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The Existence and Uniqueness of the Solution of Difference Equation in Neutrosophic Environment via Generalized Hukuhara Difference Ideology

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Abstract. In real-world scenarios, the neutrosophic set or neutrosophic numbers have been widely used to deal with the uncertain difference equations of the corresponding uncertain discrete dynamical system. A situation where discrete changes occur with vague information of a neutrosophic sense can be dealt with by the neutrosophic difference equation. In this paper, a new metric is defined for the neutrosophic set, and the sense of generalized Hukuhara difference for the fuzzy numbers is extended to the neutrosophic numbers. The generalized Hukuhara difference of the type-I and type-II and their corresponding neutrosophic parametric representation are discussed. The existence and uniqueness conditions to obtain a solution of the difference equation in a neutrosophic environment are argued by some theorems. The theoretical concept has been applied to the logistic difference equation in a neutrosophic environment. We have applied both the type-I and type-II Hukuhara differences to the two different generalized Hukuhara difference forms of the logistic difference equation. The equilibrium points and their corresponding stability criteria are established to perceive the effect of the Hukuhara differences. Finally, the numerical examples and their graphical portrayal are provided to recognize the intuition of the introduced theory in this paper.

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Keywords and Phrases: Neutrosophic set, Metric space, Hukuhara difference, Logistic difference equation.

1 Introduction

The discrete dynamical phenomena are best described mathematically using the difference equation. Due to the practical applications in various fields, including computer science, mathematical biology, economics, and control engineering, scholars have worked on the theory of difference equations and demonstrated an interest in this area [1–9]. The difference equation or set of difference equations used to represent a particular dynamical situation comprises several coefficients, parameters, and initial conditions. The solutions provided by the clean environment frequently include errors, which makes them describe reality accurately than they should. We encounter situations where data collection errors occur, experiments are challenging, and the original information or parameters must be clarified. As a result, the difference equation with ambiguous beginning circumstances and parameters captures the reality. Fuzzy sets were introduced by Zadeh [10] and

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extended in terms of theory and application by succeeding works [11–13]. Abdullayev, I. et al. [14] utilised the neutrosophic set to solve the prediction of financial distress by the relative weighted neutrosophic valued distance methodology. While the fuzzy difference equation is an extension of the crisp difference equation, it is nonetheless crucial for explaining real-world phenomena like ecological dynamics, mathematical models and finance mathematics [15–18]. Abu Arqub, O. et al. [19] discussed the analytical solution and its use. Khastan, A. [20] studied the logistic difference equation and described the overall behaviours of two alternative forms of the difference equation using the Hukuhara difference of fuzzy numbers. Further, Wang, H. et al. [21] solve the differential equation using gH-differentiability. Soroush, S. et al. [22] applied the generalized fuzzy difference method to solve the initial value problems of fuzzy equations. Shams, M. et al. [23] solved the nonlinear equations in the field of engineering and Palanikumar, M. et al. [24] utilized neutrosophic sets along with aggregated operators in the diagnostic disease problem, respectively.

The vague ambiguity had been further expanded and broadened. The fuzzy intuitionistic theory is one such refinement. The concepts of membership function and non-membership function are considered in this framework. Inside this discipline of mathematics, Atanassov [25] introduced the sleight-of-hand notion of an intuitionistic fuzzy set in 1986. The idea of a triangular intuitionistic fuzzy set was later developed by Liu and Yuan [26]. Ejegwa, P. A. et al. [27] solved the supplier selection problems using the Pythagorean fuzzy set. Ye [28] presented the trapezoidal intuitionistic fuzzy set's preliminary notion. IFSs are used in many different disciplines [29–31]. Ejegwa, P. A. et al. [32] used intuitionistic fuzzy numbers for the ascertainment of medical diagnosis assignments. Further, Kausar, N. et al. [33] solve the mechanical engineering problems using the generalized trapezoidal intuitionistic fuzzy numbers (GTrIFN) and Revathi, A. N. et al. [34] examine the students' performances using a fuzzy based uncertain model. The neutrosophic set framework encompasses the intuitionistic fuzzy set and the fuzzy notion in a more generalised form. Smarandache [35] was first introduced to the concept of a neutrosophic set in 1995. This concept considers truthiness, indeterminacies, and falseness. The neutrosophic notion is logically sound, practical, and relevant to dynamic and real-world scenarios. Türkan Y. S. et al. [36] use a neutrosophic set to select the sites for educational institute fairs. In their discussion of the single type neutrosophic set to handle any problematic issue, Wang et al. [37], Kumar, M. et al. [38] provided the triangular neutrosophic set and its categorization. Beigmohamadi, R. et al. [39] evaluate the solutions of non-periodic boundary value problems for the discrete fractional difference equations in uncertain environments. Narzary, G. et al. [40] evaluate the analytical solution to the heat equation under a neutrosophic environment using the Laplace transform. Additionally, Safikhani, L. et al. [41] solve the fuzzy differential equations using the gH-difference method.

A brief survey of the existing theory of arithmetic and calculus of neutrosophic numbers and neutrosophic valued functions points to the following lacuna. Much research has been carried out on the arithmetic properties and scoring of the neutrosophic numbers and their utilization in decision-making problems. Like fuzzy numbers, neutrosophic numbers have arithmetic properties that differ from the deterministic ones. The traditional difference cannot fulfill the purposes. The details on the metric, differences, and other related consequences are untouched in research. Some existing works describe the differentiability of neutrosophic valued functions and differential equations in the Neutrosophic arena. Discussing the theory of difference equations in neutrosophic uncertainty would be beneficial. To fill the gaps, we did the following in this paper. First, we defined a metric and generalized the Hukuhara difference [42] of neutrosophic numbers. Second, we discussed the existence and uniqueness criteria to obtain the solutions of the difference equations under the neutrosophic generalized Hukuhara difference. Next, we analyzed different cases of logistic difference equations as an application of the proposed theory.

The succeeding texts are divided into the following sections. Section 2 contains the metric, generalized Hukuhara difference in a neutrosophic environment. The existence and uniqueness conditions for solving difference equations in a neutrosophic environment are established in Section 3. Section 4 presents solutions and stability analyses of different cases of the logistic difference equation in a neutrosophic environment. The numerical examples and their graphical explanation are given in Section 5. Ending remarks are given in Section 6.

2 Metric and generalized Hukuhara difference of neutrosophic numbers

In this section, we first define a metric in the collection of neutrosophic numbers [24]. The neutrosophic numbers are applied in numerous fields, including differential equations [43], difference equations [9], optimisations [44] and many more. Then, the discussion continues with the generalised Hukuhara difference for such vague numbers.

Definition 2.1. Suppose two neutrosophic numbers $\tilde{\Omega}$ and $\tilde{\Gamma}$ are given in the parametric form (see, [48,49]) representation by

$$\begin{aligned}\tilde{\Omega} &= \Big\{ \big(\Omega(u), \Omega(v), \Omega(w)\big) \,; u, v, w \in [0,1] \Big\} \\ &= \Big\{ \big[\Omega_L(u), \Omega_R(u)\big]\,, \big[\Omega_L(v), \Omega_R(v)\big]\,, \big[\Omega_L(w), \Omega_R(w)\big] \,; u, v, w \in [0,1] \Big\}\end{aligned}$$

and

$$\begin{aligned}\tilde{\Gamma} &= \Big\{ \big(\Gamma(u), \Gamma(v), \Gamma(w)\big) \,; u, v, w \in [0,1] \Big\} \\ &= \Big\{ \big[\Gamma_L(u), \Gamma_R(u)\big]\,, \big[\Gamma_L(v), \Gamma_R(v)\big]\,, \big[\Gamma_L(w), \Gamma_R(w)\big] \,; u, v, w \in [0,1] \Big\}\end{aligned}$$

Then, the distance between $\tilde{\Omega}$ and $\tilde{\Gamma}$ will be given by,

$$D\left(\tilde{\Omega}, \tilde{\Gamma}\right) = \sup_{0 \leq u,v,w \leq 1} \Big\{ d\left(\Omega(u), \Gamma(u)\right), d\left(\Omega(v), \Gamma(v)\right), d\left(\Omega(w), \Gamma(w)\right) \Big\} \tag{1}$$

where $d\left(\Omega(\lambda), \Gamma(\lambda)\right) = \max\left\{\left\|\Omega_L(\lambda) - \Gamma_L(\lambda)\right\|, \left\|\Omega_R(\lambda) - \Gamma_R(\lambda)\right\|\right\}$ for $\lambda = u, v, w$.

In the given definition, u, v and w represent the degree of truth, indeterminacy and falsehood. Also, the symbol $\|.\|$ is used to denote the Euclidean distance of the respective parts regarding truth, indeterminacy and falsehood in parametric representations of the neutrosophic numbers. The supremum of the obtained distances in respective parts is regarded as the metric or distance between $\tilde{\Omega}$ and $\tilde{\Gamma}$. The metric properties of the defined mathematical object are established in the next theorem.

Theorem 2.2. *The function D given in Definition 2.1 is a metric for the collection of neutrosophic numbers [45].*

Proof.

(i) Suppose two neutrosophic numbers $\tilde{\Omega}$ and $\tilde{\Gamma}$ are given in the parametric form (see, [48,49]) representation by

$$\tilde{\Omega} = \Big\{ \big[\Omega_L(u), \Omega_R(u)\big]\,, \big[\Omega_L(v), \Omega_R(v)\big]\,, \big[\Omega_L(w), \Omega_R(w)\big] \,; u, v, w \in [0,1] \Big\}$$

and

$$\tilde{\Gamma} = \Big\{ \big[\Gamma_L(u), \Gamma_R(u)\big]\,, \big[\Gamma_L(v), \Gamma_R(v)\big]\,, \big[\Gamma_L(w), \Gamma_R(w)\big] \,; u, v, w \in [0,1] \Big\}$$

when $\tilde{\Omega} = \tilde{\Gamma}$ then, we have

$$\begin{cases} \Omega_L(u) = \Gamma_L(u) \\ \Omega_R(u) = \Gamma_R(u) \\ \Omega_L(v) = \Gamma_L(v) \\ \Omega_R(v) = \Gamma_R(v) \\ \Omega_L(w) = \Gamma_L(w) \\ \Omega_R(w) = \Gamma_R(w) \end{cases} \tag{2}$$

Then, $d\left(\Omega(\lambda), \Gamma(\lambda)\right) = 0$ for $\lambda = u, v, w$ and consequently,

$$D\left(\tilde{\Omega}, \tilde{\Gamma}\right) = \sup_{0 \le u,v,w \le 1} \left\{ d\left(\Omega(u), \Omega(u)\right), d\left(\Omega(v), \Omega(v)\right), d\left(\Omega(w), \Omega(w)\right) \right\} = 0 \tag{3}$$

Again, when $D\left(\tilde{\Omega}, \tilde{\Gamma}\right) = 0$, then $\sup_{(0 \le u,v,w \le 1)} \left\{ d\left(\Omega(u), \Omega(u)\right), d\left(\Omega(v), \Omega(v)\right), d\left(\Omega(w), \Omega(w)\right) \right\} = 0$ which implies

$$\begin{cases} d\left(\Omega(u), \Gamma(u)\right) \le 0 \\ d\left(\Omega(v), \Gamma(v)\right) \le 0 \\ d\left(\Omega(w), \Gamma(w)\right) \le 0 \end{cases} \tag{4}$$

for all $u, v, w \in [0, 1]$.

That is

$$\begin{cases} \left\|\Omega_L(u) - \Gamma_L(u)\right\| & \le 0 \\ \left\|\Omega_R(u) - \Gamma_R(u)\right\| & \le 0 \\ \left\|\Omega_L(v) - \Gamma_L(v)\right\| & \le 0 \\ \left\|\Omega_R(v) - \Gamma_R(v)\right\| & \le 0 \\ \left\|\Omega_L(w) - \Gamma_L(w)\right\| & \le 0 \\ \left\|\Omega_R(w) - \Gamma_R(w)\right\| & \le 0 \end{cases} \tag{5}$$

That is

$$\begin{cases} \Omega_L(u) & = \Gamma_L(u) \\ \Omega_R(u) & = \Gamma_R(u) \\ \Omega_L(v) & = \Gamma_L(v) \\ \Omega_R(v) & = \Gamma_R(v) \\ \Omega_L(w) & = \Gamma_L(w) \\ \Omega_R(w) & = \Gamma_R(w) \end{cases} \tag{6}$$

which concludes $\tilde{\Omega} = \tilde{\Gamma}$.

The trivial property of the metric is proved here for the proposed mathematical object D.

(ii) Since $\left\|\Omega_L(\lambda) - \Gamma_L(\lambda)\right\| \ge 0$ and $\left\|\Omega_R(\lambda) - \Gamma_R(\lambda)\right\| \ge 0$ for $\lambda = u, v, w$ and $u, v, w \in [0, 1]$.

Consequently, $\sup\limits_{(0 \le u,v,w \le 1)} \left\{ d(\Omega(u), \Omega(u)), d(\Omega(v), \Omega(v)), d(\Omega(w), \Omega(w)) \right\} \ge 0$, which implies that

$$D\left(\tilde{\Omega}, \tilde{\Gamma}\right) \ge 0 \tag{7}$$

The positivity property of the metric is proved in Equation (7).

(iii) Since

$$\begin{aligned} d\left(\Omega(\lambda),\Gamma(\lambda)\right) =& \max\left\{\left\|\Omega_L(\lambda)-\Gamma_L(\lambda)\right\|,\left\|\Omega_R(\lambda)-\Gamma_R(\lambda)\right\|\right\} \ ; \text{for } \lambda=u,v,w \\ =& \max\left\{\left\|\Gamma_L(\lambda)-\Omega_L(\lambda)\right\|,\left\|\Gamma_R(\lambda)-\Omega_R(\lambda)\right\|\right\} \ ; \text{for } \lambda=u,v,w \\ =& d\left(\Gamma(\lambda),\Omega(\lambda)\right). \end{aligned}$$

So,

$$\begin{aligned} D\left(\tilde{\Omega},\tilde{\Gamma}\right) =& \sup_{0\le u,v,w\le 1}\left\{d(\Omega(u),\Gamma(u)),d(\Omega(v),\Gamma(v)),d(\Omega(w),\Gamma(w))\right\} \\ =& \sup_{0\le u,v,w\le 1}\left\{d(\Gamma(u),\Omega(u)),d(\Gamma(v),\Omega(v)),d(\Gamma(w),\Omega(w))\right\} \\ =& D\left(\tilde{\Gamma},\tilde{\Omega}\right). \end{aligned}$$

The symmetric properties for metric is established here.

