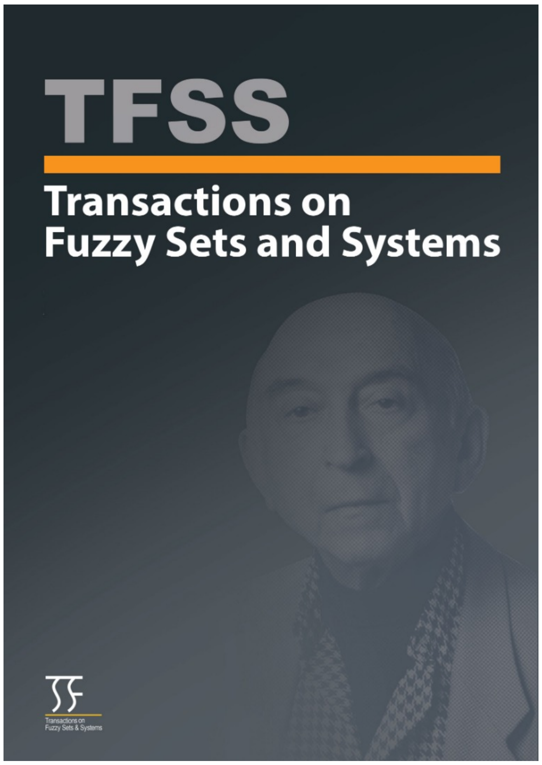
Transactions on Fuzzy Sets and Systems



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ISSN: **2821-0131**

https://sanad.iau.ir/journal/tfss/

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Vol.5, No.1, (2026), 1-25. DOI: https://doi.org/10.71602/tfss.2026.1194037

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Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://sanad.iau.ir/journal/tfss/
Online ISSN: 2821-0131
Vol.5, No.1, (2026), 1-25
DOI: https://doi.org/10.71602/tfss.2026.1194037

Article Type: Original Research Article
(Journal Pre-proof)

A Novel Generalized Interval-Valued Neutrosophic Rough Soft Set Framework for Enhanced Decision-Making: Application in Water Quality Assessment

Anjan Mukherjee , **Ajoy Kanti Das*** , **Nandini Gupta** , **Suman Patra** , **Tahir Mahmood** , **Suman Das** , **Rakhal Das**

(Special Issue on "Dynamical Systems in Fuzzy Environments".)

Abstract. This study introduces a novel framework, Generalized Interval-Valued Neutrosophic Rough Soft Sets (GIVNRS sets), designed to improve handling uncertainty, imprecision, and vagueness in complex decision-making scenarios. By integrating soft, rough, and generalized interval-valued neutrosophic set theories, the framework offers a robust methodology for addressing indeterminacy and incomplete data. The theoretical foundation of GIVNRS sets is built upon fundamental operations, including intersection, union, complement, and novel aggregation union operators tailored for multi-criteria decision-making (MCDM) applications. The practical applicability of the framework is demonstrated through a water quality assessment, where it successfully classifies river segments based on key water quality parameters such as pH, Dissolved Oxygen (DO), and Biochemical Oxygen Demand (BOD). The case study results show that the pollution scores for the river segments were computed, classifying the segments such as "Good," "Moderate," and "Poor," with corresponding pollution levels. These findings highlight the framework's ability to manage incomplete and inconsistent data, providing a reliable and comprehensive water quality evaluation. Compared to traditional models, the GIVNRS set approach offers enhanced flexibility, stability, and adaptability. This study not only contributes to the theoretical development of neutrosophic, soft, and rough set theories but also establishes GIVNRS sets as a powerful tool for water quality decision-making. Future research will explore further advancements in the application and computational efficiency of this framework.

AMS Subject Classification 2020: 90B50; 90B60; 68T01
Keywords and Phrases: Fuzzy Set, Neutrosophic Set, Soft Set, Rough Set, Generalized Neutrosophic Set, Interval-Valued Neutrosophic Set.

1 Introduction

Modeling uncertainty and imprecision in decision-making processes has been a persistent challenge across various domains, including environmental science, engineering, and medicine. Classical set theories often struggle to address ambiguity and incomplete information effectively, necessitating the development of more sophisticated mathematical frameworks. Zadeh's introduction of fuzzy set theory in 1965 was a groundbreaking step, allowing for the representation of degrees of membership rather than binary classifications [1]. This

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Received: 20 December 2024; **Revised:** 21 January 2025; **Accepted:** 29 January 2025; **Available Online:** 1 June 2025; **Published Online:** 7 May 2026

How to cite: Mukherjee A, Das AK, Gupta N, Patra S, Mahmood T, Das S, Das R. A novel generalized interval-valued neutrosophic rough soft set framework for enhanced decision-making: Application in water quality assessment. *Transactions on Fuzzy Sets and Systems.* 2026; 5(1): 1-25. DOI: https://doi.org/10.71602/tfss.2026.1194037

framework was later expanded by Atanassov's intuitionistic fuzzy sets, which incorporated membership and non-membership degrees to capture a broader spectrum of uncertainty [2]. Rough set theory, introduced by Pawlak in 1982, complemented fuzzy logic by enabling classification under vague or incomplete information through the concepts of lower and upper approximations [3]. The integration of fuzzy logic with rough sets has led to advanced hybrid models, such as soft sets, neutrosophic sets, and interval-valued neutrosophic rough sets (IVNRSs) ([4]-[7]). Notable contributions, such as those by Mukherjee and Das [8], have emphasized the potential of generalized interval-valued neutrosophic sets (GIVNSs) in addressing complex decision-making processes. These models address practical decision-making challenges, offering robust tools for multi-criteria decision-making (MCDM).

This study introduces a novel framework, Generalized Interval-Valued Neutrosophic Rough Soft Sets (GIVNRS sets), to tackle the dual challenges of indeterminacy and incompleteness in decision-making. By combining the strengths of generalized interval-valued neutrosophic sets (GIVNSs) and rough set theory, the proposed approach enhances the assessment of environmental phenomena, particularly water quality. The framework aligns with global water quality standards, such as those set by the World Health Organization (WHO) [9] and the Central Pollution Control Board (CPCB) [10], and its practical applicability is demonstrated through a case study on water quality assessment.

2 Novelty of the Research

This study introduces the GIVNRS Set framework, a novel approach integrating neutrosophic, rough, and soft set theories to handle uncertainty, indeterminacy, and incompleteness in decision-making. Key contributions include:

1. Development of the Aggregate Union Operator: A new tool to unify lower and upper approximations, enabling balanced evaluations of truth, indeterminacy, and falsity.

2. Application to Water Quality Assessment: A practical framework for classifying river segments into categories like Poor, Moderate, and Good, addressing data uncertainties and inconsistencies.

3. Versatility Across Domains: Demonstrates scalability to other fields like air quality monitoring and waste management.

This research bridges theoretical and practical gaps, offering a robust and adaptable methodology for environmental and multi-criteria decision-making challenges.

3 Literature Review

The limitations of classical set theories in handling uncertainty have spurred the evolution of advanced mathematical frameworks. Zadeh's fuzzy set theory [1] introduced a non-binary approach to uncertainty modeling, while Atanassov [2] extended this with intuitionistic fuzzy sets, addressing both membership and non-membership degrees. Pawlak's rough set theory [3] introduced a complementary perspective, classifying objects under vague information using lower and upper approximations. Hybrid approaches have further advanced uncertainty modeling. Neutrosophy, introduced by Smarandache [5, 6], provided a unifying framework for truth, indeterminacy, and falsity, inspiring developments like generalized neutrosophic sets [11] and rough neutrosophic sets [12]. Broumi and Smarandache [13] contributed interval-valued neutrosophic soft, rough sets, while Hai-Long et al. [14] advanced this with generalized interval neutrosophic rough sets for MCDM. Recent studies, such as those by Saha and Broumi [15], proposed novel operators for interval-valued neutrosophic sets, enhancing their adaptability in complex decision contexts.

In recent years, advancements in MCDM methods have provided valuable tools for addressing complex decision problems across various fields, including energy, healthcare, tourism, and environmental management. Integrating fuzzy set theory with other mathematical frameworks, such as spherical fuzzy sets, hesitant fuzzy sets, and q-rung ortho-pair fuzzy sets, has enhanced the flexibility and robustness of decision models. These innovations have paved the way for more accurate and reliable evaluations in real-world applications. For example, Sahoo and Debnath [16] proposed a novel hybrid spherical fuzzy MCDM approach for selecting the best hydroelectric power plant source in India, while Saha et al. [17] introduced q-rung ortho-pair fuzzy aggregation operators for multi-attribute decision-making, demonstrating their effectiveness in complex decision environments.

These contributions underscore the adaptability of fuzzy MCDM methods in addressing real-world challenges across diverse sectors. These mathematical tools have proven invaluable in environmental science, where uncertainty is prevalent due to data variability and measurement inconsistencies. Integrating fuzzy and rough set theories has enabled the effective assessment of water quality, a critical aspect of public health and environmental management. For instance, Das et al. [18] utilized a fuzzy MCDM model to analyze anthropogenic influences on urban river water quality. Similarly, their work on the Gomati River demonstrated the effectiveness of weighted fuzzy soft set-based models for rating water pollution [19]. Further, Das et al. [20] improved water pollution rating accuracy by introducing a water quality evaluation model using weighted hesitant fuzzy soft sets. Das and Granados [21] enhanced fuzzy soft group decision-making with weighted average ratings, while their subsequent studies examined intuitionistic fuzzy rough relations [22, 23] and new fuzzy soft set operations for decision-making [24]. Granados et al. [25, 26] further advanced neutrosophic and rough set theories by developing continuous neutrosophic distributions and weighted neutrosophic soft multisets for decision-making. Mukherjee and Das [27] contributed to the theoretical foundation of fuzzy rough models by analyzing the topological structure of intuitionistic fuzzy rough relations.

This paper introduces a novel framework, Generalized Interval-Valued Neutrosophic Rough Soft Sets (GIVNRS sets), to address the dual challenges of indeterminacy and incompleteness in data analysis. Combining the strengths of GIVNSs and rough set theory, the proposed approach provides a robust methodology for handling uncertainty in MCDM. A case study on water quality assessment illustrates the practical applicability of GIVNRS sets, showcasing their potential in addressing real-world challenges in environmental monitoring.

The remainder of this paper is structured as follows: Section 2 outlines key terminologies and foundational concepts related to GIVNRS sets. Section 3 introduces the mathematical framework and operations of GIVNRS sets. Section 4 discusses their application in water quality assessment. Section 5 provides a comparative analysis of the proposed model with an influential approach for water quality assessment and decision-making. Finally, Section 6 concludes with key findings and future research directions.

Throughout this article, we use the following short terms for the clarity of the presentation:

Abbreviation	**Full Form**
DO	Dissolved Oxygen
WHO	World Health Organization
BOD	Biochemical Oxygen Demand
MCDM	Multi-criteria decision-making
IVNS	Interval-valued Neutrosophic set
CPCB	Central Pollution Control Board
IVNRS	Interval-valued neutrosophic rough set
GIVNS	Generalized interval-valued neutrosophic set
GIVNSS	Generalized interval-valued neutrosophic soft set
GIVNRS-set	Generalized interval-valued neutrosophic rough, soft set

4 Terminologies

In this part, we outline essential definitions that establish the foundational principles for advancing our research regarding neutrosophic sets and their utilizations. Let $\mathcal{U}$ represent a universal set, or the initial universe, consisting of all possible objects under consideration. The power set of $\mathcal{U}$, denoted as $P(\mathcal{U})$, refers to the set containing all possible subsets of $\mathcal{U}$. Meanwhile, $\mathcal{G}$ represents a set of parameters that describe various aspects or attributes of the elements within $\mathcal{U}$.

