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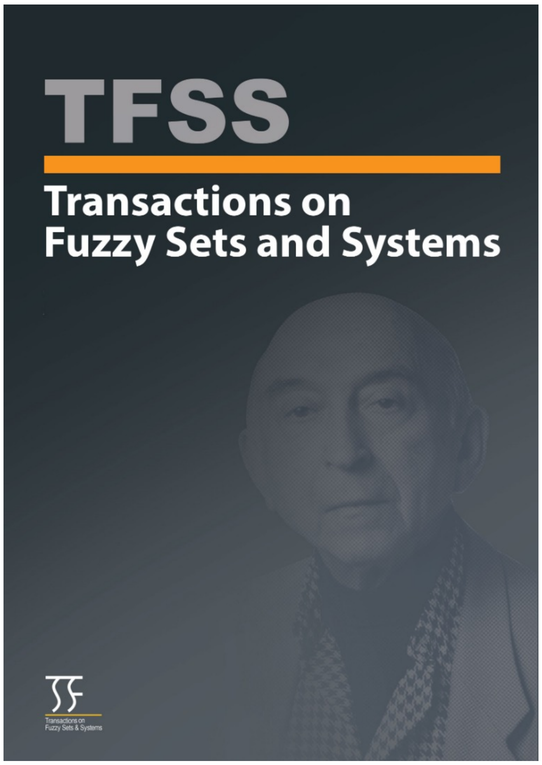


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# Solvability of Multiple Kinds of Fuzzy Fractional Hybrid Differential Equations Using Mnch's Fixed Point Theorem

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# Solvability of Multiple Kinds of Fuzzy Fractional Hybrid Differential Equations Using Mnch's Fixed Point Theorem

**Aziz El Ghazouani***, **M'hamed Elomari**

(This article is dedicated to Prof. Witold Pedrycz in recognition of his pioneering contributions to the field of Granular Computing.)

**Abstract.** The focus of this study is on hybrid differential and fractional hybrid differential equations (HFDEs). Such problems have important applications in a wide range of applied sciences. To address our models, we first study the existence theorem of fuzzy solutions under relatively weaker constraints, combining the measure of non-compactness and Mnch's fixed-point theorem. The insights provided here extend and refine several previously established findings. Subsequently, two examples are provided to demonstrate the validity of the results obtained.

**AMS Subject Classification 2020:** 34A07, 35R13
**Keywords and Phrases:** Fractional hybrid differential equations, The measure of non-compactness; Mnch's fixed-point theorem.

## 1 Introduction

Zadeh [1] pioneered the concept of fuzzy sets. Chang and Zadeh later utilized fuzzy sets to demonstrate fuzzy mapping and control [2]. A number of studies on fuzzy mappings established the framework of fundamental fuzzy calculus ([3]-[7]). In recent years, scholars have shown increasing interest in applying these concepts to fuzzy differential and integral equations in the field of physics.

Fractional integrals and fractional derivatives have existed for as long as mathematics itself. Fractional differential equations with fractional derivatives have gained significant popularity in the past few decades due to their wide range of applications in various disciplines of science and technology. Kuratowski [8] developed fixed-point theory and measures of non-compactness in 1930, which are now frequently utilized to solve many forms of differential and integral equations. Numerous academics are currently working on a significant number of new studies involving various types of analytic and differential problems. One reason for the use of fractional differential equations is that integer-order differential equations are unable to explain a wide range of phenomena. As a result, the implications of the existence of solutions to fractional-order differential equations have attracted considerable attention in recent years. These fractional differential equations are particularly useful in various industrial fields, notably in the study of polymeric viscosity materials and seismic evaluation. For further details, see ([9]-[10]).

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When studying real-world phenomena, however, uncertain features must be addressed. Under these circumstances, fuzzy set theory stands out as one of the most successful non-statistical or irregular methodologies for theoretically studying fuzzy differential equations. Several researchers have recently investigated and analyzed the existence and uniqueness of solutions to quadratic and linear fuzzy fractional differential equations (FFDEs) across various fields. For instance, the authors of [11] demonstrated the existence and uniqueness results of fractional evolution equations, while Salahshour et al. [12] established the existence and uniqueness of solutions to FFDEs. Allahviranloo et al. [13] further proved the existence and uniqueness of solutions to FFDEs under generalized Caputo Hukuhara differentiability.