(iv) Now, let $\tilde{\Gamma},\tilde{\Omega}$ and $\tilde{\Delta}$ are three neutrosophic numbers (see, [48, 49]).

$$\begin{aligned} \left\|\Omega_L(u)-\Gamma_L(u)\right\| \le& \left\|\Omega_L(u)-\Delta_L(u)\right\| + \left\|\Delta_L(u)-\Gamma_L(u)\right\| \ ; \text{for } u\in[0,1] \\ \le& \max\left\{\left\|\Omega_L(u)-\Delta_L(u)\right\|,\left\|\Omega_R(u)-\Delta_R(u)\right\|\right\} \\ & + \max\left\{\left\|\Delta_L(u)-\Gamma_L(u)\right\|,\left\|\Delta_R(u)-\Gamma_R(u)\right\|\right\} \\ =& d\left(\Omega(u),\Delta(u)\right) + d(\Delta(u),\Gamma(u)) \ ; \text{for } u\in[0,1] \end{aligned}$$

Similarly, $\left\|\Omega_R(u)-\Gamma_R(u)\right\| \le d\left(\Omega(u),\Delta(u)\right) + d\left(\Delta(u),\Gamma(u)\right)$.
Therefore, $\max\left\{\left\|\Omega_L(u)-\Gamma_L(u)\right\|,\left\|\Omega_R(u)-\Gamma_R(u)\right\|\right\} \le d\left(\Omega(u),\Delta(u)\right) + d\left(\Delta(u),\Gamma(u)\right)$.
So,

$$d\left(\Omega(u),\Gamma(u)\right) \le d\left(\Omega(u),\Delta(u)\right) + d\left(\Delta(u),\Gamma(u)\right) \tag{8}$$

Similarly,

$$d\left(\Omega(v),\Gamma(v)\right) \le d\left(\Omega(v),\Delta(v)\right) + d\left(\Delta(v),\Gamma(v)\right) \tag{9}$$

and

$$d\left(\Omega(w),\Gamma(w)\right) \le d\left(\Omega(w),\Delta(w)\right) + d\left(\Delta(w),\Gamma(w)\right) \tag{10}$$

Therefore,

$$\begin{aligned} D(\tilde{\Omega},\tilde{\Gamma}) =& \sup_{0\le u,v,w\le 1}\left\{d\left(\Omega(u),\Gamma(u)\right),d\left(\Omega(v),\Gamma(v)\right),d\left(\Omega(w),\Gamma(w)\right)\right\} \\ =& \left\{\sup_{0\le u,v,w\le 1}\left\{d\left(\Omega(u),\Gamma(u)\right)\right\},\sup_{0\le u,v,w\le 1}\left\{d\left(\Omega(v),\Gamma(v)\right)\right\},\sup_{0\le u,v,w\le 1}\left\{d\left(\Omega(w),\Gamma(w)\right)\right\}\right\} \\ \le& \left\{\sup_{0\le u,v,w\le 1}\left\{d\left(\Omega(u),\Delta(u)\right)+d\left(\Delta(u),\Gamma(u)\right)\right\},\sup_{0\le u,v,w\le 1}\left\{d\left(\Omega(v),\Delta(v)\right)+d\left(\Delta(v),\Gamma(v)\right)\right\},\right. \\ & \left.\sup_{0\le u,v,w\le 1}\left\{d\left(\Omega(w),\Delta(w)\right)+d\left(\Delta(w),\Gamma(w)\right)\right\}\right\} \\ =& \left\{\sup_{0\le u,v,w\le 1}\left\{d\left(\Omega(u),\Delta(u)\right)\right\},\sup_{0\le u,v,w\le 1}\left\{d\left(\Omega(v),\Delta(v)\right)\right\},\sup_{0\le u,v,w\le 1}\left\{d\left(\Omega(w),\Delta(w)\right)\right\}\right\} \\ & + \left\{\sup_{0\le u,v,w\le 1}\left\{d\left(\Delta(u),\Gamma(u)\right)\right\},\sup_{0\le u,v,w\le 1}\left\{d\left(\Delta(v),\Gamma(v)\right)\right\},\sup_{0\le u,v,w\le 1}\left\{d\left(\Delta(w),\Gamma(w)\right)\right\}\right\} \\ =& D\left(\tilde{\Omega},\tilde{\Delta}\right) + D\left(\tilde{\Delta},\tilde{\Gamma}\right) \end{aligned}$$

So,

$$D\left(\tilde{\Omega}, \tilde{\Gamma}\right) \leq D\left(\tilde{\Omega}, \tilde{\Delta}\right) + D\left(\tilde{\Delta}, \tilde{\Gamma}\right) \tag{11}$$

Equation (11) perceives the most significant result regarding triangle inequalities. Combining all the mentioned points, we can conclude that D is a metric on the set of neutrosophic numbers.

□

Definition 2.3. Let $\tilde{\Omega}(n)_{n=1}^{\infty}$ be a sequence of neutrosophic numbers. Then $\tilde{\Omega}(n)$ is said to be convergent if there exists a neutrosophic number $\tilde{\Omega}$ such that $D(\tilde{\Omega}(n), \tilde{\Omega}) \to 0$ as $n \to \infty$, where D is the metric in Definition 2.1.

Theorem 2.4. *Suppose, a sequence of neutrosophic numbers $\tilde{\Omega}(n)$ is given by $\tilde{\Omega}(n) = \big\{[\Omega_L(u)^n, \Omega_R(u)^n], [\Omega_L(v)^n, \Omega_R(v)^n], [\Omega_L(w)^n, \Omega_R(w)^n]\big\}$ and a neutrosophic number $\tilde{\Omega}(n) = \big\{[\Omega_L(u), \Omega_R(u)], [\Omega_L(v), \Omega_R(v)], [\Omega_L(w), \Omega_R(w)]\big\}$. Then $\tilde{\Omega}(n) \to \tilde{\Omega}$ if and only if*

$$\begin{cases} \Omega_L(u)^n & \to \Gamma_L(u) \\ \Omega_R(u)^n & \to \Gamma_R(u) \\ \Omega_L(v)^n & \to \Gamma_L(v) \\ \Omega_R(v)^n & \to \Gamma_R(v) \\ \Omega_L(w)^n & \to \Gamma_L(w) \\ \Omega_R(w)^n & \to \Gamma_R(w) \end{cases} \tag{12}$$

as $n \to \infty$.

Proof. Suppose $\tilde{\Omega}(n) \to \tilde{\Omega}$ as $n \to \infty$ in the metric space defined by D. Therefore, given $\epsilon > 0$, there exists a natural number $\mathbb{N}$ such that $D\left(\tilde{\Omega}(n), \tilde{\Omega}\right) < \epsilon$ for $n \geq \mathbb{N}$, which implies $\sup_{0 \leq u,v,w \leq 1} \Big\{ d\left(\Omega(u)^n, \Omega(u)\right), d\left(\Omega(v)^n, \Omega(v)\right), d\left(\Omega(w)^n, \Omega(w)\right) \Big\} < \epsilon$ for $n \geq N$ which implies

$$\begin{cases} d\left(\Omega(u)^n, \Omega(u)\right) < \epsilon \\ d\left(\Omega(v)^n, \Omega(v)\right) < \epsilon \\ d\left(\Omega(w)^n, \Omega(w)\right) < \epsilon \end{cases} \tag{13}$$

for all $u, v, w \in [0, 1]$, which implies

$$\begin{cases} \left\|\Omega_L(u)^n - \Omega_L(u)\right\| < \epsilon \\ \left\|\Omega_R(u)^n - \Omega_R(u)\right\| < \epsilon \\ \left\|\Omega_L(v)^n - \Omega_L(v)\right\| < \epsilon \\ \left\|\Omega_R(v)^n - \Omega_R(v)\right\| < \epsilon \\ \left\|\Omega_L(w)^n - \Omega_L(w)\right\| < \epsilon \\ \left\|\Omega_R(w)^n - \Omega_R(w)\right\| < \epsilon \end{cases}$$

for all $u, v, w \in [0, 1]$. This is equivalent to

$$\begin{cases} \Omega_L(u)^n \to \Omega_L(u) \\ \Omega_R(u)^n \to \Omega_R(u) \\ \Omega_L(v)^n \to \Omega_L(v) \\ \Omega_R(v)^n \to \Omega_R(v) \\ \Omega_L(w)^n \to \Omega_L(w) \\ \Omega_R(w)^n \to \Omega_R(w) \end{cases}$$

To prove the converse statement, suppose

$$\begin{cases} \Omega_L(u)^n \to \Omega_L(u) \\ \Omega_R(u)^n \to \Omega_R(u) \\ \Omega_L(v)^n \to \Omega_L(v) \\ \Omega_R(v)^n \to \Omega_R(v) \\ \Omega_L(w)^n \to \Omega_L(w) \\ \Omega_R(w)^n \to \Omega_R(w) \end{cases} \tag{14}$$

This implies, for given $\exists > 0$, there exist natural numbers N_1, N_2, N_3, N_4, N_5 and N_6 such that

$$\begin{cases} \left\|\Omega_L(u)^n - \Omega_L(u)\right\| < \epsilon & ; \text{if } n \geq N_1 \\ \left\|\Omega_R(u)^n - \Omega_R(u)\right\| < \epsilon & ; \text{if } n \geq N_2 \\ \left\|\Omega_L(v)^n - \Omega_L(v)\right\| < \epsilon & ; \text{if } n \geq N_3 \\ \left\|\Omega_R(v)^n - \Omega_R(v)\right\| < \epsilon & ; \text{if } n \geq N_4 \\ \left\|\Omega_L(w)^n - \Omega_L(w)\right\| < \epsilon & ; \text{if } n \geq N_5 \\ \left\|\Omega_R(w)^n - \Omega_R(w)\right\| < \epsilon & ; \text{if } n \geq N_6 \end{cases}$$

This can be rewritten as

$$\begin{cases} \max\left\{\left\|\Omega_L(u)^n - \Omega_L(u)\right\|, \left\|\Omega_R(u)^n - \Omega_R(u)\right\|\right\} < \epsilon & ; \text{for } n \geq N_u = \max\{N_1, N_2\} \\ \max\left\{\left\|\Omega_L(v)^n - \Omega_L(v)\right\|, \left\|\Omega_R(v)^n - \Omega_R(v)\right\|\right\} < \epsilon & ; \text{for } n \geq N_v = \max\{N_3, N_4\} \\ \max\left\{\left\|\Omega_L(w)^n - \Omega_L(w)\right\|, \left\|\Omega_R(w)^n - \Omega_R(w)\right\|\right\} < \epsilon & ; \text{for } n \geq N_w = \max\{N_5, N_6\} \end{cases}$$

which is equivalent to

$$\begin{cases} d\left(\Omega(u)^n, \Omega(u)\right) < \epsilon & ; \text{for } n \geq N_u \\ d\left(\Omega(v)^n, \Omega(v)\right) < \epsilon & ; \text{for} n \geq N_v \\ d\left(e\Omega(w)^n, \Omega(w)\right) < \epsilon & ; \text{for } n \geq N_w \end{cases} \tag{15}$$

Therefore, $\sup_{u,v,w \in [0,1]} \left\{ d\left(\Omega(u)^n, \Omega(u)\right), d\left(\Omega(v)^n, \Omega(v)\right), d\left(\Omega(w)^n, \Omega(w)\right) \right\} < \epsilon$, for $n \geq N = \max_{u,v,w \in [0,1]} \{N_u, N_v, N_w\}$. This implies that $D\left(\Omega(n), \Omega\right) < \epsilon$ for $n \geq N$. Consequently, $\tilde{\Omega}(n) \to \tilde{\Omega}$.

In the succeeding text, we will prove the completeness property of the metric space under the defined metric. Before proving the completeness, we first define a Cauchy sequence in neutrosophic numbers. □

Definition 2.5. A sequence of neutrosophic numbers $\tilde{\Omega}(n)_{n=1}^{\infty}$ is called a Cauchy sequence if for a given $\epsilon > 0$, there exists a natural number N such that $D\left(\tilde{\Omega}(n), \tilde{\Omega}(m)\right) < \epsilon$ for all $n, m \geq N$.

Intuitively, the sequence $\tilde{\Omega}(n)_{n=1}^{\infty}$ is said to be Cauchy when the neutrosophic numbers are clustering more tightly as phase advances. The distances of clustering values are measured using the metric D. In other words, it represents the stabilization of the system's progress. On the other hand, convergence of the sequence $\tilde{\Omega}(n)_{n=1}^{\infty}$ signifies the approach of the system towards a finite point (equilibrium). Intuitively, the existence of equilibrium must ensure stability of the system, but not vice versa. Therefore, convergence is a stronger property than the Cauchy criteria. These two properties are equivalent in a complete metric space. The succeeding theorem shows that the proposed metric provides a complete metric space of neutrosophic valued numbers.

