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On the Lattice of Filters of Intuitionistic Linear Algebras

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Abstract. In this paper, we investigate the filter theory of Intuitionistic Linear Algebra (IL-algebra, in short) with emphasis on the lattice of filters of IL-algebras and relationship between filters and congruences on IL-algebras. We characterize the filter generated by a subset and give some related properties. The prime filter for IL-algebras is characterized and the prime filter theorem for IL-algebra is established. We get that the lattice $(F(\mathbf{L}), \subseteq)$ of filters of an IL-algebra $\mathbf{L}$ is algebraic, Brouwerian, pseudocomplemented and endowed with the structure of Heyting algebra. We prove that the lattice of congruences and that of filters of any IL-algebra are isomorphic.

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Keywords and Phrases: IL-algebra, Filter, Prime filter, Congruence, Residuated lattice.

1 Introduction

Intuitionistic Linear Logic is the Intuitionistic restriction of Linear Logic, which is a substructural logic proposed by Jean-Yves Girard in [7]. Intuitionistic Linear Algebra (IL-algebra, in short) was initiated by Troelstra in [14] as an algebraic counterpart of intuitionistic Linear Logic. IL-algebra can be found as a F_{Le}-algebra (see [6, 14]). These algebras are generalizations of residuated lattices. The main difference between IL-algebras and residuated lattices are that the top of the lattice and the monoidal identity are different in IL-algebras. Some properties of IL-algebras are studied in [4, 14].

Filters, also called deductive systems play a crucial role in the study of algebraic structures and associated logic [13]. Indeed, filters correspond in logic to sets of provable formulas and have been widely studied in residuated lattices [1, 3, 5]. For an IL-algebra $\mathbf{L}$, the set $Spec(\mathbf{L})$ of all prime filters of $\mathbf{L}$, can be endowed with the Zariski topology $\tau_{\mathbf{L}}$ and $(Spec(\mathbf{L}), \tau_{\mathbf{L}})$ becomes a compact topological space as in the case of Boolean algebras (see [2]). To make this possible, it is useful to characterize prime filters and establish a prime filter theorem for IL-algebras. On the other hand, congruences play an important role in the representation, decomposition as well as classification of algebraic structures. Hence, it is necessary to study filters induced by congruences, since they imply a lot of properties on congruences.

The filter theory of IL-algebras has been introduced by Islam et al in [4, 10] and few interesting properties have been obtained. Very recently, the same authors studied the concept of fuzzy filters of IL-algebras in [11]. Using a system of affine filters, the concept of linear topology on IL-algebras is introduced by Islam et al. in [12]. Many properties of the filters in IL-algebras have not been investigated. The purpose of this paper is to investigate the filter theory of IL-algebras with an emphasis on the lattice of filters and the relationship between filters and congruences on IL-algebras.

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The structure of the paper is organized as follows: In Section 2, we recall some facts about IL-algebras and extend some existing results on residuated lattices to IL-algebras. In section 3, we characterize the filter generated by a subset and establish some related properties. Moreover, we state the prime filter theorem for IL-algebras and some properties of prime filters on IL-algebras. In the same section, we describe some properties of the lattice of filters of an arbitrary IL-algebra and we obtain that this lattice is pseudocomplemented, algebraic, Brouwerian and endowed with a structure of Heyting algebra. Finally, we introduce the negative cone $\mathbf{L}^-$ of an IL-algebra $\mathbf{L}$, we show that $\mathbf{L}^-$ is a residuated lattice, and the lattice $\mathcal{F}(\mathbf{L}^-)$ of its filters and the lattice $\mathcal{F}(\mathbf{L})$ of filters of $\mathbf{L}$ are isomorphic. In Section 4, we establish the relationship between congruences and filters on IL-algebras. We obtain that, for any IL-algebra $\mathbf{L}$, the lattice of its filters $\mathcal{F}(\mathbf{L})$ and that of its congruences $Con(\mathbf{L})$ are isomorphic.

2 Intuitionistic linear algebra

In this section, we define IL-algebras and recall some properties which will be used in the rest of the paper.

Definition 2.1. (See [11]) *An Intuitionistic Linear algebra (IL-algebra, in short) is an algebraic system* $\boldsymbol{L} = (L, \vee, \wedge, *, \rightarrow, e, \perp, \top)$ *of type* $(2, 2, 2, 2, 0, 0, 0)$ *which satisfies the following conditions:*

1. $(L, \vee, \wedge, \perp, \top)$ is a bounded lattice with least element $\perp$ and greatest element $\top$.
2. $(L, *, e)$ is a commutative monoid with unit e.
3. For any $x, y, z \in L, x * y \leq z$ if and only if $x \leq y \rightarrow z$ (residuation property).

In what follows, we denote by $\mathbf{L}$ an IL-algebra $(L, \vee, \wedge, *, \rightarrow, e, \perp, \top)$. Let $\mathbf{L}$ be an IL-algebra, in the case $e = \top$, $\mathbf{L}$ becomes a residuated lattice. The order $\leq$ in $\mathbf{L}$ is defined as follows: $x \leq y$ iff $x \wedge y = x$ (equivalently $x \leq y$ iff $x \vee y = y$). Let $n \geq 1$ be a natural number. For any $x \in L$, we define $x^n = x^{n-1} * x$ and $x^0 = e$.

The following proposition provides some known rules of calculus in IL-algebras.

Proposition 2.2. (See [9, 11]) *Let* $\boldsymbol{L}$ *be an IL-algebra,* I *a non-empty set. For all* $x, y, x_1, y_1, y_i, z \in L, i \in I$, *the following statements hold:*

(c_1) $x * (y \vee z) = (x * y) \vee (x * z)$, *if* $\bigvee_{i \in I} y_i$ *exists, then* $x * (\bigvee_{i \in I} y_i) = \bigvee_{i \in I} (x * y_i)$.

(c_2) $\perp \rightarrow \perp = \top$.

(c_3) $\top * \top = \top$.

(c_4) *If* $x, y \leq e$, *then* $x * y \leq x \wedge y$.

(c_5) *If* $e \leq x, y$, *then* $x \vee y \leq x * y$.

(c_6) $(x \rightarrow y) * (y \rightarrow z) \leq (x \rightarrow z)$.

(c_7) $e \rightarrow x = x$.

(c_8) *If* $x \leq y, x_1 \leq y_1$, *then* $x * x_1 \leq y * y_1$ *and* $y \rightarrow x_1 \leq x \rightarrow y_1$.

(c_9) $x \rightarrow (y \rightarrow z) = (x * y) \rightarrow z$.

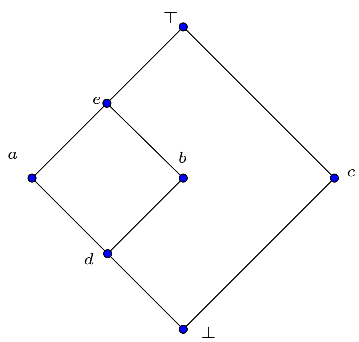
(c_{10}) $x * (x \rightarrow y) \leq y$.

(c_{11}) $e \leq x \rightarrow x$.

(c_{12}) $(z \to x) \wedge (z \to y) = z \to (x \wedge y)$.

(c_{13}) $(x \to z) \wedge (y \to z) = (x \vee y) \to z$.

Example 2.3 ([11], Example 2)**.** Let $\mathbf{L}_7$ with $L_7 = \{\perp, a, b, c, d, e, \top\}$, where the lattice diagram is given in $Fig1$, $*$ and $\to$ tables are given below:



$Fig1$

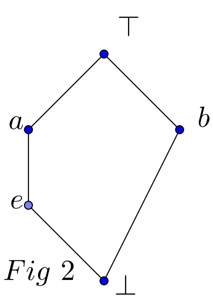
$*$	$\perp$	a	b	c	d	e	$\top$
$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$
a	$\perp$	a	d	c	d	a	$\top$
b	$\perp$	d	b	c	d	b	$\top$
c	$\perp$	c	c	d	c	c	$\top$
d	$\perp$	d	d	c	d	d	$\top$
e	$\perp$	a	b	c	d	e	$\top$
$\top$	$\perp$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

,

$\to$	$\perp$	a	b	c	d	e	$\top$
$\perp$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
a	$\perp$	e	b	c	b	e	$\top$
b	$\perp$	a	e	c	a	e	$\top$
c	$\perp$	c	c	e	c	e	$\top$
d	$\perp$	e	e	c	e	e	$\top$
e	$\perp$	a	b	c	d	e	$\top$
$\top$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\top$

Then, $\mathbf{L}_7$ is an IL-algebra which is not a residuated lattice, since we have $b \wedge c < b * c$.

Example 2.4. ([12], Example 1) Let $\mathbf{L}_5$ with $L_5 = \{\perp, a, b, e, \top\}$. The lattice diagram is given in Fig 2, and $*$ and $\to$ tables are given below:



$Fig\ 2$

$*$	$\perp$	a	b	e	$\top$
$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$
a	$\perp$	$\top$	$\top$	a	$\top$
b	$\perp$	$\top$	$\top$	b	$\top$
e	$\perp$	a	b	e	$\top$
$\top$	$\perp$	$\top$	$\top$	$\top$	$\top$

$\to$	$\perp$	a	b	e	$\top$
$\perp$	$\top$	$\top$	$\top$	$\top$	$\top$
a	$\perp$	e	$\perp$	$\perp$	$\top$
b	$\perp$	$\perp$	e	$\perp$	$\top$
e	$\perp$	a	b	e	$\top$
$\top$	$\perp$	$\perp$	$\perp$	$\perp$	$\top$

This IL-algebra is not a residuated lattice, since $a * c = a \not\leq a \wedge c = \perp$.

Example 2.5. If $(L, \vee, \wedge, *, \to, e, \perp, \top)$ is an IL-algebra and X is a non-empty set, then the set $L^X := \{f : X \to L \mid f \text{ is a map}\}$ becomes an IL-algebra $(L^X, \vee, \wedge, *, \to, \underline{\perp}, \underline{e}, \underline{\top})$ with the operations defined pointwise and $\underline{\perp}, \underline{\top}, \underline{e} : X \to L$ are the constant functions associated with $\perp, \top, e$.

In all this paper, we assume that $\mathbf{L}$ is an IL-algebra in which $e \neq \top$. In what follows, we state some useful algebraic properties of IL- algebras that are generalization of those existing in residuated lattices (see [3]).

Lemma 2.6. *Let $\boldsymbol{L}$ be an IL-algebra. For any $g, h, k \in L$ we have*
(c_{14}) $\quad (g \wedge e) \vee ((h \wedge e) * (k \wedge e)) \geq ((g \wedge e) \vee (h \wedge e)) * ((g \wedge e) \vee (k \wedge e))$.