The study of fractional derivatives plays a crucial role in numerous engineering applications, as it involves differential equations that have been widely used across various disciplines, such as chemistry, physics, and dynamical systems. Fractional-order differential equations are particularly significant because they offer greater accuracy compared to integer-order equations, thanks to their enhanced degree of freedom [14, 15]. Hybrid differential equations (HDEs), which are commonly used to model perturbations in dynamical systems, have attracted considerable attention from researchers [16, 17]. Numerous studies have explored the application of hybrid fixed-point theory to HDEs by incorporating various symmetry perturbations [18, 19]. Before delving into our research, we provide a brief overview of relevant studies addressing this problem. In 2013, Dhage demonstrated the existence and uniqueness of solutions for the following HDE:

$$[z(\varsigma)-\Lambda(\varsigma,z(\varsigma))]'=\Omega(\varsigma,z(\varsigma)),\quad \varsigma\in I=[\varsigma_0,a+\varsigma_0]\,, \tag{1}$$

with the initial condition $z(\varsigma_0)=z_0\in\mathbb{R}$, where $\Lambda,\Omega\in C(I\times\mathbb{R},\mathbb{R})$ [18, 20]. Subsequently, Lu et al. [16] generalized (1) by employing the Riemann-Liouville derivative to obtain a satisfactory relation between the analytical solution and experimental results:

$$D^{\theta}_{+0}(z(\varsigma)-\Lambda(\varsigma,z(\varsigma)))=\Omega(\varsigma,z(\varsigma)),\quad \varsigma\in I,$$

with the initial condition $z(\zeta_0)=z_0\in\mathbb{R}$. Additionally, Hilal et al. [17] proposed the boundary value problem (BVP) for fractional hybrid differential equations (FHDEs), which included Caputo's fractional-order derivative as follows:

$$\begin{cases} {}^{C}D^{\vartheta}_{+0}\left(\frac{z(\varsigma)}{\Lambda(\varsigma,z(\varsigma))}\right)=\Omega(\varsigma,z(\varsigma)),\quad \varsigma\in I,\\ T_1\left(\frac{z(0)}{\Lambda(0,z(0))}\right)+\mathrm{T}_2\left(\frac{z(\tau)}{\Lambda(\tau,z(\tau))}\right)=\mathrm{T}_3,\end{cases}$$

where $\Lambda\in C(I\times\mathbb{R},\mathbb{R}-\{0\})$, $\Omega\in C(I\times\mathbb{R},\mathbb{R})$, and $T_1,T_2$ (with $T_1+T_2\neq 0$) and $T_3$ are real constants.

In 2023, El Ghazouani and his collaborators [?] proved the existence and asymptotic behavior of nonlinear HFDEs involving the fuzzy nabla Caputo fractional difference. They provided intriguing findings on equilibrium and asymptotic equilibrium for the following problem:

$$\begin{cases} {}^{C}\nabla^{\nu}_{0}\left(\dfrac{u(t)}{f(t,u(t))}\right)=g(t,u(t))\,;\quad t\in\mathbb{N}_1,\\ u(0)=u_0\in\mathbb{R}_{\mathcal{F}},\end{cases} \tag{2}$$

where $\mathbb{N}_1$ and $\mathbb{R}_{\mathcal{F}}$ are the set of all natural integers and the set of fuzzy numbers, respectively. The functions $f,g:\mathbb{N}_1\times\mathbb{R}_{\mathcal{F}}\to\mathbb{R}_{\mathcal{F}}$ satisfy $f(t,\widehat{0})=g(t,\widehat{0})=\widehat{0}$, and ${}^{C}\nabla^{\nu}_{0}$ is the nabla-Caputo fuzzy fractional difference of order $\nu\in(0,1)$.

In [29], the authors explored the existence and stability results of fuzzy neutral fractional integrodifferential equations. The authors of [21] reported solvability and generalized Ulam-Hyers stability studies for a fuzzy nonlinear AtanganaBaleanu-Caputo fractional coupled system. For further examples, see the following references: ([22]-[25]).

To the best of our knowledge, there are no investigations in the field of science devoted to fuzzy hybrid differential equations utilizing Mnch's fixed-point theorem. As a result of the previous studies, the primary objective of this investigation is to analyze the following hybrid differential equations:

$$\frac{d}{dt}\left[\frac{u(t)}{\mathfrak{F}(t,u(t))}\right]=\Lambda(t,u(t)),\quad t\in J=[0,T] \text{ and } u(0)=\widehat{0}. \tag{3}$$

And the following equation:

$${}_{gH}^{C}D_{*}^{\alpha}\left[\frac{x(t)}{\mathfrak{F}(t,u(t))}\right]=\Lambda(t,u(t)),\quad t\in J \text{ and } u(0)=\widehat{0}. \tag{4}$$

Here, ${}_{gH}^{C}D_{*}^{\alpha}$ is the Caputo fractional generalized Hukuhara derivative of order $0<\alpha<1$, and $\mathfrak{F},\Lambda$ are two continuous functions.