Definition 4.1. ([5, 6]) *A Neutrosophic Set is characterized as a structured object denoted by* $N = \{\langle u, T_N(u), I_N(u), F_N(u)\rangle : u \in \mathcal{U}\}$*, where* $\mathcal{U}$ *is a non-empty fixed set. In this context,* $T_N(u)$*,* $I_N(u)$*, and* $F_N(u)$ *represent the degrees of membership, indeterminacy, and non-membership for each element* $u \in \mathcal{U}$*, respectively. The values are constrained within the interval* $]0^-, 1^+[$ *for* T *(truth),* I *(indeterminacy), and* F *(falsity), adhering to the condition* $0^- \leq T_N(u) + I_N(u) + F_N(u) \leq 3^+$*. This definition allows for a nuanced representation of uncertainty, accommodating various states of membership.*

Definition 4.2. [9] *A Generalized Neutrosophic Set (GNS) extends the concept of a neutrosophic set, defined similarly as* $N = \{\langle u, T_N(u), I_N(u), F_N(u)\rangle : u \in \mathcal{U}\}$*. The functions* $T_N(u)$*,* $I_N(u)$*, and* $F_N(u)$ *maintain their roles as degrees of membership, indeterminacy, and non-membership, respectively. A critical distinction is the condition* $T_N(u) \wedge I_N(u) \wedge F_N(u)$ *and* $T, I, F \rightarrow]0^-, 1^+[$ *and* $0^- \leq T_N(u) + I_N(u) + F_N(u) \leq 3^+$*, which introduces a constraint that enhances the model's applicability and robustness in decision-making scenarios.*

Definition 4.3. [13] *An IVNS is a specialized form defined as* $N = \{\langle u, T_N(u), I_N(u), F_N(u)\rangle : u \in \mathcal{U}\}$*, where each membership function is represented by intervals:* $T_N(u) = [T_N^L(u), T_N^U(u)]$*,* $I_N(u) = [I_N^L(u), I_N^U(u)]$*, and* $F_N(u) = [F_N^L(u), F_N^U(u)]$*. These intervals indicate the range of values for each function, ensuring that* $T_N(u), I_N(u), F_N(u) \in Int([0,1])$*. Here,* $Int([0,1])$ *refers to the set of all closed subintervals within the unit interval* $[0,1]$*, providing a more flexible representation of uncertainty.*

Definition 4.4. [8] *A GIVNS is articulated as* $N = \{\langle u, T_N(u), I_N(u), F_N(u)\rangle : u \in \mathcal{U}\}$*, where* $T_N(u) = [T_N^L(u), T_N^U(u)]$*,* $I_N(u) = [I_N^L(u), I_N^U(u)]$*, and* $F_N(u) = [F_N^L(u), F_N^U(u)]$*. The functions must satisfy* $\sup T_N(u) \wedge \sup I_N(u) \wedge \sup F_N(u) \leq 0.5$*, ensuring that the generalized set remains within certain bounds of uncertainty.*

Definition 4.5. [8] *The complement of a given GIVNS* $N = \{\langle u, T_N(u), I_N(u), F_N(u)\rangle : u \in \mathcal{U}\}$*, denoted by* N^c*, is defined as* $N^c = \{\langle u, T_N^c(u), I_N^c(u), F_N^c(u)\rangle : u \in \mathcal{U}\}$*, where the mappings are derived as follows:*

$T_N^c(u) = [F_N^L(u), F_N^U(u)], \quad I_N^c(u) = [1 - I_N^U(u), 1 - I_N^L(u)], \quad F_N^c(u) = [T_N^L(u), T_N^U(u)],$

where $\sup T_N(u) \wedge \sup I_N(u) \wedge \sup F_N(u) \leq 0.5$*. The maximum of a GIVNS is* $\langle [1,1], [0,0], [0,0]\rangle$*, denoted by* 1_N *, and the minimum is* $\langle [0,0], [1,1], [1,1]\rangle$*, denoted by* 0_N*.*

Definition 4.6. [8] *A GIVNS* $N = \{\langle u, T_N(u), I_N(u), F_N(u)\rangle : u \in \mathcal{U}\}$ *is considered to be a subset of another GIVNS* $K = \{\langle u, T_K(u), I_K(u), F_K(u)\rangle : u \in \mathcal{U}\}$*, represented as* $\mathcal{N} \subseteq \mathcal{K}$*, if*

$T_N^L(u) \leq T_K^L(u), \quad T_N^U(u) \leq T_K^U(u), \quad I_N^L(u) \geq I_K^L(u), \quad I_N^U(u) \geq I_K^U(u), \quad F_N^L(u) \geq F_K^L(u), \quad F_N^U(u) \geq F_K^U(u), \quad \forall u \in \mathcal{U}.$

This definition emphasizes the comparative relationships between neutrosophic sets, providing a clear framework for assessing inclusion.

Definition 4.7. [8] *The union of two GIVNSs* $N = \{\langle u, T_N(u), I_N(u), F_N(u)\rangle : u \in \mathcal{U}\}$ *and* $K = \{\langle u, T_K(u), I_K(u), F_K(u)\rangle : u \in \mathcal{U}\}$ *is represented as* $\mathcal{G} = N \cup K$*. The truth, indeterminacy, and degree of non-membership functions for* $\mathcal{G}$ *are described as:*

$$T_{\mathcal{G}}^L(u) = \sup\{T_N^L(u), T_K^L(u)\}, \quad T_{\mathcal{G}}^U(u) = \sup\{T_N^U(u), T_K^U(u)\},$$
$$I_{\mathcal{G}}^L(u) = \inf\{I_N^L(u), I_K^L(u)\}, \quad I_{\mathcal{G}}^U(u) = \inf\{I_N^U(u), I_K^U(u)\},$$
$$F_{\mathcal{G}}^L(u) = \inf\{F_N^L(u), F_K^L(u)\}, \quad F_{\mathcal{G}}^U(u) = \inf\{F_N^U(u), F_K^U(u)\}.$$

Definition 4.8. [8] *The intersection of two GIVNSs* $N = \{\langle u, T_N(u), I_N(u), F_N(u)\rangle : u \in \mathcal{U}\}$ *and* $K = \{\langle u, T_K(u), I_K(u), F_K(u)\rangle : u \in \mathcal{U}\}$ *is represented as* $\mathcal{G} = N \cap K$*. The truth, indeterminacy, and degree of non-membership functions for* $\mathcal{G}$ *are described as:*

$$T^L_{\mathcal{G}}(u) = \inf\{T^L_N(u), T^L_K(u)\}, \quad T^U_{\mathcal{G}}(u) = \inf\{T^U_N(u), T^U_K(u)\},$$

$$I^L_{\mathcal{G}}(u) = \sup\{I^L_N(u), I^L_K(u)\}, \quad I^U_{\mathcal{G}}(u) = \sup\{I^U_N(u), I^U_K(u)\},$$

$$F^L_{\mathcal{G}}(u) = \sup\{F^L_N(u), F^L_K(u)\}, \quad F^U_{\mathcal{G}}(u) = \sup\{F^U_N(u), F^U_K(u)\}.$$

Definition 4.9. [8] *A soft set is a mathematical tool for dealing with uncertainty, originally introduced to provide a flexible and simplified framework for addressing vague data, where the information may not be entirely clear or precise. The formal structure is based on the relationship between parameters and subsets of a universe of discourse.*

A soft set over a universe U *is defined as a couple* (F, G)*, where* F *is a function* $F : G \to P(U)$*. This function associates each parameter in the set* G *with a specific subset of* U*. In simpler terms, for each parameter* $p \in G$*, the mapping* $F(p)$ *identifies a subset of elements in* U *to which the parameter is applicable.*

Definition 4.10. [3] *For a subset* $B \subseteq U$*, the lower and upper approximations of* B *with respect to the approximation space* (U, K) *are denoted by* $\underline{B}_K$ *and* $\overline{B}_K$*, respectively. These are formally defined as follows:*

Lower approximation $\underline{B}_K$ *: This consists of all elements* $u \in U$ *for which the equivalence class* $[u]_K$ *is entirely contained within* B *Formally,*

$\underline{B}_K = \{u \in U : [u]_K \subseteq B\}$ *or* $\bigcup\{[u]_K : [u]_K \subseteq B\}$, $u \in U$.

Upper approximation $\overline{B_K}$*: This includes all elements* $u \in U$ *such that the intersection of* $[u]_K$ *with* B *is non-empty. That is,*

$\overline{B_K} = \{u \in U : [u]_K \cap B \neq \emptyset\}$ *or* $\bigcup\{[u]_K : [u]_K \cap B \neq \emptyset\}$, $u \in U$.

These approximations serve distinct purposes: the lower approximation $\underline{B}_K$ *captures elements that can be confidently classified within* B *based on available information, while the upper approximation* $\overline{B}_K$ *identifies elements that could potentially belong to* B*. When the lower and upper approximations coincidethat is, when* $\underline{B}_K = \overline{B}_K$*, the set* B *is termed definable, as it can be precisely described using the equivalence classes induced by* K*. However, if* $\underline{B}_K \neq \overline{B}_K$*,* B *is classified as a rough set, indicating some ambiguity or uncertainty in its boundaries.*

5 GIVNRS sets and its Properties

In this segment, we present a new notion of GIVNRS sets by combining GIVNSs, soft, and rough sets. We also study various operations including union, intersection, inclusion, and equality over these sets.

Definition 5.1. *Assume* A *is a non-empty collection, and* G *signifies a nonempty set of parameters. Let us assume* $GIVN(A)$ *denotes the collection of all GIVNSs and* K *be a relation of equivalence established on* A*. Then a generalized interval-valued neutrosophic soft set (GIVNSS) over* A *is represented by a pair* (F, G)*, where* F *is a mapping described by* $F : G \to$ *GIVNS*(A)*. A GIVNSS* (F, G) *in* A *is defined as* $\forall p \in G$*,* $F(p) = \{(u, \{(T_{F(p)}(u), I_{F(p)}(u), F_{F(p)}(u))\}) : u \in A\}$*, where*

$$T_{F(p)}(u) = [T^L_{F(p)}(u), T^U_{F(p)}(u)], I_{F(p)}(u) = [I^L_{F(p)}(u), I^U_{F(p)}(u)], F_{F(p)}(u) = [F^L_{F(p)}(u), F^U_{F(p)}(u)] \in Int([0,1]) \tag{1}$$

(Int([0, 1]) is the collection of all sub-intervals of [0, 1]). with

$$\sup T_{F(p)}(u) \wedge \sup I_{F(p)}(u) \wedge \sup F_{F(p)}(u) \leq 0.5, \, \forall u \in A, p \in G \tag{2}$$

The lower and upper approximation $\underline{(F,G)}_K$ and $\overline{(F,G)}_K$ of (F,G) in the Pawlak approximation (A,K) are as follows:

$\underline{F(p)}_K = \{(u, \{[\inf_{y\in[u]_K} T^L_{F(p)}(y), \inf_{y\in[u]_K} T^U_{F(p)}(y)],\ [\sup_{y\in[u]_K} I^L_{F(p)}(y), \sup_{y\in[u]_K} I^U_{F(p)}(y)],$
$[\sup_{y\in[u]_K} F^L_{F(p)}(y), \sup_{y\in[u]_K} F^U_{F(p)}(y)]\})\} : u \in A, p \in G.$

$\overline{F(p)}_K = \{(u, \{[\sup_{y\in[u]_K} T^L_{F(p)}(y), \sup_{y\in[u]_K} T^U_{F(p)}(y)], [\inf_{y\in[u]_K} I^L_{F(p)}(y), \inf_{y\in[u]_K} I^U_{F(p)}(y)],$
$[\inf_{y\in[u]_K} F^L_{F(p)}(y), \inf_{y\in[u]_K} F^U_{F(p)}(y)]\})\} : u \in A, p \in G.$

The relation K denotes the equivalence relation associated with the GIVNSS (F,G). In this context, $[u]_K$ represents the equivalence class of the element u.

