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$\top$-Nets and $\top$-Filters

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Abstract. In this paper, we develop a theory of $\top$-nets and study their relation to $\top$-filters. We show that convergence in strong L-topological spaces can be described by both $\top$-nets and $\top$-filters and both concepts are equivalent in the sense that definitions and proofs that are given using $\top$-filters can also be given using $\top$-nets and vice versa.

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1 Introduction

There are usually two ways in which convergence in topology is studied. One way makes use of so-called nets or Moore-Smith sequences. These were introduced by Moore and Smith [24] and made popular with the textbook of Kelley [17]. The other way uses filters and these were introduced by Cartan [4] and made popular e.g. by Kowalsky [18] and Bourbaki [3]. Bartle pointed out that both notions are equivalent in the sense that a definition, proposition, or proof based on nets can also be given using filters and vice versa [1].

In the lattice-valued case — for different lattice backgrounds — both approaches have been generalized and used from the very beginning of fuzzy topology. Lowen [22] developed a convergence theory based on prefilters and at around the same time, Pu and Liu [25] developed a convergence theory using fuzzy nets. The relationship between these two approaches was clarified in [23]. Höhle developed a theory of $\top$-filters [10] and L-filters [11, 12]. Convergence theories based on this concept were developed e.g. in [14, 15, 5, 20]. A further notable contribution is due to Yao [28] who defined and studied LM-nets and discussed the relationship to LM-filters.

Recently, new interest in Höhle's $\top$-filters evolved [7, 29, 31] as they can be used for a convergence theory for strong L-topological spaces [5, 32] or conical neighborhood spaces [21, 19]. They are also applied to study probabilistic uniform spaces [10, 7, 30] and $\top$-uniform convergence spaces [16].

In this paper, we provide a suitable theory of $\top$-nets and show with examples that this concept can also be fruitfully applied in cases where $\top$-filters have been used so far. In this sense, we again obtain equivalence between $\top$-nets and $\top$-filters.

The paper is organized as follows. In a preliminary section, we describe the lattice context used in this paper and collect the basic underlying theory and results that we use later on. The next section gives the new concepts of a $\top$-net and — most important for the equivalence mentioned above — the definition of

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a $\top$-subnet. The relationship between $\top$-nets and $\top$-filters is developed. This is followed by a section on applications of both $\top$-nets and $\top$-filters in the theory of strong L-topological spaces and a section on a diagonal principle based on $\top$-nets. Then we briefly glimpse the use of $\top$-sequences and finally we draw some conclusions.

2 Preliminaries

Let $(L, \leq)$ be a complete lattice with distinct top and bottom elements $\top \neq \bot$. We can define the *well-below relation* $\alpha \lhd \beta$ if for all subsets $D \subseteq L$ such that $\beta \leq \bigvee D$ there is $\delta \in D$ such that $\alpha \leq \delta$. A complete lattice is completely distributive if and only if we have $\alpha = \bigvee\{\beta \,:\, \beta \lhd \alpha\}$ for any $\alpha \in L$, [26]. For more details and results on lattices, we refer to [9].

The triple $\mathsf{L} = (L, \leq, *)$, where $(L, \leq)$ is a complete lattice with order relation $\leq$, is called a *commutative and integral quantale* if $(L, *)$ is a commutative semigroup with the top element of L as the unit, i.e. $\alpha * \top = \alpha$ for all $\alpha \in L$, and $*$ is distributive over arbitrary joins, i.e. $(\bigvee_{i\in J} \alpha_i) * \beta = \bigvee_{i\in J}(\alpha_i * \beta)$, see e.g. [13].

In a quantale, we can define an *implication* by $\alpha \to \beta = \bigvee\{\delta \in L \,:\, \alpha * \delta \leq \beta\}$. Then $\delta \leq \alpha \to \beta \iff \delta * \alpha \leq \beta$. A commutative and integral quantale is an *MV-algebra* [11] if $(\alpha \to \beta) \to \beta = \alpha \vee \beta$ for all $\alpha, \beta \in L$.

We will in this paper always assume that $\mathsf{L} = (L, \leq, *)$ is a commutative and integral quantale and that the lattice $(L, \leq)$ is completely distributive with the additional property that $\alpha, \beta \lhd \top$ implies $\alpha \vee \beta \lhd \top$, see [8]. While for a good part of the theory the weaker assumption $\top = \bigvee\{\alpha \,:\, \alpha \lhd \top\}$ is sufficient we will need the complete distributivity in particular for the concept of a $\top$-subnet and here, for the important Theorem 3.7.

An L-set in X is a mapping $a : X \longrightarrow L$ and we denote the set of L-sets in X by L^X. The lattice operations are extended pointwisely from L to L^X.

For $a, b \in L^X$ we denote $[a, b] = \bigwedge_{x\in X}(a(x) \to b(x))$. $[\cdot, \cdot]$ is sometimes called the *fuzzy inclusion order* [2]. We collect some of the properties that we will need later.

Lemma 2.1. *Let $a, a', b, b', c \in L^X$, $d \in L^Y$ and let $\varphi : X \longrightarrow Y$ be a mapping. Then*

(i) $a \leq b$ *if and only if* $[a, b] = \top$*;*

(ii) $a \leq a'$ *implies* $[a', b] \leq [a, b]$ *and* $b \leq b'$ *implies* $[a, b] \leq [a, b']$*;*

(iii) $[a, c] \wedge [b, c] = [a \vee b, c]$*;*

(iv) $[\varphi(a), d] = [a, \varphi^{\leftarrow}(d)]$.

Definition 2.2. [29, 10]
A subset $\mathbb{F} \subseteq L^X$ is called a $\top$-*filter* if

($\top$-F1) $\bigvee_{x\in X} b(x) = \top$ for all $b \in \mathbb{F}$;

($\top$-F2) $a, b \in \mathbb{F}$ implies $a \wedge b \in \mathbb{F}$;

($\top$-F3) $\bigvee_{b\in\mathbb{F}}[b, c] = \top$ implies $c \in \mathbb{F}$.

We denote the set of all $\top$-filters on X by $\mathsf{F}^{\top}_{\mathsf{L}}(X)$.

Example 2.3. For $x \in X$, $[x] = \{a \in L^X \,:\, a(x) = \top\}$ is a $\top$-filter.

Definition 2.4. [29, 10] A subset $\mathbb{B} \subseteq L^X$ is called a $\top$-*filter base* if

($\top$-B1) $\bigvee_{x\in X} b(x) = \top$ for all $b \in \mathbb{B}$;

(⊤-B2) $a, b \in \mathbb{B}$ implies $\bigvee_{c\in\mathbb{B}}[c, a \wedge b] = \top$.

For a ⊤-filter base $\mathbb{B}$, $[\mathbb{B}] = \{a \in L^X \,:\, \bigvee_{b\in\mathbb{B}}[b, a] = \top\}$ is the ⊤-filter generated by $\mathbb{B}$.

It is well-known, that for a ⊤-filter $\mathbb{F} \in \mathsf{F}_L^\top(X)$ and a mapping $\varphi : X \longrightarrow Y$, the set $\mathbb{B} = \{\varphi(a) \,:\, a \in \mathbb{F}\}$ is a ⊤-filter base on Y and we denote $\varphi(\mathbb{F})$ the generated ⊤-filter on Y, the *image of* $\mathbb{F}$ *under* φ, see [10].

3 ⊤-nets and their relation to ⊤-filters

A *directed set* $(D, \prec)$ is a nonvoid set with a reflexive and transitive relation which satisfies moreover that for $d, e \in D$ there is $f \in D$ such that $d, e \prec f$. We will also often write $e \succ d$ for $d \prec e$.

We denote $L^* = L \setminus \{\bot\}$. Let $(D, \prec)$ be a directed set. We consider two mappings $s_X : D \longrightarrow X$ and $s_L : D \longrightarrow L^*$. If $\bigvee_{d\prec e} s_L(e) = \top$ for all $d \in D$, then we call the pair $s = (s_X, s_L) : D \longrightarrow X \times L^*$ a ⊤*-net in* X.

Example 3.1. A constant ⊤-net with value $x \in X$ is defined by $c^x : D \longrightarrow X \times L^*$, $c^x_X(d) = x$ and $c_L(d) = \top$ for all $d \in D$.