The remainder of this paper is organized as follows: In Section 2, we review some fundamental concepts of calculus. In Sections 3 and 4, we apply Mnch's fixed-point theorem to demonstrate that systems (3) and (4) have unique solutions. Additionally, Section 5 provides examples to support the validity of our key assumptions. Finally, Section 6 offers a brief summary.

## 2 Preliminaries

In this section, we review several fundamental concepts that will be beneficial throughout the remainder of our article.

**Definition 2.1.** [26] *A fuzzy set $u$ is a map from $\mathbb{R}$ to $[0,1]$ with the subsequent possessions:*

*a. $u$ is convex, normal, and upper semi-continuous.*

*b.* supp$(u)$ *is compactly closed.*

**Definition 2.2.** [26] *Let $u$ be a fuzzy set. Then the parameterized band form of $u$ is given by*

$$u=[\underline{u}(r),\bar{u}(r)],r\in[0,1] \tag{5}$$

*and*

*a. $\underline{u}(r)$ is left-continuous and non-decreasing based on $r$.*

*b. $\overline{u}(r)$ is a right continuous and non-increasing based on $r$.*

*c. For all $r\in[0,1]$, $\bar{u}(r)\geq\underline{u}(r)$.*

**Definition 2.3.** [26] *Let $u$ and $v$ be two fuzzy sets. Therefore, there is*

$$(u\oplus v)=[\underline{u}(r)+\bar{u}(r),\underline{v}(r)+\bar{v}(r)], \tag{6}$$

$$(\lambda\odot u)=\begin{cases}[\lambda\underline{u}(r),\lambda\bar{u}(r)],\lambda\geq 0\\ [\lambda\bar{u}(r),\lambda\underline{u}(r)],\lambda<0\end{cases} \tag{7}$$

*where $r\in[0,1]$.*

The parametrization interval shape of $u(.,r)$ is computed by

$$u(.,r) = [\underline{u}(.;r), \bar{u}(.;r)], r \in [0,1]$$

For $\widehat{\rho}, \widehat{\varrho} \in \mathbb{R}_{\mathcal{F}}$, where $\mathbb{R}_{\mathcal{F}}$ is the set of all fuzzy sets. The gH difference [27] of $\widehat{\rho}$ and $\widehat{\varrho}$, as shown by $\widehat{\rho} \ominus_{gH} \widehat{\varrho}$, is expressed as

$$\widehat{\rho} \ominus_{gH} \widehat{\varrho} = \widehat{\sigma} \Longleftrightarrow \begin{cases} \texttt{(i)} \ \widehat{\rho} = \widehat{\varrho} + \widehat{\sigma} \ \texttt{or} \\ \texttt{(ii)} \ \widehat{\varrho} = \widehat{\rho} + (-1)\widehat{\sigma} \end{cases} . \tag{8}$$

In regard to $r$-cuts, we obtain

$$(\widehat{\rho} \ominus_{gH} \widehat{\varrho})^r = [\min\{\underline{\rho}(r) - \underline{\varrho}(r), \bar{\rho}(r) - \bar{\varrho}(r)\}, \max\{\underline{\rho}(r) - \underline{\varrho}(r), \bar{\rho}(r) - \bar{\varrho}(r)\}].$$

And the conditions for the existence of $\sigma = \rho \ominus_g \varrho \in \mathbb{R}_{\mathcal{F}}$ are

$$\texttt{case (i)} \quad \begin{cases} \underline{\sigma}(r) = \underline{\rho}(r) - \underline{\varrho}(r) \ \text{ and } \ \overline{\sigma}(r) = \overline{\rho}(r) - \overline{\varrho}(r) \\ \texttt{with} \ \ \underline{\sigma}(r) \ \texttt{increasing} \ , \overline{\sigma}(r) \ \texttt{decreasing} \ , \\ \underline{\sigma}(r) \leq \overline{\sigma}(r). \end{cases} \tag{9}$$

$$\texttt{case (ii)} \quad \begin{cases} \underline{\sigma}(r) = \overline{\rho}(r) - \overline{\varrho}(r) \ \text{ and } \ \overline{\sigma}(r) = \underline{\rho}(r) - \underline{\varrho}(r) \\ \texttt{with} \ \ \underline{\sigma}(r) \ \texttt{increasing} \ , \overline{\sigma}(r) \ \texttt{decreasing} \ , \\ \underline{\sigma}(r) \leq \overline{\sigma}(r). \end{cases} \tag{10}$$

for all $r \in [0,1]$.

