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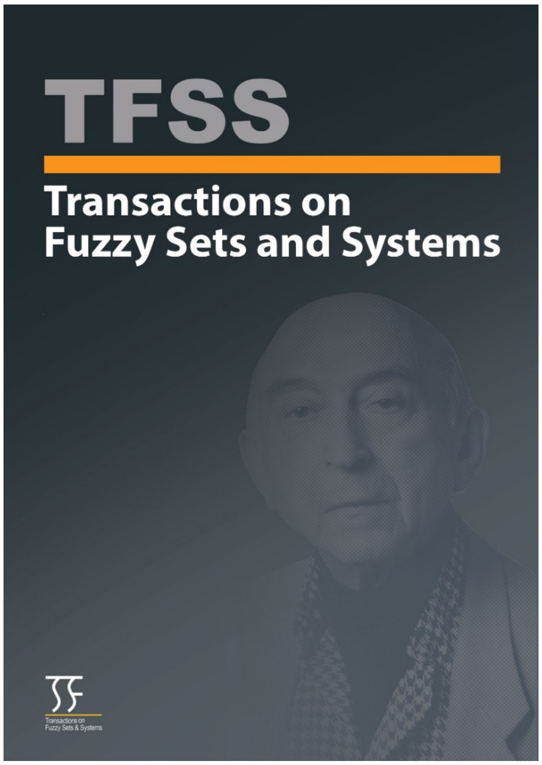

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Fixed-Point Results in Neutrosophic Normed Spaces

Nirmal Sarkar*, **Jayanta Das**, **Ashoke Das**

Abstract. This paper investigates fixed-point theorems within the framework of neutrosophic normed spaces. We provide a novel proof of the Banach Contraction Principle, offering fresh insights into its applicability in neutrosophic environments. Additionally, we extend both Caccioppolis and Kannans fixed-point theorems to neutrosophic linear spaces, establishing their validity in this generalized context. These results contribute to the theoretical advancement of neutrosophic analysis and broaden the scope of classical fixed-point theory.

AMS Subject Classification 2020: 46S40; 47H10
Keywords and Phrases: Neutrosophic normed spaces, Banach Contraction, Kannans fixed-point, Caccioppolis fixed-point.

1 Introduction

Neutrosophic sets, proposed by Smarandache [1], generalize fuzzy sets [2] and intuitionistic fuzzy sets [3] by introducing three distinct membership functions: truth, indeterminacy, and falsity. Since their introduction, these sets have gained significant attention due to their ability to handle uncertainty and incomplete information more effectively than traditional models. Several versions of fuzzy normed spaces have been explored in the literature [4, 5, 6], with notable contributions from Bag and Samanta [7], who introduced a modified fuzzy norm, leading to important fixed-point results [8, 9]. Das et al. [10] advanced the field by exploring fixed point theory within the framework of complete fuzzy normed linear spaces.

One of the notable advancements in this area is the development of neutrosophic normed spaces, which generalize both fuzzy and intuitionistic fuzzy normed spaces. The foundational work in this domain was established by Muralikrishna and Kumar [11], who investigated key structural properties. Later, Omran and Elrawy [12] explored continuous and bounded operators within neutrosophic normed spaces, broadening their applications in functional analysis. Convergence, a fundamental concept in normed spaces, has also been studied extensively, while Kirisci and imek [13] explored statistical convergence. Chaurasiya et al. [14] provided a detailed analysis of bounded operators, and Aral et al. [15] introduced ρ-strong convergence in neutrosophic normed spaces, enriching the theory further. These spaces have since garnered interest for their potential applications in decision-making, optimization, and functional analysis [16, 17, 18].

Earlier foundational work by Grabiec [19] on fixed points in fuzzy metric spaces significantly influenced later developments in neutrosophic theory. Building on this, Sowndrarajan et al. [20] extended fixed point results to neutrosophic metric spaces. In particular, Omran and Elrawy [12] extended the Banach Contraction Principle to neutrosophic normed spaces, demonstrating its viability in this generalized setting. Motivated

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by these advancements, the present work offers an alternative proof of the Banach Contraction Principle, providing new insights into its structure and implications. Furthermore, we extend Caccioppolis and Kannans fixed-point theorems to neutrosophic normed linear spaces, contributing to the ongoing development of neutrosophic functional analysis.

2 Basic Concepts

This section introduces the essential concepts and foundational results required for the subsequent development of the theory.

Definition 2.1. [11]*A binary operation* $\oplus : [0,1] \times [0,1] \to [0,1]$ *is called a continuous t-norm if it satisfies the following conditions:*

1. $\oplus$ *is continuous.*
2. $\oplus$ *is associative and commutative.*
3. *For all* $a \in [0,1]$*, it holds that* $a \oplus 1 = a$.
4. *If* $u_1 \leq u_2$ *and* $v_1 \leq v_2$*, then* $u_1 \oplus v_1 \leq u_2 \oplus v_2$ *for all* $u_1, u_2, v_1, v_2 \in [0,1]$.