$[\inf_{y\in[u]_K} T^L_{F(p)}(y), \inf_{y\in[u]_K} T^U_{F(p)}(y)] \in Int([0,1])$ *and* $[\sup_{y\in[u]_K} I^L_{F(p)}(y), \sup_{y\in[u]_K} I^U_{F(p)}(y)] \in Int([0,1]),$

$$\sup_{y\in[u]_K} F^L_{F(p)}(y), \sup_{y\in[u]_K} F^U_{F(p)}(y)] \in Int([0,1]) \tag{3}$$

$$with\ 0 \leq \inf_{y\in[u]_K} T^U_{F(p)}(y) + \sup_{y\in[u]_K} I^U_{F(p)}(y) + \sup_{y\in[u]_K} F^U_{F(p)}(y) \leq 3 \tag{4}$$

$$and\ 0 \leq (\inf_{y\in[u]_K} T^U_{F(p)}(y)) \wedge (\sup_{y\in[u]_K} I^U_{F(p)}(y)) \wedge (\sup_{y\in[u]_K} F^U_{F(p)}(y)) \leq 0.5) \tag{5}$$

then $\underline{(F,G)}_K$ is a GIVNS set.

Similarly, we have

$[\sup_{y\in[u]_K} T^L_{F(p)}(y), \sup_{y\in[u]_K} T^U_{F(p)}(y)] \in Int([0,1])$ *and* $[\inf_{y\in[u]_K} I^L_{F(p)}(y), \inf_{y\in[u]_K} I^U_{F(p)}(y)] \in Int([0,1]),$

$$[\inf_{y\in[u]_K} F^L_{F(p)}(y), \inf_{y\in[u]_{\mathbb{K}}} F^U_{F(p)}(y)] \in Int([0,1]) \tag{6}$$

$$with\ 0 \leq \sup_{y\in[u]_K} T^U_{F(p)}(y) + \inf_{y\in[u]_K} I^U_{F(p)}(y) + \inf_{y\in[u]_{\mathbb{K}}} F^U_{F(p)}(y) \leq 3 \tag{7}$$

$$and\ 0 \leq (\sup_{y\in[u]_K} T^U_{F(p)}(y)) \wedge (\inf_{y\in[u]_K} I^U_{F(p)}(y)) \wedge (\inf_{y\in[u]_K} F^U_{F(p)}(y)) \leq 0.5 \tag{8}$$

then $\overline{(F,G)}_{\mathbb{K}}$ is a GIVNS.

If $\overline{(F,G)}_R = \underline{(F,G)}_R$, i.e., $\underline{F(p)}_K = \overline{F(p)}_K$ for all $p \in G$, then (F,G) is classified as a definable set; otherwise, (F,G) is considered a GIVNRS set where $(F,G) = (\overline{(F,G)}_R, \underline{(F,G)}_{\mathbb{R}})$.

Let us assume $GIVN(A,K,G)$ denotes the collection of all GIVNRS sets over $(U,\mathbb{K})$ with a fixed parameter set $\mathbb{G}$.

Example 5.2. A water management authority monitors the quality of a river by evaluating specific water quality parameters such as pH, DO, and BOD. Measurements are often uncertain due to equipment inconsistencies, sampling errors, or seasonal variations. The GIVNRS set framework is applied to assess water quality and classify it into categories like Good, Moderate, and Poor while handling incomplete, inconsistent, and indeterminate data.

Let the river segments $U = \{w_1, w_2, w_3\}$ represent three monitoring stations along the river, and the parameters being evaluated (pH, DO, and BOD) form the parameter set $G = \{p_1 = \text{pH}, p_2 = \text{DO}, p_3 = \text{BOD}\}$. Each parameter (pH, DO, BOD) will be evaluated using the GIVNRS framework. The relation $\mathcal{K}$ is defined as an equivalence relation based on the similarity of river segments in terms of their water quality.

For each parameter $p \in G$, the GIVNS at each segment $w \in A$ is defined as:

$$F(p) = \{(w, \{(T_{F(p)}(w), I_{F(p)}(w), F_{F(p)}(w))\}) : w \in A\},$$

where the intervals for truth (T), indeterminacy (I), and falsity (F) at each segment are derived based on expert evaluations or historical data.

Table 1: Truth (T), Indeterminacy (I), and Falsity (F) at each segment

Segment	p_1 = pH (T, I, F)	p_2 = DO (T, I, F)	p_3 = BOD (T, I, F)
w_1	[0.6, 0.8], [0.1, 0.2], [0.0, 0.1]	[0.7, 0.9], [0.0, 0.1], [0.0, 0.1]	[0.5, 0.7], [0.2, 0.3], [0.0, 0.2]
w_2	[0.4, 0.6], [0.3, 0.4], [0.2, 0.3]	[0.5, 0.6], [0.2, 0.3], [0.2, 0.3]	[0.2, 0.4], [0.3, 0.4], [0.3, 0.5]
w_3	[0.7, 0.9], [0.0, 0.1], [0.0, 0.2]	[0.8, 1.0], [0.0, 0.1], [0.0, 0.1]	[0.6, 0.8], [0.1, 0.2], [0.0, 0.1]

Let $\mathcal{GIVNSS}(\mathcal{A})$ represent the collection of all GIVNSSs over $\mathcal{A}$, and let the equivalence relation $\mathcal{K}$ be defined as $U/\mathcal{K} = \{\{w_1, w_3\}, \{w_2\}\}$, where $[w_1]_\mathcal{K} = \{w_1, w_3\}$, as they have similar water quality parameters (WQPs), and $[w_2]_\mathcal{K} = \{w_2\}$.

Let:

(F, G)=

$[(p_1(\text{pH}), \{(w_1, [0.6, 0.8], [0.1, 0.2], [0.0, 0.1]),$
$(w_2, [0.4, 0.6], [0.3, 0.4], [0.2, 0.3]), (w_3, [0.7, 0.9], [0.0, 0.1], [0.0, 0.2])\}),$
$(p_2(\text{DO}), \{(w_1, [0.7, 0.9], [0.0, 0.1], [0.0, 0.1]),$
$(w_2, [0.5, 0.6], [0.2, 0.3], [0.2, 0.3]), (w_3, [0.8, 1.0], [0.0, 0.1], [0.0, 0.1])\}),$
$(p_3(\text{BOD}), \{(w_1, [0.5, 0.7], [0.2, 0.3], [0.0, 0.2]),$
$(w_2, [0.2, 0.4], [0.3, 0.4], [0.3, 0.5]), (w_3, [0.6, 0.8], [0.1, 0.2], [0.0, 0.1])\})]$, be a GIVNSS on $\mathcal{A}$

Then, by definition, the lower and upper approximations for (F, G) are calculated as:

$\underline{(F, G)}_\mathcal{K}$ =

$(p_1(\text{pH}), \{(w_1, [0.6, 0.8], [0.1, 0.2], [0.0, 0.2]),$
$(w_2, [0.4, 0.6], [0.3, 0.4], [0.2, 0.3]), (w_3, [0.6, 0.8], [0.1, 0.2], [0.0, 0.2])\}),$
$(p_2(\text{DO}), \{(w_1, [0.7, 0.9], [0.0, 0.1], [0.0, 0.1]),$
$(w_2, [0.5, 0.6], [0.2, 0.3], [0.2, 0.3]), (w_3, [0.7, 0.9], [0.0, 0.1], [0.0, 0.1])\}),$
$(p_3(\text{BOD}), \{(w_1, [0.5, 0.7], [0.2, 0.3], [0.0, 0.2]),$
$(w_2, [0.2, 0.4], [0.3, 0.4], [0.3, 0.5]), (w_3, [0.5, 0.7], [0.2, 0.3], [0.0, 0.2])\})]$

$\overline{(F, G)}_\mathcal{K}$=

$[(p_1(\text{pH}), \{(w_1, [0.7, 0.9], [0.0, 0.1], [0.0, 0.1]),$
$(w_2, [0.4, 0.6], [0.3, 0.4], [0.2, 0.3]), (w_3, [0.7, 0.9], [0.0, 0.1], [0.0, 0.1])\}),$
$(p_2(\text{DO}), \{(w_1, [0.7, 0.9], [0.0, 0.1], [0.0, 0.1]),$
$(w_2, [0.5, 0.6], [0.2, 0.3], [0.2, 0.3]), (w_3, [0.7, 0.9], [0.0, 0.1], [0.0, 0.1])\}),$
$(p_3(\text{BOD}), \{(w_1, [0.6, 0.8], [0.1, 0.2], [0.0, 0.1]),$
$(w_2, [0.2, 0.4], [0.3, 0.4], [0.3, 0.5]), (w_3, [0.6, 0.8], [0.1, 0.2], [0.0, 0.1])\})]$

Here the IVNRS set $(F, G) = (\underline{(F, G)}_K, \overline{(F, G)}_K)$ satisfying all the conditions (1), (2), (3), and (4), hence it is a GIVNRS set.

This example highlights how GIVNRS set approximations facilitate the classification of elements based on their degrees of membership, allowing us to manage both the indeterminate and incomplete aspects in decision-making. This demonstrates how the GIVNRS set can be applied to address real-world challenges in water quality assessment.

Definition 5.3. [8] *If $(F, G) = (\underline{(F, G)}_K, \overline{(F, G)}_K)$ is a GIVNRS set in (U, K), then the complement of (F, G) is a GIVNRS set denoted by*

$(F, G)^c = (\underline{(F, G)}^c_K, \overline{(F, G)}^c_K)$, where $\underline{(F, G)}^c_K$ and $\overline{(F, G)}^c_K$ are the complements of the GIVNRS sets $\underline{(F, G)}_K$ and $\overline{(F, G)}_K$, respectively.

$\underline{F(p)_K}^c = \{(u, \{[\sup_{y\in[u]_K} T_F(p)^L(y), \sup_{y\in[u]_K} T_F(p)^U(y)], [(1-\sup_{y\in[u]_K} I^U_{F(p)}(y)), (1-\sup_{y\in[u]_K} I^L_{F(p)}(y))],$
$\inf_{y\in[u]_K} F^L_{F(p)}(y), \inf_{y\in[u]_K} F^U_{F(p)}(y)]\}) : u \in A, p \in G$

In this representation:

$\sup_{y\in[u]_K} T_F(p)^L(y)$ *and* $\sup_{y\in[u]_K} T_F(p)^U(y)$

denote the supremum (maximum) of the lower and upper membership degrees of $F(p)$ *over the equivalence class* $[u]_K$.

$1 - \sup_{y\in[u]_K} I^U_{F(p)}(y)$ *and* $1 - \sup_{y\in[u]_K} I^L_{F(p)}(y)$

are the complements of the supremum (maximum) indeterminacy degrees.

$\inf_{y\in[u]_K} F^L_{F(p)}(y)$ *and* $\inf_{y\in[u]_K} F^U_{F(p)}(y)$

denote the infimum (minimum) of the truth degrees of $F(p)$ *over* $[u]_K$.

$\overline{(F(p))^c_K} = \{(u, \{[\inf_{y\in[u]_K} T_F(p)^L(y), \inf_{y\in[u]_K} T_F(p)^U(y)], [(1-\inf_{y\in[u]_K} I^U_{F(p)}(y)), (1-\inf_{y\in[u]_K} I^L_{F(p)}(y))],$
$[\sup_{y\in[u]_K} F^L_{F(p)}(y), \sup_{y\in[u]_K} F^U_{F(p)}(y)]\}) : u \in A, p \in G\}$

Here:

$\inf_{y\in[u]_K} T_F(p)^L(y)$ *and* $\inf_{y\in[u]_K} T_F(p)^U(y)$

denote the infimum (minimum) of the lower and upper membership degrees of $F(p)$ *within* $[u]_K$.

$1 - \inf_{y\in[u]_K} I^U_{F(p)}(y)$ *and* $1 - \inf_{y\in[u]_K} I^L_{F(p)}(y)$

are the complements of the infimum (minimum) indeterminacy degrees.