**Definition 2.4.** *The Hausdorff distance is computed in the following manner:*

$$d : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R} \cup \{0\}$$
$$d(\widehat{a}, \widehat{b}) = \sup_{r\in[0,1]} \max\left\{ \left|\underline{a}(r) - \underline{b}(r)\right|, \left|\bar{a}(r) - \bar{b}(r)\right| \right\}.$$

*$\mathbb{R}_{\mathcal{F}}$ denotes the set of all fuzzy numbers. Let $\mathcal{T}$ represent the area of all triangular fuzzy sets within $\mathbb{R}_{\mathcal{F}}$. Therefore, $(\mathcal{T}, d)$ is a subset of $(\mathbb{R}_{\mathcal{F}}, d)$, it is a complete metric space, and the next features are widely recognized* [13].

(1) $d(\widehat{a} \oplus \widehat{c}, \widehat{b} \oplus \widehat{c}) = d(\widehat{a}, \widehat{b}), \quad \forall \widehat{a}, \widehat{b}, \widehat{c} \in \mathbb{R}_{\mathcal{F}};$

(2) $d(\widehat{a} \oplus \widehat{b}, 0) = d(\widehat{a}, 0) + d(\widehat{b}, 0), \quad \forall \widehat{a}, \widehat{b} \in \mathbb{R}_{\mathcal{F}};$

(3) $d(\widehat{a} \oplus \widehat{b}, \widehat{a} \oplus \widehat{c}) = d(\widehat{b}, \widehat{c}), \quad \forall \widehat{a}, \widehat{b}, \widehat{c} \in \mathbb{R}_{\mathcal{F}};$

(4) $d(\widehat{a} \oplus \widehat{b}, \widehat{c} \oplus \widehat{s}) \leq d(\widehat{a}, \widehat{c}) + d(\widehat{b}, \widehat{s}), \quad \forall \widehat{a}, \widehat{b}, \widehat{c}, \widehat{s} \in \mathbb{R}_{\mathcal{F}};$

(5) $d(\widehat{a} \ominus \widehat{b}, \widehat{c} \ominus \widehat{s}) \leq d(\widehat{a}, \widehat{c}) + d(\widehat{b}, \widehat{s}), \forall \widehat{a}, \widehat{b}, \widehat{c}, \widehat{s} \in \mathbb{R}_{\mathcal{F}}$, $\widehat{a} \ominus \widehat{b}, \widehat{c} \ominus \widehat{s}$ exist;

(6) $d(\lambda \odot \widehat{a}, \lambda \odot \widehat{b}) = |\lambda| d(\widehat{a}, \widehat{b}), \quad \forall \widehat{a}, \widehat{b} \in \mathbb{R}_{\mathcal{F}}, \quad \lambda \in \mathbb{R}.$

**Definition 2.5.** [26] *The gH derivative of a fuzzy value function $u$ is described as*

$$u'_{gH}(s) = \lim_{h\to 0} \frac{u(s+h) \ominus_{gH} u(s)}{h}, \tag{11}$$

*if $u'_{gH}(s) \in \mathbb{R}_{\mathcal{F}}$, we say that $u$ is gH-differentiable,*

*Also we assert that $u$ is $[(i) - gH]$-diff if*

$$\left(u'_{gH}\right)_r(s) = \left[\underline{u}'(s,r), \bar{u}'(s,r)\right], \quad 0 \leq r \leq 1 \tag{12}$$

*and that $u$ is $[(ii) - gH]$-diff, if*

$$\left(u'_{gH}\right)_r(s) = \left[\bar{u}'(s,r), \underline{u}'(s,r)\right], \quad 0 \leq r \leq 1 \tag{13}$$

**Definition 2.6.** [10] *The Caputo fractional derivative of $u$ is written as*

$$\left({}^{C}D_{*}^{\alpha}u\right)(t)=\left(I^{n-\alpha}D^{n}u\right)(t)=\frac{1}{\Gamma(\alpha)}\int_{a}^{t}(t-s)^{(n-\alpha-1)}u^{(n)}(s)ds,\quad n-1<\alpha\leq n, n\in\mathbb{N}, s>0. \tag{14}$$

*$D$ represents a typical derivative.*

In this paper, we use the syntax $C(J,\mathcal{T})$ for the space of all continuous functions. Furthermore, consider $L(J,\mathcal{T})$ as the space of Lebesgue integrable fuzzy-valued mappings from $J$ to $\mathcal{T}$.