Definition 2.2. [11]*A binary operation* $\odot : [0,1] \times [0,1] \to [0,1]$ *is called a continuous t-conorm if it satisfies the following conditions:*

1. $\odot$ *is continuous.*
2. $\odot$ *is associative and commutative.*
3. *For all* $v \in [0,1]$*, it holds that* $v \odot 0 = v$.
4. *If* $u_1 \leq u_2$ *and* $v_1 \leq v_2$*, then* $u_1 \odot v_1 \leq u_2 \odot v_2$ *for all* $u_1, u_2, v_1, v_2 \in [0,1]$.

Definition 2.3. [1]*Let* U *be a universal set. A neutrosophic set* $\mathcal{N}$ *on* U *is defined as*

$$\mathcal{N} = \{\langle x, \mathcal{P}(x), \mathcal{Q}(x), \mathcal{R}(x)\rangle \mid x \in U\},$$

where the functions $\mathcal{P}, \mathcal{Q}, \mathcal{R} : U \to [0,1]$ *denote the degrees of truth, indeterminacy, and falsity membership associated with each element* $x \in U$*, respectively.*

Definition 2.4. [11] *Let* V *be a real vector space, and let* $\oplus$ *and* $\odot$ *denote continuous t-norm and t-conorm operations, respectively. A mapping* $\mathcal{N} = \langle \mathcal{P}, \mathcal{Q}, \mathcal{R}\rangle : V \times \mathbb{R} \to [0,1]^3$ *is called a neutrosophic norm on* V *if, for all* $u, v \in V$ *and all scalars* $s, r, t \in \mathbb{R}$*, the following conditions are satisfied:*

1. $0 \leq \mathcal{P}(u,s), \mathcal{Q}(u,s), \mathcal{R}(u,s) \leq 1$.
2. $0 \leq \mathcal{P}(u,s) + \mathcal{Q}(u,s) + \mathcal{R}(u,s) \leq 3$.
3. $\mathcal{P}(u,s) = 0$ *whenever* $s \leq 0$.
4. $\mathcal{P}(u,s) = 1$ *for* $s > 0$ *if and only if* $u = 0$.
5. *For all* $\xi \neq 0$ *and* $s > 0$*, we have* $\mathcal{P}(\xi u, s) = \mathcal{P}\left(u, \frac{s}{|\xi|}\right)$.
6. $\mathcal{P}(u,r) \oplus \mathcal{P}(v,t) \leq \mathcal{P}(u+v, r+t)$.

7. *The function* $\mathcal{P}(u,\cdot)$ *is continuous and non-decreasing for* $s>0$*, with* $\lim_{s\to\infty}\mathcal{P}(u,s)=1$.
8. $\mathcal{Q}(u,s)=1$ *whenever* $s\le 0$.
9. $\mathcal{Q}(u,s)=0$ *for* $s>0$ *if and only if* $u=0$.
10. $\mathcal{Q}(\xi u,s)=\mathcal{Q}\left(u,\frac{s}{|\xi|}\right)$ *for all* $\xi\neq 0$ *and* $s>0$.
11. $\mathcal{Q}(u,r)\odot\mathcal{Q}(v,t)\ge\mathcal{Q}(u+v,r+t)$.
12. *The function* $\mathcal{Q}(u,\cdot)$ *is continuous and non-increasing for* $s>0$*, with* $\lim_{s\to\infty}\mathcal{Q}(u,s)=0$.
13. $\mathcal{R}(u,s)=1$ *whenever* $s\le 0$.
14. $\mathcal{R}(u,s)=0$ *for* $s>0$ *if and only if* $u=0$.
15. $\mathcal{R}(\xi u,s)=\mathcal{R}\left(u,\frac{s}{|\xi|}\right)$ *for all* $\xi\neq 0$ *and* $s>0$.
16. $\mathcal{R}(u,r)\odot\mathcal{R}(v,t)\ge\mathcal{R}(u+v,r+t)$.
17. *The function* $\mathcal{R}(u,\cdot)$ *is continuous and non-increasing for* $s>0$*, with* $\lim_{s\to\infty}\mathcal{R}(u,s)=0$.

Hence, $(V,\mathcal{N},\oplus,\odot)$ *is called a neutrosophic normed linear space (NNLS).*

Example 2.5. [11] Let $(F,\|\cdot\|)$ be a normed linear space. Define binary operations on the interval $[0,1]$ by

$$a\oplus b=ab,\quad a\odot b=a+b-ab.$$

Define the neutrosophic norms $P(u,s)$, $Q(u,s)$, and $R(u,s)$ as follows:

$$P(u,s)=\begin{cases}\dfrac{s}{s+\|u\|}, & \text{if } s>\|u\|,\\ 0, & \text{if } s\le\|u\|,\end{cases}\qquad Q(u,s)=\begin{cases}\dfrac{\|u\|}{s+\|u\|}, & \text{if } s>\|u\|,\\ 1, & \text{if } s\le\|u\|,\end{cases}$$

$$R(u,s)=\begin{cases}\dfrac{\|u\|}{s}, & \text{if } s>\|u\|,\\ 1, & \text{if } s\le\|u\|.\end{cases}$$

Then, the structure $(F,N,\oplus,\odot)$, where $N:F\times\mathbb{R}^+\to[0,1]^3$ is given by

$$N(u,s)=(P(u,s),Q(u,s),R(u,s)),$$

forms a neutrosophic normed linear space (NNLS).

