic can improve that technique with more logical outputs. Moreover, the fault tree analysis (FTA) as a probabilistic risk assessment method is among the effective technique for calculating the probability of errors, faults, failures, reliability and safety integrity level (SIL) verification resulting in certain events at higher levels. The FTA also detects the main causes of events in complicated systems. Although this technique appears to be time-consuming in systems with many diverse components, it is considered a powerful tool. In this paper, the fuzzy FMEA analyzes the failure modes in a hypothetical system. After that, the process with the highest risk is selected as the input of an FTA. According to the qualitative and quantitative analysis of FTAs, a series of corrective actions will be proposed to reduce the failure probability.

AMS Subject Classification 2020: 03B52, 03E72, 68T27
Keywords and Phrases: Fault Tree Analysis (FTA), Failure Mode and Effect Analysis (FMEA), Qualitative Risk Assessment, Probabilistic Risk Assessment (PRA), Fuzzy Number.

1 Introduction

Various methods, including quantitative and qualitative, have been proposed to assess the risk. FMEA represent a preventative method with a teamwork approach. FMEA was first developed as a design methodology in the aerospace industry for needs related to reliability and safety. And then more widely, used in industry, to ensure product safety and reliability. This tool is one of the effective models for fault prediction and finding the most economical solution to prevent faults. FMEA is a structured way to start designing or reviewing and developing the product or service design in an organization [12]. FMEA mainly prevents the occurrence of faults, helps in creating and developing products, processes or serious services and records parameters and indicators in the design and development process or service [3]. FMEA results are in response to the following questions: What are the faults, bugs, or hazards? Which identified faults, bugs, or hazards are of greater importance (risk)? What are the possible solutions which could be done to reduce the occurrence of these situations? The FMEA acts systematically to answer these questions in the best way and, using the knowledge and expertise of a working group, prioritizes them in addition to identifying faults and problems or hazards at the heart of the operation.

Another comprehensive method of its kind is FTA. This method was first developed in 1961-62 in Bell Telephone Laboratories and then developed by Watson to determine and improve the reliability of the intercontinental ballistic missile control system. This method was developed by Boeing Airlines in the following

*Corresponding author: Mazdak Khodadadi-Karimvand, Email: mazdak.kh@gmail.com, ORCID: 0000-0003-2765-3784
Received: 5 March 2022; Revised: 11 July 2022; Accepted: 12 July 2022; Published Online: 7 November 2022.

How to cite: Mazdak Khodadadi-Karimvand and Sara TaheriFar, Safety Risk Assessment; Using Fuzzy Failure Mode and Effect Analysis and Fault Tree Analysis, *Trans. Fuzzy Sets Syst.*, 1(2) (2022), 90-98.

years and became legal. Since 1965, the use of the FTA method has spread to various industries such as aerospace, nuclear, chemical, etc., and it has been widely used to analyze the reliability and safety of systems [8].

FTA is an analytical technique that identifies system malfunctions, and by providing a quantitative analysis of the system, all possible paths to system malfunctions will be identified. FTA is a graphical model, which shows the sequence of events of a fault. In fact, the fault tree shows how failure occurs by introducing logical connections to events. Consider that, as the input to the fault tree analysis in the problem recognition stage, the fault mode selected in the FMEA for investigation or has a higher risk can be used. For example, items whose risk priority number is known to be higher than the allowable level, or items that have been identified in the FMEA area chart analyzed as items to review and determine corrective action, could be the input of FTA. In this article, by using a hypothetical process, the corrective action priorities are identified by the FMEA method, and the highest risk is used as the input of an FTA. In this article, the faults scenarios in FMEA are prioritized by the FRPN method, which due to the characteristics of fuzzy logic, the use of FRPN has a higher priority than RPN.

In general, FTA is a powerful analogical tool for batch analysis of a system's events. This technique is mainly used for evaluating complex systems. Today, a variety of computer programs have been developed to create the logical structure and perform the necessary calculations. The method creates connections between system events by means of logical symbols that represent the effects of an accident or hazard. The technique is robust and convenient for situations that have traditionally been decomposed in series or in parallel. This model is also used for dynamic cases, which performs qualitative and quantitative analysis and allows the analyzer to evaluate different alternatives in system design and the fault range, reduction time, measure repair and failure times and other dynamic system operations.

Fault tree analysis is very suitable in complex processes with a large variety of components and parts and leads to useful results. Although this method qualitatively evaluates a predetermined risk and adverse event, it can be quantitatively analyzed to obtain interesting and documented results and provide a solution for management decisions to be able to allocate resources and energy more confidently and ensure the system against possible damages by highlighting the safety of the system [4].

2 Literature Review

Akyildiz and Mentes (2017) used fuzzy AHP and fuzzy TOPSIS methods to assess the risks of cargo vessel accidents [1]. Khodadadi-Karimvand and Shirouyehzad (2021) use FMEA as a risk identification tool. Then, the Fuzzy Risk Priority Number (FRPN) is calculated and triangular-fuzzy numbers are prioritized through Fuzzy TOPSIS [7]. FMEA is an engineering technique that is used to identify the existing or potential failures or problems in a design, process, or service structure of a system before they occur, to prevent undesirable incidents and protect employees from occupational accidents and diseases by taking the necessary measures [13]. The severity and types of potential failures in the analyzed system are identified by FMEA, which allows decision makers to take the necessary risk-reducing measures [5].

FTA should be conducted by a team of experts on the scope to be analyzed. The method examines the causes of incidents and the conditions triggering the incident. The analysis includes the equipment and components used by the employee while performing the work, together with the components and system conditions [6].

FTA is a deduction analysis method that allows identifying and analyzing the potential causes, conditions, and factors that contribute to the occurrence of an unidentified, undesirable major incident. FTA method is used to analyze, assess, and graphically illustrate the hierarchical flow of potential incidents or situations that may negatively affect the system reliability and usability [9].

Many studies have been conducted using both FTA and FMEA methods. Li and Gao (2010) pointed out the necessity to identify the potential root causes in the system and analyze the critical situation in order to determine the maintenance operations required based on the reliability-centered maintenance and radical maintenance approaches using the FTA and failure mode effect and criticality analysis (FMECA) methods. In addition, the FTA approach is adopted to evaluate the reliability of systems and analyze the probability of failure occurrence [2].

Barozzi et al. in the paper, a representative biogas production plant was considered, and a risk assessment was carried out through the combination of Recursive Operability Analysis and Failure Mode and Effects Criticality Analysis (FMECA). The methodology is rigorous and allows for both the identification and the quantification of accidental scenarios due to procedural errors and equipment failures, which miss in the literature for the case of biogas. The analysis allows the automatic generation of the Fault Trees (FTA) for the identified Top Events, which can be numerically solved [11].

3 Methodology

In order to create a model for calculating the degree of risk priority and prioritizing faults and their effects using fuzzy logic, the following three main steps must be followed:

- Select fuzzy membership function
- Form a membership function by multiplying the membership functions by severity, probability, detection.
- De-fuzzy membership function
- Quantitative and qualitative analysis of the fault tree for the highest risk obtained

In this article, a hypothetical system with four modes of faults, failure and lost is analyzed and the highest risk calculated in the Fuzzy FMEA input of a fault tree analysis is placed.

3.1 Selection of Fuzzy Membership Function

For all the affective factors in the risk-taking degree, such as severity, probability and detection, five linguistic variables can be used VL, L, M, H and VH. Where 5 linguistic variables are assigned to rank according to table 1 [7].

Table 1: Fuzzy Numbers of Linguistic Variables Corresponding to Ranks 1 to 10

Fuzzy Number	Verbal Variable	Rank
(0.9,1,1)	VH	9,10
(0.7,0.85,1)	H	7,8,9,10
(0.4,0.6,0.8)	M	4,5,6,7,8
(0.2,0.35,0.5)	L	2,3,4,5
(0,0.15,0.3)	VL	1,2,3

$\{VL, L, M, H, VH\} = T(x) =$ Set of Linguistic Variables Values
$[0, 1] = U =$ Variation Amplitude of the Reference Set
Performing calculations with fuzzy numbers is very complex due to their special structure. To facilitate and apply fuzzy numbers, special fuzzy numbers are used in calculations. These special numbers are bell-shaped, triangular, trapezoidal, L-R trapezoidal, L-R triangular. In this paper, triangular fuzzy numbers are used.

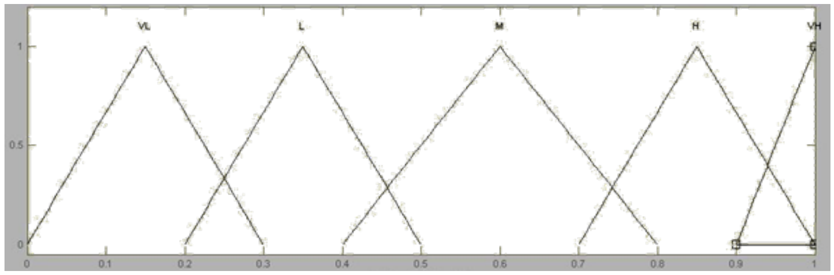


Figure 1: Membership Function of Linguistic Variables

3.2 Forming a Membership Function by Multiplying the Membership Functions of Severity, Occurrence and Detection

FRPN is calculated from the following relation by multiplying the membership functions of severity, probability and detection. If M is a linguistic variable, its triangular fuzzy number may be defined as follows [14]:

$$M = (l, m, u)$$

Where u, l and m are the upper limit, the lower limit and the mean of u, respectively where the membership degree of l is 1.
Algebraic operations rules are applied on triangular numbers as follows to calculate RPN:
RPN = Severity × Occurrence × Detection
FRPN $=(l_1, m_1, u_1) \times (l_2, m_2, u_2) \times (l_3, m_3, u_3) = (l_1 l_2 l_3, m_1 m_2 m_3, u_1 u_2 u_3)$

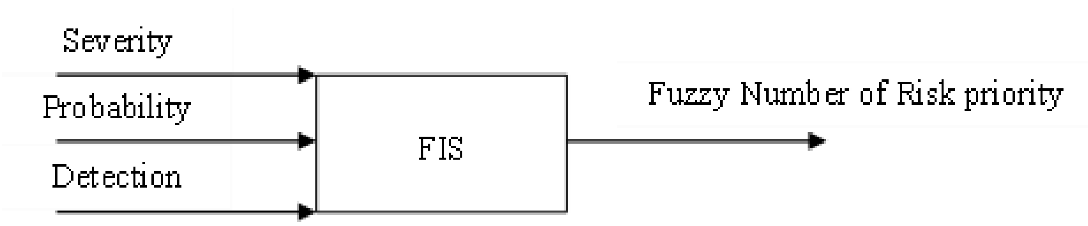


Figure 2: FRPN Model

3.3 Defuzzification

There are several ways to convert a fuzzy number to an exact value. In this paper, the values obtained from the formation of the membership function multiplied by the membership functions of severity, probability, detection using the left and right scoring method of fuzzy, non-fuzzy numbers [15].

After determining the linguistic variables instead of the values severity, probability, detection for the three potential fault modes, we replace the fuzzy values according to the Table 2.

Then, according to Table 3, the fuzzy RPN values are converted to a non-fuzzy number using the left and right scoring method, or the definite RPN resulting from de-fuzzy.

Table 2: Membership Function for the Severity, Probability and Detection

Severity	Probability	Detection	Failure Modes	#
(0.7, 0.85, 1)	(0, 0.15, 0.3)	(0.2, 0.35, 0.5)	Failure 1	1
(0.7, 0.85, 1)	(0.4, 0.6, 0.8)	(0.2, 0.35, 0.5)	Failure 2	2
(0.4, 0.6, 0.8)	(0.4, 0.6, 0.8)	(0, 0.15, 0.3)	Failure 3	3
(0.7, 0.85, 1)	(0.7, 0.85, 1)	(0, 0.15, 0.3)	Failure 4	4

Table 3: Defuzzification Using the Left and Right Scores

Total Score	Left Side Score	Right Side Score	FRPN	#
0.0517	0.9554	0.0587	(0, 0.0446, 0.15)	1
0.1877	0.9489	0.3243	(0.056, 0.1785, 0.4)	2
0.1302	0.9460	0.2064	(0, 0.054, 0.192)	3
0.1096	0.8917	0.1110	(0, 0.1083, 0.3)	4

3.4 Fault Tree Analysis

After determining the risk of activities using the FMEA method, we analyze the state of fault, failure and breakage, which has the highest risk, using the FTA method and during the following steps.

A: System definition

B: Fault tree formation

C: Qualitative analysis

D: Quantitative analysis [4].

3.5 Define the System as a Fault Tree

This includes the scope of the analysis including defining what is considered a failure. This becomes important when a system may have an element fail or a single function fails and the remainder of the system still operates. The highest risk faults, failure and failure modes are considered as the top event and the sub-events $G1, G2$ and $G3$ are on one level and the basic events $E1, E2, E3$ and $E4$ are on the next level. In fact, intermediate or sub-events and basic events are the causes of the top event that have been identified.

3.6 Fault Tree Formation

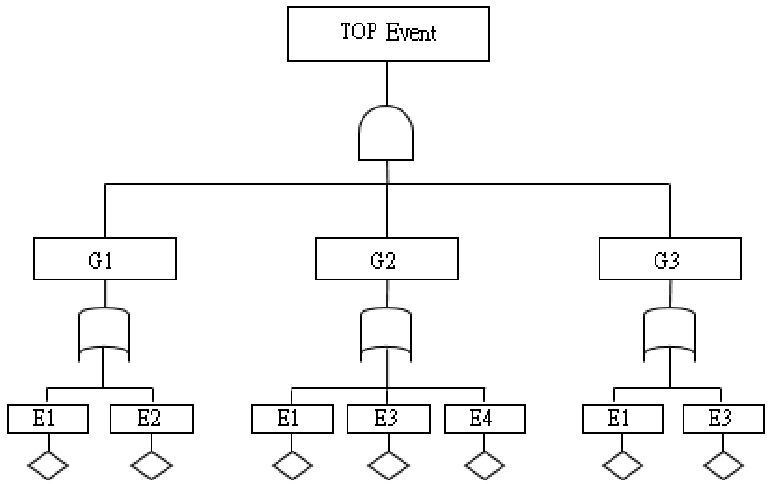


Figure 3: Drawn Fault Tree

3.7 Qualitative Analysis

Fault tree analysis is done in two stages, the first stage is qualitative analysis, which we will discuss very briefly. Qualitative analysis refers to the preparation of various combinations of events that cause system failure. In other words, in this section, the goal is to determine the minimum cut sets for the final fault tree incident.
In this fault tree the minimum cut sets are:

$$M\ cs1 = E1$$

$$M\ cs2 = E2, E3$$

Minimal cut set No. 1 is more important because of the lower floor and the importance of all events is as follows: [10]

$$E1 > E3 > E2 > E4$$

3.8 Quantitative Analysis

For a quantitative analysis, we need a list of equipment or parts in which sub-events occur due to adverse conditions and cause the process to fail.
Here we assume the equipment or part according to Table 4 and in front of this equipment we obtain the failure rate in a specified period of time using statistics and records and reports and repair instructions of the devices and then calculate the probability of failure by using the failure rate and finally, calculate the probability of process failure. In this case study, we examine the performance of equipment at 1,200 hours over 5 years.

Before calculating the probability, using Boolean Algebra, we express a method that can be used to calculate the failure rate and probability for equipment and parts. The diagram in Figure 2, which is more commonly used in maintenance topics and is known as the Bathtub Hazard Rate Curve, is divided into three sections:

Table 4: Failure Rate and Probability

Probability	Failure Rate	Symbol	Equipment
0.0004	1	$E1$	No. 1
0.0000	0	$E2$	No. 2
0.0004	1	$E3$	No. 3
0.0000	0	$E4$	No. 4

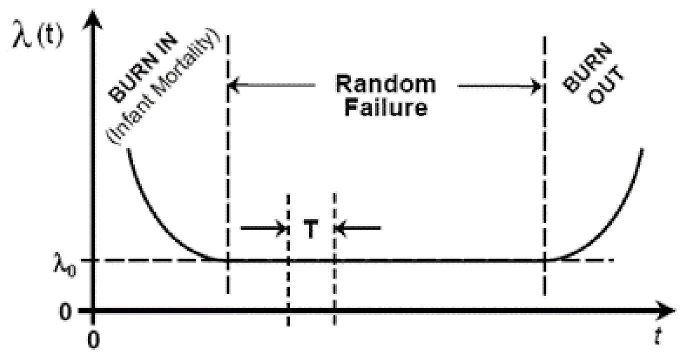


Figure 4: Bathtub Failure Rate Curve

The first section is related to the initiation of the system. At this time, the probability of failure of the system is high and during this area, the failure rate decreases with increasing time. In the second time interval, the average failure per unit time is almost constant, and failures occur randomly and unpredictably, which can have a variety of reasons, and in the third section, the device wears out and runs out. The working period is approaching and the probability of failure is high and during this area the failure rate increases with increasing time. If $\lambda(t)$ is constant, it can be shown that the probability distribution function of the random variable at the time of the failure event is an exponent with the parameter λ. Most of the time and during the operation of the system, the value of $\lambda(t)$ is independent of time and is constant. This means that failure can occur independently and accidentally at any time interval from the working area of the device. In this case we will have:

$$\lambda(t) = \lambda = etc.$$

Now with having $\lambda(t)$ and putting it in the Exponential Distribution function we have:

$$P_F = 1 - e^{-\lambda t}$$

Where P_F is the probability of failure and t is the time in the subject which we are discussing about at the failure rate [10].

3.9 Calculation of Probability and Failure Rate

$G1 = E1 + E2 = 0.0004 + 0.0000 = 0.0004$

$G2 = E1 + E3 + E4 = 0.0004 + 0.0004 + 0.0000 = 0.0008$

$G3 = E1 + E3 = 0.0004 + 0.0004 = 0.0008$

Probability of process failure $= G1 + G2 + G3 = 0.0004 + 0.0008 + 0.0008 = 0.0020$
Now, using the probability of process failure, we calculate the failure rate:

$$P_F = 1 - e^{-\lambda t}$$

$$0.0020 = 1 - e^{-\lambda t} \Longrightarrow -\lambda t = Ln(1 - 0.0020) \Longrightarrow \lambda t = 0.002002002$$

According to the 5 years time period, we have:

$$\lambda = \frac{0.002002002 \times 1200}{5} = 0.4804864$$

The above value is the number of failures per a unit year. That is, almost every two years, there is a possibility of failure once.

4 Conclusion

After the results, an expert team from all relevant groups should try to reduce or eliminate the severity of adverse events and provide suggestions. Finally, it is necessary to mention that today in all industries, resource and energy management and the right decision to allocate them to achieve full productivity, is the main concern of managers. Fault and failure state analysis and fault tree analysis are each efficient tools and due to the time-consuming method of fault tree analysis to investigate the undesirable states of the system, it seems that if the output or outputs of the analysis method and to put the breakdown and failure analysis into the input of a fault tree, we have taken steps to reduce or eliminate adverse events by considering the time and costs involved.

As it turns out, fuzzy logic in risk assessment gives us a more logical output, but in any case, outputs of the normal or fuzzy failure mode and effects analysis technique and other qualitative and Inductive methods such as Hazard and Operability Analysis (HAZOP), Hazard Identification (HAZID), Hazard Analysis (HAZAN), etc. can use as input of a fault tree analysis and other probabilistic risk assessment (PRA) or quantitative risk assessment (QRA) methods.

Conflict of Interest: The authors declare that there are no conflict of interest.

References

[1] H. Akyildiz and A. Mentes, An integrated risk assessment based on uncertainty analysis for cargo vessel safety, *Safety Science*, 92 (2017), 34-43.

[2] M. Barozzi, S. Contini, M. Raboni, V. Torretta, V. Casson Moreno and S. Copelli, Integration of Recursive Operability Analysis, FMECA and FTA for the Quantitative Risk Assessment in biogas plants: Role of procedural errors and components failures, *Journal of Loss Prevention in the Process Industries*, 71 (2021), 104468.

[3] D. H. Besterfield, Quality Control, 6th Edition, *Prentice Hall*, (2001).

[4] P. L. Clemens, Fault tree analysis, *JE Jacobs Severdurup*, 13 (2002).

[5] IEC 60812, 2018. https://webstore.iec.ch/publication/3571, Internatonal Electrotechncal Commsson.

[6] IEC 61025, 2006. https://webstore.iec.ch/publication/4311, Internatonal Electrotechncal Commsson.

[7] M. Khodadadi-Karimvand and H. Shirouyehzad, Well drilling fuzzy risk assessment using fuzzy FMEA and fuzzy TOPSIS, *Journal of fuzzy extension and application*, 2(2) (2021), 144-155.

[8] D. Kritzinger, Aircraft System Safety, *Fault Tree Analysis*, (2017), 59-99.

[9] D. Li and J. Gao, Study and application of reliability-centered maintenance considering radical maintenance, *Journal of Loss Prevention in the Process Industries*, 23(5) (2010), 622-629.

[10] M. K. Modarres and K. V. Mark, Reliability Engineering and Risk Analysis: a Practical Guide, 3rd Edition, *Boca Raton, CRC Press*, (2017).

[11] N. G. Mutlu and S. Altuntas, Risk analysis for occupational safety and health in the textile industry: Integration of FMEA, FTA, and BIFPET methods, *International Journal of Industrial Ergonomics*, 72 (2019), 222-240.

[12] N. R. Sankar and B. S. Prabhu, Modified Approach for Prioritization of Failure in a System Failure Mode and Effect Analysis, *International Journal of Quality & Reliability Management*, 18(3) (2001), 324-355.

[13] D. H. Stamatis, Failure Mode and Effect Analysis: FMEA from Theory to Execution, 2nd Edition, *ASQC Press, New York*, (2003).

[14] K. M. Tay and C. P. Lim, Fuzzy FMEA with a guided rules reduction system for prioritization of failures, *International journal of quality & reliability management*, 23(8) (2006), 1047-1066.

[15] H. J. Zimmermann, Fuzzy Set Theory and Its Applications, 4rd Edition, *Boston: Kluwer Acadamic Publishers*, (2001).

Mazdak Khodadadi-Karimvand
Department of Industrial Engineering
Najafabad Branch, Islamic Azad University, Iran &
Department of Safety Engineering
Faculty of Engineering, University of Science and Culture
Tehran, Iran
E-mail: mazdak.kh@gmail.com

Sara TaheriFar
Department of Safety Engineering, Faculty of Engineering
University of Science and Culture
Tehran, Iran
E-mail: sara.taheri.far7698@gmail.com

Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://tfss.journals.iau.ir/
Online ISSN: 2821-0131
Vol.1, No.2, (2022), 99-128
DOI: http://doi.org/10.30495/tfss.2022.1962834.1041

Article Type: Original Research Article (Invited Paper)

On Generalized Mixture Functions

Antonio Diego S. Farias, **Valdigleis S. Costa**, **Luiz Ranyer A. Lopes**, **Regivan H. N. Santiago**, **Benjamín Bedregal***

Abstract. In the literature it is very common to see problems in which it is necessary to aggregate a set of data into a single one. An important tool able to deal with these issues is the aggregation functions, which we can highlight as the OWA functions. However, there are other functions that are also capable of performing these tasks, such as the preaggregation function and mixture functions. In this paper we investigate two special types of functions, the Generalized Mixture functions and Bounded Generalized Mixture functions, which generalize both OWA and Mixture functions. We also prove some properties, constructions and examples of these functions. Both the Generalized and Bounded Generalized Mixture functions are developed in such a way that the weight vectors are variables that depend on the input vector, which generalizes the aggregation functions: *Minimum*, *Maximum*, *Arithmetic Mean* and *Median*, and are extensively used in image processing. Finally, we propose a Generalized Mixture function, denoted by **H**, and we show that **H** satisfies a series of properties in order to apply this function in an illustrative example of application: The image reduction process.

AMS Subject Classification 2020: 03E72; 47S40; 94A08
Keywords and Phrases: Aggregation functions, Preaggregation functions, OWA functions, Generalized Mixture functions, Image reduction.

1 Introduction

Some functions are able to transform a set of data into a single one, for example, aggregations functions [3, 6, 22] and mixture functions [6]. This type of function has applications in several areas; for example, we can cite [8, 17, 19, 43, 44]. Image processing used in medicine; for example, you can apply it to: detect tumors [26, 36, 40, 58]; support techniques in advancing dental treatments [14, 25, 52, 54], etc. Such images are not always obtained with suitable quality, and to detect the desired information, various methods have been developed in order to eliminate most of the noise contained in these images [29, 42, 50]. These functions can also be used to reduce the size of images (this process is called image reduction).

The methods of image reduction are used in order to decrease your resolutions, usually aiming the reduction of memory consumption required for its storage [23]. There are several techniques for image reduction to achieve this goal in the literature, among these techniques, we can cite Paternain *et al.* [45], that built a method of reduction using *weighted averaging aggregation functions*. The method proposed by Paternain *et al.* consists of: (1) Reducing a given image by using a reduction operator (based on weighted averaging

*****Corresponding author:** Benjamín Bedregal, Email: bedregal@dimap.ufrn.br, ORCID: 0000-0002-6757-7934
Received: 8 July 2022; Revised: 22 July 2022; Accepted: 22 July 2022; Published Online: 7 November 2022.

How to cite: A. D. S. Farias, V. S. Costa, L. R. A. Lopes, R. H. N. Santiago and B. Bedregal, On Generalized Mixture Functions, *Trans. Fuzzy Sets Syst.*, 1(2) (2022), 99-128.

aggregation functions); (2) Building a new image from the reduced one, and (3) Analyzing the quality of the last image be using the measures *PSNR* and *MSSIM* defined in [23].

Because of its broad capacity of applications, many researchers have invested in aggregate functions and its extensions [34, 39, 46, 48, 61, 64]. In this sense, thinking about the problem of decision-making, Yager [60] introduced a special class of aggregate functions, called Ordered Weighted Averaging - OWA, and ever since several authors have proposed generalizations for these functions [12, 33, 37, 53, 61]. Mixture functions, presented in [6], and variants of Choquet integrals in [2, 10, 15, 35] are other important examples of generalization of the OWA. These functions are not aggregate functions, but also are efficient in converting various information into a single one.

In this paper we studied a class of functions introduced in [46] and called Generalized Mixture - GM. Since then many other papers on this class of functions have been found, for example [13, 20, 21, 47, 49]. GM also generalizes the notion of OWA and consequently, also encompass functions as: *Arithmetic Mean*, *Median*, *Maximum* and *Minimum*. Besides that, it is a generalized form of another important class of functions: The *Mixture functions* - MO, which as well as OWA functions, are determined from weights $w_1, w_2, \cdots, w_n \in [0,1]$, which generally satisfy the condition $\sum\limits_{i=1}^{n} w_i = 1$. The GM functions,as well as the MO functions, are weighted averaging means with dynamic weights, i.e., the weights of these functions depend on the input variables. This characteristic of more flexible weights of OWA′ allows us to define functions whose weights are suited for each input, which does not occur in OWA's. However, we ended up losing the property of monotonicity, which can be replaced by directional monotonicity [9] in order to obtain preaggregation functions.

Later, in this work, we weaken the condition of the vector of weights $\left(\sum\limits_{i=1}^{n} w_i = 1 \text{ to } \sum\limits_{i=1}^{n} w_i \leq 1\right)$, thereby obtaining in another generalization of OWA, called the *Bounded Generalized Mixture* - BGM function, we propose a special GM function (denoted by **H**). This way, we provide a wide range of their properties such as: idempotence, symmetry, homogeneity and directional monotonicity. To finalize this work, we apply **H** in a method of image reduction [4, 7, 44, 51, 56, 59] and we compare this function with *Minimum*, *Maximum*, *Arithmetic Mean*, *Median* and cOWA. The method adopted was the same as Paternain *et al.* [45].

This work is structured in the following way: The next section provides the basic concepts of Aggregation functions theory; In Section 3, we introduce the concepts of Generalized Mixture - GM and Bounded Generalized Mixture - BGM operators, we show properties, constructions, examples and propose a particular GM function (called **H**). Also in Section 3, we show that **H** is idempotent, homogeneous, shift-invariant, symmetric, self dual and directionally monotonic, which is important to the image reduction field [45]. In Section 4, we provide an illustrative application for GM's. in image reduction and finally in Section 5 we close this paper with some final remarks.

2 Aggregation Functions

Aggregation functions are important mathematical tools for applications in various fields, such as: Information fuzzy [17, 19, 24, 32]; Decision making [8, 11, 41, 44, 64]; Image processing [4, 26, 45] and Engineering [31, 43]. In this section we introduce them together with examples and properties. We also present a special family of aggregation functions called *Ordered Weighted Averaging* (OWA), showing some of its features and the notion of *Mixture Operator* (MO), a generalized form of OWA.

2.1 Definition and Examples

Aggregation functions are n-ary operations on the unit interval $[0,1]$ which are able to summarize an n-dimensional information $\mathbf{x} = (x_1, \ldots, x_n) \in [0,1]^n$ into a unique data $\mathbf{x} \in [0,1]$. Formally, they are the following functions:

Definition 2.1. *An n-ary aggregation function is a mapping $A : [0,1]^n \to [0,1]$, which associates each n-dimensional vector $\mathbf{x} = (x_1, \ldots, x_n)$ to a single value $A(\mathbf{x})$ in the interval $[0,1]$ which satisfies the mononicity condition*[2] *and also the boundary condition*[3]*:*

Example 2.2. *Given $\mathbf{x} = (x_1, \ldots, x_n)$,*

(a) Arithmetic Mean: $Arith(\mathbf{x}) = \frac{1}{n}(x_1 + x_2 ... + x_n)$

(b) Minimum: $Min(\mathbf{x}) = min\{x_1, x_2, ..., x_n\}$;

(c) Maximum: $Max(\mathbf{x}) = max\{x_1, x_2, ..., x_n\}$;

(d) Product: $Prod(\mathbf{x}) = \prod_{i=1}^{n} x_i$;

(e) Weighted Average: For $(w_1, \cdots, w_n) \in [0,1]^n$, with $\sum_{i=1}^{n} w_i = 1, WAvg(\mathbf{x}) = \sum_{i=1}^{n} w_i \cdot x_i$.

Remark 2.3. *From now on we will use the short term "aggregation" instead of "n-ary aggregation function".*

Aggregations can be divided into four distinct classes: *Averaging, Conjunctive, Disjunctive* and *Mixed*. Since this paper focus on averaging aggregations, we will define only this class.

