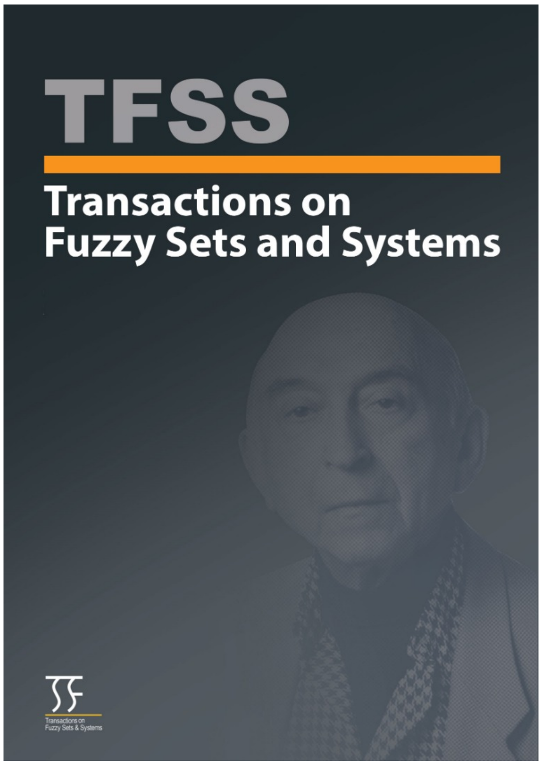
Transactions on Fuzzy Sets and Systems



Transactions on Fuzzy Sets and Systems

ISSN: **2821-0131**

https://sanad.iau.ir/journal/tfss/

Triangle Algebras and Relative Co-annihilators

Vol.3, No.1, (2024), 43-56. DOI: http://doi.org/10.30495/tfss.2024.2000172.1088

Author(s):

Emile Djomgoue Nana, Department of Mathematics and Computer Science, University of Dschang, Dschang, Cameroon. E-mail: emiledjomgouenana@gmail.com
Ariane Gabriel Tallee Kakeu, Department of Mathematics and Computer Science, University of Dschang, Dschang, Cameroon. E-mail: arianekakeugabie@yahoo.fr
Blaise Bleriot Koguep Njionou, Department of Mathematics and Computer Science, University of Dschang, Dschang, Cameroon. E-mail: blaise.koguep@univ-dschang.org
Lele Celestin, Department of Mathematics and Computer Science, University of Dschang, Dschang, Cameroon. E-mail: celestinlele@yahoo.com

Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://sanad.iau.ir/journal/tfss/
Online ISSN: 2821-0131
Vol.3, No.1, (2024), 43-56
DOI: http://doi.org/10.30495/tfss.2024.2000172.1088

Article Type: Original Research Article

Triangle Algebras and Relative Co-annihilators

Emile Djomgoue Nana, **Ariane Gabriel Tallee Kakeu***, **Blaise Bleriot Koguep Njionou**, **Lele Celestin**

Abstract. Triangle algebras are an important variety of residuated lattices enriched with two approximation operators as well as a third angular point (different from 0 and 1). They provide a well-defined mathematical framework for formalizing the use of closed intervals derived from a bounded lattice as truth values, with a set of structured axioms. This paper introduces the concept of relative co-annihilator of a subset within the framework of triangle algebras. As filters of triangle algebras, these relative co-annihilators are explored and some of their properties and characterizations are given. A meaningful contribution of this work lies in its proof that the relative co-annihilator of a subset T with respect to another subset Y in a triangle algebra $\mathcal{L}$ inherits specific filter's characteristics of Y. More precisely, if Y is a Boolean filter of the second kind, then the co-annihilator of T with respect to Y is also a Boolean filter of the second kind. The same statement applies when we replace the Boolean filter of the second kind with an implicative filter, pseudo complementation filter, Boolean filter, prime filter, prime filter of the third kind, pseudo-prime filter, or involution filter, respectively. Finally, we establish some conditions under which the co-annihilator of T relative to Y is a prime filter of the second kind.

AMS Subject Classification 2020: MSC 03G10; MSC 03G25; MSC 08A72
Keywords and Phrases: Triangularization, Triangle algebra, Filter, Co-annihilator, Relative co-annihilator.

1 Introduction

George Boole's endeavor to formalize propositional logic led to the concept of Boolean algebra ([1]). Unfortunately, the discrete nature of the truth values fails to handle situations in which the accuracy of statements is not precisely known. In his attempt to solve this problem, Zadeh ([13]) proposed the idea of working with the unit interval $[0, 1]$ equipped with the usual order, giving rise to fuzzy logic. Considering the potential non-comparability of elements within the set of truth values, a substantial advancement occurred in 1967 when Goguen [2] brought in a novel approach: replacing the unit interval with a bounded lattice, and using triangular norms and co-norms to extend the concepts of logical conjunction and disjunction. Among the significant features of triangular norms and co-norms, their compatibility with the principle of residuation stands out, resulting in the algebraic structure called *residuated lattice* (see [12]). In 2008, Van Gasse et al. ([8]) established residuated lattices based on lattices of closed intervals, also known as triangular lattices, thereby introducing the concept of *Interval-valued residuated lattices (IVRLs)*. Subsequently, they equipped the latter with two approximation operators and with a third angular point, leading to the so called *extended Interval-valued residuated lattices*, which are equationally represented by *triangle algebras* [10].

*__Corresponding Author:__ Ariane Gabriel Tallee Kakeu, Email: arianekakeugabie@yahoo.fr, ORCID: 0000-0003-4420-1835
Received: 31 October 2023; **Revised**: 30 December 2023; **Accepted**: 23 January 2024; **Available Online**: 23 January 2024; **Published Online**: 7 May 2024.

How to cite: Djomgoue Nana E, Gabriel Tallee Kakeu A, Bleriot Koguep Njionou B, Celestin L. Triangle algebras and relative co-annihilators. *Transactions on Fuzzy Sets and Systems.* 2024; 3(1): 43-56. DOI: http://doi.org/10.30495/tfss.2024.2000172.1088

A crucial concept in algebraic structures used for formal fuzzy logic, is that of a filter, since filters have a natural interpretation as sets of provable formulas, and therefore are important in the proof of the completeness of these logics. Indeed, the theory of triangle algebras has been endowed with the filter theory (see [11, 16, 15]). In 2017, Zahiri et al. [16] conducted an investigation into a particular class of filters in triangle algebras, namely, co-annihilators. Our main purpose is to introduce and thoroughly explore relative co-annihilators in triangle algebras, as a generalization of co-annihilators.

In the literature, the concept of co-annihilator of an element a relative to a filter F was introduced in BL-algebra by Meng and Xin [6]. Following this, Maroof et al. ([5]) and Rasouli ([7]) extended this notion to residuated lattices. In [5], they examinined the co-annihilator of an arbitrary subset T with respect to another subset Y within a residuated lattice. Nevertheless, the concept of relative co-annihilator remains unexplored in triangle algebras.