Definition 2.6. [11] *A sequence* $\{u_n\}$ *in an NNLS* $(Y,N,\oplus,\odot)$ *is said to* ***converge*** *to an element* $u\in Y$ *if, for every* $s>0$,

$$\lim_{n\to\infty}\mathcal{P}(u_n-u,s)=1,\quad\lim_{n\to\infty}\mathcal{Q}(u_n-u,s)=0,\quad\lim_{n\to\infty}\mathcal{R}(u_n-u,s)=0.$$

Definition 2.7. [11] *A sequence* $\{u_n\}$ *in an NNLS* $(Y,N,\oplus,\odot)$ *is called a* ***Cauchy sequence*** *if, for every* $s>0$ *and for each* $m\in\mathbb{N}$,

$$\lim_{n\to\infty}\mathcal{P}(u_{n+m}-u_n,s)=1,\quad\lim_{n\to\infty}\mathcal{Q}(u_{n+m}-u_n,s)=0,\quad\lim_{n\to\infty}\mathcal{R}(u_{n+m}-u_n,s)=0.$$

Definition 2.8. [11] *An NNLS* $(Y,N,\oplus,\odot)$ *is said to be* ***complete*** *if every Cauchy sequence in* Y *converges to a limit in* Y.

3 Main Results

In 1988, Grabiec [19] generalized the classical Banach Contraction Principle within the framework of fuzzy metric spaces. Later, Das et al. [10] extended this principle to fuzzy normed linear spaces. More recently, Omran and Elrawy [12] demonstrated a similar principle in neutrosophic normed spaces. In this work, we present an alternative proof of this principle within the same framework.

Theorem 3.1. *Let $(X, \mathcal{N}, \oplus, \odot)$ be a complete NNLS, and let $F : X \to X$ be a mapping satisfying the following conditions for all $\xi, \eta \in X$ and for some constants $0 < k, l, m < 1$:*

$$\mathcal{P}(F\xi - F\eta, t) \geq \mathcal{P}(\xi - \eta, \tfrac{t}{k}), \tag{1}$$
$$\mathcal{Q}(F\xi - F\eta, t) \leq \mathcal{Q}(\xi - \eta, \tfrac{t}{l}), \tag{2}$$
$$\mathcal{R}(F\xi - F\eta, t) \leq \mathcal{R}(\xi - \eta, \tfrac{t}{m}). \tag{3}$$

Then F admits a unique fixed point in X.

Proof. Choose an arbitrary element $\xi_0 \in X$, and define $\xi_1 = F(\xi_0)$. If $\xi_1 = \xi_0$, then ξ_0 is a fixed point of F, and the proof is complete.

Otherwise, if $\xi_1 \neq \xi_0$, define $\xi_2 = F(\xi_1)$ and continue iteratively. This process generates a sequence $\{\xi_n\}$ in X, given by

$$\xi_{n+1} = F(\xi_n) = F^{n+1}(\xi_0), \quad \text{where} \quad \xi_n \neq \xi_{n+1}, \quad n = 0, 1, 2, \ldots$$

Using inequality (1), we obtain

$$\begin{aligned}\mathcal{P}(\xi_r - \xi_{r+1}, t) &= \mathcal{P}(F^r\xi_0 - F^{r+1}\xi_0, t)\\ &\geq \mathcal{P}(F^{r-1}\xi_0 - F^r\xi_0, \tfrac{t}{k}).\end{aligned}$$

Repeating this process r times yields

$$\mathcal{P}(\xi_r - \xi_{r+1}, t) \geq \mathcal{P}(\xi_0 - \xi_1, \tfrac{t}{k^r}). \tag{4}$$

Similarly, from (2) and (3), we have

$$\mathcal{Q}(\xi_r - \xi_{r+1}, t) \leq \mathcal{Q}(\xi_0 - \xi_1, \tfrac{t}{l^r}), \tag{5}$$
$$\mathcal{R}(\xi_r - \xi_{r+1}, t) \leq \mathcal{R}(\xi_0 - \xi_1, \tfrac{t}{m^r}). \tag{6}$$