Definition 2.4. *A function $f : [0,1]^n \longrightarrow [0,1]$ satisfies the* **averaging property**, *if for all $\mathbf{x} \in [0,1]^n$ we have:*

$$Min(\mathbf{x}) \leq f(\mathbf{x}) \leq Max(\mathbf{x}).$$

When an aggregation f satisfies the averaging property we say that f is a **averaging function**. *Futhermore, if a aggregation that satisfies the averaging property is called of* **averaging aggregation function**. *As in this paper we are dedicated to studying only functions that satisfy the averaging property, we will not detail the Conjunctive, Disjuntive and Mixed functions. A wider approach in aggregation can be found in [1, 3, 6, 16, 22].*

Example 2.5. *The functons Min, Max, $Arith$ and $WAvg$ are averaging aggregations.*

In the definition below we describe a series of properties that the aggregations functions (like any other function) can satisfy.

Definition 2.6. *Let $f : [0,1]^n \to [0,1]$ be a function. We say that f*

(1) is **Idempotent** *if, and only if, $f(x, ..., x) = x$ for all $x \in [0,1]$.*

(2) is **Homogeneous** *of order k if, and only if, for all $\lambda \in [0,1]$ and $\mathbf{x} \in [0,1]^n$, $f(\lambda x_1, \lambda x_2, ..., \lambda x_n) = \lambda^k f(x_1,$
$x_2, ..., x_n)$. When f is homogeneous of order 1 we simply say that f is homogeneous.*

(3) is **Shift-invariant** *if, and only if, $f(x_1 + r, x_2 + r, .., x_n + r) = f(x_1, x_2, .., x_n) + r$, for all $r \in [-1,1]$, $\mathbf{x} \in [0,1]^n$, $(x_1 + r, x_2 + r, ..., x_n + r) \in [0,1]^n$ and $f(x_1, x_2, ..., x_n) + r \in [0,1]$.*

(4) is **Monotonic** *if, and only if, $f(\mathbf{x}) \leq f(\mathbf{y})$ whenever $x_i \leq y_i$, for all $i \in \{1, \cdots, n\}$.*

(5) is **Strictly Monotone** *if, and only if, $f(\mathbf{x}) < f(\mathbf{y})$ whenever $\mathbf{x} < \mathbf{y}$, i.e., $\mathbf{x} \leq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$.*

[2] If $\mathbf{x} \leq \mathbf{y}$, i.e., $x_i \leq y_i$, for all $i = 1, 2, ..., n$, then $A(\mathbf{x}) \leq A(\mathbf{y})$.
[3] $A(0, ..., 0) = 0$ and $A(1, ..., 1) = 1$.

(6) has a **Neutral Element** $e \in [0,1]$*, if for all* $t \in [0,1]$ *it has to be:*

$$f(e, ..., e, t, e, ..., e) = t.$$

(7) is **Symmetric** *if, and only if, its value is not changed under the permutations of coordinate for any input vector, i.e.:*

$$f(x_1, x_2, ..., x_n) = f(x_{p_{(1)}}, x_{p_{(2)}}, \cdots, x_{p_{(n)}})$$

for all vector $\mathbf{x} = (x_1, x_2, ..., x_n)$ *and any permutation* $p : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$*.*

(8) has an **Absorbing Element** ***(Annihilator)*** $a \in [0,1]$*, if:*

$$f(x_1, x_2, ..., x_{i-1}, a, x_{i+1}, ..., x_n) = a.$$

(9) has a **Zero Divisor** $a \in\ ]0,1[$*, if for all* $i \in \{1, 2, \cdots, n\}$ *there is some vector* $\mathbf{x} \in]0,1]^n$*, of the form* $(x_1, ..., x_{i-1},$
$a, x_{i+1}, ..., x_n)$*, such that* $f(\mathbf{x}) = 0$*.*

(10) has a **One Divisor** $a \in]0,1[$*, if for any* $i \in \{1, 2, \cdots, n\}$ *there is some vector* $\mathbf{x} \in [0,1[^n$*, of the form* $(x_1, ..., x_{i-1},$
$a, x_{i+1}, ..., x_n)$*, such that* $f(\mathbf{x}) = 1$*.*

Example 2.7.

(i) The functions: $Arith, Min$ *and* Max *are examples of idempotent, homogeneous, shift-invariant and symmetric aggregations.*

(ii) Min *and* Max *have the elements* 0 *and* 1 *as its respective annihilators, but* $Arith$ *does not have annihiladors.*

(iii) Min*,* Max *and* $Arith$ *does not have zero divisors and one divisors.*

2.2 Ordered Weighted Averaging - OWA Functions

In the field of aggregations there is a very important kind of function in which the aggregation behavior is provided parametrically; they are called: ***Ordered Weighted Averaging*** or simply OWA [60]. More precisely, they are average aggregation whose behavior is in function of a vector of weights. Observe the following definition.

Definition 2.8. *Let be an input vector* $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in [0,1]^n$ *and a vector of weights* $\mathbf{w} = (w_1, \ldots, w_n) \in [0,1]^n$*, such that* $\sum_{i=1}^{n} w_i = 1$*. Assuming the permutation of* $\mathbf{x}$*:*

$$Sort(\mathbf{x}) = (x_{(1)}, x_{(2)}, \ldots, x_{(n)})$$

such that $x_{(i)} \geq x_{(i+1)}$*, i.e.,* $x_{(1)} \geq x_{(2)} \geq \cdots \geq x_{(n)}$*. The Ordered Weighted Averaging (OWA) function with respect to* $\mathbf{w}$*, is the function* $\mathit{OWA}_{\mathbf{w}} : [0,1]^n \to [0,1]$ *such that:*

$$\mathit{OWA}_{\mathbf{w}}(\mathbf{x}) = \sum_{i=1}^{n} w_i \cdot x_{(i)}$$

Remark 2.9. *In what follows we remove* $\mathbf{w}$ *from* $\mathit{OWA}_{\mathbf{w}}(\mathbf{x})$ *and write only OWA.*

The main properties of OWA functions are:

(a) For any vector of weights $\mathbf{w}$, the function $\mathsf{OWA}_{\mathbf{w}}(\mathbf{x})$ is an idempotent aggregation function. Moreover, OWA's are strictly increasing if all weights $\mathbf{w}$ are positive;

(b) The dual of a $\mathsf{OWA}_{\mathbf{w}}$, denoted by $(\mathsf{OWA})^d$, is an OWA with the vector of weights dually ordered, i.e. $(\mathsf{OWA}_{\mathbf{w}})^d = \mathsf{OWA}_{\mathbf{w}^d}$, where $\mathbf{w}^d = (w_{p(n)}, w_{p(n-1)}, ..., w_{p(1)})$.

(c) OWA are continuous, symmetric and shift-invariant;

(d) They do not have neutral or absorption elements, on exception for the second and third case below.

Following is a series of examples of OWA functions

Example 2.10.

(1) If $\mathbf{w} = (0, 0, 0, ..., 1)$, then $OWA(\mathbf{x}) = Min(\mathbf{x})$;

(2) If $\mathbf{w} = (1, 0, 0, ..., 0)$, then $OWA(\mathbf{x}) = Max(\mathbf{x})$;

(3) If all weight vector components are equal to $\frac{1}{n}$, then $OWA(\mathbf{x}) = Arith(\mathbf{x})$;

(4) if $w_i = 0$, for all i, with the exception of a k-th member, i.e, $w_k = 1$, then this OWA is called **static** *and $OWA_{\mathbf{w}}(x) = x_{(k)}$;*

(5) Given a vector $\mathbf{x}$ and its ordered permutation $Sort(\mathbf{x}) = (x_{(1)}, \ldots, x_{(n)})$, the Median function

$$Med(\mathbf{x}) = \begin{cases} \frac{1}{2}(x_{(k)} + x_{(k+1)}), & \text{if } n = 2k \\ x_{(k+1)}, & \text{if } n = 2k+1 \end{cases}$$

is an OWA function in which the vector of weights is defined by:

- *If n is odd, then $w_i = 0$ for all $i \neq \lceil \frac{n}{2} \rceil$ and $w_{\lceil n/2 \rceil} = 1$.*
- *If n is even, then $w_i = 0$ for all $i \neq \lfloor \frac{n+1}{2} \rfloor$ and $i \neq \lceil \frac{n+1}{2} \rceil$, and $w_{\lceil (n+1)/2 \rceil} = w_{\lfloor (n+1)/2 \rfloor} = \frac{1}{2}$.*

In addition to the above functions, another important example of OWA, which we will use later in this work, is the **centered OWA** or cOWA[61].

Example 2.11. *The n-dimensional cOWA function is the OWA operator, with weighted vector defined by:*

- *If n is even, then $w_j = \frac{2(2j-1)}{n^2}$, for $1 \leq j \leq \frac{n}{2}$, and $w_{n/2+i} = w_{n/2-i+1}$, for $1 \leq i \leq \frac{n}{2}$.*
- *If n is odd, then $w_j = \frac{2(2j-1)}{n^2}$, for $1 \leq j \leq \frac{n-1}{2}$, $w_{n/2+i} = w_{n/2-i+1}$, for $1 \leq i \leq \frac{n}{2}$, and $w_{(n+1)/2} = 1 - 2\sum_{j=1}^{(n-1)/2} w_i$.*

The OWA functions are defined in terms of a predetermined vector of weights; namely this vector of wights is fixed previously by the user. In the next section present a generalized form of OWA in order to relax this situation. The vector of weights will be in function of the vector of inputs $\mathbf{x} = (x_1, \ldots x_n)$. To achieve that we replace, in the OWA expression, the vector of weights by a family of functions, called **Weighted functions**.

3 Weighted functions

As mentioned, the OWA functions are means with previously fixed weights. In the literature we can find some kind of functions that overcome this situation, by providing variable weights. These functions are called here *weighted functions*. An important example of that is the Mean of Bajraktarevic, presented in [6].

Definition 3.1 (Mean of Bajraktarevic)**.** *Let* $\mathbf{w}(t) = (w_1(t), \cdots, w_n(t))$ *be a vector of weights functions* $w_i : [0,1] \to [0,+\infty)$*, and let* $g : [0,1] \to (-\infty,+\infty)$ *be a strictly monotone function. The* ***mean of Bajraktarevic*** *is the function:*

$$f(\mathbf{x}) = g^{-1}\left(\frac{\sum\limits_{i=1}^{n} w_i(x_i)g(x_i)}{\sum\limits_{i=1}^{n} w_i(x_i)}\right)$$

In the case of $g(t) = t$, the mean of Bajraktarevic is also called **Mixture function**, in other words, the mixture functions have the form:

$$M(\mathbf{x}) = \frac{\sum\limits_{i=1}^{n} w_i(x_i) \cdot x_i}{\sum\limits_{i=1}^{n} w_i(x_i)} \tag{1}$$

Generally, the mixture functions are not aggregation functions in general, since they do not always satisfy monotonicity, however [38, 39, 48] provides sufficient conditions to overcome this situation.

Remark 3.2. *Note in Equation (1) that each weight* $w_i(x_i)$ *is the value of a single variable function; namely the weight is the value of a function* w_i *applied to the* i*-th position of the input vector* $\mathbf{x} = (x_1, \ldots, x_n)$*. However, this restriction can be relaxed in order to obtain a weight* $w_i(\mathbf{x})$*, i.e. weight which is in function of the whole input. This generalization of mixture operators were done by Pereira [46, 47] and the resulting functions were called of* ***Generalized Mixture Functions (GMF)****.*

Although Pereira has introduced GMFs he did not provide a deep investigation about them. In what follows we provide some results about such functions; their relation with OWA's, Mixture Functions and Preaggregations. We finally generalize GMF's to the notion of ***Bounded Generalized Mixture Functions (BGMF)*** *and provide some relations of them with the notions of monotonicity, directional monotonicity, Weak-dual and Weak-conjugate functions.*

3.1 Weighted Averaging Functions

Before defining the notion of Weighted Averaging functions, we need to establish the notion of **weight-function**.

Definition 3.3. *A finite family of functions* $\Gamma = \{f_i : [0,1]^n \to [0,1] \mid 1 \leq i \leq n\}$ *such that* $\sum\limits_{i=1}^{n} f_i(\mathbf{x}) = 1$ *is called family of* **weight-functions (FWF)***.*

The **Generalized Mixture Function***, or simply GM, associated to a FWF* Γ *is the function* $GM_\Gamma : [0,1]^n \to [0,1]$ *given by:*

$$GM_\Gamma(\mathbf{x}) = \sum_{i=1}^{n} f_i(\mathbf{x}) \cdot x_i$$

In the Examples 3.4-3.10 we present GM functions.

Example 3.4. *Let be* $\Gamma = \left\{ f_i(\mathbf{x}) = \frac{1}{n} \mid 1 \leq i \leq n \right\}$*. The* GM *operator associated to* Γ*,* $\mathsf{GM}_\Gamma(\mathbf{x})$*, is* $Arith(\mathbf{x})$*.*

Example 3.5. *The function Minimum can be obtained from* $\Gamma = \{f_i \mid 1 \leq i \leq n\}$*, where for all* $\mathbf{x} \in [0,1]^n$*,* $f_{(n)}(\mathbf{x}) = 1$ *and* $f_i(\mathbf{x}) = 0$*, if* $i \neq (n)$ *.*

Example 3.6. *Similarly, the function Maximum is also of type* GM *with* Γ *dually defined.*

Example 3.7. *For any vector of weights* $\mathbf{w} = (w_1, w_2, ..., w_n)$*, A function* $\mathsf{OWA}_{\mathbf{w}}(\mathbf{x})$ *is a* GM *in which the weight-function are given by:* $f_i(\mathbf{x}) = w_{p(i)}$*, where* $p : \{1, 2, \cdots, n\} \longrightarrow \{1, 2, \cdots, n\}$ *is the permutation, such that* $p(i) = j$ *with* $x_i = x_{(j)}$*. For example: If* $\mathbf{w} = (0.3, 0.4, 0.3)$*, then for* $\mathbf{x} = (0.1, 1.0, 0.9)$ *we have* $x_1 = x_{(3)}$*,* $x_2 = x_{(1)}$ *and* $x_3 = x_{(2)}$*. Thus,* $f_1(\mathbf{x}) = 0.3$*,* $f_2(\mathbf{x}) = 0.3$*,* $f_3(\mathbf{x}) = 0.4$*, and* $\mathsf{GM}(\mathbf{x}) = 0.3 \cdot 0.1 + 0.3 \cdot 1.0 + 0.4 \cdot 0.9 = 0.69$

Remark 3.8. *Example 3.7 shows that any* OWA *function is* GM*. However, there are* GM *functions which are not* OWA*:*

Example 3.9. *Let* $\Gamma = \{\sin(x) \cdot y, 1 - \sin(x) \cdot y\}$*. The respective* GM *function is*

$$\mathsf{GM}(x, y) = (\sin(x) \cdot y) \cdot x + (1 - \sin(x) \cdot y) \cdot y,$$

which is not an OWA *function.*

The following example shows that the *mixture functions* are also special types of GM function.

Example 3.10. *If* $\mathbf{w}(t) = (w_1(t), \cdots, w_n(t))$ *is a vector of weight functions* $w_i : [0,1] \to [0, +\infty)$*, and the mixture operator is* $M(\mathbf{x}) = \frac{\sum_{i=1}^{n} w_i(x_i) \cdot x_i}{\sum_{i=1}^{n} w_i(x_i)}$*, then* M *is also a* GM *function, with weight-functions given by* $f_i(\mathbf{x}) = \frac{w_i(x_i)}{\sum_{i=1}^{n} w_i(x_i)}$*.*

Remark 3.11. *Observe that the* GM *function at Example 3.9 can not be characterized as a mixture function, since* w_1 *is not a function that depends only of variable* x *and* w_2 *is not a function that depends only of variable* y*.*

At this point of paper, we relax the condition $\sum_{i=1}^{n} f_i(\mathbf{x}) = 1$ to $\sum_{i=1}^{n} f_i(\mathbf{x}) \leq 1$, thus obtaining a new family of generalized mixture functions.

Definition 3.12. *Let* $\Gamma = \{f_i : [0,1]^n \to [0,1] \mid 1 \leq i \leq n\}$ *such that:*

(I) $\sum_{i=1}^{n} f_i(\mathbf{x}) \leq 1$*, and*

(II) $\sum_{i=1}^{n} f_i(1, \cdots, 1) = 1$*, for all* $i \in \{1, 2, \cdots, n\}$*.*

A **Bounded Ganeralized Mixture** *(*BGM*) operator associated to a* Γ *is a function* $\mathsf{BGM}_\Gamma : [0,1]^n \to [0,1]$ *given by:*

$$\mathsf{BGM}_\Gamma(\mathbf{x}) = \sum_{i=1}^{n} f_i(\mathbf{x}) \cdot x_i$$

Remark 3.13.

1. *Note that* GM *functions are* BGM *operators subject to the condition:*

   (III) $\sum_{i=1}^{n} f_i(\mathbf{x}) = 1$, *for any* $\mathbf{x} \in [0,1]^n$,

2. *Let* $\Gamma = \{f_i(x,y) = \frac{x}{n} : 1 \leq i \leq n\}$. *Then,* $BGM_\Gamma = \sum_{i=1}^{n} \frac{x_i^2}{n}$ *is not a* GM *operator, because, for example,* $\sum_{i=1}^{n} f_i(0, \cdots, 0) = 0$.

3. *As* BGM *is a generalized form of* GM*, it follows that the functions defined in the Examples 3.4-3.10 are also* BGM *function. In this sense, is worth emphasizing that* BGM *generalize both:* OWA *and* GM *operators.*

Now, we establish several properties of GM and BGM functions.

3.2 Properties of GM and BGM Functions

As we have said previously, GM and BGM are generalized forms of OWA, which in turn belongs to the class of avegaring functions. However, we can not always guarantee that a BGM is an averaging function, while then GM functions are averaging function. The next proposition gives us a sufficient condition to achieve that.

Proposition 3.14. *If* Γ *is a FWF with* $\sum_{i=1}^{n} f_i(\mathbf{x}) = 1$, *then* GM_Γ *is an averaging function, i.e.:*

$$Min(\mathbf{x}) \leq GM_\Gamma(\mathbf{x}) \leq Max(\mathbf{x})$$

Proof. For all $\mathbf{x} = (x_1, ..., x_n)$,

$$Min(\mathbf{x}) \leq x_i \leq Max(\mathbf{x}),\ \forall i = 1, 2, ..., n.$$

So,

$$\sum_{i=1}^{n} f_i(\mathbf{x}) \cdot Min(\mathbf{x}) \leq \sum_{i=1}^{n} f_i(\mathbf{x}) \cdot x_i \leq \sum_{i=1}^{n} f_i(\mathbf{x}) \cdot Max(\mathbf{x}),$$

but as $\sum_{i=1}^{n} f_i(\mathbf{x}) = 1$, it follows that

$$Min(\mathbf{x}) \leq \sum_{i=1}^{n} f_i(\mathbf{x}) \cdot x_i \leq Max(\mathbf{x}).$$

□

Remark 3.15. *Observe that the restriction of condition* $\sum_{i=1}^{n} f_i(\mathbf{x}) = 1$ *can not be removed, i.e.,* BGM *not always are averaging functions, since for* $f_1(x,y) = \frac{x}{2}$ *and* $f_2(x,y) = \frac{y}{2}$, *we have* $BGM(0.5, 0.5) = 0.25 < Min(0.5, 0.5)$.

Proposition 3.16. *Let* Γ *be a FWF. Then, the* BGM_Γ *is idempotent if, and only, if* $\sum_{i=1}^{n} f_i(x, \cdots, x) = 1$ *for any* $x \in [0,1]$.

Proof. If $\sum_{i=1}^{n} f_i(\mathbf{x}) = 1$ and $\mathbf{x} = (x, ..., x)$, then:

$$\mathsf{BGM}_\Gamma(\mathbf{x}) = \sum_{i=1}^{n} f_i(\mathbf{x}) \cdot x = x \cdot \sum_{i=1}^{n} f_i(\mathbf{x}) = x$$

Reciprocally, if BGM is an idempotent function and $\sum_{i=1}^{n} f_i(x, \cdots, x) < 1$ for some $x \in [0,1]$ we have to

$$\mathsf{BGM}_\Gamma(\mathbf{x}) = \sum_{i=1}^{n} f_i(\mathbf{x}) \cdot x < x \cdot 1 = x.$$

Thus, the condition $\sum_{i=1}^{n} f_i(x, \cdots, x) = 1$ can not be removed. □

Corollary 3.17. *Any GM function is idempotent.*

Proof. Straightforward.

□

Example 3.18. *We can not always guarantee that a BGM is idempotent, because if we take $f_1(x, y) = \frac{x}{2}$ and $f_2(x, y) = \frac{y}{2}$, then $BGM(0.5, 0.5) = 0.25 \neq 0.5$.*

Proposition 3.19. *If Γ is a FWF invariant under translations, i.e, $f_i(x_1+\lambda, x_2+\lambda, ..., x_n+\lambda) = f_i(x_1, x_2, ..., x_n)$ for any $\mathbf{x} \in [0,1]^n$, for $i \in \{1, 2, \cdots, n\}$, satisfying 1 and $\lambda \in [-1, 1]$, then BGM_Γ is shift-invariant.*

Proof. Let $\mathbf{x} = (x_1, ..., x_n) \in [0,1]^n$ and $\lambda \in [-1, 1]$ such that $(x_1 + \lambda, x_2 + \lambda, ..., x_n + \lambda) \in [0,1]^n$. then,

$$\begin{aligned}
\mathsf{BGM}_\Gamma(x_1 + \lambda, ..., x_n + \lambda) &= \sum_{i=1}^{n} f_i(x_1 + \lambda, ..., x_n + \lambda) \cdot (x_i + \lambda) \\
&= \sum_{i=1}^{n} f_i(x_1 + \lambda, ..., x_n + \lambda) \cdot x_i + \sum_{i=1}^{n} f_i(x_1 + \lambda, ..., x_n + \lambda) \cdot \lambda \\
&= \sum_{i=1}^{n} f_i(x_1, ..., x_n) \cdot x_i + \lambda \\
&= \mathsf{BGM}_\Gamma(x_1, ..., x_n) + \lambda
\end{aligned}$$

□

Remark 3.20. *The condition 1 is also important to preserve shift-invariance, since if we define $f_1(x, y) = f_2(x, y) = \frac{|x-y|}{2}$, for $(x, y) \neq (1, 1)$, and $f_1(1, 1) = f_2(1, 1) = \frac{1}{2}$, then f_1 and f_2 are invariant under translations, but $BGM(0, 0.1) = 0.005$ and $BGM(0 + 0.1, 0.1 + 0.1) = 0.015 \neq 0.005 + 0.1$.*

Proposition 3.21. *If Γ is homogeneous of order k (i.e. if each f_i is homogeneous of order k), then $BGM_\Gamma(\mathbf{x})$ is homogeneous of order $k + 1$.*

Proof. Of course that, if $\lambda = 0$, then $\mathsf{BGM}_\Gamma(\lambda x_1, ..., \lambda x_n) = \lambda f(x_1, ..., x_n)$. Now, to $\lambda \neq 0$ we have:

$$\begin{aligned}
\mathsf{BGM}_\Gamma(\lambda x_1, ..., \lambda x_n) &= \sum_{i=1}^{n} f_i(\lambda x_1, ..., \lambda x_n) \cdot \lambda x_i \\
&= \lambda \cdot \sum_{i=1}^{n} \lambda^k f_i(x_1, ..., x_n) x_i \\
&= \lambda^{k+1} \cdot \mathsf{BGM}_\Gamma(x_1, ..., x_n)
\end{aligned}$$

□

Remark 3.22. *Note that if* $\sum_{i=1}^{n} f_i(\mathbf{x}) = 1$*, then* f_i *cannot be homogeneous of order* $k > 0$*, since*

$$1 = \sum_{i=1}^{n} f_i(\lambda x_1, \cdots, \lambda x_n) = \lambda^k \sum_{i=1}^{n} f_i(\mathbf{x}) = \lambda^k,$$

i.e., there are no GM*'s homogeneous of order* $k > 1$*. However, if we remove this restriction, then we can have* Γ *with homogeneous* f_i*s of order* $k > 0$*. For example,* $f_i(\mathbf{x}) = \frac{x_i}{n}$ *is homogeneous of order* 1*, and so, according to Proposition 3.21,* BGM_Γ *is homogeneous of order* 2*.*

The next example shows a GM function which is not a mixture operator.

Example 3.23. *Let* Γ *be defined by*

$$f_i(x_1, ..., x_n) = \begin{cases} \frac{1}{n}, & \text{if } x_1 = \cdots = x_n = 0 \\ \frac{x_i}{\sum_{j=1}^{n} x_j}, & \text{otherwise} \end{cases}$$

Then,

$$\mathsf{GM}_\Gamma(\mathbf{x}) = \begin{cases} 0, & \text{if } x_1, ..., x_n = 0 \\ \frac{\sum_{i=1}^{n} x_i^2}{\sum_{i=1}^{n} x_i}, & \text{otherwise} \end{cases}$$

Observe that this function, like that in Example 3.9, cannot be characterized as a mixture function, since f_i does not depend exclusively from x_i. This GM_Γ is idempotent, homogeneous and shift-invariant, but is not monotonic, since $\mathsf{GM}_\Gamma(0.5, 0.2, 0.1) = 0.375$ and $\mathsf{GM}_\Gamma(0.5, 0.22, 0.2) = 0.368$.

Proposition 3.24. *The* N*-dual*[4]*, with respect to stantard fuzzy negation*[5]*, of a* GM *function is also a* GM *function.*

Proof. If Γ is a FWF, then

$$\begin{aligned}
\mathsf{GM}_\Gamma^N(x_1, \cdots, x_n) &= 1 - \sum_{i=1}^{n} f_i(1 - x_1, \cdots, 1 - x_n) \cdot (1 - x_i) \\
&= 1 - \sum_{i=1}^{n} f_i(1 - x_1, \cdots, 1 - x_n) + \sum_{i=1}^{n} f_i(1 - x_1, \cdots, 1 - x_n) \cdot x_i \\
&= \sum_{i=1}^{n} f_i(1 - x_1, \cdots, 1 - x_n) \cdot x_i \\
&= \sum_{i=1}^{n} g_i(x_1, \cdots, x_n) \cdot x_i,
\end{aligned}$$

where $g_i(x_1, \cdots, x_n) = f_i(1 - x_1, \cdots, 1 - x_n)$. □

Proposition 3.25. *If* $\Gamma = \{f_1, \cdots, f_n\}$ *is a FWF, then* $\Gamma^R = \{f_n, \cdots, f_1\}$ *also is a FWF. Besides that,* $\mathsf{GM}_\Gamma^R = \mathsf{GM}_{\Gamma^R}$

[4] The N-dual of a function $F : [0, 1]^n \longrightarrow [0, 1]$ is $F^N(x_1, \cdots, x_n) = N(F(N(x_1), \cdots, N(x_n))$, where N is a fuzzy negation, i.e., a function decreasing function $N : [0, 1] \longrightarrow [0, 1]$ with $N(0) = 1$ and $N(1) = 0$.

[5] The standard fuzzy negation if $N(x) = 1 - x$.

Proof. Direct from the definition. □

Examples 3.9 and 3.10 show that GM functions encompass both: OWA and Mixture functions, and thus these functions are special cases GM proposed here. It is also important to note that GM and BGM functions, as well as Mixture functions, are not generally aggregations since it fails to satisfy the monotonicity condition. In examples 3.4, 3.5, 3.6, 3.7 and 3.9 the respective GM's are monotonic, but in Example 3.23 (that we bring forward) the function there is not monotonic. When the GM is monotonic, obviously, this function is an aggregation, since the boundary condition is trivially satisfied.

Some conditions for monotonicity of GM functions were studied by Pereira *et al.* in [46, 47, 48]. In this work we will not study monotonicity criteria, but a more weakened form, called **weak monotonicity** or **directional monotonicity**.

3.3 Directional Monotonicity

There are many n-ary functions that do not satisfy the monotonicity condition, but its restriction to certain directions are monotonic functions. In this sense, Wilkin and Beliakov in [57] introduce the concept of **weakly monotonicity** (see also [5]), which was generalized by Bustince *et al.* in [9], which defines the notion of **directional monotonicity**.

Definition 3.26. *Let* $\mathbf{r} = (r_1, \cdots, r_n)$ *a not null* n*-dimentional vector. A function* $F : [0,1]^n \longrightarrow [0,1]$ *is* **r-increasing** *if fo all* $\mathbf{x} = (x_1, \cdots, x_n)$ *and* $t > 0$ *such that* $(x_1 + tr_1, \cdots, x_n + tr_n) \in [0,1]^n$*, we have*

$$F(x_1, \cdots, x_n) \leq F(x_1 + tr_1, \cdots, x_n + tr_n),$$

that is, F *is increasing in the direction of vector* $\mathbf{r}$*.*

Definition 3.27. *A function* $F : [0,1]^n \longrightarrow [0,1]$ *is an* n*-ary* **preaggregation** *function (or simply preaggregation) if satisfies the boundary condition,* $F(0, \cdots, 0) = 0$ *and* $F(1, \cdots, 1) = 1$*, and is* $\mathbf{r}$*-increasing for some direction* $\mathbf{r} \in [0,1]^n$*.*

In [34], Lucca *et al.* was presented properties, constructions and application for preaggregations function. They show that the following functions are examples of preaggregations.

Example 3.28.