This paper is organized as follows: In **Section 2**, we recall some preliminary notions in order to make the document self-contained. **Section 3** is devoted to the notion of relative co-annihilator in triangle algebras, with some of its properties. In **Section 4**, we provide more properties of relative co-annihilators through filters of triangle algebras. We prove that for any two nonempty subsets T and Y of a triangle algebras $\mathcal{L}$, if Y is a Boolean filter of the second kind (respectively, pseudo-complementation filter, implicative filter, Boolean filter, prime filter, prime filter of the third kind, pseudo-prime filter, involution filter), then, so is the co-annihilator of T relative to Y. Finally, we highlight some conditions under which the co-annihilator of T relative to Y is a prime filter of the second kind.

2 Preliminaries

In this section, we recall some notions that will be useful in this paper.

Definition 2.1. [3, 12] A *residuated lattice* is an algebra $\mathcal{L} = (L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ with four binary operations and two constants such that:

(R1) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice;

(R2) $(L, \odot, 1)$ is a commutative monoid;

(R3) $x \odot y \leq z$ iff $x \leq y \rightarrow z$, for all x, y and z in L.

Unless otherwise specified, by $\mathcal{L}$ we will denote the residuated lattice $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$. The negation $\neg$ in $\mathcal{L}$ is defined by $\neg x = x \rightarrow 0$, for all x in L.

Theorem 2.2. *[4, 5, 10, 12] Let $\mathcal{L}$ be a residuated lattice. Then, the following properties are valid, for all $x, x_1, x_2, y, y_1, y_2, z \in L$:*

(RL1) $1 \rightarrow x = x, x \rightarrow x = 1$, $\neg 1 = 0$, *and* $\neg 0 = 1$;

(RL2) $x \odot y \leq x, y$ *hence* $x \odot y \leq x \wedge y$, $y \leq x \rightarrow y$ *and* $x \odot 0 = 0$;

(RL3) $x \odot y \leq x \rightarrow y$, *and* $x \odot y = 0$ *iff* $x \leq \neg y$;

(RL4) $x \leq y$ *iff* $x \rightarrow y = 1$;

(RL5) $x \odot (x \rightarrow y) \leq y$, $x \leq (x \rightarrow y) \rightarrow y$, $((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y$;

(RL6) $x \rightarrow y \leq (x \odot z) \rightarrow (y \odot z)$;

(RL7) $x \leq y$ *implies* $(x \odot z) \leq (y \odot z)$, $z \rightarrow x \leq z \rightarrow y$, $y \rightarrow z \leq x \rightarrow z$, *and* $\neg y \leq \neg x$;

(RL8) $x \to y \leq (z \to x) \to (z \to y)$, $x \to y \leq (y \to z) \to (x \to z)$;

(RL9) $x \odot (y \to z) \leq y \to (x \odot z) \leq (x \odot y) \to (z \odot z)$;

(RL10) $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z)$;

(RL11) $x_1 \to y_1 \leq (y_2 \to x_2) \to [(y_1 \to y_2) \to (x_1 \to x_2)]$;

(RL12) $(x \to z) \vee (y \to z) \leq x \wedge y \to z$;

(RL13) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$, $z \vee (x \odot y) \geq (z \vee x) \odot (z \vee y)$;

(RL14) $x \leq \neg\neg x \leq \neg x \to x$, $\neg\neg\neg x = \neg x$;

(RL15) $\neg(x \odot y) = x \to \neg y, y \to \neg x = \neg\neg x \to \neg y$, *and* $x \to y \leq \neg y \to \neg x$.

Recall from [11] that a *filter* of a residuated lattice $\mathcal{L}$ is a nonempty subset F of L such that for all $x, y \in L$:

(F1) if $x \in F$ and $x \leq y$, then $y \in F$;

(F2) if $x, y \in F$, then $x \odot y \in F$.

We now recall the notion of interval-valued residuated lattices, which are residuated lattices on triangularizations. This has led to the development of triangle algebras through the use of approximation operators, describing the aspect of incompleteness inherent in interval-valued residuated lattices.

Definition 2.3. [8, 10] Let $\mathcal{L} = (L, \vee, \wedge, 0, 1)$ be a bounded lattice. We call *triangularization or triangular lattice* of $\mathcal{L}$ the bounded lattice, $\mathbb{T}(\mathcal{L})$ of the closed intervals of L defined by

$$\mathbb{T}(\mathcal{L}) = (Int(\mathcal{L}), \vee_{Int(\mathcal{L})}, \wedge_{Int(\mathcal{L})}, [0,0], [1,1])$$

such that $Int(\mathcal{L}) = \{[x_1, x_2] : x_1, x_2 \in L \text{ and } x_1 \leq x_2\}$, and for all $x_1, x_2, y_1, y_2 \in L$,

- $[x_1, x_2] \vee_{Int(\mathcal{L})} [y_1, y_2] = [x_1 \vee y_1, x_2 \vee y_2]$;
- $[x_1, x_2] \wedge_{Int(\mathcal{L})} [y_1, y_2] = [x_1 \wedge y_1, x_2 \wedge y_2]$;
- $[x_1, x_2] \leq_{Int(\mathcal{L})} [y_1, y_2]$ iff $x_1 \leq y_1$ and $x_2 \leq y_2$.

The set $D(\mathcal{L}) = \{[x, x] : x \in L\}$ is called *diagonal* of $\mathbb{T}(\mathcal{L})$.

From [8, 10], an *interval-valued residuated lattice (IVRL)* is a residuated lattice $(\text{Int}(\mathcal{L}), \vee, \wedge, \odot, \to_{\odot}, [0,0], [1,1])$ on the triangularization $\mathbb{T}(\mathcal{L})$ of a bounded lattice $\mathcal{L}$, in which the diagonal $D(\mathcal{L})$ is closed under $\odot$ and $\to_{\odot}$, i.e., $[x,x] \odot [y,y] \in D(\mathcal{L})$ and $[x,x] \to_{\odot} [y,y] \in D(\mathcal{L})$, for all x, y in L.

Definition 2.4. [10, 9] An *extended IVRL* is a structure $(Int(\mathcal{L}), \vee, \wedge, \odot, \to, pr_v, pr_h, [0,0], [0,1], [1,1])$ where $u = [0,1]$ is a constant interval, pr_v and pr_h are maps from $Int(\mathcal{L})$ to $Int(\mathcal{L})$, respectively called vertical and horizontal projections defined by $pr_v([x_1, x_2]) = [x_1, x_1]$ and $pr_h([x_1, x_2]) = [x_2, x_2]$, for all $[x_1, x_2] \in Int(\mathcal{L})$.

The following definition presents the concept of triangle algebra, which serves as an equational representation of interval-valued residuated lattices.