$\sup_{y\in[u]_K} F^L_{F(p)}(y)$ *and* $\sup_{y\in[u]_K} F^U_{F(p)}(y)$ *indicate the supremum (maximum) of the truth degrees over the equivalence class* $[u]_K$.

This complement structure provides an extended framework for assessing membership, indeterminacy, and truth values in the rough approximation of GIVNRS set, allowing for more granular analysis of uncertainty within neutrosophic contexts. The combination of upper and lower approximations in both the original and complementary GIVNRS set aids in a comprehensive assessment of set characteristics under relational dependencies.

Example 5.4. Here, we extend the analysis from Example 5.2 by exploring the concept of the complement of the GIVNRS set approximations. Given the GIVNRS set (F, G) and the equivalence relation K over the universe U, we aim to determine the upper and lower approximations of the complement $\overline{(F, G)^c_K}$ and $\underline{(F, G)^c_K}$. The lower approximation of the complement, $\underline{(F, G)^c_K}$, is defined as follows:

$\underline{(F, G)}^c_K = \{(p_1(\text{pH}), \{(w_1, [0.7, 0.9], [0.9, 1], [0.0, 0.1]),$
$(w_2, [0.4, 0.6], [0.6, 0.7], [0.2, 0.3]), (w_3, [0.7, 0.9], [0.9, 1], [0.0, 0.1])\}),$
$(p_2(\text{DO}), \{(w_1, [0.7, 0.9], [0.9, 1], [0.0, 0.1]),$
$(w_2, [0.5, 0.6], [0.7, 0.8], [0.2, 0.3]), (w_3, [0.7, 0.9], [0.9, 1], [0.0, 0.1])\}),$
$(p_3(\text{BOD}), \{(w_1, [0.6, 0.8], [0.8, 0.9], [0.0, 0.1]),$
$(w_2, [0.2, 0.4], [0.6, 0.7], [0.3, 0.5]), (w_3, [0.6, 0.8], [0.8, 0.9], [0.0, 0.1])\})\}$

This lower approximation $\underline{(F, G)^c_K}$ represents elements within U that do not fully satisfy the membership conditions of (F, G) but are part of the complement set with defined intervals for truth, indeterminacy, and falsity. Each interval value here provides an adjusted degree of membership that accounts for the complements characteristics in K.

Similarly, the upper approximation of the complement, $\overline{(F, G)^c_K}$, is given by:

$\overline{(F, G)^c_K} = \{(p_1(\text{pH}), \{(w_1, [0.6, 0.8], [0.8, 0.9], [0.0, 0.2]),$
$(w_2, [0.4, 0.6], [0.6, 0.7], [0.2, 0.3]),$
$(w_3, [0.6, 0.8], [0.8, 0.9], [0.0, 0.2])\}),$
$(p_2(\text{DO}), \{(w_1, [0.7, 0.9], [0.9, 1], [0.0, 0.1]), (w_2, [0.5, 0.6], [0.7, 0.8], [0.2, 0.3]),$
$(w_3, [0.7, 0.9], [0.9, 1], [0.0, 0.1])\}),$
$(p_3(\text{BOD}), \{(w_1, [0.5, 0.7], [0.7, 0.8], [0.0, 0.2]), (w_2, [0.2, 0.4], [0.6, 0.7], [0.3, 0.5]),$
$(w_3, [0.5, 0.7], [0.7, 0.8], [0.0, 0.2])\})\}$

In this upper approximation $\overline{(F,G)}^c_K$, the set includes additional elements from the equivalence classes of K that partially meet the criteria of (F,G)s complement, encompassing a broader scope of indeterminate and falsity values. This upper approximation reflects potential elements of $(F,G)^c$ based on the expanded intervals for each parameter, thereby handling cases where uncertainty and partial indeterminacy play a role.

Together, $\underline{(F,G)}^c_K$ and $\overline{(F,G)}^c_K$ illustrate how the GIVNRS set approach can manage and distinguish between complete and partial membership in a sets complement, providing nuanced approximations that can handle both high and low degrees of membership uncertainty. This dual approximation approach allows for robust decision-making applications where elements' inclusion or exclusion from a set is based on varying degrees of truth, indeterminacy, and falsity across intervals, adapting to the dynamic and often imprecise nature of real-world data.

Definition 5.5. *Assume (F_1,G) and (F_2,G) are GIVNRS sets, $\underline{(F_1,G)_K}$ and $\overline{(F_1,G)_K}$ are the lower and upper approximation of a GIVNRS set (F_1,G) with respect to the approximation space (U,K), respectively, and $\underline{(F_2,G)_K}$ and $\overline{(F_2,G)_K}$ are the lower and upper approximation of a GIVNRS set (F_2,G) with respect to the approximation space (U,K), respectively. The union of these two GIVNRS sets (F_1,G) and (F_2,G) is represented as $(F,G) = (\underline{(F_1,G)_K}, \overline{(F_2,G)_K}) = (\underline{(F_1,G)\cup(F_2,G)_K}, \overline{(F_1,G)\cup(F_2,G)_K})$.*

The truth, indeterminacy, and degree of non-membership functions for $\underline{(F_1,G)\cup(F_2,G)}_K$ and $\overline{(F_1,G)\cup(F_2,G)}_K$ are described as:

For all $u\in U$:

$T^L_{\underline{(F_1\cup F_2)}(p)}(u) = \inf\{T^L_{(F_1\cup F_2)(p)}(t) : t\in[u]_K\}$

$T^U_{\underline{(F_1\cup F_2)}(p)}(u) = \inf\{T^U_{(F_1\cup F_2)(p)}(t) : t\in[u]_K\}$

$I^L_{\underline{(F_1\cup F_2)}(p)}(u) = \sup\{I^L_{(F_1\cup F_2)(p)}(t) : t\in[u]_K\}$

$I^U_{\underline{(F_1\cup F_2)}(p)}(u) = \sup\{I^U_{(F_1\cup F_2)(p)}(t) : t\in[u]_K\}$

$F^L_{\underline{(F_1\cup F_2)}(p)}(u) = \sup\{F^L_{(F_1\cup F_2)(p)}(t) : t\in[u]_K\}$

$F^U_{\underline{(F_1\cup F_2)}(p)}(u) = \sup\{F^U_{(F_1\cup F_2)(p)}(t) : t\in[u]_K\}$

and

$$T^L_{\overline{(F_1\cup F_2)}(p)}(u) = \sup\{T^L_{(F_1\cup F_2)(p)}(t) : t\in[u]_K\}$$

$$T^U_{\overline{(F_1\cup F_2)}(p)}(u) = \sup\{T^U_{(F_1\cup F_2)(p)}(t) : t\in[u]_K\}$$

$$I^L_{\overline{(F_1\cup F_2)}(p)}(u) = \inf\{I^L_{(F_1\cup F_2)(p)}(t) : t\in[u]_K\}$$

$$I^U_{\overline{(F_1\cup F_2)}(p)}(u) = \inf\{I^U_{(F_1\cup F_2)(p)}(t) : t\in[u]_K\}$$

$$F^L_{\overline{(F_1\cup F_2)}(p)}(u) = \inf\{F^L_{(F_1\cup F_2)(p)}(t) : t\in[u]_K\}$$

$$F^U_{\overline{(F_1\cup F_2)}(p)}(u) = \inf\{F^U_{(F_1\cup F_2)(p)}(t) : t\in[u]_K\}$$

Definition 5.6. *The intersection of two GIVNRS sets (F_1,G) and (F_2,G) is represented as $(F,G) = (\underline{(F,G)}_K, \overline{(F,G)}_K) = (\underline{(F_1,G)\cap(F_2,G)}_K, \overline{(F_1,G)\cap(F_2,G)}_K)$.*

The truth, indeterminacy, and degree of non-membership functions for $\underline{(F_1,G)\cap(F_2,G)}_K$ and $\overline{(F_1,G)\cap(F_2,G)}_K$ are described as follows:

For all $u\in U$:

$T^L_{\underline{(F_1\cap F_2)}(p)}(u) = \inf\{T^L_{(F_1\cup F_2)(p)}(t) : t\in[u]_K\}$,

$T^U_{\underline{(F_1\cap F_2)}(p)}(u) = \inf\{T^U_{(F_1\cup F_2)(p)}(t) : t\in[u]_K\}$,

$I^L_{\underline{(F_1\cap F_2)(p)}}(u) = \sup\{I^L_{(F_1\cup F_2)(p)}(t) : t \in [u]_K\}$,
$I^U_{\underline{(F_1\cap F_2)(p)}}(u) = \sup\{I^U_{(F_1\cup F_2)(p)}(t) : t \in [u]_K\}$,
$F^L_{\underline{(F_1\cap F_2)(p)}}(u) = \sup\{F^L_{(F_1\cup F_2)(p)}(t) : t \in [u]_K\}$,
$F^U_{\underline{(F_1\cap F_2)(p)}}(u) = \sup\{F^U_{(F_1\cup F_2)(p)}(t) : t \in [u]_K\}$.
And for $\overline{(F_1 \cap F_2)}(p)$, *the following hold:*
$T^L_{\overline{(F_1\cap F_2)(p)}}(u) = \sup\{T^L_{(F_1\cup F_2)(p)}(t) : t \in [u]_K\}$,
$T^U_{\overline{(F_1\cap F_2)(p)}}(u) = \sup\{T^U_{(F_1\cup F_2)(p)}(t) : t \in [u]_K\}$,
$I^L_{\overline{(F_1\cap F_2)(p)}}(u) = \inf\{I^L_{(F_1\cup F_2)(p)}(t) : t \in [u]_K\}$,
$I^U_{\overline{(F_1\cap F_2)(p)}}(u) = \inf\{I^U_{(F_1\cup F_2)(p)}(t) : t \in [u]_K\}$,
$F^L_{\overline{(F_1\cap F_2)(p)}}(u) = \inf\{F^L_{(F_1\cup F_2)(p)}(t) : t \in [u]_K\}$,
$F^U_{\overline{(F_1\cap F_2)(p)}}(u) = \inf\{F^U_{(F_1\cup F_2)(p)}(t) : t \in [u]_K\}$.

Theorem 5.7. *Assume that* (F_1, G) *and* (F_2, G) *are GIVNRS sets. Let* $\underline{(F_1, G)_K}$ *and* $\overline{(F_1, G)_K}$ *be the lower and upper approximations of the GIVNRS set* (F_1, G) *with respect to the approximation space* (U, K), *respectively. Similarly, let* $\underline{(F_2, G)_K}$ *and* $\overline{(F_2, G)_K}$ *be the lower and upper approximations of the GIVNRS set* (F_2, G) *with respect to the approximation space* (U, K), *respectively. Then the following hold:*

1. $\underline{(F_1, G)}_K \subseteq (F_1, G) \subseteq \overline{(F_1, G)}_K$
2. $\overline{(F_1, G) \cup (F_2, G)}_K = \overline{(F_1, G)}_K \cup \overline{(F_2, G)}_K$, *and* $\underline{(F_1, G) \cap (F_2, G)}_K = \underline{(F_1, G)}_K \cap \underline{(F_2, G)}_K$
3. $\overline{(F_1, G) \cap (F_2, G)}_K = \overline{(F_1, G)}_K \cap \overline{(F_2, G)}_K$, *and* $\underline{(F_1, G) \cup (F_2, G)}_K = \underline{(F_1, G)}_K \cup \underline{(F_2, G)}_K$
4. *If* $(F_1, G) \subseteq (F_2, G)$, *then* $\overline{(F_1, G)}_K \subseteq \overline{(F_2, G)}_K$ *and* $\underline{(F_1, G)}_K \subseteq \underline{(F_2, G)}_K$
5. $\underline{(F_1, G)}^c_K = (\overline{(F_1, G)})^c_K$ *and* $\overline{(F_1, G)^c}_K = (\underline{(F_1, G)}_K)^c$

Proof.