Theorem 2.6. *The metric space under the metric D is a complete metric space.*

Proof. Let $\tilde{\Omega}(n)$ be a Cauchy sequence in the metric space. Thus, for given $\epsilon > 0$,
$D\left(\tilde{\Omega}(n), \tilde{\Omega}(m)\right) < \epsilon$ for all $n, m \geq N$, which implies that

$$\sup_{u,v,w\in[0,1]} \left\{ d\left(\Omega(u)^n, \Omega(u)^m\right), d\left(\Omega(v)^n, \Omega(v)^m\right), d\left(\Omega(w)^n, \Omega(w)^m\right) \right\} < \epsilon \tag{16}$$

for all $n, m \geq N$. This gives,

$$\begin{cases} d\left(\Omega(u)^n, \Omega(u)^m\right) < \epsilon \\ d\left(\Omega(v)^n, \Omega(v)^m\right) < \epsilon \\ d\left(\Omega(w)^n, \Omega(w)^m\right) < \epsilon \end{cases}$$

for all $n, m \geq N$. This implies that

$$\begin{cases} \max\left\{\left\|\Omega_L(u)^n - \Omega_L(u)^m\right\|, \left\|\Omega_R(u)^n - \Omega_R(u)^m\right\|\right\} & < \epsilon \\ \max\left\{\left\|\Omega_L(v)^n - \Omega_L(v)^m\right\|, \left\|\Omega_R(v)^n - \Omega_R(v)^m\right\|\right\} & < \epsilon \\ \max\left\{\left\|\Omega_L(w)^n - \Omega_L(w)^m\right\|, \left\|\Omega_R(w)^n - \Omega_R(w)\right\|^m\right\} & < \epsilon \end{cases}$$

for all $n, m \geq N$ and consequently

$$\begin{cases} \left\|\Omega_L(u)^n - \Omega_L(u)^m\right\| < \epsilon \\ \left\|\Omega_R(u)^n - \Omega_R(u)^m\right\| < \epsilon \\ \left\|\Omega_L(v)^n - \Omega_L(v)^m\right\| < \epsilon \\ \left\|\Omega_R(v)^n - \Omega_R(v)^m\right\| < \epsilon \\ \left\|\Omega_L(w)^n - \Omega_L(w)^m\right\| < \epsilon \\ \left\|\Omega_R(w)^n - \Omega_R(w)^m\right\| < \epsilon \end{cases} \tag{17}$$

which implies that $\left\{\Omega_L(u)^n\right\}, \left\{\Omega_R(u)^n\right\}, \left\{\Omega_L(v)^n\right\}, \left\{\Omega_R(v)^n\right\}, \left\{\Omega_L(w)^n\right\}$ and $\left\{\Omega_R(w)^n\right\}$ are all Cauchy sequences in $\mathbb{R}$. Since, Cauchy sequence in real numbers are convergent, $\left\{\Omega_L(u)^n\right\}, \left\{\Omega_R(u)^n\right\}, \left\{\Omega_L(v)^n\right\}$, $\left\{\Omega_R(v)^n\right\}, \left\{\Omega_L(w)^n\right\}$ and $\left\{\Omega_R(w)^n\right\}$ are all convergent sequences. It can be shown that there exists a limit of the Cauchy sequence $\tilde{\Omega}(n)$ in the metric space, taking limits of these convergent sequences. This proves the completeness property. □

Lemma 2.7. *Let $\tilde{\phi}$ and $\tilde{X}$ be two neutrosophic numbers. Then, $\tilde{\psi}$ will be the Hukuhara difference when*

$\tilde{\phi} = \tilde{X} + \tilde{\psi}$, that is, when

$$\begin{cases} \phi_L(u) = X_L(u) + \psi_L(u) \\ \phi_R(u) = X_R(u) + \psi_R(u) \\ \phi_L(v) = X_L(v) + \psi_L(v) \\ \phi_R(v) = X_R(v) + \psi_R(v) \\ \phi_L(w) = X_L(w) + \psi_L(w) \\ \phi_R(w) = X_R(w) + \psi_R(w) \end{cases} \tag{18}$$

or, equivalently, when

$$\begin{cases} \psi_L(u) = \phi_L(u) - X_L(u) \\ \psi_R(u) = \phi_R(u) - X_R(u) \\ \psi_L(v) = \phi_L(v) - X_L(v) \\ \psi_R(v) = \phi_R(v) - X_R(v) \\ \psi_L(w) = \phi_L(w) - X_L(w) \\ \psi_R(w) = \phi_R(w) - X_R(w) \end{cases} \tag{19}$$

for all $n, v, w \exists [0, 1]$. In notation, it can be expressed as

$$\tilde{\phi} \ominus_H \tilde{X} = \tilde{\psi} \tag{20}$$

Lemma 2.8. *Let $\tilde{\phi}$ and $\tilde{X}$ be two neutrosophic numbers. Then, $\tilde{\psi}$ will be the generalized Hukuhara difference [46] if either $\tilde{\phi} = \tilde{X} + \tilde{\psi}$ or, $\tilde{X} = \tilde{\phi} + (-1)\tilde{\psi}$.*

For the first case, we say $\tilde{\psi}$ is the generalized Hukuhara difference of type I and for the second case, it is said to be the generalized Hukuhara difference of type II. In case of type II, by the impact of neutrosophic environment, the parametric split up of the equation $\tilde{X} = \tilde{\phi} + (-1)\tilde{\psi}$, which implies

$$\begin{aligned} &\Big\{ \big[X_L(u), X_R(u)\big], \ \big[X_L(v), X_R(v)\big], \big[X_L(w), X_R(w)\big] \Big\} \\ &\quad = \Big\{ \big[\phi_L(u), \phi_R(u)\big], \big[\phi_L(v), \phi_R(v)\big], \big[\phi_L(w), \phi_R(w)\big] \Big\} \\ &\qquad + (-1) \Big\{ \big[\psi_L(u), \psi_R(u)\big], \big[\psi_L(v), \psi_R(v)\big], \big[\psi_L(w), \psi_R(w)\big] \Big\} \end{aligned}$$

or,

$$\begin{aligned} &\Big\{ \big[X_L(u), X_R(u)\big], \ \big[X_L(v), X_R(v)\big], \big[X_L(w), X_R(w)\big] \Big\} \\ &\quad = \Big\{ \big[\phi_L(u), \phi_R(u)\big], \big[\phi_L(v), \phi_R(v)\big], \big[\phi_L(w), \phi_R(w)\big] \Big\} \\ &\qquad + \Big\{ \big[-\psi_L(u), -\psi_R(u)\big], \big[-\psi_L(v), -\psi_R(v)\big], \big[-\psi_L(w), -\psi_R(w)\big] \Big\} \end{aligned}$$

Therefore, the generalized Hukuhara difference (gH) $\tilde{\psi}$ can be obtained as

$$\begin{cases} \psi_L(u) = \phi_R(u) - X_R(u) \\ \psi_R(u) = \phi_L(u) - X_L(u) \\ \psi_L(v) = \phi_R(v) - X_R(v) \\ \psi_R(v) = \phi_L(v) - X_L(v) \\ \psi_L(w) = \phi_R(w) - X_R(w) \\ \psi_R(w) = \phi_L(w) - X_L(w) \end{cases} \tag{21}$$

Since there is no notion of classical subtraction for the set valued numbers, like fuzzy, interval or neutrosophic numbers, the notion of generalized Hukuhara difference is necessary. It includes two distinct cases. Type I corresponds with the phenomena of additive increment. On the contrary, the subtractive increment is given by Type II of the generalized Hukuhara difference.

Theorem 2.9. *If $\tilde{\phi}$ and $\tilde{X}$ are two neutrosophic numbers, then the following statements are true.*

(i) If gH difference exists, then it is unique.

(ii) $\tilde{\phi} \ominus_{gH} \tilde{X} = \tilde{\phi} \ominus_H \tilde{X}$ or $\tilde{\phi} \ominus_{gH} \tilde{X} = -(\tilde{X} \ominus_H \tilde{\phi})$.

(iii) If $\tilde{\phi} \ominus_{gH} \tilde{X}$ exist in type I sense, then $\tilde{X} \ominus_{gH} \tilde{\phi}$ exist in type-II sense.

(iv) $(\tilde{\phi} + \tilde{X}) \ominus_{gH} \tilde{\phi} = \tilde{X}$.

(v) $\tilde{\phi} \ominus_{gH} \tilde{X} = \tilde{X} \ominus_{gH} \tilde{\phi} = \tilde{\psi}$ if and only if $\tilde{\psi} = -\tilde{\psi}$.

Proof.

(i) Suppose, the gH difference of $\tilde{\phi}$ and $\tilde{X}$ exist and it is equal to $\tilde{\psi}$. Then, $\tilde{\phi} \ominus_H \tilde{X} = \tilde{\psi}$. Then either $\tilde{\phi} = \tilde{X} + \tilde{\psi}$ or $\tilde{X} = \tilde{\phi} + (-1)\tilde{\psi}$. If possible, let $\tilde{Z}$ be another gH difference of $\tilde{\phi}$ and $\tilde{X}$, then $\tilde{\phi} \ominus_{gH} \tilde{X} = \tilde{Z}$, which implies that either $\tilde{\phi} = \tilde{X} + \tilde{Z}$ or $\tilde{X} = \tilde{\phi} + (-1)\tilde{Z}$, uniqueness of the gH difference is proved straightforward. Similarly, in case of $\tilde{X} = \tilde{\phi} + (-1)\tilde{\psi}$ and $\tilde{X} = \tilde{\phi} + (-1)\tilde{Z}$, the conclusion can be made easily. Therefore, we consider the third case where $\tilde{\phi} = \tilde{X} + \tilde{\psi}$ and $\tilde{X} = \tilde{\phi} + (-1)\tilde{Z}$. Then, $\tilde{\phi} = \tilde{\phi} + (-1)\tilde{Z} + \tilde{\psi}$ which implies that

$$\begin{cases} \psi_L(u) = Z_L(u) \\ \psi_R(u) = Z_R(u) \\ \psi_L(v) = Z_L(v) \\ \psi_R(v) = Z_R(v) \\ \psi_L(w) = Z_L(w) \\ \psi_R(w) = Z_R(w) \end{cases} \tag{22}$$

This shows that $\tilde{\psi} = \tilde{Z} = \{K\}$ a crisp singleton set. Hence, the generalized Hukuhara difference is unique in this case also.

It is worth mentioning that generalized Hukuhara difference also possesses limitations in the context of existence, even refining the notion of Hukuhara difference. However, this first property ensures the uniqueness of generalized Hukuhara difference, provided it exists.

(ii) when $\tilde{\phi} \ominus_{gH} \tilde{X} = \tilde{\psi}$, then either $\tilde{\phi} = \tilde{X} + \tilde{\psi}$ or $\tilde{X} = \tilde{\phi} + (-1)\tilde{\psi}$. If $\tilde{\phi} = \tilde{X} + \tilde{\psi}$, then $\tilde{\phi} \ominus_H \tilde{X} = \tilde{\psi}$. In case of $\tilde{X} = \tilde{\phi} + (-1)\tilde{\psi}$, we have $\Big\{\big[X_L(u), X_R(u)\big], \big[X_L(v), X_R(v)\big], \big[X_L(w), X_R(w)\big]\Big\} = \Big\{\big[\phi_L(u) - \psi_R(u), \phi_R(u) - \phi_L(u)\big], \big[\phi_L(v) - \psi_R(v), \phi_R(v) - \phi_L(v)\big] + \big[\phi_L(w) - \psi_R(w), \phi_R(w) - \phi_L(w)\big]\Big\}$. This is equivalent to the following system.

$$\begin{cases} \psi_R(u) = -\big[X_L(u) - \phi_L(u)\big] \\ \psi_L(u) = -\big[X_R(u) - \phi_R(u)\big] \\ \psi_R(v) = -\big[X_L(v) - \phi_L(v)\big] \\ \psi_L(v) = -\big[X_R(v) - \phi_R(v)\big] \\ \psi_R(w) = -\big[X_L(w) - \phi_L(w)\big] \\ \psi_L(w) = -\big[X_R(w) - \phi_R(w)\big] \end{cases} \tag{23}$$

Therefore, the generalized Hukuhara difference $\tilde{\phi}$ is given by $\tilde{\psi} = -\left(\tilde{X} \ominus_H \tilde{\phi}\right)$.

This second property incorporates the forward and backwards difference concepts together. The generalized Hukuhara difference coincides with the Hukuhara difference when the first fuzzy number is greater than the latter in a fuzzy sense. The generalized Hukuhara difference is equal to the negative of the Hukuhara difference for the reverse intuition of superiority between fuzzy numbers.

(iii) Suppose, $\tilde{\phi} \ominus_{gH} \tilde{X} = \tilde{\psi}$ in type-I sense. Then, $\tilde{\phi} = \tilde{X} + \tilde{\psi}$, that is,

$$\begin{cases} \phi_L(u) = X_L(u) + \psi_L(u) \\ \phi_R(u) = X_R(u) + \psi_R(u) \\ \phi_L(v) = X_L(v) + \psi_L(v) \\ \phi_R(v) = X_R(v) + \psi_R(v) \\ \phi_L(w) = X_L(w) + \psi_L(w) \\ \phi_R(w) = X_R(w) + \psi_R(w) \end{cases}$$

This system can be written as

$$\begin{cases} \phi_L(u) = X_L(u) + (-1)\left(-\psi_L(u)\right) \\ \phi_R(u) = X_R(u) + (-1)\left(-\psi_R(u)\right) \\ \phi_L(v) = X_L(v) + (-1)\left(-\psi_L(v)\right) \\ \phi_R(v) = X_R(v) + (-1)\left(-\psi_R(v)\right) \\ \phi_L(w) = (-1)\left(-X_L(w) + \psi_L(w)\right) \\ \phi_R(w) = (-1)\left(-X_R(w) + \psi_R(w)\right) \end{cases} \tag{24}$$

The last system of equations implies that $\tilde{\phi} = \tilde{X} + (-1)\tilde{Z}$, where $\tilde{Z} = \big\{[-z_2(\alpha), -z_1(\alpha)]; [-z_2(\beta), -z_1(\beta)]; [-z_2(\gamma), -z_1(\gamma)]\big\} = -\tilde{\psi}$. So, $\tilde{X} \ominus_{gH} \tilde{\phi}$ exist in type-II sense.

It provides the interconnection between two cases of difference covered in the generalized Hukuhara sense.

(iv) When $(\tilde{\phi} + \tilde{X}) \ominus_{gH} \tilde{\phi} = \tilde{\psi}$ exist in type-I sense and its value is $\tilde{\psi}$, then $\tilde{X} = \tilde{\psi}$ is trivial. In case where $(\tilde{\phi} + \tilde{X}) \ominus_{gH} \tilde{\phi} = \tilde{\psi}$ exist in type-II sense, $\tilde{\phi} = \tilde{\phi} + \tilde{X} + (-1)\tilde{\psi}$. This concludes that $\tilde{X} = \tilde{\psi} = \{k\}$, a crisp number.