Theorem 3.2. *Let* $s = (s_X, s_L) : D \longrightarrow X \times L^*$ *be a* ⊤*-net in* X*.*

(i) The set $\mathbb{B}_s = \{b^s_d \,:\, d \in D\}$*, with* $b^s_d = \bigvee_{d\prec e} s_L(e) * \top_{s_X(e)}$ *a "tail" of the* ⊤*-net* s*, is a* ⊤*-filter basis.*

(ii) For the generated ⊤*-filter* $\mathbb{F}_s = [\mathbb{B}_s]$ *we have* $a \in \mathbb{F}_s$ *if and only if*

$$\bigvee_{d\in D} \bigwedge_{d\prec e} (s_L(e) \to a(s_X(e))) = \top.$$

Proof. We first show (1). We have $\bigvee_{z\in X} b_d(z) \geq \bigvee_{d\prec e} b_d(s_X(e)) = \bigvee_{d\prec e} s_L(e) = \top$ for each $d \in D$ and hence (⊤-B1) is satisfied. For (⊤-B2), let $b_d, b_e \in \mathbb{B}_s$. For $d, e \in D$ there is $f \in D$ with $d, e \prec f$. Then $b_f \leq b_d \wedge b_e$ and we conclude $\bigvee_{b\in\mathbb{B}_s}[b, b_d \wedge b_e] \geq [b_f, b_d \wedge b_e] = \top$.

To show (2), we note that for $d \in D$ and $a \in L^X$ we have

$$[b_d, a] = \bigwedge_{z\in X} (b_d(z) \to a(z)) = \bigwedge_{z\in X} \bigwedge_{d\prec e} (s_L(e) * \top_{s_X(e)}(z) \to a(z)) = \bigwedge_{d\prec e} s_L(e) \to a(s_X(e)).$$

□

It is a simple exercise to show that $\mathbb{F}_{c^x} = [x]$ for a constant ⊤-net.

Remark 3.3. For the special case that $(D, \prec) = (\mathbb{N}, \leq)$ we obtain the concept of a ⊤*-sequence.*

Proposition 3.4. *Let* $s = (s_X, s_L) : D \longrightarrow X \times L^*$ *be a* ⊤*-net and let* $\varphi : X \longrightarrow Y$ *be a mapping. We define the* image of s under φ *by* $\varphi(s) = (\varphi \circ s_X, s_L) : D \longrightarrow Y \times L^*$*. Then* $\mathbb{F}_{\varphi(s)} = \varphi(\mathbb{F}_s)$*.*

Proof. We have $a \in \varphi(\mathbb{F}_s)$ if and only if $\varphi^{\leftarrow}(a) \in \mathbb{F}_s$. This is equivalent to

$$\top = \bigvee_{d\in D} \bigwedge_{d\prec e} (s_L(e) \to \varphi^{\leftarrow}(a)(s_X(e))) = \bigvee_{d\in D} \bigwedge_{d\prec e} (s_L(e) \to a(\varphi \circ s_X(e))),$$

i.e. to $a \in \mathbb{F}_{\varphi(s)}$. □

Let now $\mathbb{F} \in \mathsf{F}_L^\top(X)$ be a ⊤-filter. We define

$$D_{\mathbb{F}} = \{((x, \alpha), f) \,:\, \bot \neq \alpha \lhd \top, f \in \mathbb{F}, f(x) \geq \alpha\}$$

and for $((x, \alpha), f), ((y, \beta), g) \in D_{\mathbb{F}}$ we define $((x, \alpha), f) \prec ((y, \beta), g)$ if and only if $g \leq f$.

Proposition 3.5. *Let* $\mathbb{F} \in \mathsf{F}_L^{\top}(X)$*. Then* $(D_{\mathbb{F}}, \prec)$ *is a directed set.*

Proof. We note that $D_{\mathbb{F}}$ is not empty because $\mathbb{F}$ is a $\top$-filter. The reflexivity and transitivity of $\prec$ are obvious. Let $d_1 = ((x, \alpha), f), d_2 = ((y, \beta), g) \in D_{\mathbb{F}}$. Then $f, g \in \mathbb{F}$ and $f(x) \geq \alpha \neq \perp$ and $g(y) \geq \beta \neq \perp$ and $\alpha, \beta \lhd \perp$. Then $f \wedge g \in \mathbb{F}$ and, by our assumption on the quantale, also $\alpha \vee \beta \lhd \top$. From $\alpha \vee \beta \lhd \top = \bigvee_{z \in X} f \wedge g(z)$ we conclude that there is $z \in X$ such that $\alpha \vee \beta \leq f \wedge g(z)$. Hence, $d_3 = ((z, \alpha \vee \beta), f \wedge g) \in D_{\mathbb{F}}$ and clearly $d_1, d_2 \prec d_3$. □

We define now the mapping $s_{\mathbb{F}} : D_{\mathbb{F}} \longrightarrow X \times L^*$ by $s_{\mathbb{F}}((x, \gamma), f) = (x, \gamma)$. For simplicity, we denote $s_{\mathbb{F}} = (s_X, s_L)$. We note that if $((x, \alpha), f) \in D_{\mathbb{F}}$, then, as $f \in \mathbb{F}$, for each $\beta \lhd \top$, we have $\bigvee_{z \in X} f(z) = \top \rhd \beta$ and thus there is $z_\beta \in X$ such that $f(z_\beta) \geq \beta$. Therefore $((z_\beta, \beta), f) \in D_{\mathbb{F}}$ and clearly $((x, \alpha), f) \prec ((z_\beta, \beta), f)$. We conclude

$$\bigvee_{((x,\alpha),f) \prec ((Y,\beta),g)} s_L((y, \beta), g) \geq \bigvee_{\beta \lhd \top} \beta = \top$$

and $s_{\mathbb{F}}$ is a $\top$-net on X.

Proposition 3.6. *Let* $\mathbb{F} \in \mathsf{F}_L^{\top}(X)$*. Then* $\mathbb{F}_{(s_{\mathbb{F}})} = \mathbb{F}$*.*

Proof. Let first $a \in \mathbb{F}$. For $\perp \neq \alpha \leq a(x)$ with $\alpha \lhd \top$ then $d = ((x, \alpha), a) \in D_{\mathbb{F}}$. If $((x, \alpha), a) \prec ((y, \beta), g) \in D_{\mathbb{F}}$ then $\beta \leq g(y) \leq a(y)$ and hence

$$\bigwedge_{((x,\alpha),a) \prec ((y,\beta),g)} s_L(((y, \beta), g)) \to a(s_X(((y, \beta), g))) = \bigwedge_{((x,\alpha),a) \prec ((y,\beta),g)} \beta \to a(y) = \top.$$

Therefore

$$\bigvee_{d \in D_{\mathbb{F}}} \bigwedge_{d \prec e} s_L(e) \to a(s_X(e)) = \top$$

and we have $a \in \mathbb{F}_{s_{\mathbb{F}}}$.

Conversely, let $a \in \mathbb{F}_{s_{\mathbb{F}}}$. Then

$$\begin{aligned}
\top &= \bigvee_{d \in D_{\mathbb{F}}} \bigwedge_{d \prec e} s_L(e) \to a(s_X(e)) \\
&\leq \bigvee_{f \in \mathbb{F}} \bigwedge_{((x,\gamma),f) \prec ((y,\delta),f)} (\delta \to a(y)) \\
&\leq \bigvee_{f \in \mathbb{F}} \bigwedge_{y \in X} \bigwedge_{\delta : f(y) \geq \delta} (\delta \to a(y)) \\
&= \bigvee_{f \in \mathbb{F}} \bigwedge_{y \in X} ((\bigvee_{\delta : f(y) \geq \delta} \delta) \to a(y)) \\
&= \bigvee_{f \in \mathbb{F}} \bigwedge_{y \in X} (f(y) \to a(y)) \;=\; \bigvee_{f \in \mathbb{F}} [f, a],
\end{aligned}$$

and hence $a \in \mathbb{F}$. □

Clearly, for a $\top$-net $s : D \longrightarrow X \times L^*$ we do not have that $s_{(\mathbb{F}_s)}$ equals s as $D_{\mathbb{F}_s}$ does not coincide with the original directed set D. This is similar to the classical relation between nets and filters. For the "equivalence" of both concepts with regards to theories and applications of convergence, we need the notion of a $\top$-subnet.

Let $s : D \longrightarrow X \times L^*$ and $t : E \longrightarrow X \times L^*$ be two $\top$-nets on X. We call t a $\top$-*subnet* of s if there is a mapping $\phi : E \longrightarrow D$ with $t_X = s_X \circ \phi$, $t_L \leq s_L \circ \phi$ and if for all $d \in D$ there is $e \in E$ such that $e \prec h$ implies $d \prec \phi(h)$.