**Definition 2.7.** [28] *Suppose $u\in L(J,\mathcal{T})$. The fuzzy Riemann Liouville integral of $u$ is defined as:*

$$\left(I_{a}^{\alpha}u\right)(t)=\frac{1}{\Gamma(\alpha)}\int_{0}^{t}\frac{u(s)ds}{(t-s)^{1-\alpha}},\quad a<s<t,\quad 0<\alpha\leq 1. \tag{15}$$

**Definition 2.8.** [13] *Letting $x\in A(J,\mathcal{T})$. The fractional gH Caputo derivative of $u$ is:*

$$\left({}_{gH}^{C}D_{*}^{\alpha}u\right)(\tau)=I_{a}^{1-\alpha}\left(u'_{gH}\right)(\tau)=\frac{1}{\Gamma(1-\alpha)}\int_{a}^{\tau}\frac{\left(u'_{gH}\right)(s)ds}{(\tau-s)^{\alpha}},\quad a<s<\tau,\quad 0<\alpha\leq 1. \tag{16}$$

*We further argue that $u$ is ${}^{C}[(i)-gH]$.-diff at $\tau_0$ if*

$$\left({}_{gH}^{C}D_{*}^{\alpha}u\right)_{r}(\tau_0)=\left[{}^{C}D_{*}^{\alpha}\underline{u}(\tau_0,\gamma),{}^{C}D_{*}^{\alpha}\bar{u}(\tau_0,r)\right],\quad 0\leq\alpha\leq 1 \tag{17}$$

*and $u$ is ${}^{C}[(ii)-gH]$-diff at $\tau_0$ if*

$$\left({}_{gH}^{C}D_{*}^{\alpha}u\right)_{r}(\tau_0)=\left[{}^{C}D_{*}^{\alpha}\bar{u}(\tau_0,r),{}^{C}D_{*}^{\alpha}\underline{u}(\tau_0,r)\right],\quad 0\leq\alpha\leq 1 \tag{18}$$

**Lemma 2.9.** [13] *Let $u\in A(J,\mathcal{T})$, we have*

$$I_{0}^{\alpha}\left({}_{gH}^{C}D_{*}^{\alpha}u\right)(\tau)=u(\tau)\ominus_{gH}u(0) \tag{19}$$

Following that, the Kuratowski measure of non compactness is defined, and some of its key aspects are examined.

**Definition 2.10.** [29] *The Kuratowski measure of non compactness (K.m.n.c) $\mathcal{M}(\cdot)$ constructed on the bound subset $\mathcal{V}$ of $E$ is indeed:*

$$\mathcal{M}(\mathcal{V}):=\inf\left\{\varepsilon>0:\mathcal{V}=\cup_{i=1}^{n}\mathcal{V}_{i}\ \textit{and}\ \operatorname{diam}(\mathcal{V}_{i})\leq\varepsilon\ \texttt{for}\ i=1,2,\ldots,n\right\}.$$

The K.m.n.c has the very next well-known features.

**Lemma 2.11.** [29] *Letting $\mathcal{E}$ be a Banach space and $\mu,\nu\subset\mathcal{E}$ be bounded. The next aspects are met:*

(1) $\mathcal{M}(\mu)\leq\mathcal{M}(\nu)$ *if* $\mu\subset\nu$*;*

(2) $\mathcal{M}(\mu)=\mathcal{M}(\bar{\mu})=\mathcal{M}(\overline{\operatorname{conv}}\mu)$

(3) $\mathcal{M}(\mu)=0$ *iff $\mu$ is relatively compact;*

(4) $\mathcal{M}(\lambda\mu)=|\lambda|\mathcal{M}(\mu)$, *where* $\lambda\in\mathbb{R}$*;*

(5) $\mathcal{M}(\mu\cup\nu)=\max\{\mathcal{M}(\mu),\mathcal{M}(\nu)\}$*;*

(6) $\mathcal{M}(\mu+\nu) \leq \mathcal{M}(\mu)+\mathcal{M}(\nu)$, *where* $\mu+\nu=\{w \mid w=m+n, m\in\mu, n\in\nu\}$;

(7) $\mathcal{M}(\mu+y)=\mathcal{M}(\mu)$, $\forall y \in E$.