(v) Suppose $\tilde{\phi} \ominus_{gH} \tilde{X} = \tilde{X} \ominus_{gH} \tilde{\phi} = \tilde{\psi}$ both exist in type-I sense. Now, $\tilde{\phi} \ominus_{gH} \tilde{X} = \tilde{\psi}$ implies that $\tilde{\phi} = \tilde{X} + \tilde{\psi}$. Then the following system of equations is true:

$$\begin{cases} \phi_L(u) = X_L(u) + \psi_L(u) \\ \phi_R(u) = X_R(u) + \psi_R(u) \\ \phi_L(v) = X_L(v) + \psi_L(v) \\ \phi_R(v) = X_R(v) + \psi_R(v) \\ \phi_L(w) = X_L(w) + \psi_L(w) \\ \phi_R(w) = X_R(w) + \psi_R(w) \end{cases} \tag{25}$$

Again, $\tilde{X} \ominus_{gH} \tilde{\phi} = \tilde{\psi}$ implies that $\tilde{X} = \tilde{\phi} + \tilde{\psi}$. This is equivalent to the system

$$\begin{cases} X_L(u) = \phi_L(u) + \psi_L(u) \\ X_R(u) = \phi_R(u) + \psi_R(u) \\ X_L(v) = \phi_L(v) + \psi_L(v) \\ X_R(v) = \phi_R(v) + \psi_R(v) \\ X_L(w) = \phi_L(w) + \psi_L(w) \\ X_R(w) = \phi_R(w) + \psi_R(w) \end{cases} \tag{26}$$

The above two system of equations conclude that $\tilde{\psi}_i(u) = 0$, $\tilde{\psi}_i(v) = 0$ and $\tilde{\psi}_i(w) = 0$; $i \in \{L, R\}$. Therefore, $\tilde{\psi}$ is the zero element in the collection of neutrosophic numbers. A similar result will appear when we consider the Hukuhara difference in the type-II sense. So, we discuss the third phenomenon here in which $\tilde{\phi} \ominus_{gH} \tilde{X}$ exist in a type-I sense but $\tilde{X} \ominus_{gH} \tilde{\phi}$ is in type-II. Then, $\tilde{\phi} = \tilde{X} + \tilde{\psi}$ and $\tilde{\phi} = \tilde{X} + (-1)\tilde{\psi}$. That is, $\tilde{\psi} = -\tilde{\psi}$. Conversely, when $\tilde{\psi} = -\tilde{\psi}$ then, $\tilde{\phi} \ominus_{gH} \tilde{X}$ implies and is implied by $\tilde{\phi} = \tilde{X} + \tilde{\psi}$ or $\tilde{X} = \tilde{\phi} + (-1)\tilde{\psi}$ implies and is implied by $\tilde{\phi} = \tilde{X} + (-1)\tilde{\psi}$ or $\tilde{X} = \tilde{\phi} + \tilde{\psi}$ implies and is implied by $\tilde{X} \ominus_{gH} \tilde{\phi}$.

This completes the proof.

The last two properties are synonymous with the outputs in deterministic arithmetic systems.

□

With this theorem, we bring an end to the present subsection containing the theories of metric space and generalized Hukuhara difference for neutrosophic numbers. In the next section, we will discuss the existence and uniqueness criteria for the solution of the imprecise difference equation under neutrosophic uncertainty.

3 Existence and uniqueness conditions for solution of difference equation in neutrosophic environment

The difference equation, with coefficients and initial values considered as neutrosophic numbers, is referred to as a neutrosophic fuzzy difference equation. The theory will differ when the generalized Hukuhara difference is taken instead of the Hukuhara difference. Before going into the existence and uniqueness criteria of solvability of the imprecise difference equation in the generalised Hukuhara sense, we discuss the following theorem in the Hukuhara sense.

Theorem 3.1. *Suppose a neutrosophic Hukuhara difference equation with its initial value is given in the following system.*

$$\begin{cases} \tilde{X}(t+h) \ominus_H \tilde{X}(t) = \tilde{\Lambda}\left(t, \tilde{X}(t)\right) \\ \tilde{X}(t_0) = \tilde{X}_0 \end{cases} \tag{27}$$

The system is considered for $t \in [t_0, T]$ and Λ is taken as a continuous valued neutrosophic function, provided the existence of the Hukuhara difference. Then, the system (27) is equivalent to

$$\tilde{X}(t_{n+1}) = \tilde{X}_0 + \sum_{i=0}^{n} \tilde{\Lambda}\left(t_i, \tilde{X}(t_i)\right) \tag{28}$$

In Equation (28), the partition of the interval $[t_0, T]$ is given by, $0 = t_0 < t_1 < \cdots < t_n = T, h = \frac{T-t_0}{n}$ and $t_i = t_0 + ih$ for $i = 1, 2, \ldots, n$.

In Equation (27), h represents a small generalized increment of the independent variable t. The imprecise difference equation addresses the imprecise rate of change or growth governed by the function $\tilde{\Lambda}$, with initial information for such increment h. Now, the value of h can be fixed anticipating the number of recurring phases n and partitioning the interval $[t_0, T]$. Subsequent values of t are given in terms of the initial value t_0, increment h and step i. In this context, the n-th value of the independent variable in a partitioned system is $t_n = t_0 + nh$. In the proposed theory of discrete systems, we will use t_n for $t + h$.

Proof. Suppose the given system (27) in which H-difference exists and $\tilde{\Lambda}$ is a continuous function for all $t \exists [t_0, T]$. Then, $\tilde{X}(t_n + h) \ominus_H \tilde{X}(t_n) = \tilde{\Lambda}\left(t_n, \tilde{X}(t_n)\right)$, which implies that

$$\begin{cases} X_L\left(t_{n+1}, u\right) = X_L(t_n, u) + \Lambda_L\left(t_n, X_L(t_n, u), X_R(t_n, u), u\right) \\ X_R\left(t_{n+1}, u\right) = X_R(t_n, u) + \Lambda_R\left(t_n, X_L(t_n, u), X_R(t_n, u), u\right) \\ X_L\left(t_{n+1}, v\right) = X_L(t_n, v) + \Lambda_L\left(t_n, X_L(t_n, v), X_R(t_n, v), v\right) \\ X_R\left(t_{n+1}, v\right) = X_R(t_n, v) + \Lambda_R\left(t_n, X_L(t_n, v), X_R(t_n, v), v\right) \\ X_L\left(t_{n+1}, w\right) = X_L(t_n, w) + \Lambda_L\left(t_n, X_L(t_n, w), X_R(t_n, w), w\right) \\ X_R\left(t_{n+1}, w\right) = X_R(t_n, w) + \Lambda_R\left(t_n, X_L(t_n, w), X_R(t_n, w), w\right) \end{cases} \tag{29}$$

If we focus on the first equation of the above system, we get the following result.

$$\begin{aligned} X_L\left(t_{n+1}, u\right) &= X_L(t_n, u) + \Lambda_L\left(t_n, X_L(t_n, u), X_R(t_n, u), u\right) \\ &= X_L(t_{n-1}, u) + \Lambda_L\left(t_{n-1}, X_L(t_{n-1}, u), X_R(t_{n-1}, u), u\right) \\ &= X_L(t_0, u) + \sum_{i=0}^{n-1} \Lambda_L\left(t_i, X_L(t_i, u), X_R(t_i, u), u\right) \end{aligned}$$

Similarly, the following results are obtained.

$$\begin{aligned} X_R\left(t_{n+1}, u\right) &= X_R(t_0, u) + \sum_{i=0}^{n-1} \Lambda_R\left(t_i, X_L(t_i, u), X_R(t_i, u), u\right) \\ X_L\left(t_{n+1}, v\right) &= X_L(t_0, v) + \sum_{i=0}^{n-1} \Lambda_L\left(t_i, X_L(t_i, v), X_R(t_i, v), v\right) \\ X_R\left(t_{n+1}, v\right) &= X_R(t_0, v) + \sum_{i=0}^{n-1} \Lambda_R\left(t_i, X_L(t_i, v), X_R(t_i, v), v\right) \\ X_L\left(t_{n+1}, w\right) &= X_L(t_0, w) + \sum_{i=0}^{n-1} \Lambda_L\left(t_i, X_L(t_i, w), X_R(t_i, w), w\right) \\ X_R\left(t_{n+1}, w\right) &= X_R(t_0, w) + \sum_{i=0}^{n-1} \Lambda_R\left(t_i, X_L(t_i, w), X_R(t_i, w), w\right) \end{aligned}$$

Therefore, the above six crisp equations can be written as

$$\tilde{X}\left(t_{n+1}\right) = \tilde{X}_0 + \sum_{i=0}^{n} \tilde{\Lambda}\left(t_i, \tilde{X}(t_i)\right) \tag{30}$$

a neutrosophic equation addressing iterations. Equation (27) represents the imprecise rate of change or growth governed by the function $\tilde{\Lambda}$, with initial information, ignoring the cumulative numbers in successive steps of discrete phenomena, using a neutrosophic difference equation. In other words, Equation (27) addresses the dynamical behaviour of the system with a known initial state. On the contrary, the systems' response (or cumulative credit) through successive phases based on the available information in a particular phase can be perceived by the corresponding recurrence equation. In this context, Equation (30) provides information about the population size in the $(n+1)$-th step, based on the initial information $\tilde{X}_0$ and the dynamics-controlling function $\tilde{\Lambda}$ in the mentioned imprecise environment. Therefore, the two mentioned phenomena of distinct perspectives are equivalent through sharing the ultimate goal of forecasting population size in arbitrary phases. This present theorem has proved the mathematical equivalence of Equation (27) and Equation (30).

Imprecise differences are not merely an extension of differences in deterministic senses. A neutrosophic number C is called a Hukuhara difference of two neutrosophic numbers A and B when $A = B + C$. However, the Hukuhara difference has limitations. For instance, if A is inferior to B, in a fuzzy number sense, then the Hukuhara difference of two neutrosophic numbers A and B does not exist. In fact, $A + (-1)A \neq 0$. Therefore, the generalized Hukuhara difference between two fuzzy numbers was introduced to enable forward and backwards differences together. In the next theorem, we generalize the preceding theorem in the case of the generalized Hukuhara difference. □

Theorem 3.2. *Suppose we consider the generalized Hukuhara difference in a neutrosophic environment as follows:*

$$\begin{cases} \tilde{X}(t+h) \ominus_{gH} \tilde{X}(t) = \tilde{\Lambda}\left(t, \tilde{X}(t)\right) \\ \tilde{X}(t_0) = \tilde{X}_0 \end{cases} \tag{31}$$

In Equation (31), $\tilde{\Lambda}$ is a continuous function. Then, Equation (31) is equivalent to either of the following equations.

$$\tilde{X}\left(t_{n+1}\right) = \tilde{X}(t_0) + \sum_{i=0}^{n} \tilde{\Lambda}\left(t_i, \tilde{X}(t_i)\right) \tag{32}$$

$$\text{Or, } \tilde{X}\left(t_0\right) = \tilde{X}(t_{n+1}) + (-1)\sum_{i=0}^{n} \tilde{\Lambda}\left(t_i, \tilde{X}(t_i)\right) \tag{33}$$

Proof. Equation (31) is equivalent to either of the following systems.

$$\begin{cases} \tilde{X}(t+h) = \tilde{X}(t) + \tilde{\Lambda}\left(t, \tilde{X}(t)\right) \\ \tilde{X}(t_0) = \tilde{X}_0 \end{cases} \tag{34}$$

$$\text{Or , } \begin{cases} \tilde{X}(t) = \tilde{X}(t+h) + (-1)\tilde{\Lambda}\left(t, \tilde{X}(t)\right) \\ \tilde{X}(t_0) = \tilde{X}_0 \end{cases} \tag{35}$$

Case I. when the System (27) is true. Then by Theorem 2.9, Equation (31) implies Equation (32).
Case II.. When the system (31) is true in the Type-II Hukuhara difference sense.

Then, $\tilde{X}(t) = \tilde{X}(t+h) + (-1)\tilde{\Lambda}\left(t, \tilde{X}(t)\right)$ and consequently, $\tilde{X}\left(t_n\right) = \tilde{X}(t_{n+1}) + (-1)\tilde{\Lambda}\left(t_n, \tilde{X}(t_n)\right)$.