Definition 2.5. [8, 18] A *triangle algebra* is a structure $\mathcal{L} = (L, \vee, \wedge, \odot, \rightarrow, \nu, \mu, 0, u, 1)$ in which $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a residuated lattice, ν and μ are unary operations on L, u $(0 \neq u \neq 1)$ a constant, all satisfying the following conditions:

$(T.1)\nu x \leq x;$	$(T.1')\, x \leq \mu x;$
$(T.2)\nu x \leq \nu\nu x;$	$(T.2')\, \mu\mu x \leq \mu x;$
$(T.3)\nu(x \wedge y) = \nu x \wedge \nu y;$	$(T.3')\, \mu(x \wedge y) = \mu x \wedge \mu y;$
$(T.4)\nu(x \vee y) = \nu x \vee \nu y;$	$(T.4')\, \mu(x \vee y) = \mu x \vee \mu y;$
$(T.5)\nu u = 0;$	$(T.5')\, \mu u = 1;$
$(T.6)\nu\mu x = \mu x;$	$(T.6')\, \mu\nu x = \nu x;$
$(T.7)\nu(x \rightarrow y) \leq \nu x \rightarrow \nu y;$	
$(T.8)(\nu x \leftrightarrow \nu y) \odot (\mu x \leftrightarrow \mu y) \leq (x \leftrightarrow y);$	
$(T.9)\nu x \rightarrow \nu y \leq \nu(\nu x \rightarrow \nu y).$	

Note that the statement $x \leftrightarrow y$ stands for $(x \rightarrow y) \wedge (y \rightarrow x)$.

Remark 2.6. $\nu 0 = \mu 0 = 0$ and $\nu 1 = \mu 1 = 1$.

Unless otherwise specified, the triangle algebra $(L, \vee, \wedge, \odot, \rightarrow, \nu, \mu, 0, u, 1)$ will be denoted by $\mathcal{L}$.

Proposition 2.7. *[17] Let $\mathcal{L}$ be a triangle algebra. Then, for all $x, y \in L$ we have:*

1. $\nu(x \odot y) = \nu x \odot \nu y;$
2. $\mu(x \odot y) \leq \mu x \odot \mu y.$

Lemma 2.8. *Let $\mathcal{L}$ be a triangle algebra. For all $x, y \in L$, if $\nu x \vee y = 1$, then $x \odot y = x \wedge y$.*

Proof.

Let $x, y \in L$. We already know from $(RL2)$ of Theorem 2.2 that $x \odot y \leq x \wedge y$. All we need to prove is $x \wedge y \leq x \odot y$. We have:

$$\begin{aligned} x \wedge y &= 1 \odot (x \wedge y) & \\ &= (\nu x \vee y) \odot (x \wedge y), & \text{as } \nu x \vee y = 1 \\ &= [\nu x \odot (x \wedge y)] \vee [y \odot (x \wedge y)], & \text{from } (RL13) \\ &\leq (\nu x \odot y) \vee (x \odot y), & \text{as } x \wedge y \leq x, y \\ &\leq (x \odot y) \vee (x \odot y), & \text{as } \nu x \leq x \\ &= x \odot y. & \end{aligned}$$

□

Definition 2.9. [11, 18] A *filter* (or *IVRL-filter*) of a triangle algebra $\mathcal{L}$ is a nonempty subset F of $\mathcal{L}$ satisfying:

(F1) if $x \in F$, $y \in L$ and $x \leq y$, then $y \in F$;

(F2) if $x, y \in F$, then $x \odot y \in F$;

(F3) if $x \in F$, then $\nu x \in F$.

It is worth noticing that, for every filter F of a triangle algebra $\mathcal{L}$, $1 \in F$,
and [$x \in F$ if and only if $\nu x \in F$], see [14].

Definition 2.10. [15, 16] Let F be a filter of a triangle algebra $\mathcal{L}$. Then, F is said to be:

1. a *Boolean filter* (BF) if for all $x \in L$, $\nu(x \vee \neg x) \in F$.
2. a *Boolean filter of the second kind* (BF2) if for all $x \in L, \nu x \in F$ or $\nu(\neg x) \in F$.
3. a *prime filter* (PF) if for all $x, y \in L$, $\nu(x \to y) \in F$ or $\nu(y \to x) \in F$ (or both).
4. a *prime filter of the second kind* (PF2) if for all $x, y \in L$, $\nu(x \vee y) \in F$ implies $\nu x \in F$ or $\nu y \in F$ (or both).
5. a *prime filter of the third kind* (PF3) if for all $x, y \in L$, $\nu[(x \to y) \vee (y \to x)] \in F$.
6. a *pseudo-prime filter* (PPF) if for all $x, y \in L$, $\nu x \to \nu y \in F$ or $\nu y \to \nu x \in F$ (or both).
7. an *implicative filter* (IF) if for all $x, y, z \in L$, $\nu[x \to (y \to z)] \in F$ and $\nu(x \to y) \in F$ imply $\nu(x \to z) \in F$ (first form) or equivalently, $\nu[x \to (x \to z)] \in F$ implies that $\nu(x \to z) \in F$ (second form).
8. a *pseudocomplementation filter* (PSF) if for all $x \in L$, $\nu[\neg(x \wedge \neg x)] \in F$.
9. an *involution filter* (VF) iff for all $x \in L, \nu(\neg\neg x \to x) \in F$.

Proposition 2.11. *[16] Let F be a filter of $\mathcal{L}$. Then, F is an implicative filter iff $\nu(x \to x^2) \in F$, for all $x \in L$.*

Definition 2.12. [18]
Let A be a nonempty subset of a triangle algebra $\mathcal{L}$. Then, the *co-annihilator* of A, denoted by $A^\top$ is the filter defined by $A^\top = \{x \in L \mid \nu x \vee a = 1, \text{ for all } a \in A\}$.

3 Relative Co-annihilators in Triangle Algebras

In this section, we introduce the notion of relative co-annihilator in a triangle algebra $\mathcal{L}$ and investigate some of its properties.

Definition 3.1. Let $\mathcal{L}$ be a triangle algebra, A and B be subsets of L. The *co-annihilator of A relative to B* is the set $(A^\top, B) = \{a \in L \mid (\forall b \in A), \nu a \vee b \in B\}$.

If $B = \{x\}$, then we will denote $(A^\top, \{x\})$ by $(A^\top, x)$.
In a similar way, If $A = \{a\}$, then we will denote $(\{a\}^\top, B)$ by $(a^\top, B)$.

Remark 3.2. For any subset A of L, $(A^\top, 1) = A^\top$.

Example 3.3.
Let $L = \{[0,0], [0,a], [0,b], [a,a], [b,b], [0,1], [a,1], [b,1], [1,1]\}$ be the lattice whose associated Hasse diagram is depicted in Figure 1. Define $\odot$ and $\Rightarrow$ as presented in Table 1.