1. $Mode(x_1, \cdots, x_n)$*, that is* $(1,1)$*-increasing;*
2. $F(x, y) = x - (max\{0, x - y\})^2$*, tha is* $(0,1)$*-increasing;*
3. *The weighted Lehmer mean (with convention* $0/0 = 0$*)*

$$L_\lambda(x, y) = \frac{\lambda x^2 + (1-\lambda)y^2}{\lambda x + (1-\lambda)y}, \text{ where } 0 < \lambda < 1$$

   is $(1-\lambda, \lambda)$*-increasing;*

4. $$A(x, y) = \begin{cases} x(1-x), & \text{if } y \leq 3/4 \\ 1, & \text{otherwise} \end{cases}$$

   is $(0, a)$*-increasing for any* $a > 0$*, but for no other direction;*

5. $$B(x, y) = \begin{cases} y(1-y), & \text{if } x \leq 3/4 \\ 1, & \text{otherwise} \end{cases}$$

   is $(b, 0)$*-increasing for any* $b > 0$*, but for no other direction.*

Remark 3.29. *Any aggregation functions is also a preaggregation function.*

Proposition 3.30. *If BGM_Γ is shift-invariant, then BGM_Γ is a preaggregation function $(k, k, \cdots, k)$-increasing.*

Proof. Just see that for all $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in [0,1]^n$ and any $t > 0$ such that $(x_1 + tk, x_2 + tk, \cdots, x_n + tk) \in [0,1]$ we have

$$\mathsf{BGM}_\Gamma(x_1 + tk, \cdots, x_n + tk) = \mathsf{BGM}_\Gamma(x_1, \cdots, x_n) + tk,$$

and so

$$\mathsf{BGM}_\Gamma(x_1, \cdots, x_n) \leq \mathsf{BGM}(x_1 + tk, \cdots, x_n + tk)$$

□

Corollary 3.31. *If Γ is a FWF invariant under translations, i.e, $f_i(x_1+\lambda, x_2+\lambda, ..., x_n+\lambda) = f_i(x_1, x_2, ..., x_n)$, for $i \in \{1, 2, \cdots, n\}$, for any $\mathbf{x} = (x_1, \cdots, x_n) \in [0,1]^n$ and $\lambda \in [0,1]$ such that $(x_1 + \lambda, x_2 + \lambda, ..., x_n + \lambda) \in [0,1]^n$ satisfying 1, BGM_Γ is a preaggregation function $(k, k, \cdots, k)$-increasing.*

Proof. By Proposition 3.19, BGM_Γ is shift-invariant, and so, by Proposition 3.30, BGM_Γ is a preaggregation function $(k, k, \cdots, k)$-increasing. □

In fact, the conditions required by Corollary 3.31 are very strong. In the following proposition, we relax these conditions:

Proposition 3.32. *If Γ is a FWF with $f_i(x_1, \cdots, x_n) \leq f_i(x_1 + \lambda, \cdots, x_i + \lambda)$, for $i \in \{1, 2, \cdots, n\}$, for any $\mathbf{x} = (x_1, \cdots, x_n) \in [0,1]^n$ and $\lambda \in [0,1]$ such that $(x_1 + \lambda, x_2 + \lambda, ..., x_n + \lambda) \in [0,1]^n$, then BGM_Γ is a preaggregation function $(k, k, \cdots, k)$-increasing.*

Proof. For any $\mathbf{x} = (x_1, \cdots, x_n) \in [0,1]^n$ and $\lambda \in [0,1]$ such that $(x_1 + \lambda, x_2 + \lambda, ..., x_n + \lambda) \in [0,1]^n$ we observe that

$$\begin{aligned}
\mathsf{BGM}_\Gamma(x_1 + \lambda, ..., x_n + \lambda) &= \sum_{i=1}^{n} f_i(x_1 + \lambda, ..., x_n + \lambda) \cdot (x_i + \lambda) \\
&= \sum_{i=1}^{n} f_i(x_1 + \lambda, ..., x_n + \lambda) \cdot x_i \; + \; \sum_{i=1}^{n} f_i(x_1 + \lambda, ..., x_n + \lambda) \cdot \lambda \\
&\geq \sum_{i=1}^{n} f_i(x_1, ..., x_n) \cdot x_i + \lambda \\
&\geq \mathsf{BGM}_\Gamma(x_1, ..., x_n)
\end{aligned}$$

□

Example 3.33. *Let Γ whose functions are given by*

$$f_i(x_1, \cdots, x_n) = \begin{cases} \frac{1}{n}, & \text{if } x_1 = \cdots = x_n \\ \frac{x_{(1)} - x_i}{\sum\limits_{j=1}^{n} (x_{(1)} - x_j)}, & \text{otherwise} \end{cases}.$$

We can easily prove that satisfies

$$f_i(x_1 + \lambda, x_2 + \lambda, \cdots, x_n + \lambda) = f_i(x_1, x_2, \cdots, x_n).$$

More generally, for any $\alpha \geq 1$

$$f_i(x_1, \cdots, x_n) = \begin{cases} \frac{1}{n}, & \text{if } x_1 = \cdots = x_n \\ \frac{x_{(1)} - x_i}{\sum\limits_{j=1}^{n} (x_{(1)} - x_j)^\alpha}, & \text{otherwise} \end{cases}$$

is such that

$$f_i(x_1, x_2, \cdots, x_n) \leq f_i(x_1 + \lambda, x_2 + \lambda, \cdots, x_n + \lambda).$$

Thus, the corresponding BGM is $(k, \cdots, k)$-increasing. In additon, note that, for $\alpha > 1$, $\Gamma = \{f_i\}$ does not satisfies $\sum_{i=1}^{n} f_i(\mathbf{x}) = \mathbf{1}$.

We can also establish a criterion analogous to the Proposition 3.32, substituting the vector $(k, \cdots, k)$ for any direction $\mathbf{r}$, as follow:

Proposition 3.34. *If Γ is a FWF such that there is a diretional vector $\mathbf{r} = (r_1, r_2, \cdots, r_n) \in [0,1]^n$ with $f_i(x_1, \cdots, x_n) \leq f_i(x_1 + \lambda \cdot r_1, \cdots, x_i + \lambda \cdot r_n)$, for $i \in \{1, 2, \cdots, n\}$, for any $\mathbf{x} = (x_1, \cdots, x_n) \in [0,1]^n$ and $\lambda \in [0,1]$ such that $(x_1 + \lambda \cdot r_1, x_2 + \lambda \cdot r_2, ..., x_n + \lambda \cdot r_n) \in [0,1]^n$, then BGM_Γ is a preaggregation function $\mathbf{r}$-increasing.*

Proof. Is similar to what was done in Proposition 3.32. □

Corollary 3.35. *If Γ is a FWF such that there is a diretional vector $\mathbf{r}$ with $\frac{\partial f_i}{\partial \mathbf{r}}(\mathbf{x}) \geq 0$ for any $f_i \in \Gamma$ and $\mathbf{x} \in [0,1]^n$, then BGM_Γ is a preaggregation function $\mathbf{r}$-increasing.*

Note that this condition can not be satisfied, in the case that $\sum_{i=1}^{n} f_i(\mathbf{x}) = 1$, for all $\mathbf{x} \in [0,1]^n$, unless that the functions f_i are constant in the direction of vector $\mathbf{r}$, because:

$$\sum_{i=1}^{n} f_i(\mathbf{x}) = 1 \implies \sum_{i=1}^{n} \frac{\partial f_i(\mathbf{x})}{\partial \mathbf{r}} = 0$$

and so,

$$\frac{\partial f_i(\mathbf{x})}{\partial \mathbf{r}} \geq 0 \Longrightarrow \frac{\partial f_i(\mathbf{x})}{\partial \mathbf{r}} = 0$$

Example 3.36. *Obviously, if $f_i = w_i$ is constant, then BGM_Γ is $\mathbf{r}$-increaing for any direction $\mathbf{r}$. Now, given a direction $\mathbf{r} = (r_1, \cdots, r_n) \in [0,1]^n$ we can build a $\mathbf{r}$-increaing BGM function defining:*

$$f_i(x_1, \cdots, x_n) = \begin{cases} 0, & \text{if } min\{x_1, \cdots, x_n\} = 0 \\ \frac{min\left\{\frac{x_i}{r_i}, 1\right\}}{n}, & \text{otherwise} \end{cases},$$

we obtain a BGM $\mathbf{r}$-increasing.

As previously mentioned, both the Aggregation functions $(Min, Max, Med, Arith, \mathsf{OWA}, \cdots)$ and generalized mixture functions (and also bounded generalized mixture functions) can be used in many applications. To finalize this paper we bring an illustrative example of application, where we apply some functions in the scope of image processing. More precisely, we will use generalized mixture functions in the image reduction process.

Before presenting this example of application, we propouse a special GM function, which satisfies several interesting properties, as we will show in this paper, and will be used in the application.

Definition 3.37. *Consider the family Γ of functions*

$$f_i(\mathbf{x}) = \begin{cases} \frac{1}{n}, & \text{if } \mathbf{x} = (x, ..., x) \\ \frac{1}{n-1}\left(1 - \frac{|x_i - Med(\mathbf{x})|}{\sum_{j=1}^{n} |x_j - Med(\mathbf{x})|}\right), & \text{otherwise} \end{cases}$$

Γ is a FWF, with $\sum_{i=1}^{n} f_i(\mathbf{x}) = 1$ for all $\mathbf{x} \in [0,1]^n$, i.e., BGM_Γ is a GM, that will be denoted by $\mathbf{H}$. The computation of $\mathbf{H}$ can be performed using the following expressions:

$$\mathbf{H}(\mathbf{x}) = \begin{cases} x, & \text{if } \mathbf{x} = (x, ..., x) \\ \frac{1}{n-1} \sum_{i=1}^{n} \left(x_i - \frac{x_i |x_i - Med(\mathbf{x})|}{\sum_{j=1}^{n} |x_j - Med(\mathbf{x})|} \right), & \text{otherwise} \end{cases}$$

Example 3.38. *Let be $n = 5$. So, for $\mathbf{x} = (0.1, 0.25, 0.3, 0, 1)$ we have*

$$f_1(\mathbf{x}) = 0.21875, \;\; f_2(\mathbf{x}) = 0.25, \;\; f_3(\mathbf{x}) = 0.2395, f_4(\mathbf{x}) = 0.198, f_5(\mathbf{x}) = 0.09375$$

And

$$\mathbf{H}(\mathbf{x}) = 0.249975.$$

Note that the larger weights occur in the coordinates closest to the median. Besides, if we take the fixed vector of weights $\mathbf{w} = (0.21875, 0.25, 0.2395, 0.198, 0.09375)$, then $\mathsf{OWA}_\mathbf{w}(0.1, 0.25, 0.3, 0, 1) = 0.249975 = \mathbf{H}(0.1, 0.25, 0.3, 0, 1)$. In other words, the function $\mathbf{H}$ can be seen as a function that transforms each input $\mathbf{x}$ into the output of an OWA. More precisely,

$$\mathbf{H}(\mathbf{x}) = \mathsf{OWA}_{(f_1(\mathbf{x}), \cdots, f_n(\mathbf{x}))}(\mathbf{x})$$

It is not difficult to see that the above equation holds for all $n \in \mathbb{N}$ and $\mathbf{x} \in [0,1]^n$.

In the next subsection we discuss others properties of the function $\mathbf{H}$.

3.4 Properties of H

In this part of paper we will discuss about the properties of operator $\mathbf{H}$. It is easy to check that $\sum_{i=1}^{n} f_i(\mathbf{x}) = 1$ for any $\mathbf{x} \in [0,1]^n$ and therefore, by Propositions 3.14 and 3.16, $\mathbf{H}$ is an averaging and idempotent function. Furthermore,

Proposition 3.39. *The weight-functions at Definition 3.37 are invariant under translations and is also homogeneous of order 0.*

Proof. Let $\mathbf{x} = (x_1, ..., x_n) \in [0,1]^n$ and $\lambda \in [0,1]$ such that $\mathbf{x}' = (x_1 + \lambda, ..., x_n + \lambda) \in [0,1]^n$. Then, since $Med(\mathbf{x}') = Med(\mathbf{x}) + \lambda$ we have, for $\mathbf{x} \neq (x, ..., x)$:

$$\begin{aligned} f_i(\mathbf{x}') &= \frac{1}{n-1}\left(1 - \frac{|x_i + \lambda - Med(\mathbf{x}')|}{\sum_{j=1}^{n} |x_j + \lambda - Med(\mathbf{x}')|}\right) \\ &= \frac{1}{n-1}\left(1 - \frac{|x_i + \lambda - (Med(\mathbf{x}) + \lambda)|}{\sum_{j=1}^{n} |x_j + \lambda - (Med(\mathbf{x}) + \lambda)|}\right) \\ &= \frac{1}{n-1}\left(1 - \frac{|x_i - Med(\mathbf{x})|}{\sum_{j=1}^{n} |x_j - Med(\mathbf{x})|}\right) \\ &= f_i(\mathbf{x}). \end{aligned}$$

Therefore, $(f_1(\mathbf{x}'), ..., f_n(\mathbf{x}')) = (f_1(\mathbf{x}), ..., f_n(\mathbf{x}))$. The case in which $\mathbf{x} = (x, ..., x)$ is immediate.

To check the second property, make $\mathbf{x}'' = (\lambda x_1, ..., \lambda x_n)$, note that $Med(\mathbf{x}'') = \lambda Med(\mathbf{x})$ and for $\mathbf{x} \neq (x, ..., x)$

$$\begin{aligned}
f_i(\mathbf{x}'') &= \tfrac{1}{n-1}\left(1 - \frac{|\lambda x_i - Med(\lambda \mathbf{x})|}{\sum\limits_{j=1}^{n} |\lambda x_j - Med(\lambda \mathbf{x})|}\right) \\
&= \tfrac{1}{n-1}\left(1 - \frac{|\lambda x_i - \lambda Med(\mathbf{x})|}{\sum\limits_{j=1}^{n} |\lambda x_j - \lambda Med(\mathbf{x})|}\right) \\
&= \tfrac{1}{n-1}\left(1 - \frac{|\lambda| \cdot |x_i - Med(\mathbf{x})|}{|\lambda| \cdot \sum\limits_{j=1}^{n} |x_j - Med(\mathbf{x})|}\right) \\
&= \tfrac{1}{n-1}\left(1 - \frac{|x_i - Med(\mathbf{x})|}{\sum\limits_{j=1}^{n} |x_j - Med(\mathbf{x})|}\right) \\
&= f_i(\mathbf{x})
\end{aligned}$$

Hence, $(f_1(\mathbf{x}''), ..., f_n(\mathbf{x}'')) = (f_1(\mathbf{x}), ..., f_n(\mathbf{x})) = f(\mathbf{x})$. The case in which $\mathbf{x} = (x, ..., x)$ is also immediately. $\square$

Corollary 3.40. **H** *is shift-invariant and homogeneous.*

Proof. Straightforward for Propositions 3.19 and 3.21. $\square$

In addition to idempotency, homogeneity and shift-invariance **H** has the following proprerties.

Proposition 3.41. **H** *has no neutral element.*

Proof. Suppose **H** has a neutral element e, find the vector of weight for $\mathbf{x} = (e, ..., e, x, e, ..., e)$. Note that if $n \geq 3$, then $Med(\mathbf{x}) = e$ and therefore,

$$\begin{aligned}
f_i(\mathbf{x}) &= \tfrac{1}{n-1}\left(1 - \frac{|x_i - Med(\mathbf{x})|}{\sum\limits_{j=1}^{n} |x_j - Med(\mathbf{x})|}\right) \\
&= \tfrac{1}{n-1}\left(1 - \frac{|x_i - e|}{\sum\limits_{j=1}^{n} |x_j - e|}\right) \\
&= \tfrac{1}{n-1}\left(1 - \tfrac{|x_i - e|}{|x - e|}\right).
\end{aligned}$$

So,

$$f_i(\mathbf{x}) = \begin{cases} \frac{1}{n-1}, & \text{if } x_i = e \\ 0, & \text{if } x_i = x \end{cases}, \text{ to } n \geq 3$$

i.e.,

$$f(\mathbf{x}) = \left(\tfrac{1}{n-1}, ..., \tfrac{1}{n-1}, 0, \tfrac{1}{n-1}, ..., \tfrac{1}{n-1}\right)$$

and

$$\mathbf{H}(\mathbf{x}) = (n-1) \cdot \frac{e}{n-1} = e$$

But since e is a neutral element of $\mathbf{H}$, $\mathbf{H}(\mathbf{x}) = x$. Absurd, since we can always take $x \neq e$.

For $n = 2$, we have $Med(\mathbf{x}) = \frac{x+e}{2}$, where $\mathbf{x} = (x, e)$ or $\mathbf{x} = (e, x)$. In both cases it is not difficult to show that $f(\mathbf{x}) = (0.5, 0.5)$ and $\mathbf{H}(\mathbf{x}) = \frac{x+e}{2}$. Thus, taking $x \neq e$, again we have $\mathbf{H}(x, e) \neq x$. □

Proposition 3.42. $\mathbf{H}$ *has no absorbing elements.*

Proof. To $n = 2$, we have $\mathbf{H}(\mathbf{x}) = \frac{x_1+x_2}{2}$, which has no absorbing elements. Now for $n \geq 3$ we have to $\mathbf{x} = (a, 0, ..., 0)$ with $Med(\mathbf{x}) = 0$ therefore,

$$f_1(\mathbf{x}) = \frac{1}{n-1}\left(1 - \frac{a}{a}\right) = 0 \text{ and } f_i(\mathbf{x}) = \frac{1}{n-1}, \forall i = 2, ..., n.$$

therefore,

$$\mathbf{H}(a, 0, ..., 0) = 0 \cdot a + \frac{1}{n-1} \cdot 0 + ... + \frac{1}{n-1} \cdot 0 = a \Rightarrow a = 0,$$

but to $\mathbf{x} = (a, 1, ..., 1)$ we have to $Med(\mathbf{x}) = 1$. Furthermore,

$$f_1(\mathbf{x}) = \frac{1}{n-1}\left(1 - \frac{1-a}{1} - a\right) = 0$$

and

$$f_i(\mathbf{x}) = \frac{1}{n-1} \text{ for } i = 2, 3, ..., n.$$

therefore,

$$\mathbf{H}(a, 1, ..., 1) = 0 \cdot a + \frac{1}{n-1} \cdot 1 + ... + \frac{1}{n-1} \cdot 1 = a \Rightarrow a = 1.$$

With this we prove that $\mathbf{H}$ does note have annihiladors. □

Proposition 3.43. $\mathbf{H}$ *has no zero divisors.*

Proof. Let $a \in]0, 1[$ and consider $\mathbf{x} = (a, x_2, ..., x_n) \in]0, 1]^n$. In order to have $\mathbf{H}(\mathbf{x}) = \sum_{i=1}^{n} f_i(\mathbf{x}) \cdot x_i = 0$ we have $f_i(\mathbf{x}) \cdot x_i = 0$ for all $i = 1, 2, ..., n$. But as $a \neq 0$ and we can always take $x_2, x_3, ..., x_n$ also different from zero, then for each $i = 1, 2, ..., n$ there remains only the possibility of terms:

$$f_i(\mathbf{x}) = 0 \text{ for } i = 1, 2, ..., n.$$

This is absurd, for $f_i(\mathbf{x}) \in [0, 1]$ and $\sum_{i=1}^{n} f_i(\mathbf{x}) = 1$. like this, $\mathbf{H}$ has no zero divisors. □

Proposition 3.44. $\mathbf{H}$ *does not have one divisors*

Proof. Just to see that $a \in]0, 1[$, we have to $\mathbf{H}(a, 0, ..., 0) = f_1(\mathbf{x}).a \leq a < 1$. □

Proposition 3.45. $\mathbf{H}$ *is symmetric.*

Proof. Let $P : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ be a permutation. So we can easily see that

$$Med(x_{P(1)}, x_{P(2)}, ..., x_{P(n)}) = Med(x_1, x_2, ..., x_n)$$

for all $\mathbf{x} = (x_1, x_2, ..., x_n) \in [0, 1]^n$. We also have to $\sum_{i=1}^{n} |x_{P(i)} - Med(x_{P(1)}, x_{P(2)}, ..., x_{P(n)})| = \sum_{i=1}^{n} |x_i - Med(\mathbf{x})|$. Thus, it suffices to consider the case where $(x_{P(1)}, x_{P(2)}, ..., x_{P(n)}) \neq (x, x, ..., x)$. But $(x_{P(1)}, x_{P(2)}, ..., x_{P(n)}) \neq (x, x, ..., x)$ we have to:

$$\begin{aligned}
\mathbf{H}(x_{P(1)}, x_{P(2)}, ..., x_{P(n)}) &= \frac{1}{n-1} \sum_{i=1}^{n} \left(x_{P(i)} - \frac{x_{P(i)}|x_{P(i)} - Med(x_{P(1)}, ..., x_{P(n)})|}{\sum_{j=1}^{n} |x_{P(i)} - Med(x_{P(1)}, ..., x_{P(n)})|} \right) \\
&= \frac{\sum_{i=1}^{n} x_{P(i)}}{n-1} - \frac{1}{n-1} \cdot \sum_{i=1}^{n} \frac{x_{P(i)}|x_{P(i)} - Med(x_1, ..., x_n)|}{\sum_{j=1}^{n} |x_{P(i)} - Med(x_1, ..., x_n)|} \\
&= \frac{\sum_{i=1}^{n} x_i}{n-1} - \frac{1}{n-1} \cdot \sum_{i=1}^{n} \frac{x_{P(i)}|x_{P(i)} - Med(x_1, ..., x_n)|}{\sum_{j=1}^{n} |x_i - Med(x_1, ..., x_n)|} \\
&= \frac{\sum_{i=1}^{n} x_i}{n-1} - \frac{1}{n-1} \cdot \sum_{i=1}^{n} \frac{x_i|x_i - Med(x_1, ..., x_n)|}{\sum_{j=1}^{n} |x_i - Med(x_1, ..., x_n)|} \\
&= \mathbf{H}(x_1, ..., x_n).
\end{aligned}$$

□

Proposition 3.46. *If $N : [0, 1] \longrightarrow [0, 1]$ is the standard fuzzy negation, then $\mathbf{H}^N = \mathbf{H}$.*

Proof. If $\mathbf{x} = (x, \cdots, x)$, then

$$\mathbf{H}^N(\mathbf{x}) = 1 - \mathbf{H}(1 - x, 1 - x, \cdots, 1 - x) = 1 - (1 - x) = x = \mathbf{H}(\mathbf{x})$$

For $\mathbf{x} \neq (x, \cdots, x)$, we have:

$$\begin{aligned}
\mathbf{H}^N(\mathbf{x}) &= 1 - \frac{1}{n-1} \sum_{i=1}^{n} \left(1 - x_i - \frac{(1-x_i)|1 - x_i - Med(1 - x_1, \cdots, 1 - x_n)|}{\sum_{j=1}^{n} |1 - x_i - Med(1 - x_1, \cdots, 1 - x_n)|} \right) \\
&= 1 - \frac{1}{n-1} \sum_{i=1}^{n} \left(1 - x_i - \frac{(1-x_i)|1 - x_i - 1 + Med(x_1, \cdots, x_n)|}{\sum_{j=1}^{n} |1 - x_i - 1 + Med(x_1, \cdots, x_n)|} \right) \\
&= 1 - \frac{1}{n-1} \sum_{i=1}^{n} \left(1 - x_i - \frac{(1-x_i)|-x_i + Med(x_1, \cdots, x_n)|}{\sum_{j=1}^{n} |-x_i + Med(x_1, \cdots, x_n)|} \right)
\end{aligned}$$

$$
\begin{aligned}
&= 1-\frac{1}{n-1}\sum_{i=1}^{n}\left(1-x_i-\frac{(1-x_i)|x_i-Med(x_1,\cdots,x_n)|}{\sum_{j=1}^{n}|x_i-Med(x_1,\cdots,x_n)|}\right)\\
&= 1-\frac{1}{n-1}\left[n-\sum_{i=1}^{n}\left(x_i-\frac{x_i|x_i-Med(x_1,\cdots,x_n)|}{\sum_{j=1}^{n}|x_i-Med(x_1,\cdots,x_n)|}\right)-\sum_{i=1}^{n}\frac{|x_i-Med(x_1,\cdots,x_n)|}{\sum_{j=1}^{n}|x_i-Med(x_1,\cdots,x_n)|}\right]\\
&= 1-\frac{1}{n-1}\left[n-1-\sum_{i=1}^{n}\left(x_i-\frac{x_i|x_i-Med(x_1,\cdots,x_n)|}{\sum_{j=1}^{n}|x_i-Med(x_1,\cdots,x_n)|}\right)\right]\\
&= \frac{1}{n-1}\sum_{i=1}^{n}\left(x_i-\frac{x_i|x_i-Med(x_1,\cdots,x_n)|}{\sum_{j=1}^{n}|x_i-Med(x_1,\cdots,x_n)|}\right)\\
&= \mathbf{H}(\mathbf{x})
\end{aligned}
$$

□

Therefore, **H** satisfies the following properties:

- Idempotence
- Homogeneity
- Shift-invariance
- Symmetry.
- has no neutral element
- has no absorbing elements
- has no zero divisors
- does not have one divisors
- is self dual

Although we have not been able to demonstrate that **H** is an aggregation function, in the next proposition we show that **H** is $(k,\cdots,k)$-increasing (for $k>0$), so **H** is a preaggregation function.

Proposition 3.47. *If $k>0$, then* **H** *is $(k,\cdots,k)$-increasing.*

Proof. As **H** is shift-invariant, its follow of Proposition 3.30 that **H** is $(k,\cdots,k)$-increasing. □

Corollary 3.48. **H** *is a preaggregation function.*

The aggregation functions are very important for computing science, since in many applications the expected result is a single data, and therefore these applications use an aggregation function to convert this set of data into a unique output. In fact, a preaggregation can often be applied in place of aggregation. In this sense, we will apply the function **H** (which is a GM function) (in an illustrative example) to reduce images and then we compare the obtained results with the results obtained by some aggregations.

4 The Image Reduction by GM functions

In this part of our work we use the GM functions Min, Max, $Arith$, Med, cOWA and **H** to build image reduction operators and is an improvement of the done in [18]. But first, we will introduce some important concepts of image processing.

Definition 4.1. *An image is a matrix $m \times n$, $M = A(i,j)$, where each $A(i,j) \in [0,1]$ represents a pixel. More specifically, the value $A(i,j)$ is proportional to the light intensity at the considered point.*

In essence, a reduction operator reduces a given image $m \times n$ to another $m' \times n'$, such that $m' < m$ and $n' < n$. For example,

$$\begin{bmatrix} 0.1 & 0.2 & 0 & 0.5 \\ 0.3 & 0.3 & 0.2 & 0.8 \\ 1 & 0.5 & 0.6 & 0.4 \\ 0 & 0.3 & 0.5 & 0.7 \end{bmatrix} \longmapsto \begin{bmatrix} 0.1 & 0 \\ 1 & 0.6 \end{bmatrix}$$

There are several possible ways to reduce a given image, as shown in the following example:

Example 4.2. *The image*

$$M = \begin{bmatrix} 0.8 & 0.7 & 0.2 & 1 & 0.5 & 0.5 \\ 0.6 & 0.2 & 0.3 & 0.1 & 1 & 0 \\ 0 & 0 & 0.6 & 0.4 & 0.9 & 1 \\ 0.1 & 0.2 & 0.3 & 0.4 & 0.5 & 0.6 \end{bmatrix},$$

can be reduced to another 2×3 by partitioning M in blocks 2×2 and applying to each block, for example, the function $f(x,y,z,w) = Max(x,y,z,w)$. In this case, we obtain the image:

$$M_* = \begin{bmatrix} 0.8 & 1 & 1 \\ 0.2 & 0.6 & 1 \end{bmatrix}$$

The Figure 1 illustrates the reduction process of an image.

Figure 1: Example of image redction.

In fact, if we apply any other function, we get a new image, usually different from the previous one, but what is the best?

One possible answer to this question involves a method called **magnification** or **extension** (see [27, 62, 63]), which is a method which magnifies the reduced image to another with the same size of the original one. The magnified image is then compared with the original input image.

Example 4.3. *From M_* we can build a 4×6 image imply cloning each pixel (also known as nearest neighbor interpolation), as below:*

$$\left[\begin{array}{c} x \end{array}\right] \longmapsto \left[\begin{array}{cc} x & x \\ x & x \end{array}\right]$$

Thus, we obtain the following image:

$$M_1 = \left[\begin{array}{cccccc} 0.8 & 0.8 & 1 & 1 & 1 & 1 \\ 0.8 & 0.8 & 1 & 1 & 1 & 1 \\ 0.2 & 0.2 & 0.6 & 0.6 & 1 & 1 \\ 0.2 & 0.2 & 0.6 & 0.6 & 1 & 1 \end{array}\right]$$

This simple magnification method is also called of nearest neighbor interpolation. The Figure 2 illustrates the magnification process.

Figure 2: Example of magnification.

Given two different reductions of the same image (let's say M' and M^*), We compare the reductions following the steps: (1) Use a magnification method to magnify M' and M^* for the original size; (2) Compare each obtained image with the original one, using a some similarity measure.

There are several similarity measures, as for example, the measure PSNR (see [23]), that is calculated as follows:

$$PSNR(I,K) = 10 \cdot log_{10}\left(\frac{MAX_I^2}{MSE(I,K)}\right),$$

where $I = I(i,j)$ and $K = K(i,j)$ are two images, $MSE(I,K) = \frac{1}{nm}\sum_{i=1}^{m}\sum_{j=1}^{n}[I(i,j) - K(i,j)]^n$ and MAX_I is the maximum possible pixel value of pixel.

The degree of similarity between two images is proportional to the value of the PSNR, i.e., how much larger if the PSNR, more approximated are the analyzed images[6].

In what follows, we use the GM functions: Min, Mix, Med, $Arith$ cOWA and $\mathbf{H}$ to reduce images in grayscale[7], applying the following method:

[6]In particular, if the input image are equal, then the MSE value is zero and the PSNR will be infinity.

[7]The reduction of color images is similar.

Method 1

1. Reduce the input images using the Min, Max, $Arith$, Med, cOWA and $\mathbf{H}$;
2. Magnify the reduced images to the original size using the nearest neighbor interpolation;
3. Compare the last image with the original one using the measure $PSNR$.

Remark 4.4. *This is a general method which can be applied to any kind of image. In this work we applied it to 10 images in grayscale of size* 512×512[8] *(as shown in Figure 3).*

Figure 3: Imput images

In the Tables 1 and 2 (see Appendix) we present the PSNR values between the output images provided by Method 1 and original inputs.

According to PSNR, $Arith$ provided the higher quality images. However, the reduction operators generated by $\mathbf{H}$ and cOWA provide with us quite similar images to those given by $Arith$.

Note that although the magnification method by cloning of pixels is a simple and quick method (in running time) it brings us some limitations. The results obtained by this method are not good, in addition, the method itself causes that the $Arith$ operator is better than other operators, since by reducing a set of pixels x_1, x_2, x_3, x_4 to a single pixel y, and then compare $MSE = (x_1, y)^2 + (x_2 - y)^2 + (x_3 - y)^2 + (x_4 - y)^2$ (because each pixel y is repeated 4 times in the process of magnification), so of course $y = \frac{1}{4}(x_1 + x_2 + x_3 + x_4)$ has the lowest measurement error.

For this reason we also analyze two other methods of magnification: (1) Bilinear interpolation and (2) Bicubic interpolation (see [23, 28, 30, 55]). Thus, we have two other methods: Method 2 and Method 3, respectively

In Tables 3, 4, 5 and 6 (see Appendix), we present the results obtained with the use of these others magnification methods.