Taking (u, v, w)-cuts in the above neutrosophic equation, we get

$$\begin{cases} X_L(t_n, u) = X_L(t_{n+1}, u) + \Lambda_R\left(t_n, X_L(t_n, u), X_R(t_n, u), u\right) \\ X_R(t_n, u) = X_R(t_{n+1}, u) + \Lambda_L\left(t_n, X_L(t_n, u), X_R(t_n, u), u\right) \\ X_L(t_n, v) = X_L(t_{n+1}, v) + \Lambda_R\left(t_n, X_L(t_n, v), X_R(t_n, v), v\right) \\ X_R(t_n, v) = X_R(t_{n+1}, v) + \Lambda_L\left(t_n, X_L(t_n, v), X_R(t_n, v), v\right) \\ X_L(t_n, w) = X_L(t_{n+1}, w) + \Lambda_R\left(t_n, X_L(t_n, w), X_R(t_n, w), w\right) \\ X_R(t_n, w) = X_R(t_{n+1}, w) + \Lambda_L\left(t_n, X_L(t_n, w), X_R(t_n, w), w\right) \end{cases} \tag{36}$$

The leading equation of the preceding system implies

$$\begin{aligned} X_L(t_n, u) =& X_L(t_{n+1}, u) + \Lambda_R\left(t_n, X_L(t_n, u), X_R(t_n, u), u\right) \\ =& X_L(t_{n+1}, u) + \sum_{i=0}^{n} \Lambda_R\left(t_i, X_L(t_i, u), X_R(t_i, u), u\right) \end{aligned}$$

which is equivalent to

$$X_L(t_0, u) = X_L(t_{n+1}, u) + (-1)\sum_{i=0}^{n} \Lambda_R\left(t_i, X_L(t_i, u), X_R(t_i, u), u\right)$$

Proceeding similarly, we obtain the following results

$$\begin{aligned} X_R(t_0, u) =& X_R(t_{n+1}, u) + (-1)\sum_{i=0}^{n} \Lambda_R\left(t_i, X_L(t_i, u), X_R(t_i, u), u\right) \\ X_L(t_0, v) =& X_L(t_{n+1}, v) + (-1)\sum_{i=0}^{n} \Lambda_L\left(t_i, X_L(t_i, v), X_R(t_i, v), v\right) \\ X_R(t_0, v) =& X_R(t_{n+1}, v) + (-1)\sum_{i=0}^{n} \Lambda_R\left(t_i, X_L(t_i, v), X_R(t_i, v), v\right) \\ X_L(t_0, w) =& X_L(t_{n+1}, w) + (-1)\sum_{i=0}^{n} \Lambda_L\left(t_i, X_L(t_i, w), X_R(t_i, w), w\right) \\ X_R(t_0, w) =& X_R(t_{n+1}, w) + (-1)\sum_{i=0}^{n} \Lambda_R\left(t_i, X_L(t_i, w), X_R(t_i, w), w\right) \end{aligned}$$

Combining all of the above six results, we get

$$\tilde{X}(t_0) = \tilde{X}(t_{n+1}) + (-1)\sum_{i=0}^{n} \tilde{\Lambda}\left(t_i, \tilde{X}(t_i)\right) \tag{37}$$

Therefore, the theorem is proved for both cases. □

Equation (31) represents the imprecise rate of change or growth governed by the function $\tilde{\Lambda}$, with initial information, enabling both forward and backwards growth. In other words, Equation (31) addresses the bi-directional dynamical behaviour of the system with a known initial state. On the contrary, the systems' response (or cumulative credit) through successive phases based on the available information in a particular phase can be perceived by the corresponding recurrence equation. In this context, Equation (32) provides information about the population size in the $(n+1)$-th step, based on the initial information $\tilde{X}_0$ and the dynamics-controlling function $\tilde{\Lambda}$ in the mentioned imprecise environment. On the contrary, he initial information $\tilde{X}_0$ is given in terms of population size in the $(n+1)$-th step and the dynamics-controlling function $\tilde{\Lambda}$ in (33). Therefore, the two mentioned phenomena of distinct perspectives are equivalent through sharing the ultimate goal of forecasting population size in arbitrary phases. This present theorem has proved the mathematical equivalence of Equation (31) with Equation (32) and (33) in forward and backwards growth, respectively.

Definition 3.3. A Neutrosophic valued function $\tilde{\Lambda}$ is Lipschitz continuous if there exists a real number $L > 0$ such that $D\left(\tilde{\Lambda}(r), \tilde{\Lambda}(s)\right) \leq L \mid r - s \mid$, for all deterministic (real or complex) numbers r and s.

In the above definition $\mid . \mid$ represent Euclidean distance between crisp numbers and D is the metric in Definition 2.1. This property is stronger than continuity and uniform continuity. In fact, Lipschitz continuity implies uniform continuity and uniform continuity implies continuity.

Theorem 3.4. *Suppose a neutrosophic difference equation with an initial condition is given as below:*

$$\begin{cases} \tilde{X}(t+h) \ominus_{gH} \tilde{X}(t) = \tilde{\Lambda}\left(t, \tilde{X}(t)\right) \\ \tilde{X}(t_0) = \tilde{X}_0 \end{cases} \tag{38}$$

In Equation (38), $\tilde{\Lambda}$ is a continuous function. Then, Equation (38) is equivalent to either of the following equations.

Proof. Without loss of generality, let Equation (38) imply

$$\tilde{X}\left(t_{n+1}\right) = \tilde{X}(t_0) + \sum_{i=0}^{n} \tilde{\Lambda}\left(t_i, \tilde{X}(t_i)\right) \tag{39}$$

To avoid complexity, let us write the equation as

$$\tilde{X}_{t_{n+1}} = \tilde{X}_{t_0} + \sum_{i=0}^{n} \tilde{\Lambda}\left(t_i, \tilde{X}_{t_i}\right) \tag{40}$$

We consider a sequence $\tilde{X}^k_{t_{n+1}}$ satisfying the Equation (40).
Then, $\tilde{X}^k_{t_{n+1}} = \tilde{X}_{t_0} + \sum_{i=0}^{n} \tilde{\Lambda}\left(t_i, \tilde{X}^k_{t_i}\right)$.
Then,

$$\begin{aligned} D\left(\tilde{X}_{t_0} + \sum_{i=0}^{n} \tilde{\Lambda}\left(t_i, \tilde{X}^k_{t_i}\right), \tilde{X}_{t_0}\right) &\leq \sum_{i=0}^{n} D\left(\tilde{\Lambda}\left(t_i, \tilde{X}^k_{t_i}\right), 0\right) \\ &\leq M\left(\frac{t_n - t_0}{h}\right) \end{aligned}$$

Therefore,

$$\begin{aligned} D\left(\tilde{X}^{k+1}_{t_{n+1}}, \tilde{X}^k_{t_{n+1}}\right) &= D\left(\tilde{X}_{t_0} + \sum_{i=0}^{n} \tilde{\Lambda}\left(t_i, \tilde{X}^k_{t_i}\right), \tilde{X}_{t_0} + \sum_{i=0}^{n} \tilde{\Lambda}\left(t_i, \tilde{X}^{k-1}_{t_i}\right)\right) \\ &= D\left(\sum_{i=0}^{n} \tilde{\Lambda}\left(t_i, \tilde{X}^k_{t_i}\right), \sum_{i=0}^{n} \tilde{\Lambda}\left(t_i, \tilde{X}^{k-1}_{t_i}\right)\right) \\ &\leq L^k M\left(\frac{t_n - t_0}{h}\right)^n \end{aligned} \tag{41}$$

From this, it can be proved that the sequence $\left\{\tilde{X}^k_{t_{n+1}}\right\}$ is a Cauchy sequence for each n. The sequence $\left\{\tilde{X}^k_{t_{n+1}}\right\}$ is a Cauchy sequence in the concept of a complete metric space given by the metric D for all $t \in [t_0, T]$. So, $\left\{\tilde{X}^k_{t_{n+1}}\right\}$ converges uniformly to a limit process $\tilde{X}_{t_{n+1}}$, which is the solution of system (38) and it proves the existence of such solution. To prove the uniqueness let $\tilde{X}'_{t_{n+1}}$ and $\tilde{X}_{t_{n+1}}$ are the two limit process then,

$$\begin{aligned} D\left(\tilde{X}'_{t_{n+1}}, \tilde{X}_{t_{n+1}}\right) =& D\left(\tilde{X}_{t_0} + \sum_{i=0}^{n} \tilde{\Lambda}\left(t_i, \tilde{X}'_{t_i}\right), \tilde{X}_{t_0} + \sum_{i=0}^{n} \tilde{\Lambda}\left(t_i, \tilde{X}_{t_i}\right)\right) \\ =& D\left(\sum_{i=0}^{n} \tilde{\Lambda}\left(t_i, \tilde{X}'_{t_i}\right), \sum_{i=0}^{n} \tilde{\Lambda}\left(t_i, \tilde{X}_{t_i}\right)\right) \\ \leq& L \sum_{i=0}^{n} D\left(\tilde{X}'_{t_i}, \tilde{X}_{t_i}\right) \end{aligned} \tag{42}$$

So, Grownwall inequality implies that $D\left(\tilde{X}'_{t_{n+1}}, \tilde{X}_{t_{n+1}}\right) = 0$. This proves the uniqueness of the theorem. □

Lemma 3.5. *If $\tilde{f}$ is continuous function, $\tilde{X}$ and $\tilde{Z}$ are two neutrosophic sets then, followed by [47], we can write $[\tilde{f}(\tilde{X}, \tilde{Z})]_{(u,v,w)} = \tilde{f}([\tilde{X}]_{(u,v,w)}, [\tilde{Z}]_{(u,v,w)})$; $\forall u, v, w \in (0, 1]$.*

4 Logistic Neutrosophic difference equation as an application

Let us consider the logistic neutrosophic difference equation

$$\begin{cases} \tilde{\phi}_{n+1} = \tilde{\rho}\tilde{\phi}_n\left(1 - \tilde{\phi}_n\right) \\ \tilde{\phi}_{n=0} = \tilde{\phi}_0 \end{cases} \tag{43}$$

At first, we express the Equation (43) into two different forms in the sense of neutrosophic number and generalized Hukuhara difference as

$$\begin{cases} \tilde{\phi}_{n+1} \ominus_{gH} \tilde{\rho}\tilde{\phi}_n\left(1 - \tilde{\phi}_n\right) = 0 \\ \tilde{\phi}_{n=0} = \tilde{\phi}_0 \end{cases} \tag{44}$$

$$\begin{cases} \tilde{\phi}_{n+1} = \tilde{\rho}\tilde{\phi}_n \ominus_{gH} \tilde{\rho}(\tilde{\phi}_n)^2 \\ \tilde{\phi}_{n=0} = \tilde{\phi}_0 \end{cases} \tag{45}$$

Case I. Applying both type-I and type-II gH-difference on the Equation (44) then, we have the difference equations as

$$\tilde{\phi}_{n+1} = \tilde{\rho}\tilde{\phi}_n\left(1 - \tilde{\phi}_n\right) + 0 \tag{46}$$

and

$$\text{and } \tilde{\rho}\tilde{\phi}_n\left(1 - \tilde{\phi}_n\right) = \tilde{\phi}_{n+1} + (-1)0 \tag{47}$$

respectively. Thus, for both types of gH-difference cases and regarding 0 as a crisp quantity, we have the difference equation as

$$\tilde{\phi}_{n+1} = \tilde{\rho}\tilde{\phi}_n \left(1 - \tilde{\phi}_n\right) \tag{48}$$

with the initial neutrosophic value $\tilde{\phi}_{n=0} = \tilde{\phi}_0$. Let the neutrosophic (u, v, w)-cut of the population at the n-th generation $\tilde{\phi}_n$ and growth rate $(\tilde{\rho})$ are denoted by

$$\left[\tilde{\phi}_n\right]_{(u,v,w)} = \left\{\left[\phi_{L,n}(u), \phi_{R,n}(u)\right], \left[\phi_{L,n}(v), \phi_{R,n}(v)\right], \left[\phi_{L,n}(w), \phi_{R,n}(w)\right]\right\}$$

and

$$[\tilde{\rho}]_{(u,v,w)} = \left\{\left[\rho_L(u), \rho_R(u)\right], \left[\rho_L(v), \rho_R(v)\right], \left[\rho_L(w), \rho_R(w)\right]\right\}$$

respectively.

Using Equation (48) and the Lemma 3.5, in terms of parametric neutrosophic numbers, we have the following systems of crisp difference equations:

$$\begin{cases} \phi_{L,n+1}(u) = \rho_L(u)\phi_{L,n}(u)\left(1 - \phi_{R,n}(u)\right) \\ \phi_{R,n+1}(u) = \rho_R(u)\phi_{R,n}(u)\left(1 - \phi_{L,n}(u)\right) \\ \phi_{L,n+1}(v) = \rho_L(v)\phi_{L,n}(v)\left(1 - \phi_{R,n}(v)\right) \\ \phi_{R,n+1}(v) = \rho_R(v)\phi_{R,n}(v)\left(1 - \phi_{L,n}(v)\right) \\ \phi_{L,n+1}(w) = \rho_L(w)\phi_{L,n}(w)\left(1 - \phi_{R,n}(w)\right) \\ \phi_{R,n+1}(w) = \rho_R(w)\phi_{R,n}(w)\left(1 - \phi_{L,n}(w)\right) \end{cases} \tag{49}$$

In order to investigate the dynamical behaviour of the solution $\tilde{\phi}_n$ of the neutrosophic difference Equation (48), we find out the equilibrium points of the System (49). Then, we analyze the nature of the solution in the vicinity of the equilibrium points, whether the equilibrium points are stable or not.

Let, $\left[\tilde{\phi}\right]_{(u,v,w)} = \left\{\left[\phi_L(u), \phi_R(u)\right], \left[\phi_L(v), \phi_R(v)\right], \left[\phi_L(w), \phi_R(w)\right]\right\}$ is an equilibrium point of the System (49), then we have the following equilibrium points:

1. $\tilde{E}^0 = \{[0,0],[0,0],[0,0]\}$,
2. $\tilde{E}^1 = \{[0,\phi_R(u)],[0,\phi_R(v)],[0,\phi_R(w)]\}$,
3. $\tilde{E}^2 = \{[\phi_L(u),0],[\phi_L(v),0],[\phi_L(w),0]\}$,
4. $\tilde{E}^3 = \left\{\left[\frac{\rho_R(u)-1}{\rho_R(u)}, \frac{\rho_L(u)-1}{\rho_L(u)}\right], \left[\frac{\rho_R(v)-1}{\rho_R(v)}, \frac{\rho_L(v)-1}{\rho_L(v)}\right], \left[\frac{\rho_R(w)-1}{\rho_R(w)}, \frac{\rho_L(w)-1}{\rho_L(w)}\right]\right\}$.

Lemma 4.1. *Consider the system of difference Equation (49) with positive neutrosophic initial value $\tilde{\phi}_{n=0} = \tilde{\phi}_0$ and growth rate $\tilde{\rho}$, then the following statements of the equilibrium points based on the stability criterion are true:*

(i) The neutrosophic equilibrium point $\{[0,0],[0,0],[0,0]\}$ is stable if $\rho_i(j) < 1; i = L, R$ and $j = u, v, w$.

(ii) There exist infinite number of equilibrium points $[0,\phi_R(u)],[0,\phi_R(v)],[0,\phi_R(w)]$ if $\rho_R(j) = 1$ and it is Lyapunov stable if $\mid \rho_L(j)(1-\phi_R(j)) \mid < 1$ where $j = u, v, w$.

(iii) The equilibrium points $[\phi_L(u),0],[\phi_L(v),0],[\phi_L(w),0]$ exist infinitely if $\rho_L(j) = 1$ and the equilibrium point is Lyapunov stable if $\mid \rho_R(j)(1-\phi_L(j)) \mid < 1$ for $j = u, v, w$.