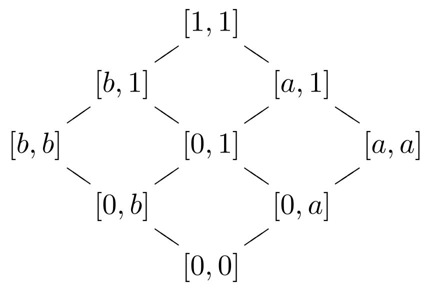


Figure 1: Hasse diagram of $\mathcal{L}$ in Example 3.3

Table 1: Operation tables of $\odot$ and $\Rightarrow$ for $\mathcal{L}$ in Example 3.3

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\Rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Consider the actions on L of ν, μ, $\odot$ and $\rightarrow$ defined as follows: for all $[x_1, x_2]$, $[y_1, y_2] \in L$, $\nu\,[x_1, x_2] = [x_1, x_1]$; $\mu\,[x_1, x_2] = [x_2, x_2]$; $[x_1, x_2] \odot_L [y_1, y_2] = [x_1 \odot y_1, x_2 \odot y_2]$; $[x_1, x_2] \rightarrow [y_1, y_2] = [(x_1 \Rightarrow y_1) \wedge (x_2 \Rightarrow y_2), x_2 \Rightarrow y_2]$.

Then, $\mathcal{L} = (L, \vee, \wedge, \odot_L, \rightarrow, \nu, \mu, [0, 0], [0, 1], [1, 1])$ is a triangle algebra [18].

Set $A = \{[b, b], [b, 1], [1, 1]\}$ and $B = \{[0, 0], [a, a], [1, 1]\}$. One easily verifies that $(A^\top, B) = \{[a, a], [a, 1], [1, 1]\}$. In a similar manner, $(B^\top, A) = \{[b, b], [b, 1], [1, 1]\}$.

Some of the following properties of relative co-annihilators have been established within the framework of residuated lattices (see [5]). Nevertheless, the formulations presented here are specific to triangle algebras, since the approximation operator ν is involved.

Proposition 3.4. *Let $\mathcal{L}$ be a triangle algebra. Let A and B be subsets of L. Then,*

(1) $A = \emptyset$, *implies* $(A^\top, B) = L$;

(2) With $A \neq \emptyset$:

(i) if $B = \emptyset$, *then* $(A^\top, B) = \emptyset$;

(ii) for $A \neq \{0\}$, $(A^\top, 0) = \emptyset$;

(iii) $(A^\top, 1) \subseteq \{x \in L \mid (\forall y \in A), x \odot y = x \wedge y\}$;

(iv) $(0^\top, 1) = \{1\}$, $(1^\top, 0) = \emptyset$, $(L^\top, 1) = \{1\}$, $(1^\top, 1) = L$, $(0^\top, 0) = \{x \in L \mid \nu x = 0\}$;

(v) $(L^\top, A) = \emptyset$ *iff* $1 \notin A$;

(vi) $(L^\top, A) \subseteq A$.

Proof.

(1) We write $(A^{\top}, B)$ in a more logical form as $\{x \in L \mid (\forall y)(y \in A \text{ implies } \nu x \vee y \in B)\}$. Thus, $(\emptyset^{\top}, B) = \{x \in L \mid (\forall y)(y \in \emptyset \text{ implies } \nu x \vee y \in B)\} = L$, since the statement "$(\forall y)(y \in \emptyset \text{ implies } \nu x \vee y \in B)$" is always true, for all $x \in L$.

2 Consider $A \neq \emptyset$:

(i) if $B = \emptyset$, then :

$$\begin{aligned}(A^{\top}, B) &= (A^{\top}, \emptyset)\\ &= \{x \in L \mid (\forall y \in A), \nu x \vee y \in \emptyset\}\\ &= \emptyset.\end{aligned}$$

(ii) If $A \neq \{0\}$, then,

$$\begin{aligned}(A^{\top}, 0) &= \{x \in L \mid (\forall y \in A), \nu x \vee y = 0\}\\ &= \emptyset\end{aligned}$$

(iii) For all $z \in L$,

$$\begin{aligned}z \in (A^{\top}, 1) &\Rightarrow \forall y \in A, \nu z \vee y = 1\\ &\Rightarrow \forall y \in A, z \odot y = z \wedge y, \quad \textit{(by Lemma 2.8)}\\ &\Rightarrow z \in \{x \in L \mid (\forall y \in A), x \odot y = x \wedge y\}\end{aligned}$$

Thus, $(A^{\top}, 1) \subseteq \{x \in L \mid (\forall y \in A), x \odot y = x \wedge y\}$.

(iv) We have:

$$\begin{aligned}(0^{\top}, 1) &= \{x \in L \mid \nu x \vee 0 = 1\}\\ &= \{1\};\\ (1^{\top}, 0) &= \{x \in L \mid \nu x \vee 1 = 0\}\\ &= \emptyset;\\ (L^{\top}, 1) &= L^{\top}, \quad \textit{(by Remark 3.2)}\\ &= \{x \in L \mid (\forall y \in L), \nu x \vee y = 1\}\\ &= \{1\};\\ (0^{\top}, 0) &= \{x \in L \mid \nu x \vee 0 = 0\}\\ &= \{x \in L \mid \nu x = 0\};\\ (1^{\top}, 1) &= \{x \in L \mid \nu x \vee 1 = 1\}\\ &= L.\end{aligned}$$

(v) Let $(L^{\top}, A) = \emptyset$. Then $1 \notin A$, otherwise we would have $(L^{\top}, 1) = \emptyset$, which implies from *(iv)* that $\{1\} = \emptyset$, a contradiction.

Conversely,

suppose by contrary that $(L^{\top}, A) \neq \emptyset$ and let $x \in (L^{\top}, A)$. Then, for all $y \in L, \nu x \vee y \in A$. Since $1 \in L$, then $1 = \nu x \vee 1 \in A$, which contradicts the fact that $1 \notin A$.

It follows that $(L^{\top}, A) = \emptyset$ iff $1 \notin A$.

(vi) Suppose by contrary that $(L^\top, A) \nsubseteq A$. Then, there is $x \in (L^\top, A)$ such that $x \notin A$, i.e., for all $y \in L$, $\nu x \vee y \in A$ and $x \notin A$. In particular, for $y = x$, we have $\nu x \vee x \in A$ and $x \notin A$, i.e., $x \in A$ (since $\nu x \leq x$) and $x \notin A$, which is absurd. Thus, $(L^\top, A) \subseteq A$.

□

The reverse inclusion in Proposition 3.4 (*vi*) is not always true, as it is deduced from Proposition 3.4 (*v*) that $A \nsubseteq \emptyset = (L^\top, A)$ whenever $1 \notin A$.