Proposition 3.7. *Let $t = (t_X, t_L) : E \longrightarrow X \times L^*$ be a ⊤-subnet of $s = (s_X, s_L) : D \longrightarrow X \times L^*$. Then $\mathbb{F}_t \geq \mathbb{F}_s$.*

Proof. Let $d \in D$ and let $b^s_d = \bigvee_{d \prec f} s_L(f) * \top_{s_X(f)}$ be an element of the ⊤-basis of $\mathbb{F}_s$. We choose $e \in E$ such that $e \prec h$ implies $d \prec \phi(h)$. Then for the element b^t_e of the ⊤-basis of $\mathbb{F}_t$ we have

$$b^t_e = \bigvee_{e \prec h} t_L(h) * \top_{t_X(h)} \leq \bigvee_{d \prec \phi(h)} s_L(\phi(h)) * \top_{s_X(\phi(h))} \leq \bigvee_{d \prec f} s_L(f) * \top_{s_X(f)} = b^s_d.$$

Hence, $b^s_d \in \mathbb{F}_t$ and we have $\mathbb{F}_s \leq \mathbb{F}_t$. □

Crucial for us is the following result.

Theorem 3.8. *Let $s = (s_X, s_L) : D \longrightarrow X \times L^*$ be a ⊤-net and let $\mathbb{G} \geq \mathbb{F}_s$. Then there is a ⊤-subnet $t = (t_X, t_L) : E \longrightarrow X \times L^*$ of s such that $\mathbb{G} = \mathbb{F}_t$.*

Proof. We define the set

$$E = \{(e, d, g, \varepsilon) \ : \ d, e \in D, d \prec e, g \in \mathbb{G}, \varepsilon \lhd \top, g(s_X(e)) \wedge s_L(e) \geq \varepsilon\}.$$

We note that for $\varepsilon \lhd \top, d \in D$ we have $b^s_d \in \mathbb{F}_s \leq \mathbb{G}$ and hence $b^s_d \wedge g \in \mathbb{G}$. From $\varepsilon \lhd \top = \bigvee_{z \in X} b^s_d \wedge g(z)$ we conclude that there is $z \in X$ such that $\varepsilon \lhd b^s_d(z) = \bigvee_{d \prec e} s_L(e) * \top_{s_X(e)}(z)$ and $\varepsilon \lhd g(z)$. Hence there is $e \succ d$ such that $s_X(e) = z$, $s_L(e) \geq \varepsilon$ and we conclude $g(s_X(e)) \wedge s_L(e) \geq \epsilon$. Therefore, the set E is not empty and for each $d \in D, \varepsilon \lhd \top, g \in \mathbb{G}$ there is an element $(e, d, g, \varepsilon) \in E$.

We define an order on E as follows:

$$(e_1, d_1, g_1, \varepsilon_1) \prec (e_2, d_2, g_2, \varepsilon_2) \iff d_1 \prec d_2 \text{ and } g_1 \geq g_2.$$

It is not difficult to see that $\prec$ is a reflexive and transitive relation on E. We show that $(E, \prec)$ is directed. Let $(e_1, d_1, g_1, \varepsilon_1), (e_2, d_2, g_2, \varepsilon_2) \in E$. We choose $d_3 \succ d_1, d_2$, $\varepsilon_3 \leq \varepsilon_1 \wedge \varepsilon_2$ and $g_3 = g_1 \wedge g_2 \in \mathbb{G}$. As we have just seen, for $\varepsilon_3 \lhd \top$ there is $e_3 \succ d_3$ such that $g_3(s_X(e_3)) \wedge s_L(e_3) \geq \varepsilon_3$ and hence $(e_3, d_3, g_3, \varepsilon_3) \in E$ and $\succ (e_1, d_1, g_1, \varepsilon_1), (e_2, d_2, g_2, \varepsilon_2)$.

We define now $\phi : E \longrightarrow D$ by $\phi(e, d, g, \varepsilon) = e$ and we put $t_X(e, d, g, \varepsilon) = s_X(e)$, $t_L(e, d, g, \varepsilon) = \epsilon$. Then $t_X = s_X \circ \Phi$ and $t_L \leq s_L \circ \Phi$. For $d \in D$ we choose $(e, d, g, \varepsilon) \in E$. If $(e_1, d_1, g_1, \varepsilon_1) \succ (e, d, g, \varepsilon)$ then by the definition of E we have $e_1 \succ d_1$ and from the order we get moreover $d_1 \succ d$. Hence $\Phi(e_1, d_1, g_1, \varepsilon_1) = e_1 \succ d$. In order to conclude that $t : E \longrightarrow X \times L^*$ is a ⊤-subnet of s, we need only to show that t is a ⊤-net. To this end, let $(e_0, d_0, g_0, \varepsilon_0) \in E$. For $\varepsilon_1 \lhd \top$ we choose, as $b^s_{d_0} \wedge g_0 \in \mathbb{G}$, as before $e \succ d_0$ such that $s_X(e) = z, s_L(e) \geq \varepsilon_1, g_0(z) \geq \varepsilon_1$. Then $(e, d_0, g_0, \varepsilon_1) \in E$ and is $\succ (e_0, d_0, g_0, \varepsilon_0)$. Hence

$$\bigvee_{(e_0, d_0, g_0, \varepsilon_0) \prec (e, d, g, \varepsilon)} t_L(e) \geq \varepsilon_1.$$

This is true for all $\varepsilon_1 \lhd \top$ and hence $\bigvee_{(e_0, d_0, g_0, \varepsilon_0) \prec (e, d, g, \varepsilon)} t_L(e, d, g, \varepsilon) = \top$. Hence t is a ⊤-subnet of s.

We will now show that $\mathbb{G} = \mathbb{F}_t$. Consider a "tail" of $t = (t_X, t_L)$,

$$b^t_{(e_0, d_0, g_0, \varepsilon_0)} = \bigvee_{(e_0, d_0, g_0, \varepsilon_0) \prec (e, d, g, \varepsilon)} t_L(e, d, g, \varepsilon) * \top_{t_X(e, d, g, \varepsilon)} = \bigvee_{(e_0, d_0, g_0, \varepsilon_0) \prec (e, d, g, \varepsilon)} \varepsilon * \top_{s_X(e)}.$$

If $(e_0, d_0, g_0, \varepsilon_0) \prec (e, d, g, \varepsilon)$ then $e \succ d$, $d \succ d_0$, $g \leq g_0$, $\varepsilon \lhd \top$ and $g(s_X(e)) \wedge s_L(e) \geq \varepsilon$ and we have

$$b^s_{d_0}(s_X(e)) = \bigvee_{\overline{e} \succ d} s_(\overline{e}) * \top_{s_X(\overline{e})}(s_X(e)) \geq s_L(e)$$

and

$$g_0(s_X(e)) \wedge s_L(e) \geq g(s_X(e)) \wedge s_L(e) \geq \varepsilon = \varepsilon * \top_{s_X(e)}(s_X(e)).$$

Hence we conclude $g_0 \wedge b^s_{d_0}(z) \geq \varepsilon * \top_{s_X(e)}(z)$ for all $z \in X$ and we have $b^t_{(e_0,d_0,g_0,\varepsilon_0)} \leq g_0 \wedge b_{d_0}$.