Tables 1, 2, 3, 4, 5 and 6 (see Appendix) show us that among the analyzed GM, the averaging functions ($Arith$, Med, cOWA and $\mathbf{H}$) are responsible for generating better quality images. However it is difficult to determine the most appropriate function to reduce images, since each particular function may be more suitable for a certain method of magnification, for example: $Arith$ is closer to magnifying by pixels cloning.

We can also observe that a more complex method of magnification, interpolation, are able to reconstruct images with higher quality. Obviously, the computational cost (running time) of these methods are also higher.

[8] In this paper we made two reductions: using 2×2 blocks and 4×4 blocks.

Ii is worth to emphasize that the reduction with H operator together with magnification by bicubic interpolation scored the highest quality among all analyzed methods (function together magnification) or both reduction: In scale as 2×2 and in scale 4×4.

This shows that in some applications, the use of a generating function of weights (i.e., a weight-function) in order to obtain a GM function may be more interesting than the use of a single weight vector.

This idea of replacing the weight vector by a weight function may also be used in others areas of computing, for example: In decision making and in artificial intelligence. These publications will be investigated in future work.

5 Final Remarks

In this paper we study two generalized forms of Ordered Weighted Averaging function and Mixture function, calls respectively of **Generalized Mixture** and **Bounded Generalized Mixture** functions. These functions are defined by weights, which are obtained dynamically from each input vector $\mathbf{x} \in [0,1]^n$. We demonstrated, among other results, that OWA and mixture functions are particular cases of GM and BGM functions, and thus we obtain that functions such as *Arithmetic Mean*, *Median*, *Maximum*, *Minimum* and cOWA are also examples of GM functions.

In the second part of this work, we present some properties as well as constructs and examples of GM functions. In particular we define a special GM function, called **H**, and show that **H** satisfies important properties for image applications: Idempotence, symmetry, homogeneity, shift-invariance, and moreover, it has no zero divisors and one divisors, and also does not have neutral elements. We further prove that **H** is a preaggregation function $(k, \cdots, k)$-increasing, and then we use GM functions ($Min, Max, Med, Arith$, cOWA and **H**) to verify the applicability of these functions, in this paper for image reduction.

To determine whether these functions are good reducers of images, we need a method of magnification. In Method 1, we magnify images by simply cloning the pixels. However this method brings some limitations, therefore also analyzes the other two magnification methods (bilinear and bicubic interpolation), giving rise to Methods 2 and 3. This other methods are more suitable, and we see that **H** is a fine function to perform this task, using Method 3.

Note that the generalized mixture functions can also be used in others fields of application, for example in data classification [13] and decision making [49]. In this paper, your focus is on just one of this possibility of applications. However, other applications will be investigated in future works.

Acknowledgements: This work is partially supported by the Brazilian National Council for Scientific and Technological Development CNPq under the Processes 311429/2020-3 and 312053/2018-5.

Conflict of Interest: The authors declare that there are no conflict of interest.

References

[1] M. Baczyński and B. Jayaram, Fuzzy Implications, Vol. 231 of Studies in Fuzziness and Soft Computing, *Springer, Berlin*, (2008). doi: 10.1007/978-3-540-69082-5.

[2] T. V. V. Batista, B. Bedregal and R. M. Moraes, Constructing multi-layer classifier ensembles using the Choquet integral based on overlap and quasi-overlap functions, *Neurocomputing*, 500 (2022), 413–421.

[3] G. Beliakov, A. Pradera and T. Calvo, Aggregation Functions: A Guide for Practitioners, Vol. 221 of Studies in Fuzziness and Soft Computing, *Springer, Berlin*, (2007).

[4] G. Beliakov, H. Bustince and D. Paternain, Image reduction using means on discrete product lattices, *IEEE Trans. Image Process.*, 21 (2012), 1070–1083.

[5] G. Beliakov, T. Calvo and T. Wilkin, On the weak monotonicity of Gini means and other mixture functions, *Inf. Sci.*, 300 (2015), 70–84.

[6] G. Beliakov, H. Bustince and T. Calvo, A Practical Guide to Averaging Functions, Vol. 329 of Studies in Fuzziness and Soft Computing, *Springer, Berlin*, (2016).

[7] G. Beliakov, G. Das, H. Q. Vu, T. Wilkin and Y. Xiang, Fuzzy connectives for efficient image reduction and speeding up image analysis, *IEEE Access*, 6 (2018), 68403–68414.

[8] H. Bustince, M. Galar, B. Bedregal, A. Kolesárová and R. Mesiar, A new approach to interval-valued Choquet integrals and the problem of ordering in interval-valued fuzzy set applications, *IEEE Trans. Fuzzy Syst.*, 21 (2013), 1150–1162.

[9] H. Bustince, J. Fernandez, A. Kolesárová and R. Mesiar, Directional monotonicity of fusion functions, *Eur. J. Oper. Res.*, 244 (2015), 300–308.

[10] H. Bustince, R. Mesiar, J. Fernández, M. Galar, D. Paternain, A. H. Altalhi, G. P. Dimuro, B. R. C. Bedregal and Z. Takác, d-Choquet integrals: Choquet integrals based on dissimilarities, *Fuzzy Sets Syst.*, 414 (2021), 1–27.

[11] H. Bustince, B. Bedregal, M.J. Campión, I.A. da Silva, J. Fernández, E. Induráin, A. Raventós-Pujol, R. H. N. Santiago, Aggregation of Individual Rankings Through Fusion Functions: Criticism and Optimality Analysis, *IEEE Trans. Fuzzy Syst.*, 30(3) (2022), 638–648.

[12] C.-H. Cheng and J.-R. Chang, MCDM aggregation model using situational ME-OWA and ME-OWGA operators, *Int. J. Uncertain. Fuzziness Knowl. Based Syst*, 14 (2006), 421–443.

[13] V. S. Costa, A. D. S. Farias, B. R. C. Bedregal, R. H. N. Santiago and A. M. de P. Canuto, Combining multiple algorithms in classifier ensembles using generalized mixture functions, *Neurocomputing*, 313 (2018), 402–414.

[14] S. C. Dighe and R. Shriram, Dental biometrics for human identification based on dental work and image properties in Periapical radiographs, *TENCON 2012 - 2012 IEEE Region 10 Conference*, (2012), 1–6. doi: 10.1109/TENCON.2012.6412216.

[15] G. P. Dimuro, J. Fernández, B. R. C. Bedregal, R. Mesiar, J. A. Sanz, G. Lucca and H. Bustince, The state-of-art of the generalizations of the Choquet integral: From aggregation and pre-aggregation to ordered directionally monotone functions, *Inf. Fusion*, 57 (2020), 27–43.

[16] D. Dubois and H. Prade, Fundamentals of Fuzzy Sets, Vol. 7 of The Handbooks of Fuzzy Sets, 1 ed., *Springer, New York*, (2000).

[17] D. Dubois and H. Prade, On the use of aggregation operations in information fusion processes, *Fuzzy Sets Syst.*, 142 (2004), 143–161.

[18] A. D. S. Farias, V. S. Costa, R. H. N. Santiago and B. R. C. Bedregal, The image reduction process based on generalized mixture functions, *2016 Annual Conference of the North American Fuzzy Information Processing Society (NAFIPS), 31 October 2016 – 04 November 2016, El Paso, TX, USA* , (2016), 1–6. doi: 10.1109/NAFIPS.2016.7851591.

[19] A. D. S. Farias, L. R. A. Lopes, B. C. Bedregal and R. H. N. Santiago, Closure properties for fuzzy recursively enumerable languages and fuzzy recursive languages, *J. Intell. Fuzzy Syst.*, 31 (2016), 1795–1806.

[20] A. D. S. Farias, R. H. N. Santiago and B. R. C. Bedregal, Some properties of generalized mixture functions, 2016 IEEE International Conference on Fuzzy Systems, FUZZ-IEEE, July 24-29, 2016, Vancouver, BC, Canada, (2016), 288–293. doi: 10.1109/FUZZ-IEEE.2016.7737699.

[21] A. D. S. Farias, C. Callejas, R. H. N. Santiago and B. R. C. Bedregal, Directional and ordered directional monotonicity of generalized and bounded generalized mixture functions, *2018 IEEE International Conference on Fuzzy Systems, FUZZ-IEEE, July 8-13, 2018, Rio de Janeiro, Brazil*, (2018), 1–7. doi: 10.1109/FUZZ-IEEE.2018.8491541.

[22] M. Grabisch, J. L. Marichal, R. Mesiar and E. Pap, Aggregation Functions, Vol. 127 of Encyclopedia of Mathematics and its Applications, *University Press Cambridge, Cambridge*, (2009).

[23] R. C. Gonzales and R. E. Woods, Digital Image Processing, 3rd ed., *Pearson, New Jersey*, (2008).

[24] E. Hancer, B. Xue, M. Zhang, D. Karaboga and B. Akay, A multi-objective artificial bee colony approach to feature selection using fuzzy mutual information, *2015 IEEE Congress on Evolutionary Computation (CEC)*, (2015), 2420–2427. doi: 10.1109/CEC.2015.7257185.

[25] G. Jaffino, A. Banumathi, U. Gurunathan, J. P. Jose, Dental work extraction for different radiographic images in human forensic identification, *2014 International Conference on Communication and Network Technologies (ICCNT)*, (2014), 52–56. doi: 10.1109/CNT.2014.7062724.

[26] R. P. Joseph, C. S. Singh and M. Manikandan, Brain tumor MRI image segmentation and detection in image processing, *Int. J. Res. Technol.*, 3 (2014), 1–5.

[27] A. Jurio, M. Pagola, R. Mesiar, G. Beliakov and H. Bustince, Image magnification using interval information, *IEEE Trans. Image Process.*, 20 (2011), 3112–3123.

[28] R. Keys, Cubic convolution interpolation for digital image processing, *IEEE Trans. Acoust. Speech Signal Process.*, 29 (1981).

[29] J. H. Kim, I. J. Ahn, W. H. Nam and J. B. Ra, An effective post-filtering framework for 3-D PET image denoising based on noise and sensitivity characteristics, *IEEE Trans. Nucl. Sci.*, 62 (2015), 137–147.

[30] T. M. Lehmann, C. Gonner and K. Spitzer, Survey: interpolation methods in medical image processing, *IEEE Trans. Medical Imaging*, 18 (1999), 1049–1075.

[31] X. Liang and W. Xu, Aggregation method for motor drive systems, *Electric Power Syst. Research*, 117 (2014), 27–35.

[32] L. Lingling, Z. Xian, H. Pengju and L. Zhigang, The research on the method of fuzzy information processing, *2012 3rd International Conference on System Science, Engineering Design and Manufacturing Informatization (ICSEM)*, 2 (2012), 47–50. doi: 10.1109/ICSSEM.2012.6340804.

[33] I. Lizasoain and C. Moreno, OWA operators defined on complete lattices, *Fuzzy Sets Syst.*, 224 (2013), 36 – 52.

[34] G. Lucca, J. A. Sanz, G. P. Dimuro, B. Bedregal, R. Mesiar, A. Kolesárová and H. Bustince, Preaggregation functions: Construction and an application, *IEEE Trans. Fuzzy Syst.*, 24 (2016), 260–272.

[35] G. Lucca, J. A. Sanz, G. P. Dimuro, B. R. C. Bedregal and H. Bustince, A proposal for tuning the α parameter in C_αC-integrals for application in fuzzy rule-based classification systems, *Nat. Comput.*, 19 (2020), 533–546.

[36] N. V. Manokar, V. Manokar, Rinesh, K. P. Sridhar and L. M. Patnaik, Wavelets based decomposition and classification of diseased fMRI brain images for inter racial disease types of Alzheimer's vs tumors using SOFM and enhancement by LVQ neural networks, *2012 2nd IEEE International Conference on Parallel Distributed and Grid Computing (PDGC)*, (2012), 822–827. doi: 10.1109/PDGC.2012.6449929.

[37] J. M. Merigó and A. M. Gil-Lafuente, The induced generalized OWA operator, *Inf. Sci.*, 179 (2009), 729–741.

[38] R. Mesiar and J. Špirková, Weighted means and weighting functions, *Kybernetika*, 42 (2006), 151–160.

[39] R. Mesiar, J. Špirková and L. Vavríková, Weighted aggregation operators based on minimization, *Inf. Sci.*, 178 (2008), 1133–1140.

[40] J. Mihailović, A. Savić, J. Bogdanović-Pristov and K. Radotić, MRI brain tumors images by using independent component analysis, *2011 IEEE 9th International Symposium on Intelligent Systems and Informatics*, (2011), 433–435. doi: 10.1109/SISY.2011.6034366.

[41] T. Milfont, I. Mezzomo, B. Bedregal, E. Mansilla and H. Bustince, Aggregation functions on n-dimensional ordered vectors equipped with an admissible order and an application in multi-criteria group decision-making, *Int. J. Approx. Reason.*, 137 (2021), 34–50.

[42] N. Mustafa, S. A. Khan, J. P. Li, M. Khalil, K. Kumar and M. Giess, Medical image de-noising schemes using wavelet transform with fixed form thresholding, *2014 11th International Computer Conference on Wavelet Active Media Technology and Information Processing (ICCWAMTIP)*, (2014), 397–402. doi: 10.1109/ICCWAMTIP.2014.7073435.

[43] D. Nolasco, F. Costa, E. Palmeira, D. Alves, B. Bedregal, T. Rocha, R. Ribeiro and J. Silva, Wavelet-fuzzy power quality diagnosis system with inference method based on overlap functions: Case study in an AC microgrid, *Eng. Appl. Artif. Intell.*, 85 (2019), 284–294.

[44] D. Paternain, A. Jurio, E. Barrenechea, H. Bustince, B. Bedregal and E. Szmidt, An alternative to fuzzy methods in decision-making problems, *Expert Syst. Appl.*, 39 (2012), 7729–7735.

[45] D. Paternain, J. Fernández, H. Bustince, R. Mesiar and G. Beliakov, Construction of image reduction operators using averaging aggregation functions, *Fuzzy Sets Syst.*, 261 (2015), 87–111.

[46] R. A. M. Pereira and G. Pasi, On non-monotonic aggregation: mixture operators, *Proc. 4th Meeting of the EURO Working Group on Fuzzy Sets (EUROFUSE'99) and 2nd Internat. Conf. on Soft and Intelingent Computing (SIC'99), Budapest, Hungary*, (1999).

[47] R. A. M. Pereira, The orness of mixture operators: the exponential case, *Proc. 8th Internat. Conf. on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU'2000), Madrid, Spain*, (2000).

[48] R. A. M. Pereira and R. A. Ribeiro, Aggregation with generalized mixture operators using weighting functions, *Fuzzy Sets Syst.*, 137 (2003), 43–58.

[49] R. A. Ribeiro and R. A. M. Pereira, Generalized mixture operators using weighting functions: A comparative study with WA and OWA, *Eur. J. Oper. Res.*, 145 (2003), 329–342.

[50] S. Sahu, N. Loya and A. G. Keskar, Restoration and enhancement of impulse noise image for human visual system, *2015 IEEE International Conference on Electronics, Computing and Communication Technologies (CONECCT)*, (2015), 1–6. doi: 10.1109/CONECCT.2015.7383913.

[51] B. Z. Shaick and L. Yaroslavsky, Image reduction for object recognition, *4th EURASIP-IEEE Region 8 International Symposium on Video/Image Processing and Multimedia Communications VIPromCom*, (2002), 333–338. doi: 10.1109/VIPROM.2002.1026678.

[52] A. J. Solanki, K. R. Jain and N. P. Desai, ISEF based identification of RCT/filling in dental caries of decayed tooth, *Int. J. Image Process.*, 7 (2013), 149–162.

[53] Z.-X. Su, G.-P. Xia, M.-Y. Chen and L. Wang, Induced generalized intuitionistic fuzzy OWA operator for multi-attribute group decision making, *Expert Syst. Appl.*, 39 (2012), 1902–1910.

[54] Suprijanto, Gianto, E. Juliastuti, Azhari and L. Epsilawati, Image contrast enhancement for film-based dental panoramic radiography, *2012 International Conference on System Engineering and Technology (ICSET)*, (2012), 1–5. doi:10.1109/ICSEngT.2012.6339321.

[55] P. Thevenaz, T. Blu and M. Unser, Interpolation revisited, *IEEE Trans. Medical Imaging*, 19 (2000), 739–758.

[56] T. Wilkin, Image reduction operators based on non-monotonic averaging functions, *2013 IEEE Int. Conf. on Fuzzy Systems, (FUZZ)*, (2013), 1–8. doi: 10.1109/FUZZ-IEEE.2013.6622458.

[57] T. Wilkin and G. Beliakov, Weakly monotonic averaging functions, *Int. J. Intell. Syst.*, 30(2) (2015), 144–169.

[58] S. K. Woo, K. M. Kim, T. S. Lee, J. H. Jung, J. G. Kim, J. S. Kim, T. H. Choi, G. I. An and G. J. Cheon, Registration method for the detection of tumors in lung and liver using multimodal small animal imaging, *IEEE Trans. Nucl. Sci.*, 56 (2009), 1454–1458.

[59] B. Y. Wu and X. Q. Sheng, A complex image reduction technique using genetic algorithm for the MoM solution of half-space MPIE, *IEEE Trans. Antennas Propag.*, 63 (2015), 3727–3731.

[60] R. R. Yager, On ordered weighted averaging aggregation operators in multicriteria decisionmaking, *IEEE Trans. Syst. Man Cybern.*, 18 (1988), 183–190.

[61] R. R. Yager, Centered OWA operators, *Soft Comput.*, 11 (2006), 631–639.

[62] J. Yang, J. Wright, T. Huang and Y. Ma, Image super-resolution as sparse representation of raw image patches, IEEE Conference on Computer Vision and Pattern Recognition, CVPR 2008, (2008), 1–8. doi: 10.1109/CVPR.2008.4587647.

[63] J. Yang, J. Wright, T. S. Huang and Y. Ma, Image super-resolution via sparse representation, *IEEE Trans. Image Process.*, 19 (2010), 2861–2873.

[64] S.-M. Zhou, F. Chiclana, R. I. John and J. M. Garibaldi, Type-1 OWA operators for aggregating uncertain information with uncertain weights induced by type-2 linguistic quantifiers, *Fuzzy Sets Syst.*, 159 (2008), 3281–3296.

Antonio Diego Silva Farias
Department of Exact and Natural Sciences

Federal Rural University of Semi-Arid–UFERSA
Pau dos Ferros-RN, Brazil
E-mail: antonio.diego@ufersa.edu.br

Valdigleis da Silva Costa
Collegiate of Computer Science
Federal University of Vale do São Francisco–UNIVASF
Salgueiro-PE, Brazil
E-mail: valdigleis.costa@univasf.edu.br

Luiz Ranyer A. Lopes
Federal Institute of Rio Grande Norte–IFRN
Natal-RN, Brazil
E-mail: ranyer.lopes@gmail.com

Regivan Hugo Nunes Santiago
Department of Informatics and Applied Mathematics
Federal University of Rio Grande do Norte–UFRN
Natal-RN, Brazil
E-mail: regivan@dimap.ufrn.br

Benjamín Bedregal
Department of Informatics and Applied Mathematics
Federal University of Rio Grande do Norte–UFRN
Natal-RN, Brazil
E-mail: bedregal@dimap.ufrn.br

6 Appendix

Table 1: $PSNR$ values of reconstruction of imagens of Figure 3 by nearest neighbor interpolation. The underline value represents the second high quality image

USING 2×2 BLOCKS

	Min	Max	Med	$Arith$	cOWA	**H**
Img 01	26,68848	26,60371	30,66996	**30,89667**	30,73823	$\underline{30,75448}$
Img 02	33,50403	33,46846	37,51525	**37,64240**	37,57713	$\underline{37,58138}$
Img 03	26,80034	26,74460	30,47904	**30,55504**	$\underline{30,52128}$	30,51564
Img 04	28,90415	28,83284	32,88120	**33,01225**	$\underline{32,94828}$	32,94146
Img 05	25,04896	25,04438	28,75582	**28,85475**	$\underline{28,81506}$	28,79901
Img 06	38,10156	38,07248	42,08612	**42,13003**	$\underline{42,12316}$	42,11653
Img 07	24,48520	24,38872	28,31229	**28,45667**	28,35114	$\underline{28,37668}$
Img 08	23,69576	23,73464	27,41557	**27,51579**	$\underline{27,46383}$	27,45864
Img 09	26,19262	26,09448	30,06427	**30,22940**	30,11893	$\underline{30,13332}$
Img 10	21,48459	21,41350	25,37475	**25,58054**	25,43016	$\underline{25,45073}$
Avg	27,49057	27,43978	31,35543	**31,48735**	31,40872	$\underline{31,41279}$

Table 2: $PSNR$ values of reconstruction of imagens of Figure 1 by nearest neighbor interpolation. The underline value represents the second high quality image

USING 4×4 BLOCKS

	Min	Max	Med	$Arith$	cOWA	**H**
Img 01	21,37117	20,83960	26,73708	**27,07854**	27,01270	$\underline{27,07067}$
Img 02	19,70858	19,54290	23,92198	**24,07786**	24,05762	$\underline{24,07478}$
Img 03	20,46198	20,82576	25,64113	**26,16092**	26,08186	$\underline{26,14607}$
Img 04	22,59335	22,24354	27,94347	**28,26449**	28,19574	$\underline{28,25700}$
Img 05	18,86628	19,55278	24,12507	**24,68962**	24,58713	$\underline{24,67322}$
Img 06	29,48308	29,26559	34,89670	**35,11481**	35,09436	$\underline{35,11023}$
Img 07	18,95771	18,72670	24,18918	**24,55073**	24,48373	$\underline{24,54269}$
Img 08	17,71071	18,59348	23,11305	**23,54332**	23,43522	$\underline{23,53119}$
Img 09	20,97846	20,44416	26,23824	**26,53197**	26,42064	$\underline{26,52562}$
Img 10	16,47636	16,22205	21,89755	**22,22614**	22,10356	$\underline{22,21825}$
Avg	20,66077	20,62565	25,87034	**26,22384**	26,14726	$\underline{26,21497}$

Table 3: $PSNR$ values of reconstruction of imagens of Figure 3 by bilinear interpolation. The underline value represents the second high quality image

USING 2×2 BLOCKS

	Min	Max	Med	$Arith$	cOWA	**H**
Img 01	27,25658	27,41249	<u>31, 70137</u>	31,66148	31,64818	**31,70944**
Img 02	29,07393	29,09065	29,98667	**30,00618**	<u>29, 99790</u>	29,99295
Img 03	28,07377	27,53953	**31,96271**	31,87901	31,87085	<u>31, 94673</u>
Img 04	29,70934	29,78913	**34,39128**	34,28215	34,31414	<u>34, 37504</u>
Img 05	26,30684	25,74955	**30,17965**	30,08193	30,05530	<u>30, 16533</u>
Img 06	40,09734	39,94107	**48,99047**	48,55730	48,52986	<u>48, 86710</u>
Img 07	25,10689	25,04408	<u>28, 93328</u>	28,92340	28,89276	**28,94254**
Img 08	24,63619	24,10410	<u>28, 19100</u>	28,17758	28,16818	**28,19312**
Img 09	26,60297	26,71398	30,54028	**30,56126**	30,52693	<u>30, 55733</u>
Img 10	21,93973	21,90280	25,71329	**25,74295**	25,69402	<u>25, 73353</u>
Avg	27,88036	27,72874	**32,05900**	31,98732	31,96981	<u>32, 04831</u>

Table 4: $PSNR$ values of reconstruction of imagens of Figure 3 by bilinear interpolation. The underline value represents the second high quality image

USING 4×4 BLOCKS

	Min	Max	Med	$Arith$	cOWA	**H**
Img 01	21,84394	21,46624	<u>28, 12885</u>	28,03911	**28,13262**	28,08806
Img 02	20,22210	19,99324	24,09349	24,09114	<u>24, 09696</u>	**24,10058**
Img 03	21,36383	21,65788	27,34577	27,53279	**27,57114**	<u>27, 56163</u>
Img 04	23,23057	22,96007	**29,81717**	29,65596	<u>29, 77096</u>	29,71475
Img 05	19,54307	20,06159	25,32192	25,47922	<u>25, 51400</u>	**25,51442**
Img 06	30,92215	30,60188	**42,72668**	41,77064	<u>41, 99358</u>	41,97442
Img 07	19,43662	19,19604	24,96897	25,00413	**25,05911**	<u>25, 02899</u>
Img 08	18,28578	18,86696	23,87169	<u>24, 09781</u>	24,07356	**24,10310**
Img 09	21,32747	20,91360	27,09762	27,10526	**27,16280**	<u>27, 13073</u>
Img 10	16,77848	16,57833	22,58040	22,61488	<u>22, 63949</u>	**22,63987**
Avg	21,29540	21,22958	<u>27, 59525</u>	27,53909	**27,60142**	27,58566

Table 5: $PSNR$ values of reconstruction of imagens of Figure 3 by bicubic interpolation. The underline value represents the second high quality image

USING 2×2 BLOCKS

	Min	Max	Med	$Arith$	cOWA	**H**
Img 01	27,39667	27,45993	32,53367	**32,62657**	32,52946	<u>32,58602</u>
Img 02	30,06149	30,00816	31,28820	**31,31873**	<u>31,30611</u>	31,29877
Img 03	28,09952	27,62931	<u>32,92967</u>	32,90897	32,87767	**32,93859**
Img 04	29,92114	29,94430	<u>35,70586</u>	35,70361	35,68906	**35,73313**
Img 05	26,38597	25,93655	<u>31,32017</u>	31,30790	31,25508	**31,33640**
Img 06	40,05229	40,02173	**51,35284**	51,07478	51,01447	<u>51,31081</u>
Img 07	25,23188	25,16984	29,85564	**29,93609**	29,85733	<u>29,89915</u>
Img 08	24,72669	24,32047	29,10402	**29,15066**	29,11737	<u>29,12822</u>
Img 09	26,73252	26,79140	31,27454	**31,38274**	31,29368	<u>31,32452</u>
Img 10	22,04218	21,98136	26,39147	**26,52171**	26,41585	<u>26,44659</u>
Avg	28,06504	27,92630	33,17561	<u>33,19318</u>	33,13561	**33,20022**

Table 6: $PSNR$ values of reconstruction of imagens of Figure 3 by bicubic interpolation. The underline value represents the second high quality image

USING 4×4 BLOCKS

	Min	Max	Med	$Arith$	cOWA	**H**
Img 01	21,83423	21,39364	28,64265	28,74908	**28,80893**	<u>28,78768</u>
Img 02	20,20038	19,88701	24,49596	<u>24,56989</u>	24,56761	**24,57359**
Img 03	21,25132	21,55589	27,82091	<u>28,31402</u>	28,28961	**28,32229**
Img 04	23,22310	22,89860	30,47704	30,54773	**30,60332**	<u>30,59348</u>
Img 05	19,45423	20,06391	25,74518	<u>26,18606</u>	26,15139	**26,20092**
Img 06	30,81953	30,48357	**44,31891**	43,83439	44,03526	<u>44,05492</u>
Img 07	19,36949	19,11221	25,29211	25,49221	<u>25,49999</u>	**25,50641**
Img 08	18,21007	18,91559	24,17857	**24,57330**	24,49174	<u>24,56575</u>
Img 09	21,32252	20,85345	27,41366	<u>27,56839</u>	27,55860	**27,58354**
Img 10	16,76501	16,53815	22,82004	<u>23,00025</u>	22,96201	**23,01459**
Avg	21,24499	21,17020	28,12050	28,28353	<u>28,29685</u>	**28,32032**

Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://tfss.journals.iau.ir/
Online ISSN: 2821-0131
Vol.1, No.2, (2022), 129-141
DOI: http://doi.org/10.30495/tfss.2022.1956811.1023

Article Type: Original Research Article

Fuzzy (Soft) Quasi-Interior Ideals of Semirings

Arsham Borumand Saeid*, **Marapureddy Murali Krishna Rao**,
Rajendra Kumar Kona, **Noorbhasha Rafi**

Abstract. In this paper, as a further generalization of fuzzy ideals, we introduce the notion of a fuzzy (soft) quasi-interior ideals of semirings and characterize regular semiring in terms of fuzzy (soft) quasi-interior ideals of semirings. We prove that (μ, A) is a fuzzy soft left quasi-interior ideal over a regular semiring M, if and only if (μ, A) is a fuzzy soft quasi-ideal over a semiring M, and study some of the properties.

AMS Subject Classification 2020:16Y60; 16Y99; 03E72
Keywords and Phrases: Semiring, Regular semiring, Quasi-interior ideal, Fuzzy quasi-interior ideal, Fuzzy soft quasi-interior ideal.

1 Introduction

The notion of ideals introduced by Dedekind for the theory of algebraic numbers was generalized by E. Noether for associative rings. The one and two-sided ideals presented by her are still central concepts in ring theory. We know that the notion of a one-sided ideal of any algebraic structure is a generalization of the notion of an ideal. The quasi ideals are the generalization of left and right ideals, whereas the bi-ideals are the generalization of quasi ideals. The notion of bi-ideals in semigroups was introduced by Lajos [8]. Iseki introduced the concept of quasi ideal for semiring [4, 5, 6]. M. Henriksen studied ideals in semirings [3]. As a further generalization of ideals, Steinfeld first introduced the notion of quasi ideals for semigroups and then for rings. We know that the notion of the bi-ideal in semirings is a special case of the (m, n) ideal introduced by S. Lajos. The concept of bi-ideals was first introduced by R. A. Good and D. R. Hughes for a semigroup[2]. Lajos and Szasz introduced the concept of bi-ideals for rings[9].

Many real-world problems are complicated due to various uncertainties. In addressing them, classical methods may not be the best option. To overcome such, several theories like randomness, rough set, and fuzzy set were introduced. L. A. Zadeh developed the fuzzy set theory in 1965 [18]. Many papers on fuzzy sets appeared, showing the importance of the concept and its applications to logic, set theory, group theory, ring theory, real analysis, topology, measure theory etc. N. Kuroki studied fuzzy interior ideals in semigroups [7].

Molodtsov introduced the concept of soft set theory as a new mathematical tool for dealing with uncertainties, only partially resolving the problem because objects in a universal set often do not precisely satisfy the parameters associated to each of the elements in the set[11]. Soft set theory has wide applications in fields like game theory, operations research, data analysis, decision making, probability theory, and measurement

*__Corresponding author:__ Arsham Borumand Saeid, Email: arsham@uk.ac.ir, ORCID: 0000-0001-9495-6027
Received: 16 April 2022; Revised: 6 July 2022; Accepted: 7 August 2022; Published Online: 7 November 2022.