Proof. Let $g, h, k \in L$, we have

$$\begin{aligned}
& [(g \wedge e) \vee (h \wedge e)] * [(g \wedge e) \vee (k \wedge e)] \\
\overset{(c_1)}{=} \; & [((g \wedge e) \vee (h \wedge e)) * (g \wedge e)] \vee [((g \wedge e) \vee (h \wedge e)) * (k \wedge e)] \\
\overset{(c_1)}{=} \; & [((g \wedge e) * (g \wedge e)) \vee ((g \wedge e) * (h \wedge e))] \vee [((g \wedge e) * (k \wedge e)) \vee ((h \wedge e) * (k \wedge e))] \\
\overset{(c_4)}{\leq} \; & [(g \wedge e) \vee (g \wedge h \wedge e)] \vee [(g \wedge k \wedge e) \vee ((h \wedge e) * (k \wedge e))] \\
\overset{(c_4)}{\leq} \; & (g \wedge e) \vee [(g \wedge e) \vee ((h \wedge e) * (k \wedge e))] \\
\leq \; & (g \wedge e) \vee ((h \wedge e) * (k \wedge e)) \text{ (by associative and idempotent laws of } \vee \text{).}
\end{aligned}$$

□

Corollary 2.7. *Let $\boldsymbol{L}$ be an IL-algebra. For any positive integer $n \geq 2$ and $g, h_1, \ldots, h_n \in L$, the following statement holds:*

$$(c_{15}) \; (g \wedge e) \vee [(h_1 \wedge e) * \ldots * (h_n \wedge e)] \geq [(g \wedge e) \vee (h_1 \wedge e)] * \ldots * [(g \wedge e) \vee (h_n \wedge e)].$$

Proof. Follows from Lemma 2.6 and induction on n. □

Corollary 2.8. *If $\boldsymbol{L}$ is an IL-algebra and $g, h \in L$, $n \geq 1$, then*

$$(c_{16}) \quad (g \wedge e) \vee ((h \wedge e)^n) \geq ((g \wedge e) \vee (h \wedge e))^n.$$

Proof. It is a consequence of Corollary 2.7 taking $h_i = h$. □

Corollary 2.9. *If $\boldsymbol{L}$ is an IL-algebra and $g, h \in L$, $m, n \geq 1$, then*

$$(c_{17}) \quad (g \wedge e)^n \vee (h \wedge e)^m \geq ((g \wedge e) \vee (h \wedge e))^{mn}.$$

Proof. Let $g, h \in L$, then by Corollary 2.8 we have

$$(g \wedge e)^n \vee (h \wedge e)^m \geq ((g \wedge e)^n \vee (h \wedge e))^m.$$

Since $(g \wedge e)^n \vee (h \wedge e) \geq ((g \wedge e) \vee (h \wedge e))^m$, we get by (c_8)

$$((g \wedge e)^n \vee (h \wedge e))^m \geq ((g \wedge e) \vee (h \wedge e))^{mn}.$$

Hence $(g \wedge e)^m \vee (h \wedge e)^n \geq ((g \wedge e) \vee (h \wedge e))^{mn}$. □

Lemma 2.10. *For all $a, b, c \in L$, the following inequality holds:*

$$(c_{18}) \qquad a \to b \leq (a * c) \to (b * c).$$

Proof. Let $a, b, c \in L$, by (c_{10}) $a * (a \to b) \leq b$, and applying (c_8) we get $(a * c) * (a \to b) \leq b * c$. Using the associativity of $*$ and the residuation property we obtain $a \to b \leq (a * c) \to (b * c)$. □

3 The Lattice of filters in IL-algebras

In order to study the properties of the set of filters of an IL-algebra, the characterization of a filter generated by a subset plays a central role. In this section, we first characterize the filter generated by a subset and establish some related properties. Moreover we state the prime filter theorem for IL-algebras.

3.1 Filters generated by a subset

In this subsection, we study filters of IL-algebras with respect to their order structures and generating subsets.

Definition 3.1. (See [11]) *Let $\mathbf{L}$ be an IL-algebra. A nonempty subset F of $\mathbf{L}$ is called a filter if the following conditions are satisfied:*

1. $e \in F$.
2. If $x, y \in F$, then $x * y \in F$ and $x \wedge y \in F$.
3. If $x \leq y$ and $x \in F$, then $y \in F$.

Definition 3.2. (See [12]) *Let $\mathbf{L}$ be an IL-algebra. A nonempty subset D of L is called **deductive system** if the following conditions are satisfied:*

1. $e \in D$.
2. If $e \leq x$, then $x \in F$.
3. If $x, x \to y \in F$, then $y \in F$.

It is easy to check that filters on IL-algebras are deductive systems. A filter F of $\mathbf{L}$ is called **proper** if $F \neq L$, in that case $\perp \notin F$. A filter F is called **maximal** if if is proper and is not contained in another proper filter of $\mathbf{L}$.

Example 3.3. Consider the IL-algebra $\mathbf{L}_7$. It is easy to see that the sets: $F_1 = \{d, a, b, e, \top\}, F_2 = \{a, e, \top\}$, $F_3 = \{b, e, \top\}$, $F_4 = \{e, \top\}$ and L_7 are filters of $\mathbf{L}_7$. Clearly F_1 is a maximal filter.

Proposition 3.4. (See [11], Proposition 2) *If F is a filter of an IL-algebra $\mathbf{L}$, then the following implication holds for all $x, y \in L$.*

$$(c_{19}) \qquad \text{If } x \leq y, \text{ then } x \to y \in F.$$

Remark 3.5. From (c_{10}), $x * (x \to y) \leq y$. Hence, taking $x = e$ and $y = \perp$ we get $\perp \leq e * (e \to \perp) \leq \perp$, that is $\perp \leq e \to \perp \leq \perp$. Thus $(a) \quad e \to \perp = \perp$.

For any $x \in L$, we denote by $\neg x$ the negation of x in L defined by $\neg x = x \to \perp$. An important subset of lattice with zero is the dense set. For an IL-algebra $\mathbf{L}$ we consider the set

$$D(L) = \{x \in L \mid \neg(x \wedge e) = \perp\}.$$

Proposition 3.6. *Let $\mathbf{L}$ be an IL-algebra and $D(L)$ as above. Then $D(L)$ is a proper filter of $\mathbf{L}$.*

Proof. From Remark 3.5 we have $\neg(e \wedge e) = e \to \perp = \perp$, therefore $e \in D(L)$. Assume that $x \in D(L)$ and $x \leq y$, then $x \wedge e \leq y \wedge e$ and $(x \wedge e) \to \perp = \perp$ (1). Using (c_8) and (1) we get $(y \wedge e) \to \perp \leq (x \wedge e) \to \perp$, therefore $\neg(y \wedge e) = \perp$. Hence $y \in D(L)$.

Let $x, y \in D(L)$, then $(x \wedge e) \to \perp = \perp$ (2) and $(y \wedge e) \to \perp = \perp$ (3). We have to show that $x * y, x \wedge y \in D(L)$. Since $x \wedge e \leq x$ and $y \wedge e \leq y$, by (c_8) and (c_4) we have $(x \wedge e) * (y \wedge e) \leq x * y,\ x \wedge y$. Clearly we have $(x \wedge e) * (y \wedge e) \leq e$. Therefore $[(x \wedge e) * (y \wedge e)] \wedge e = (x \wedge e) * (y \wedge e)$. We have

$[(x \wedge e) * (y \wedge e)] \to \perp \overset{(c_9)}{=} (x \wedge e) \to [(y \wedge e) \to \perp] \overset{(3)}{=} (x \wedge e) \to \perp \overset{(2)}{=} \perp$. Therefore $(x \wedge e) * (y \wedge e) \in D(L)$. Since $(x \wedge e) * (y \wedge e) \leq x * y,\ x \wedge y$, we deduce that $x * y, x \wedge y \in D(L)$. Hence $D(L)$ is a filter. Since $\neg(\perp \wedge e) = \perp \to \perp = \top \neq \perp$, we have $\perp \notin D(L)$. Thus $D(L)$ is a proper filter. $\square$

Example 3.7. For the IL-algebra $\mathbf{L}_7$, $D(L_7) = \{d, a, b, e, \top\}$ is a maximal filter.

The elements of $D(L)$ are called **dense elements** of $\mathbf{L}$.

To show that the collection of filters of $\mathbf{L}$ forms a complete lattice, we need the following lemma.

Lemma 3.8. *Let $\boldsymbol{L}$ be an IL-algebra and K be a non-emty set. If $(F_k)_{k\in K}$ is a family of filters of $\boldsymbol{L}$, then $F := \bigcap_{k\in K} F_k$ is a filter of $\boldsymbol{L}$.*

Proof. Straightforward. $\square$

We denote by $F(\mathbf{L})$ the set of filters of $\mathbf{L}$. From Lemma 3.8 $(F(\mathbf{L}), \subseteq)$ is a complete lattice denoted by $\mathcal{F}(\mathbf{L})$. For every subset $S \subseteq L$, the smallest filter of $\mathbf{L}$ containing S (i.e. the intersection of all filters $F \in F(\mathbf{L})$ such that $S \subseteq F$) denoted by $\langle S\rangle$, is called the **filter generated by** S. A **principal filter** is a filter generated by a singleton and is denoted by $\langle a\rangle$. For a filter F of $\mathbf{L}$ and $x \in L$ we set $F(x)$ for the filter generated by $F \cup \{x\}$. For $F_1, F_2 \in F(\mathbf{L})$, we define $F_1 \wedge F_2 = F_1 \cap F_2$ and $F_1 \vee F_2 = \langle F_1 \cup F_2\rangle$. Clearly $(F(\mathbf{L}), \wedge, \vee)$ is a complete lattice. For a more precise characterization of this lattice we need an explicit characterization of the filter generated by a subset. For any $a, b \in L$ with $a \leq b$ we set $[a, b] = \{x \in L \mid a \leq x \leq b\}$ and $[a, b[= \{x \in L \mid a \leq x < b\}$.

For each non-empty subset S of L, the filter generated by S and an explicit characterization of the join of two filters F_1, F_2 of $\mathbf{L}$ is described in the following proposition.

Proposition 3.9. *Let $\boldsymbol{L}$ be an IL-algebra, $F, F_1, F_2 \in F(\boldsymbol{L})$, $a \in L$ and $S \subseteq L, S \neq \emptyset$. Then the following statements hold:*

1. $\langle a\rangle = \{x \in L \mid (a \wedge e)^n \leq x \wedge e \textit{ for some } n \geq e\}$.
2. $\langle \emptyset\rangle = \langle e\rangle = \{x \in L \mid e \leq x\}$.
3. $\langle S\rangle = \{x \in L \mid (s_1 \wedge e) * \ldots * (s_n \wedge e) \leq x \wedge e, \textit{ for some } s_1, \ldots, s_n \in S, n \geq 1\}$.
4. $F_1 \vee F_2 = \langle F_1 \cup F_2\rangle = \{x \in L \mid \exists f_1 \in F_1, f_2 \in F_2, (f_1 \wedge e) * (f_2 \wedge e) \leq x \wedge e\}$.
5. $\langle F \cup \{a\}\rangle = \{x \in L \mid \exists f \in F, n \geq 1, (f \wedge e) * (a \wedge e)^n \leq x \wedge e\}$
   $= \{x \in L : (a \wedge e)^n \to (x \wedge e) \in F\}$.

Proof.

(1) Let

$$J = \{x \in L \mid \exists n \geq 1, (a \wedge e)^n \leq x \wedge e\}.$$

It is easy to show that J is a filter of $\mathbf{L}$ containing a. Let M be a filter of L such that $a \in M$. We will show that $J \subseteq M$. Let $x \in J$, then there exists $n \geq 1$ such that $(a \wedge e)^n \leq x \wedge e$. Since M is a filter and $a \in M$, we get $(a \wedge e)^n \in M$, hence $x \in M$ and $J \subseteq M$. Thus $J = \langle a\rangle$.

(2) For $a = e$ in (1) we get $\langle e\rangle = \{x \in L \mid e \leq x\}$. Let F be a filter of $\mathbf{L}$, then $e \in F$ and for any $x \in \langle e\rangle$, $e \leq x$, hence $x \in F$. Therefore $\langle \emptyset\rangle = [e, \top] \subseteq F$. Thus $\langle e\rangle = \langle \emptyset\rangle = \{x \in L \mid e \leq x\}$.