1. Assume $(F_1, G) = \{(p, (u, [[T^L_{F_1(p)}(u), T^U_{F_1(p)}(u)],$
$[I^L_{F_1(p)}(u), I^U_{F(p)}(u)], [F^L_{F_1(p)}(u), F^U_{F_1(p)}(u)]]) : u \in A) : p \in G\}$ is a GIVNRS set satisfying the conditions $[T^L_{F_1(p)}(u), T^U_{F_1(p)}(u)]$,
$[I^L_{F_1(p)}(u), I^U_{F_1(p)}(u)]$, and $[F^L_{F_1(p)}(u), F^U_{F_1(p)}(u)] \in \mathrm{Int}([0, 1])$
and $\sup[T^L_{F_1(p)}(u), T^U_{F_1(p)}(u)] \wedge \sup[I^L_{F_1(p)}(u), I^U_{F_1(p)}(u)] \wedge \sup[F^L_{F_1(p)}(u), F^U_{F_1(p)}(u)] \leq 0.5$, for all $u \in A$ and $p \in G$. Now, from Definition 5.1, for all $u \in U$, and for all $p \in G$,
$T^L_{\underline{F_1(p)}}(u) = \inf_{y\in[u]_K}\{T^L_{F_1(p)}(y)\} \leq T^L_{F_1(p)}(y) \leq \sup_{y\in[u]_K}\{T^L_{F_1(p)}(y)\} = T^L_{\overline{F_1(p)}}(u)$
$T^U_{\underline{F_1(p)}}(u) = \inf_{y\in[u]_K}\{T^U_{F_1(p)}(y)\} \leq T^U_{F_1(p)}(y) \leq \sup_{y\in[u]_K}\{T^U_{F_1(p)}(y)\} = T^U_{\overline{F_1(p)}}(u)$
$I^L_{\underline{F_1(p)}}(u) = \sup_{y\in[u]_K}\{I^L_{F_1(p)}(y)\} \geq I^L_{F_1(p)}(y) \geq \inf_{y\in[u]_K}\{I^L_{F_1(p)}(y)\} = I^L_{\overline{F_1(p)}}(u)$
$I^U_{\underline{F_1(p)}}(u) = \sup_{y\in[u]_K}\{I^U_{F_1(p)}(y)\} \geq I^U_{F_1(p)}(y) \geq \inf_{y\in[u]_K}\{I^U_{F_1(p)}(y)\} = I^U_{\overline{F_1(p)}}(u)$
$F^L_{\underline{F_1(p)}}(u) = \sup_{y\in[u]_K}\{F^L_{F_1(p)}(y)\} \geq F^L_{F_1(p)}(y) \geq \inf_{y\in[u]_K}\{F^L_{F_1(p)}(y)\} = F^L_{\overline{F_1(p)}}(u)$
$F^U_{\underline{F_1(p)}}(u) = \sup_{y\in[u]_K}\{F^U_{F_1(p)}(y)\} \geq F^U_{F_1(p)}(y) \geq \inf_{y\in[u]_K}\{F^U_{F_1(p)}(y)\} = F^U_{\overline{F_1(p)}}(u)$

Satisfying the conditions (5) to (8).

Thus

$[\underline{T_{F_1(p)}}^L(u), \underline{T_{F_1(p)}}^U(u)],\ [\underline{I_{F_1(p)}}^L(u), \underline{I_{F_1(p)}}^U(u)],\ [\underline{F_{F_1(p)}}^L(u), \underline{F_{F_1(p)}}^U(u)]$

$\subseteq [T^L_{F_1(p)}(u), T^U_{F_1(p)}(u)], [I^L_{F_1(p)}(u), I^U_{F_1(p)}(u)], [F^L_{F_1(p)}(u), F^U_{F_1(p)}(u)]$

$\subseteq [\overline{T_{F_1(p)}}^L(u), \overline{T_{F_1(p)}}^U(u)],\ [\overline{I_{F_1(p)}}^L(u), \overline{I_{F_1(p)}}^U(u)], [\overline{F_{F_1(p)}}^L(u), \overline{F_{F_1(p)}}^U(u)]$

for all $u \in U$, and for all $p \in G$,

Hence, we conclude that:

$\underline{(F_1, G)_K} \subseteq (F_1, G) \subseteq \overline{(F_1, G)_K}$

2. Let $(F_1, G) = \{(p, (u, [T^L_{F_1(p)}(u), T^U_{F_1(p)}(u)], [I^L_{F_1(p)}(u), I^U_{F_1(p)}(u)], [F^L_{F_1(p)}(u), F^U_{F_1(p)}(u)])) : u \in A, p \in G\}$, be a GIVNRS set satisfying the conditions

   $[T^L_{F_1(p)}(u), T^U_{F_1(p)}(u)], [I^L_{F_1(p)}(u), I^U_{F_1(p)}(u)], [F^L_{F_1(p)}(u), F^U_{F_1(p)}(u)] \in \text{Int}([0, 1])$,

   and

   $\sup[T^L_{F_1(p)}(u), T^U_{F_1(p)}(u)] \wedge \sup[I^L_{F_1(p)}(u), I^U_{F_1(p)}(u)] \wedge \sup[F^L_{F_1(p)}(u), F^U_{F_1(p)}(u)] \leq 0.5,\ \forall u \in A, p \in G.$

   $(F_2, G) = \{(p, (u, [T^L_{F_2(p)}(u), T^U_{F_2(p)}(u)], [I^L_{F_2(p)}(u), I^U_{F_2(p)}(u)], [F^L_{F_2(p)}(u), F^U_{F_2(p)}(u)])) : u \in A, p \in G\}$,

   $\sup[T^L_{F_2(p)}(u), T^U_{F_2(p)}(u)] \wedge \sup[I^L_{F_2(p)}(u), I^U_{F_2(p)}(u)] \wedge \sup[F^L_{F_2(p)}(u), F^U_{F_2(p)}(u)] \leq 0.5, \quad \forall u \in A, p \in G$

   be two GIVNRS sets. $((F_1, G) \cup (F_2, G))_K = \{(p, (u, [T^L_{\overline{F_1 \cup F_2(p)}}(u), T^U_{\overline{F_1 \cup F_2(p)}}(u)], [I^L_{\overline{F_1 \cup F_2(p)}}(u), I^U_{\overline{F_1 \cup F_2(p)}}(u)],$

   $[F^L_{\overline{F_1 \cup F_2(p)}}(u), F^U_{\overline{F_1 \cup F_2(p)}}(u)])) : u \in A\} : p \in G$

   where $[T^L_{\overline{F_1 \cup F_2(p)}}(u), T^U_{\overline{F_1 \cup F_2(p)}}(u)], [I^L_{\overline{F_1 \cup F_2(p)}}(u), I^U_{\overline{F_1 \cup F_2(p)}}(u)], [F^L_{\overline{F_1 \cup F_2(p)}}(u), F^U_{\overline{F_1 \cup F_2(p)}}(u)])) \in \text{Int}([0, 1])$,

   And $\sup[T^L_{\overline{F_1 \cup F_2(p)}}(u), T^U_{\overline{F_1 \cup F_2(p)}}(u)] \wedge \sup[I^L_{\overline{F_1 \cup F_2(p)}}(u), I^U_{\overline{F_1 \cup F_2(p)}}(u)] \wedge \sup[F^L_{\overline{F_1 \cup F_2(p)}}(u), F^U_{\overline{F_1 \cup F_2(p)}}(u)] \leq 0.5$

   Now $\overline{(F_1, G)}_K \cup \overline{(F_2, G)}_K = \{(p, (u, [\sup(T^L_{\overline{(F_1(p))}}(u), T^L_{\overline{(F_2(p))}}(u)), \sup(T^U_{\overline{(F_1(p))}}(u), T^U_{\overline{(F_2(p))}}(u))],$

   $[\inf(I^L_{\overline{(F_1(p))}}(u), I^L_{\overline{(F_2(p))}}(u)), \inf(I^U_{\overline{(F_1(p))}}(u), I^U_{\overline{(F_2(p))}}(u))],$

   $[\inf(F^L_{\overline{(F_1(p))}}(u), F^L_{\overline{(F_2(p))}}(u)), \inf(F^U_{\overline{(F_1(p))}}(u), F^U_{\overline{(F_2(p))}}(u))])\} : u \in A) : p \in G$

   $T_{\overline{F_1 \cup F_2(p)^L}(u) = \sup\{T_{(F_1 \cup F_2(p))^L}(t) : t \in [u]_K\}}$

   $= \sup\{T^L_{F_1(p)}(t) \vee \{T^L_{F_2(p)}(t) : t \in [u]_K\}\} = \{\sup\{T^L_{F_1(p)}(t) : t \in [u]_K\} \vee \{\sup\{T^L_{F_2(p)}(t) : t \in [u]_K\}$

   $= (T_{\overline{F_1(p)}^L} \vee T_{\overline{F_2(p)}^L})(u)$

   $T^U_{\overline{F_1 \cup F_2(p)}}(u) = \sup\{\{T^U_{(F_1 \cup F_2(p))}(t) : t \in [u]_K\}$

   $= \sup\{\{T^U_{F_1(p)}(t) \vee \{T^L_{F_2(p)}(t) : t \in [u]_K\} = \{\sup\{T^U_{F_1(p)}(t) : t \in [u]_K\} \vee \{\sup\{T^U_{F_2(p)}(t) : t \in [u]_K\}$

   $= (T^U_{\overline{F_1(p)}} \vee T^U_{\overline{F_2(p)}})(u)$

   $I^L_{\overline{F_1 \cup F_2(p)}}(u) = \inf\{\{I^L_{(F_1 \cup F_2(p))}(t) : t \in [u]_K\}$

   $= \inf\{\{I^L_{F_1(p)}(t) \wedge \{I^L_{F_2(p)}(t) : t \in [u]_K\} = \{\inf\{I^L_{F_1(p)}(t) : t \in [u]_K\} \wedge \{inf\{I^L_{F_2(p)}(t) : t \in [u]_K\}$

   $= (I^L_{\overline{F_1(p)}} \wedge I^L_{\overline{F_2(p)}})(u)$

$I^{U}_{\overline{F_1 \cup F_2(p)}}(u) = \inf\{\{I^{L}_{(F_1 \cup F_2(p))}(t) : t \in [u]_K\}$
$= \inf\{\{I^{U}_{F_1(p)}(t) \wedge \{I^{U}_{F_2(p)}(t) : t \in [u]_K\} = \{\inf\{I^{U}_{F_1(p)}(t) : t \in [u]_K\} \wedge \{inf\{I^{L}_{F_2(p)}(t) : t \in [u]_K\}$
$= (I^{U}_{\overline{F_1(p)}} \wedge I^{U}_{\overline{F_2(p)}})(u)$
$F^{L}_{\overline{F_1 \cup F_2(p)}}(u) = \inf\{\{F^{L}_{(F_1 \cup F_2(p))}(t) : t \in [u]_K\}$
$= \inf\{\{F^{L}_{F_1(p)}(t) \wedge \{F^{L}_{F_2(p)}(t) : t \in [u]_K\} = \{\inf\{F^{L}_{F_1(p)}(t) : t \in [u]_K\} \wedge \{inf\{F^{L}_{F_2(p)}(t) : t \in [u]_K\}$
$= (F^{L}_{\overline{F_1(p)}} \wedge F^{L}_{\overline{F_2(p)}})(u)$
$F_{\overline{F_1 \cup F_2(p)}^U}(u) = \inf\{\{F^{U}_{(F_1 \cup F_2(p))}(t) : t \in [u]_K\}$
$= \inf\{\{F^{U}_{F_1(p)}(t) \wedge \{F^{U}_{F_2(p)}(t) : t \in [u]_K\} = \{\inf\{F^{U}_{F_1(p)}(t) : t \in [u]_K\} \wedge \{inf\{F^{U}_{F_2(p)}(t) : t \in [u]_K\}$
$= (F^{U}_{\overline{F_1(p)}} \wedge F^{U}_{\overline{F_2(p)}})(u)$
Hence $\overline{(F_1, G) \cup (F_2, G)}_K = \overline{(F_1, G)}_K \cup \overline{(F_2, G)}_K$, and similarly $\underline{(F_1, G) \cap (F_2, G)}_K = \underline{(F_1, G)}_K \cap \underline{(F_2, G)}_K$ for all $u \in U$,

3. The proof for this property follows a similar reasoning as stated in property (2).
4. The proof is obvious.
5. The proof is obvious.