Conversely, let $\eta \lhd g_0 \wedge b^s_{d_0}(z) = g_0(z) \wedge \bigvee_{e \succ d_0} s_L(e) * \top_{s_X(e)}$. Then $g_0(z) \geq \eta$ and there is $e \succ d_0$ such that $z = s_X(e)$ and $s_L(e) \geq \eta$. We conclude $(e, d_0, g_0, \eta) \in E$ and $\succ (e_0, d_0, g_0, \varepsilon_0)$. Hence, $b^t_{(e_0,d_0,g_0,\varepsilon_0)}(z) \geq s_L(e) \wedge \eta \wedge \top_{s_X(e)}(z) = \eta$ and we have $g_0 \wedge b^s_{d_0} \leq b^t_{(e_0,d_0,g_0,\varepsilon_0)}$. Together, we have shown $g_0 \wedge b^s_{d_0} = b^t_{(e_0,d_0,g_0,\varepsilon_0)}$. As the "tails" $b^t_{(e_0,d_0,g_0,\varepsilon_0)}$ are a $\top$-basis of $\mathbb{F}_t$, we finally show that the set $\mathbb{B} = \{g \wedge b^s_d \,:\, g \in \mathbb{G}, d \in D\}$ is a $\top$-basis of $\mathbb{G}$. The property ($\top$-B1) follows, as $b^s_d \in \mathbb{F}_s \leq \mathbb{G}$ and therefore $g \wedge b^s_d \in \mathbb{G}$. The property ($\top$-B2) can be seen as follows. Let $g_1 \wedge b^s_{d_1}, g_2 \wedge b^s_{d_2} \in \mathbb{B}$. We choose $d_3 \succ d_1, d_2$. Then $b^s_{d_3} \leq b^s_{d_1} \wedge b^s_{d_2}$ and also $g_3 = g_1 \wedge g_2 \in \mathbb{G}$. Hence $g_3 \wedge b^s_{d_3} \leq (g_1 \wedge b^s_{d_1}) \wedge (g_2 \wedge b^s_{d_2})$ and we conclude $\bigvee_{g \in \mathbb{G}, d \in D}[g \wedge b^s_d, (g_1 \wedge b_{d^s_1}) \wedge (g_2 \wedge b^s_{d_2})] = \top$. Hence $\mathbb{B}$ is in fact a $\top$-basis. Let now $g \in \mathbb{G}$, then $g \wedge b^s_d \leq g$ and hence $\bigwedge_{h \in \mathbb{G}, d \in D}[h \wedge b^s_d, g] = \top$ and we have $g \in \mathbb{G}$. Conversely, if $\top = \bigvee_{h \in \mathbb{G}, d \in D}[h \wedge b^s_d, g]$, then, as $h \wedge b^s_d \in \mathbb{G}$, also $\bigvee_{h \in \mathbb{G}}[h, g] = \top$ which implies $g \in \mathbb{G}$. Therefore, $\mathbb{B}$ is a $\top$-basis of $\mathbb{G}$ and the proof is complete. $\square$

4 The equivalence of $\top$-filter and $\top$-net convergence in L-topology

A subset $\tau \subseteq L^X$ is called a *strong L-topology* [32] (or a probabilistic topology [10]) if the following conditions are satisfied.

(ST1) $\perp_X, \top_X \in \tau$,

(ST2) $f \wedge g \in \tau$ whenever $f, g \in \tau$,

(ST3) $\bigvee_{j \in J} f_j \in \tau$ whenever $f_j \in \tau$ for all $j \in J$,

(ST4) $\alpha * f \in \tau$ whenever $f \in \tau$ and $\alpha \in L$,

(ST5) $\alpha \to f \in \tau$ whenever $f \in \tau$ and $\alpha \in L$.

The pair (X, τ) is called a *strong L-topological space*. For a strong L-topological space (X, τ) and $x \in X$ we define the $\top$-*neighbourhood filter of* x [10] by

$$\mathbb{U}^x_\tau = \{u \in L^X \;:\; \bigvee_{g \in \tau, g(x) = \top} [g, u] = \top\}$$

and we call a $\top$-filter $\mathbb{F} \in \mathsf{F}(X)$ *convergent to* x if $\mathbb{F} \geq \mathbb{U}^x_\tau$ and we write $\mathbb{F} \xrightarrow{\tau} x$ in this case. A mapping $\varphi : (X, \tau) \longrightarrow (Y, \sigma)$ between the strong L-topological spaces (X, τ) and (Y, σ) is called *continuous* if for all $x \in X$ we have $\mathbb{U}^{\varphi(x)}_\sigma \leq \varphi(\mathbb{U}^x_\tau)$.

We call a $\top$-net $s : (s_X, s_L) : D \longrightarrow X \times L^*$ *convergent to* x if for all $u \in \mathbb{U}^x_\tau$ we have $\top = \bigvee_{d \in D} \bigwedge_{e \succ d}(s_L(e) \to u(s_X(e)))$. This is equivalent to the fact that $\mathbb{F}_s$ is convergent to x and we write $s \xrightarrow{\tau} x$ in this case.

A strong L-topological space (X, τ) can be charaterized by an interior operator, $\mathrm{int}(a) = \bigvee_{g \in \tau}[g, a] * g$ for all $a \in L^X$, [32]. It is shown in [5] that $\mathrm{int}(a) = \bigvee_{g \in \tau, g \leq a} g$. The interior operator has the following properties [32, 5]. For $a, b \in L^X$ and $\alpha \in L$ we have

(I1) $[a, b] \leq [\mathrm{int}(a), \mathrm{int}(b)]$;

(I2) $\mathrm{int}(a) \leq a$;

(I3) $\mathrm{int}(\alpha \to a) = \alpha \to \mathrm{int}(a)$;

(I4) $\operatorname{int}(a \wedge b) = \operatorname{int}(a) \wedge \operatorname{int}(b)$;

(I5) $\operatorname{int}(int(a)) = \operatorname{int}(a)$.

The strong L-topology τ consists of the fixed-points of int, i.e. we have $g \in \tau \iff \operatorname{int}(g) = g$. For the $\top$-neighbourhood filter $\mathbb{U}_\tau^x$ we have $u \in \mathbb{U}_\tau^x$ if and only if $\operatorname{int}(u)(x) = \top$. For $u \in \mathbb{U}_\tau^x$ we have on the one hand $\operatorname{int}(u)(x) \geq \bigvee_{g \in \tau, g(x)=\top} [g,u] * g(x) = \bigvee_{g \in \tau, g(x)=\top} [g,u] = \top$ and if $\operatorname{int}(u)(x) = \top$ we have, on the other hand, $\bigvee_{g \in \tau, g(x)=\top} [g,u] \geq \bigvee_{g \in \tau, g(x)=\top} g(x) \to u(x) = u(x) \geq \operatorname{int}(u)(x) = \top$ by (I2) and hence $u \in \mathbb{U}_\tau^x$.

We first characterize the interior operator by convergence.

Proposition 4.1. *Let (X,τ) be a strong topological space and let $a \in L^X$. Then*

$$\operatorname{int}(a)(x) = \bigvee_{u \in \mathbb{U}^x} [u,a] = \bigwedge_{\mathbb{F} \xrightarrow{\tau} x} \bigvee_{f \in \mathbb{F}} [f,a] = \bigwedge_{s \xrightarrow{\tau} x} \bigvee_{d \in D} [b_d^s, a].$$

In the last equality, the meet is taken over all convergent $\top$-nets $s : D \longrightarrow X \times L^*$.

Proof. We first show the first equality. We have on the one hand

$$\begin{aligned} \bigvee_{u \in \mathbb{U}_\tau^x} [u,a] &= \bigvee_{u \in \mathbb{U}_\tau^x} \bigvee_{g \in \tau, g(x)=\top} [g,u] * [u,a] \leq \bigvee_{g \in \tau, g(x)=\top} [g,a] \\ &\leq \bigvee_{g \in \tau} [g,a] * g(x) = \operatorname{int}(a)(x). \end{aligned}$$

On the other hand, we define an L-set $b \in L^X$ by $b(z) = \operatorname{int}(a)(x) \to a(z)$ for $z \in X$. Then, using (I3), $\operatorname{int}(b)(x) = \top$, i.e. $b \in \mathbb{U}_\tau^x$ and we conclude

$$\bigvee_{u \in \mathbb{U}_\tau^x} [u,a] \geq \bigwedge_{z \in X} ((\operatorname{int}(a)(x) \to a(z)) \to a(z)) \geq \operatorname{int}(a)(x).$$

For the second equality, we get $\bigwedge_{\mathbb{F} \xrightarrow{\tau} x} \bigvee_{f \in \mathbb{F}} [f,a] \leq \bigvee_{u \in \mathbb{U}^x} [u,a]$ as $\mathbb{U}_\tau^x \xrightarrow{\tau} x$. Let now $\eta \lhd \bigvee_{u \in \mathbb{U}_\tau^x} [u,a]$. Then there is $u \in \mathbb{U}_\tau^x$ such that $\eta \leq [u,a]$. If $\mathbb{F} \xrightarrow{\tau} x$, then $u \in \mathbb{F}$ and hence $\eta \leq \bigvee_{f \in \mathbb{F}} [f,a]$ and hence $\eta \leq \bigwedge_{\mathbb{F} \xrightarrow{\tau} x} \bigvee_{f \in \mathbb{F}} [f,a]$. This shows $\bigvee_{u \in \mathbb{U}^x} [u,a] \leq \bigwedge_{\mathbb{F} \xrightarrow{\tau} x} \bigvee_{f \in \mathbb{F}} [f,a]$.