(iv) *The coexistence equilibrium points* $\left\{\left[\frac{\rho_R(u)-1}{\rho_R(u)}, \frac{\rho_L(u)-1}{\rho_L(u)}\right], \left[\frac{\rho_R(v)-1}{\rho_R(v)}, \frac{\rho_L(v)-1}{\rho_L(v)}\right], \left[\frac{\rho_R(w)-1}{\rho_R(w)}, \frac{\rho_L(w)-1}{\rho_L(w)}\right]\right\}$ *is feasible if all* $\rho_i(j) = 1$ *and unstable always where* $i = L, R$ *and* $j = u, v, w$.

Proof. After linearization in the neighbourhood of the neutrosophic equilibrium point

$$\left[\tilde{\phi}\right]_{(u,v,w)} = \left\{\left[\phi_L(u), \phi_R(u)\right], \left[\phi_L(v), \phi_R(v)\right], \left[\phi_L(w), \phi_R(w)\right]\right\}$$

the system of difference Equation (49) can be written as

$$\tilde{\phi}_{n+1} = D\tilde{\phi}_n \tag{50}$$

where, $\tilde{\phi}_n = \left\{\left[\phi_{L,n}(u), \phi_{R,n}(u), \phi_{L,n}(v), \phi_{R,n}(v), \phi_{L,n}(w), \phi_{R,n}(w)\right]^T\right\}$ and D is a the Jacobian matrix at $\left[\tilde{\phi}\right]_{(u,v,w)}$ and it is expressed by

$$D_{|\left[\tilde{\phi}\right]_{(u,v,w)}} = \begin{bmatrix} \left(A(u)\right)_{2\times 2} & & \\ & \left(A(v)\right)_{2\times 2} & \\ & & \left(A(w)\right)_{2\times 2} \end{bmatrix} \tag{51}$$

where each block matrix is $\left(A(j)\right)_{2\times 2} = \begin{bmatrix} \rho_L(j)(1-\phi_R(j)) & -\rho_L(j)\phi_L(j) \\ -\rho_R(j)\phi_R(j) & \rho_R(j)(1-\phi_L(j)) \end{bmatrix}$; $j = u, v, w \in [0,1]$.

(i) Now using Equation (51), the Jacobian matrix at the neutrosophic equilibrium point

$$D_{|\tilde{E}^0} = \begin{bmatrix} \left(A_1(u)\right)_{2\times 2} & & \\ & \left(A_1(v)\right)_{2\times 2} & \\ & & \left(A_1(w)\right)_{2\times 2} \end{bmatrix}$$

where $\left(A(j)\right)_{2\times 2} = \begin{bmatrix} \rho_L(j) & 0 \\ 0 & \rho_R(j) \end{bmatrix}$. The eigenvalues of the matrix $D_{|[\tilde{0}]_{(u,v,w)}}$ are $\rho_L(j)$, $\rho_R(j)$; $j = u, v, w \in [0,1]$.

Therefore, the equilibrium point $\tilde{E}^0$ is stable if all the eigen values lies inside the unit disk i.e., $\mid \rho_L(j) \mid < 1$ and $\mid \rho_R(j) \mid < 1$ for $j = u, v, w \in [0,1]$.

Note 1. The stability of the neutrosophic equilibrium point $\tilde{E}^0$ of the logistic growth model depends upon the parametric values of the neutrosophic growth rate $\tilde{\rho}$ only.

(ii) Here the Jacobian matrix at the neutrosophic equilibrium point $\tilde{E}^1$ is

$$D_{|\tilde{E}^1} = \begin{bmatrix} \left(A_2(u)\right)_{2\times 2} & & \\ & \left(A_2(v)\right)_{2\times 2} & \\ & & \left(A_2(w)\right)_{2\times 2} \end{bmatrix}$$

Where the block matrices are $\left(A_2(j)\right)_{2\times 2} = \begin{bmatrix} \rho_L(j)(1-\phi_R(j)) & 0 \\ -\phi_R(j) & 1 \end{bmatrix}$; $j = u, v, w \in [0,1]$.

Since the matrix is a lower triangle matrix, the eigenvalues are 1 and $\rho_L(j)(1-\phi_R(j))$. Therefore, 1 is an eigen value of algebraic multiplicity 3 of the matrix $D_{|\tilde{E}^1}$ and other three eigen values are $\rho_L(j)(1-\phi_R(j))$ for $j = u, v, w$.

Therefore, the equilibrium point $\tilde{E}^1$ is Lyapunov stable if $\mid \rho_L(j)(1-\phi_R(j)) \mid < 1$.

(iii) The proof is similar with (ii).

(iv) Again, the Jacobian matrix near the neutrosophic equilibrium point $\tilde{E}^3$ is given by

$$D_{|\tilde{E}^3} = \begin{bmatrix} \big(A_4(u)\big)_{2\times2} & & \\ & \big(A_4(v)\big)_{2\times2} & \\ & & \big(A_4(w)\big)_{2\times2} \end{bmatrix}$$

where the block matrices are $\big(A_4(j)\big)_{2\times2} = \begin{bmatrix} 1 & \frac{\rho_L(j)}{\rho_R(j)}(1-\rho_R(j)) \\ \frac{\rho_R(j)}{\rho_L(j)}(1-\rho_L(j)) & 1 \end{bmatrix}$; $j = u, v, w \in [0,1]$.

Therefore, the corresponding characteristic equation is

$$\det\left(D_{|\tilde{E}^3} - \lambda I_6\right) = 0 \tag{52}$$

$$\text{or, } \begin{bmatrix} \big(A_4(u)\big)_{2\times2} - \lambda I_2 & & \\ & \big(A_4(v)\big)_{2\times2} - \lambda I_2 & \\ & & \big(A_4(w)\big)_{2\times2} - \lambda I_2 \end{bmatrix} = 0$$

$$\begin{aligned} &\text{or, } \mid \big(A_4(u)\big)_{2\times2} - \lambda I_2 \mid = 0 \ ; \ j = u, v, w \in [0,1] \\ &\text{or, } \lambda^2 - 2\lambda + \left\{1 - \frac{\rho_L(j)}{\rho_R(j)}(1-\rho_R(j)) \cdot \frac{\rho_R(j)}{\rho_L(j)}(1-\rho_L(j))\right\} = 0 \\ &\text{or, } \lambda^2 - 2\lambda + (\rho_L(j) + \rho_R(j) - \rho_L(j)\rho_r(j)) = 0 \\ &\text{or, } \lambda_{1,2}(j) = 1 \pm \sqrt{1 + \rho_L(j)\rho_r(j) - \rho_L(j)\rho_r(j)} \end{aligned} \tag{53}$$

Clearly, whatever be the values of $\sqrt{1 + \rho_L(j)\rho_r(j) - \rho_L(j)\rho_r(j)}$, there exist three eigenvalues of modulus greater than 1. Since all the eigenvalues of the matrix $D_{|\tilde{E}^3}$ do not lie inside the unit disk, hence the equilibrium point $\tilde{E}^3$ is unstable.

□

Case II. Applying both the type-I and type-II gH-difference on the neutrosophic difference Equation (45), we have the following equations accordingly as

$$\begin{cases} \tilde{\rho}\tilde{\phi}_n = \tilde{\rho}(\tilde{\phi}_n)^2 + \tilde{\phi}_{n+1} \\ \tilde{\phi}_{n=0} = \tilde{\phi}_0 \end{cases} \tag{54}$$

and

$$\begin{cases} \tilde{\rho}(\tilde{\phi}_n)^2 = \tilde{\rho}\tilde{\phi}_n + (-1)\tilde{\phi}_{n+1} \\ \tilde{\phi}_{n=0} = \tilde{\phi}_0 \end{cases} \tag{55}$$

Now, using the Lemma 2.7 on the neutrosophic difference Equation (54), gives the parametric system of difference equation as

$$\begin{cases} \phi_{L,n+1}(u) = \rho_L(u)\phi_{L,n}(u)\left(1 - \phi_{L,n}(u)\right) \\ \phi_{R,n+1}(u) = \rho_R(u)\phi_{R,n}(u)\left(1 - \phi_{R,n}(u)\right) \\ \phi_{L,n+1}(v) = \rho_L(v)\phi_{L,n}(v)\left(1 - \phi_{L,n}(v)\right) \\ \phi_{R,n+1}(v) = \rho_R(v)\phi_{R,n}(v)\left(1 - \phi_{R,n}(v)\right) \\ \phi_{L,n+1}(w) = \rho_L(w)\phi_{L,n}(w)\left(1 - \phi_{L,n}(w)\right) \\ \phi_{R,n+1}(w) = \rho_R(w)\phi_{R,n}(w)\left(1 - \phi_{R,n}(w)\right) \end{cases} \tag{56}$$

Finding the exact solution of the non-linear System (56) is too difficult. In order to investigate the dynamical behaviour of the system, we mainly focus on the dynamics of the system near the equilibrium points. This is why, solving the system of difference Equation (56), we have the following neutrosophic equilibrium points:

1. $\tilde{E}_1^0 = \{[0,0],[0,0],[0,0]\}$
2. $\tilde{E}_1^1 = \left\{\left[0, \frac{\rho_R(u)-1}{\rho_R(u)}\right], \left[0, \frac{\rho_R(v)-1}{\rho_R(v)}\right], \left[0, \frac{\rho_R(w)-1}{\rho_R(w)}\right]\right\}$
3. $\tilde{E}_1^2 = \left\{\left[\frac{\rho_L(u)-1}{\rho_L(u)}, 0\right], \left[\frac{\rho_L(v)-1}{\rho_L(v)}, 0\right], \left[\frac{\rho_L(w)-1}{\tilde{\rho}_L(w)}, 0\right]\right\}$
4. $\tilde{E}_1^3 = \left\{\left[\frac{\rho_L(u)-1}{\rho_L(u)}, \frac{\rho_R(u)-1}{\rho_R(u)}\right], \left[\frac{\rho_L(v)-1}{\rho_L(v)}, \frac{\rho_R(v)-1}{\rho_R(v)}\right], \left[\frac{\rho_L(w)-1}{\rho_L(w)}, \frac{\rho_R(w)-1}{\rho_R(w)}\right]\right\}$

Lemma 4.2. *Consider the system of difference Equation (56), then the following results are true:*

(i) The trivial neutrosophic equilibrium point $\{[0,0],[0,0],[0,0]\}$ is stable if $\rho_L(j)$, $\rho_R(j) \in (0,1)$ for $j = u,v,w$.

(ii) The equilibrium point $\tilde{E}_1^1$ is stable if $\rho_L(j) \in (0,1)$ and $1 < \rho_R(j) < 3$ where $j = u,v,w$.

(iii) The equilibrium point $\tilde{E}_1^2$ is unstable.

(iv) The coexistence equilibrium point is feasible if all $\rho_i(j) > 1$ and stable if $1 < \rho_i(j) < 3$ where $i = L,R$ and $j = u,v,w$.

Proof. Linearizing the system of difference Equation (56) in the neighbourhood of the neutrosophic equilibrium point

$$\tilde{\phi}_{n+1} = D^1_{|[\tilde{\phi}^*]_{(u,v,w)}} \tilde{\phi}_n \tag{57}$$

where, $\tilde{\phi}_n = \left\{\left[\phi_{L,n}(u), \phi_{R,n}(u), \phi_{L,n}(v), \phi_{R,n}(v), \phi_{L,n}(w), \phi_{R,n}(w)\right]^T\right\}$ and $D^1_{|[\tilde{\phi}^*]_{(u,v,w)}}$ is the Jacobian matrix. Therefore, the matrix

$$D^1_{|[\tilde{\phi}^*]_{(u,v,w)}} = \begin{bmatrix} (B(u))_{2\times 2} & & \\ & (B(v))_{2\times 2} & \\ & & (B(w))_{2\times 2} \end{bmatrix} \tag{58}$$

where each block matrix is of the form $(B(j))_{2\times 2} = \begin{bmatrix} \rho_L(j) - 2\rho_L(j)\phi_L^*(j) & 0 \\ 0 & \rho_R(j) - 2\rho_R(j)\phi_R^*(j) \end{bmatrix}$; $j = u,v,w \in [0,1]$.

(i) The Jacobian matrix at the neutrosophic equilibrium point $\tilde{E}_1^0$ is

$$D^1_{|\tilde{E}_1^0} = \begin{bmatrix} (B_1(u))_{2\times 2} & & \\ & (B_1(v))_{2\times 2} & \\ & & (B_1(w))_{2\times 2} \end{bmatrix}$$

where $\left(B_1(j)\right)_{2\times 2} = \begin{bmatrix} \rho_L(j) & 0 \\ 0 & \rho_R(j) \end{bmatrix}$. Clearly, the eigenvalues of the matrix $D_{|\tilde{E}_1^0}$ are $\rho_L(j)$ and $\rho_R(j)$ where $j = u, v, w$. Now, all the eigen values of this matrix lies inside the unit disc, i.e., $\mid \rho_L(j) \mid < 1$ and $\mid \rho_R(j) \mid < 1$ for $j = u, v, w \in [0, 1]$. Since, $\tilde{\rho}$ is a positive neutrosophic growth rate, hence $\rho_L(j) > 0$ and $\rho_R(j) > 0$. Therefore, both the inequation $\mid \rho_L(j) \mid < 1$ and $\mid \rho_R(j) \mid < 1$ implies that $\rho_L(j), \rho_R(j) \in (0, 1)$ for $j = u, v, w \in [0, 1]$.

Therefore, the equilibrium point $\tilde{E}_1^0$ is stable if $\rho_L(j), \rho_R(j) \in (0, 1)$ where $j = u, v, w \in [0, 1]$.

(ii) Using the Equation (58), the eigen values of the matrix $D^1_{|\tilde{E}_1^1}$ are $\rho_L(j)$ and $2 - \rho_R(j)$. Therefore, the equilibrium point is a stable if $\rho_L(j) < 1$ and $\mid 2 - \rho_L(j) \mid < 1$. On simplification of the inequations $\rho_L(j) < 1$ and $\mid 2 - \rho_R(j) \mid < 1$ implies that $\rho_L(j) \in (0, 1)$ and $1 < \rho_R(j) < 3$; where $j = u, v, w$.