How to cite: A. Borumand Saeid, M. Murali Krishna Rao, K. Rajendra Kumar and N. Rafi, Fuzzy (soft) quasi-interior ideals of semirings, *Trans. Fuzzy Sets Syst.*, 1(2) (2022), 129-141.

theory. Acar et. al. gave the basic concept of soft rings. Feng et al. studied soft semirings using the soft set theory[1]. Then Maji et. al.[10] extended soft set theory to fuzzy soft set theory. Fuzzy soft set theory is a unification of fuzzy set theory and soft set theory. Aktas and Cagman defined the soft set and soft groups. Fuzzy soft set theory has wide applications in medical diagnosis.

M. Murali Krishna Rao introduced the notion of (quasi-interior, bi-interior, bi-quasi, tri, and tri-quasi interior) ideals as a generalization of (quasi, bi and interior) ideals of a semiring, semigroup, $\Gamma-$semiring, $\Gamma-$semigroup and studied their properties[12, 13, 14, 15, 17]. M. Murali Krishna Rao studied fuzzy bi-interior ideals of $\Gamma-$semiring [16].

This paper aims to introduce the notion of fuzzy quasi-interior ideal and fuzzy soft quasi-interior ideal of a semiring. We prove that every fuzzy soft left quasi-interior ideal over a regular semiring if and only if it is a fuzzy soft quasi ideal over a semiring. Regular semiring is characterized in terms of fuzzy(soft) quasi-interior ideals of a semiring. We study, M is regular if and only if $\mu_a = \chi_M \circ \mu_a \circ \chi_M \circ \mu_a, a \in A$, for any fuzzy left quasi-interior ideals of fuzzy soft quasi-interior ideals (μ, A) over a semiring M.

2 Preliminaries

In this section, we recall some of the fundamental concepts and definitions which are necessary for this paper.

Definition 2.1. [13] A set M together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$ respectively) will be called semiring provided

(i) addition is a commutative operation.

(ii) multiplication distributes over addition both from the left and from the right.

(iii) there exists $0 \in M$ such that $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in M$.

Example 2.2. Let M be the set of all natural numbers. Then $(M,\ max,\ min)$ is a semiring.

Definition 2.3. [13] A non-empty subset A of a semiring M is called:

(i) a *subsemiring* of M, if $(A, +)$ is a subsemigroup of $(M, +)$ and $AA \subseteq A$,

(ii) a *quasi ideal* of M, if A is a subsemiring of M and $AM \cap MA \subseteq A$,

(iii) a *bi-ideal* of M, if A is a subsemiring of M and $AMA \subseteq A$,

(iv) an *interior ideal* of M, if A is a subsemiring of M and $MAM \subseteq A$,

(v) a $left$ $(right)$ *ideal* of M, if A is a subsemiring of M and $MA \subseteq A(AM \subseteq A)$,

(vi) an *ideal*, if A is a subsemiring of $M, AM \subseteq A$ and $MA \subseteq A$,

(vii) a $left(right)$ *bi-quasi ideal* of M, if A is a subsemiring of M and $MA \cap AMA(AM \cap AMA) \subseteq A$,

(ix) a *bi- quasi ideal* of M, if A is a left bi- quasi ideal and a right bi- quasi ideal of M.

(x) a left(right) quasi-interior ideal of M, if A is a subsemiring of M and $MAMA(AMAM) \subseteq A$.

Definition 2.4. [13] An element a of a semiring M is called a regular element if there exists an element b of M such that $a = aba$.

Definition 2.5. [13] A semiring M is called a regular semiring if every element of M is a regular element.

Definition 2.6. [16] Let A be a non-empty subset of M. The *characteristic function* of A is a fuzzy subset of M, defined by

$$\chi_{_A}(x) = \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{if } x \notin A. \end{cases}$$

Definition 2.7. [16] A function $f : R \to M$, where R and M are semirings. Then f is called a semiring homomorphism, if $f(a + b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$ for all $a, b \in R$.

Definition 2.8. [16] Let U be an initial Universe set and E be the set of parameters. Let $P(U)$ denotes the power set of U. A pair (μ, E) is called a soft set over U where μ is a mapping given by $\mu : E \to P(U)$.

Definition 2.9. [16] Let U be an initial Universe set and E be the set of parameters, $A \subseteq E$. A pair (μ, A) is called fuzzy soft set over U where μ is a mapping given by $\mu : A \to I^U$ where I^U denotes the collection of all fuzzy subsets of U. $\mu(a)$, $a \in A$, be a fuzzy subset and is denoted by μ_a.

Definition 2.10. [16] Let $(\mu, A), (\lambda, B)$ be fuzzy soft sets over U then (μ, A) is said to be a fuzzy soft subset of (λ, B), denoted by $(\mu, A) \subseteq (\lambda, B)$ if $A \subseteq B$ and $\mu_a \subseteq \lambda_a$ (μ_a, λ_a are fuzzy subsets) for all $a \in A$.

Definition 2.11. [16] Let (μ, A), (λ, B) be fuzzy soft sets. The *intersection* of (μ, A) and (λ, B), denoted by $(\mu, A) \cap (\lambda, B) = (\gamma, C)$, where $C = A \cup B$, is defined as:

$$\gamma_c = \begin{cases} \mu_c, & \text{if } c \in A \setminus B; \\ \lambda_c, & \text{if } c \in B \setminus A; \\ \mu_c \cap \lambda_c, & \text{if } c \in A \cap B. \end{cases}$$

Definition 2.12. [16] Let (μ, A), (λ, B) be fuzzy soft sets. The *union* of (μ, A) and (λ, B), denoted by $(\mu, A) \cup (\lambda, B) = (\gamma, C)$, where $C = A \cup B$, is defined as:

$$\gamma_c = \begin{cases} \mu_c, & \text{if } c \in A \setminus B; \\ \lambda_c, & \text{if } c \in B \setminus A; \\ \mu_c \cup \lambda_c, & \text{if } c \in A \cap B. \end{cases}$$

Definition 2.13. [16] Let M be a semiring, E be a parameter set and $A \subseteq E$. Let $\mu : A \to [0, 1]^M$ be a mapping, where $[0, 1]^M$ denotes the collection of all fuzzy subsets of M. Then (μ, A) is called a *fuzzy soft left* (*right*) *ideal over* M, if for each $a \in A$, the corresponding fuzzy subset $\mu_a : M \to [0, 1]$ is a fuzzy left(right) ideal of M, i.e., for all $x, y \in M$,

(i) $\mu_a(x + y) \geq \min\{\mu_a(x), \mu_a(y)\}$, (ii) $\mu_a(xy) \geq \mu_a(y)(\mu_a(x))$.

(μ, A) is called a *fuzzy soft ideal* over M, if

(i) $\mu_a(x + y) \geq \min\{\mu_a(x), \mu_a(y)\}$, (ii) $\mu_a(xy) \geq \max\{\mu_a(x), \mu_a(y)\}$.

Definition 2.14. [16] Let M be a semiring, E be a parameter set, let $A \subseteq E$ and let $\mu : A \to [0, 1]^M$ be a mapping. Then (μ, A) is called a *fuzzy soft quasi ideal* over M, if for each $a \in A$, the corresponding fuzzy subset $\mu_a : M \to [0, 1]$ is a fuzzy quasi ideal of M, i.e. for all $x, y \in M$,

(i) $\mu_a(x + y) \geq \min(\mu_a(x), \mu_a(y))$, (ii) $\mu_a \circ \chi_M \wedge \chi_M \circ \mu_a \subseteq \mu_a$.

(μ, A) is called a *fuzzy soft interior ideal* over M, if for each $a \in A$, the corresponding fuzzy subset $\mu_a : M \to [0, 1]$ is a fuzzy interior ideal of M, i.e., for all $x, y \in M$, (i) $\mu_a(x + y) \geq \min\{\mu_a(x), \mu_a(y)\}$, (ii) $\mu_a(xyz) \geq \max\{\mu_a(y)\}$.

3 Fuzzy quasi interior ideals

In this section, we introduce the notion of fuzzy (right, left) quasi interior ideal and study the properties of fuzzy (right, left) quasi interior ideals of semirings.

Definition 3.1. A fuzzy subset μ of a semiring M, is called a fuzzy left (right) quasi interior ideal if μ satisfies the following conditions

(i) $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in M$.

(ii) $\chi_M \circ \mu \circ \chi_M \circ \mu \subseteq \mu\big(\mu \circ \chi_M \circ \mu \circ \chi_M \subseteq \mu\big)$.

A fuzzy subset μ of a semiring M, is called a fuzzy quasi interior ideal if it is both left fuzzy quasi interior ideal and right fuzzy quasi interior ideal of M.

Theorem 3.2. *Let I be a non-empty subset of a semiring M and χ_I be the characteristic function of I. Then I is a right quasi interior ideal of a semiring M if and only if χ_I is a fuzzy right quasi interior ideal of a semiring M.*

Proof. Let I be a non-empty subset of the semiring M and χ_I be the characteristic function of I. Suppose I is a right quasi interior ideal of the semiring M. Obviously, χ_I is a fuzzy subsemiring of M. We have $IMIM \subseteq I$. Then $\chi_I \circ \chi_M \circ \chi_I \circ \chi_M = \chi_{IMIM} \subseteq \chi_I$. Therefore χ_I is a fuzzy right quasi interior ideal of the semiring M.
Conversely, suppose that χ_I is a fuzzy right quasi interior ideal of M. Then I is a subsemiring of M. We have $\chi_I \circ \chi_M \circ \chi_I \circ \chi_M \subseteq \chi_I$, implies that $\chi_{IMIM} \subseteq \chi_I$. Therefore $IMIM \subseteq I$. Hence I is a right quasi interior ideal of the semiring M. $\square$

Theorem 3.3. *Let I be a right quasi interior ideal of a semiring M and μ be a fuzzy subset of M, is defined by* $\mu(x) = \begin{cases} \alpha_0 & \text{if } x \in I, \\ \alpha_1, & \text{otherwise.} \end{cases}$
for all $x \in M$, $\alpha_0, \alpha_1 \in [0,1]$ such that $\alpha_0 > \alpha_1$. Then μ is a fuzzy right quasi interior ideal of M and $\mu_{\alpha_0} = I$.

Theorem 3.4. *Every fuzzy (right, left) ideal of a semiring M is a fuzzy right quasi-interior ideal of M.*

Proof. Let μ be a fuzzy right ideal of the semiring M and $x \in M$.

$$\begin{aligned}
\mu \circ \chi_M(x) &= \sup_{x=ab} \min\{\mu(a), \chi_M(b)\} \ a, b \in M. \\
&= \sup_{x=ab} \mu(a) \\
&\leq \sup_{x=ab} \mu(ab) \\
&= \mu(x).
\end{aligned}$$

Therefore $\mu \circ \chi_M(x) \leq \mu(x)$. Then

$$\begin{aligned}
\mu \circ \chi_M \circ \mu \circ \chi_M(x) &= \sup_{x=uvs} \min\{\mu \circ \chi_M(uv), \mu \circ \chi_M(s)\} \\
&\leq \sup_{x=uvs} \min\{\mu(uv), \mu(s)\} \\
&= \mu(x).
\end{aligned}$$

Hence μ is a fuzzy right quasi-interior ideal of M.

Let μ be a fuzzy left ideal of the semiring M and $x \in M$.

$$\begin{aligned}\chi_M \circ \mu(x) &= \sup_{x=ab} \min\{\chi_M(a), \mu(b)\}\ a, b \in M.\\ &= \sup_{x=ab} \mu(b)\\ &\leq \sup_{x=ab} \mu(ab)\\ &= \mu(x).\end{aligned}$$

Therefore $\chi_M \circ \mu(x) \leq \mu(x)$. Then

$$\begin{aligned}\chi_M \circ \mu \circ \chi_M \circ \mu(x) &= \sup_{x=uvs} \min\{\chi_M \circ \mu(u), \chi_M \circ \mu(vs)\}\\ &\leq \sup_{x=uvs} \min\{\mu(u), \mu(vs)\}\\ &= \mu(x).\end{aligned}$$

Hence μ is a fuzzy left quasi-interior ideal of M. □

Example 3.5. Let $M = \{a, b, c, d\}$. The binary operation is defined by the following tables

$+$	a	b	c	d
a	a	b	c	d
b	b	b	b	b
c	c	b	c	d
d	d	b	d	d

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	b	b
c	a	c	c	c
d	a	b	b	a

then $(M, +, \cdot)$ is a semiring.
I) Let $J = \{a, d\}$, then J is a subsemiring.
J is not a(ideal, left, right, bi, quasi, interior) ideal.
J is a right quasi-interior ideal.
1) Define $\mu = M \to [0, 1]$

$$\mu(x) = \begin{cases} 1 & \text{if } x \in J, \\ 0, & \text{otherwise.} \end{cases}$$

Then μ is a fuzzy right quasi-interior ideal of M and μ is not a fuzzy ideal.
2) Define $\mu = M \to [0, 1]$

$$\mu(x) = \begin{cases} 0.7 & \text{if } x \in J, \\ 0.4, & \text{otherwise.} \end{cases}$$

Then μ is a fuzzy right quasi-interior ideal of M and μ is not a fuzzy ideal.

II) Let $J_1 = \{a, c\}$, then J_1 is a subsemiring.
J_1 is not a(ideal, left, right, bi, quasi, interior) ideal.
J_1 is a left quasi-interior ideal.
1) Define $\mu = M \to [0, 1]$

$$\mu(x) = \begin{cases} 1 & \text{if } x \in J_1, \\ 0, & \text{otherwise.} \end{cases}$$

Then μ is a fuzzy left quasi-interior ideal of M and μ is not a fuzzy left ideal.
2) Define $\mu = M \to [0, 1]$

$\mu(x) = \begin{cases} 0.6 & \text{if } x \in J_1, \\ 0.3, & \text{otherwise.} \end{cases}$
Then μ is a fuzzy right quasi-interior ideal of M and μ is not a fuzzy ideal.

Theorem 3.6. *Let M be a semiring and μ be a non-empty fuzzy subset of M. A fuzzy subset μ is a fuzzy left quasi interior ideal of a semiring M if and only if the level subset μ_t of μ is a left quasi interior ideal of a semiring M for every $t \in [0,1]$, where $\mu_t \neq \phi$.*

Proof. Let M be a semiring and μ be a non-empty fuzzy subset of M. Suppose μ is a fuzzy left quasi interior ideal of the semiring M, $\mu_t \neq \phi, t \in [0,1]$ and $a, b \in \mu_t$. Then $\mu(a) \geq t, \mu(b) \geq t,$ *so* $\mu(a+b) \geq \min\{\mu(a), \mu(b)\} \geq t$, therefore $a+b \in \mu_t$ *and* $\mu(ab) \geq \min\{\mu(a), \mu(b)\} \geq t$, hence $ab \in \mu_t$.
Let $x \in M\mu_t M\mu_t$. Then $x = badc$, where $b, d \in M, a, c \in \mu_t$, thus
$\chi_M \circ \mu \circ \chi_M \circ \mu(x) \geq t,$ so $\mu(x) \geq \chi_M \circ \mu \circ \chi_M \circ \mu(x) \geq t.$
Therefore $x \in \mu_t$. Hence μ_t is a left quasi interior ideal of M.
Conversely, suppose that μ_t is a left quasi interior ideal of the semiring M, for all $t \in Im(\mu)$. Let $x, y \in M, \mu(x) = t_1, \mu(y) = t_2$ and $t_1 \geq t_2$. Then $x, y \in \mu_{t_2}$, so $x+y \in \mu_{t_2}$ and $xy \in \mu_{t_2}$, then $\mu(x+y) \geq t_2 = \min\{t_1, t_2\} = \min\{\mu(x), \mu(y)\}$. Therefore $\mu(x+y) \geq t_2 = \min\{\mu(x), \mu(y)\}$ and $\mu(xy) \geq t_2 = \min\{t_1, t_2\} = \min\{\mu(x), \mu(y)\}$. Therefore $\mu(xy) \geq t_2 = \min\{\mu(x), \mu(y)\}$. We have $M\mu_l M\mu_l \subseteq \mu_l$, for all $l \in Im(\mu)$. Suppose $t = \min\{Im(\mu)\}$. Then $M\mu_t M\mu_t \subseteq \mu_t$. Therefore $\chi_M \circ \mu \circ \chi_M \circ \mu \subseteq \mu$. Hence μ is a fuzzy left quasi interior ideal of M. □

Corollary 3.7. *Let M be a semiring and μ be a non-empty fuzzy subset of M. A fuzzy subset μ is a fuzzy (right) quasi interior ideal of a semiring if and only if the level subset μ_t of μ is a (right) quasi-interior ideal of a semiring M for every $t \in [0,1]$, where $\mu_t \neq \phi$.*

Theorem 3.8. *If μ and λ are fuzzy left quasi interior ideals of a semiring M, then $\mu \cap \lambda$ is a fuzzy left quasi interior ideal of a semiring M.*

Proof. Let μ and λ be fuzzy left quasi interior ideals of M and $x, y \in M$.

$$\begin{aligned} \mu \cap \lambda(x+y) &= \min\{\mu(x+y), \lambda(x+y)\} \\ &\geq \min\{\min\{\mu(x), \mu(y)\}, \min\{\lambda(x), \lambda(y)\}\} \\ &= \min\{\min\{\mu(x), \lambda(x)\}, \min\{\mu(y), \lambda(y)\}\} \\ &= \min\{\mu \cap \lambda(x), \mu \cap \lambda(y).\} \end{aligned}$$

$$\begin{aligned} \mu \cap \lambda(xy) &= \min\{\mu(xy), \lambda(xy)\} \\ &\geq \min\{\min\{\mu(x), \mu(y)\}, \min\{\lambda(x), \lambda(y)\}\} \\ &= \min\{\min\{\mu(x), \lambda(x)\}, \min\{\mu(y), \lambda(y)\}\} \\ &= \min\{\mu \cap \lambda(x), \mu \cap \lambda(y)\} \end{aligned}$$

Then $\mu \cap \lambda$ is a fuzzy subsemiring. And

$$\begin{aligned} \chi_M \circ \mu \cap \lambda(x) &= \sup_{x=ab} \min\{\chi_M(a), \mu \cap \lambda(b)\} \\ &= \sup_{x=ab} \min\left\{\chi_M(a), \min\{\mu(b), \lambda(b)\}\right\} \\ &= \sup_{x=ab} \min\left\{ \min\{\chi_M(a), \mu(b)\}, \min\{\chi_M(a), \lambda(b)\}\right\} \\ &= \min\left\{ \sup_{x=ab} \min\{\chi_M(a), \mu(b)\}, \sup_{x=ab} \min\{\chi_M(a), \lambda(b)\}\right\} \\ &= \min\{\chi_M \circ \mu(x). \chi_M \circ \lambda(x)\} \\ &= \chi_M \circ \mu \cap \chi_M \circ \lambda(x). \end{aligned}$$

Therefore $\chi_M \circ \mu \cap \chi_M \circ \lambda = \chi_M \circ \mu \cap \lambda$.

$$\begin{aligned}
&(\chi_M \circ \mu \cap \lambda) \circ (\chi_M \circ \mu \cap \lambda)(x)\\
&= \sup_{x=abc} \min\{\chi_M \circ \mu \cap \chi_M \circ \lambda(a), \chi_M \circ \mu \cap \lambda(bc)\}\\
&= \sup_{x=abc} \min\{\chi_M \circ \mu \cap \lambda(a), \chi_M \circ \mu \cap \chi_M \circ \lambda(bc)\}\\
&= \sup_{x=abc} \min\{\chi_M \circ \mu(a), \chi_M \circ \lambda(a)\}\}, \min\{\chi_M \circ \mu(bc), \chi_M \circ \lambda(bc)\}\}\\
&= \sup_{x=abc} \min\Big\{ \min\{\chi_M \circ \mu(a), \chi_M \circ \mu(bc)\}, \min\{\chi_M \circ \lambda(a), \chi_M \circ \lambda(bc)\}\Big\}\\
&= \min\Big\{ \sup_{x=abc} \min\{\chi_M \circ \mu(a), \chi_M \circ \mu(bc)\}, \sup_{x=abc} \min\{\chi_M \circ \lambda(a), \chi_M \circ \lambda(bc)\}\Big\}\\
&= \min\{\chi_M \circ \mu \circ \chi_M \circ \mu(x), \chi_M \circ \lambda \circ \chi_M \circ \lambda(x)\}\\
&= \chi_M \circ \mu \circ \chi_M \circ \mu \cap \chi_M \circ \lambda \circ \chi_M \circ \lambda(x).
\end{aligned}$$

Then $\chi_M \circ \mu \cap \lambda \circ \chi_M \circ \mu \cap \lambda = \chi_M \circ \mu \circ \chi_M \circ \mu \cap \chi_M \circ \lambda \circ \chi_M \circ \lambda$
Therefore $\chi_M \circ \mu \cap \lambda \circ \chi_M \circ \mu \cap \lambda = \chi_M \circ \mu \circ \chi_M \circ \mu \cap \chi_M \circ \lambda \circ \chi_M \circ \lambda \subseteq \mu \cap \lambda$.
Hence $\mu \cap \lambda$ is the fuzzy left quasi-interior ideal of M. □

Corollary 3.9. *If μ and λ are fuzzy (right) quasi interior ideals of a semiring M, then $\mu \cap \lambda$ is a fuzzy (right) quasi interior ideal of a semiring M.*

Theorem 3.10. *If μ and λ are fuzzy left quasi interior ideals of a semiring M, then $\mu \cup \lambda$ is a fuzzy left quasi interior ideal of a semiring M.*

Proof. Let μ and λ be fuzzy left quasi interior ideals of M and $x, y \in M$.

$$\begin{aligned}
\mu \cup \lambda(x+y) &= \max\{\mu(x+y), \lambda(x+y)\}\\
&\geq \max\{\min\{\mu(x), \mu(y)\}, \min\{\lambda(x), \lambda(y)\}\}\\
&= \min\{\max\{\mu(x), \lambda(x)\}, \max\{\mu(y), \lambda(y)\}\}\\
&= \min\{\mu \cup \lambda(x), \mu \cup \lambda(y).\}
\end{aligned}$$

$$\begin{aligned}
\mu \cup \lambda(xy) &= \max\{\mu(xy), \lambda(xy)\}\\
&\geq \max\{\min\{\mu(x), \mu(y)\}, \min\{\lambda(x), \lambda(y)\}\}\\
&= \min\{\max\{\mu(x), \lambda(x)\}, \max\{\mu(y), \lambda(y)\}\}\\
&= \min\{\mu \cup \lambda(x), \mu \cup \lambda(y).\}
\end{aligned}$$

Then $\mu \cup \lambda$ is a fuzzy subsemiring. And

$$\begin{aligned}
\chi_M \circ \mu \cup \lambda(x) &= \sup_{x=ab} \min\{\chi_M(a), \mu \cup \lambda(b)\}\\
&= \sup_{x=ab} \min\{\chi_M(a), \max\{\mu(b), \lambda(b)\}\}\\
&= \sup_{x=ab} \max\{\min\{\chi_M(a), \mu(b)\}, \min\{\chi_M(a), \lambda(b)\}\}\\
&= \max\{\sup_{x=ab} \min\{\chi_M(a), \mu(b)\}, \sup_{x=ab} \min\{\chi_M(a), \lambda(b)\}\}\\
&= \max\{\chi_M \circ \mu(x), \chi_M \circ \lambda(x)\}\\
&= \chi_M \circ \mu \cup \chi_M \circ \lambda(x).
\end{aligned}$$

Therefore $\chi_M \circ \mu \cup \chi_M \circ \lambda = \chi_M \circ \mu \cup \lambda$.

$$
\begin{aligned}
&(\chi_M \circ \mu \cup \lambda) \circ (\chi_M \circ \mu \cup \lambda)(x) \\
&= \sup_{x=abc} \min\{\chi_M \circ \mu \cup \chi_M \circ \lambda(a), \chi_M \circ \mu \cup \chi_M \circ \lambda(bc)\} \\
&= \sup_{x=abc} \min\Big\{\max\{\chi_M \circ \mu(a), \chi_M \circ \lambda(a)\}, \max\{\chi_M \circ \mu(bc), \chi_M \circ \lambda(bc)\}\Big\} \\
&= \sup_{x=abc} \min\Big\{\max\{\chi_M \circ \mu(a), \chi_M \circ \lambda(a)\}, \max\{\chi_M \circ \mu(bc), \chi_M \circ \lambda(bc)\}\Big\} \\
&= \sup_{x=abc} \min\Big\{\max\{\chi_M \circ \mu(a), \chi_M \circ \mu(bc)\}, \max\{\chi_M \circ \lambda(a), \chi_M \circ \lambda(bc)\}\Big\} \\
&= \max\Big\{\sup_{x=abc} \min\{\chi_M \circ \mu(a), \chi_M \circ \mu(bc)\}, \sup_{x=abc} \min\{\chi_M \circ \lambda(a), \chi_M \circ \lambda(bc)\}\Big\} \\
&= \max\{\chi_M \circ \mu \circ \chi_M \circ \mu(x), \chi_M \circ \lambda \circ \chi_M \circ \lambda(x)\} \\
&= \chi_M \circ \mu \circ \chi_M \circ \mu \cup \chi_M \circ \lambda \circ \chi_M \circ \lambda(x).
\end{aligned}
$$

Then $\chi_M \circ \mu \cup \lambda \circ \chi_M \circ \mu \cup \lambda = \chi_M \circ \mu \circ \chi_M \circ \mu \cup \chi_M \circ \lambda \circ \chi_M \circ \lambda$
Therefore $\chi_M \circ \mu \cup \lambda \circ \chi_M \circ \mu \cup \lambda = \chi_M \circ \mu \circ \chi_M \circ \mu \cup \chi_M \circ \lambda \circ \chi_M \circ \lambda \subseteq \mu \cup \lambda$.
Hence $\mu \cup \lambda$ is a fuzzy left quasi-interior ideal of M. $\square$

Corollary 3.11. *If μ and λ are fuzzy (right) quasi-interior ideals of a semiring M, then $\mu \cup \lambda$ is a fuzzy (right) quasi-interior ideal of a semiring M.*

Theorem 3.12. *Let M be a semiring. Then M is a regular if and only if $\mu = \chi_M \circ \mu \circ \chi_M \circ \mu$, for any fuzzy left quasi interior ideal μ of a semiring M.*

Proof. Let μ be a fuzzy left quasi interior ideal of the regular semiring M and $x, y \in M$. Then $\chi_M \circ \mu \circ \chi_M \circ \mu \subseteq \mu$.

$$
\begin{aligned}
\chi_M \circ \mu \circ \chi_M \circ \mu(x) &= \sup_{x=xyx} \Big\{\min\{\chi_M \circ \mu(x), \chi_M \circ \mu(yx)\}\Big\} \\
&\geq \sup_{x=xyx} \Big\{\min\{\mu(x), \mu(x)\}\Big\} \\
&= \mu(x).
\end{aligned}
$$

Therefore $\mu \subseteq \chi_M \circ \mu \circ \chi_M \circ \mu$. Hence $\chi_M \circ \mu \circ \chi_M \circ \mu = \mu$.
Conversely suppose that $\mu = \chi_M \circ \mu \circ \chi_M \circ \mu$, for any fuzzy left quasi interior ideal μ of the semiring M. Let B be a left quasi interior ideal of the semiring M.
By Theorem 3.2, χ_B is a fuzzy left quasi interior ideal of the semiring M.
Then $\chi_B = \chi_M \circ \chi_B \circ \chi_M \circ \chi_B = \chi_{MBMB}$. Therefore $B = MBMB$.
Hence M is the regular semiring . $\square$

4 Fuzzy Soft (left, right) Quasi Interior Ideals

In this section, we introduce the notion of fuzzy soft right(left) quasi interior ideal, fuzzy soft quasi interior ideal of a semiring and study their properties.

Definition 4.1. Let M be a semiring, E be a parameter set and $A \subseteq E$. Let μ be a mapping given by $\mu : A \to [0,1]^M$ where $[0,1]^M$ denotes the collection of all fuzzy subsets of M. Then (μ, A) is called a fuzzy soft left (right) quasi interior ideal over M if and only if for each $a \in A$, the corresponding fuzzy subset satisfies the following conditions

(i) $\mu_a(x+y) \geq \min\{\mu_a(x), \mu_a(y)\}$ for all $x, y \in M$.

(ii) $\chi_M \circ \mu_a \circ \chi_M \circ \mu_a \subseteq \mu_a(\mu_a \circ \chi_M \circ \mu_a \circ \chi_M \subseteq \mu_a)$.

A fuzzy soft set(μ, A) over a semiring M, is called a fuzzy soft quasi interior ideal if it is both fuzzy soft left quasi interior ideal and fuzzy soft right quasi interior ideal over M.

Example 4.2. Let $M = \{0, a, b, c\}$, define the binary operations " + " and " $\cdot$ " on M, with the following tables

$+$	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	b	c
c	b	b	b	c

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
c	0	a	b	c

Then $(M, +, \cdot)$ is a semiring.
Let $E = \{e_1, e_2, e_3\}$. Choose the fuzzy set (F, E) over M.

Define

	0	a	b	c
f_{e_1}	0.7	0.4	0.6	0
f_{e_2}	0.8	0.5	0.7	0
f_{e_3}	0.9	0.6	0.8	0

$\{f_{e_i}\}, i = 1, 2, 3$ is a fuzzy right quasi interior ideal of M, and $\{f_{e_i}\}$ is not a fuzzy right ideal of M. Therefore (F, E) is not a fuzzy soft right ideal and (F, E) is a fuzzy soft right quasi interior ideal over M.