Proposition 3.5. *Let $\mathcal{L}$ be a triangle algebra. Let T, T_1, T_2, Y_1, Y_2, Y and Z be nonempty subsets of L. Then,*

(i) $T_1 \subseteq T_2$ *implies* $\left(T_1^\top, Y\right) \subseteq \left(T_2^\top, Y\right)$;

(ii) $Y_1 \subseteq Y_2$ *implies* $\left(T^\top, Y_1\right) \subseteq \left(T^\top, Y_2\right)$;

(iii) $\left(T_1^\top, Y\right) \cap \left(T_2^\top, Z\right) \subseteq \left((T_1 \cap T_2)^\top, Y \cap Z\right)$;

(iv) $(T^\top, (T^\top, Y \cap Z)) \subseteq (T^\top, (T^\top, Y)) \cap (T^\top, (T^\top, Z))$;

(v) $\left(T^\top, \bigcap\limits_{i \in I} Y_i\right) \subseteq \bigcap\limits_{i \in I} \left(T^\top, Y_i\right) \subseteq \left(T^\top, \bigcup\limits_{i \in I} Y_i\right) \subseteq \bigcup\limits_{i \in I} \left(T^\top, Y_i\right)$;

(vi) $\left(\bigcap\limits_{i \in I} T_i^\top, Y\right) \subseteq \bigcap\limits_{i \in I} \left(T_i^\top, Y\right) \subseteq \left(\bigcup\limits_{i \in I} T_i^\top, Y\right) \subseteq \bigcup\limits_{i \in I} \left(T_i^\top, Y\right)$;

(vii) $T \cap (T^\top, Y) \subseteq Y$;

(viii) $(T^\top, Y) = \bigcap\limits_{t \in T} (t^\top, Y)$.

Proof.

(i) Suppose that $T_1 \subseteq T_2$ and let $x \in \left(T_1^\top, Y\right)$. Then, for all $t_1 \in T_1 \subseteq T_2$, $\nu x \vee y \in Y$. Thus, $x \in \left(T_2^\top, Y\right)$ and consequently, $\left(T_1^\top, Y\right) \subseteq \left(T_2^\top, Y\right)$.

(ii) Let $x \in \left(T^\top, Y_1\right)$. Then, for all $z \in T$, $\nu x \vee z \in Y_1 \subseteq Y_2$, that is, $x \in \left(T^\top, Y_2\right)$. Therefore, $\left(T^\top, Y_1\right) \subseteq \left(T^\top, Y_2\right)$.

(iii) Let $x \in L$. Then, $x \in \left(T_1^\top, Y\right) \cap \left(T_2^\top, Z\right)$ implies that for all $y \in T_1$ and $z \in T_2$, $\nu x \vee y \in Y$ and $\nu x \vee z \in Z$. Given that $T_1 \cap T_2 \subseteq T_1, T_2$, we deduce that for all $y \in T_1 \cap T_2$, $\nu x \vee t \in Y \cap Z$, i.e., $x \in \left((T_1 \cap T_2)^\top, Y \cap Z\right)$.

(iv) We have $Y \cap Z \subseteq Y, Z$. Then by *(ii)*, $(T^\top, Y \cap Z) \subseteq (T^\top, Y), (T^\top, Z)$. Applying *(ii)* again, $(T^\top, (T^\top, Y \cap Z)) \subseteq (T^\top, (T^\top, Y)), (T^\top, (T^\top, Z))$. Therefore, $(T^\top, (T^\top, Y \cap Z)) \subseteq (T^\top, (T^\top, Y)) \cap (T^\top, (T^\top, Z))$.

(v) *(*)* Let us prove that $\left(T^\top, \bigcap\limits_{i \in I} Y_i\right) \subseteq \bigcap\limits_{i \in I} \left(T^\top, Y_i\right)$.

Since $\bigcap\limits_{i \in I} Y_i \subseteq Y_i$ for all $i \in I$, by *(ii)*, we have $\left(T^\top, \bigcap\limits_{i \in I} Y_i\right) \subseteq \left(T^\top, Y_i\right)$, for all $i \in I$.

Thus, $\left(T^\top, \bigcap\limits_{i \in I} Y_i\right) \subseteq \bigcap\limits_{i \in I} \left(T^\top, Y_i\right)$.

(**) To show that $\bigcap\limits_{i \in I} \left(T^\top, Y_i\right) \subseteq \left(T^\top, \bigcup\limits_{i \in I} Y_i\right)$, for all $i \in I$, we have $Y_i \subseteq \bigcup\limits_{i \in I} Y_i$. Thus, by *(ii)*, we

obtain that for all $i \in I$, $(T^\top, Y_i) \subseteq \left(T^\top, \bigcup_{i\in I} Y_i\right)$, that is, $\bigcap_{i\in I}(T^\top, Y_i) \subseteq \left(T^\top, \bigcup_{i\in I} Y_i\right)$.

(***) Now we prove that $\left(T^\top, \bigcup_{i\in I} Y_i\right) \subseteq \bigcup_{i\in I}(T^\top, Y_i)$. Let $x \in \left(T^\top, \bigcup_{i\in I} Y_i\right)$. Then, for all $y \in T$, there is $i \in I$ such that $\nu x \vee y \in Y_i$. Thus, there is $i \in I$ such that $x \in (T^\top, Y_i)$, that is $x \in \bigcup_{i\in I}(T^\top, Y_i)$. Therefore, $\left(T^\top, \bigcup_{i\in I} Y_i\right) \subseteq \bigcup_{i\in I}(T^\top, Y_i)$.

(vi) *(*)* We have $\bigcap_{i\in I} T_i^\top \subseteq T_i^\top$, for all $i \in I$. Then by *(i)*, we obtain that $\left(\bigcap_{i\in I} T_i^\top, Y\right) \subseteq (T_i^\top, Y)$, for all $i \in I$. Thus, $\left(\bigcap_{i\in I} T_i^\top, Y\right) \subseteq \bigcap_{i\in I}(T_i^\top, Y)$.

(**) For all $i \in I$, $T_i^\top \subseteq \bigcup_{i\in I} T_i^\top$. By applying *(i)*, we have $(T_i^\top, Y) \subseteq \left(\bigcup_{i\in I} T_i^\top, Y\right)$, for all $i \in I$. Therefore, $\bigcap_{i\in I}(T_i^\top, Y) \subseteq \left(\bigcup_{i\in I} T^\top, Y\right)$.

(***) Let $x \in \left(\bigcup_{i\in I} T_i^\top, Y\right)$. Then, there exists $i \in I$ such that $\nu x \vee y \in Y$, for all $y \in T_i$. Thus, there exist $i \in I$ such that $x \in (T_i^\top, Y)$, that is, $x \in \bigcup_{i\in I}(T_i^\top, Y)$. Therefore, $\left(\bigcup_{i\in I} T_i^\top, Y\right) \subseteq \bigcup_{i\in I}(T_i^\top, Y)$.

(vii) If $x \in T \cap (T^\top, Y)$, then $x \in T$ and $\nu x \vee y \in Y$, for all $y \in T$. In particular, $\nu x \vee x \in Y$, which implies that $x \in Y$, as $\nu x \leq x$. Thus, $T \cap (T^\top, Y) \subseteq Y$.

(viii) Let $x \in L$. Then, $x \in (T^\top, Y)$ iff for all $t \in T, \nu x \vee y \in Y$ iff for all $t \in T, x \in (t^\top, Y)$ iff $x \in \bigcap_{t\in T}(t^\top, Y)$. Therefore, $(T^\top, Y) = \bigcap_{t\in T}(t^\top, Y)$.

□

4 Relative Co-annihilators as Filters of Triangle Algebras.

Exploring the relative co-annihilator $(A^\top, B)$, where A and B are arbitrary subsets of L, prompts a natural query: what happens when B is a filter of $\mathcal{L}$? This section examines relative co-annihilators with respect to filters of triangle algebras, providing additional properties.