The last equality can finally be shown as follows. If $s \xrightarrow{\tau} x$, then $\mathbb{F}_s \xrightarrow{\tau} x$ and the "tails" b_d^s form a $\top$-basis of $\mathbb{F}_s$. Hence $\bigwedge_{\mathbb{F} \xrightarrow{\tau} x} \bigvee_{f \in \mathbb{F}} [f,a] \leq \bigwedge_{s \xrightarrow{\tau} x} \bigvee_{d \in D} [b_d^s, a]$. On the other hand, if $\mathbb{F} \xrightarrow{\tau} x$, then $s_{\mathbb{F}} \xrightarrow{\tau} x$ and we have $\mathbb{F} = \mathbb{F}_{(s_{\mathbb{F}})}$. Hence $\bigwedge_{\mathbb{F} \xrightarrow{\tau} x} \bigvee_{f \in \mathbb{F}} [f,a] = \bigwedge_{s_{\mathbb{F}} \xrightarrow{\tau} x} \bigvee_{f \in \mathbb{F}_{(s_{\mathbb{F}})}} [f,a] \geq \bigwedge_{s \xrightarrow{\tau} x} \bigvee_{f \in \mathbb{F}_s} [f,a] \geq \bigwedge_{s \xrightarrow{\tau} x} \bigvee_{d \in D} [b_d^s, a]$. □

Corollary 4.2. *Let (X,τ) be a strong L-topological space. Then the following assertions are equivalent.*

1. *$g \in \tau$;*
2. *$g(x) \leq \bigwedge_{\mathbb{F} \xrightarrow{\tau} x} \bigvee_{f \in \mathbb{F}} [f,a]$ for all $x \in X$;*
3. *$g(x) \leq \bigwedge_{s \xrightarrow{\tau} x} \bigvee_{d \in D} [b_d^s, a]$ for all $x \in X$.*

We define the *closure* of an L-set $a \in L^X$ in a strong L-topological space in accordance with [27] by

$$\overline{a}(x) = \bigvee_{\mathbb{G} \geq \mathbb{U}_\tau^x} \bigvee_{g \in \mathbb{G}} [g,a], \qquad x \in X.$$

This is an L-valued interpretation of the closure of a subset A in a topological space X: A point $x \in X$ belongs to the closure of A if and only if there is a filter converging to x which contains A.

We can also characterize the closure of an L-set using $\top$-nets.

Proposition 4.3. *Let (X, τ) be a strong L-topological space and let $a \in L^X$. Then*

$$\overline{a}(x) = \bigvee_{s \to x} \bigvee_{d \in D} [b^s_d, a], \qquad x \in X.$$

Proof. We have $s \to x$ if and only if $\mathbb{F}_s \geq \mathbb{U}^x_\tau$. Hence

$$\overline{a}(x) = \bigvee_{\mathbb{F} \geq \mathbb{U}^x_\tau} \bigvee_{f \in \mathbb{F}} [f, a] \geq \bigvee_{s \to x} \bigvee_{f \in \mathbb{F}_s} [f, a] \geq \bigvee_{s \to x} \bigvee_{d \in D} [b^s_d, a].$$

On the other hand, for $f \in \mathbb{F}_s$ we have $\bigvee_{d \in D} [b^s_d, f] = \top$. Using $\mathbb{F} = \mathbb{F}_{(s_\mathbb{F})}$ we conclude

$$\overline{a}(x) = \bigvee_{\mathbb{F} \geq \mathbb{U}^x_\tau} \bigvee_{f \in \mathbb{F}} [f, a] = \bigvee_{s_\mathbb{F} \to x} \bigvee_{f \in \mathbb{F}_{(s_\mathbb{F})}} [f, a]$$

$$\leq \bigvee_{s \to x} \bigvee_{f \in \mathbb{F}_s} [f, a] = \bigvee_{s \to x} \bigvee_{f \in \mathbb{F}_s} \bigvee_{d \in D} [b^s_d, f] * [f, a] \leq \bigvee_{s \to x} \bigvee_{d \in D} [b^s_d, a]$$

and the proof is complete. $\square$

Next we turn to the concept of a cluster point.

For a $\top$-filter $\mathbb{F} \in \mathsf{F}^L_\top(X)$ a point $x \in X$ is called a *cluster point* of $\mathbb{F}$ if $\mathbb{F} \vee \mathbb{U}^x_\tau$ exists or, equivalently, if for all $f \in \mathbb{F}$ and all $u \in \mathbb{U}^x_\tau$ we have $\bigvee_{x \in X} f(x) \wedge u(x) = \top$. In [10] a cluster point of a $\top$-filter is called an *adherent point* of the $\top$-filter.

Lemma 4.4. *Let (X, τ) be a strong L-topological space and let $\mathbb{F}$ be a $\top$-filter in X and let $x \in X$. Then x is a cluster point of $\mathbb{F}$ if and only if there is a $\top$-filter $\mathbb{G} \geq \mathbb{F}$ which converges to x.*

Proof. If x is a cluster point of $\mathbb{F}$, then we can choose $\mathbb{G} = \mathbb{F} \vee \mathbb{U}^x_\tau$, which clearly converges to x. If there is $\mathbb{G} \geq \mathbb{F}$ converging to x, then $\mathbb{G} \geq \mathbb{U}^x_\tau$ and hence $\mathbb{F} \vee \mathbb{U}^x_\tau$ exists and x is a cluster point of $\mathbb{F}$. $\square$

Similarly, for a $\top$-net $s = (s_X, s_L) : D \longrightarrow X$ a point $x \in X$ is called a *cluster point* of s if $\bigvee_{d \prec e} s_L(e) \wedge u(s_X(e)) = \top$ for all $d \in D$ and all $u \in \mathbb{U}^x_\tau$.

Proposition 4.5. *Let (X, τ) be a strong L-topological space and let $s = (s_X, s_L) : D \longrightarrow X$ be a $\top$-net in X and let $x \in X$. Then x is a cluster point of s if and only if x is a cluster point of $\mathbb{F}_s$.*

Proof. Let first x be a cluster point of s and let $f \in \mathbb{F}_s$ and $u \in \mathbb{U}^x_\tau$. Then $\bigvee_{d \in D} \bigwedge_{d \prec e} (s_L(e) \to f(s_X(e))) = \top$, because $f \in \mathbb{F}_s$, and $\bigvee_{d \prec h} s_L(h) \wedge u(s_X(h)) = \top$. We conclude, using the inequality $(\alpha \wedge \beta) * \gamma \leq \alpha \wedge (\beta * \gamma)$,

$$\begin{aligned}
\top &= \bigvee_{d \in D} \left(\left[\bigwedge_{d \prec e} (s_L(e) \to f(s_X(e))) \right] * \left[\bigvee_{d \prec h} s_L(h) \wedge u(s_X(h)) \right] \right) \\
&= \bigvee_{d \in D} \bigvee_{d \prec h} \left((u(s_X(h)) \wedge s_L(h)) * \bigwedge_{d \prec e} (s_L(e) \to f(s_X(e))) \right) \\
&\leq \bigvee_{d \in D} \bigvee_{d \prec h} u(s_X(h)) \wedge (s_L(h) * (s_L(h) \to f(s_X(h)))) \\
&\leq \bigvee_{d \in D} \bigvee_{d \prec h} u(s_X(h)) \wedge f(s_X(h)) \\
&\leq \bigvee_{x \in X} u(x) \wedge f(x).
\end{aligned}$$

Hence x is a cluster point of $\mathbb{F}_s$.

For the converse, we choose $f = \bigvee_{d\prec e} s_L(e) * \top_{s_X(e)} \in \mathbb{F}_s$. Then, x being a cluster point of $\mathbb{F}_s$ we obtain

$$\top = \bigvee_{x\in X} (\bigvee_{d\prec e} s_L(e) * \top_{s_X(e)}(x) \wedge u(x) = \bigvee_{d\prec e} s_L(e) \wedge u(s_X(e))$$

which means that x is a cluster point of s. □

Corollary 4.6. *Let (X,τ) be a strong L-topological space and let $\mathbb{F}$ be a $\top$-filter in X and let $x \in X$. Then x is a cluster point of $\mathbb{F}$ if and only if x is a cluster point of $s_{\mathbb{F}}$.*

Proof. By Proposition 4.3, x is a cluster point of $s_{\mathbb{F}}$ if and only if x is a cluster point of $\mathbb{F}_{(s_{\mathbb{F}})} = \mathbb{F}$. □

Proposition 4.7. *Let (X,τ) be a strong L-topological space and let $s = (s_X, s_L) : D \longrightarrow X$ be a $\top$-net in X and let $x \in X$. Then x is a cluster point of s if and only if there is a $\top$-subnet t of s which converges to x.*

Proof. Proposition 4.3 shows that x is a cluster point of s if and only if x is a cluster point of $\mathbb{F}_s$. This is by Lemma 4.2 equivalent to the existence of $\mathbb{G} \geq \mathbb{F}_s$, converging to x. Theorem 3.7 shows that this is equivalent to the existence of a $\top$-subnet t of s such that $\mathbb{G} = \mathbb{F}_t$, converging to x. But this means that the subnet t converges to x. □

We now characterize cluster points using the closure.