□

Definition 5.8. *If* $(F, G) = (\underline{(F, G)_K}, \overline{(F, G)_K})$ *is a GIVNRS set in* (U, K), *then the aggregate union operator, represented by* $\overline{(F, G)_K}\ \Theta\ \underline{(F, G)_K}$, *is characterized in the following manner:*

$\overline{((F, G))_K}\ \Theta\ \underline{((F, G))_K} =$
$\{(u, \{[T^{L}_{\overline{F}\Theta\underline{F}(p)}(u), T^{U}_{\overline{F}\Theta\underline{F}(p)}(u)],$
$[I^{L}_{\overline{F}\Theta\underline{F}(p)}(u), I^{U}_{\overline{F}\Theta\underline{F}(p)}(u)],$
$[F^{L}_{\overline{F}\Theta\underline{F}(p)}(u), F^{U}_{\overline{F}\Theta\underline{F}(p)}(u)]\}) : u \in A, p \in G\}$
$= \{(u, \{[\sup_{y \in [u]_K} T_F(p)^L(y) + \inf_{y \in [u]_K} T_F(p)^L(y) - \sup_{y \in [u]_K} T_F(p)^L(y) \cdot \inf_{y \in [u]_K} T_F(p)^L(y)],$
$[\inf_{y \in [u]_K} I_F(p)^L(y) \cdot \sup_{y \in [u]_K} I_F(p)^L(y), \inf_{y \in [u]_K} I_F(p)^U(y) \cdot \sup_{y \in [u]_K} I_F(p)^U(y)],$
$[\inf_{y \in [u]_K} F^{L}_{F(p)}(y) \cdot \sup_{y \in [u]_K} F^{L}_{F(p)}(y), \inf_{y \in [u]_K} F^{U}_{F(p)}(y) \cdot \sup_{y \in [u]_K} F^{U}_{F(p)}(y)]\}) : u \in A, p \in G\}$

The formula for the Aggregate Union Operator $\overline{((F, G))_K}\Theta\underline{((F, G))_K}$ *involves the combination of three key componentstruth (T), indeterminacy (I), and falsity (F):*

- ***Truth Component:*** *The truth values of the two approximations are added together, but their overlap is subtracted to avoid double-counting:*
  $T = [T^{L}_{\overline{F}\Theta\underline{F}(p)}(u), T^{U}_{\overline{F}\Theta\underline{F}(p)}(u)]$
  $=[\sup_{y \in [u]_K} T_F(p)^L(y) + \inf_{y \in [u]_K} T_F(p)^L(y) - \sup_{y \in [u]_K} T_F(p)^L(y) \cdot \inf_{y \in [u]_K} T_F(p)^L(y)]$
  This reflects the aggregation of truth information while considering the overlap between the two sets.

- ***Indeterminacy Component:*** *The indeterminacy values of the two approximations are multiplied to ensure that the resulting uncertainty accounts for both sets' indeterminate information:*
  $I = [I^{L}_{\overline{F}\Theta\underline{F}(p)}(u), I^{U}_{\overline{F}\Theta\underline{F}(p)}(u)]$

$= [\inf_{y\in[u]_K} I_F(p)^L(y) \cdot \sup_{y\in[u]_K} I_F(p)^L(y), \inf_{y\in[u]_K} I_F(p)^U(y) \cdot \sup_{y\in[u]_K} I_F(p)^U(y)]$.

This highlights the combined uncertainty from both approximations.

- ***Falsity Component:*** *The falsity values are similarly combined by multiplying them, reflecting the false information present in both approximations:*

  $F = [F^L_{\overline{F}\Theta\underline{F}(p)}(u), F^U_{\overline{F}\Theta\underline{F}(p)}(u)]$

  $= [\inf_{y\in[u]_K} F^L_{F(p)}(y) \cdot \sup_{y\in[u]_K} F^L_{F(p)}(y), \inf_{y\in[u]_K} F^U_{F(p)}(y) \cdot \sup_{y\in[u]_K} F^U_{F(p)}(y)]$

  This ensures that the false elements are accounted for in the union of the two sets.

The primary goal of the Aggregate Union Operator is to combine the results from two different approximations ($\overline{(F,G)_K}$ and $\underline{(F,G)_K}$) into a single, unified set that reflects the truth, indeterminacy, and falsity of the data in a balanced manner. This is especially useful in cases where data is uncertain or incomplete, as it captures both the definite and ambiguous aspects of the data.

Advantages:

- **Comprehensive Representation:** This operator allows for a more complete representation of data by considering both certain and uncertain elements in a unified framework.
- **Handling of Uncertainty:** The operator is specifically designed to address uncertainty in data, which is common in real-world applications.
- **Flexibility:** The Aggregate Union Operator can be applied to various types of data, including environmental, medical, and financial data, making it a versatile tool for many domains.

The Aggregate Union Operator provides a robust and flexible method for combining uncertain data from different approximations. Its ability to handle truth, indeterminacy, and falsity makes it an ideal tool for applications that involve incomplete, inconsistent, or ambiguous information. By incorporating this operator into decision-making processes, users can obtain more reliable and comprehensive results, ensuring better-informed decisions in complex scenarios.

Example 5.9. Now consider the example 5.2, then

$\overline{(F,G)_K}\ \Theta\underline{(F,G)_K}$

$= \{[p_1(\text{pH}), [w_1, [0.88, 0.98], [0.0, 0.02], [0.0, 0.02]],$

$[w_2, [0.64, 0.84], [0.09, 0.16], [0.04, 0.09]],\ [w_3, [0.88, 0.98], [0.0, 0.02], [0.0, 0.02]]\})],$

$[p_2\ (\text{DO}), [w_1, [0.91, 0.99], [0.0, 0.01], [0.0, 0.01]],$

$[w_2, [0.75, 0.84], [0.04, 0.09], [0.04, 0.09]], [w_3, [0.91, 0.99], [0.0, 0.01], [0.0, 0.01]]\})],$

$[p_3\ (\text{BOD}), [w_1, [0.8, 0.94], [0.02, 0.06], [0.0, 0.02]],$

$[w_2, [0.36, 0.64], [0.09, 0.16], [0.09, 0.25]], [w_3, [0.8, 0.94], [0.02, 0.06], [0.0, 0.02]]\})]\}\}$

6 Water Quality Assessment Using GIVNRS set and the Aggregate Union Operator

In our study, the thresholds used for classifying water quality into "Good," "Moderate," and "Poor" categories were determined by a combination of expert judgment and standardized regulations. These two approaches were chosen to ensure that the classification system was both contextually relevant to the local conditions and aligned with internationally recognized standards for water quality assessment. Expert judgment played a significant role in establishing the thresholds for key water quality parameters. Environmental scientists, water quality specialists and hydrologists familiar with the region's ecological and hydrological conditions were consulted. Their knowledge of local water quality issues, pollution sources, and the environmental sensitivity of the study area allowed for the development of thresholds that were practical and realistic for the specific context of the study. Experts considered factors such as:

- Local water body characteristics (e.g., river flow, seasonal variations),
- Historical data and past pollution levels,
- Impacts on aquatic life and human health.

For example, experts may have determined that a dissolved oxygen level of 5 mg/L is suitable for maintaining a healthy aquatic ecosystem in the study area, classifying it as "Good", while lower levels would indicate varying levels of pollution.

In addition to expert judgment, existing water quality standards provided a key foundation for the classification thresholds. The guidelines set by international bodies such as the World Health Organization (WHO) and the Central Pollution Control Board (CPCB) in India were referenced to ensure that the classification system aligns with recognized criteria for safe and healthy water quality. These standards are based on extensive scientific research and are designed to safeguard public health and protect aquatic ecosystems.

For instance, the WHO guidelines for pH, DO, and BOD were used to establish upper limits for "Good" water quality. Water bodies with levels exceeding these thresholds would be classified as "Moderate" or "Poor" depending on the severity of the exceedance. The use of these internationally recognized standards ensured that the classification system was not only locally relevant but also consistent with global water quality assessment practices.

The thresholds for classifying water quality were therefore developed by integrating expert recommendations with standardized regulations. This combination ensured that the thresholds were both scientifically robust and regionally tailored, reflecting both the global best practices for water quality and the specific conditions of the study area. This approach also allowed us to make informed decisions about water quality management, ensuring that our findings would be directly applicable to local environmental policy and sustainable water management strategies. By using a combination of expert judgment and standardized regulations, we aimed to create a classification system that was both accurate and practical for addressing water quality challenges in the region.

6.1 Algorithm

In this section, we introduce the algorithm developed for the water quality assessment, which is rooted in the principles of GIVNRS sets. This algorithm is specifically tailored for MCDM applications, especially in environmental management scenarios where data is often uncertain, incomplete, and inconsistent. The algorithm was developed as a solution to the challenge of managing uncertainty and imprecision in decision-making processes, particularly in environmental monitoring and water quality assessment. Traditional decision-making models often struggle with incomplete or inconsistent data, which is frequently encountered in real-world environmental studies due to the complex nature of water quality indicators. The algorithm is built on a combination of soft set theory, rough set theory, and IVNS, which are integrated within the GIVNRS set framework. These set theories are known for their ability to manage and model uncertainty, vagueness, and indeterminacy in decision-making. The GIVNRS set framework allows for the consideration of multiple sources of uncertainty simultaneously, providing a more accurate representation of real-world data where exact values may not be available, or where there are inconsistencies and contradictions in the data.

This algorithm integrates the GIVNRS set and the aggregate union operator to assess water quality across multiple river segments. The steps are designed to classify water quality into categories like Good, Moderate, or Poor by addressing uncertainty, indeterminacy, and falsity in the measurements of water quality parameters such as pH, DO, and BOD.

Step 1: Define the Sets Let us consider

$$U = \{u_1, u_2, u_3, \ldots, u_n\}$$

representing the sampling stations along river segments.

A set of water quality parameters

$$G = \{p_1, p_2, p_3, \ldots, p_m\}$$

for each parameter $p_k \in G$, the intervals for truth, indeterminacy, and falsity (T, I, F) at each segment are given.

Step 2: Define GIVNS set For each water quality parameter $p_k \in G$, define the GIVNS set (F, G) for each river segment $U = \{u_1, u_2, u_3, \ldots, u_n\}$. This includes truth, indeterminacy, and falsity intervals for each segment:

$$F(p_k) = \{(u, \{T_F(p_k)(u), I_F(p_k)(u), F_F(p_k)(u)\}) : u \in A\}$$

Where:

$$T_F(p_k)(u) \text{ is the truth interval for segment } u \in A \text{ for parameter } p_k \in G,$$

$$I_F(p_k)(u) \text{ is the indeterminacy interval,}$$

$$F_F(p_k)(u) \text{ is the falsity interval.}$$

Step 3: Establish the Equivalence Relation $\mathcal{K}$ Define the equivalence relation $\mathcal{K}$ to group river segments that have similar water quality characteristics (based on parameters $p_1, p_2, \ldots, p_m$). In this context, $[u]_{\mathcal{K}}$ represents the equivalence class of the element u.

Step 4: Calculate Lower and Upper Approximations For each parameter $p_k \in G$, calculate the lower and upper approximations of the GIVNSS (F, G) These approximations are used to classify the river segments into more certain or uncertain categories based on their water quality values.