Proposition 4.1. *Let A and B be two nonempty subsets of a triangle algebra $\mathcal{L}$. If B is a filter of $\mathcal{L}$, then $(A^\top, B)$ is a filter of $\mathcal{L}$.*

Proof. Since B is a filter of L, then $1 \in B$. Also, for all $a \in A$, $\nu 1 \vee a = 1 \in B$ (by Remark 2.6). Thus, $1 \in (A^\top, B)$, and therefore $(A^\top, B)$ is nonempty.
Let $x \in (A^\top, B)$ and $y \in L$ such that $x \leq y$. Then, $\nu x \vee a \in B$. But $x \leq y$ implies $x = x \wedge y$. By $(T.3)$, we have $\nu x = \nu(x \wedge y) = \nu x \wedge \nu y$, that is, $\nu x \leq \nu y$, which implies that $\nu x \vee a \leq \nu y \vee a$, for all $a \in A$. But since B is a filter of $\mathcal{L}$, we deduce that $\nu y \vee a \in B$. Thus, $y \in (A^\top, B)$.
Now, let $x, y \in (A^\top, B)$. Then, for all $a \in A$, we have $\nu x \vee a \in B$ and $\nu y \vee a \in B$. Since B is a filter of $\mathcal{L}$, then $(\nu x \vee a) \odot (\nu y \vee a) \in B$. But by $(RL13)$ and Proposition 2.7 (1), $(\nu x \vee a) \odot (\nu y \vee a) \leq a \vee \nu(x \odot y)$ and since B is a filter of $\mathcal{L}$, we have $a \vee \nu(x \odot y) \in B$, for all $a \in A$. Hence, $x \odot y \in (A^\top, B)$.
Moreover, let $x \in (A^\top, B)$. This implies that $\nu x \vee a \in B$, that for all $a \in A$. But $\nu x \leq \nu\nu x$, which implies

that $\nu x \vee a \leq \nu\nu x \vee a$, for all $a \in A$. It follows that $\nu\nu x \vee a \in B$, since B is a filter of $\mathcal{L}$. Hence, $\nu x \in (A^\top, B)$. □

The converse of Proposition 4.1 is not necessarily true. Indeed, consider the triangle algebra $\mathcal{L}$ from Example 3.3. For $X = \{[a,1]\}$ and $Y = \{[a,a],[1,1]\}$, we observe that $(X^\top, Y) = \{[b,b],[b,1],[1,1]\}$ which is a filter of $\mathcal{L}$. However, Y is not a filter, as $[a,a] \leq [a,1] \notin Y$.

Proposition 4.2. *Let T be a filter of a triangle algebra $\mathcal{L}$, and Y a nonempty subset of L. Then,*

(i) $T \subseteq (Y^\top, T)$;

(ii) $(Y^\top, T) = L$ *iff* $Y \subseteq T$ *(specifically,* $(Y^\top, L) = L$, *and* $(T^\top, T) = L$*)*;

(iii) $(L^\top, T) = T$;

(iv) $((T^\top, T)^\top, T)) = T$ *and* $((T^\top, (T^\top, T)) = L$;

(v) $Y \cap (Y^\top, T) = Y \cap T$;

(vi) $T \subseteq Y$ *implies* $(Y^\top, T) \cap Y = T$;

(vii) If $Y \subseteq T$, *then* $((Y^\top, T)^\top, T) = T$;

(viii) $(Y^\top, T)^\top, T) \cap (Y^\top, T) = T$.

Proof.

(i) Let $x \in T$. Then, $\nu x \in T$ since T is a filter. We have $\nu x \leq \nu\nu x \leq \nu\nu x \vee y$, for all $y \in Y$. Therefore, $\nu\nu x \vee y \in T$, as T is a filter. Thus, $\nu x \in (Y^\top, T)$ and consequently, $T \subseteq (Y^\top, T)$.

(ii) Suppose that $(Y^\top, T) = L$ and $y \in Y$. Since $0 \in L = (Y^\top, T)$, then $y = \nu 0 \vee y \in T$. Therefore, $Y \subseteq T$.
Reciprocally, for any $y \in Y \subseteq T$, $\nu 0 \vee y = y \in T$, i.e., $0 \in (Y^\top, T)$. Hence, $L = (Y^\top, T)$.

(iii) By *(i)*, we have $T \subseteq (L^\top, T)$.
Also, from Proposition 3.4 *(vi)*, we have $(L^\top, T) \subseteq T$. Thus, $(L^\top, T) = T$.

(iv) $(T^\top, T) = L$ (by *(ii)*). This implies that $((T^\top, T)^\top, T) = (L^\top, T) = T$, by *(iii)*.
Also, $(T^\top, (T^\top, T)) = (T^\top, T) = L$, by *(ii)*.

(v) Clearly, $T \subseteq (Y^\top, T)$ by *(i)* , which implies that $Y \cap T \subseteq Y \cap (Y^\top, T)$.
In addition, $Y \cap (Y^\top, T) \subseteq T$ by Proposition *3.5 (vii)*. Thus, $Y \cap (Y^\top, T) = Y \cap [Y \cap (Y^\top, T)] \subseteq Y \cap T$, i.e., $Y \cap (Y^\top, T) \subseteq Y \cap T$. Therefore, $Y \cap (Y^\top, T) = Y \cap T$.

(vi) Assume that $T \subseteq Y$. Then, $Y \cap T = T$. Thus, *(v)* becomes $Y \cap (Y^\top, T) = T$.

(vii) Since $Y \subseteq T$, then by *(ii)*, $(Y^\top, T) = L$. We obtain from *(iii)* that $((Y^\top, T)^\top, T) = (L^\top, T) = T$.

(viii) From *(i)*, we have $T \subseteq (Y^\top, T)$. Then, from *(vi)*, we deduce that $((Y^\top, T)^\top, T) \cap (Y^\top, T) = T$.

□

Lemma 4.3. *Let $\mathcal{L}$ be a triangle algebra, T, Y and Z, nonempty subsets of L. If Z is a filter, then* $(T^\top, (Y^\top, Z)) \subseteq \bigcap_{t \in T, y \in Y} ((\nu t \vee y)^\top, Z)$.