Proposition 4.8. *Let (X,τ) be a strong L-topological space, let $\mathbb{F}$ be a $\top$-filter on X and let $s = (s_X, s_L) : D \longrightarrow X$ be a $\top$-net in X.*

1. *x is a cluster point of $\mathbb{F}$ if and only if $\overline{f}(x) = \top$ for all $f \in \mathbb{F}$;*
2. *x is a cluster point of s if and only if $\overline{b^s_d}(x) = \top$ for all $d \in D$.*

Proof. (1) Let first x be a cluster point of $\mathbb{F}$ and let $f \in \mathbb{F}$. Then $\mathbb{F} \vee \mathbb{U}^x_\tau$ exists and converges to x. Also $f \wedge u$ is in $\mathbb{F} \vee \mathbb{U}^x_\tau$ for all $u \in \mathbb{U}^x_\tau$. We conclude

$$\overline{f}(x) \geq \bigvee_{g\in\mathbb{F}\vee\mathbb{U}^x_\tau} [g,f] \geq \bigvee_{u\in\mathbb{U}^x_\tau} [f\wedge u, f] = \top.$$

Conversely, let $\overline{f}(x) = \top$ for all $f \in \mathbb{F}$. We fix $f \in \mathbb{F}$. Then

$$\top = \bigvee_{\mathbb{G}\geq\mathbb{U}^x_\tau} \bigvee_{g\in\mathbb{G}} [g,f] = \bigvee_{\mathbb{G}\geq\mathbb{U}^x_\tau} \bigvee_{g\in\mathbb{G}} \bigwedge_{z\in X} (g(z) \to f(z))$$

Let $\alpha \lhd \top$. Then there is $\mathbb{G} \geq \mathbb{U}^x_\tau$ and $g \in \mathbb{G}$ such that for all $z \in X$ we have $\alpha * g(z) \leq f(z)$. Let $u \in \mathbb{U}^x_\tau$. Then $g \wedge u \in \mathbb{G}$ and hence $\bigvee_{z\in X} g \wedge u(z) = \top$. We conclude $(g \wedge u(z)) * \alpha \leq f \wedge u(z)$ for all $z \in X$ and hence

$$\alpha = \alpha * \bigvee_{z\in X} g \wedge u(z) \leq \bigvee_{z\in X} f \wedge u(z).$$

The complete distributivity then yields $\top = \bigvee_{z\in X} f \wedge u(z)$. Hence $\mathbb{F} \vee \mathbb{U}^x_\tau$ exists and x is a cluster point of $\mathbb{F}$.

(2) A point x is a cluster point of s if and only if it is a cluster point of $\mathbb{F}_s$. According to (1) this is equivalent to $\overline{f}(x) = \top$ for all $f \in \mathbb{F}_s$ and this implies, the "tails" b^s_d being members of $\mathbb{F}_s$, that $\overline{b^s_d}(x) = \top$.

Conversely, if $\overline{b_d}(x) = \top$ for all $d \in D$, then for $f \in \mathbb{F}_s$ we conclude

$$\top = \bigvee_{d\in D} [b^s_d, f] \leq \bigvee_{d\in D} [\overline{b^s_d}, \overline{f}] \leq \bigvee_{d\in D} \overline{b^s_d}(x) \to \overline{f}(x) = \overline{f}(x).$$

Hence x is a cluster point of $\mathbb{F}_s$, which means that x is a cluster point of s. □

We can characterize continuity by convergence.

Proposition 4.9. *Let $(X,\tau),(Y,\sigma)$ be strong L-topological spaces and let $\varphi : X \longrightarrow Y$ be a mapping. The following assertions are equivalent.*

1. *φ is continuous;*
2. *for all $\mathbb{F} \in \mathsf{F}^L_\top(X)$, $\varphi(\mathbb{F})$ converges to $\varphi(x)$ whenever $\mathbb{F}$ converges to x;*
3. *for all $\top$-nets s on X, $\varphi(s)$ converges to $\varphi(x)$ whenever s converges to x.*

Proof. The equivalence of (1) and (2) is not difficult and not shown. We show the equivalence of (2) and (3). If the $\top$-net s converges to x, then $\mathbb{F}_s \geq \mathbb{U}^x_\tau$ and hence, using Proposition 3.3 and (2), $\mathbb{F}_{\varphi(s)} = \varphi(\mathbb{F}_s) \geq \mathbb{U}^{\varphi(x)}_\sigma$. This means that $\varphi(s)$ converges to $\varphi(x)$. Conversely, if (3) is valid and $\mathbb{F}$ converges to x, then with Proposition 3.5 we get $\mathbb{F}_{(s_\mathbb{F})} = \mathbb{F} \geq \mathbb{U}^x_\tau$, i.e. the $\top$-net $s_\mathbb{F}$ converges to x. With (3) then also $\varphi(s)$ converges to $\varphi(x)$ which means $\varphi(\mathbb{F}) = \varphi(\mathbb{F}_{(s_\mathbb{F})}) = \mathbb{F}_{\varphi(s_\mathbb{F})} \geq \mathbb{U}^{\varphi(x)}_\sigma$, i.e. $\varphi(\mathbb{F})$ converges to $\varphi(x)$. □

We now turn our attention to separation. A strong L-topological space (X,τ) is called *$\top$-Hausdorff separated* [10] if for $x,y \in X$, $x \neq y$ there are $u \in \mathbb{U}^x_\tau$, $v \in \mathbb{U}^y_\tau$ such that $\bigvee_{z\in X} u \wedge v(z) \neq \top$.

Proposition 4.10. *Let (X,τ) be a strong L-topological space. Then*

1. *(X,τ) is $\top$-Hausdorff separated if and only if each $\top$-filter converges to at most one point;*
2. *(X,τ) is $\top$-Hausdorff separated if and only if each $\top$-net converges to at most one point.*

Proof. We only prove (2). Let (X,τ) be $\top$-Hausdorff separated and assume that the $\top$-net converges to x and y. Then $\mathbb{F}_s \geq \mathbb{U}^x_\tau$ and $\mathbb{F}_s \geq \mathbb{U}^y_\tau$ and hence $\mathbb{U}^x_\tau \vee \mathbb{U}^y_\tau$ exists. Therefore, for all $u \in \mathbb{U}^x_\tau$ and all $v \in \mathbb{U}^y_\tau$ we have $\bigvee_{z\in X} u \wedge v(z) = \top$, a contradiction.

Conversely, let each $\top$-net converge to only one point and assume that $\bigvee_{z\in X} u \wedge v(z) = \top$ for all $u \in \mathbb{U}^x_\tau$ and all $v \in \mathbb{U}^y_\tau$. Then $\mathbb{F} = \mathbb{U}^x_\tau \vee \mathbb{U}^y_\tau$ exists and, as $\mathbb{F}_{(s_\mathbb{F})} = \mathbb{F}$, $s_\mathbb{F}$ converges to both x and y. Hence $x = y$. □

Without going into more details we have shown in this section that $\top$-nets, like $\top$-filters, are versatile tools for the theory of strong L-topological spaces. We would simply like to mention that compactness of a space can be defined by the requirement that each $\top$-net has a cluster point or, equivalently, that each $\top$-net has a convergent $\top$-subnet.

5 A diagonal principle

We first need some preparations, where we follow the work of Fang and Yue [7]. Let J be a set. For a "selection function" $\sigma : J \longrightarrow \mathsf{F}^\top_L(X)$ and $f \in L^X$ we define $\widehat{\sigma}(f) \in L^J$ by $\widehat{\sigma}(j) = \bigvee_{h\in\sigma(j)}[h,f]$ for $j \in J$. Then, for $\mathbb{G} \in \mathsf{F}^\top_L(J)$ we define $\kappa\sigma\mathbb{G} \in \mathsf{F}^\top_L(X)$ by $f \in \kappa\sigma\mathbb{G}$ if and only if $\widehat{\sigma}(f) \in \mathbb{G}$. The $\top$-filter $\kappa\sigma\mathbb{G}$ is called the *$\top$-diagonal filter* of $(\mathbb{G},\sigma)$.

The next property of the $\top$-neighborhood filters is well-known but we shall provide a proof because it is important for us later and to point out that the assumption of a complete MV-algebra, which is usually assumed in the corresponding papers, is not needed here.