(3) Let $S \neq \emptyset$. Set

$$K = \{x \in L \mid (s_1 \wedge e) * \ldots * (s_n \wedge e) \leq x \wedge e, \text{ for some } s_1, \ldots, s_n \in S, n \geq 1\}.$$

We have to show that K is the least filter of L containing S.

For all $x \in S$ we have $x \wedge e \leq x \wedge e$ and $x \wedge e \leq e \wedge e$, hence $S \subseteq K$ and $e \in K$. Let $x \in K$ and $y \in L$ such that $x \leq y$. Then there exist $n \geq 1$ and $s_1, \ldots, s_n$ such that $(s_1 \wedge e) * \ldots * (s_n \wedge e) \leq x \wedge e \leq y \wedge e$, therefore $y \in K$. Assume that $x, y \in K$, then there exist $n, m \geq 1$ and $x_1, \ldots, x_n, y_1, \ldots, y_m \in S$ such that

$$(x_1 \wedge e) * \ldots * (x_n \wedge e) \leq (x \wedge e) \quad (1.1) \text{ and } (y_1 \wedge e) * \ldots * (y_n \wedge e) \leq (y \wedge e) \quad (1.2).$$

From (1.1), (1.2) combining with (c_7) we get

$$(x_1 \wedge e) * \ldots * (x_n \wedge e) * (y_1 \wedge e) * \ldots * (y_m \wedge e) \leq (x \wedge e) * (y \wedge e).$$

Since $x \wedge e, y \wedge e \leq e$, by (c_8) we have $(x \wedge e) * (y \wedge e) \leq (x \wedge y) \wedge e, (x * y) \wedge e$. Therefore $x \wedge y, x * y \in K$ and K is a filter containing S.

Let G be a filter of L containing S, we have to show that $K \subseteq G$. Let $x \in K$, then there exist $n \geq 1, s_1, \ldots, s_n \in S$ such that $(s_1 \wedge e) * \ldots * (s_n \wedge e) \leq x \wedge e$. Sinc G is a filter and $S \subseteq G$, we get $(s_1 \wedge e) * \ldots * (s_n \wedge e) \in G$, therefore $x \wedge e \in G$. Hence $x \in G$ and $K \subseteq G$. Thus $J = \langle S \rangle$.

(4) Set

$$F := \{x \in L \mid \exists f_1 \in F_1, f_2 \in F_2, (f_1 \wedge e) * (f_2 \wedge e) \leq x \wedge e\}.$$

We have to show that $F = F_1 \vee F_2$. It is easy to see that $F_1 \cup F_2 \subseteq F$. A similar argument used in the proof of (3) shows that F is a filter. Hence F is a filter containing $F_1 \cup F_2$. Let G be a filter such that $F_1 \cup F_2 \subseteq G$. Let $x \in F$, then there are $f_1 \in F_1, f_2 \in F_2$ such that $(f_1 \wedge e) * (f_2 \wedge e) \leq x \wedge e$. Since $F_1 \cup F_2 \subseteq G$ and G is a filter, we have $(f_1 \wedge e) * (f_2 \wedge e) \in G$, hence $x \wedge e \in G$. Therefore $x \in G$. Thus $F \subseteq G$ and $F = F_1 \vee F_2$.

(5) Let $F \in F(\mathbf{L})$ and $a \in L$. Set

$$J = \{x \in L \mid \exists f \in F, n \geq 1, (f \wedge e) * (a \wedge e)^n \leq x \wedge e\}.$$

By (4) and (1) $F \vee \langle a \rangle = \langle F \cup \langle a \rangle \rangle = J$. It remains to show that $\langle F \cup \{a\} \rangle = \langle F \cup \langle a \rangle \rangle$. Clearly we have $\langle F \cup \{a\} \rangle \subseteq \langle F \cup \langle a \rangle \rangle$. Let $x \in \langle F \cup \langle a \rangle \rangle$, then from (4) and (1) there are $f \in F$ and an integer $n \geq 1$ such that $(f \wedge e) * (a \wedge e)^n \leq x \wedge e$. Since $f, a \in F \cup \{a\}$, using (3) we deduce that $x \in \langle F \cup \{a\} \rangle$, therefore $\langle F \cup \langle a \rangle \rangle \subseteq F(a)$. Thus $F(a) = \langle F \cup \langle a \rangle \rangle = J$. Set

$$K = \{x \in L \mid \exists n \geq 1, (a \wedge e)^n \to (x \wedge e) \in F\}.$$

Our aim is to show that $K = J$. Let $x \in J = \langle F \cup \{a\} \rangle$, then by (5) there exist $f \in F$ and an integer $n \geq 1$ such that $(f \wedge e) * (a \wedge e)^n \leq x \wedge e$. Applying residuation law we obtain $(f \wedge e) \leq (a \wedge e)^n \to (x \wedge e)$. Since $f \in F$ and F is a filter, we deduce that $(a \wedge e) \to (x \wedge e) \in F$, therefore $x \in J$. Hence $J \subseteq K$.

Conversely, assume that $x \in K$. Then there exists $n \geq 1$ such that $(a \wedge e)^n \to (x \wedge e) \in F$. Set $f = (a \wedge e)^n \to (x \wedge e)$. By (c_{10}) we have

$$(a \wedge e)^n * ((a \wedge e)^n \to (x \wedge e)) \leq x \wedge e.$$

Since $f \in F$ and $*$ is commutative, by (4) we have $x \in J$, hence $K \subseteq J$. Thus $J = K$. □

We denote by $F_p(\mathbf{L})$ the set of principal filters of $\mathbf{L}$. In order to explore some properties of $F_p(\mathbf{L})$ we state the following lemma.

Lemma 3.10. *Let $\mathbf{L}$ be an IL-algebra and $a, b \in L$. Then the following statements hold.*

1. *If $a \leq b$, then $\langle b \rangle \subseteq \langle a \rangle$.*
2. *$\langle a \rangle = \langle a \wedge e \rangle$.*
3. *$\langle a * b \rangle \subseteq \langle a \rangle \vee \langle b \rangle = \langle a \wedge b \rangle$.*
4. *$\langle a \vee b \rangle \subseteq \langle a \rangle \cap \langle b \rangle = \langle (a \wedge e) \vee (b \wedge e) \rangle$.*
5. *$F(a) \wedge F(b) = F((a \wedge e) \vee (b \wedge e))$.*

Proof. (1) Assume that $a \leq b$. Let $x \in \langle b\rangle$, then there exists $n \geq 1$ such that $(b \wedge e)^n \leq x \wedge e$. Using (c_8) and $a \leq b$ we get $(a \wedge e)^n \leq (b \wedge e)^n \leq x \wedge e$, hence $x \in \langle a\rangle$. Thus $\langle b\rangle \subseteq \langle a\rangle$.

(2) Since $a \wedge e \leq a$, we have $\langle a\rangle \subseteq \langle a \wedge e\rangle$ (by (1)). Since $\langle a\rangle$ is a filter and $a \wedge e \in \langle a\rangle$, we have $\langle a \wedge e\rangle \subseteq \langle a\rangle$. Hence $\langle a\rangle = \langle a \wedge e\rangle$.

(3) Since $a \wedge b \leq a, b$, we have $\langle a\rangle, \langle b\rangle \subseteq \langle a \wedge b\rangle$ (by (1)). Therefore $\langle a\rangle \vee \langle b\rangle \subseteq \langle a \wedge b\rangle$. Since $a \in \langle a\rangle, b \in \langle b\rangle$ and $\langle a\rangle \cup \langle b\rangle \subseteq \langle a\rangle \vee \langle b\rangle$, we get $a \wedge b \in \langle a\rangle \vee \langle b\rangle$, hence $\langle a \wedge b\rangle \subseteq \langle a\rangle \vee \langle b\rangle$. Thus $\langle a \wedge b\rangle = \langle a\rangle \vee \langle b\rangle$. Since $a, b \in \langle a \wedge b\rangle$, then $a * b \in \langle a \wedge b\rangle$, therefore $\langle a * b\rangle \subseteq \langle a \wedge b\rangle$.

(4) Since $a, b \leq a \vee b$, using (1) we get $\langle a \vee b\rangle \subseteq \langle a\rangle, \langle b\rangle$, hence $\langle a \vee b\rangle \subseteq \langle a\rangle \cap \langle b\rangle$.

It remains to show that $\langle a\rangle \cap \langle b\rangle = \langle (a \wedge e) \vee (b \wedge e)\rangle$. Since $(a \wedge e) \vee (b \wedge e) \geq (a \wedge e), (b \wedge e)$, using (1) and (2) we obtain $\langle (a \wedge e) \vee (b \wedge e)\rangle \subseteq \langle a \wedge e\rangle \wedge \langle b \wedge e\rangle = \langle a\rangle \cap \langle b\rangle$. Let $x \in \langle a\rangle \cap \langle b\rangle$, then there are $n, m \geq 1$ such that $x \wedge e \geq (a \wedge e)^n$ and $x \wedge e \geq (b \wedge e)^m$. We have

$$x \wedge e \geq (a \wedge e)^m \vee (b \wedge e)^n \overset{(c_{17})}{\geq} ((a \wedge e) \vee (b \wedge e))^{mn}.$$

Hence $x \in \langle (a \wedge e) \vee (b \wedge e)\rangle$ and $\langle a\rangle \cap \langle b\rangle \subseteq \langle (a \wedge e) \vee (b \wedge e)\rangle$. Thus $\langle a\rangle \cap \langle b\rangle = \langle (a \wedge e) \vee (b \wedge e)\rangle$.

The proof of (5) is similar to that of (4). □

Proposition 3.11. *The algebra $\mathcal{F}_p(\boldsymbol{L})$ is a bounded sublattice of $\mathcal{F}(\boldsymbol{L})$ with least element $\langle e\rangle$ and greatest element $L = \langle \perp\rangle$.*

Proof. Clearly $F_p(\mathbf{L}) \subseteq F(\mathbf{L})$. Using (3) and (4) of Lemma 3.10 we obtain the result. □

3.2 Prime filter theorem

Prime filter theorem plays an important role in the factorization, representation and the study of topology in ordered algebraic structures. In the case of residuated lattices, prime deductive system theorem was established in [3]. In what follows, we extend the result to IL-algebras.