Proof. Let $x \in L$, then,

$$\begin{aligned} x \in (T^\top, (Y^\top, Z)) &\Rightarrow \forall t \in T, \nu x \vee t \in (Y^\top, Z) \\ &\Rightarrow \forall t \in T, \forall y \in Y, \nu(\nu x \vee t) \vee y \in Z \\ &\Rightarrow \forall t \in T, \forall y \in Y, (\nu\nu x \vee \nu t) \vee y \in Z \quad \text{(by (T.4))} \\ &\Rightarrow \forall t \in T, \forall y \in Y, \nu\nu x \vee (\nu t \vee y) \in Z \quad \text{(by associativity)} \\ &\Rightarrow \nu x \in \bigcap_{t \in T, y \in Y} ((\nu t \vee y)^\top, Z). \end{aligned}$$

But since $\nu x \leq x$ and Z is a filter, then by Proposition *4.1*, $\bigcap_{t \in T, y \in Y} ((\nu t \vee y)^\top, Z)$ is also a filter and we have $x \in \bigcap_{t \in T, y \in Y} ((\nu t \vee y)^\top, Z)$.
Consequently, $(T^\top, (Y^\top, Z)) \subseteq \bigcap_{t \in T, y \in Y} ((\nu t \vee y)^\top, Z)$. □

Theorem 4.4. *Let T and Y be two nonempty subsets of a triangle algebra $\mathcal{L}$. If Y is a BF2 (respectively PSF, IF, BF, PF, PF3, PPF, VF), then so is $(T^\top, Y)$.*

Proof. *We establish the first three properties, and the remaining ones are demonstrated in a similar manner.*

Let T and Y be two nonempty subsets of a triangle algebra $\mathcal{L}$:

(i) Suppose that Y is a (BF2) and that for all $x \in L$, $\nu(\neg x) \notin (T^\top, Y)$. Let us show that $\nu x \in (T^\top, Y)$. Since Y is a (BF2), we have $\nu x \in Y$ or $\nu(\neg x) \in Y$. But since Y is a filter of triangle algebra, and that $\nu x \leq \nu x \vee a$ and $\nu(\neg x) \leq \nu(\neg x) \vee a$ for all $a \in T \subseteq L$, then, we have $\nu x \vee a \in Y$ or $\nu(\neg x) \vee a \in Y$, for all $a \in T$. That is, $x \in (T^\top, Y)$ or $\neg x \in (T^\top, Y)$. But, $(T^\top, Y)$ is a filter and $\nu(\neg x) \notin (T^\top, Y)$ by assumption, therefore $x \in (T^\top, Y)$, and hence, $\nu x \in (T^\top, Y)$.

(ii) Suppose that Y is a (PSF). For all $x \in L$, let us show that $\nu[\neg(x \wedge \neg x)] \in (T^\top, Y)$. Now, since Y is a (PSF), then $\nu[\neg(x \wedge \neg x)] \in Y$. But $\nu[\neg(x \wedge \neg x)] \leq \nu[\neg(x \wedge \neg x)] \vee a$ for all $a \in T \subseteq L$. And since Y is a filter of $\mathcal{L}$, we have $\nu[\neg(x \wedge \neg x)] \vee a \in Y$, for all $a \in T$. It yields that, $\neg(x \wedge \neg x) \in (T^\top, Y)$. Hence, since $(T^\top, Y)$ is a filter, we have $\nu[\neg(x \wedge \neg x)] \in (T^\top, Y)$.

(iii) Suppose that Y is an (IF). let us prove that $(T^\top, Y)$ is also an (IF).
Let x be an arbitrary element of L. By Proposition *2.11*, it is sufficient to show that $\nu(x \to x^2) \in (T^\top, Y)$. We have $\nu(x \to x^2) \in Y$ and $\nu(x \to x^2) \leq \nu\nu(x \to x^2) \leq \nu\nu(x \to x^2) \vee a$, for all $a \in T$. Since Y is a filter, we have $\nu\nu(x \to x^2) \vee a \in Y$ for all $a \in T$. Hence, $\nu(x \to x^2) \in (T^\top, Y)$.

□

The following property is specific to PF2 (Prime filter of second kind).

Proposition 4.5. *Let $\mathcal{L}$ be a triangle algebra, T be PF2 of $\mathcal{L}$, and Y a subset of L such that $Y \nsubseteq T$. Then, $(Y^\top, T) = T$ (and hence $(Y^\top, T)$ is PF2).*

Proof.

Since T is a filter of $\mathcal{L}$, then $T \subseteq (Y^\top, T)$, by Proposition 4.2 *(i)*.

For the converse, let us suppose by contrary that $(Y^\top, T) \nsubseteq T$. Then, there is $x \in L$ such that $x \in (Y^\top, T)$ and $x \notin T$. This means that for all $a \in Y, \nu x \vee a \in T$ and $\nu x \notin T$ (as T is a filter), which implies that for all $a \in Y, \nu(\nu x \vee a) \in T$ and $\nu x \notin T$ (from (F3)).
Since T is a PF2, it follows that for all $a \in Y, [\nu\nu x \in T$ or $\nu a \in T]$ and $\nu x \notin T$.

This implies that for all $a \in Y, [\nu x \in T$ or $a \in T$ (as T is a filter)] and $\nu x \notin T$. Which is absurd, since $a \in Y \nsubseteq T$ from hypothesis. Therefore, $(Y^\top, T) \subseteq T$. $\square$

It is evident that the converse of Proposition 4.5 may not always hold. Specifically, consider the triangle algebra $\mathcal{L}$ from Example 3.3:

- Let $T = [1, 1]$. We have $(L^\top, T) = T$, but T does not satisfy the PF2 property.
- For $X = \{[a, 1]\}$ and $Y = \{[a, a], [1, 1]\}$, we obtain $(X^\top, Y) = \{[b, b], [b, 1], [1, 1]\}$ which is $PF2$. But Y is not even a filter.

Proposition 4.6. *Let $\mathcal{L}$ be a triangle algebra, T a filter of $\mathcal{L}$ and Y a nonempty subset of L. If $\mathcal{L}$ is linear, then $(Y^\top, T) = T$ or $(Y^\top, T) = L$.*

Proof. Let us suppose that $(Y^\top, T) \neq L$ and prove that $(Y^\top, T) = T$. Since T is a filter of $\mathcal{L}$, then $T \subseteq (Y^\top, T)$, from Proposition 4.2 *(i)*.
Since $(Y^\top, T) \neq L$ then by *(ii)* of Proposition 4.2, we have $Y \nsubseteq T$. Thus, there exists $b \in L$ such that $b \in Y$ and $b \notin T$. Let $a \in (Y^\top, T)$; then for all $y \in Y$, $\nu a \vee y \in T$, which implies that, $\nu a \vee b \in T$ as, $b \in Y$. Also, since $\nu a \leq a$, and that $\nu a \vee b \leq a \vee b$, then $a \vee b \in T$ due to the fact that T is a filter. Now, since $\mathcal{L}$ is linear, then either $a \leq b$ or $b \leq a$. We claim that $a \nleq b$ otherwise, we would have $b = a \vee b \in T$ which is a contradiction. So, $b \leq a$. Consequently, $a = a \vee b \in T$. This shows that $(Y^\top, T) \subseteq T$. It yields that $(Y^\top, T) = T$. $\square$

The converse of Proposition 4.6 is not always guaranteed. Revisiting Example 3.3, if we set ($Y = L$ and $T = \{[1, 1]\}$) or ($Y = \{[1, 1]\}$ and $T = \{[1, 1]\}$), in both cases, we find that $(Y^\top, T) = T$ or $(Y^\top, T) = L$, whereas $\mathcal{L}$ is not linear.