To verify that $\{\xi_n\}$ is a Cauchy sequence in X, consider:

$$\begin{aligned}\mathcal{P}(\xi_n - \xi_{n+p}, t) &\geq \mathcal{P}(\xi_n - \xi_{n+1}, \tfrac{t}{2}) \oplus \mathcal{P}(\xi_{n+1} - \xi_{n+p}, \tfrac{t}{2})\\ &\geq \mathcal{P}(\xi_n - \xi_{n+1}, \tfrac{t}{2}) \oplus \mathcal{P}(\xi_{n+1} - \xi_{n+2}, \tfrac{t}{2^2}) \oplus \mathcal{P}(\xi_{n+2} - \xi_{n+p}, \tfrac{t}{2^2})\\ &\vdots\\ &\geq \mathcal{P}(\xi_n - \xi_{n+1}, \tfrac{t}{2}) \oplus \mathcal{P}(\xi_{n+1} - \xi_{n+2}, \tfrac{t}{2^2}) \oplus \cdots \oplus \mathcal{P}(\xi_{n+p-1} - \xi_{n+p}, \tfrac{t}{2^{p-1}})\\ &\geq \mathcal{P}(\xi_0 - \xi_1, \tfrac{t}{2k^n}) \oplus \mathcal{P}(\xi_0 - \xi_1, \tfrac{t}{2^2k^{n+1}}) \oplus \cdots \oplus \mathcal{P}(\xi_0 - \xi_1, \tfrac{t}{2^{p-1}k^{n+p-1}}) \qquad \text{by (4)}\end{aligned}$$

As $n \to \infty$, the right-hand side converges to 1, implying that

$$\lim_{n\to\infty} \mathcal{P}(\xi_n - \xi_{n+p}, t) = 1, \quad \forall\ t > 0,\ p \geq 1,\ p \in \mathbb{N}.$$

Similarly, we obtain

$$\begin{aligned}\mathcal{Q}(\xi_n - \xi_{n+p}, t) &\leq \mathcal{Q}(\xi_n - \xi_{n+1}, \tfrac{t}{2}) \odot \mathcal{Q}(\xi_{n+1} - \xi_{n+2}, \tfrac{t}{2^2}) \odot \cdots \odot \mathcal{Q}(\xi_{n+p-1} - \xi_{n+p}, \tfrac{t}{2^{p-1}}) \\ &\leq \mathcal{Q}(\xi_0 - \xi_1, \tfrac{t}{2l^n}) \odot \mathcal{Q}(\xi_0 - \xi_1, \tfrac{t}{2^2 l^{n+1}}) \odot \cdots \odot \mathcal{Q}(\xi_0 - \xi_1, \tfrac{t}{2^{p-1} l^{n+p-1}}). \qquad \text{by (5)}\end{aligned}$$

As $n \to \infty$, the right-hand side converges to 0, implying that

$$\lim_{n\to\infty} \mathcal{Q}(\xi_n - \xi_{n+p}, t) = 0, \quad \forall\, t > 0,\, p \geq 1,\, p \in \mathbb{N}.$$

Again, from (6), we get

$$\lim_{n\to\infty} \mathcal{R}(\xi_n - \xi_{n+p}, t) = 0.$$

Hence, the sequence $\{\xi_n\}$ is Cauchy. Given that the space $(X, \mathcal{N}, \oplus, \odot)$ is complete, there exists an element $\xi \in X$ such that ξ_n converges to ξ.
Next, we show that $F\xi = \xi$. Indeed,

$$\begin{aligned}\mathcal{P}(F\xi - \xi, t) &\geq \mathcal{P}(F\xi - F\xi_n, \tfrac{t}{2}) \oplus \mathcal{P}(F\xi_n - \xi, \tfrac{t}{2}) \\ &\geq \mathcal{P}(\xi - \xi_n, \tfrac{t}{2k}) \oplus \mathcal{P}(\xi_{n+1} - \xi, \tfrac{t}{2}).\end{aligned}$$

Taking the limit as $n \to \infty$, it follows that $\mathcal{P}(F\xi - \xi, t) = 1$ for all $t > 0$, and hence $F\xi = \xi$.
To establish uniqueness, assume that $F\eta = \eta$ for some $\eta \in X$. Then, using (1), we obtain:

$$\mathcal{P}(\xi - \eta, t) = \mathcal{P}(F\xi - F\eta, t) \geq \mathcal{P}(\xi - \eta, \tfrac{t}{k^n}).$$

Taking the limit as $n \to \infty$, we conclude that $\mathcal{P}(\xi - \eta, t) = 1$ for every $t > 0$, implying $\xi = \eta$. Therefore, the fixed point of F is unique. □

In 2015, Das et al. [10] presented a version of Caccioppolis fixed point theorem within the framework of fuzzy normed linear spaces. In the following theorem, we extend this result to neutrosophic normed spaces, thereby broadening its applicability to a more generalized setting.