Lemma 3.12. *Let F be a filter of $\boldsymbol{L}$ and $a, b \in L$. If $(a \wedge e) \vee (b \wedge e) \in F$, then $\langle F \cup \{a\}\rangle \cap \langle F \cup \{b\}\rangle = F$.*

Proof. Clearly $F \subseteq \langle F \cup \{a\}\rangle \cap \langle F \cup \{b\}\rangle$. Let $x \in \langle F \cup \{a\}\rangle \cap \langle F \cup \{b\}\rangle$, then there are $f_1, f_2 \in F, n, m \geq 1$ such that $x \wedge e \geq (f_1 \wedge e) * (a \wedge e)^n$ and $x \wedge e \geq (f_2 \wedge e) * (b \wedge e)^m$. We have

$$\begin{aligned}
x \geq\ & [((f_1 \wedge e) * (a \wedge e)^n)] \vee [(f_2 \wedge e) * (b \wedge e)^m] \\
\overset{(c_{14})}{\geq}\ & [((f_1 \wedge e) * (a \wedge e)^n) \vee (f_2 \wedge e)] * [((f_1 \wedge e) * (a \wedge e)^n) \vee ((b \wedge e)^m)] \\
\overset{(c_{14})}{\geq}\ & [(f_2 \wedge e) \vee (f_1 \wedge e)] * [(f_2 \wedge e) \vee ((a \wedge e)^n)] * [(f_1 \wedge e) \vee ((b \wedge e)^m)] * [(a \wedge e)^n \vee (b \wedge e)^m] \\
\overset{(c_{16}),(c_{17})}{\geq}\ & [(f_2 \wedge e) \vee (f_1 \wedge e)] * [(f_2 \wedge e) \vee (a \wedge e)]^n * [(f_1 \wedge e) \vee (b \wedge e)]^m * [(a \wedge e) \vee (b \wedge e)]^{mn} \in F,
\end{aligned}$$

because f_1, f_2 and $(a \wedge e) \vee (b \wedge e)$ belong to F. Therefore $\langle F \cup \{x\}\rangle \cap \langle F \cup \{b\}\rangle = F$. □

Proposition 3.13. *If $F \in F(\boldsymbol{L})$, then the following conditions are equivalent.*

(i) *If $F = F_1 \cap F_2$ with $F_1, F_2 \in F(\boldsymbol{L})$, then $F = F_1$ or $F = F_2$.*

(ii) *If $a, b \in L$, with $(a \wedge e) \vee (b \wedge e) \in F$, then $a \in F$ or $b \in F$.*

(iii) *If $F_1 \cap F_2 \subseteq F$ with $F_1, F_2 \in F(\boldsymbol{L})$, then $F_1 \subseteq F$ or $F_2 \subseteq F$.*

Proof. $(i) \Rightarrow (ii)$ Assume that $a, b \in L$ such that $(a \wedge e) \vee (b \wedge e) \in F$, then by Lemma 3.12 we have $\langle F \cup \{a\}\rangle \cap \langle F \cup \{b\}\rangle = F$, therefore $F = \langle F \cup \{a\}\rangle$ or $F = \langle F \cup \{b\}\rangle$. Hence $a \in F$ or $b \in F$.
$(ii) \Rightarrow (i)$ Let $F_1, F_2 \in F(\mathbf{L})$ such that $F = F_1 \cap F_2$. If by contrary $F_1 \neq F$ and $F_2 \neq F$, then there are $a \in F_1 \setminus F$ and $b \in F_2 \setminus F$. Let $c = (a \wedge e) \vee (b \wedge e)$, then $c \in F_1 \cap F_2 = F$. Hence $a \in F$ or $b \in F$, contradiction.
$(ii) \Leftrightarrow (iii)$ is obtained with similar arguments. □

Definition 3.14. *Let $\mathbf{L}$ be an IL-algebra. A filter F of $\mathbf{L}$ is called a **prime filter** if $F \neq L$ and satisfies one of the conditions of Proposition 3.13.*

Example 3.15. It is easy to check that $F = \{a, e, \top\}$, $G = \{b, e, \top\}$ and $H = \{d, a, b, e, \top\}$ are prime filters of the IL-algebra $\mathbf{L}_7$.

We denote by $Spec(\mathbf{L})$ the set of all prime filters of $\mathbf{L}$. We denote by $L(\mathbf{L})$ the lattice reduct of $\mathbf{L}$ (i.e. $L(\mathbf{L}) = (L, \vee, \wedge)$). We recall that a subset I of a lattice $\mathbf{L}$ is called an ideal if I is a $\vee$-closed subset (i.e. if $a, y \in I$, then $x \vee y \in I$) and $x \leq y$ implies $x \in I$ for any $y \in I$ and $x \in L$.

Theorem 3.16. *(Prime filter theorem for IL-algebras). Let $\mathbf{L}$ be an IL-algebra. If $F \in F(\mathbf{L})$ and I is an ideal of the lattice $L(\mathbf{L})$ such that $F \cap I = \emptyset$, then there is a prime filter P of L such that $F \subseteq P$ and $P \cap I = \emptyset$.*

Proof. Let $F_F := \{H \in F(L) \mid F \subseteq H \text{ and } H \cap I = \emptyset\}$. A routine application of Zorn's lemma shows that F_F has a maximal element P. Suppose that P is not a prime filter, that is there are $a, b \in L$ such that $(a \wedge e) \vee (b \wedge e) \in P$ but $a \notin P, b \notin P$. By the maximality of P we deduce that $\langle P \cup \{a\}\rangle$ and $\langle P \cup \{b\}\rangle$ are not in F_F, hence $\langle P \cup \{a\}\rangle \cap I \neq \emptyset$ and $\langle P \cup \{b\}\rangle \cap I \neq \emptyset$, that is there are $p_1 \in \langle P \cup \{a\}\rangle \cap I \neq \emptyset$ and $p_2 \in \langle P \cup \{b\}\rangle \cap I \neq \emptyset$. By (4) of Proposition 3.9, there exist $f, g \in P, n, m \geq 1$ such that $p_1 \geq (f \wedge e) * (a \wedge e)^n$ and $p_2 \geq (g \wedge e) * (b \wedge e)^m$. Therefore

$$\begin{aligned} p_1 \vee p_2 &\geq ((f \wedge e) * (a \wedge e)^n) \vee ((g \wedge e) * (b \wedge e)^m) \\ &\overset{(c_{14})}{\geq} [(f \wedge e) \vee (g \wedge e)] * [(g \wedge e) \vee (a \wedge e)^n] * [(f \wedge e) \vee (b \wedge e)^m] * [(b \wedge e)^m \vee (a \wedge e)^n)] \\ &\overset{(c_{16}),(c_{17})}{\geq} [(f \wedge e) \vee (g \wedge e)] * [(g \wedge e) \vee (a \wedge e)]^n * [(f \wedge e) \vee (b \wedge e)]^m * [(a \wedge e) \vee (b \wedge e)]^{mn}. \end{aligned}$$

Since $(f \wedge e) \vee (g \wedge e), (g \wedge e) \vee (a \wedge e)^n, (f \wedge e) \vee (b \wedge e)^m \in P$, we deduce that $p_1 \vee p_2 \in P$, but $p_1 \vee p_2 \in I$. Hence $P \cap I \neq \emptyset$, which is a contradiction. Thus P is a prime filter. □

Corollary 3.17. *If $\mathbf{L}$ is a nontrivial IL-algebra, then any proper filter of $\mathbf{L}$ can be extended to a prime filter of $\mathbf{L}$.*

Proposition 3.18. *For a filter $P \in F(\mathbf{L})$, the following conditions are equivalent.*

- (i) *$P \in Spec(\mathbf{L})$.*
- (ii) *For every $x, y \in L \setminus P$, there is $z \in L \setminus P$ such that $x \wedge e \leq z$ and $y \wedge e \leq z$.*

Proof. $(i) \Rightarrow (ii)$ Let $P \in Spec(\mathbf{L})$ and $x, y \in L \setminus P$. By contrary, we suppose that for every $a \in L$, if $x \wedge e \leq a$ and $y \wedge e \leq a$, then $a \in P$. Since $x \wedge e, y \wedge e \leq (x \wedge e) \vee (y \wedge e)$, we have $(x \wedge e) \vee (y \wedge e) \in P$, hence $x \in P$ or $y \in P$, a contradiction.

$(ii) \Rightarrow (i)$ Suppose by contrary that there exist $F_1, F_2 \in F(\mathbf{L})$ such that $F_1 \cap F_2 = P$ and $F_1 \neq P$, $F_2 \neq P$. Then, there exist $x \in F_1 \setminus P$ and $y \in F_2 \setminus P$. By hypothesis, there is $z \in L \setminus P$ such that $x \wedge e \leq z \wedge e$ and $y \wedge e \leq z \wedge e$, we deduce that $z \in F_1 \cap F_2 = P$, contradiction. □

Corollary 3.19. *For a proper filter $P \in F(\mathbf{L})$, the following conditions are equivalent:*

(i) *$P \in Spec(\boldsymbol{L})$;*

(ii) *If $x, y \in L$ and $\langle x\rangle \cap \langle y\rangle \subseteq P$, then $x \in P$ or $y \in P$.*

Proof. $(i) \Rightarrow (ii)$ Let $x, y \in L$ such that $\langle x\rangle \cap \langle y\rangle \subseteq P$, then $(x \wedge e) \vee (y \wedge e) \in P$. Since P is a prime filter we deduce that $x \in P$ or $y \in P$.

$(ii) \Rightarrow (i)$ Let $x, y \in L$ such that $(x \wedge e) \vee (y \wedge e) \in P$, then $\langle (x \wedge e) \vee (y \wedge e)\rangle = \langle x\rangle \cap \langle x\rangle \subseteq P$, hence by hypothesis $x \in P$ or $y \in P$. Hence $P \in Spec(\mathbf{L})$. $\square$

Corollary 3.20. *Let $F \in F(\boldsymbol{L})$ and $a \in L \setminus F$, then the following statements hold:*

(i) *There is $P \in Spec(\boldsymbol{L})$ such that $F \subseteq P$ and $a \notin P$.*

(ii) *F is the intersection of those prime filters which contain F.*

(iii) *$\cap Spec(\boldsymbol{L}) = \langle e\rangle$.*

Proof. (i) Consider $I = \{x \in L \mid x \leq a\}$. Clearly I is an ideal of the lattice $L(\mathbf{L})$.
Suppose that $F \cap I \neq \emptyset$, that is there is an $x \in F \cap I$. It follows that $x \leq a$ and $x \in F$, hence $a \in F$, contradiction. Thus $F \cap I = \emptyset$. According to the prime filter theorem, there is a prime filter P of $\mathbf{L}$ such that $F \subseteq P$ and $a \notin P$.

(ii) We have to show that $F = \cap\{P \in Spec(\mathbf{L}) \mid F \subseteq P\}$. Let $G = \cap\{P \in Spec(\mathbf{L}) \mid F \subseteq P\}$. Assume that $F \subsetneq G$, then there exists $a \in G$ such that $a \notin F$. According to (i), there is a prime filter P such that $a \notin P$ and $F \subseteq P$, contradiction with $a \in G$. Therefore $G = F$.

(iii) Clearly $\langle e\rangle \subseteq \cap Spec(\mathbf{L})$. Assume that $\cap Spec(\mathbf{L}) \nsubseteq \langle e\rangle$, then there exists $a \in \cap Spec(\mathbf{L})$ such that $a \notin \langle e\rangle$. According to (i) there exists $P \in Spec(\mathbf{L})$ such that $\langle e\rangle \subseteq P$ and $a \notin P$, since by assumption $a \in P$ we obtain a contradiction. $\square$

Corollary 3.21. *If $\boldsymbol{L}$ is a nontrivial IL-algebra, then $Max(\boldsymbol{L}) \subseteq Spec(\boldsymbol{L})$.*

Proof. Obvious. $\square$

3.3 The lattice of filters of IL-algebras

It is known that given a residuated lattice $\mathbf{L}$, the lattice $\mathcal{F}(\mathbf{L})$ of its filters is algebraic and form a Heyting algebra (see [3]). In this subsection we show that for an IL-algebra $\mathbf{L}$, the lattice $\mathcal{F}(\mathbf{L})$ of its filters has these properties.