5 Conclusion and Future Work

This article is in the general framework of the study of filters in triangle algebras. We have introduced the notion of relative co-annihilator, established some of its properties, and built the link with filters in triangle algebras. In addition, we proved that the co-annihilator of a nonempty subset T of a triangle algebra $\mathcal{L}$ relative to a filter Y of $\mathcal{L}$ preserves certain characteristics of the filter Y. In particular, if Y is a Boolean filter of the second kind, then the co-annihilator of T with respect to Y is also a Boolean filter of the second kind; the same applies to an implicative filter, a pseudo complementation filter, a Boolean filter, a prime filter, a prime filter of the third kind, a pseudo-prime filter or an involution filter, respectively. Moreover, we have presented certain conditions under which the co-annihilator of T with respect to Y is a prime filter of the second type.

Filters are particularly interesting as they are closely related to congruence relations, which are used to construct quotient algebras: from each filter, a congruence relation can be defined (see [11]). However, we have identified some inaccuracies in [18]: the relation θ given in [18, Example 4.2] is not a congruence relation since it is not reflexive. Also, contrary to what they claimed in [18, Example 4.3], the congruence relation θ does preserve co-annihilators. In the process of asserting that every congruence relation on an MTL-triangle algebra preserves co-annihilators, they claimed that the relation $\theta(Y) = \{(a, b) \in A \times A; \varphi(a) \cap Y = \varphi(b) \cap Y\}$ is a congruence relation, which is not always true. Hence, this remains an open problem in the framework of triangle algebras for further examination in future works.

In our forthcoming work, we will extend our exploration of algebraic structures, with a specific focus on triangle algebras. More precisely, since *ideals* also represent sets of provable formulas within algebraic structures, and knowing that *ideals* and *filters* are not dual notions in residuated lattices, it follows that they will not be dual notions in triangle algebras either, given that triangle algebras are enriched residuated lattices. A subsequent paper introducing *ideals in Triangle algebras* and proving soundness and completeness with respect to triangle algebras is in preparation.

Another challenge for the future is the investigation of the concepts of *annihilator and relative annihilator* in triangle algebras, viewed as special types of ideals.

Acknowledgements: We express our gratitude to the reviewers for their insightful recommendations, which have been carefully integrated to enhance the quality of the paper.

Conflict of Interest: The authors declare no conflict of interest.

References

[1] Boole G. *The Mathematical Analysis of Logic: Being an Essay Towards a Calculus of Deductive Reasoning*. Cambridge: Macmillan, Barclay and Macmillan; 1947.

[2] Goguen JA. L-fuzzy sets. *Journal of Mathematical Analysis and Application.* 1967; 18(1): 145-174. DOI: https://doi.org/10.1016/0022-247X(67)90189-8

[3] Hajek P. *Metamathematics of Fuzzy Logic*. Kluwer Acad Publishing; 1998.

[4] Holdon CL, Borumand Saeid A. Regularity in residuated lattices. *Iranian Journal of Fuzzy Systems.* 2019; 16(6): 107-126. DOI: https://doi.org/10.22111/IJFS.2019.5023

[5] Maroof FG, Borumand Saeid A, Eslami E. On co-annihilators in residuated lattices. *Journal of Intelligent and Fuzzy Systems.* 2016; 31(3): 1263-1270. DOI: https://doi.org/10.3233/ifs-162192

[6] Meng BL, Xin XL. Generalized co-annihilator of BL-algebras. *Open Math.* 2015; 13: 639-654. DOI: https://doi.org/10.1515/math-2015-0060

[7] Rasouli S. Generalized co-annihilators in residuated lattices. *Annals of the University of Craiova-Mathematics and Computer Science Series.* 2018; 45(2): 190-207. DOI: https://doi.org/10.52846/ami.v45i2.827

[8] Van Gasse B, Cornelis C, Deschrijver G, Kerre EE. Triangle algebras: A formal logic approach to interval-valued residuated lattices. *Fuzzy Sets and Systems.* 2008; 159: 1042-1060. DOI: https://doi.org/10.1016/j.fss.2007.09.003

[9] Van Gasse B, Cornelis C, Deschrijver G, Kerre EE. A characterization of interval-valued residuated lattices. *International Journal of Approximate Reasoning.* 2008; 49: 478-487. DOI: https://doi.org/10.1016/j.ijar.2008.04.006

[10] Van Gasse B, Cornelis C, Deschrijver G, Kerre EE. Triangle algebras: Towards an axiomatization of interval-valued residuated lattices. In: Information Sciences. 2010. p.117-126. DOI: https://doi.org/10.1007/11908029_14

[11] Van Gasse B, Deschrijver G, Cornelis C, Kerre EE. Filters of residuated lattices and triangle algebras. *Information Sciences.* 2010; 180: 3006-3020. DOI: https://doi.org/10.1016/j.ins.2010.04.010

[12] Ward M, Dilworth R. Residuated lattices. *Transactions of the American Mathematical Society.* 1939; 45(3): 335-354. DOI: https://doi.org/10.1090/s0002-9947-1939-1501995-3

[13] Zadeh LA. Fuzzy sets. *Information and Control.* 1964; 8: 338-353. DOI: https://doi.org/10.21236/ad0608981

[14] Zahiri S, Borumand Saeid A. An investigation on the n-fold IVRL-filters in triangle algebras. *Mathematica Bohemica.* 2020; 145(1): 75-91. DOI: https://doi.org/10.21136/MB.2019.0104-17

[15] Zahiri S, Borumand Saeid A, Eslami E. On maximal filters in triangle algebras. *Journal of Intelligent and Fuzzy Systems.* 2016; 30(2): 1181-1193. DOI: https://doi.org/10.3233/ifs-151842

[16] Zahiri S, Borumand Saeid A, Eslami E. A new approach to filters in triangle algebras. *Publications de l'Institut Mathematique (Belgrade).* 2017; 101(115): 267-283. DOI: https://doi.org/10.2298/pim1715267z

[17] Zahiri S, Borumand Saeid A, Turunen E. On local triangle algebras. *Journal of Intelligent and Fuzzy Systems.* 2021; 418: 126-138. DOI: https://doi.org/10.1016/j.fss.2020.07.001

[18] Zahiri S, Borumand Saeid A, Zahiri M. An investigation on the Co-annihilators in triangle algebras. *Iranian Journal of Fuzzy Systems.* 2018; 15(7): 91-102. DOI: https://doi.org/10.22111/IJFS.2018.4287

Emile Djomgoue Nana
Department of Mathematics and Computer Science
University of Dschang
Dschang, Cameroon
E-mail: emiledjomgouenana@gmail.com

Ariane Gabriel Tallee Kakeu
Department of Mathematics and Computer Science
University of Dschang
Dschang, Cameroon
E-mail: arianekakeugabie@yahoo.fr

Blaise Bleriot Koguep Njionou
Department of Mathematics and Computer Science
University of Dschang
Dschang, Cameroon
E-mail: blaise.koguep@univ-dschang.org

Lele Celestin
Department of Mathematics and Computer Science
University of Dschang
Dschang, Cameroon
E-mail: celestinlele@yahoo.com