Theorem 3.2. *Let $(X, \mathcal{N}, \oplus, \odot)$ be a complete neutrosophic normed space, and let $F : X \to X$ be a self-mapping. Assume that for every pair $\xi, \eta \in X$, and for sequences $\{k_n\}, \{l_n\}, \{m_n\}$, the following conditions are satisfied:*

$$P(F^n\xi - F^n\eta, t) \geq P(\xi - \eta, \frac{t}{k_n}), \tag{7}$$

$$Q(F^n\xi - F^n\eta, t) \leq Q(\xi - \eta, \frac{t}{l_n}), \tag{8}$$

$$R(F^n\xi - F^n\eta, t) \leq R(\xi - \eta, \frac{t}{m_n}), \tag{9}$$

where each of the sequences $\{k_n\}$, $\{l_n\}$, and $\{m_n\}$ consists of strictly positive real numbers. If

$$\lim_{n\to\infty} k_n = 0, \quad \lim_{n\to\infty} l_n = 0 \quad \text{and} \quad \lim_{n\to\infty} m_n = 0.$$

then the mapping F has a unique fixed point in X.

Proof. Let $\xi_0 \in X$ be chosen arbitrarily, and define $\xi_1 = F(\xi_0)$. If $\xi_1 = \xi_0$, then ξ_0 is a fixed point of F, and the proof is complete.

Otherwise, if $\xi_1 \neq \xi_0$, we proceed by setting $\xi_2 = F(\xi_1)$, and continue this process to generate a sequence $\{\xi_n\}$ in X, defined recursively by

$$\xi_{n+1} = F(\xi_n) = F^{n+1}(\xi_0), \quad \text{where} \quad \xi_n \neq \xi_{n+1}, \quad n = 0, 1, 2, \ldots$$

Using inequality (7), we obtain

$$\begin{aligned}\mathcal{P}(\xi_n-\xi_{n+p},t) &\geq \mathcal{P}(\xi_n-\xi_{n+1},\frac{t}{2})\oplus\mathcal{P}(\xi_{n+1}-\xi_{n+2},\frac{t}{2^2})\oplus\cdots\oplus\mathcal{P}(\xi_{n+p-1}-\xi_{n+p},\frac{t}{2^{p-1}})\\ &\geq \mathcal{P}(\xi_0-\xi_1,\frac{t}{2k_n})\oplus\mathcal{P}(\xi_0-\xi_1,\frac{t}{2^2k_{n+1}})\oplus\cdots\oplus\mathcal{P}(\xi_0-\xi_1,\frac{t}{2^{p-1}k_{n+p-1}})\\ &\to 1\oplus 1\oplus\cdots\oplus 1=1,\quad \text{as } n\to\infty \text{ and } k_n\to 0.\end{aligned}$$

Therefore, $\lim_{n\to\infty}\mathcal{P}(\xi_n-\xi_{n+p},t)=1,\quad$ for every $t>0$, $p\geq 1$, $p\in\mathbb{N}$.
Similarly, from inequality (8), we have

$$\begin{aligned}\mathcal{Q}(\xi_n-\xi_{n+p},t) &\leq \mathcal{Q}(\xi_n-\xi_{n+1},\frac{t}{2})\odot\mathcal{Q}(\xi_{n+1}-\xi_{n+2},\frac{t}{2^2})\odot\cdots\odot\mathcal{Q}(\xi_{n+p-1}-\xi_{n+p},\frac{t}{2^{p-1}})\\ &\leq \mathcal{Q}(\xi_0-\xi_1,\frac{t}{2l_n})\odot\mathcal{Q}(\xi_0-\xi_1,\frac{t}{2^2l_{n+1}})\odot\cdots\odot\mathcal{Q}(\xi_0-\xi_1,\frac{t}{2^{p-1}l_{n+p-1}})\\ &\to 0\odot 0\odot\cdots\odot 0=0,\quad \text{as } n\to\infty \text{ and } l_n\to 0.\end{aligned}$$

As a result, we obtain

$$\lim_{n\to\infty}\mathcal{Q}(\xi_n-\xi_{n+p},t)=0,\quad \text{for every } t>0,\ p\geq 1,\ p\in\mathbb{N}.$$

Likewise,

$$\lim_{n\to\infty}\mathcal{R}(\xi_n-\xi_{n+p},t)=0,\quad \text{for every } t>0,\ p\geq 1,\ p\in\mathbb{N}.$$

This confirms that the sequence $\{\xi_n\}$ is Cauchy in the NNLS $(X,\mathcal{N},\oplus,\odot)$. Since the space is assumed to be complete, it follows that there exists an element $\xi\in X$ such that $\xi_n\to\xi$.
Next, we prove that $F\xi=\xi$:

$$\begin{aligned}\mathcal{P}(\xi-F\xi,t) &\geq \mathcal{P}(\xi-\xi_{n+1},\frac{t}{2})\oplus\mathcal{P}(\xi_{n+1}-F\xi,\frac{t}{2})\\ &\geq \mathcal{P}(\xi-\xi_{n+1},\frac{t}{2})\oplus\mathcal{P}(\xi_n-\xi,\frac{t}{2k_1})\quad \text{by (7)}\\ &\to 1\oplus 1=1,\quad \text{as } n\to\infty,\ \text{for every } t>0.\end{aligned}$$