Definition 3.22. (See [2], p.17) *Let $\boldsymbol{L} = (L, \wedge, \vee)$ be a lattice.*

1. *An element $a \in L$ is **compact** if whenever $\vee A$ exists and $a \leq \vee A$ for $A \leq L$, then $a \leq \vee B$ for some finite $B \subseteq A$.*

2. *The lattice $\mathbf{L}$ is **compactly generated** iff every element in L is a supremum of compact elements.*

3. *The lattice $\mathbf{L}$ is **algebraic** if $\mathbf{L}$ is complete and compactly generated.*

4. *The lattice $\mathbf{L}$ is called **brouwerian** if it satifies the identity*

$$a \wedge (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \wedge b_i) \text{ (whenever the arbitrary joins exist).}$$

Remark 3.23. If $\mathbf{L}$ is an IL-algebra, and $F_i \in F(\mathbf{L}), i \in I$, then the following equality holds:

$$\underset{i\in I}{\vee} F_i = \underset{(i_1,\ldots,i_n)\in I^\star}{\cup} (F_{i_1} \vee \ldots \vee F_{i_n})$$

where $I^\star$ is the set of finite tuples $(i_1, \ldots, i_n) \in I^n, n \geq 1, n \in \mathbb{N}$.

Theorem 3.24. *The lattice $\mathcal{F}(\boldsymbol{L})$ is algebraic, Brouwerian and the compact elements are the principal filters.*

Proof. From Lemma 3.8, $\mathcal{F}(\mathbf{L})$ is a complete lattice. Let F be a compact element of $F(\mathbf{L})$, then $F \subseteq \vee_{a\in F}\langle a\rangle$. Since F is compact, there exist $a_1, \ldots, a_n$ such that $F \subseteq \langle a_1\rangle \vee \ldots \vee \langle a_n\rangle = \langle a_1 \wedge \ldots \wedge a_n\rangle \subseteq F$. Therefore $F = \langle b\rangle$ with $b = a_1 \wedge \ldots \wedge a_n$. Hence, each compact element of $\mathcal{F}(\mathbf{L})$ is a principal filter.

Conversely, let $a \in L$, we show that $\langle a\rangle$ is a compact element. Let $\{F_j, j \in J\} \subseteq F(\mathbf{L})$ with J an arbitrary non-empty set such that $\langle a\rangle \subseteq \underset{j\in J}{\vee} F_j$. Since $a \in \langle a\rangle$, from Remark 3.23, there are $n \geq 1$ and $i_1, \ldots, i_n \in J$ such that $a \in F_{i_1} \vee \ldots \vee F_{i_n}$. Therefore $\langle a\rangle \subseteq F_{i_1} \vee \ldots \vee F_{i_n}$. Hence $\langle a\rangle$ is a compact element. Since $\mathcal{F}(\mathbf{L})$ is a complete lattice, it remains to show that each member of $F(\mathbf{L})$ is a join of compact elements. Let $F \in F(\mathbf{L})$, then $F = \underset{a\in F}{\vee}\langle a\rangle$, as each principal filter is a compact element we are done. Therefore $\mathcal{F}(\mathbf{L})$ is algebraic.

Let's show that $\mathcal{F}(\mathbf{L})$ is brouwerian. Let $F, F_i \in \mathcal{F}(\mathbf{L})$, $i \in K$ where K is a non-empty set. We have to show that $F \wedge (\underset{k\in K}{\vee} F_k) = \underset{k\in K}{\vee}(F \wedge F_k)$, that is

$$F \cap (\underset{k\in K}{\vee} F_k) = \langle \underset{k\in K}{\cup}(F \cap F_k)\rangle = \underset{i\in I}{\vee}(F \cap F_i).$$

Clearly, $\underset{i\in I}{\vee}(F \cap F_i) \subseteq F \cap (\underset{k\in K}{\vee} F_k)$. Let $x \in F \cap (\underset{k\in K}{\vee} F_k) = F \cap [\underset{(i_1,\ldots,i_n)\in I^\star}{\cup}(F_{i_1} \vee \ldots \vee F_{i_n}))]$, then there exist $n \geq 1, i_1, \ldots, i_n \in I, a_{i_j} \in F_{i_j}, j = 1, \ldots, n$ such that $x \wedge e \geq (x_{i_1} \wedge e) * (x_{i_2} \wedge e) * \ldots * (x_{i_n} \wedge e)$. Furthermore we have

$$\begin{aligned} x \wedge e &\geq (x \wedge e) \vee [(x_{i_1} \wedge e) * \ldots * (x_{i_n} \wedge e)] \\ &\overset{(c_{15})}{\geq} ((x \wedge e) \vee (x_{i_1} \wedge e)) * \ldots * ((x \wedge e) \vee (x_{i_n} \wedge e)). \end{aligned}$$

Since $(x \wedge e) \in F$ and $(x_{i_j} \wedge e) \in F_{i_j}, 1 \leq j \leq n$, we have $(x \wedge e) \vee (x_{i_j} \wedge e) \in F \cap F_{i_j}$, hence $x \in (F \cap F_{i_1}) \vee \ldots \vee (F \cap F_{i_n}) \subseteq \underset{i\in I}{\vee}(F \cap F_i)$. Thus $F \cap (\underset{k\in K}{\vee} F_k) \subseteq \underset{i\in K}{\vee}(F \cap F_i)$. $\square$

The following result is a consequence of Theorem 3.24.

Corollary 3.25. *The lattice $\mathcal{F}(\boldsymbol{L})$ is distributive.*

Let $\mathbf{L}$ be an IL-algebra. For any $F_1, F_2 \in F(L)$, we set

$$F_1 \to F_2 = \{x \in L \mid \langle x\rangle \cap F_1 \subseteq F_2\}.$$

Lemma 3.26. *If $\boldsymbol{L}$ is an IL-algebra and $F_1, F_2 \in F(\boldsymbol{L})$, then the following statements hold:*

(i) $F_1 \to F_2 \in F(\boldsymbol{L})$.

(ii) For any $F \in F(\boldsymbol{L})$, $F_1 \cap F \subseteq F_2$ *if and only if* $F \subseteq F_1 \to F_2$ *(that is* $F_1 \to F_2 = \sup\{F \in F(\boldsymbol{L}) : F_1 \cap F \subseteq F_2\}$*).*

Proof. (i) Let $F_1, F_2 \in F(\mathbf{L})$. We have to show that $F_1 \to F_2$ is a filter of $\mathbf{L}$.

We have $\langle e\rangle \cap F_1 = \langle e\rangle \subseteq F_1$, hence $e \in F_1 \to F_2$. Let $x \in F_1 \to F_2$ and $y \in L$ such that $x \leq y$. We show that $y \in F_1 \to F_2$. Since $x \leq y$, by (1) of Lemma 3.10 we get $\langle y\rangle \subseteq \langle x\rangle$. In addition, $x \in F_1 \to F_2$ implies $\langle x\rangle \cap F_1 \subseteq F_2$. Hence $\langle y\rangle \cap F_1 \subseteq \langle x\rangle \cap F_1 \subseteq F_2$. Thus $y \in F_1 \to F_2$.

Let $x, y \in F_1 \to F_2$, then $\langle x\rangle \cap F_1 \subseteq F_1$ and $\langle y\rangle \cap F_1 \subseteq F_2$, so $((\langle x\rangle \cap F_1) \vee (\langle y\rangle \vee F_1) \subseteq F_2$. From Corollary 3.25 the lattice $\mathcal{F}(\mathbf{L})$ is distributive, hence $((\langle x\rangle \cap F_1) \vee ((\langle y\rangle \cap F_1) = ((\langle x\rangle \vee \langle y\rangle) \cap F_1$. By (3) of Lemma 3.10 we get $\langle x\rangle \vee \langle y\rangle = \langle x \wedge y\rangle$, therefore $\langle x \wedge y\rangle \cap F_1 \subseteq F_2$, hence $x \wedge y \in F_1 \to F_2$. From (3) of Lemma 3.10 we get $\langle x * y\rangle \subseteq \langle x \wedge y\rangle$. Since $\langle x \wedge y\rangle \cap F_1 \subseteq F_2$, we deduce that $\langle x * y\rangle \cap F_1 \subseteq F_2$, hence $x * y \in F_1 \to F_2$. Thus $F_1 \to F_2$ is a filter.

(ii) Let $F, F_1, F_2 \in F(\mathbf{L})$. Assume that $F_1 \cap F \subseteq F_2$. To show that $F \subseteq F_1 \to F_2$, let $x \in F$ and $t \in \langle x\rangle \cap F_1$, then $t \in F_1$ and there exists $n \geq 1$ such that $t \wedge e \geq (x \wedge e)^n$. Since $x \in F$, we have $(x \wedge e)^n \in F$, hence $t \in F$ and $t \in F_1$. In addition, $F \wedge F_1 \subseteq F_2$, we deduce that $t \in F_2$. Hence $x \in F_1 \to F_2$.

Conversely, assume that $F \subseteq F_1 \to F_2$. We will show that $F_1 \cap F \subseteq F_2$. Let $x \in F_1 \cap F$. Since $F \subseteq F_1 \to F_2$, we get $\langle x\rangle \cap F_1 \subseteq F_2$. Since $x \in \langle x\rangle$ and $x \in F$, we have $x \in \langle x\rangle \cap F_1 \subseteq F_2$. Hence $x \in F_2$. □

We recall that an **Heyting algebra** (see [2]) is a lattice $(L, \wedge, \vee)$ with 0 such that for every $a, b \in L$, there exists an element $a \to b \in L$ (called **pseudocomplement** of a with respect to b) such that for every $x \in L$, $a \wedge x \leq b$ if and only if $x \leq a \to b$ (that is, $a \to b = \sup\{x \in L \mid a \wedge x \leq b\}$).

Theorem 3.27. *Let L be an IL-algebra. The algebra $(F(\boldsymbol{L}), \wedge, \vee, \to, \langle e\rangle)$ is a complete Heyting algebra, where for all $F_1, F_2 \in F(L)$,*

$$F_1 \wedge F_2 = F_1 \cap F_2,\ \ F_1 \vee F_2 = \langle F_1 \cup F_2\rangle$$

$$F_1 \to F_2 = \{x \in L : F_1 \cap \langle x\rangle \subseteq F_2\}.$$

Set $F^\star = F \to \langle e\rangle = \{x \in L : \langle x\rangle \cap F = \langle e\rangle\}$.