Proposition 5.1. *Let (X,τ) be a strong L-topological space. We define a selection function $\sigma_N : X \longrightarrow \mathsf{F}^\top_L(X)$ by $\sigma_N(y) = \mathbb{U}^y_\tau$ for $y \in X$. Then we have $\mathbb{U}^x_\tau \leq \kappa\sigma_N\mathbb{U}^x_\tau$ for all $x \in X$.*

Proof. From Proposition 4.1 we know that for $u \in \mathbb{U}^x_\tau$ we have $\mathrm{int}(u) = \widehat{\sigma_N}(u)$. Hence, using (I5), we have for $u \in \mathbb{U}^x_\tau$ that $\mathrm{int}(\mathrm{int}(u))(x) = \top$, i.e. that $\mathrm{int}(u) = \widehat{\sigma_N}(u) \in \mathbb{U}^x_\tau$, which means that $u \in \kappa\sigma_N\mathbb{U}^x_\tau$. □

We note that the other inequality is always true [6], i.e. that we have $\mathbb{U}^x_\tau = \kappa\sigma_N\mathbb{U}^x_\tau$ for all $x \in X$. Fang and Yue [7] show that Proposition 5.1 implies the following result. Again an MV-algebra is not needed here.

Proposition 5.2 ([7]). *Let (X,τ) be a strong* L*-topological space. Then the following axiom (⊤-F) is true. For any selection function $\sigma : J \longrightarrow \mathsf{F}_L^\top(X)$, $\mathbb{G} \in \mathsf{F}_L^\top(J)$ and mapping $\varphi : J \longrightarrow X$ we have: if $\sigma(j) \xrightarrow{\tau} \varphi(j)$ for all $j \in J$ and if $\varphi(\mathbb{G}) \xrightarrow{\tau} x$ then $\kappa\sigma\mathbb{G} \xrightarrow{\tau} x$.*

We will now use this result and show a diagonal principle for ⊤-nets in a strong L-topological space (X,τ). Again, we first need some preparations.

If $s : D \longrightarrow X \times L^*$ is a ⊤-net and $d \in D$, then also $D^d = \{e \in D \ : \ e \succ d\}$ is directed and $s^d : D^d \longrightarrow X \times L^*$ defined by $s^d_X(e) = s_X(e), s^d_L(e) = s_L(e)$ for $e \in D^d$ is a ⊤-net. If $s \xrightarrow{\tau} x$, then we have $\bigvee_{d\in D} \bigwedge_{e\succ d}(s_(e) \to u(s_X(e))) = \top$ for all $u \in \mathbb{U}^x_\tau$. If $\eta \lhd \top$ there is $d_0 \in D$ such that for all $e \succ d_0$ we have $\eta \le s_L(e) \to u(s_X(e))$. We choose $d_1 \succ d, d_0$. Then $d_1 \in D^d$ and for all $e \succ d_1$ we have $\eta \le s_L(e) \to u(s_X(e))$. Hence

$$\eta \le \bigwedge_{e\succ d_1} (s_L(e) \to u(s_X(e))) \le \bigvee_{d_1\in D^d} \bigwedge_{e\succ d_1} (s_L(e) \to u(s_X(e))).$$

The complete distributivity then yields $\top = \bigvee_{d_1\in D^d} \bigwedge_{e\succ d_1}(s_L(e) \to u(s_X(e)))$ for all $u \in \mathbb{U}^x_\tau$ which means that also $s^d \xrightarrow{\tau} x$.

If $(D_j, \prec_j)$ are directed sets for all $j \in J$, then also the product $\prod_{j\in J} D_j$ becomes directed by the product order, i.e. $(d_j)_{j\in J} \prec (e_j)_{j\in J}$ if and only if for all $j \in J$ we have $d_j \prec_j e_j$. We will in the sequel, to simplify the notation, write $\prec$ for all orders and hope that the set, on which this order is defined, will be clear from the context.

Let D and E_d be directed sets for each $d \in D$ and denote $J = \bigcup_{d\in D}(\{d\} \times E_d)$. For $(d, e), (\overline{d}, \overline{e}) \in J$ we define $(\overline{d}, \overline{e}) \succ (d, e)$ if $\overline{d} \succ d$ or if $\overline{d} = d$ and $\overline{e} \succ e$. It is not difficult to show that $(J, \prec)$ is a directed set.

We consider now a ⊤-net $s : J \longrightarrow X \times L^*$, $(d, e) \longmapsto (s_X(d, e), s_L(d, e))$ such that for all $d \in D$, $s^d : E_d \longrightarrow X \times L$, $e \longmapsto (s^d_X(e) = s_X(d, e), s^d_L(e) = s_L(d, e))$ is a ⊤-net which converges to a point $y_d \in X$, i.e. $s^d \xrightarrow{\tau} y_d$. Furthermore, the ⊤-net $y : D \longrightarrow X \times L^*$, defined by $y_X(d) = y_d$, $y_L(d) = \top$ for $d \in D$ shall converge to $x \in X$, i.e. we have $y \xrightarrow{\tau} x$. We shall write $(y_d, \top)$ for y.

We denote $F = D \times \prod_{d\in D} E_d$ and define the ⊤-net $r : F \longrightarrow J \times L^*$ by $r_X(d, (e_j)) = (d, e_d)$ and $r_L(d, (e_j)) = \top$. This ⊤-net is used to select a "diagonal ⊤-net" from s, defined by

$$s \circ r : \left\{ \begin{array}{ccc} F & \longrightarrow & X \times L^* \\ (d, (e_j)) & \longmapsto & (s_X(d, e_d), s_L(d, e_d)) \end{array} \right. .$$

We note that $s \circ r$ is a ⊤-net. We are now in the position to state the "diagonal principle".

Theorem 5.3. *Let (X,τ) be a strong* L*-topological space and define, as above, $J = \bigcup_{d\in D}(\{d\} \times E_d)$ and $F = D \times \prod_{d\in D} E_d$ and the ⊤-nets $s : J \longrightarrow X \times L^*$, $s^d : E_d \longrightarrow X \times L^*$, $r : F \longrightarrow J \times L^*$ and $s \circ r : F \longrightarrow X \times L^*$.*

If $s^d \xrightarrow{\tau} y_d$ for each $d \in D$ and $(y_d, \top) \xrightarrow{\tau} x$, then there is a ⊤-subnet t of $s \circ r$, a "diagonal ⊤-net", with $t \xrightarrow{\tau} x$.

Proof. For $e \in E_d$ we define $s^{de} : E^e_d = \{f \in E_d \ : \ f \succ e\} \longrightarrow X \times L^*$, $f \longmapsto (s^d_X(f), s^d_L(f))$. With this we define the selection mapping $\sigma : J \longrightarrow \mathsf{F}^\top_L(X)$ by $\sigma(d, e) = \mathbb{F}_{s^{de}}$. Furthermore we define $\varphi : J \longrightarrow X$ by $\varphi(d, e) = y_d$. Then $\sigma(d, e) \xrightarrow{\tau} \varphi(d, e)$ for all $(d, e) \in J$. For $\mathbb{F}_r \in \mathsf{F}^\top_L(J)$ we have $\varphi(\mathbb{F}_r) = \mathbb{F}_{\varphi(r)}$ with $\varphi(r) = (\varphi \circ r_X, r_L)$, i.e. $\varphi(r)(d, (e_j)) = (\varphi(d, e_d), \top) = (y_d, \top)$ for $(d, (e_j)) \in F$. Hence $\varphi(\mathbb{F}_r) = \mathbb{F}_y \xrightarrow{\tau} x$. The axiom (⊤-F) then yields $\kappa\sigma\mathbb{F}_r \xrightarrow{\tau} x$.