Hence, $\mathcal{P}(\xi-F\xi,t)=1$ for all $t>0$, implying that $F\xi=\xi$. To establish uniqueness, assume that there exists another point $\eta\in X$ such that $F\eta=\eta$. In this case, it follows that $F^n\xi=\xi$ and $F^n\eta=\eta$ for every $n\in\mathbb{N}$. Utilizing condition (7), we obtain

$$\mathcal{P}(\xi-\eta,t)=\mathcal{P}(F^n\xi-F^n\eta,t)\geq\mathcal{P}(\xi-\eta,\frac{t}{k_n})\to 1,\quad \text{as } n\to\infty.$$

Consequently, we have $\xi=\eta$, which confirms that the fixed point of the mapping F is unique. □

Example 3.3. Consider the complete neutrosophic normed space $X=\mathbb{R}$ endowed with the usual absolute value norm. Let $(\mathbb{R},N,\oplus,\odot)$ denote the corresponding complete neutrosophic normed linear space. Define a self-mapping $F:\mathbb{R}\to\mathbb{R}$ by $F(\xi)=\frac{\xi}{5}$.

We begin by evaluating $\mathcal{P}$ for the iterates of F:

$$
\begin{aligned}
\mathcal{P}(F^n\xi - F^n\eta, s) &= \frac{s}{s + \|F^n\xi - F^n\eta\|} \\
&= \frac{s}{s + \frac{1}{5^n}|\xi - \eta|} \\
&\geq \frac{t}{t + \left(\frac{1}{2}\right)^n |\xi - \eta|} \\
&= \frac{\frac{s}{k_n}}{\frac{s}{k_n} + |\xi - \eta|} = \mathcal{P}(\xi - \eta, \frac{s}{k_n}), \quad \text{where } k_n = \frac{1}{2^n}.
\end{aligned}
$$

Next, we compute the corresponding expression for $\mathcal{Q}$:

$$
\begin{aligned}
\mathcal{Q}(F^n\xi - F^n\eta, s) &= \frac{\|F^n\xi - F^n\eta\|}{s + \|F^n\xi - F^n\eta\|} \\
&= \frac{\frac{1}{5^n}|\xi - \eta|}{s + \frac{1}{5^n}|\xi - \eta|} \\
&= \frac{|\xi - \eta|}{5^n s + |\xi - \eta|} \\
&\leq \frac{|\xi - \eta|}{3^n s + |\xi - \eta|} \\
&= \frac{|\xi - \eta|}{\frac{s}{l_n} + |\xi - \eta|} = \mathcal{Q}(\xi - \eta, \frac{s}{l_n}), \quad \text{where } l_n = \frac{1}{3^n}.
\end{aligned}
$$

Finally, for $\mathcal{R}$, we find:

$$
\begin{aligned}
\mathcal{R}(F^n\xi - F^n\eta, s) &= \frac{\|F^n\xi - F^n\eta\|}{s} \\
&= \frac{1}{5^n}\frac{|\xi - \eta|}{s} \\
&\leq \frac{1}{4^n}\frac{|\xi - \eta|}{s} \\
&= \frac{|\xi - \eta|}{\frac{s}{m_n}} = \mathcal{R}(\xi - \eta, \frac{s}{m_n}), \quad \text{where } m_n = \frac{1}{4^n}.
\end{aligned}
$$

These results hold for all $\xi, \eta \in \mathbb{R}$, $s > 0$, and for each $n \in \mathbb{N}$. Additionally, the sequences k_n, l_n, and m_n are strictly positive and converge to zero as $n \to \infty$. Therefore, the hypotheses of Theorem 3.2 are satisfied, implying that F has a unique fixed point in $\mathbb{R}$, namely 0.

Indian mathematician Kannan made a significant breakthrough in 1968 by proving a fixed point theorem without requiring continuity and introducing a unique contraction modulus, $0 < \beta < \frac{1}{2}$. Decades later, in 2015, Das et al. [10] extended Kannans theorem to fuzzy normed linear spaces using the minimum t-norm. Building upon these advancements, our next result further generalizes this theorem within the framework of neutrosophic linear spaces.