Proof. Clearly $(F(\mathbf{L}), \wedge, \vee)$ is a complete lattice with least element $\langle e\rangle$. By Lemma 3.26 for any $F_1, F_2 \in F(\mathbf{L})$, $F_1 \to F_2 \in F(\mathbf{L})$ and $F_1 \to F_2$ is a pseudocoplement of F_1 with respect to F_2.
□

A residuated lattice $(L, \wedge, \vee, *, \to, \perp, \top)$ is called a **Gödel algebra** if $x^2 = x * x = x$, for all $x \in L$. Taking $* = \cap$ on $F(\mathbf{L})$ and $\to$ as above, we consider the algebra

$$\mathcal{F}(\mathbf{L}) = (F(\mathbf{L}), \wedge, \vee, *, \to, \langle e\rangle, L).$$

Corollary 3.28. *Let $\boldsymbol{L}$ be an IL-algebra, $*$ and $\to$ be the binary operations defined on $F(\boldsymbol{L})$ as above, then the algebra $\mathcal{F}(\boldsymbol{L}) = (F(\boldsymbol{L}), \wedge, \vee, *, \to, \langle e\rangle, L)$ is a commutative residuated lattice.*

Proof. (1) It is clear that $(F(\mathbf{L}), \cap, L)$ is a commutative monoid and $(\mathcal{F}(\mathbf{L}), \cap, \vee, \langle e\rangle, L)$ is a bounded lattice. In addition the law of residuation holds in $\mathcal{F}(\mathbf{L})$ by (ii) of Lemma 3.26, $\cap$ is commutative and $F \cap L = F$ for all $F \in \mathcal{F}(\mathbf{L})$. Hence $\mathcal{F}(\mathbf{L})$ is a residuated lattice. □

For $F_1, F_2 \in F(\mathbf{L})$, we set

$$F_1 \oplus F_2 = \{x \in L \mid (x \wedge e) \vee (y \wedge e) \in F_2, \text{for all } y \in F_1\}.$$

Proposition 3.29. *For all $F_1, F_2 \in F(\boldsymbol{L})$, $F_1 \oplus F_2 = F_1 \to F_2$.*

Proof. First we show that $F_1 \oplus F_2 \subseteq F_1 \to F_2$. Let $x \in F_1 \oplus F_2$ and $z \in \langle x\rangle \cap F_1$, then $z \in F_1$ and $z \wedge e \geq (x \wedge e)^n$ for some integer $n \geq 1$ and $x \vee z \in F_2$. We have $z \geq (z \wedge e) \vee (x \wedge e)^n \geq ((z \wedge e) \vee (x \wedge e))^n$ by (c_{16}). It follows that $z \in F_2$ (due to $(z \wedge e) \vee (x \wedge e) \in F_2$), hence $x \in F_1 \to F_2$. Thus $F_1 \oplus F_2 \subseteq F_1 \to F_2$.

For the converse inclusion, let $x \in F_1 \to F_2$, then $\langle x\rangle \cap F_1 \subseteq F_2$. If $y \in F_1$, then $(x \wedge e) \vee (y \wedge e) \in \langle x\rangle \cap F_1$, therefore $(x \wedge e) \vee (y \wedge e) \in F_2$. Hence $x \in F_1 \oplus F_2$ and $F_1 \to F_2 \subseteq F_1 \oplus F_2$. Thus $F_1 \to F_2 = F_1 \oplus F_2$. □

Following the result by G. Grätzer on pseudocomplemented lattices ([8], p.99) any lattice that satisfies the join Infinite Distributive Identity (JID) is a pseudocomplemented distributive lattice. Hence, every distributive algebraic lattice is pseudocomplemented. An element $a^\star \in L$ is a **pseudocomplement** of $a \in L$

if $a \wedge a^* = 0$ and $a \wedge x = 0$ implies $x \leq a^\star$. A **pseudocomplemented lattice** is one in which every element has a pseudo complement.

We characterize the pseudo-complement of any filter F of $\mathbf{L}$ in the sequel. For any filter F of $\mathbf{L}$ we set

$$F^\star = \{x \in L \mid \text{for any } y \in F, (x \wedge e) \vee (y \wedge e) = e\}.$$

Lemma 3.30. *Let $\boldsymbol{L}$ be an IL-algebra, $F, G \in F(\boldsymbol{L})$ and $F^\star$, $G^\star$ as above, then the following statements hold.*

1. *$F^\star = F \rightarrow \langle e \rangle \in F(\boldsymbol{L})$.*
2. *If $F \subseteq G$, then $G^\star \subseteq F^\star$.*
3. *$F \cap F^\star = \langle e \rangle$.*
4. *If $F \cap G = \langle e \rangle$, then $G \subseteq F^\star$.*
5. *$L^\star = \langle e \rangle$ and $\langle e \rangle^\star = L$.*

Proof. For (1), we just need to show the equality $F^\star = F \rightarrow \langle e \rangle$. Let $y \in F \rightarrow \langle e \rangle$, then $\langle y \rangle \cap F \subseteq \langle e \rangle$. Let $x \in F$, then $(x \wedge e) \vee (y \wedge e) \in \langle y \rangle \cap F = \langle e \rangle$ (since $y \wedge e \in \langle y \rangle$ and $x \wedge e \in F$), hence $(x \wedge e) \vee (y \wedge e) = e$ and $y \in F^*$. Therefore $F \rightarrow \langle e \rangle \subseteq F^\star$. Let $y \in F^\star$, we show that $y \in F \rightarrow \langle e \rangle$, that is $\langle y \rangle \cap F \subseteq \langle e \rangle$. Let $t \in \langle y \rangle \cap F$, then $t \in F$ and there exists $n \geq 1$ such that $t \wedge e \geq (y \wedge e)^n$. Since $y \in F^\star$, we get $(t \wedge e) \vee (y \wedge e) = e$, $(t \wedge e) \vee (y \wedge e)^n = (t \wedge e) \geq [(t \wedge e) \vee (y \wedge e)]^n = e$ by (c_{16}) and assumption on y, hence $(t \wedge e) \geq e$, therefore $t \geq e$ and $t \in \langle e \rangle$. Hence $F^* \subseteq F \rightarrow \langle e \rangle$. From Lemma 3.26 $F \rightarrow \langle e \rangle$ is a filter.

(2) The proof of (2) is easy to check.

(3) Let $x \in F \cap F^\star$, then $(x \wedge e) \vee (x \wedge e) = e$ by definition of $F^\star$, hence $x \wedge e = e$ and $x \in \langle e \rangle$. Thus $F \cap F^\star = \langle e \rangle$.

(4) Assume that $F \cap G = \langle e \rangle$. Let's us show that $G \subseteq F^*$. Let $x \in G$ and $y \in F$, then $(x \wedge e) \vee (y \wedge e) \in F \cap G = \langle e \rangle$, that is $e \geq (x \wedge e) \vee (y \wedge e) \geq e$, hence $(x \wedge e) \vee (y \wedge e) = e$ and $x \in F^\star$. Thus $G \subseteq F^\star$.

(5) By (1) we have $L^\star = L \rightarrow \langle e \rangle = \langle e \rangle$ and $\langle e \rangle^\star = \langle e \rangle \rightarrow \langle e \rangle = L$ (because L is the greatest element and $\langle e \rangle$ the least element in the residuated lattice $(F(L), \wedge, \vee, *, \rightarrow, \langle e \rangle, L)$. □

Proposition 3.31. *If $\boldsymbol{L}$ is an IL-algebra, then the algebra $(F(\boldsymbol{L}), \wedge, \vee, ^\star, \langle e \rangle, L)$ is a bounded pseudocomplemented distributive lattice.*

Proof. Using (1),(2),(3) and (4) of Lemma 3.30 we get that for any $F \in F(\mathbf{L})$, $F^\star$ is a pseudocomplement of F. Furthermore, by Corollary 3.25 $F(\mathbf{L})$ is distributive. Therefore $F(\mathbf{L})$ is a distributive pseudocomplemented lattice. □

Theorem 3.32. *If every $F \in F(\boldsymbol{L})$ has a unique representation as an intersection of elements of $Spec(\boldsymbol{L})$, then $(F(\boldsymbol{L}), \wedge, \vee, ^\star, \langle e \rangle, L)$ is a Boolean algebra.*

Proof. Since $(F(\mathbf{L}), \subseteq)$ is a distributive lattice, it remains to show that it is complemented. Let $F \in F(\mathbf{L})$. If $F \in \{\langle e \rangle, L\}$, then F is complemented. Assume that $F \neq L$ and $F \neq \langle e \rangle$. Then by (iii) of Corollary 3.20 there exists $Q \in Spec(\mathbf{L})$ such that $F \nsubseteq Q$. Therefore $F_1 = \cap\{P \in Spec(\mathbf{L}) \mid F \nsubseteq P\} \in F(\mathbf{L})$. Clearly $F \cap F_1 = \langle e \rangle$ (by using (ii) and (iii) of Corollary 3.20). We claim that $F \vee F_1 = L$. Suppose by contrary that $F \vee F_1 \neq L$, then by (i) of Corollary 3.20 there exists $P \in Spec(\mathbf{L})$ such that $F_1 \subseteq P$ and $P \neq L$. Consequently

$$F_1 = \cap\{M \in Spec(\mathbf{L}) \mid F \nsubseteq M\} = P \cap [\cap\{K \in Spec(\mathbf{L}) \mid F \nsubseteq K\}]$$

which is in contradiction with the assumption. Therefore $F \vee F_1 = L$ and $F^\star = F_1$ is the complement of F. Hence $F(\mathbf{L})$ is complemented. Thus $(F(\mathbf{L}), \subseteq)$ is a Boolean algebra. □

3.4 Negative cone in IL-algebra

In ([6], p.142), given a non-bounded residuated lattice $(L, \wedge, \vee, e, *, \rightarrow)$, an element a of L is positive if $a \geq e$ and negative if $a \leq e$ and the positive part of $\mathbf{L}$ is defined as the set $L^+ = \{x \in L \mid x \geq e\}$ of all positive elements of $\mathbf{L}$ and the negative part is $L^- = \{x \in L \mid x \leq e\}$.

The **negative cone** of an IL-algebra $\mathbf{L}$ is the algebra $\mathbf{L}^- = (L^-, \wedge, \vee, *_{\mathbf{L}^-}, \rightarrow_{\mathbf{L}}, e)$, where

$$x \rightarrow_L y = (x \rightarrow y) \wedge e, \quad x *_L y = (x * y) \wedge e.$$

For the IL-algebra $\mathbf{L}_7$, we have $L_7^- = \{\perp, d, a, b, e\}$.

Proposition 3.33. *If $\boldsymbol{L}$ is an IL-algebra which is not a residuated lattice, then the negative cone of $\boldsymbol{L}$ is a commutative residuated lattice.*

Proof. It is clear that $\mathbf{L}^-$ is a bounded lattice with least element $\perp$ and greatest element e. In addition $(L^-, *_L, e)$ is a commutative submonoid of $(L, *, e)$ with $a *_L b = a * b$. The residuation law holds in $\mathbf{L}^-$ because it holds in $\mathbf{L}$. $\square$

Definition 3.34. *A nonempty subset S of L is called a **base** of a filter F if*

$$F = \{x \in L \mid \exists x_0 \in S, x_0 \leq x\}.$$

Example 3.35. We consider the IL-algebra $\mathbf{L}_7$, the set $S_1 = \{a, e\}$ is a base of the filter $F_1 = \{a, e, \top\}$ and $S_2 = \{b, e\}$ is a base of the filter $G = \{b, e, \top\}$.

We state the relationship between filter of $\mathcal{F}(\mathbf{L}^-)$ and those of $\mathcal{F}(\mathbf{L})$. For any $H \in F(\mathbf{L}^-)$ we set

$$F_H = \{x \in L \mid \exists x_0 \in H, x_0 \leq x\}.$$

Lemma 3.36. *Let $\boldsymbol{L}$ be an IL-algebra and $\boldsymbol{L}^-$ as above. Let $H, H_1, H_2 \in F(\boldsymbol{L}^-)$ and $G \in F(\boldsymbol{L})$ and $F_{H_i}, i = 1, 2$ and F_H as above, then the following statements hold:*

1. *$F_H \in F(\boldsymbol{L})$.*
2. *$G \cap L^- \in F(\boldsymbol{L}^-)$.*
3. *$H_1 \subseteq H_2$ if and only if $F_{H_1} \subseteq F_{H_2}$.*
4. *If $F_{H_1} = F_{H_2}$, then $H_1 = H_2$.*

Proof. (1) Clearly $e \in F_H$. Let $x \in F_H$ and $y \in L$ such that $x \leq y$. We have to show that $y \in F_H$. Since $x \in F_H$, there exists $x_0 \in H$ such that $x_0 \leq x \leq y$, therefore $y \in G$. Let $x, y \in F_H$, then there are $x_1, x_2 \in H$ such that $x_1 \leq x, x_2 \leq y$. By (c_8) we have $x_1 * x_2 \leq x * y$, and $x_1 \wedge x_2 \leq x \wedge y$. Since H is a filter we get $x_1 \wedge x_2, x_1 * x_2 \in H$, hence $x * y, x \wedge y \in F_H$. Thus F_H is a filter of $\mathbf{L}$.