Theorem 4.3. *Let M be a semiring, E be a parameter set and $A \subseteq E$. If (μ, A) is a fuzzy soft right ideal over M, then (μ, A) is a fuzzy soft right quasi interior ideal over M.*

Proof. Let μ_a be a fuzzy soft right ideal of the semiring M and $x \in M$.

$$\begin{aligned}
\mu_a \circ \chi_M(x) &= \sup_{x=ab} \min\{\mu_a(a), \chi_M(b)\}\ a, b \in M. \\
&= \sup_{x=ab} \mu_a(a) \\
&\leq \sup_{x=ab} \mu_a(ab) \\
&= \mu_a(x).
\end{aligned}$$

Therefore $\mu_a \circ \chi_M(x) \leq \mu_a(x)$. Now

$$\begin{aligned}
\mu_a \circ \chi_M \circ \mu_a \circ \chi_M(x) &= \sup_{x=uvs} \min\{\mu_a \circ \chi_M(uv), \mu_a \circ \chi_M(s)\} \\
&\leq \sup_{x=uvs} \min\{\mu_a(uv), \mu_a(s)\} \\
&= \mu_a(x).
\end{aligned}$$

Thus μ_a is fuzzy right quasi-interior ideal of M. Hence (μ, A) is a fuzzy soft right quasi-interior ideal over M.
$\square$

Corollary 4.4. *Every fuzzy soft (left) ideal of a semiring M is a fuzzy soft(left) quasi interior ideal over M.*

Theorem 4.5. *Let M be a semiring, $A \subseteq E$ and (η, A) be a non-empty fuzzy soft over M. Then (η, A) is a fuzzy soft left quasi interior ideal over M, if and only if the level subset $(\eta_a)_k$ of (η, A) is a left quasi interior ideal of M, $a \in A$, for every $k \in [0, 1]$, where $(\eta_a)_k \neq \phi$.*

Proof. The proof of the following theorem is similar to Theorem 3.6, so we omit the proof. □

Theorem 4.6. *Let M be a semiring, E be a parameter set and $A \subseteq E$, $B \subseteq E$. If (μ, A) and (λ, B) are fuzzy soft left quasi interior ideals over M, then $(\mu, A) \cap (\lambda, B)$ is a fuzzy soft left quasi interior ideal over M.*

Proof. Let (μ, A) and (λ, B) are fuzzy soft left quasi interior ideals of a semiring M. By Definition 2.11, we have that $(\mu, A) \cap (\lambda, B) = (\gamma, C)$ where $C = A \cup B$.
Case (i): If $c \in A \setminus B$, then $\gamma_c = \mu_c$. Thus γ_c is a fuzzy left quasi interior ideal of M, since (μ, A) is a fuzzy soft left quasi interior ideal over M.
Case (ii): If $c \in B \setminus A$, then $\gamma_c = \lambda_c$. Therefore γ_c is a fuzzy left quasi interior ideal of M, since (λ, B) is a fuzzy soft left quasi interior ideal over M.
Case (iii): If $c \in A \cap B$, and $x, y \in M$, then $\gamma_c = \mu_c \cap \lambda_c$ and
Therefore By Theorem 3.8, γ_c is a fuzzy left quasi interior ideal of M. Hence $(\mu, A) \cap (\lambda, B)$ is a fuzzy soft left quasi interior ideal over M. □

Corollary 4.7. *If (μ, A) and (λ, B) are fuzzy soft(right) quasi interior ideals over semiring M, then $(\mu, A) \cap (\lambda, B)$ is a fuzzy soft(right) quasi-interior ideal over M.*

Theorem 4.8. *Let M be a semiring, E be a parameter set and $A \subseteq E$, $B \subseteq E$. If (μ, A) and (λ, B) are fuzzy soft left quasi-interior ideals of M, then $(\mu, A) \cup (\lambda, B)$ is a fuzzy soft left quasi interior ideal over M.*

Proof. Let (μ, A) and (λ, B) are fuzzy soft left quasi interior ideals over the semiring M. By Definition 2.12, we have that $(\mu, A) \cup (\lambda, B) = (\gamma, C)$ where $C = A \cup B$.
Case (i): If $c \in A \setminus B$, then $\gamma_c = \mu_c$. Thus γ_c is a fuzzy left quasi-interior ideal of M, since (μ, A) is a fuzzy soft left quasi-interior ideal over M.
Case (ii): If $c \in B \setminus A$, then $\gamma_c = \lambda_c$. Therefore γ_c is a fuzzy left quasi-interior ideal of M, since $(\lambda, B)$is a fuzzy soft left quasi-interior ideal over M.
Case (iii): If $c \in A \cup B$, and $x, y \in M$, then $\gamma_c = \mu_c \cup \lambda_c$.
Therefore By Theorem 3.10, γ_c is a fuzzy left quasi-interior ideal of M. Hence $(\mu, A) \cup (\lambda, B)$ is a fuzzy soft left quasi-interior ideal over M. □

Corollary 4.9. *If (μ, A) and (λ, B) are fuzzy soft(right) quasi-interior ideals over semiring M, then $(\mu, A) \cup (\lambda, B)$ is a fuzzy soft(right) quasi-interior ideal over M.*

Theorem 4.10. *Let M be a semiring, E be a parameter set and $A \subseteq E$. Then (μ, A) is a fuzzy soft left quasi-interior ideal over a regular semiring M if and only if (μ, A) is a fuzzy soft quasi ideal over a semiring M.*

Proof. Let (μ, A) be a fuzzy soft left quasi interior ideal over the regular semiring M and $x \in M$. Then for each $a \in A$, $\chi_M \circ \mu_a \circ \chi_M \circ \mu_a \subseteq \mu_a$.
Suppose $\chi_M \circ \mu_a(x) > \mu_a(x)$ and $\mu_a \circ \chi_M(x) > \mu_a(x)$.

Since M is regular, there exists $y \in M$, such that $x = xyx$.

$$\begin{aligned}
\mu_a \circ \chi_M(x) &= \sup_{x=xyx} \min\{\mu_a(x), \chi_M(yx)\} \\
&= \sup_{x=xyx} \min\{\mu_a(x), 1\} \\
&= \sup_{x=xyx} \mu_a(x) \\
&> \mu_a(x). \textit{ And} \\
\mu_a \circ \chi_M \circ \mu_a \circ \chi_M(x) &= \sup_{x=xyx} \min\{\mu_a \circ \chi_M(x), \mu_a \circ \chi_M(yx)\} \\
&> \sup_{x=xyx} \min\{\mu_a(x), \mu_a(yx)\} \\
&= \mu_a(x)
\end{aligned}$$

Which is a contradiction. Hence (μ, A) is a fuzzy soft quasi ideal of M.
Let (μ, A) be a soft quasi interior ideal over the semiring M, and $a \in A$.
Then $\mu_a \circ \chi_M \wedge \chi_M \circ \mu_a \subseteq \mu_a$.
$\mu_a \circ \chi_M \circ \mu_a \circ \chi_M \subseteq \mu_a \circ \chi_M$, and $\chi_M \circ \mu_a \circ \chi_M \circ \mu_a \subseteq \chi_M \circ \mu_a$.
Therefore $\chi_M \circ \mu_a \circ \chi_M \circ \mu_a \wedge \mu_a \circ \chi_M \circ \mu_a \circ \chi_M \subseteq \chi_M \circ \mu_a \wedge \mu_a \circ \chi_M \subseteq \mu_a$.
Hence (μ, A) is the fuzzy soft quasi interior ideal over M. □

Corollary 4.11. *Let M be a regular semiring, E be a parameter set and $A \subseteq E$. Then (μ, A) is a fuzzy soft(right) quasi interior ideal over a semiring M if and only if (μ, A) is a fuzzy soft quasi ideal over a semiring M.*

Theorem 4.12. *Let M be a semiring, E be a parameter set and $A \subseteq E$. Then M is a regular if and only if $\mu_a = \chi_M \circ \mu_a \circ \chi_M \circ \mu_a, a \in A$, for any fuzzy left quasi interior ideal of fuzzy soft quasi interior ideal (μ, A) over a semiring M.*

Proof. Let (μ, A) be a fuzzy soft left quasi interior ideal over the regular semiring M and $x, y \in M$. Then to each $a \in A$, $\chi_M \circ \mu_a \circ \chi_M \circ \mu_a \subseteq \mu_a$.

$$\begin{aligned}
\chi_M \circ \mu_a \circ \chi_M \circ \mu_a(x) &= \sup_{x=xyx} \{\min\{\chi_M \circ \mu_a(x), \chi_M \circ \mu_a(yx)\}\} \\
&\geq \sup_{x=xyx} \{\min\{\mu_a(x), \mu_a(x)\}\} \\
&= \mu_a(x).
\end{aligned}$$

Therefore $\mu_a \subseteq \chi_M \circ \mu_a \circ \chi_M \circ \mu_a$. Hence $\chi_M \circ \mu_a \circ \chi_M \circ \mu_a = \mu_a$.
Conversely suppose that $\mu_a = \chi_M \circ \mu_a \circ \chi_M \circ \mu_a$, for any fuzzy soft quasi interior ideal (μ, A) over the semiring M and $a \in A$. Let B be a quasi interior ideal of the semiring M. By Theorem 3.2, χ_B be a fuzzy quasi interior ideal of the semiring M. Then $\chi_B = \chi_M \circ \chi_B \circ \chi_M \circ \chi_B = \chi_{MBMB}$. Thus $B = MBMB$.
Hence M is the regular semiring . □

Theorem 4.13. *Let M be a semiring, E be a parameter set and $A \subseteq E, B \subseteq E$. Then M is a regular if and only if $\mu_b \cap \gamma_a \subseteq \mu_b \circ \gamma_a \circ \mu_b \circ \gamma_a$, for every fuzzy soft left quasi interior ideal (γ, A) and every fuzzy soft ideal (μ, B) over a semiring M, $a \in A, b \in B$.*

Proof. Let M be a regular semiring and $x \in M$. Then there exist $y \in M$ such that $x = xyx$, for each $a \in A, b \in B$, γ_a is a fuzzy left quasi interior ideal, μ_b is a fuzzy ideal of the semiring M. Then

$$\begin{aligned}
&\mu_b \circ \gamma_a \circ \mu_b \circ \gamma_a(x) = \sup_{x=xyx} \Big\{ \min\{\mu_b \circ \gamma_a(xy),\ \mu_b \circ \gamma_a(x)\} \Big\} \\
&= \min\Big\{ \sup_{xy=xyxy} \{\min\{\mu_b(x), \gamma_a(yxy)\}, \sup_{xy=xyxy} \{\min\{\mu_b(x), \gamma_a(yxy)\}\} \Big\} \\
&\geq \min\Big\{ \min\{\mu_b(x), \gamma_a(x)\}, min\{\mu_b(x), \gamma_a(x)\} \Big\} \\
&= \min\{\mu_b(x), \gamma_a(x)\} = \mu_b \cap \gamma_a(x).
\end{aligned}$$

Hence $\mu_b \cap \gamma_a \subseteq \mu_b \circ \gamma_a \circ \mu_b \circ \gamma_a$.
Conversely, suppose that the condition holds. Let (μ, B) be a fuzzy soft left quasi interior ideal of the semiring M. Then to each $b \in B$, $\mu_b \cap \chi_M \subseteq \chi_M \circ \mu_b \circ \chi_M \circ \mu_b, \mu_b \subseteq \chi_M \circ \mu_b \circ \chi_M \circ \mu_b$. Hence M is the regular semiring. $\square$

5 Conclusion

In this paper, we discussed the algebraic properties of fuzzy right(left) quasi interior ideal and fuzzy soft quasi interior ideal of a semiring. Regular semiring is characterized in terms of fuzzy quasi interior ideals and fuzzy soft quasi-interior ideals. We proved, that if M is a semiring, E be a parameter set and $A \subseteq E,\ B \subseteq E$ and if (μ, A) and (λ, B) are fuzzy soft left quasi interior ideals over M, then $(\mu, A) \cap (\lambda, B), ((\mu, A) \cup (\lambda, B))$ are fuzzy soft left quasi interior ideal over M. One can extend this work by studying the other algebraic structures.

Acknowledgment: We would like to thank the reviewers for their thoughtful comments and efforts towards improving our manuscript.

Conflict of Interest: The authors declare that there are no conflict of interest.

References

[1] F. Feng, Y. B. Jun and X. Zhao, Soft semirings, *Comput. and Math. with Appli.,* 56 (2008), 2621-2628.

[2] R. A. Good and D. R. Hughes, Associated groups for a semigroup, *Bull. Amer. Math. Soc.,* 58 (1952), 624-625.

[3] M. Henriksen, Ideals in semirings with commutative addition, *Amer. Math. Soc. Notices*, 5 (1958), 321.

[4] K. Iseki, Quasi-ideals in semirings without zero, *Proc. Japan Acad.*, 34 (1958), 79-84.

[5] K. Iseki, Ideal theory of semiring, *Proc. Japan Acad.*, 32 (1956), 554-559.

[6] K. Iseki, Ideal in semirings, *Proc. Japan Acad.*, 34 (1958), 29-31.

[7] N. Kuroki, On fuzzy semigroups, *Information Sciences*, 53(3) (1991), 203-236.

[8] S. Lajos, On the bi-ideals in semigroups, *Proc. Japan Acad.*, 45 (1969), 710-712.

[9] S. Lajos and F. A. Szasz, On the bi-ideals in associative ring, *Proc. Japan Acad.*, 46 (1970), 505-507.

[10] P. K. Maji, R. Biswas and A. R. Roy, Fuzzy soft sets, *The J. of Fuzzy Math.*, 9(3) (2001), 589-602.

[11] D. Molodtsov, Soft set theory-First results, *Comput. Math. Appl.*, 37 (1999), 19-31.

[12] M. Murali Krishna Rao, Bi-Interior Ideals of $\Gamma-$ semirings, *Disc. Math. General Algebra and Appli.*, 38 (2018), 239-254.

[13] M. Murali Krishna Rao, A study of quasi-interior ideal of semiring, *Bull. Int. Math. Virtual Inst*, 2 (2019), 287-300.

[14] M. Murali Krishna Rao, Bi-quasi Ideals of $\Gamma-$semirings, *Disc. Math. General Algebra and Appli.*, 38 (2018), 69-78.

[15] M. Murali Krishna Rao, Tri-Ideals Of Semirings, *Bull. Int. Math. Virtual Inst.*, 10(1) (2020), 145-155.

[16] M. Murali Krishna Rao, Fuzzy Soft Bi-interior Ideals over $\Gamma-$semirings, *Journal of Hyperstructures*, 10(1) (2021), 47-62.

[17] M. Murali Krishna Rao, Tri-quasi Ideals Of $\Gamma-$semirings, *Disc. Math. General Algebra and Appli.*, 41 (2021), 33-44.

[18] L. A. Zadeh, Fuzzy sets, *Information and control*, 8 (1965), 338-353.

Arsham Borumand Saeid
Department of Pure Mathematics,
Faculty of Mathematics and Computer,
Shahid Bahonar University of Kerman,
Kerman, Iran.
E-mail: arsham@uk.ac.ir

Marapureddy Murali Krishna Rao
Department of Mathematics,
Sankethika Institute of Tech. and Management,
Visakhapatnam-530 041, A.P., India.
E-mail: mmarapureddy@gmail.com

Rajendra Kumar Kona
Department of Mathematics,
GIS, GITAM (Deemed to be University),
Visakhapatnam-530 045, A.P., India.
E-mail: rkkona1972@gmail.com

Noorbhasha Rafi
Department of Mathematics,
Bapatla Engineering College,
Bapatla, A. P., India-522 101
E-mail: rafimaths@gmail.com

Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://tfss.journals.iau.ir/
Online ISSN: 2821-0131
Vol.1, No.2, (2022), 142-159
DOI: http://doi.org/10.30495/tfss.2022.1959857.1034

Article Type: Original Research Article (Invited Paper)

The Category of L-algebras

Lavinia Corina Ciungu

Abstract. In this paper, we define and study the category of L-algebras, proving that this category has equalizers, coequalizers, kernel pairs and products. We investigate the existence of injective objects in this category and show that an object in the subcategory of cyclic L-algebras is injective if and only if it is a complete and divisible cyclic L-algebra.

AMS Subject Classification 2020: 18A20; 18B35; 03G25; 06D35
Keywords and Phrases: L-algebra, Cyclic L-algebra, MV-algebra, Equalizer, Product, Co-product.

1 Introduction

The Yang–Baxter equation first appeared in theoretical physics, and in statistical mechanics. Finding solutions of this equation represents a research topic of current interest. W. Rump proved in [19] that every set A with a binary operation $\cdot$ satisfying equation (L) $(x\cdot y)\cdot(x\cdot z)=(y\cdot x)\cdot(y\cdot z)$ corresponds to a solution of the quantum Yang–Baxter equation if the left multiplication is bijective. Equation (L) also appears in algebraic logic, classical or intuitionistic logic, as well as in infinite-valued Łukasiewicz logic (see [20] for details). Based on equation (L), W. Rump developed in [20] the concept of L-algebras, proving that for every L-algebra A there exists a self-similar closure $S(A)$, unique up to isomorphism, with an embedding of A to $S(A)$. The self-similar closure $S(A)$ admits a left group of fractions $G(A)$ with a natural map $A\hookrightarrow S(A)\longrightarrow G(A)$ and, if A is a semiregular L-algebra, then the structure group $G(A)$ is an ℓ-group ([20, Th. 3, Th. 4]). W. Rump and Y. Yang proved that an L-algebra is representable as an interval in an ℓ-group if and only if it is semiregular with the smallest element and bijective negation ([21, Th. 3.11]), and that the pseudo MV-algebras can be characterized as semiregular L-algebras with negation ([30]). Since L-algebras have applications in many areas such as number theory ([24]), group theory ([22], [23], [25]), lattice theory ([26]), the study of these algebras is a topic of great interest nowadays (see for example [7], [14], [28], [29]). The categories of algebras of fuzzy logic have been investigated for Hilbert algebras ([3], [4], [13]), BCI-algebras ([1]), p-semisimple BCI-algebras ([32]), BCH-algebras ([6]), EQ-algebras ([2]), pseudo BCI-algebras ([11], [12]).
Motivated by the fact that the studies on L-algebras are of current interest, in this paper we study the category **Lalg** of L-algebras and prove that the category **Lalg** has equalizers, and coequalizers, kernel pairs and products. We also prove that any coequalizer is surjective and it is a coequalizer of its kernel pair. We construct the product of two particular objects in **Lalg**, and finally we give an example of two objects in **Lalg** having a co-product. We introduce the notion of divisible cyclic L-algebras and prove that the cyclic L-algebras and MV-algebras are categorial equivalent. We also investigate the existence of injective objects

∗**Corresponding author:** Lavinia Corina Ciungu, Email: lciungu@sfc.edu, ORCID: 0000-0002-9423-921X
Received: 28 May 2022; Revised: 31 August 2022; Accepted: 31 August 2022; Published Online: 7 November 2022.

How to cite: L. C. Ciungu, The Category of L-algebras, *Trans. Fuzzy Sets Syst.*, 1(2) (2022), 142-159.

in the category **Lalg** and prove that $\{1\}$ is the only injective object in this category. The main result consists of proving that an object X in the category **CyLalg** of cyclic L-algebras is injective if and only if X is a complete and divisible cyclic L-algebra.

2 Preliminaries

In this section we recall some basic notions and results regarding L-algebras that we use in this paper (see [20]).

Magma is a structure $(A, \to)$, where $\to$ is a binary operation of a set A. In a magma $(A, \to)$, an element $e \in A$ is a *logical unit* if

$$e \to x = x,\ x \to x = x \to e = e. \tag{U}$$

The logical unit is unique. Indeed, if e, e' are logical units, then $e = e \to e = e'$. We denote the logical unit by 1. Then $(A, \to, 1)$ is called a *unital* magma. A magma $(A, \to)$ is a *cycloid* such that

$$(x \to y) \to (x \to z) = (y \to x) \to (y \to z). \tag{L}$$

A *unital cycloid* is a cycloid with logical unit (see [20]). If a unital cycloid $(A, \to, 1)$ satisfies

$$x \to y = y \to x = 1 \text{ implies } x = y, \tag{An}$$

then it is called an *L-algebra*. If an L-algebra $(A, \to, 1)$ satisfies

$$x \to (y \to x) = 1, \tag{K}$$

then it is called a *KL-algebra*. A *CL-algebra* is an L-algebra $(A, \to, 1)$ such that

$$(x \to (y \to z)) \to (y \to (x \to z)) = 1. \tag{C}$$

It follows that in any L-algebra A satisfying condition (C) we have $x \to (y \to z) = y \to (x \to z)$, for all $x, y, z \in A$. Given an L-algebra $(A, \to, 1)$, a binary relation $\leq$ is defined by $x \leq y$ iff $x \to y = 1$, for all $x, y \in A$.

The notion of a self-similar closure was introduced by W. Rump in [20] and it proved to play a crucial role in the study of L-algebras. Let H be a self-similar L-algebra and let A be a subalgebra of H. As we mentioned, H is a left hoop. If the monoid H is generated by A, we call H a *self-similar closure* of A and it is denoted by $S(A)$. According to [20, Th. 3], for any L-algebra A, the self-similar closure $S(A)$ exists and it is unique, up to isomorphism. Obviously, if H is a self-similar left hoop, then $S(H) = H$. By condition (H), any self-similar left hoop H satisfies the *left Ore condition* (for each pair of elements $a, b \in H$, there are $c, d \in H$ such that $ca = db$ - see [20]), hence the self-similar closure $S(A)$ of an L-algebra A has the left Ore condition. Due to the left Ore condition, $S(A)$ admits a *left group of fractions* $G(S(A))$ (consisting of left fractions $x^{-1}y$, for all pairs $x, y \in S(A)$), denoted by $G(A)$. The morphism $A \hookrightarrow S(A) \longrightarrow G(A)$ defines a natural map $q : A \longrightarrow G(A)$ with $q(x) = q(y)$ if and only if there is $c \in S(A)$ such that $cx = cy$ (see [20, Def. 5]). By [20, Prop. 10], if A is a KL-algebra, then $S(A)$ is also a KL-algebra. A monoid H with an additional operation $\to$ is a *left hoop* if the following hold for all $a, b, c \in H$: (E) $a \to a = 1$, (A) $ab \to c = a \to (b \to c)$, (H) $(a \to b)a = (b \to a)b$. It was proved in [20, Prop. 3] that every left hoop is an L-algebra. An L-algebra A is said to be *self-similar* if and only if for any $x \in A$, the map $\rho : \downarrow x = \{y \in X \mid y \leq x\} \longrightarrow A$, defined by $\rho(y) = x \to y$ is a bijection. It is easy to see that ρ is isotone, more precisely, it is monotone increasing. Based on the bijective map ρ we define a new operation on A, namely, for all $x, y \in A$, the product xy is defined as the inverse image of x. In other words, xy is unique, and it is determined by $xy \leq y$ and $y \to xy = x$. By [20, Th. 1], every self-similar L-algebra with the new product operation is a left-hoop. An L-algebra A is called *commutative* if $S(A)$ is commutative as a monoid. According to [20, Prop. 19], $S(A)$ is commutative if and only if A is a KL-algebra and, for all $x, y \in A$:

$$(x \to y) \to y = (y \to x) \to x. \tag{Com}$$

In this case $S(A) \cong G(A)^-$ and $x \vee y = (x \to y) \to y$ holds for all $x, y \in S(A)$. Indeed, since L is a KL-algebra

we have $x \leq (y \to x) \to x = x \vee y$ and $y \leq (x \to y) \to y = x \vee y$. If $u \in L$ such that $x \leq u$ and $y \leq u$, then $u \to y \leq x \to y$, so that $x \vee y = (x \to y) \to y \leq (u \to y) \to y = (y \to u) \to u = 1 \to u = 1$. Similarly, $u \to x \leq y \to x$ and $x \vee y = (y \to x) \to x \leq (u \to x) \to x = (x \to u) \to u = 1 \to u = u$. Hence $x \vee y$ is the lower upper bound of $\{x, y\}$.
Let $(A, \to, 1)$ be an L-algebra. We call $I \subseteq A$ an *ideal* of A if it satisfies the following conditions for all $x, y \in A$ ([20]): (I_0) $1 \in I$; (I_1) $x, x \to y \in I$ imply $y \in I$; (I_3) $x \in I$ implies $y \to x, y \to (x \to y) \in I$. Denote by $\mathcal{ID}(A)$ the set of all ideals of A. Obviously $\{1\}, A \in \mathcal{ID}(A)$.
Let $(A, \to, 1)$ be an L-algebra and let $I \in \mathcal{ID}(A)$. According to [20], [7] we have:
(1) If A satisfies condition (K), then (I_3) can be omitted.
(2) If A satisfies condition (D), then (I_2) can be omitted.
(3) If A satisfies condition (C), then (I_2) and (I_3) can be omitted.
Let A be an L-algebra. For every subset $B \subseteq A$, the smallest ideal of A containing B (i.e. the intersection of all ideals $I \in \mathcal{ID}(A)$ such that $B \subseteq I$) is called the *ideal generated by* B and it will be denoted by $[B)$. If $B = \{x\}$ we write $[x)$ instead of $[\{x\})$. In this case $[x)$ is called a *principal ideal* of A. Let $(A, \to, 1)$ be an L-algebra. Then every ideal I of A defines a congruence:

$$x \sim y \text{ iff } x \to y, y \to x \in I.$$

Conversely, each congruence $\sim$ of A defines an ideal $I := \{x \in X \mid x \sim 1\}$.
A congruence $\sim$ of A is called a *relative congruence* if the quotient algebra $(A/\sim, \to, [1]_{\sim})$ is an L-algebra. According to [20, Cor. 1], for an L-algebra X, there is a bijective correspondence between ideals and relative congruences. We denote by $\theta_I = \sim_I$ a relative congruence defined by an ideal I, and $(A/I, \to, [1]_I)$ the corresponding quotient algebra. We write $[x]_{\sim_I} = x/I$ and obviously $I = 1/I$. The function $\pi_I : A \longrightarrow A/I$ defined by $\pi_I(x) = x/I$ for any $x \in A$ is a surjective homomorphism which is called the *canonical projection* from A to A/I. One can easily prove that $\mathrm{Ker}\,(\pi_I) = I$. If A is a self-similar L-algebra and I is an ideal of A, then by [20, Cor. 3] A/I is a self-similar L-algebra.

Let $(A, \to, 1)$ and $(B, \to, 1)$ be two L-algebras. A map $f : A \longrightarrow B$ is called a *morphism* if $f(x \to y) = f(x) \to f(y)$, for all $x, y \in A$. Denote by $\mathrm{HOM}\,(A, B)$ the set of all morphisms from A to B. If $f \in \mathrm{HOM}\,(A, B)$, then $\mathrm{Ker}\,(f) = \{x \in A \mid f(x) = 1\}$ is called the *kernel* of f.
For any $f \in \mathrm{HOM}\,(A, B)$ the following hold: (i) $f(1) = 1$, (ii) $f(x) \leq f(y)$, whenever $x, y \in A$, $x \leq y$, (iii) $\mathrm{Ker}\,(f) \in \mathcal{ID}(A)$.

Proposition 2.1. *Let A, B be two self-similar L-algebras. If $f \in \mathrm{HOM}\,(A, B)$, then $f(xy) = f(x)f(y)$, for all $x, y \in A$.*

Proof. For all $x, y \in A$ we have $xy \leq y$ and $y \to xy = x$. It follows that $f(y) \to f(xy) = f(x)$. On the other hand, $f(x)f(y) \leq f(y)$ and $f(y) \to f(x)f(y) = f(x)$. Since the product is unique we get $f(xy) = f(x)f(y)$. $\square$

3 MV-algebras as L-algebras

We recall the definition and certain results on MV-algebras, and we define the notion of cyclic L-algebras. The main result consists of proving that an algebra $(A, \oplus, 0)$ is an MV-algebra if and only if $(A, \to, 1)$ is a cyclic L-algebra.

Let A be an L-algebra having a smallest element 0, and denote $x^- = x \to 0$, for all $x \in A$. We say that A has a *negation* if the map $^- : A \longrightarrow A$, defined by $x \mapsto x^-$ is bijective. Using the inverse of negation $^-$, denoted by $^\sim$, we define the second implication on A by $x \rightsquigarrow y = y^\sim \to x^\sim$. Clearly, $x \rightsquigarrow 0 = x^\sim$ and

$x^{-\sim} = x^{\sim -} = x$, for any $x \in A$. By [21, Prop. 2.8], if A is a semiregular L-algebra with negation, then $x \leq y$ iff $x^- \geq y^-$. According to [21, Th. 3.8], for any semiregular L-algebra with a negation $(A, \rightarrow, 1)$, the structure $A^{op} := (A, \rightsquigarrow, 1)$ is a semiregular L-algebra with negation such that $(A^{op})^{op} = A$. For a semiregular L-algebra with negation A, a product operation $\cdot$ was defined in [21] by $x \cdot y = (x \rightarrow y^-)^\sim$, for all $x, y \in A$, and it is proved that $x \cdot y \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \rightsquigarrow z$, for all $x, y, z \in A$ ([21, Prop. 3.2]). Moreover, from $x \rightarrow y \leq x \rightarrow y$ and $x \rightsquigarrow y \leq x \rightsquigarrow y$ we get $x \leq (x \rightarrow y) \rightsquigarrow y$ and $x \leq (x \rightsquigarrow y) \rightarrow y$, respectively. It follows that a semiregular L-algebra with negation is a CL-algebra. According to [21, Prop. 3.5], a semiregular L-algebra with negation is a left hoop, so that the operation $\cdot$ is associative. For a semiregular L-algebra with negation A we set:

$$x \wedge y := ((x \rightarrow y) \rightarrow x^-)^\sim,\ x \vee y = (x^\sim \rightarrow y^\sim) \rightarrow x,$$

for all $x, y \in A$. It is proved in [21, Prop. 2.9] that $(A, \wedge, \vee)$ is a lattice.

Proposition 3.1. *([8]) Let $(A, \rightarrow, 1)$ be a semiregular L-algebra with negation. Then the following hold for all $x, y \in A$:*
(1) $x \cdot 0 = 0 \cdot x = 0$, $x \cdot 1 = 1 \cdot x = x$;
(2) $x^- \cdot x == 0$;
(3) $x \rightarrow y = y^- \rightarrow x^-$;
(4) $x^- \rightarrow y = y^- \rightarrow x$;
(5) $y \leq x \rightarrow y$.

Let $(A, \rightarrow, 0, 1)$ be a semiregular L-algebra with negation. We define the sum of the elements x and y of A:

$$x + y := y^- \rightarrow x = x^- \rightarrow y.$$

Proposition 3.2. *([8]) Let A be a semiregular L-algebra with negation. Then the following hold for all $x, y \in A$:*
(1) $0 + x = x + 0 = x$;
(2) $1 + x = x + 1 = 1$;
(3) $x + x^- = 1$;
(4) $x \cdot y = (y^- + x^-)^-$;
(5) $x + y = (y^- \cdot x^-)^-$;
(6) $x + y = y + x$.

Proof. The proof is straightforward. $\square$

Definition 3.3. A semiregular L-algebra with negation A is said to be *cyclic* if $x^- = x^\sim$, for all $x \in A$.