Theorem 3.4. *Let $(X, \mathcal{N}, \oplus, \odot)$ be a complete neutrosophic normed space, and let $F : X \to X$ be a self-mapping. Suppose there exist constants $0 < k, l, m < \frac{1}{2}$ such that for all $\xi, \eta \in X$, the following conditions* *hold:*

$$\mathcal{P}(F\xi - F\eta, t) \geq \mathcal{P}(\xi - F\xi, \frac{t}{k}) \oplus \mathcal{P}(\eta - F\eta, \frac{t}{k}), \tag{10}$$

$$\mathcal{Q}(F\xi - F\eta, t) \leq \mathcal{Q}(\xi - F\xi, \frac{t}{l}) \odot \mathcal{Q}(\eta - F\eta, \frac{t}{l}), \tag{11}$$

$$\mathcal{R}(F\xi - F\eta, t) \leq \mathcal{R}(\xi - F\xi, \frac{t}{m}) \odot \mathcal{R}(\eta - F\eta, \frac{t}{m}), \tag{12}$$

where $\oplus$ *denotes a minimum-type t-norm and* $\odot$ *denotes a maximum-type t-conorm. Then the mapping* F *has a unique fixed point in* X*.*

Proof. Let us choose an initial point $\xi_0 \in X$ and set $\xi_1 = F(\xi_0)$. If it happens that $\xi_1 = \xi_0$, then ξ_0 is clearly a fixed point of F, and the result follows immediately. Otherwise, assume that $\xi_1 \neq \xi_0$, and define $\xi_2 = F(\xi_1)$. Proceeding recursively, we generate a sequence $\{\xi_n\}$ in X defined by

$$\xi_{n+1} = F(\xi_n) = F^{n+1}(\xi_0), \quad \text{where} \quad \xi_n \neq \xi_{n+1}, \quad n = 0, 1, 2, \ldots$$

Using inequality (10), we obtain

$$\begin{aligned}\mathcal{P}(\xi_r - \xi_{r+1}, t) &= \mathcal{P}(F^r\xi_0 - F^{r+1}\xi_0, t)\\ &\geq \mathcal{P}(F^{r-1}\xi_0 - F^r\xi_0, \frac{t}{k}) \oplus \mathcal{P}(F^r\xi_0 - F^{r+1}\xi_0, \frac{t}{k})\\ &= \mathcal{P}(\xi_{r-1} - \xi_r, \frac{t}{k}) \oplus \mathcal{P}(\xi_r - \xi_{r+1}, \frac{t}{k}).\end{aligned}$$

Since the operation $\oplus$ corresponds to the minimum t-norm, we proceed by examining two separate cases.
Case I: Suppose that $\mathcal{P}(\xi_r - \xi_{r+1}, t) \geq \mathcal{P}(\xi_r - \xi_{r+1}, \frac{t}{k})$ holds. Applying this inequality iteratively n times, we obtain

$$\mathcal{P}(\xi_r - \xi_{r+1}, t) \geq \mathcal{P}(\xi_r - \xi_{r+1}, \frac{t}{k^n}).$$

By passing to the limit as $n \to \infty$, we conclude that $\mathcal{P}(\xi_r - \xi_{r+1}, t) = 1$, implying $\xi_r = \xi_{r+1}$, which leads to a contradiction.
Case II: Suppose

$$\mathcal{P}(\xi_r - \xi_{r+1}, t) \geq \mathcal{P}(\xi_{r-1} - \xi_r, \frac{t}{k}).$$

Repeating the above process, we obtain

$$\mathcal{P}(\xi_r - \xi_{r+1}, t) \geq \mathcal{P}(\xi_0 - \xi_1, \frac{t}{k^r}). \tag{13}$$

Further, using the above inequality, we establish

$$\begin{aligned}\mathcal{P}(\xi_n - \xi_{n+p}, t) &\geq \mathcal{P}(\xi_n - \xi_{n+1}, \frac{t}{2}) \oplus \mathcal{P}(\xi_{n+1} - \xi_{n+2}, \frac{t}{2^2}) \oplus \cdots \oplus \mathcal{P}(\xi_{n+p-1} - \xi_{n+p}, \frac{t}{2^{p-1}})\\ &\geq \mathcal{P}(\xi_0 - \xi_1, \frac{t}{2k^n}) \oplus \mathcal{P}(\xi_0 - \xi_1, \frac{t}{2^2k^{n+1}}) \oplus \cdots \oplus \mathcal{P}(\xi_0 - \xi_1, \frac{t}{2^{p-1}k^{n+p-1}})\\ &\to 1 \oplus 1 \oplus \cdots \oplus 1 = 1, \quad \text{as } n \to \infty, \text{ for every } t > 0.\end{aligned}$$

Therefore, $\lim_{n\to\infty} \mathcal{P}(\xi_{n+p} - \xi_n, t) = 1, \quad \text{for all } t > 0,\ p \geq 1,\ p \in \mathbb{N}.$
Using inequality (11), we similarly obtain

$$\begin{aligned}\mathcal{Q}(\xi_r - \xi_{r+1}, t) &= \mathcal{Q}(F^r\xi_0 - F^{r+1}\xi_0, t)\\ &\leq \mathcal{Q}(F^{r-1}\xi_0 - F^r\xi_0, \frac{t}{k}) \oplus \mathcal{Q}(F^r\xi_0 - F^{r+1}\xi_0, \frac{t}{k})\\ &= \mathcal{Q}(\xi_{r-1} - \xi_r, \frac{t}{k}) \oplus \mathcal{Q}(\xi_r - \xi_{r+1}, \frac{t}{k}).\end{aligned}$$