One observes that for any $H \in F(\mathbf{L}^-)$, $F_H \cap L^- = H$. Using this observation one can easily shows that (2)-(4) hold. $\square$

Theorem 3.37. *The lattices $\mathcal{F}(\boldsymbol{L}^-)$ and $\mathcal{F}(\boldsymbol{L})$ are isomorphic.*

Proof. By Lemma 3.36, the map $\Phi : F(\mathbf{L}^-) \mapsto F(\mathbf{L}), H \mapsto \Phi(H) = F_H$ is an order-isomorphism with the inverse isomorphism Ψ given by $\Psi(F) = F \cap L^-$ for all $F \in F(\mathbf{L})$. $\square$

4 Congruences and filters on IL-algebras

In this section, we establish the relations between congruences and filters in IL-algebras and some related properties.

Definition 4.1. *Let L be an IL-algebra. The equivalence relation $\theta \subseteq L^2$ is called a* ***congruence*** *in* $\boldsymbol{L}$ *if θ is compatible with $\to, *, \wedge$ and $\vee$, that is for any $(x,y),(a,b) \in \theta$, $(x \wedge a, y \wedge b), (x \to a, y \to b), (x * a, y * b), (x \vee a, y \vee b) \in \theta$.*

The set of all congruences of $\mathbf{L}$ denoted by $Con(\mathbf{L})$ is a complete lattice ordered by $\subseteq$ ([2]). For any $\theta \in Con(\mathbf{L})$, we denote by $[a]_\theta$ the set $\{x \in L \mid (a,x) \in \theta\}$.

Example 4.2. The equivalence relation defined by its classes $\theta \cong \{\perp\}, \{d,a,b,e\}, \{c\}, \{\top\}$ is a congruence on the IL-algebra $\mathbf{L}_7$.

We have $[\top]_\theta = \{\top\}$ and $[e]_\theta = \{e,a,b,c,d\}$. The class $[e]_\theta$ is not a filter (since $\top \notin [e]_\theta$), and $[\top]_\theta$ also is not a filter, since $e \notin [\top]_\theta$.

Remark 4.3. For an arbitrary congruence θ on an IL-algebra $\mathbf{L}$, the class $[\top]_\theta$ (resp. $[e]_\theta$) is not always a filter of $\mathbf{L}$.

Let $\mathbf{L}$ be an IL-algebra and θ be a congruence on $\mathbf{L}$, set

$$F_\theta := \{x \in L \mid (x \wedge e, e) \in \theta\}.$$

Proposition 4.4. *If θ and β are congruences on $\boldsymbol{L}$ and F_θ, F_β as above, then the following statements hold:*

1. *F_θ is a filter of L.*

2. *If $\theta \subseteq \beta$, then $F_\theta \subseteq F_\beta$.*

Proof. (1) We have to show that $F_\theta \in F(\mathbf{L})$.

Clearly $e \in F_\theta$. Assume that $x \in F_\theta$ and $y \in L$ such that $x \leq y$. Then $x \wedge e \leq y \wedge e$ and $(x \wedge e) \vee (y \wedge e) = y \wedge e$. Since $(x \wedge e, e) \in \theta$ and $(y \wedge e, y \wedge e) \in \theta$ we have $(y \wedge e, e) \in \theta$, therefore $y \in F_\theta$. Let $x, y \in F_\theta$, then $(x \wedge e, e), (y \wedge e, e) \in \theta$ (a). Let's show that $x \wedge y, x * y \in F$. Since θ is compatible with $\wedge$, we get $((x \wedge e) \wedge (y \wedge e), e \wedge e) = ((x \wedge y) \wedge e, e) \in \theta$, hence $x \wedge y \in F_\theta$.

To finish we show that $x * y \in F_\theta$. We have $(x \wedge e, e), (y \wedge e, e) \in \theta$ imply by compatibility with $*$ that $((x \wedge e) * (y \wedge e), e * e) = ((x \wedge e) * (y \wedge e), e) \in \theta$. Furthermore, using (c_4) and $x \wedge e, y \wedge e \leq e$ we have $(x \wedge e) * (y \wedge e) \leq (x \wedge y) \wedge e \leq e$, therefore $((x \wedge e) * (y \wedge e), e) \in \theta$ and $(x \wedge e) * (y \wedge e) \in F_\theta$. Since $x \wedge e, y \wedge e \leq x, y$, by (c_8) we have $(x \wedge e) * (y \wedge e) \leq x * y$, and $[(x \wedge e) * (y \wedge e)] \wedge e = (x \wedge e) * (y \wedge e) \leq (x * y) \wedge e$ and $(x \wedge e) * (y \wedge e) \in F_\theta$, therefore $x * y \in F_\theta$. Thus F_θ is a filter of L.

(2) is obvious. $\square$

For the above congruence θ in the Example 4.2 we get $F_\theta = \{e, a, b, d, \top\} = F$. One can check that $(x,y) \in \theta$ if and only if $x \to y, y \to x \in F$.

We need the following lemma for the sequel.

Lemma 4.5. *Let $\boldsymbol{L}$ be an IL-algebra and $\theta \in Con(\boldsymbol{L})$, then the following statements are equivalent.*

1. *$(a,b) \in \theta$.*

2. *$((a \to b) \wedge e, e) \in \theta$ and $((b \to a) \wedge e, e) \in \theta$.*

Proof. (1)$\Rightarrow$ (2) Let $(a,b) \in \theta$, then $(a \to a, b \to a) \in \theta$. By (c_{11}) we have $e \leq a \to a$, therefore $e = (a \to a) \wedge e$ and $(e, (b \to a) \wedge e)) \in \theta$. Since $e \leq b \to b$ and $(a \to b, b \to b) \in \theta$, by compatibility of θ with $\wedge$ we get $((a \to b) \wedge e, (b \to b) \wedge e) = ((a \to b) \wedge e, e) \in \theta$. Therefore $((a \to b) \wedge e, (b \to a) \wedge e) \in \theta$ (since θ is transitive). Hence (1) $\Rightarrow$ (2) holds.

(2)$\Rightarrow$(1) Assume that $((a \to b) \wedge e, e) \in \theta$ (1.1) and $((b \to a) \wedge e, e) \in \theta$ (1.2).

Let's show that $(a,b) \in \theta$. Using compatibility of θ with $*$, (1.1) and (1.2), we have $(((a \to b) \wedge e) * a, a) \in \theta$ (2.1) and $(((b \to a) \wedge e) * b, b) \in \theta$ (2.2). Set $r = [(a \to b) \wedge e] * a$ and $s = [(b \to a) \wedge e] * b$. Since $(a \to b) \wedge e \leq a \to b$, by (c_8) $((a \to b) \wedge e) * a \leq (a \to b) * a \leq b$ by (c_{10}), therefore $r \leq b$. A similar argument show that $s \leq a$. It follows from $(r,a) \in \theta, (s,b) \in \theta$ and θ compatible with $\wedge$ that $(r \wedge b, b \wedge a) = (r, b \wedge a) \in \theta$ and $(s \wedge a, b \wedge a) = (s, b \wedge a) \in \theta$, hence $(r,s) \in \theta$. Since $(r,a) \in \theta$, $(s,b) \in \theta$ and $(r,s) \in \theta$, by transitivity of θ we have $(a,b) \in \theta$. Thus $(a,b) \in \theta$ and (2)$\Rightarrow$ (1) holds. $\square$

Lemma 4.6. *If $\theta \in Con(\boldsymbol{L})$ and $c \in L$, then $[c]_\theta$ is a convex subset of $\boldsymbol{L}$.*

Proof. Straightforward. $\square$

Proposition 4.7. *If $\theta \in Con(\boldsymbol{L})$, then $\langle [e]_\theta \rangle = F_\theta = \{x \in L \mid (x \wedge e, e) \in \theta\}$.*

Proof. From Proposition 4.4 F_θ is a filter. Let $x \in [e]_\theta$, then $(x,e) \in \theta$ and using compatibility of θ with $\wedge$ and $(e,e) \in \theta$ we get $(x \wedge e, e) \in \theta$, hence $x \in F_\theta$. Thus $[e]_\theta \subseteq F_\theta$. Since F_θ is a filter we deduce that $\langle [e]_\theta \rangle \subseteq F_\theta$. Let $x \in F_\theta$, then $(x \wedge e, e) \in \theta$. Since $x \wedge e \leq e$ and $x \wedge e \in [e]_\theta$, we deduce by (3) of Proposition 3.9 that $x \in \langle [e]_\theta \rangle$. Hence $F_\theta \subseteq \langle [e]_\theta \rangle$. Thus $F_\theta = \langle [e]_\theta \rangle$. $\square$ For a filter H of L we set

$$\theta_H = \{(a,b) \in L^2 \mid \exists h \in H, h * a \leq b \text{ and } h * b \leq a\},$$

$$\Theta(H) = \{(a,b) \in L^2 \mid a \to b \in H \text{ and } b \to a \in H\}.$$

Proposition 4.8. *Let $\boldsymbol{L}$ be an IL-algebra, $H, F_1, F_2 \in F(\boldsymbol{L})$, $\theta_1, \theta_2, \theta \in Con(\boldsymbol{L})$ and θ_H and $\Theta(H)$ as above, then the following statements hold:*

1. *$\theta_H = \Theta(H)$.*
2. *$\Theta(H)$ is a congruence relation on $\boldsymbol{L}$.*
3. *If $[e]_{\theta_1} = [e]_{\theta_2}$, then $\theta_1 = \theta_2$.*
4. *$\langle [e]_{\Theta(H)} \rangle = H$.*
5. *If $\Theta(F_1) = \Theta(F_2)$, then $F_1 = F_2$.*
6. *$\Theta(F_1) \subseteq \Theta(F_2)$ if and only if $F_1 \subseteq F_2$.*

Proof. (1) Let $(a,b) \in \Theta(H)$, then $(a \to b) \wedge e \in H$ and $(b \to a) \wedge e \in H$. Set $h = (a \to b) \wedge (b \to a) \wedge e$. Clearly $h \in H$ (since H is a filter). Let's show that $h * a \leq b$ and $h * b \leq a$. Since $[(a \to b) \wedge e \wedge (b \to a) \wedge e] \leq (a \to b), (b \to a)$ (3.1) by (c_8) $h * a \leq (a \to b) * a \leq b$ by (c_{10}) and $h * b \leq (b \to a) * b \leq a$ by (c_{10}), we have $h * b \leq a$ and $h * a \leq b$, therefore $(a,b) \in \theta_H$ and $\Theta(H) \subseteq \theta_H$.

Conversely, suppose that $(a,b) \in \theta_H$, then there exists $h \in H$ such that $h * a \leq b$ and $h * b \leq a$. By the residuation property we have $h * a \leq b$ implies $h \leq a \to b$ and $h \wedge e \leq (a \to b) \wedge e$. Since $h, e \in H$, we get $h \wedge e \in H$, therefore $(a \to b) \wedge e \in H$. From $h * b \leq a$, by the residuation property we get $h \leq b \to a$, hence $(b \to a) \wedge e \in H$ (since $e, h \in H$ and H is a filter). Therefore $(a,b) \in \Theta(H)$ and $\theta_H \subseteq \Theta(H)$. Thus $\theta_H = \Theta(H)$.