Lower Approximation $\underline{F(p)}_{\mathcal{K}}$:

$\underline{F(p)}_{\mathcal{K}} = \{(u, [\inf_{y\in[u]_K} T_F(p)^L(y), \inf_{y\in[u]_K} T_F(p)^U(y)],$
$[\sup_{y\in[u]_K} I^L_{F(p)}(y), \sup_{y\in[u]_K} I^U_{F(p)}(y)], [\sup_{y\in[u]_K} F^L_{F(p)}(y), \sup_{y\in[u]_K} F^U_{F(p)}(y)]) : u \in A\}, p \in G\}$

Upper Approximation $\overline{(F(p)}_{\mathcal{K}}$: $\overline{(F(p)}_{\mathcal{K}} = \{(u, [\sup_{y\in[u]_K} T_F(p)^L(y), \sup_{y\in[u]_K} T_F(p)^U(y)],$
$[\inf_{y\in[u]_K} I^L_{F(p)}(y), \inf_{y\in[u]_K} I^U_{F(p)}(y)], [\inf_{y\in[u]_K} F^L_{F(p)}(y), \inf_{y\in[u]_K} F^U_{F(p)}(y)]\ ) : u \in A\}, p \in G\}$

Step 5: Apply Aggregate Union Operator Obtain the result by applying the aggregate union operator Θ to the lower and upper approximations:

$$\overline{(F, G)}_{\mathcal{K}} \Theta \underline{(F, G)}_{\mathcal{K}}$$

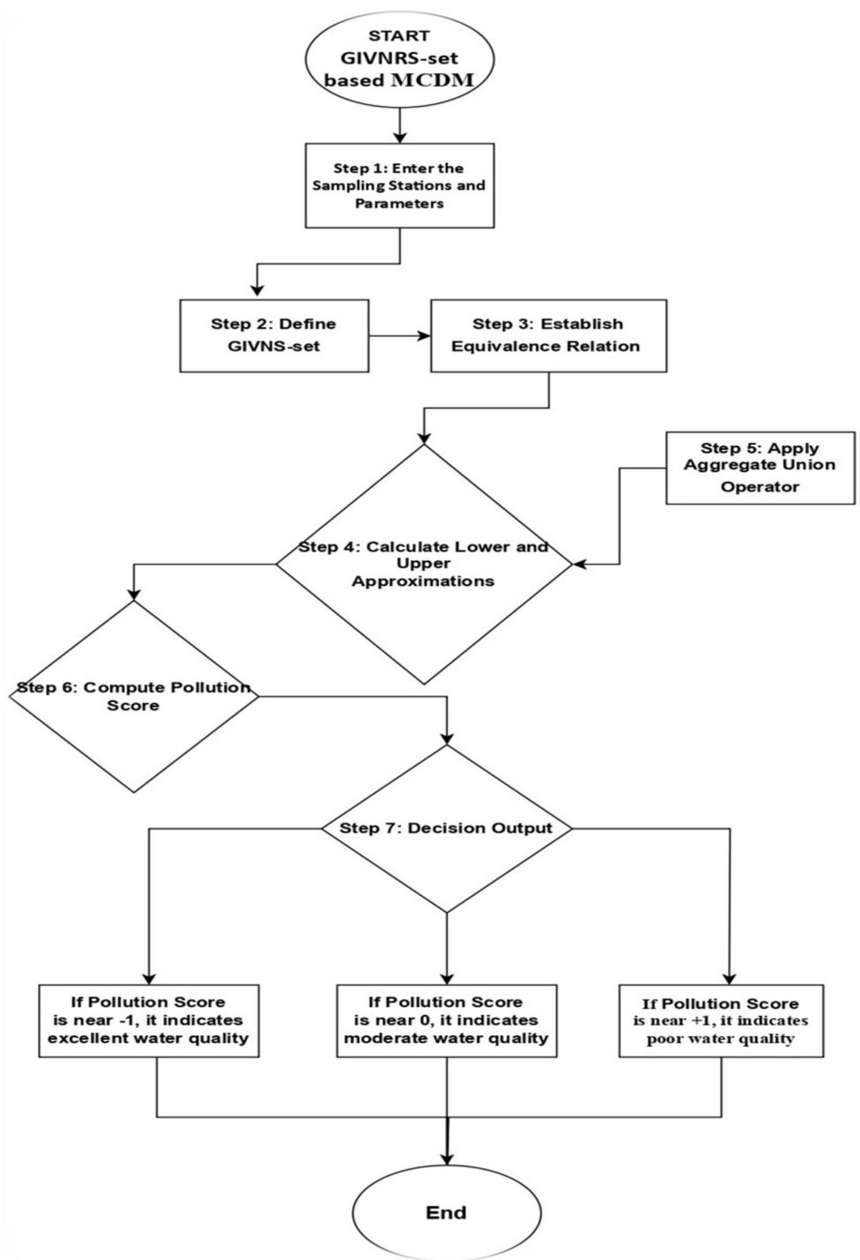
This operator aggregates the truth, indeterminacy, and falsity intervals to create a unified representation of the water quality for each segment.

Step 6: Obtain the Pollution-Score $W(u)$ Compute the pollution-score $W(u)$ for each sampling station $u \in A$ using the formula:

$$W(u) = \sum_{p\in G} \frac{\left(\frac{T^L_{\overline{F}\Theta\underline{F}(p)}(u) + T^U_{\overline{F}\Theta\underline{F}(p)}(u)}{2} - \frac{F^L_{\overline{F}\Theta\underline{F}(p)}(u) + F^U_{\overline{F}\Theta\underline{F}(p)}(u)}{2}\right)}{1 + \frac{I^L_{\overline{F}\Theta\underline{F}(p)}(u) + I^U_{\overline{F}\Theta\underline{F}(p)}(u)}{2}}$$

Step 7: Decision Output If $W(u)$ values near -1 indicate excellent water quality, while near +1 indicate poor water quality.

The flowchart below outlines the step-by-step process for assessing water quality using a fuzzy soft set-based decision-making model. Each step systematically defines, calculates, and evaluates key components to compute the pollution score, leading to a final decision on water quality classification.



(1).jpg (1).bb

Figure 1: Flowchart depicting the stepwise methodology for water quality assessment using the GIVNRS set-based MCDM

7 Case Study and Application Example

7.1 Overview

A water management authority monitors the quality of a river by evaluating specific water quality parameters such as pH, DO, and BOD. These parameters are essential indicators of water quality but are often subject to uncertainties caused by equipment inconsistencies, sampling errors, and seasonal variations. The proposed GIVNRS set framework is employed to assess water quality, classifying it into categories such as Good, Moderate, and Poor, while effectively handling incomplete, inconsistent, and indeterminate data.

7.2 Methodology

Let the river segments $U = \{w_1, w_2, w_3\}$ represent three monitoring stations along the river, and the parameters being evaluated (pH, DO, and BOD) form the parameter set $G = \{p_1 = \text{pH}, p_2 = \text{DO}, p_3 = \text{BOD}\}$. Each parameter (pH, DO, BOD) will be evaluated using the GIVNRS framework. The relation $\mathcal{K}$ is defined as an equivalence relation based on the similarity of river segments in terms of their water quality.

For each parameter $p \in G$, the GIVNS at each segment $w \in A$ is defined as:

$$F(p) = \{(w, (T_{F(p)}(w), I_{F(p)}(w), F_{F(p)}(w))) : w \in A\}$$

The intervals for truth (T), indeterminacy (I), and falsity (F) at each segment are derived based on expert evaluations or historical data.

7.3 Results and Analysis

Let $\mathcal{GIVNSS}(\mathcal{A})$ represent the collection of all GIVNSSs over $\mathcal{A}$ and let the equivalence relation $\mathcal{K}$ be defined as $U/\mathcal{K} = \{\{w_1, w_3\}, \{w_2\}\}$, establishing the classification of river segments based on similarity in water quality characteristics (e.g., ranges of pH, DO, and BOD), where

$[w_1]_{\mathcal{K}} = \{w_1, w_3\}$,as they have similar WQPs. $[w_2]_{\mathcal{K}} = \{w_2\}$.

Let $(F, G) =$

$\{(p_1(\text{pH}), ((w_1, [0.6, 0.8], [0.1, 0.2], [0.0, 0.1]),$
$(w_2, [0.4, 0.6], [0.3, 0.4], [0.2, 0.3]),\ (w_3, [0.7, 0.9], [0.0, 0.1], [0.0, 0.2]))),$
$(p_2(\text{DO}), ((w_1, [0.7, 0.9], [0.0, 0.1], [0.0, 0.1]),$
$(w_2, [0.5, 0.6], [0.2, 0.3], [0.2, 0.3]), (w_3, [0.8, 1.0], [0.0, 0.1], [0.0, 0.1]))),$
$(p_3(\text{BOD}), ((w_1, [0.5, 0.7], [0.2, 0.3], [0.0, 0.2]),$
$(w_2, [0.2, 0.4], [0.3, 0.4], [0.3, 0.5]), (w_3, [0.6, 0.8], [0.1, 0.2], [0.0, 0.1])))\}$

be a GIVNSS ON $\mathcal{A}$

Then by definition the lower and upper approximations for (F, G) are calculated as:

$\underline{(F, G)}_{\mathcal{K}} = \{(p_1(\text{pH}), ((w_1, [0.6, 0.8], [0.1, 0.2], [0.0, 0.2]),$
$(w_2, [0.4, 0.6], [0.3, 0.4], [0.2, 0.3]), (w_3, [0.6, 0.8], [0.1, 0.2], [0.0, 0.2]))),$
$(p_2(\text{DO}), ((w_1, [0.7, 0.9], [0.0, 0.1], [0.0, 0.1]),$
$(w_2, [0.5, 0.6], [0.2, 0.3], [0.2, 0.3]), (w_3, [0.7, 0.9], [0.0, 0.1], [0.0, 0.1]))),$
$(p_3(\text{BOD}), ((w_1, [0.5, 0.7], [0.2, 0.3], [0.0, 0.2]),$
$(w_2, [0.2, 0.4], [0.3, 0.4], [0.3, 0.5]), (w_3, [0.5, 0.7], [0.2, 0.3], [0.0, 0.2])))\}$

And

$\overline{(F, G)}_{\mathcal{K}} =$

$\{(p_1(\text{pH}), ((w_1, [0.7, 0.9], [0.0, 0.1], [0.0, 0.1]),$
$(w_2, [0.4, 0.6], [0.3, 0.4], [0.2, 0.3]), (w_3, [0.7, 0.9], [0.0, 0.1], [0.0, 0.1]))),$

$(p_2(\text{DO}), ((w_1, [0.7, 0.9], [0.0, 0.1], [0.0, 0.1]),$
$(w_2, [0.5, 0.6], [0.2, 0.3], [0.2, 0.3]), (w_3, [0.7, 0.9], [0.0, 0.1], [0.0, 0.1]))),$
$(p_3(\text{BOD}), ((w_1, [0.6, 0.8], [0.1, 0.2], [0.0, 0.1]),$
$(w_2, [0.2, 0.4], [0.3, 0.4], [0.3, 0.5]), (w_3, [0.6, 0.8], [0.1, 0.2], [0.0, 0.1])))\}$

Now we obtain $\overline{(F, G)}_{\mathcal{K}} \Theta \underline{(F, G)}_{\mathcal{K}}$ using the aggregate union operator Θ as

$\overline{(F, G)}_{\mathcal{K}} \Theta \underline{(F, G)}_{\mathcal{K}} =$
$\{(p_1(\text{pH}), (w_1, [0.88, 0.98], [0.0, 0.02], [0.0, 0.02]), (w_2, [0.64, 0.84], [0.09, 0.16], [0.04, 0.09]),$
$(w_3, [0.88, 0.98], [0.0, 0.02], [0.0, 0.02])))))$
$(p_2(\text{DO}), ((w_1, [0.91, 0.99], [0.0, 0.01], [0.0, 0.01]),$
$(w_2, [0.75, 0.84], [0.04, 0.09], [0.04, 0.09]),$
$(w_3, [0.91, 0.99], [0.0, 0.01], [0.0, 0.01]))))),$
$(p_3(\text{BOD}), ((w_1, [0.8, 0.94], [0.02, 0.06], [0.0, 0.02]),$
$(w_2, [0.36, 0.64], [0.09, 0.16], [0.09, 0.25]),$
$(w_3, [0.8, 0.94], [0.02, 0.06], [0.0, 0.02]))))))\}\}$

Based on the above combined results of the Aggregate Union Operator $\overline{(F, G)}_{\mathcal{K}} \Theta \underline{(F, G)}_{\mathcal{K}}$, we compute the pollution-score $W(w_i)$ for each sampling station $w_i \in U$ as follows:

$W(w_1) = 0.893, \quad W(w_2) = 0.526, \quad W(w_3) = 0.893.$

The decision output classifies each river segment as follows:

- Segment w_1: Classified as Poor.
- Segment w_2: Classified as Moderate.
- Segment w_3: Classified as Poor.