(iii) Proceeding in the similar way as the part (i), the eigen values of the matrix $D^1_{|\tilde{E}_1^2}$ are $\rho_R(j)$ and $2-\rho_L(j)$. Now, both the eigenvalues are belong within the unit disk if $\rho_R(j) < 1$ and $\mid 2 - \rho_L(j) \mid < 1$. The second inequality implies that $1 < \rho_L(j) < 3$. As the parametric quantities $0 < \rho_L(j) < \rho_R(j)$, therefore no values of $\rho_L(j), \rho_R(j)$ found such that the modulas of the eigen values are less than unity and hence the equilibrium point is unstable.

(iv) The proof is similar to (i).

Applying Lemma 2.8 related to type-II generalized Hukuhara difference and its parametric representation on the difference Equation (55), we have the following system of difference equations as

$$\begin{cases} \phi_{L,n+1}(u) = \rho_R(u)\phi_{R,n}(u)\left(1 - \phi_{R,n}(u)\right) \\ \phi_{R,n+1}(u) = \rho_L(u)\phi_{L,n}(u)\left(1 - \phi_{L,n}(u)\right) \\ \phi_{L,n+1}(v) = \rho_R(v)\phi_{R,n}(v)\left(1 - \phi_{R,n}(v)\right) \\ \phi_{R,n+1}(v) = \rho_L(v)\phi_{L,n}(v)\left(1 - \phi_{L,n}(v)\right) \\ \phi_{L,n+1}(w) = \rho_R(w)\phi_{R,n}(w)\left(1 - \phi_{R,n}(w)\right) \\ \phi_{R,n+1}(w) = \rho_L(w)\phi_{L,n}(w)\left(1 - \phi_{L,n}(w)\right) \end{cases} \tag{59}$$

In order to investigate the dynamical behaviour in the vicinity of the equilibrium points, we solve the system of Equation (59). The neutrosophic equilibrium points are given below:

- $\tilde{E}_2^0 = \left\{[0, 0], [0, 0], [0, 0]\right\}$,
- $\tilde{E}_2^1 = \left\{[\phi_L(u), \phi_R(u)], [\phi_L(v), \phi_R(v)], [\phi_R(w), \phi_R(w)]\right\}$, the coexistence equilibrium point. Where the neutrosophic components $\tilde{\phi_i}(j)$ are the roots (4.14), $i = L, R$ and $j = u, v, w$.

□

Lemma 4.3. *Consider the system of difference Equation (59), then the following results are true:*

(i) The neutrosophic trivial equilibrium point $\tilde{E}_2^0$ is stable if $\sqrt{\mid \rho_L(j)\rho_R(j) \mid} < 1$ when $j = u, v, w$.

(ii) The coexistence equilibrium point $\tilde{E}_2^1$ is stable if $\mid \lambda^*_{1,2}(j) \mid < 1$, where
$\lambda^*_{1,2}(j) = 1 \pm \sqrt{\left\{\rho_L(j)\left(1 - 2\phi_L(j)\right)\right\}\left\{\rho_R(j)\left(1 - 2\phi_R(j)\right)\right\}}$ *for $j = u, v, w$.*

Proof. The system of difference Equation (59), after the linearization in the neighbourhood of the neutrosophic equilibrium point $\tilde{E}_2^1$, can be written as

$$\tilde{\phi}_{n+1} = D^2_{|\tilde{E}_2^1}\tilde{\phi}_n \tag{60}$$

where, $\tilde{\phi}_n = \left[\phi_{L,n}(u), \phi_{R,n}(u), \phi_{L,n}(v), \phi_{R,n}(v), \phi_{L,n}(w), \phi_{R,n}(w)\right]^T$ and $D^2_{|\tilde{E}_2^1}$ is the Jacobian matrix, where

$$D^2_{|\tilde{E}_2^1} = \begin{bmatrix} \big(C(u)\big)_{2\times 2} & & \\ & \big(C(v)\big)_{2\times 2} & \\ & & \big(C(w)\big)_{2\times 2} \end{bmatrix} \tag{61}$$

In which the block matrices are represented by $\big(C(j)\big)_{2\times 2} = \begin{bmatrix} 0 & \rho_R(j)\left(1-2\phi_R(j)\right) \\ \rho_L(j)\left(1-2\phi_L(j)\right) & 0 \end{bmatrix}$; $j = u, v, w \in [0,1]$.

(i) The proof is simple.

(ii) The characteristic equation of the matrix (61) is

$$\begin{aligned}
&\det\left(D_{|\tilde{E}_2^1} - \lambda I_6\right) = 0 \\
\text{or, } &\begin{bmatrix} \big(C(u)\big)_{2\times 2} - \lambda I_2 & & \\ & \big(C(v)\big)_{2\times 2} - \lambda I_2 & \\ & & \big(C(w)\big)_{2\times 2} - \lambda I_2 \end{bmatrix} = 0 \\
\text{or, } &\mid \big(C(j)_{2\times 2}\big) - \lambda I_2 \mid = 0 \ ;\ j = u, v, w \in [0,1] \\
\text{or, } &\lambda^2 - \Big\{\rho_L(j)\left(1-2\phi_L(j)\right)\Big\}\Big\{\rho_R(j)\left(1-2\phi_R(j)\right)\Big\} = 0 \\
\text{or, } &\lambda^*_{1,2}(j) = \pm\sqrt{\Big\{\rho_L(j)\left(1-2\phi_L(j)\right)\Big\}\Big\{\rho_R(j)\left(1-2\phi_R(j)\right)\Big\}}
\end{aligned}$$

Therefore, all the eigenvalues belong within the unit disk if $\mid \lambda^*_{1,2}(j) \mid < 1$.

Hence, the equilibrium point $\tilde{E}^1_{2Neu}$ is stable if $\mid \lambda^*_{1,2}(j) \mid < 1$, where $\lambda^*_{1,2}(j) = \pm\sqrt{\Big\{\rho_L(j)\left(1-2\phi_L(j)\right)\Big\}\Big\{\rho_R(j)\left(1-2\phi_R(j)\right)\Big\}}$ for $j = u, v, w \in [0,1]$.

□

5 Numerical examples and graphical correspondence

The numerical examples and graphical representation of the neutrosophic solutions are presented in detail. Three examples are considered for this study, as follows:

Example 5.1. Consider the logistic difference Equation (48)

$$\tilde{\phi}_{n+1} = \tilde{\rho}\tilde{\phi}_n\left(1-\tilde{\phi}_n\right)$$

with the initial population $\tilde{\phi}_{n=0} = \tilde{\phi}_0$ and growth rate $\tilde{\rho}$ are taken as triangular neutrosophic numbers such that

$$\begin{cases} \tilde{\phi}_0 = [0.45, 0.50, 0.55, 0.35, 0.50, 0.65, 0.40, 0.50, 0.60] \\ \tilde{\rho} = [2.70, 2.75, 2.80; 2.6, 2.75, 2.9; 2.65, 2.75, 2.85] \end{cases} \tag{62}$$

The parametric neutrosophic representation of the Equation (62) is given by

$$\begin{cases} \left[\tilde{\phi}_0\right]_{(u,v,w)} = [0.45 + 0.05u, 0.50 - 0.05u], [0.5 - 0.15v, 0.5 + 0.15v], [0.5 - 0.1w, 0.5 + 0.1w] \\ [\tilde{\rho}]_{(u,v,w)} = [2.7 + 0.05u, 2.8 - 0.05u], [2.75 - 0.15v, 2.75 + 0.15v], [2.75 - 0.1w, 2.75 + 0.1w] \end{cases} \tag{63}$$

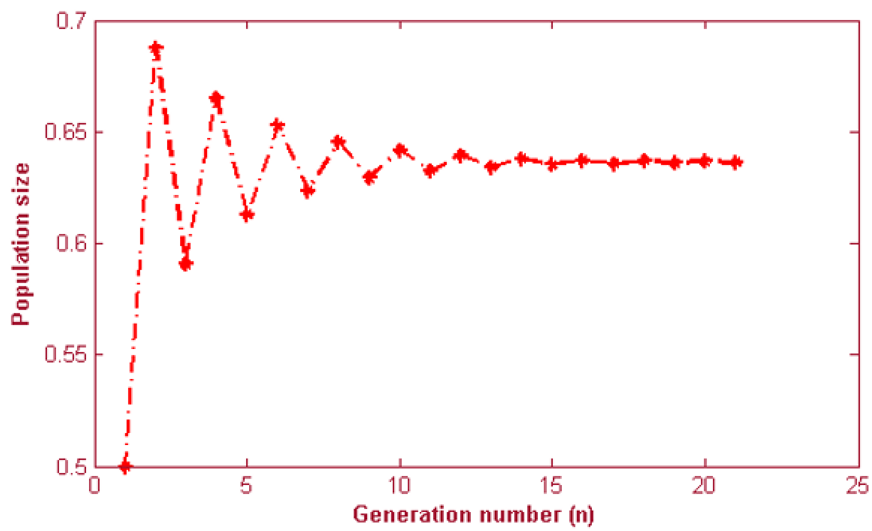

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Figure 1: Illustrates the dynamical stability of the crisp logistic difference equation

$$\tilde{\phi}_{n+1} = \tilde{\rho}\tilde{\phi}_n\left(1 - \tilde{\phi}_n\right)$$

with growth rate $\tilde{\rho} = 2.75$ and the initial population $\tilde{\phi}_0 = 0.5$. As the generation increases, the population size approaches towards a fixed quantity ($\cong 0.64$) with respect to the mentioned initial input.

Then using Equation (62) and Equation (63), the Equation (48) gives the system

$$\begin{cases} \phi_{L,n+1}(u) = (2.7 + 0.05u)\phi_{L,n}(u)\left(1 - \phi_{R,n}(u)\right) \\ \phi_{R,n+1}(u) = (2.8 - 0.05u)\phi_{R,n}(u)\left(1 - \phi_{L,n}(u)\right) \\ \phi_{L,n+1}(v) = (2.75 - 0.15v)\phi_{L,n}(v)\left(1 - \phi_{R,n}(v)\right) \\ \phi_{R,n+1}(v) = (2.75 + 0.15v)\phi_{R,n}(v)\left(1 - \phi_{L,n}(v)\right) \\ \phi_{L,n+1}(w) = (2.75 - 0.1w)\phi_{L,n}(w)\left(1 - \phi_{R,n}(w)\right) \\ \phi_{R,n+1}(w) = (2.75 + 0.1w)\phi_{R,n}(w)\left(1 - \phi_{L,n}(w)\right) \end{cases} \tag{64}$$

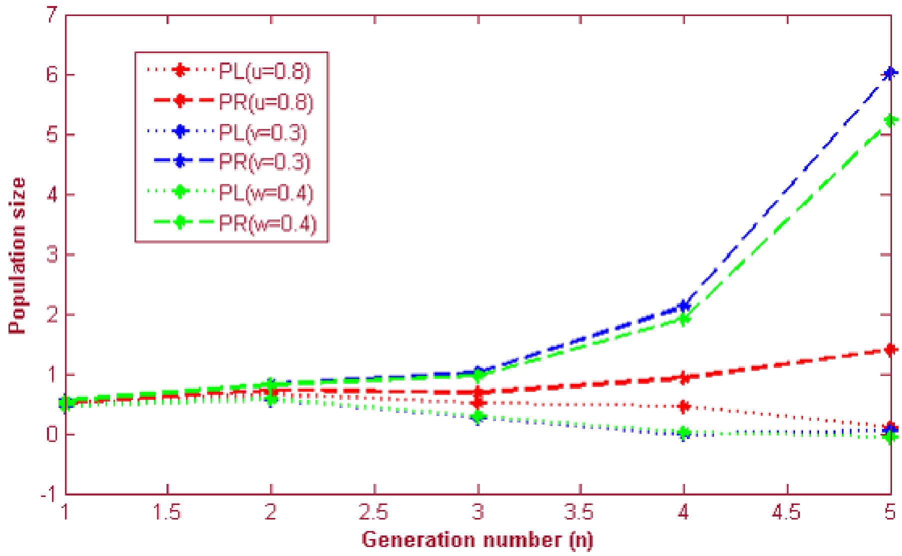

Figure 2: The discrete change-over the population size and the generation number of the System (64) with the triangular neutrosophic parametric population size is given by Equation (63) for $u = 0.8, v = 0.3$ and $w = 0.4$.

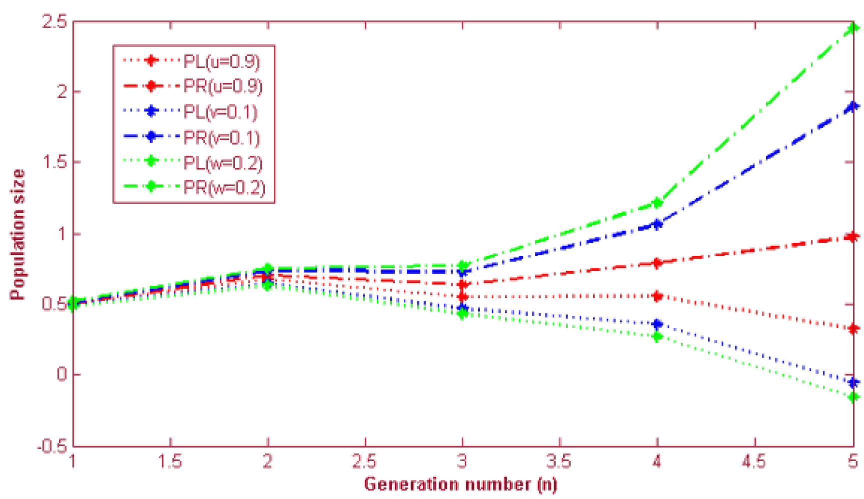

Figure 3: The discrete change-over the population size and the generation number of the System (64) with the triangular neutrosophic parametric population size is given by Equation (63) for $u = 0.9, v = 0.1$ and $w = 0.2$.