**Lemma 2.12.** [30] *Let* $\mathcal{V} \subset C(I,E)$ *be a bounded and equi-continuous subset. Then, the function* $s \to \mathcal{M}(\mathcal{V}(s))$ *is continuous on* $I$, *and the following hold:*

$$\mathcal{M}_C(\mathcal{V}) = \max_{s\in I} \mathcal{M}(\mathcal{V}(s)),$$

*and*

$$\mathcal{M}\left(\int_I u(s)\,\mathrm{d}s\right) \leq \int_I \mathcal{M}(\mathcal{V}(s))\,\mathrm{d}s,$$

*where* $\mathcal{V}(s)=\{u(s): u\in\mathcal{V}\}$, *for* $s\in I$.

The preceding is a beneficial fixed point outcome for our aims:

**Theorem 2.13.** (Monch's fixed point theorem [31]) *Let* $\mathcal{Y}$ *be a convex, bounded, and closed subset of a Banach space such that* $0\in\mathcal{Y}$. *Let* $\mathcal{Z}$ *be a continuous map from* $\mathcal{Y}$ *into itself. If, for every subset* $\mathcal{V}\subset\mathcal{Y}$, *the implication*

$$\mathcal{V}=\overline{\text{conv}}\mathcal{Z}(\mathcal{V}) \quad or \quad \mathcal{V}=\mathcal{Z}(\mathcal{V})\cup\{0\} \quad \Rightarrow \quad \mathcal{M}(\mathcal{V})=0,$$

*holds, then* $\mathcal{Z}$ *has a fixed point.*

# 3 Solvability of a fuzzy hybrid differential equation

This section investigates the existence of a solution to the following hybrid differential equation (HDE) in $\mathcal{C}(J,\mathcal{T})$:

$$\frac{d}{dt}\left[\frac{u(t)}{\mathfrak{F}(t,u(t))}\right] = \Lambda(t,u(t)), \quad t\in[0,T]=J \text{ and } u(0)=\widehat{0}. \tag{20}$$

Equation (20) is equivalent to the hybrid integral equations (21) and (22), given as follows:

- If $u$ is $(i)-gH$ differentiable, then

$$u(t)=\mathfrak{F}(t,u(t))\int_0^t \Lambda(\eta,u(\eta))\,d\eta. \tag{21}$$

- If $u$ is $(ii)-gH$ differentiable, then

$$u(t)=\ominus(-1)\mathfrak{F}(t,u(t))\int_0^t \Lambda(\eta,u(\eta))\,d\eta. \tag{22}$$

**Definition 3.1.** *A **fuzzy solution of type 1** for Equation (20) refers to a function* $u\in\mathcal{C}(J,\mathcal{T})$ *that satisfies (21). Similarly, a **fuzzy solution of type 2** for (20) refers to a function* $x\in\mathcal{C}(J,\mathcal{T})$ *that satisfies (22).*

Let's define $\mathrm{P}_\mu \subset \mathcal{C}([0;1],\mathcal{T})$ as the set closed of $u\in\mathcal{T}$ such that $d(u,\widehat{0})\leq\mu$ for some $\mu>0$. Our objective is to demonstrate the existence of a fixed point for a constructed operator $\mathcal{Q}$ from (21) or (22) within the subset $\mathrm{P}_\mu$:

$$\mathrm{P}_\mu=\{u\in\mathcal{T} \mid d(u,\widehat{0})\leq\mu\}.$$

To prove the existence of a solution to (20), we require the following set of assumptions:

($\mathcal{H}_1$) $\mathfrak{F} : J \times \mathcal{T} \to \mathcal{T}$ is continuous, and there exists a constant $\xi_1 > 0$ such that

$$d\left(\mathfrak{F}(t, u(t)), \mathfrak{F}(t, v(t))\right) \leq \xi_1 d(u(t), v(t)),$$

for all $t \in J$ and $u, v \in \mathcal{T}$.

Additionally, for every $t \in J$,

$$\mathfrak{F}(t, \widehat{0}) = z_0.$$

($\mathcal{H}_2$) $\Lambda : J \times \mathcal{T} \to \mathcal{T}$ is continuous, and there exists a constant $\Lambda_1 > 0$ such that

$$d\left(\Lambda(t, u(t)), \Lambda(t, v(t))\right) \leq \Lambda_1 d(u(t), v(t)),$$

for all $t \in J$ and $u, v \in \mathcal{T}$.

Additionally, for every $t \in J$,

$$\Lambda(t, \widehat{0}) = \widehat{0}.$$

($\mathcal{H}_3$) There exists a positive value $\mu$ such that

$$T\left(\xi_1 \mu + z_0\right) \Lambda_1 < 1.$$

**Theorem 3.2.** *Under the assumptions* ($\mathcal{H}_1$)-($\mathcal{H}_3$), *the problem* (20) *has at least one solution of type 1 in* $\mathcal{C}(J, \mathcal{T})$.