Since $\oplus$ corresponds to the maximum t-conorm, we distinguish two cases:
Case I: Suppose $\mathcal{Q}(\xi_r - \xi_{r+1}, t) \leq \mathcal{Q}(\xi_r - \xi_{r+1}, \frac{t}{k})$. Repeating the inequality, we have

$$\mathcal{Q}(\xi_r - \xi_{r+1}, t) \leq \mathcal{Q}(\xi_r - \xi_{r+1}, \frac{t}{k^n}).$$

Passing to the limit as $n \to \infty$, we obtain $\mathcal{Q}(\xi_r - \xi_{r+1}, t) = 0$, implying $\xi_r = \xi_{r+1}$, which leads to a contradiction.
Case II: Suppose

$$\mathcal{Q}(\xi_r - \xi_{r+1}, t) \leq \mathcal{Q}(\xi_{r-1} - \xi_r, \frac{t}{k}).$$

Then,

$$\begin{aligned}\mathcal{Q}(\xi_n - \xi_{n+p}, t) &\leq \mathcal{Q}(\xi_n - \xi_{n+1}, \frac{t}{2}) \odot \mathcal{Q}(\xi_{n+1} - \xi_{n+2}, \frac{t}{2^2}) \odot \cdots \odot \mathcal{Q}(\xi_{n+p-1} - \xi_{n+p}, \frac{t}{2^{p-1}}) \\ &\leq \mathcal{Q}(\xi_0 - \xi_1, \frac{t}{2l^{n+1}}) \odot \mathcal{Q}(\xi_0 - \xi_1, \frac{t}{2^2 l^{n+2}}) \odot \cdots \odot \mathcal{Q}(\xi_0 - \xi_1, \frac{t}{2^{p-1} l^{n+p}}) \\ &\to 0 \odot 0 \odot \cdots \odot 0 = 0, \quad \text{as } n \to \infty.\end{aligned}$$

Thus,

$$\lim_{n\to\infty} \mathcal{Q}(\xi_{n+p} - \xi_n, t) = 0, \quad \text{for every } t > 0,\ p \geq 0,\ p \in \mathbb{N}.$$

Similarly,

$$\lim_{n\to\infty} \mathcal{R}(\xi_n - \xi_{n+p}, t) = 0.$$

This confirms that the sequence $\{\xi_n\}$ is Cauchy in a NNLS $(X, \mathcal{N}, \oplus, \odot)$. Since the space is assumed to be complete, it follows that there exists an element $\xi \in X$ such that $\xi_n \to \xi$.
Now, to verify that ξ is a fixed point of F, observe:

$$\begin{aligned}\mathcal{P}(\xi - F\xi, t) &\geq \mathcal{P}(\xi - \xi_n, \frac{t}{2}) \oplus \mathcal{P}(F\xi_{n-1} - F\xi, \frac{t}{2}) \\ &\geq \mathcal{P}(\xi - \xi_n, \frac{t}{2}) \oplus \mathcal{P}(\xi_{n-1} - \xi_n, \frac{t}{2k}) \oplus \mathcal{P}(\xi - F\xi, \frac{t}{2k}) \quad \text{by (10)}.\end{aligned}$$

By considering the limit as $n \to \infty$, we obtain

$$\mathcal{P}(\xi - F\xi, t) \geq \mathcal{P}(\xi - F\xi, \frac{t}{2k}) \geq \cdots \geq \mathcal{P}(\xi - F\xi, \frac{t}{2^r k^r}).$$

Letting $r \to \infty$ and using the fact that $0 < k < \frac{1}{2}$, we deduce that $F\xi = \xi$.
To prove uniqueness, suppose there exists another fixed point $\eta \in X$. Then

$$\begin{aligned}\mathcal{P}(\xi - \eta, t) &= \mathcal{P}(F\xi - F\eta, t) \\ &\geq \mathcal{P}(\xi - F\xi, \frac{t}{k}) \oplus \mathcal{P}(\eta - F\eta, \frac{t}{k}) \\ &= 1 \oplus 1 = 1.\end{aligned}$$

Thus, $\xi = \eta$, proving uniqueness. $\square$

4 Conclusion

In this work, we have generalized key fixed-point theorems within the framework of neutrosophic normed spaces, thereby strengthening their mathematical foundation and broadening their theoretical applications. Our new proof of the Banach contraction principle in neutrosophic normed space provides a fresh perspective on contraction mappings under uncertainty. Additionally, the generalization of Caccioppolis and Kannans fixed-point theorems further strengthens the analytical foundation of neutrosophic normed spaces. Given their wide-ranging applications in functional analysis, optimization, and decision sciences, future research can explore their role in solving real-world problems. This work lays the groundwork for further advancements in neutrosophic mathematical analysis, encouraging deeper exploration into its practical applications and theoretical extensions.

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