(2) Obviously $\Theta(H)$ is reflexive and symmetric. Let's show that $\Theta(H)$ is transitive. Assume that $(x,y), (y,z) \in \Theta(H)$. Then $x \to y, y \to x, y \to z, z \to y \in H$. We have $(x \to y) * (y \to z) \leq x \to z \in F$ and

$(z \to y) * (y \to x) \leq z \to x \in F$ by (c_6) and F is an upper set. Therefore $(x,z) \in \Theta(H)$. Hence $\Theta(H)$ is transitive. It remains to show that $\Theta(H)$ is compatible with $\vee, \wedge, \to, *$. Assume that $(x,y),(u,v) \in \Theta(H)$. Then $x \to y, y \to x, u \to v, v \to u \in F$. We have to show that $(x\alpha u, y\alpha v) \in \Theta(H)$ with $\alpha \in \{*, \to, \vee, \wedge\}$.

We have $(x \wedge u) \to (y \wedge v) \overset{(c_{12})}{=} [(x \wedge u) \to y] \wedge [(x \wedge u) \to v] \geq (x \to y) \wedge (u \to v)$ by (c_8), hence $(x \wedge u) \to (y \wedge v) \in F$ (since $x \to y, u \to v \in F$ and F is an upper set). A similar argument shows that $(y \wedge v) \to (x \wedge u) \in F$. Thus $(x \wedge u, y \wedge v) \in \Theta(H)$.

For $\to$, we have $(y \to x)*(x \to u)*(u \to v) \leq (y \to v)$ by (c_8) and (c_6) and using the residuation property we get $(z \to x) * (u \to v) \leq (x \to u) \to (y \to v)$. Since $y \to x, u \to v \in F$, we deduce $(y \to x) * (u \to v) \in H$, so $(x \to u) \to (y \to v) \in H$. A similar argument shows that $(y \to v) \to (x \to u) \in F$. Hence $\Theta(H)$ is compatible with $\to$.

For $\vee$, we have

$$(x \vee u) \to (y \vee v) \overset{(c_{13})}{=} (x \to (y \vee v)) \wedge (u \to (y \vee v)) \geq (x \to y) \wedge (u \to v) \in H.$$

Therefore $(x \vee u) \to (y \vee v) \in H$. Similarly $(y \vee v) \to (x \vee u) \in H$. Hence $\Theta(H)$ is compatible with $\vee$.

For $*$, first we show that $(x * z, y * z) \in \Theta(H)$ for any $z \in L$. We have $x \to y \leq x * z \to y * z$ (by (c_{18}). Since $x \to y \in H$ and H is an upper set, we deduce that $x * z \to y * z \in H$. A similar argument shows that $y * z \to x * z \in H$, hence $(x * z, y * z) \in \Theta(H)$ for any $z \in L$. Therefore, taking $z = u$ we have $(x * u, y * u) \in \Theta(H)$, $(u * y, v * y) \in \Theta(H)$. Since $\Theta(H)$ is transitive, we deduce that $(x * u, y * v) \in \Theta(H)$. Thus $\Theta(H) \in Con(\mathbf{L})$.

(3) For $\theta_1, \theta_2 \in Con(\mathbf{L})$, assume that $[e]_{\theta_1} = [e]_{\theta_2}$. We will show that $\theta_1 = \theta_2$. Let $(a,b) \in \theta_1$, then by Lemma 4.5 we have $((a \to b) \wedge e, e) \in \theta_1$ and $((b \to a) \wedge e, e) \in \theta_1$, that is $(a \to b) \wedge e \in [e]_{\theta_1} = [e]_{\theta_2}$ and $(b \to a) \wedge e \in [e]_{\theta_1} = [e]_{\theta_2}$, hence $((a \to b) \wedge e, e), ((b \to a) \wedge e, e) \in \theta_2$ and we deduce by Lemma 4.5 that $(a,b) \in \theta_2$, therefore $\theta_1 \subseteq \theta_2$. A similar argument show that $\theta_2 \subseteq \theta_1$, hence $\theta_1 = \theta_2$.

(4) Let H be a filter of L and $\Theta(H)$ as above. We will show that $\langle [e]_{\Theta(H)} \rangle = H$. Set $F = \langle [e]_{\Theta(H)} \rangle$. Let $h \in F$, then $h, e \in H$, hence $h \wedge e \in H$. Since $h \wedge e \leq e$, we get $(h \wedge e) \to e \in H$ due to H is a filter, therefore $((h \wedge e) \to e) \wedge e \in H$ $(\star_1)$. From (c_7), $e \to (h \wedge e) = h \wedge e$, this implies $(e \to (h \wedge e)) \wedge e = (h \wedge e) \wedge e \in H$ $(\star_2)$. From $(\star_1)$, $(\star_2)$ and Lemma 4.5 we get $h \wedge e \in [e]_{\Theta(H)}$. Since $[e]_{\Theta(H)} \subseteq F$ and F a filter, we deduce that $h \in F$ due to $h \wedge e \leq h$. Hence $H \subseteq F$.

Now we show that $F \subseteq H$. If we show that $[e]_{\Theta(H)} \subseteq H$ we are done. Let $a \in [e]_{\Theta(H)}$, then $(a,e) \in \Theta(H)$, that is $(a \to e) \wedge e \in H$ and $(e \to a) \wedge e \in H$. By (c_7) we have $e \to a = a$, hence $(e \to a) \wedge e = a \wedge e \in H$. Since $a \wedge e \leq a$ and H is a filter, we have $a \in H$, hence $[e]_{\Theta(H)} \subseteq H$. Therefore $F \subseteq H$. Thus $F = H$ and (4) holds.

(5) Assume that $\Theta(F_1) = \Theta(F_2)$. We show that $F_1 = F_2$. Since $\Theta(F_1) = \Theta(F_2)$, $[e]_{\Theta(F_1)} = [e]_{\Theta(F_2)}$, and using (4) we deduce that $F_1 = F_2$.

(6) Assume that $\Theta(F_1) \subseteq \Theta(F_2)$. We have to show that $F_1 \subseteq F_2$. Let $h \in F_1$, then $h \wedge e \in [e]_{\Theta(F_1)} \subseteq [e]_{\Theta(F_2)} \subseteq F_2$ (by (4)), hence $F_1 \subseteq F_2$. Conversely, assume that $F_1 \subseteq F_2$, using Lemma 4.5 we can easily see that $\Theta(F_1) \subseteq \Theta(F_2)$. $\square$

Example 4.9. Consider the IL-algebra $\mathbf{L}_7$ and $F = \{d, a, b, e, \top\}$. Clearly F is a filter and the congruence $\Theta(F)$ is given by its classes

$$[\top]_{\Theta(F)} = \{\top\}, [e]_{\Theta(F)} = \{e, a, b, d\}, [c]_{\Theta(F)} = \{c\}.$$

Definition 4.10. (See [6], p.28) *Let $\boldsymbol{L}$ be an algebra. If there is an element $a \in L$ such that for any $\theta, \eta \in Con(\boldsymbol{L})$ we have $[a]_\theta = [a]_\eta$ implies $\theta = \eta$, then $\boldsymbol{L}$ is called a-**regular**.*

One of consequences of Proposition 4.8 is:

Corollary 4.11. *Any IL-algebra* $\boldsymbol{L}$ *is e-regular.*

Proof. According to (3) of Proposition 4.8 we are done. $\square$

Theorem 4.12. *The lattices* $Con(\boldsymbol{L})$ *and* $F(\boldsymbol{L})$ *are isomorphic, the isomorphism is given by*

$$\Phi : F(\boldsymbol{L}) \to Con(\boldsymbol{L}), H \mapsto \Theta(H)$$

with the inverse given by

$$\Psi : Con(\boldsymbol{L}) \to F(\boldsymbol{L}), \theta \mapsto \ \Psi(\theta) = \langle [e]_\theta \rangle.$$

Proof. Using (1), (2), (4), (5) and (6) of Proposition 4.8, we get that Φ is an order-isomorphism with the inverse the map Ψ. $\square$

Definition 4.13. (See [2], p.57) *An algebra* $\boldsymbol{L}$ *is a* ***subdirect product*** *of an indexed family* $(\boldsymbol{L}_i)_{i\in I}$ *of algebras if*

(i) $\boldsymbol{L} \leq \Pi_{i\in I}\boldsymbol{L}_i$,

(ii) $\pi_i(\boldsymbol{L}) = L_i$ *for each* $i \in I$.

An algebra $\boldsymbol{L}$ *is* ***subdirectly irreducible*** *if for every subdirect embedding*

$$\alpha : \boldsymbol{L} \to \Pi_{i\in I}\boldsymbol{L}_i$$

there is an $i \in I$ *such that* $\pi_i \circ \alpha : \boldsymbol{L} \to \boldsymbol{L}_i$ *is an isomorphism.*

Theorem 4.14. (See [2], p.57) *An algebra* $\boldsymbol{L}$ *is subdirectly irreducible if and only if* $\boldsymbol{L}$ *is trivial or there is a minimum congruence in* $Con(\boldsymbol{L}) \setminus \{\Delta\}$ *(where* $\Delta = \{(x,x) \mid x \in L\}$*).*

We end this paper by the following theorem that characterizes subdirectly irreducible IL-algebras.

Theorem 4.15. *Let* $\boldsymbol{L}$ *be a non-trivial IL-algebra that is not a residuated lattice* ($e \neq \top$)*. Then* $\boldsymbol{L}$ *is subdirectly irreducible if it has a unique subcover of* e*. Hence it is simple iff* $[\perp, e] = \{\perp, e\}$.

Proof. According to Theorem 4.14 and Theorem 4.12, an IL-algebra $\mathbf{L}$ is subdirectly irreducible if it has a unique minimal filter F. Obviously, any such filter F must be a principal filter $\langle d\rangle$ for some $d \leq e$, $d \neq e$ and moreover $d \in \langle a\rangle$ for any $a \leq e$, $a \neq e$. That means d is the greatest element in the set $[\perp, e[$. In the case $[\perp, e] = \{\perp, e\}$, $F(\mathbf{L}) = \{\langle e\rangle, L\}$ and by Theorem 4.12 $\mathbf{L}$ has exactly two congruences, L^2 and Δ. $\square$

5 Conclusion

We have described the filter generated by an arbitrary subset of an IL-algebra, and shown that the set of principal filters forms a bounded sublattice of the lattice of filters. Prime filters are characterized and the prime filter theorem for IL-algebras is established. We have also shown that the lattice $(F(\mathbf{L}), \subseteq)$ is algebraic, pseudocomplemented, brouwerian and endowed with a structure of Heyting algebra. Given an IL-algebra $\mathbf{L}$ which is not a residuated lattice, we defined the negative cone $\mathbf{L}^-$ of $\mathbf{L}$ which is a residuated lattice and we established that, the lattices $\mathcal{F}(\mathbf{L})$ and $\mathcal{F}(\mathbf{F}^-)$ are isomorphic. Finally, we established the relationship between filters and congruences by showing that the corresponding lattices are isomorphic.

Our future work is concerned with the study of some subclass of particular filters in IL-algebras and extends some existing results on spectral topology in residuated lattices to that of IL-algebras.

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