Example 5.2. Consider the neutrosophic logistic difference equation

$$\tilde{\phi}_{n+1} = \tilde{\rho}\tilde{\phi}_n \ominus_{gH} \tilde{\rho}(\tilde{\phi}_n)^2$$

in the type-I gH-difference concept with the initial neutrosophic population and growth rate information are as (63). Then combined impact of the Equation (56) and (63) implies the system,

$$\begin{cases} \phi_{L,n+1}(u) = (2.7+0.05u)\phi_{L,n}(u)\left(1-\phi_{L,n}(u)\right) \\ \phi_{R,n+1}(u) = (2.8-0.05u)\phi_{R,n}(u)\left(1-\phi_{R,n}(u)\right) \\ \phi_{L,n+1}(v) = (2.75-0.15v)\phi_{L,n}(v)\left(1-\phi_{L,n}(v)\right) \\ \phi_{R,n+1}(v) = (2.75+0.15v)\phi_{R,n}(v)\left(1-\phi_{R,n}(v)\right) \\ \phi_{L,n+1}(w) = (2.75-0.1w)\phi_{L,n}(w)\left(1-\phi_{L,n}(w)\right) \\ \phi_{R,n+1}(w) = (2.75+0.1w)\phi_{R,n}(w)\left(1-\phi_{R,n}(w)\right) \end{cases} \tag{65}$$

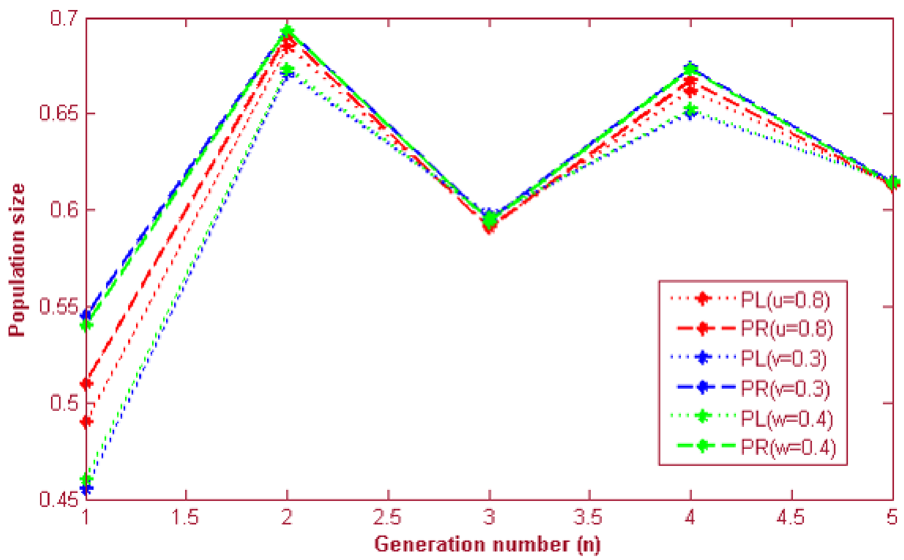

Figure 4: The discrete change-over the population size and the generation number of the System (65) for $u = 0.8, v = 0.3$ and $w = 0.4$.

Example 5.3. Consider the neutrosophic logistic difference equation

$$\tilde{\phi}_{n+1} = \tilde{\rho}\tilde{\phi}_n \ominus_{gH} \tilde{\rho}(\tilde{\phi}_n)^2$$

in the type-II gH-difference concept with the initial neutrosophic population and growth rate values as Equation (63).

Then from Equation (59) and using the Equation (63) we have,

$$\begin{cases} \phi_{L,n+1}(u) = (2.8-0.05u)\phi_{R,n}(u)\left(1-\phi_{R,n}(u)\right) \\ \phi_{R,n+1}(u) = (2.7+0.05u)\phi_{L,n}(u)\left(1-\phi_{L,n}(u)\right) \\ \phi_{L,n+1}(v) = (2.75+0.15v)\phi_{R,n}(v)\left(1-\phi_{R,n}(v)\right) \\ \phi_{R,n+1}(v) = (2.75-0.15v)\phi_{L,n}(v)\left(1-\phi_{L,n}(v)\right) \\ \phi_{L,n+1}(w) = (2.75+0.1w)\phi_{R,n}(w)\left(1-\phi_{R,n}(w)\right) \\ \phi_{R,n+1}(w) = (2.75-0.1w)\phi_{L,n}(w)\left(1-\phi_{L,n}(w)\right) \end{cases} \tag{66}$$

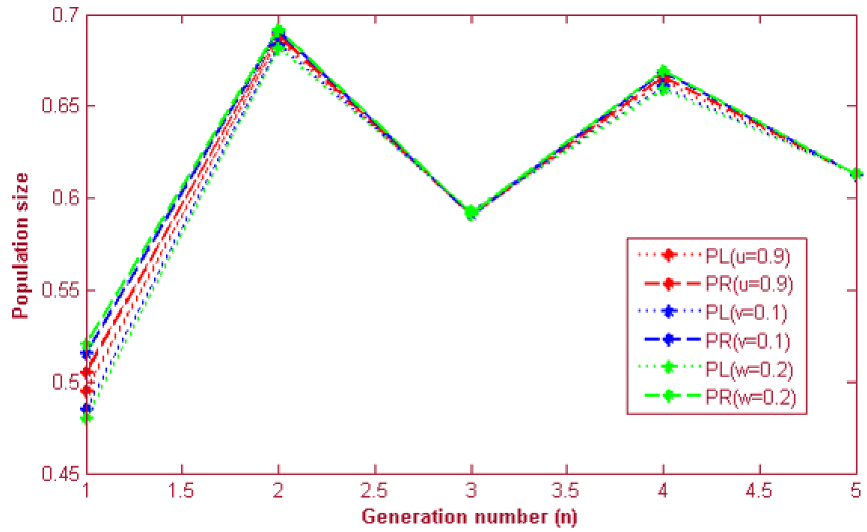

Figure 5: The discrete change-over the population size and the generation number of the System (65) for $u = 0.9, v = 0.1$ and $w = 0.2$.

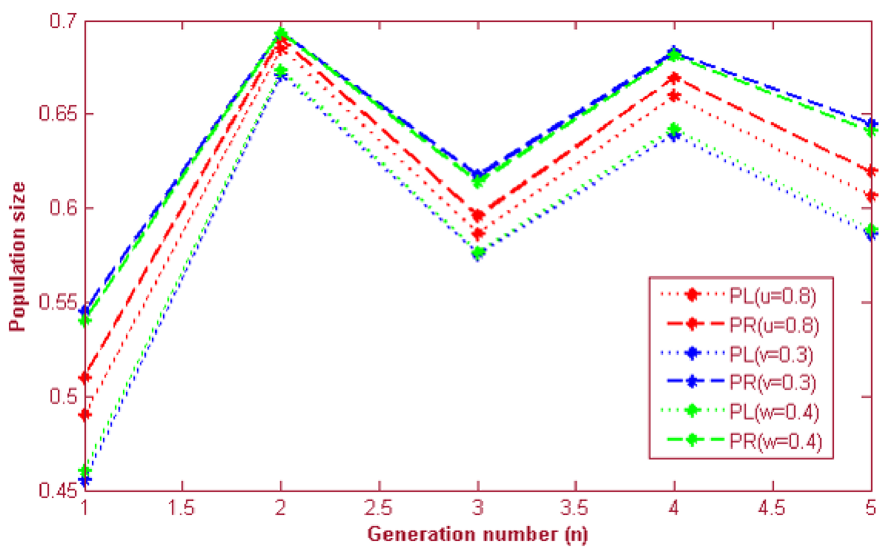

Figure 6: The discrete change-over the population size and the generation number of the System (66) for $u = 0.8, v = 0.3$ and $w = 0.4$.

5.1 Discussion on the numerical results:

All the figures depict the dynamical state of the systems (62) [see the figures Fig. 2 and Fig. 3], (64) [see the figures Fig. 4 and 5.] and the system (65) [see the figures 6. and 7.] with the supplied initial inputs consisting of the triangular initial population size and the growth rate given by (63) in the two dimensional reference frame of the population size and the generation number. In each frame, there are six number of plots (three

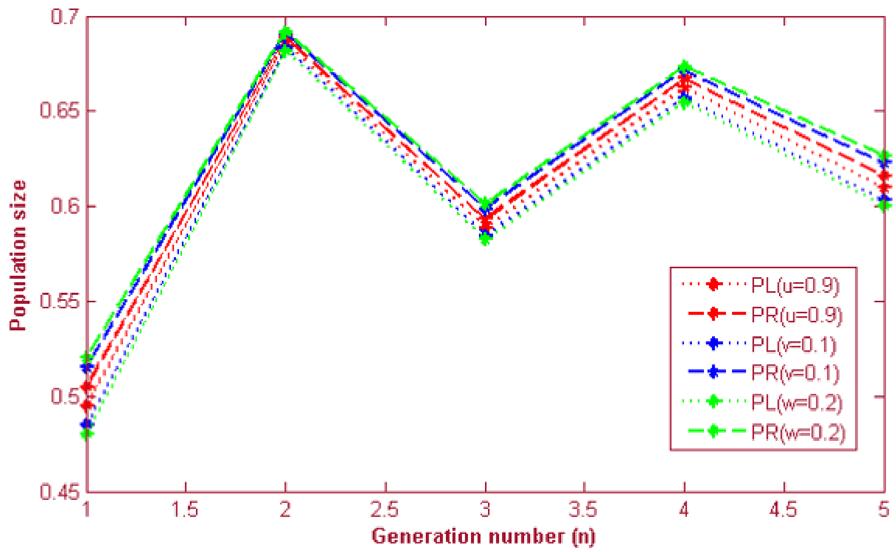

Figure 7: The discrete change-over the population size and the generation number of the System (66) for $u = 0.9, v = 0.1$ and $w = 0.2$.

for lower parametric neutrosophic (u, v, w)-cut and three for upper parametric neutrosophic (u, v, w)-cut) in neutrosophic environment for some fixed parametric values which are indicated within the legend box as well as in the figures descriptions. The following significant decision may be taken on the basis of the numerical results:

(a) If the generation number increases, then the population size tends to either zero or an unbounded solution (see the figures Fig. 2 and Fig. 3). In this case, the dynamical state is unstable and the direction of lower branches refers to the possibility of population extinction. Although, we have taken the membership functions taking $\left[\tilde{\phi}_0\right]_u = [0.495, 0.505]$ and $[\tilde{\rho}]_u = [0.2745, 0.2755]$ which are near about the crisp population size $\tilde{\phi}_0 = 0.5$ and the logistic growth $\tilde{\rho} = 2.75$, the logistic difference Equation (48) unstable (see the Fig. 3.) where as it is stable in crisp environment (see the Fig. 1.). Through this particular Example 5.1, the neutrosophic logistic model (43) in the form of the generalized Hukuhara difference (44) in the light of both the generalized Hukuhara difference (type-I gH and type-II gH) is unstable but stable in a crisp environment.

(b) The neutrosophic logistic difference equation (43) in the generalized Hukuhara difference equation of the form $\tilde{\phi}_{n+1} = \tilde{\rho}\tilde{\phi}_n \ominus_{gH} \tilde{\rho}(\tilde{\phi}_n)^2$ in type-I gH-difference concept is stable for the initial information (63) and for the parameters values $u = 0.8, v = 0.3$ and $w = 0.4$ and $u = 0.9, v = 0.1$ and $w = 0.2$, respectively (see the Fig. 4. and Fig. 5.). If the number of generations increases, the population size oscillates slowly and finally approaches towards a stable population (≈ 0.63) (see the tendency in crisp result Fig. 1.). Therefore, there is no chance of population extinction in this numerical result and it is stable in crisp as well as neutrosophic environments.

(c) The neutrosophic logistic difference equation (43) in the generalized Hukuhara difference equation of the form $\tilde{\phi}_{n+1} = \tilde{\rho}\tilde{\phi}_n \ominus_{gH} \tilde{\rho}(\tilde{\phi}_n)^2$ in type-II gH-difference concept is unstable for the same initial information (63) and for the parameters values $u = 0.8, v = 0.3$ and $w = 0.4$ and $u = 0.9, v = 0.1$ and $w = 0.2$ respectively (see the Fig. 6. and Fig. 7.). If the number of generations increases, the population size oscillates slowly but not stable in neutrosophic environments but there is no chance of population extinction in this numerical result. The significant outcomes are observed that the same model shows a stable situation by type-I gH-difference concept but unstable in type-II gH-difference concept (compare the figures Fig. 4. and Fig. 5. with Fig. 6. and Fig. 7.).

6 Conclusion

The difference equation reflects discrete changes with mathematical notations. Neutrosophic sets and numbers describe uncertain phenomena with truthfulness, indeterminacy, and falsehood functions. A situation where discrete changes occur with vague information of a neutrosophic sense can be dealt with by the neutrosophic difference equation. In this context, the solvability of such difference equations and the uniqueness of solutions (if they exist) are significant concerns. This paper consists of two parts, namely (i) the theoretical part and (ii) the application part. Through the theoretical part, we have extended the fuzzy metric and the corresponding Lemmas, Theorems to the neutrosophic metric theory and discussed the different theorems, lemmas associated with the Hukuhara difference, generalized Hukuhara difference of both type-I and type-II in a neutrosophic environment with proper justification. In fact, this paper contributes a background analysis discussing the properties of metrics and the Hukuhara differences of neutrosophic numbers in different perspectives before establishing the uniqueness and existence criteria for the solvability of the neutrosophic difference equation. We have discussed the practical significance of the proposed theory, taking the Logistic growth model under neutrosophic uncertainty (see Section 4). Numerical examples and graphical results are provided to correctly perceive this studys meaning.

The significantly extended and generalized established theory of neutrosophic uncertainty from existing fuzzy theory reduces the gaps in the neutrosophic environment. The existence and uniqueness criteria can be utilized to check the solvability of any given difference equation under a neutrosophic environment. Thus, the present study opens a large scope for investigations of more complicated mathematical models based on specific insights towards discrete dynamics and imprecise decision phenomena in the future. Besides such application-based investigations, the proposed theory can be modified for higher orders of difference equations and a more generalized sense of uncertainty. If the difference equations are rational type under neutrosophic under uncertainty, then the established theory may face into the challenge to analyse the dynamical stability condition through the eigenvalue method. For such a challenge, the researcher may try to analyse a discrete Michaelis-Menten harvesting model through the established theory of this paper.

Conflict of Interest: "The authors declare no conflict of interest."

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