7.4 Discussion

- **Insights from Results:** Segments w_1 and w_3 exhibit similar water quality, categorized as "Poor," likely due to similar pollutant sources or geographical proximity. In contrast, w_2, with a "Moderate," classification, may reflect localized variations in pollution control or natural water flow dynamics.
- **Practical Implications:** The classification provides actionable insights for water resource management, prioritizing pollution mitigation efforts in segments w_1 and w_3.

7.5 Broader Applications

The methodology presented in this study, based on the GIVNRS set framework, showcases its robustness in addressing complex decision-making scenarios characterized by uncertainty, indeterminacy, and incomplete data. While this study primarily applies the framework to water quality assessment, its adaptability allows it to be extended to other domains, such as air quality monitoring and waste management, to validate its broader applicability.

7.5.1 Air Quality Monitoring

Air quality monitoring is a critical domain where data often contains uncertainties due to fluctuating atmospheric conditions and sensor inaccuracies. By leveraging the GIVNRS methodology, regions can be classified based on their air quality levels, enabling more precise interventions for pollution control.

7.5.2 Application of GIVNRS Framework

Parameters: The methodology can process key air quality indicators such as PM2.5, PM10, SO_2,NO_2, and Air Quality Index (AQI).

Approach:

1. Monitoring stations across an urban area can serve as the segments $U = \{u_1, u_2, ..., u_n\}$.

2. Parameters G (e.g., PM2.5, PM10) are evaluated using the GIVNRS framework, accounting for uncertainties from sensor measurements and seasonal variations.

3. Truth (T), indeterminacy (I), and falsity (F) intervals for each parameter are derived from historical and real-time datasets.

Outcome: The aggregate union operator can classify regions into categories like Good Air Quality, Moderate Air Quality, or Severely Polluted, providing targeted guidance for deploying pollution control measures like emission restrictions or public health advisories.

Validation Example: In a city like Delhi, India, where pollution levels vary across neighborhoods due to traffic density and industrial zones, the framework could provide fine-grained classifications, helping prioritize interventions in the most polluted areas while continuously monitoring improvements in air quality.

7.5.3 Waste Management

Effective waste management requires understanding the performance of recycling programs, landfill usage, and waste segregation practices. The GIVNRS methodology can handle the inherent uncertainties in waste management data, such as variations in citizen compliance, seasonal waste production, and monitoring errors.

7.5.4 Application of GIVNRS Framework

Parameters: Key indicators such as the percentage of waste recycled, amount of segregated waste, landfill capacity usage, and illegal dumping occurrences can be analyzed.

Approach:

1. Urban and rural areas are segmented into zones $U = \{z_1, z_2, ..., z_n\}$, with parameters G representing waste management metrics.

2. Expert evaluations and historical data establish T, I, and F intervals for each parameter.

3. The aggregate union operator combines approximations, generating a comprehensive score for each zone.

Outcome: Zones are categorized into Efficient, Moderately Efficient, or Inefficient waste management regions, providing a roadmap for targeted infrastructure upgrades, policy adjustments, and awareness campaigns.

Validation Example: In Agartala, Tripura, where urbanization has increased waste generation, the framework could pinpoint areas with inefficient waste management practices. For instance, regions with high landfill usage but low recycling rates might require new composting facilities or stricter enforcement of waste segregation rules.

7.6 Broader Applicability and Validation

The successful application of the GIVNRS framework to domains like air quality monitoring and waste management validates its versatility and scalability. These domains share common challenges with water quality assessment, such as:

- Uncertainty in Measurements: Variability in sensor data or human reporting.
- Incomplete Data: Missing or inconsistent records due to technical or operational limitations.
- Complex Decision-Making: There is a need to balance multiple criteria and prioritize interventions.

By demonstrating its adaptability to air quality and waste management, the GIVNRS methodology establishes itself as a powerful decision-making tool applicable across diverse environmental and societal challenges.

Future work can extend the validation by applying the framework to other critical areas such as climate resilience, urban planning, and public health risk assessment, further solidifying its role in advancing sustainable development goals.

7.7 Research Implications

The proposed Generalized Interval-Valued Neutrosophic Rough Soft Set (GIVNRS) framework offers significant implications for various domains, particularly in the fields of water resource management, environmental policy-making, and sustainable development. Below are key areas where this research can make a meaningful impact:

7.7.1 Water Resource Management

Enhanced Decision-Making: The GIVNRS framework provides a structured methodology to handle uncertainty, indeterminacy, and incomplete data in water quality assessments. By computing pollution scores and categorizing river segments, the framework enables water resource managers to:

- Prioritize high-risk areas for remediation.
- Allocate resources efficiently for pollution control interventions.

Case Study Validation: The case study demonstrates the practical applicability of the framework, allowing authorities to classify river segments into Poor, Moderate, and Good categories. This classification directly supports:

- Identifying pollutant sources.
- Planning long-term water quality improvement strategies.

7.7.2 Policy-Making

Evidence-Based Policies: By offering a robust, data-driven approach to assessing water quality, the GIVNRS framework can inform policymakers on critical issues, such as:

- Regulating industrial discharges based on real-time water quality data.
- Implementing stricter pollution control measures in the most vulnerable regions (e.g., segments classified as "Poor").

Environmental Standards Compliance: The framework aligns with global water quality standards, such as those outlined by the World Health Organization (WHO), helping governments to ensure compliance and improve public health outcomes.

Scalability: The methodology's adaptability to other domains, such as air quality monitoring and waste management, allows policymakers to use a unified framework across environmental sectors for more integrated decision-making.

7.7.3 Environmental Sustainability

Promoting Sustainable Practices: By identifying specific areas and parameters contributing to poor water quality, this research supports initiatives to:

- Reduce pollution at its source, such as managing agricultural runoff or untreated wastewater.
- Encourage sustainable practices, like adopting eco-friendly farming methods or improving urban waste treatment infrastructure.

Long-Term Monitoring: The GIVNRS framework can be integrated with real-time monitoring systems (e.g., Internet of Things-enabled sensors) for continuous water quality assessment, ensuring sustainable water resource management over time.

7.7.4 Broader Applications and Research Potential

Framework Versatility: The study establishes the GIVNRS framework as a versatile decision-making tool applicable beyond water quality, such as:

- Assessing air quality, enabling urban planners to mitigate air pollution risks.
- Evaluating waste management systems to improve recycling efficiency and landfill usage.
- Monitoring climate change impacts on ecosystems through environmental parameter analysis.

This research lays the groundwork for exploring advanced applications, such as integrating machine learning and predictive analytics into the GIVNRS framework for enhanced accuracy and automation. Expanding the methodology to analyze multi-temporal datasets, allowing for the study of trends and seasonal variations. The GIVNRS framework represents a significant advancement in decision-making under uncertainty, with practical implications for water resource management, policy formulation, and environmental sustainability. By addressing key limitations in traditional methodologies, this research provides a scalable and robust tool to support data-driven strategies in managing environmental challenges.

8 Comparative Analysis

In this section, we compare the proposed model with an influential approach for water quality assessment and decision-making. Specifically, we examine the work of Patel and Chitnis [19], who modeled the dynamics of Sabarmati River water quality using fuzzy logic, considering the impacts of industrialization and climate change.

The comparative analysis includes a comparison of the fuzzy Water Quality Index (WQI) values derived from both the Patel-Chitnis model and the proposed model. In the Patel-Chitnis model, fuzzy data summaries of water quality parameters were processed to generate fuzzy WQI values for different river segments. The average fuzzy WQI values for the river segments at w_1, w_2, and w_3 were found to be:

$$\mathrm{WQI}(w_1) = 0.7, \quad \mathrm{WQI}(w_2) = 0.45, \quad \mathrm{WQI}(w_3) = 0.8.$$

The decision output classifies each river segment as follows:

- Segment w_1: Classified as Poor.
- Segment w_2: Classified as Moderate.
- Segment w_3: Classified as Poor.

Both the Patel-Chitnis model and the proposed model generate similar optimal choices and rankings (Table 2), but differences in the score values arise due to the distinct methods used. Our model utilizes the GIVNRS pollution-score, whereas the Patel-Chitnis model employs fuzzy WQI values. One of the main innovations of our model is the use of the GIVNRS pollution-score instead of the average score applied in the Patel-Chitnis model. The GIVNRS pollution-score offers greater stability and feasibility in managing uncertainty. Additionally, the flexibility and adaptability of the GIVNRS pollution-score function W in our Decision-Making Model surpass those of the Patel-Chitnis model, making our approach more robust and applicable in various decision-making scenarios.

Table 2: Comparative water quality assessment for pollution rating

Models	Best Optimal Choice	Ranking According to Good Quality
Patel-model [19]	w_2	$w_2 \gg w_1 \gg w_3$
Proposed model	w_2	$w_2 \gg w_1 \gg w_3$

9 Conclusion

This study introduced the GIVNRS set framework, a novel approach to addressing the challenges of uncertainty, imprecision, and indeterminacy in decision-making processes. By integrating the principles of soft and rough set theories with GIVNSs, the GIVNRS set framework provides a robust and flexible tool for analyzing complex datasets in diverse applications, such as environmental monitoring, medical diagnostics, and risk assessment. The proposed methodology's theoretical foundations, including fundamental operations and novel operators like the aggregate union operator, were thoroughly examined. Their practical applicability was demonstrated through a water quality assessment case study, showcasing the framework's ability to classify river segments effectively, even when confronted with incomplete, inconsistent, or vague data. The GIVNRS set framework addresses key limitations of traditional decision-making models by offering a dual capability to manage both indeterminacy and incompleteness. This makes it a significant advancement in the field of multi-criteria decision-making. The case study further validates its utility in real-world scenarios, emphasizing its potential to enhance decision-making accuracy and reliability.

9.1 Limitations of the Study

This study has several limitations that should be acknowledged. The water quality thresholds used for classification were based on expert judgment, which can be subjective and may vary across different regions or experts. Additionally, the study's data was limited by the temporal scope and availability of samples, which may not capture long-term or seasonal variations in water quality. The geographical focus on the study area restricts the generalizability of the findings to other regions with different environmental conditions. Lastly, the fuzzy decision-making model used, while effective in handling uncertainty, may not fully capture all complexities in multi-source pollution scenarios. Future research should address these limitations by expanding the dataset, refining thresholds, and including more parameters and regions for broader applicability.

Future research can explore integrating machine learning techniques with GIVNRS sets to improve their predictive and analytical capabilities. Additionally, further theoretical advancements, such as developing efficient computational algorithms and exploring new application domains, will strengthen the framework's versatility and impact. Ultimately, the GIVNRS set framework is positioned as a powerful tool for tackling uncertainty and imprecision in decision-making, paving the way for more informed and effective strategies across various disciplines.

Conflict of Interest: "The authors declare no conflict of interest."

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