If A is cyclic, then we can easily see that $x \vee y = (x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$, for all $x, y \in A$.
The *MV-algebras* were defined by Chang in 1958 ([5]) as algebraic counterparts of $\aleph_0$-valued Łukasiewicz logic. For details on MV-algebras we refer the reader to [9].
An *MV-algebra* is an algebra $(A, \oplus, ^-, 0)$ with a binary operation $\oplus$, a unary operation $^-$ and a constant 0 satisfying the following equations, for all $x, y, z \in A$: (MV_1) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
(MV_2) $x \oplus y = y \oplus x$;
(MV_3) $x \oplus 0 = x$;
(MV_4) $(x^-)^- = x$;
(MV_5) $x \oplus 0^- = 0^-$;
(MV_6) $(x^- \oplus y)^- \oplus y = (y^- \oplus x)^- \oplus x$.
Axioms (MV_1)-(MV_3) state that $(A, \oplus, 0)$ is a commutative monoid. As a consequence, in any MV-algebra A we have $1^- = 0$ and $x \oplus x^- = 1$, for all $x \in A$. We can easily see that the map $x \mapsto x^-$ is bijective. Indeed, if $x_1, x_2 \in A$ with $x_1^- = x_2^-$, then $x_1^{--} = x_2^{--}$, and by (MV_4) we get $x_1 = x_2$. Moreover, since $x = (x^-)^-$, the

map $x \mapsto x^-$ is bijective. If $(A, \oplus, ^-, 0)$ is an MV-algebra, we define the following operations, for all $x, y \in A$: $x \odot y = (x^- \oplus y^-)^-$, $x \to y = x^- \oplus y = (x \odot y^-)^-$, $1 = 0^-$. We can see that $x^- = x \to 0$. A partial order relation $\le$ is defined on A by $x \le y$ iff $x^- \oplus y = 1$. Two auxiliary operations $\vee$ and $\wedge$ are defined, by setting $x \vee y = x \oplus y \odot x^- = y \oplus x \odot y^-$ and $x \wedge y = x \odot (y \oplus x^-) = y \odot (x \oplus y^-)$. Then $(A, \wedge, \vee, 0, 1)$ is a lattice.

Lemma 3.4. *If $(A, \oplus, ^-, 0)$ is an MV-algebra, then the following hold for all $x, y \in A$:*
(1) $x \le y$ *iff* $y^- \le x^-$;
(2) $(x \to y) \vee (y \to x) = 1$;
(3) $y \to x \odot y = x^- \vee y$.

Proof. (3) Replacing y by y^- in (MV_6), we get $y^- \oplus (y^- \oplus x^-)^- = (x \oplus y)^- \oplus x$, so that $y^- \oplus x \odot y = (x \oplus y)^- \oplus x$. It follows that $y \to x \odot y = (x^- \to y) \to x = x^- \vee y$. □

A monoid $(H, \odot, 1)$ with an additional binary operation $\to$ will be called a *left hoop* if the following are satisfied for $x, y, z \in H$: (h_1) $x \to x = 1$, (h_2) $x \to (y \to z) = x \odot y \to z$, (h_3) $(x \to y) \odot x = (y \to x) \odot y$ ([20, Def. 3]).

Lemma 3.5. *If $(A, \oplus, ^-, 0)$ is an MV-algebra, then $(A, \odot, \to, 1)$ is a left hoop.*

Proof. For all $x, y, z \in A$, we have: $x \to x = x^- \oplus x = 1$, $x \odot y \to z = (x \odot y)^- \oplus z = (y^- \oplus x^-) \odot z = x^- \oplus (y^- \oplus z) = x \to (y \to z)$, and $(x \to y) \odot x = (x^- \oplus y) \odot x = x \wedge y = (y^- \oplus x) \odot y = (y \to x) \odot y$. Hence $(A, \odot, \to, 1)$ is a left hoop. □

Proposition 3.6. *If $(A, \oplus, ^-, 0)$ is an MV-algebra, then $(A, \to, 0, 1)$ is a cyclic L-algebra.*

Proof. We check axioms (U), (L) and (An) from the definition of L-algebras.
Since $x \to x = x^- \oplus x = 1$, $1 \to x = 1^- \oplus x = 0 \oplus x = x$, and $x \to 1 = x^- \oplus 1 = 1$, axiom (U) is satisfied. If $x \to y = y \to x = 1$, then $x^- \oplus y = y^- \oplus x = 1$, so that $x \le y$ and $y \le x$. It follows that $x = y$, that is axiom (An) is also verified. Let $x, y, z \in A$. Replacing x by x^- and y by y^- in (MV_6) we get $(x \oplus y^-)^- \oplus y^- = (y \oplus x^-)^- \oplus x^-$, so that $(y \oplus x^-)^- \oplus (x^- \oplus z) = (x \oplus y^-)^- \oplus (y^- \oplus z)$. It follows that $(x \to y)^- \oplus (x \to z) = (y \to x)^- \oplus (y \to z)$, that is $(x \to y) \to (x \to z) = (y \to x) \to (y \to z)$, and so, axiom (L) is satisfied. It follows that $(A, \to, 1)$ is an L-algebra. By Lemma 3.5, A is a left hoop and according to [21, Thm. 3.7], an L-algebra with negation is semiregular if and only if it is a left hoop satisfying conditions from Lemma 3.4. We conclude that $(A, \to, 0, 1)$ is a cyclic L-algebra. □

Proposition 3.7. *If $(A, \to, 0, 1)$ is a cyclic L-algebra, then $(A, +, 0)$ is an MV-algebra.*

Proof. We check axioms (MV_1)-(MV_6) from the definition of MV-algebras. Since $(x + y) + z = (x^- \to y) + z = z^- \to (x^- \to y) = x^- \to (z^- \to y) = x + (y + z)$, axiom (MV_1) is satisfied. Axioms (MV_2), (MV_3) and (MV_5) follow from Proposition 3.2(6),(1),(2), respectively, while axiom (MV_4) is true by the definition of negation. Finally, the identity $(x \to y) \to y = (y \to x) \to x (= x \vee y)$ implies (MV_6), so that $(A, +, 0)$ is an MV-algebra. □

Theorem 3.8. *An algebra $(A, \oplus, 0)$ is an MV-algebra if and only if $(A, \to, 1)$ is a cyclic L-algebra.*

Proof. It follows by Propositions 3.6 and 3.7. □

Example 3.9. Consider the set $A = \{0, a, b, 1\}$ and the operation $\to$ given by the following table:

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

.

The structure $(A, \rightarrow, 1)$ is a cyclic L-algebra. The negation $^-$ and the operations $\cdot$, $+$ are given in the tables below.

x	0	a	b	1
x^-	1	b	a	0

$\cdot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$+$	0	a	b	1
0	0	a	b	1
a	0	a	1	1
b	b	1	b	1
1	1	1	1	1

.

Then $(A, +, 0)$ is an MV-algebra.

Proposition 3.10. *The meets and unions $(\wedge_{MV}, \vee_{MV})$ of an MV-algebra coincide with the meets and unions $(\wedge_L, \vee_L)$ of its corresponding cyclic L-algebra $(A, \rightarrow, 0, 1)$.*

Proof. Recall that:

$$x \wedge_{MV} y = x \odot (y \oplus x^-) = y \odot (x \oplus y^-),\ x \vee_{MV} y = x \oplus y \odot x^- = y \oplus x \odot y^-,$$
$$x \wedge_L y = ((x \rightarrow y) \rightarrow x^-)^- = ((y \rightarrow x) \rightarrow y^-)^-,\ x \vee_L y = (x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x,$$

for all $x, y \in A$. Then we have:

$$\begin{aligned} x \wedge_{MV} y &= x \odot (y \oplus x^-) = (x^- \oplus y) \odot x = ((x^- \oplus y) \odot x)^{--} = ((x^- \oplus y)^- \oplus x^-)^- \\ &= ((x \rightarrow y)^- \oplus x^-)^- = ((x \rightarrow y) \rightarrow x^-)^- = x \wedge_L y, \\ x \vee_{MV} y &= y \oplus x \odot y^- = x \odot y^- \oplus y = (x^- \oplus y)^- \oplus y = (x \rightarrow y)^- \oplus y \\ &= (x \rightarrow y) \rightarrow y = x \vee_L y. \end{aligned}$$

Hence the two pairs of lattice operations coincide. $\square$

4 The Category of L-algebras

In this section, we define the category **Lalg** of L-algebras and prove that this category has equalizers, co-equalizers, and kernel pairs. We also prove that any coequalizer is surjective and it is a coequalizer of its kernel pair.

We consider the category of L-algebras, denoted by **Lalg** whose objects are L-algebras and whose morphisms are L-algebras homomorphisms. Denote by **Ob**(Lalg) the class of objects of **Lalg**, and for any $X, Y \in$ **Lalg**, we denote by **Lalg**(X,Y) the class of morphisms of **Lalg**. For details regarding the notions and results of category theory we refer the reader to [17], [18], [16], [3].

In a category $\mathcal{C}$, an object **0** is called an *initial object* if, for every object X of $\mathcal{C}$, there is exactly one morphism from **0** to X. And dually, an object **1** is called a *terminal or final object* if, for every object X, there is exactly one morphism from X to **1**. If an object is simultaneously an initial and a final object, it is called a *nullary object* or a *zero object*.

Proposition 4.1. *The category* **Lalg** *has an initial and final object.*

Proof. We can see that in the category **Lalg**, $\mathbf{0} = \mathbf{1} = (\{1\}, \rightarrow, 1)$ is an initial object as well as a final object. Indeed, for any $X \in \mathbf{Ob}(Lalg)$ there is a unique morphism $f : \{1\} \longrightarrow X$ and there is a unique morphism $f : X \longrightarrow \{1\}$. Hence $\{1\}$ is a nullary object of **Lalg**. $\square$

Generally speaking, if $\mathcal{C}$ is an algebraic category and $X, Y \in \mathbf{Ob}(\mathcal{C})$, then $f \in \mathcal{C}(X, Y)$ is a *monomorphism* if for any $Z \in \mathbf{Ob}(\mathcal{C})$ and $g, h \in \mathcal{C}(Z, X)$ such that $f \circ g = f \circ h$, we have $g = h$. Similarly, if $g \circ f = h \circ f$ implies $g = h$ for any $g, h \in \mathcal{C}(Y, Z)$, then f is called an *epimorphism*.

In this section we extend to the case of **Lalg** some results proved in [11] and [6] for the categories of pseudo-BCI algebras and pseudo-BCH algebras, respectively.

Theorem 4.2. *In the category* **Lalg** *monomorphisms and injective morphisms coincide.*

Proof. Let $X, Y \in \mathbf{Ob}(Lalg)$ and let $f \in \mathbf{Lalg}(X, Y)$ injective. Consider $X' \in \mathbf{Ob}(Lalg)$ and $g, h \in \mathbf{Lalg}(X', X)$ such that $f \circ g = f \circ h$, that is $(f(g(x)) = f(h(x))$, for any $x \in X'$. Since f is injective, we get $f(x) = g(x)$ for all $x \in X'$, hence $g = h$. It follows that f is a monomorphism of **Lalg**. Conversely, suppose that f is a monomorphism, so that $f \circ g = f \circ h$ implies $g = h$. It is enough to prove that $\mathrm{Ker}\,(f) = \{1\}$. Let $\mathrm{Ker}\,(f)$ such that $x \neq 1$, and define $g, h : \mathrm{Ker}\,(f) \longrightarrow X$, by $g(x) = x$, $h(x) = 1$, for all $x \in \mathrm{Ker}\,(f)$. We have $f(x) = f(1) = 1$, hence $f \circ g = f \circ h$). Since f is a monomorphism, we get $g = h$, a contradiction. Thus $\mathrm{Ker}\,(f) = \{1\}$, that is f is injective. (Indeed, if $x_1, x_2 \in A$ such that $f(x_1) = f(x_2)$, we have $f(x_1 \to x_2) = f(x_1) \to f(x_2) = 1$ and $f(x_2 \to x_1) = f(x_2) \to f(x_1) = 1$. It follows that $x_1 \to x_2, x_2 \to x_1 \in \mathrm{Ker}\,(f) = \{1\}$, that is $x_1 \to x_2 = x_2 \to x_1 = 1$. We get $x_1 \leq x_2$ and $x_2 \leq x_1$, hence by (L_3) we have $x_1 = x_2$). $\square$

Proposition 4.3. *In the category* **Lalg** *surjective morphisms are epimorphisms.*

Proof. Let $X, Y \in \mathbf{Ob}(Lalg)$ and let $f \in \mathbf{Lalg(X, Y)}$ surjective. Consider $Z \in \mathbf{Ob}(Lalg)$ and $g, h \in \mathbf{Lalg(Y, Z)}$ such that $g \circ f = h \circ f$. Let $y \in Y$. Since f is surjective, there is $x \in X$ such that $f(x) = y$. It follows that $g(y) = g(f(x)) = h(f(x)) = h(y)$, for all $y \in Y$, that is $g = h$. We conclude that f is an epimorphism in **Lalg**. $\square$

Remark 4.4. The converse of Proposition 4.3 is not always true. Indeed, in [4, Ex. 4.1] is given an example of an epimorphism of Hilbert algebras which is not surjective. Since by [7, Rem. 4.12] any Hilbert algebra is an L-algebra, it follows that not any surjective morphism in **Lalg** is an epimorphism.

We recall that $f \in \mathbf{Lang}(X, Y)$ is a *bimorphism* if it is both monomorphism and epimorphism. If any bimorphism in a category is an isomorphism, the category is called *balanced* or *perfect*.

Corollary 4.5. *The category* **Lang** *is not perfect.*

Proposition 4.6. *Let $f : X \longrightarrow Y$ be an epimorphism of L-algebras. Then $[\mathrm{Im}\,(f)) = Y$.*

Proof. Let $I = [\mathrm{Im}\,(f))$ and suppose that $I \neq Y$. Consider the map $\mathbf{1}_Y : Y \longrightarrow Y/I$ defined by $\mathbf{1}_Y(x) = 1/I$, for all $x \in Y$. Since $f(x) \in \mathrm{Im}\,(f) \subseteq I$, for any $x \in X$, we have $(\pi_I \circ f)(x) = \pi_I(f(x)) = 1/I = \mathbf{1}_Y(f(x)) = (\mathbf{1}_Y \circ f)(x)$. Hence $\pi_I \circ f = \mathbf{1}_Y \circ f$. On the other hand, $\pi_I(x) = \mathbf{1}_B(x)$ if and only if $x \in I \neq Y$. It follows that f is not an epimorphism, a contradiction. We conclude that $[\mathrm{Im}\,(f)) = Y$. $\square$

Corollary 4.7. *If $f : X \longrightarrow Y$ is an epimorphism of L-algebras such that $\mathrm{Im}\,(f) \in \mathcal{ID}(Y)$, then f is surjective.*

Definition 4.8. A homomorphism $f : X \longrightarrow Y$ of L-algebras satisfying $\mathrm{Im}\,(f) \in \mathcal{ID}(Y)$ is said to be *regular*. A category has *ES property* (*epimorphism surjectivity property*) if all its epimorphisms are surjective.

Corollary 4.9. *The category* **Lalg** *does not have ES property.*

Let $\mathcal{C}$ be a category, and let $X, Y \in \mathbf{Ob}(\mathcal{C})$ and $f, g \in \mathcal{C}(X, Y)$. An *equalizer* of the couple (f, g) is a pair (E, e) with $E \in \mathbf{Ob}(\mathcal{C})$ and $e \in \mathcal{C}(E, X)$ such that:
(i) $f \circ e = g \circ e$;
(ii) if (E', e') is another pair that satisfies (i), then there exists a unique morphism $u \in \mathcal{C}(E', E)$ such that $e' = e \circ u$.

$$\begin{tikzcd} E \arrow[r, "e"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\ E' \arrow[u, "u"] \arrow[ur, "e'"'] & & \end{tikzcd}$$

If a couple of morphisms in $\mathcal{C}$ has an equalizer (E, e), then it is unique up to an isomorphism ([3, Rem. 4.2.14]) and e is a monomorphism in $\mathcal{C}$([3, Rem. 4.2.16]). We say that the category $\mathcal{C}$ has equalizers if any couple of morphisms in $\mathcal{C}$ has an equalizer.

Theorem 4.10. *The category* **Lalg** *has equalizers.*

Proof. Let $X, Y \in \mathbf{Ob}(Lalg)$ and let $f, g \in \mathbf{Lalg}(X, Y)$. Then $E = \{x \in X \mid f(x) = g(x)\}$ is a nonempty subalgebra of X and consider the embedding $e : E \longrightarrow X$ ($e(x) = x$, for any $x \in E$). Obviously, $E \in \mathbf{Ob}(Lalg)$, $e \in \mathbf{Lalg}(E, X)$ and $f \circ e = g \circ e$. Moreover, it is easy to see that e is a monomorfism in **Lalg**. Let $E' \in \mathbf{Ob}(Lalg)$ and let $e' \in \mathbf{Lalg}(E', X)$ such that $f \circ e' = g \circ e'$. Define $u : E' \longrightarrow X$, by $u(x) = e'(x)$ for any $x \in E'$. Since $f(e'(x)) = g(e'(x))$, it follows that $e'(x) \in E$ for all $x \in E'$, hence u is well defined. We have $e(u(x)) = e(e'(x)) = e'(x)$ for any $x \in E'$, so that $e \circ u = e'$. By the fact that e is a monomorphism, it follows that u is unique. We conclude that (E, e) is an equalizer of the couple (f, g), that is **Lalg** has equalizers. $\square$

Corollary 4.11. *If a couple of morphisms in the category* **Lalg** *has an equalizers* (E, e)*, then* e *is injective.*

Proof. It follows by Theorem 4.2, since e is a monomorphism in **Lalg**. $\square$

Example 4.12. Let $X_1 = \{0, 1\}$, $Y_1 = \{0, 1, 2\}$ and consider the following binary operations $\rightarrow_1, \odot_1$ and $\rightarrow_2, \odot_2$ defined on X_1, Y_1, respectively.

$\rightarrow_1$	0	1
0	1	1
1	0	1

$\odot_1$	0	1
0	0	0
1	0	1

$\rightarrow_2$	0	1	2
0	2	2	2
1	1	2	2
2	0	1	2

$\odot_2$	0	1	2
0	0	0	0
1	0	0	1
2	0	1	2

Then the structures $(X_1, \odot_1, \rightarrow_1, 1), (Y_1, \odot_2, \rightarrow_2, 1)$ are BL-algebras ([15, Ex. 7.1]), and according to to [7, Prop. 4.7], $X = (X_1, \rightarrow_1, 1), Y = (Y_1, \rightarrow_2, 1)$ are L-algebras. Hence $X, Y \in \mathbf{Ob}(Lalg)$, and let $f, g \in \mathbf{Lalg}(X, Y)$ defined by $f(0) = 0, f(1) = 2$, $g(0) = 1, g(1) = 2$. Consider $E = \{x \in X \mid f(x) = f(y)\} = \{1\} \in \mathbf{Ob}(Lalg)$, and let $e \in \mathbf{Lalg}(E, X)$ defined by $e(x) = x$. Then (E, e) is an equalizer of the pair (f, g).

Let $\mathcal{C}$ be a category, and let $X, Y \in \mathbf{Ob}(\mathcal{C})$ and $f, g \in \mathcal{C}(X, Y)$. A *coequalizer* of the couple (f, g) is a pair (Q, q) with $Q \in \mathbf{Ob}(\mathcal{C})$ and $q \in \mathcal{C}(Y, Q)$ such that:
(i) $q \circ f = q \circ g$;
(ii) if (Q', q') is another pair which satisfies (i), then there exists a unique morphism $u \in \mathcal{C}(Q, Q')$ such that $q' = u \circ q$.

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "q"] \arrow[dr, "q'"'] & Q \arrow[d, "u"] \\ & & Q' \end{tikzcd}$$

We say that the category $\mathcal{C}$ has coequalizers if any couple of morphisms in $\mathcal{C}$ has a coequalizer.

Theorem 4.13. *The category* **Lalg** *has coequalizers.*

Proof. Let $X, Y \in \mathbf{Ob}(Lalg)$ and let $f, g \in \mathbf{Lalg}(X, Y)$. Denote $Z = \{(f(x), g(x)) \in Y \times Y \mid x \in X\}$ and let $Q = Y/[Z) \in \mathbf{Ob}(Lalg)$ (by [20, Cor. 1]). If $q : Y \longrightarrow Q$ is the canonical projection, then $q \in \mathbf{Lalg}(Y, Q)$, and we prove that (Q, q) is a equalizator for (f, g). Obviously, $(f(x), g(x)) \in \theta_{[Z)}$ for all $x \in X$, so that $(q \circ f)(x) = q(f(x)) = [f(x)]_{\theta_{[Z)}} = [g(x)]_{\theta_{[Z)}} = q(g(x)) = (q \circ g)(x)$, for all $x \in X$. Hence $q \circ f = q \circ g$. Let $Q' \in \mathbf{Ob}(Lalg)$ and let $q' \in \mathbf{Lalg}(Y, Q')$ such that $q' \circ f = q' \circ g$, that is $q'(f(x)) = q'(g(x))$ for all $x \in X$. It follows that $f(x) \to g(x), g(x) \to f(x) \in \mathrm{Ker}\,(q')$, hence $(f(x), g(x)) \in \theta_K$, where $K = \mathrm{Ker}\,(q')$. Thus $Z \subseteq \theta_K$, that is $\theta_{[Z)} \subseteq \theta_K$. Define the morphism $u : Q \longrightarrow Q'$ by $u(y/[Z)) = q'(y)$ (u is well defined, since $y_1/[Z) = y_2/[Z)$ implies $(y_1, y_2) \in \theta_{[Z)} \subseteq \theta_K$, that is $q'(y_1) = q'(y_2)$). Obviously $u \circ q = q'$. Since q is surjective, it is an epimorphism, that is u is unique. We conclude that (Q, q) is a coequalizator for the couple (f, g). $\Box$

Example 4.14. Consider $X, Y \in \mathbf{Ob}(Lalg)$ and $f, g \in \mathbf{Lalg}(X, Y)$ from Example 4.12. Let $Z = \{(f(x), g(x)) \in Y \times Y \mid x \in X\} = \{(f(0), g(0)), (f(1), g(1))\} = \{(0, 1), (2, 2)\}$.
Then $[Z) = \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 2)\}$ and $Q = Y/[Z) = \{[0] = [1] = \{0, 1\}, [2] = \{2\}\}$. We have $Q \in \mathbf{Ob}(Lalg)$ and let $q \in \mathbf{Lalg}(Y, Q)$ be the canonical projection: $q(0) = q(1) = [0] = [1]$, $q(2) = [2]$. Then (Q, q) is a coequalizator for the pair (f, g).

Let $\mathcal{C}$ be a category, and let $X, Y \in \mathbf{Ob}(\mathcal{C})$ and $f \in \mathcal{C}(X, Y)$. A *kernel pair* of the f is a system (P, p_1, p_2) with $P \in \mathbf{Ob}(\mathcal{C})$ and $p_1, p_2 \in \mathcal{C}(P, X)$ such that:
(i) $f \circ p_1 = f \circ p_2$;
(ii) if (Q, q_1, q_2) is another system which satisfies (i), then there exists a unique morphism $u \in \mathcal{C}(Q, P)$ such that $p_1 \circ u = q_1$ and $p_2 \circ u = q_2$.

$$\begin{tikzcd}
P \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & X \arrow[r, "f"] & Y \\
Q \arrow[u, "u"] \arrow[ur, shift left, "q_1"] \arrow[ur, shift right, "q_2"'] & &
\end{tikzcd}$$

We say that the category $\mathcal{C}$ has kernel pairs if any morphisms in $\mathcal{C}$ has a kernel pair.

Theorem 4.15. *The category* **Lalg** *has kernel pairs.*

Proof. Let $X, Y \in \mathbf{Ob}(Lalg)$ and let $f \in \mathbf{Lalg}(X, Y)$. Obvioulsy, the structure $(X \times X, \to, (1, 1))$ is an L-algebra, where $(x_1, y_1) \to (x_2, y_2) = (x_1 \to x_2, y_1 \to y_2)$. Denote $P = \{(x_1, x_2) \in X \times X \mid f(x_1) = f(x_2)\}$, and clearly P is an L-subalgebra of $X \times X$, that is $P \in \mathbf{Ob}(Lalg)$. Let $p_1, p_2 : P \longrightarrow X$ be the canonical projections, that is $p_1(x_1, x_2) = x_1$, $p_2(x_1, x_2) = x_2$, for all $(x_1, x_2) \in X \times X$. Obviously $p_1, p_2 \in \mathbf{Lalg}(P, X)$ such that $f \circ p_1 = f \circ p_2$. Consider now $Q \in \mathbf{Ob}(Lalg)$ and $q_1, q_2 \in \mathbf{Lalg}(Q, X)$ such that $f \circ q_1 = f \circ q_2$, and define $u : Q \longrightarrow P$ by $u(x) = (q_1(x), q_2(x))$, for all $x \in Q$. Since $f(q_1(x)) = f(q_2(x))$ implies $(q_1(x), q_2(x)) \in P$ for all $x \in Q$, it follows that u is well defined. Moreover, $u(x_1 \to x_2) = (q_1(x_1 \to x_2), q_2(x_1 \to x_2)) = (q_1(x_1) \to q_1(x_1), q_2(x_1) \to q_2(x_1)) = (q_1(x_1), q_2(x_1)) \to q_1(x_2), q_2(x_2)) = u(x_1) \to u(x_2)$, that is $u \in \mathbf{Lalg}(Q, P)$. For any $x \in Q$, we have $(p_1 \circ u)(x) = p_1(u(x)) = p_1((q_1(x), q_2(x))) = q_1(x)$ and $(p_2 \circ u)(x) = p_2(u(x)) = p_2((q_1(x), q_2(x))) = q_2(x)$, that is $p_1 \circ u = q_1$ and $p_2 \circ u = q_2$. For another $u' \in \mathbf{Lalg}(Q, P)$ such that $p_1 \circ u' = q_1$ and $p_2 \circ u' = q_2$, let $u'(x) = (x_1, x_2)$. From $p_1 \circ u' = p_1 \circ u$ and

$p_2 \circ u' = p_2 \circ u$ we get $p_1(x_1, x_2) = p_1(q_1(x), q_2(x)) = q_1(x)$, $p_2(x_1, x_2) = p_2(q_1(x), q_2(x)) = q_2(x)$, hence $x_1 = q_1(x)$ and $x_1 = q_2(x)$. It follows that $u'(x) = (x_1, x_2) = (q_1(x), q_2(x)) = u(x)$ for all $x \in Q$. Thus u is unique, and we conclude that (P, p_1, p_2) is a kernel pair of f. $\square$

Example 4.16. Consider $X, Y \in \mathbf{Ob}(Lalg)$ and $f \in \mathbf{Lalg}(X, Y)$ from Example 4.12, that is $f(0) = 0$, $f(1) = 2$. We have $X \times X = \{(0,0), (0,1), (1,0), (1,1)\}$ and $P = \{(x_1, x_2) \mid f(x_1) = f(x_2)\} = \{(0,0), (1,1)\}$. Let $p_1, p_2 : P \longrightarrow X$ be the canonical projections, that is $p_1(0,0) = 0$, $p_1(1,1) = 1$, $p_2(0,0) = 0$, $p_2(1,1) = 1$. Then $p_1, p_2 \in \mathbf{Lalg}(P, X)$ and $f \circ p_1 = f \circ p_2$, hence (P, p_1, p_2) is a kernel pair of f.

Let $f : X \longrightarrow Y$ be a morphism in the category $\mathcal{C}$, and let $f \in \mathcal{C}(X, Y)$. If there exists $Z \in \mathbf{Ob}(\mathcal{C})$ and $\varphi, \psi \in \mathcal{C}(Z, X)$ such that (Y, f) is a coequalizer of the couple (φ, ψ), then we say that f is a coequalizer in $\mathcal{C}$.