We now show $\mathbb{F}_{s\circ r} \le \kappa\sigma\mathbb{F}_r$. First, let $f \in L^X$. Then $\widehat{\sigma}(f) \in L^J$ is defined by

$$\widehat{\sigma}(f)(d, e) = \bigvee_{h\in\mathbb{F}_{s^{de}}} [h, f] = \bigvee_{\overline{e}\in E^e_d} [b^{s^{de}}_{\overline{e}}, f] = \bigvee_{\overline{e}\in E^e_d} \bigwedge_{\overline{\overline{e}}\succ\overline{e}} (s_L(d, \overline{\overline{e}}) \to f(s_X(d, \overline{\overline{e}}))).$$

Hence we have $f \in \kappa\sigma\mathbb{F}_r$ if and only if $\widehat{\sigma}(f) \in \mathbb{F}_r$ if and only if

$$\top = \bigvee_{(d,(e_j))\in F} \bigwedge_{(\overline{d},(\overline{e}_j))\succ(d,(e_j))} \widehat{\sigma}(f)(\overline{d}, \overline{e}_{\overline{d}}) = \bigvee_{(d,(e_j))\in F} \bigwedge_{(\overline{d},(\overline{e}_j))\succ(d,(e_j))} \bigvee_{\widetilde{e}\in E^{\overline{e}_{\overline{d}}}_{\overline{d}}} \bigwedge_{\widetilde{\widetilde{e}}\succ\widetilde{e}} (s_L(d,\widetilde{\widetilde{e}}) \to f(s_X(d,\widetilde{\widetilde{e}}))).$$

Let now $f \in \mathbb{F}_{s\circ r}$. Then

$$\top = \bigvee_{(d,(e_j))\in F} \bigwedge_{(\overline{d},(\overline{e}_j))\succ(d,(e_j))} (s_L(\overline{d}, \overline{e}_{\overline{d}}) \to f(s_X(\overline{d}, \overline{e}_{\overline{d}}))).$$

Let $\eta \lhd \top$. Then there is $(d,(e_j)) \in F$ such that for all $(\overline{d},(\overline{e}_j)) \succ (d,(e_j))$ we have $\eta \leq s_L(\overline{d}, \overline{e}_{\overline{d}}) \to f(s_X(\overline{d}, \overline{e}_{\overline{d}}))$. Then $\overline{e}_{\overline{d}} \in E^{\overline{e}_{\overline{d}}}_{\overline{d}}$ and for $\widetilde{\widetilde{e}} \succ \overline{e}_{\overline{d}}$ we define $(\overline{d},(e^*_j)) \in F$ by $e^*_j = e_j$ for $j \neq d$ and $e^*_{\overline{d}} = \widetilde{\widetilde{e}}$. Then $(\overline{d},(e^*_j)) \succ (d,(e_j))$ and hence

$$\eta \leq s_L(\overline{d}, e^*_{\overline{d}}) \to f(s_X(\overline{d}, e^*_{\overline{d}})) = s_L(\overline{d}, \widetilde{\widetilde{e}}) \to f(s_X(\overline{d}, \widetilde{\widetilde{e}})).$$

Therefore we obtain

$$\eta \leq \bigwedge_{\widetilde{\widetilde{e}}\succ\overline{e}_{\overline{d}}} (s_L(\overline{d}, \widetilde{\widetilde{e}}) \to f(s_X(\overline{d}, \widetilde{\widetilde{e}}))) \leq \bigvee_{\overline{e}\in E^{\overline{e}_{\overline{d}}}_{\overline{d}}} \bigwedge_{\widetilde{\widetilde{e}}\succ\overline{e}_{\overline{d}}} (s_L(\overline{d}, \widetilde{\widetilde{e}}) \to f(s_X(\overline{d}, \widetilde{\widetilde{e}}))).$$

This holds for all $(\overline{d},(\overline{e}_j)) \succ (d,(e_j))$ and we get

$$\eta \leq_{(d,(e_j))\in F} \bigwedge_{(\overline{d},(\overline{e}_j))\succ(d,(e_j))} \bigvee_{\widetilde{e}\in E^{\overline{e}_{\overline{d}}}_{\overline{d}}} \bigwedge_{\widetilde{\widetilde{e}}\succ\widetilde{e}} (s_L(d,\widetilde{\widetilde{e}}) \to f(s_X(d,\widetilde{\widetilde{e}}))).$$

This is true for all $\eta \lhd \top$ and the complete distributivity then yields $f \in \kappa\sigma\mathbb{F}_r$.

Hence we have shown $\mathbb{F}_r \leq \kappa\sigma\mathbb{F}_r$ and we conclude from Theorem 3.7 that there is a $\top$-subnet t of $s \circ r$ with $\mathbb{F}_t = \kappa\sigma\mathbb{F}_r$, i.e. $t \xrightarrow{\tau} x$. □

6 First countable spaces and $\top$-sequences

We call a strong L-topological space *first countable* if for each $x \in X$ the $\top$-neighborhood filter $\mathbb{U}^x_\tau$ has a countable $\top$-basis.

In first countable spaces, $\top$-sequences suffice for the definition and study of most concepts. We shall illustrate this with one example.

Proposition 6.1. *Let the lattice L have a sequence $\bot \neq \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq ...$ with $\alpha_k \lhd \top$ for all $k = 1, 2, 3, ...$ and $\bigvee_{k=1}^{\infty} \alpha_k = \top$ and let (X, τ) be a first countable, strong L-topological space. Then for $a \in L^X$ and $x \in X$ we have*

$$\overline{a}(x) = \bigvee_{t\to x, t\ \top\text{-sequence}} \bigvee_{n\in\mathbb{N}} [b^t_n, a],$$

where $b^t_n = \bigvee_{k\geq n} t_L(k) * \top_{t_X(k)}$ is a "tail" of the $\top$-sequence $t = (t_X, t_L) : \mathbb{N} \longrightarrow X \times L^*$.

Proof. As $\top$-sequences are $\top$-nets, we obtain from Proposition 4.7 that $\bigvee_{t\to x} \bigvee_{n\in\mathbb{N}} [b^t_n, a] \leq \overline{a}(x)$, where the first join extends of all $\top$-sequences t that converge to x. For the converse, let $\eta \lhd \overline{a}(x)$. Then there is a $\top$-net $s = (s_X, s_L) : D \longrightarrow X \times L^*$ converging to x and a $d \in D$ such that $[b^s_d, a] \geq \eta$. We consider a countable $\top$-basis $v_1, v_2, v_3, ...$ of $\mathbb{U}^x_\tau$. Then $b^s_d \in \mathbb{F}_s \geq \mathbb{U}^x_\tau$ and hence we have $b^s_d \wedge v_1, b^s_d \wedge v_2, ... \in \mathbb{F}_s$. For $\alpha_k \lhd \top = \bigvee_{x\in X} b^s_d \wedge v_k(x)$ we choose $x_k \in X$ such that $b^s_d(x_k) \wedge v_k(x_k) \geq \alpha_k$ for $k = 1, 2, 3, ...$ and we consider the $\top$-sequence $t = (x_k, \alpha_k)$. As $\alpha_1, \alpha_2, ... \leq \alpha_n$ for each $n \in \mathbb{N}$, we have $\bigvee_{k\geq n} \alpha_k = \bigvee_{k=1}^\infty \alpha_k = \top$, i.e. t is in fact a $\top$-sequence. For a "tail" $b^t_k = \bigvee_{n\geq k} \alpha_k * \top_{x_k}$ we have

$$\begin{aligned} b^t_k(x) &\leq \bigvee_{n\geq k} (b^s_d(x_n) \wedge v_k(x_n)) * \top_{x_n}(x) \\ &= \begin{cases} \bot & \text{if } x \neq x_n \text{ for all } n \geq k \\ \bigvee_{n\geq k, x=x_n} b^s_d(x_n) \wedge v_k(x_n) & \text{if } x = x_n \text{ for some } n \geq k \end{cases} \\ &= \begin{cases} \bot & \text{if } x \neq x_n \text{ for all } n \geq k \\ b^s_d(x) \wedge v_k(x) & \text{if } x = x_n \text{ for some } n \geq k \end{cases} \\ &\leq b^s_d(x) \wedge v_k(x). \end{aligned}$$

Hence $b^t_k \leq b^s_d \wedge v_k$ and we therefore conclude that $v_k \in \mathbb{F}_t$ for all $k = 1, 2, ...$, i.e. $\mathbb{U}^x_\tau \leq \mathbb{F}_t$ and $t \to x$. Moreover we have $\bigvee_{n=1}^\infty [b^t_k, a] \geq [b^s_d, a] \geq \eta$. This is true for all $\eta \lhd \overline{a}(x)$ and the missing inequality follows. □

7 Conclusions

We have shown in this paper that besides a convergence theory based on $\top$-filters, also a convergence theory based on $\top$-nets is available in strong L-topological spaces. Both concepts seem equivalent to one another in the sense that definitions and proofs that are given using one concept can also be given using the other. This was demonstrated with some examples like interior and closure of an L-set, cluster points of $\top$-filters or $\top$-nets, continuity, and Hausdorff separation.

It was shown in [7] that $\top$-filters can be used to develop an abstract theory of $\top$-convergence spaces and, similarly, for a theory of $\top$-uniform convergence spaces [16]. It seems that also $\top$-nets could be used for such a purpose. This research question is left open at this stage.

Important for the theory may be the concept of a $\top$-sequence as a special case of a $\top$-net. This concept will allow to naturally extend and study notions like countable compactness or countable completeness and so on. We will postpone this, however, to future work.

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