**Proof.** Let $\mathcal{Q} : \mathcal{C}(J, \mathcal{T}) \to \mathcal{C}(J, \mathcal{T})$ be an operator defined as

$$(\mathcal{Q}u)(t) = \mathfrak{F}(t, u(t)) \int_0^t \Lambda(\eta, u(\eta))\, d\eta.$$

**Step (1):** We show that $\mathcal{Q}$ maps $\mathrm{P}_\mu$ into $\mathrm{P}_\mu$. Let $u \in \mathrm{P}_\mu$. Then,

$$\begin{aligned}
d\left((\mathcal{Q}u)(t), \widehat{0}\right) &\leq d\left(\mathfrak{F}(t, u(t)) \int_0^t \Lambda(\eta, u(\eta))\, d\eta, \widehat{0}\right) \\
&\leq d\left(\mathfrak{F}(t, u(t)), \widehat{0}\right) \int_0^t d\left(\Lambda(\eta, u(\eta)), \widehat{0}\right)\, d\eta \\
&\leq \left(d\left(\mathfrak{F}(t, u(t)), \mathfrak{F}(t, \widehat{0})\right) + d\left(\widehat{0}, \mathfrak{F}(t, \widehat{0})\right)\right) \int_0^t d\left(\Lambda(\eta, u(\eta)), \Lambda(\eta, \widehat{0})\right)\, d\eta \\
&\leq \left(\xi_1 d(u, \widehat{0}) + z_0\right) \int_0^t \Lambda_1 d(u, \widehat{0})\, d\eta \\
&\leq \left(\xi_1 \mu + z_0\right) t \Lambda_1 \mu \\
&\leq T\left(\xi_1 \mu + z_0\right) \Lambda_1 \mu.
\end{aligned}$$

Since $d(u, \widehat{0}) < \mu$, it follows from assumption ($\mathcal{H}_3$) that

$$d\left((\mathcal{Q}u)(t), \widehat{0}\right) < \mu.$$

Thus, $\mathcal{Q}$ maps $\mathrm{P}_\mu$ into $\mathrm{P}_\mu$.

**Step (2):** We establish the continuity of $\mathcal{Q}$ on $\mathrm{P}_\mu$. Let $\varepsilon > 0$ and $u, v \in \mathrm{P}_\mu$ such that $d(u,v) < \varepsilon$. Then,

$$\begin{aligned} d\left((\mathcal{Q}u)(t), (\mathcal{Q}v)(t)\right) &\leq d\left(\mathfrak{F}(t,u(t))\int_0^t \Lambda(\eta,u(\eta))\,d\eta, \mathfrak{F}(t,v(t))\int_0^t \Lambda(\eta,v(\eta))\,d\eta\right) \\ &\leq d\left(\mathfrak{F}(t,u(t)),\mathfrak{F}(t,v(t))\right)\int_0^t d\left(\Lambda(\eta,u(\eta)),\Lambda(\eta,v(\eta))\right)\,d\eta \\ &\leq \xi_1 d(u,v)\int_0^t \Lambda_1 d(u,v)\,d\eta \\ &\leq \xi_1\Lambda_1 T d(u,v) \\ &\leq \xi_1\Lambda_1 T\varepsilon. \end{aligned}$$

As $\varepsilon \to 0$, $d\left((\mathcal{Q}u)(t), (\mathcal{Q}v)(t)\right) \to 0$. Therefore, $\mathcal{Q}$ is continuous on $\mathrm{P}_\mu$.

**Step (3):** Let $\Omega(\neq \emptyset) \subseteq \mathrm{P}_\mu$. Let $\varpi > 0$ be arbitrary, and take $u \in \Omega$ and $t_1, t_2 \in J$ such that $d(t_2,t_1) \leq \varpi$ with $t_2 \geq t_1$. Then,