Proposition 4.17. *Any surjective morphism in* **Lalg** *is a coequalizer of its kernel pair.*

Proof. Let $X, Y \in \mathbf{Ob}(Lalg)$ and let $f \in \mathbf{Lalg}(X, Y)$ be a surjective morphism. According to Theorem 4.15, f has a kernel pair (P, p_1, p_2), where $P = \{(x_1, x_2) \in X \times X \mid f(x_1) = f(x_2)\}$ and $p_1, p_2 : P \longrightarrow X$ are the canonical projections. We prove that the pair (Y, f) is a coequalizer of (p_1, p_2). Obviously, $f \circ p_1 = f \circ p_2$. Suppose that there exists $Y' \in \mathbf{Ob}(Lalg)$ and $f' \in \mathbf{Lalg}(X, Y')$ auch that $f' \circ p_1 = f' \circ p_2$. Let $y \in Y$. Since f is surjective, there exists $x \in X$ such that $f(x) = y$. Consider $u : Y \longrightarrow Y'$ defined by $u(y) = f'(x)$. If $x_1, x_2 \in X$ such that $f(x_1) = f(x_2) = y$, then $(x_1, x_2) \in P$ and $u(y) = f'(x_1) = (f' \circ p_1)(x_1, x_2) = (f' \circ p_2)(x_1, x_2) = f'(x_2)$, so that u is well defined. Consider $y_1, y_2 \in Y$, so that there exist $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. It follows that $f'(x_1) = u(y_1)$, $f'(x_2) = u(y_2)$ and $y_1 \to y_2 = f'(x_1) \to f'(x_2) = f'(x_1 \to x_2)$. We get $u(y_1 \to y_2) = f'(x_1 \to x_2) = f'(x_1) \to f'(x_2) = u(y_1) \to u(y_2)$, so that $u \in \mathbf{Lalg}(Y, Y')$. We can easy chck that $u \circ f = f'$, while u is unique, since f is an epimorphism. We conclude that f is a coequalizer of its pair kernel. $\square$

Proposition 4.18. *Any coequalizer in* **Lalg** *is a coequalizer of its kernel pair.*

Proof. Let $X, Y \in \mathbf{Ob}(Lalg)$ and let $f \in \mathbf{Lalg}(X, Y)$ be a coequalizer in **Lalg**, that is there exists $Z \in \mathbf{Ob}(\mathcal{C})$ and $\varphi, \psi \in \mathcal{C}(Z, X)$ such that f is a coequalizer of the couple (φ, ψ). According to Theorem 4.15, f has a kernel pair (P, p_1, p_2), where $P = \{(x_1, x_2) \in X \times X \mid f(x_1) = f(x_2)\}$ and $p_1, p_2 : P \longrightarrow X$ are the canonical projections. We have $f \circ p_1 = f \circ p_2$, so that it is enough to prove that for any other morphism $f' \in \mathbf{Lalg}(X, Y')$ such that $f' \circ p_1 = f' \circ p_2$, there exists a unique morphism $u \in \mathbf{Lalg}(Y, Y')$ such that $f' = u \circ f$. Since (P, p_1, p_2) is a kernel pair of f and $f \circ \varphi = f \circ \psi$, there exists a unique morphism $v \in \mathbf{Lalg}(Z, P)$ such that $\varphi = p_1 \circ v$ and $\psi = p_2 \circ v$.

$$\begin{tikzcd}
Z \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\psi"'] \arrow[d, "v"'] & X \arrow[r, "f"] \arrow[dr, "f'"'] & Y \arrow[d, "u"] \\
P \arrow[ur, shift left, "p_1"] \arrow[ur, shift right, "p_2"'] & & Y'
\end{tikzcd}$$

We have $f' \circ \varphi = (f' \circ p_1) \circ v = (f' \circ p_2) \circ v = f' \circ \psi$. Since f is a coequalizer of the couple (φ, ψ), there exists a unique morphism $u \in \mathbf{Lalg}(Y, Y')$ such that $f' = u \circ f$. We conclude that f is a coequalizer of its kernel pair (P, p_1, p_2). $\square$

Lemma 4.19. *Let $X, Y, Z \in \mathbf{Ob}(Lalg)$ and let $f \in \mathbf{Lalg}(X, Y)$, $g \in \mathbf{Lalg}(X, Z)$. If f is surjective and* $\mathrm{Ker}\,(f) \subseteq \mathrm{Ker}\,(g)$*, then there exists a unique morphism $h \in \mathbf{Lalg}(Y, Z)$ such that $h \circ f = g$.*

Proof. According to Theorem 4.15, f has a kernel pair (P, p_1, p_2), where $P = \{(x_1, x_2) \in X \times X \mid f(x_1) = f(x_2)\}$ and $p_1, p_2 : P \longrightarrow X$ are the canonical projections. Since f is surjective, by Theorem 4.17, (Y, f) is a coequalizer of (p_1, p_2). For any $(x_1, x_2) \in P$, we have $f(x_1) = f(x_2)$, so that $x_1 \to x_2, x_2 \to x_1 \in \mathrm{Ker}\,(f) \subseteq \mathrm{Ker}\,(g)$, that is $g(x_1) = g(x_2)$. It follows that $g \circ p_1 = g \circ p_2$. Since f is a coequalizer of (p_1, p_2), then there exists a unique morphism $h \in \mathbf{Lalg}(Y, Z)$ such that $h \circ f = g$. $\square$

Theorem 4.20. *Any coequalizer in* **Lalg** *is surjective.*

Proof. Let f be a coequalizer **Lalg**. According to Theorem 4.18, f is a coequalizer of its kernel pair (P, p_1, p_2), where $P = \{(x_1, x_2) \in X \times X \mid f(x_1) = f(x_2)\} = \{(x_1, x_2) \in X \times X \mid x_1 \to x_2, x_2 \to x_1 \in \mathrm{Ker}\,(f)\}$. Since $\mathrm{Ker}\,(f) \in \mathcal{ID}(X)$, then $X/\mathrm{Ker}\,(f)\} \in \mathbf{Ob}(Lalg)$, and let $p : P \longrightarrow X/\mathrm{Ker}\,(f)$ be the canonical projection. We can see that $(p \circ p_1)(x_1, x_2) = x_1/\mathrm{Ker}\,(f) = x_2/\mathrm{Ker}\,(f) = (p \circ p_2)(x_1, x_2)$, for any $(x_1, x_2) \in X \times X$, that is $p \circ p_1 = p \circ p_2$. Since (Y, f) is a coequalizer of the couple (p_1, p_2), there exists a unique morphism $u : Y \longrightarrow X/\mathrm{Ker}\,(f)$ such that $u \circ f = p$.

$$\begin{tikzcd}
P \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & X \arrow[r, "f"] \arrow[dr, "p"'] & Y \arrow[d, shift right, "u"'] \\
& & X/\mathrm{Ker}\,(f) \arrow[u, shift right, "v"']
\end{tikzcd}$$

For any $x \in \mathrm{Ker}\,(p)$ we have $p(x) = 1/\mathrm{Ker}\,(f)$, so that $p(x) = x/\mathrm{Ker}\,(f) = 1/\mathrm{Ker}\,(f)$. It follows that $x \in \mathrm{Ker}\,(f)$, that is $\mathrm{Ker}\,(p) \subseteq \mathrm{Ker}\,(f)$. According to Lemma 4.19, there exists a unique morphism $v : X/\mathrm{Ker}\,(f) \longrightarrow Y$ such that $v \circ p = f$. It follows that $(u \circ v) \circ p = u \circ f = p = 1_{X/\mathrm{Ker}\,(f)} \circ p$ and $(v \circ u) \circ f = v \circ p = f = 1_Y \circ f$. But p and f are epimorphisms (p is surjective, while f is a coequalizer), so that $u \circ v = 1_{X/\mathrm{Ker}\,(f)}$ and $v \circ u = 1_Y$. It follows that v is an isomorphism (u the inverse of v, and v the inverse of u), that is v is surjective. Hence $f = v \circ p$ is surjective. $\square$

5 Products and co-products in the Category Lalg

We prove that the category **Lalg** has products, and the subcategory **CLalg** of CL-algebras has co-products. As an example, we construct the product of two objects in **Lalg**, and finally we give an example of two objects in **Lalg** having co-product.

Let $\mathcal{C}$ be a category, and let $(X_i)_{i\in I}$ be a family of objects in $\mathcal{C}$. A *direct product* of the family $(X_i)_{i\in I}$ is a pair $(X, (p_i)_{i\in I})$, with $X \in \mathbf{Ob}(\mathcal{C})$ and $p_i \in \mathcal{C}(X, X_i)$, for any $i \in I$, such that for any other pair $(X', (p'_{i\in I}))$ with $X' \in \mathbf{Ob}(\mathcal{C})$ and $p'_i \in \mathcal{C}(X', X_i)$, there is a unique $u \in \mathcal{C}(X', X)$ such that $p_i \circ u = p'_i$, for any $i \in I$, that is the following diagram is commutative, for any $i \in I$.

$$\begin{tikzcd}
X = \prod_{i\in I} X_i \arrow[r, "p_i"] & X_i \\
X' \arrow[u, "u"] \arrow[ur, "p'_i"'] &
\end{tikzcd}$$

If the direct product of a family $(X_i)_{i\in I}$ of objects in $\mathcal{C}$ exists, then it is unique up to an isomorphism ([3, Rem. 4.6.2]), and it is denoted by $\prod_{i\in I} X_i$. The map $p_j : \prod_{i\in I} X_i \longrightarrow X_j$ will be called the j-th *canonical projection*. We say that a category $\mathcal{C}$ *has products* if there exists the direct product of any family of objects in $\mathcal{C}$.

Theorem 5.1. *The category* **Lalg** *has products.*

Proof. Let $(X_i)_{i\in I}$ be a family of objects in **Lalg** and let $X = \prod_{i\in I} X_i$ be the set of all maps $f : I \longrightarrow \bigcup_{i\in I} X_i$ such that $f(i) \in X_i$ for all $i \in I$. If 1_i and $\rightarrow_i$ are the logical unit and the implication in L-algebra X_i, we consider the map $1 : I \longrightarrow \bigcup_{i\in I} X_i$ defined by $1(i) = 1_i$. For any $f, g \in X$ define the operation $\rightarrow$ on X by $(f \rightarrow g)(i) = f(i) \rightarrow_i g(i)$, for all $i \in I$. It is easy to check that $(X, \rightarrow, 1)$ is an L-algebra, so that it is an object in **Lalg**. For $i \in I$, the projection $p_i : X \longrightarrow X_i$ is defined by $p_i(f) = f(i)$, for all $f \in X$. For any $X' \in \mathbf{Ob}(Lalg)$ and $p'_i \in \mathbf{Lalg}(X', X_i)$, define $u : X' \longrightarrow X$ by $(u(x))(i) = p'_i(x)$, for all $x \in X'$ and $i \in I$. Then we have $(u(x \rightarrow y))(i) = p'_i(x) \rightarrow_i p'_i(y) = (u(x))(i) \rightarrow_i (u(y))(i)$, for all $x, y \in X'$ and $i \in I$, that is u is an L-algebras homomorphism. Moreover, $(p_i \circ u)(x) = p_i(u(x)) = (u(x))(i) = p'_i(x)$, for all $x \in X'$, that is $p_i \circ u = p'_i$. Suppose that there exists another morphism $v : X' \longrightarrow X$ such that $p_i \circ v = p'_i$ for all $i \in I$. It follows that $(p_i \circ v)(x) = p'_i(x) = (p_i \circ u)(x)$ for all $i \in I$ and $x \in X'$. Hence $(v(x))(i) = (u(x))(i)$ for all $i \in I$, so that $v(x) = u(x)$ for all $x \in X'$, that is $v = u$. We conclude that the category **Lalg** has products. $\square$

Example 5.2. Let $X_1 = \{a, b, c, 1_1\}$, $X_2 = \{0, x, y, 1_2\}$ and let $\rightarrow_1, \rightarrow_2$ be binary operation on X_1, X_2 given in the following tables.

$\rightarrow_1$	a	b	c	1_1
a	1_1	a	c	1_1
b	1_1	1_1	c	1_1
c	1_1	1_1	1_1	1_1
1_1	a	b	c	1_1

$\rightarrow_2$	0	x	y	1_2
0	1_2	1_2	1_2	1_2
x	y	1_2	y	1_2
y	x	x	1_2	1_2
1_2	0	x	y	1_2

Then $(X_1, \rightarrow_1, 1_1), (X_2, \rightarrow_2, 1_2) \in \mathbf{Ob}(Lalg)$, and let $I = \{1, 2\}$. Let $X = \{f_1, f_2, \ldots, f_{15}, 1\}$ be the set all functions $f : I \longrightarrow X_1 \cup X_2$ with $f(1) \in X_1$, $f(2) \in X_2$, and define $p_i : X \longrightarrow X_i$, by $p_i(f) = f(i)$, for $i \in I$ (see the tables below).

i	1	2
$f_1(i)$	a	0
$f_2(i)$	a	x
$f_3(i)$	a	y
$f_4(i)$	a	1_2
$f_5(i)$	b	0
$f_6(i)$	b	x
$f_8(i)$	b	y
$f_8(i)$	b	1_2
$f_9(i)$	c	0
$f_{10}(i)$	c	x
$f_{11}(i)$	c	y
$f_{12}(i)$	c	1_2
$f_{13}(i)$	1_1	0
$f_{14}(i)$	1_1	x
$f_{15}(i)$	1_1	y
$1(i)$	1_1	1_2

f	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}	1
$p_1(f)$	a	a	a	a	b	b	b	b	c	c	c	c	1_1	1_1	1_1	1_1
$p_2(f)$	0	0	0	0	x	x	x	x	y	y	y	y	1_2	1_2	1_2	1_2

We have $(X, \rightarrow, 1) \in \mathbf{Ob}(Lalg)$ and $X_1 \prod X_2 = (X, p_1, p_2)$.

Let $\mathcal{C}$ be a category, and let $(X_i)_{i\in I}$ be a family of objects in $\mathcal{C}$. A *co-product* (also called a *direct sum*) of the family $(X_i)_{i\in I}$ is a pair $((\alpha_i)_{i\in I}), X)$, with $X \in \mathbf{Ob}(\mathcal{C})$ and $\alpha_i \in \mathcal{C}(X_i, X)$, for any $i \in I$, such that for any other pair $((\alpha'_{i\in I}), X')$ with $X' \in \mathbf{Ob}(\mathcal{C})$ and $\alpha'_i \in \mathcal{C}(X_i, X')$, there is a unique $f \in \mathcal{C}(X, X')$ such that $f \circ \alpha_i = \alpha'_i$, for any $i \in I$, that is the following diagram is commutative, for any $i \in I$.

$$\begin{tikzcd}
X_i \arrow[r, "\alpha_i"] \arrow[dr, "\alpha'_i"'] & X = \coprod_{i\in I} X_i \arrow[d, "f"] \\
& X'
\end{tikzcd}$$

If the co-product of a family $(X_i)_{i\in I}$ of objects in $\mathcal{C}$ exists, then it is unique up to an isomorphism ([3, Rem. 4.6.7]), and it is denoted by $\coprod_{i\in I} X_i$. The map $\alpha_j : X_j \longrightarrow \coprod_{i\in I} X_i$ will be called the j-th *canonical injection*. We say that a category $\mathcal{C}$ *has co-products* if there exists the co-product of any family of objects in $\mathcal{C}$.

We give the following example using an idea from [2].

Example 5.3. Let $X_1 = \{u, a, b, 1\}$, $X_2 = \{0, u\}$, $X = X_1 \cup X_2$ and let $\rightarrow_1, \rightarrow_2, \rightarrow$ be binary operations on X_1, X_2, X given in the following tables.

$\rightarrow_1$	u	a	b	1
u	1	1	1	1
a	u	1	1	1
b	u	b	1	1
1	u	a	b	1

$\rightarrow_2$	0	u
0	u	u
u	0	u

$\rightarrow$	0	u	a	b	1
0	1	1	1	1	1
u	0	1	1	1	1
a	0	u	1	1	1
b	0	u	b	1	1
1	0	u	a	b	1

Then $(X_1, \rightarrow_1, u), (X_2, \rightarrow_2, 1), (X, \rightarrow, 1) \in \mathbf{Ob}(Lalg)$, and let $I = \{1, 2\}$. Define $\alpha_1 : X_1 \longrightarrow X$ by by $\alpha_1(x) = x$ for all $x \in X_1$, and $\alpha_2 : X_2 \longrightarrow X$, by $\alpha_2(x) = x$ for all $x \in X_2$. Then (X, α_1, α_2) is the co-product of X_1 and X_2.

$$\begin{tikzcd}
& X' & \\
X_1 \arrow[ur, "\alpha'_1"] \arrow[r, "\alpha_1"'] & X \arrow[u, "f"] & X_2 \arrow[ul, "\alpha'_2"'] \arrow[l, "\alpha_2"]
\end{tikzcd}$$

Indeed, suppose that X' is another L-algebra with two homomorphisms $\alpha'_1 : X_1 \longrightarrow X'$, $\alpha'_2 : X_2 \longrightarrow X'$.

$$f(x) \quad = \begin{cases} \alpha'_1(x) & x \in X_1 \\ \alpha'_2(x) & x \in X_2. \end{cases}$$

Since α_1' and α_2' are homomorphism, then f is homomorphism. We can easily check that $f \circ \alpha_1 = \alpha_1'$ and $f \circ \alpha_2 = \alpha_2'$. Suppose that there exists another homomorphism $g : X \longrightarrow X'$ such that $g \circ \alpha_1 = \alpha_1'$ and $g \circ \alpha_2 = \alpha_2'$, that is $g(\alpha_1(x)) = f(\alpha_1(x))$ for all $x \in X_1$ and $g(\alpha_2(x)) = f(\alpha_2(x))$ for all $x \in X_2$. It follows that $g(x) = f(x)$ for all $x \in X$, hence f is unique. We conclude that (X, α_1, α_2) is the co-product of X_1 and X_2.

Example 5.4. Consider the elements $0 \leq c \leq u \leq a \leq b \leq 1$ and the sets $X_1 = \{u, a, b, 1\}$, $X_2 = \{0, u\}$, $X = X_1 \cup X_2$, $Y = \{0, c, u\}$. Let $\rightarrow_1, \rightarrow_2, \rightarrow, \rightarrow'$ be binary operations on X_1, X_2, X, Y given in the following tables.

$\rightarrow_1$	u	a	b	1
u	1	1	1	1
a	u	1	1	1
b	u	b	1	1
1	u	a	b	1

$\rightarrow_2$	0	u
0	u	u
u	0	u

$\rightarrow$	0	u	a	b	1
0	1	1	1	1	1
u	0	1	1	1	1
a	0	u	1	1	1
b	0	u	b	1	1
1	0	u	a	b	1

$\rightarrow'$	0	c	u
0	u	u	u
c	0	u	u
u	0	c	u

Then $(X_1, \rightarrow_1, 1), (X_2, \rightarrow_2, u), (X, \rightarrow, 1), (Y, \rightarrow', u) \in \mathbf{Ob}(Lalg)$. Let $\alpha_1 \in \mathbf{Lalg}(X_1, X)$ and $\alpha_2 \in \mathbf{Lalg}(X_2, X)$ defined by $\alpha_1(u) = a$, $\alpha_1(a) = \alpha_1(b) = \alpha_1(1) = 1$, $\alpha_2(0) = b$, $\alpha_2(u) = 1$. We show that the pair (X, α_1, α_2) is not a co-product of the family (X_1, X_2).
Consiter $\alpha_1' \in \mathbf{Lalg}(X_1, Y)$ and $\alpha_2' \in \mathbf{Lalg}(X_2, Y)$ defined by $\alpha_1'(u) = c$, $\alpha_1'(a) = \alpha_1(b) = \alpha_1(1) = u$, $\alpha_2'(0) = c$, $\alpha_2'(u) = u$. We must prove that there exists $f \in \mathbf{Lalg}(X, Y)$ such that $f \circ \alpha_1 = \alpha_1'$ and $f \circ \alpha_2 = \alpha_2'$.

$$\begin{tikzcd}
 & Y & \\
X_1 \arrow[ur, "\alpha_1'"] \arrow[r, "\alpha_1"'] & X \arrow[u, "f"] & X_2 \arrow[ul, "\alpha_2'"'] \arrow[l, "\alpha_2"]
\end{tikzcd}$$

The homomorphisms $\mathbf{Lalg}(X, Y)$ are given in the following table.

x	0	u	a	b	1
$f_1(x)$	0	c	u	u	u
$f_2(x)$	0	u	u	u	u
$f_3(x)$	c	u	u	u	u
$f_4(x)$	u	u	u	u	u

For any $i = 1, 2, 3, 4$, we have $(f_i \circ \alpha_1)(u) = f_i \circ (\alpha_1(u)) = f_i(a) = u \neq c = \alpha_1'(u)$, and $(f_i \circ \alpha_2)(0) = f_i \circ (\alpha_2(0)) = f_i(b) = u \neq c = \alpha_2'(0)$. It follows that $f \circ \alpha_1 \neq \alpha_1'$ and $f \circ \alpha_2 \neq \alpha_2'$, for all $f \in \mathbf{Lalg}(X, Y)$, so that the pair (X, α_1, α_2) is not a co-product of the family (X_1, X_2).

A category $\mathbf{C}'$ is a subcategory of a category $\mathbf{C}$ if the following conditions are satisfied: (i) $\mathbf{Ob}(C') \subseteq \mathbf{Ob}(C)$; (ii) $\mathbf{C}'(X, Y) \subseteq \mathbf{C}(X, Y)$, for all $X, Y \in \mathbf{Ob}(C')$; (iii) the composition of any two morphisms in $\mathbf{C}'$ is the same as their composition in $\mathbf{C}$; (iv) $\mathbf{1}_X$ is the same in $\mathbf{C}'$ as in $\mathbf{C}$, for all $X \in \mathbf{Ob}(C')$ ([3, Def. 4.1.3]). We can easily check that the category **CLalg** of CL-algebras is a subcategory of **Lalg**.

Theorem 5.5. *The subcategory* **CLalg** *of CL-algebras has co-products.*

Proof. According to [7, Prop. 2.3], any CL-algebra is a BCK-algebra, so that **CLalg** is also a subcategory of the category **BCK** of BCK-algebras. It was proved in [31] that the category **BCK** has co-products, hence the sucategory **CLalg** also has co-products. $\square$

Open problem 5.6. Investigate whether the category **Lalg** has co-products or not.

6 On the Injective Objects in the Category Lalg

In this section, we introduce the notion of divisible cyclic L-algebras and prove that the cyclic L-algebras and MV-algebras are categorial equivalent. The main result consists of proving that an object X in the category **CyLalg** of cyclic L-algebras is injective if and only if X is a complete and divisible cyclic L-algebra.
Using an idea from [12] we prove that $\{1\}$ is the only injective object in the category **Lalg**.

An object Q in a category **C** is called *injective* if for any morphism $f : X \longrightarrow Q$ and any monomorphism $g : X \longrightarrow Y$, there is a morphism $h : Y \longrightarrow Q$ such that $h \circ g = f$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Q \\ Y \arrow[ur, "h"'] & \end{tikzcd}$$

A *retraction* of a morphism $f : X \longrightarrow Y$ is a morphism $g : Y \longrightarrow X$ such that $f \circ g = Id_Y$. If f has a retraction, then f is a monomorphism ([3, Def. 4.2.6, Prop. 4.2.7]).

Lemma 6.1. *Let $(X, \rightarrow, 1)$ be an L-algebra and let $0 \notin X$. Then $(X \cup \{0\}, \rightarrow, 1)$ is an L-algebra with 0 as the smallest element, where $x \rightarrow 0 = 0$, $0 \rightarrow x = 1$, $0 \rightarrow 0 = 1$, for any $x \in X$.*

Proof. The proof is straightforward. $\square$

Lemma 6.2. *$\{1\}$ is an injective object in* **Lalg**.

Proof. Obviously, if $f : X \longrightarrow \{1\}$ is a morphism, then $f(x) = 1$, for all $x \in X$. For any monomorphism $g : X \longrightarrow Y$, define the morphism $h : Y \longrightarrow \{1\}$, by $h(y) = 1$, for all $y \in Y$. Then, for any $x \in X$ we have $(h \circ g)(x) = h(g(x)) = 1 = f(x)$, that is $h \circ g = f$. Hence $\{1\}$ is an injective object in **Lalg**. $\square$

Theorem 6.3. *An object X in* **Lalg** *is injective if and only if $X = \{1\}$.*

Proof. By Lemma 6.2, $\{1\}$ is an injective object in **Lalg**. Conversely, assume that X is an injective object in **Lalg**. Consider the L-algebra $X \cup \{0\}$ from Lemma 6.1 and let $i : X \longrightarrow X \cup \{0\}$ be the inclusion map. Obviously i is injective, so that i is a monomorphism. Since X is an injective object, there exists a retraction $r : X \cup \{0\} \longrightarrow X$ such that $r \circ i = Id_X$.

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "Id_X"'] & X \cup \{0\} \arrow[dl, "r"] \\ X & \end{tikzcd}$$

Then $r(x) = x$ for any $x \in X$, and let $y = r(0)$. It follows that $y = r(0) = r(y \to 0) = r(y) \to r(0) = y \to y = 1$, that is $r(0) = 1$. For any $x \in X$, we have $1 = r(0) = r(0 \to x) = r(0) \to r(x) = 1 \to x = x$. We conclude that $X = \{1\}$. $\square$

Theorem 6.4. *The cyclic L-algebras and MV-algebras are categorial equivalent.*

Proof. Denote by **MValg** and **CyLalg** the categories of MV-algebras and cyclic L-algebras, respectively. In order to prove the categorial equivalence, with the notations from Section 3 we define two functors $\mathbf{\Phi} : \mathbf{MValg} \longrightarrow \mathbf{CyLalg}$, $\mathbf{\Psi} : \mathbf{CyLalg} \longrightarrow \mathbf{MValg}$. by $\mathbf{\Phi}(X, \oplus, 0) = (X, \to, 0, 1)$, $\mathbf{\Psi}(X, \to, 0, 1) = (X, \oplus, 0)$, $\mathbf{\Phi}(f)(x) = f(x)$, $\mathbf{\Psi}(g)(x) = g(x)$, for any $(X, \oplus, 0) \in \mathbf{Ob}(MValg)$, $(X, \to, 0, 1) \in \mathbf{Ob}(CyLalg)$, $f \in \mathbf{MValg}(X, Y)$, $g \in \mathbf{CyLalg}(X, Y)$, $x \in X$. By Theorem 3.8, $\mathbf{\Phi}$ and $\mathbf{\Psi}$ are mutually inverse, hence **MValg** and **CyLalg** are categorial equivalent. $\square$

Let $(X, \oplus, 0)$ be an MV-algebra. For any $x \in X$ and $n \in \mathbb{N}$, define $0x = 0$ and $nx = x \oplus (n-1)x$, for $n \geq 1$. An MV-algebra X is called *divisible* if for any $a \in X$ and for any $n \in \mathbb{N}$, there is $x \in X$ such that $nx = a$ and $a^- \oplus (n-1)x = x^-$.

Theorem 6.5. ([27]) *For any MV-algebra X the following are equivalent:*
(a) X is an injective object in the category **MValg***;*
(b) X is complete and divisibile MV-algebra.

Definition 6.6. A cyclic L-algebra $(X, \to, 0, 1)$ is called *divisible* if its corresponding MV-algebra $(X, \oplus, 0)$ is divisible.

Theorem 6.7. *For any cyclic L-algebra X the following are equivalent:*
(a) X is an injective object in the category **CyLalg***;*
(b) X is a complete and divisibile cyclic L-algebra.

Proof. It follows by Theorems 6.7 and 6.4. $\square$

7 Concluding Remarks

Studying the L-algebras is a topic of great current interest; motivated by this fact, in this paper we define and study the category **Lalg** of L-algebras. We prove that this category has equalizers, coequalizers, kernel pairs and products, and we investigate the existence of injective objects in **Lalg**. We prove that an object of the subcategory of cyclic L-algebras is injective if and only it is a complete and divisible cyclic L-algebra. It was proved in [7, Rem. 4.12] that any Hilbert algebra is an L-algebra, so the category **Halg** of Hilbert algebras is a subcategory of **Lalg**. According to [13], the category **Halg** has co-products. We give an example of two L-algebras having a co-product, but we leave as an open problem whether the category of L-algebras has co-products or not.

Dvurečenskij and Zahiri studied the epicomplete objects in the category of MV-algebras ([10]), and they found a relation between injective MV-algebras and epicomplete MV-algebras. As another topic of research, one could investigate the epicomplete objects in various subcategories of **Lalg**.

Conflict of Interest: The author declares that there are no conflict of interest.

References

[1] S. S. Ahn, On injective BCK-algebras, *Honam Math. J.,* 29 (2007), 289-297.

[2] N. Akhlaghinia, R. A. Borzooei, M. A. Kologani and X. L. Xin, On the category of EQ-algebras, *Bull. Sect. Logic*, https://doi.org/10.18778/0138-0680.2021.01, (2021).

[3] B. Buşneag, Categories of algebraic logic, *Editura Academiei Romane, Bucharest,* (2006).

[4] B. Buşneag and M. Ghiţă, Some properties of epimorphisms of Hilbert algebras, *Centr. Eur. J. Math.,* 8 (2010), 41-52.

[5] C. C. Chang, Algebraic analysis of many-valued logic, *Trans. Amer. Math. Soc.,* 88 (1958), 467-490.

[6] M. A. Chaudhry and H. Fakhar-Ud-Din, Some categorical aspects of BCH-algebras, *IJMMS,* 27 (2003), 1739-1750.

[7] L. C. Ciungu, Results in L-algebras, *Algebra Univ.,* 82(1) (2021), Paper no. 7.

[8] L. C. Ciungu, Quantifiers on L-algebras, *Math. Slovaca*, accepted for publication.

[9] A. Dvurečenskij and S. Pulmannova, New trends in Quantum Structures, *Kluwer Academic Publishers, Dordrecht, Ister Science, Bratislava,* (2000).

[10] A. Dvurečenskij and O. Zahiri, On epicomplete MV-algebras, *FLAP,* 5 (2018), 165-184.

[11] G. Dymek, On the category of pseudo BCI-algebras, *Demonstr. Math.,* 4 (2013), 631-644.

[12] G. Dymek, An injective pseudo-BCI algebra is trivial, *Discuss. Math. Gen. Algebra Appl.,* 39 (2019) 221-229.

[13] M. Ghiţă, Some categorical properties of Hilbert algebras, *An. Univ. Craiova Ser. Mat. Inform.,* 36 (2009), 95-104.

[14] X. J. Hua, State L-algebras and derivations of L-algebras, *Soft Comput.,* 25 (2021), 4201-4212.

[15] A. Iorgulescu, Classes of BCK-algebras - Part III, *Preprint Series of the Institute of Mathematics of the Romanian Academy,* 3 (2004), 1-42.

[16] B. Mitchell, Theory of Categories, *Pure and Applied Mathematics, vol. XVII, Academic Press, New York,* (1965).

[17] N. Popescu and A. Radu, Theory of categories and sheaves, *Editura Ştiinţifică, Bucharest, (in Romanian),* (1971).

[18] L. Popescu and N. Popescu, Theory of categories, *Editura Academiei Romane, Bucharest and Sijthoff & Noordhoff International Publishers,* (1979).

[19] W. Rump, A decomposition theorem for square-free unitary solutions of the quantum Yang–Baxter equations, *Adv. Math.,* 193 (2005), 40-55.

[20] W. Rump, L-algebras, self-similarity, and ℓ-groups, *J. Algebra,* 320 (2008), 2328-2348.

[21] W. Rump and Y. Yang, Intervals in ℓ-groups as L-algebras, *Algebra Univ.,* 67 (2012), 121-130.

[22] W. Rump, Right ℓ-groups, geometric Garside groups, and solutions of the quantum Yang–Baxter equation, *J. Algebra,* 439 (2015), 470-510.

[23] W. Rump, The structure group of an L-algebra is torsion-free, *J. Group Theory,* 20 (2017), 309-324.

[24] W. Rump, The L-algebra of Hurwitz primes, *J. Number Theory,* 190 (2018), 394-413.

[25] W. Rump, Von Neumann algebras, L-algebras, Baer*-monoids, and Garside groups, *Forum Math.,* 30 (2018), 973-995.

[26] W. Rump, The structure group of a generalized orthomodular lattice, *Stud. Logica,* 106 (2018), 85-100.

[27] E. Turunen, Mathematics Behind Fuzzy Logic, *Physica-Verlag,* (1999).

[28] Y. Wu, J. Wang and Y. Yang, Lattice-ordered effect algebras and L-algebras, *Fuzzy Sets Syst.,* 369 (2019), 103-113.

[29] Y. Wu and Y. Yang, Orthomodular lattices as L-algebras, *Soft Comput.,* 24 (2020), 14391-14400.

[30] Y. Yang and W. Rump, Pseudo-MV algebras as L-algebras, *J. Mult.-Valued Logic Soft Comput.,* 19 (2012), 621-632.

[31] H. Yutani, Colimits in the category of BCK-algebras, *Math. Japonica,* 30 (1985), 527-534.

[32] S. M. A. Zaidi and S. Khan, The Category of p-semisimple BCI-algebra, *Kyungpook Math. J.,* 43 (2003), 279-291.

Lavinia Corina Ciungu
Department of Mathematics
St Francis College
Brooklyn Heights, USA
E-mail: lciungu@sfc.edu