$$\begin{aligned} d\left((\mathcal{Q}u)(t_1), (\mathcal{Q}u)(t_2)\right) &\leq d\left(\mathfrak{F}(t_1,u(t_1))\int_0^{t_1} \Lambda(\eta,u(\eta))\,d\eta, \mathfrak{F}(t_2,u(t_2))\int_0^{t_2} \Lambda(\eta,u(\eta))\,d\eta\right) \\ &\leq d\left(\mathfrak{F}(t_1,u(t_1)),\mathfrak{F}(t_2,u(t_2))\right)\int_0^{t_1} d\left(\Lambda(\eta,u(\eta)),\widehat{0}\right)\,d\eta \\ &+ d\left(\mathfrak{F}(t_1,u(t_2)),\widehat{0}\right)\int_{t_1}^{t_2} d\left(\Lambda(\eta,u(\eta)),\widehat{0}\right)\,d\eta \\ &\leq \xi_1 d(u(t_1),u(t_2))T\Lambda_1\mu + d\left(\mathfrak{F}(t_1,u(t_2)),\widehat{0}\right)\varpi\Lambda_1\mu \\ &+ d\left(\mathfrak{F}(t_2,u(t_2)),\mathfrak{F}(t_1,u(t_2))\right)T\Lambda_1\mu. \end{aligned}$$

Let

$$\mathcal{M}(u,\varpi) = \sup\left\{d(u(t_1),u(t_2)) \mid d(t_2,t_1) \leq \varpi; t_1,t_2 \in J\right\}.$$

Since $\mathfrak{F}$ is continuous, $d\left(\mathfrak{F}(t_2,u(t_2)),\mathfrak{F}(t_1,u(t_2))\right) \to 0$ as $\varpi \to 0$. Therefore,

$$\mathcal{M}(\mathcal{Q},\varpi) \leq \xi_1 T\Lambda_1\mu\mathcal{M}(u,\varpi) + \sup_{d(t_2,t_1)\leq\varpi} d\left(\mathfrak{F}(t_1,u(t_2)),\widehat{0}\right)\varpi\Lambda_1\mu + \sup_{d(t_2,t_1)\leq\varpi} d\left(\mathfrak{F}(t_2,u(t_2)),\mathfrak{F}(t_1,u(t_2))\right)T\Lambda_1\mu.$$

As $\varpi \to 0$, taking the supremum over $u \in \Omega$, we obtain

$$\mathcal{M}_0(\Omega) \leq \xi_1\Lambda_1 T\mu\mathcal{M}_0(\Omega).$$

Hence, $\mathcal{M}(\Omega(t)) \leq \mathcal{M}(\Omega(t)) = 0$, which implies that $\Omega(t)$ is relatively compact in $\mathrm{P}_\mu$. By the Ascoli-Arzela theorem, $\Omega$ is relatively compact in $\mathrm{P}_\mu$. Based on Theorem 2.13, there exists a fixed point $u$ of $\mathcal{Q}$ in $\Omega \subseteq \mathrm{P}_\mu$, i.e., Equation (20) has a solution in $\mathcal{C}(J,\mathcal{T})$. This completes the proof. □

**Remark 3.3.** *The same procedure can be applied to the other case* (22).

# 4 Solvability of fuzzy fractional hybrid differential equation

In this section, the existence of a solution to a type of fuzzy fractional hybrid differential equation (FFHDE) is demonstrated.

Consider the following equation:

$$
\begin{aligned}
&{}^{C}_{gH}D^{\alpha}_{*}\left[\frac{u(t)}{\mathfrak{F}(t,u(t))}\right] = \Lambda(t,u(t)), \quad 0<\alpha<1, \varrho \in J=[0,T],\\
&\text{and } u(0)=\widehat{0}.
\end{aligned} \tag{23}
$$

Equation (23) corresponds to the following hybrid integral equations:

- If $u$ is $(i)-gH$ differentiable, then

$$
u(t) = \frac{\mathfrak{F}(t,u(t))}{\Gamma(\alpha)} \int_0^t (t-\eta)^{\alpha-1}\Lambda(\eta,u(\eta))\,d\eta. \tag{24}
$$

- If $u$ is $(ii)-gH$ differentiable, then

$$
u(t) = \ominus(-1)\frac{\mathfrak{F}(t,u(t))}{\Gamma(\alpha)} \int_0^t (t-\eta)^{\alpha-1}\Lambda(\eta,u(\eta))\,d\eta. \tag{25}
$$

**Definition 4.1.** *A **fuzzy solution of type 1** for Equation (23) refers to a function $u \in \mathcal{C}(J,\mathcal{T})$ that satisfies (24). Similarly, a **fuzzy solution of type 2** for (23) refers to a function $x \in \mathcal{C}(J,\mathcal{T})$ that satisfies (25).*

Let's define $\mathrm{P}_\mu \subset \mathcal{C}([0;1],\mathcal{T})$ as the set closed of $u \in \mathcal{T}$ such that $d(u,\widehat{0}) \le \mu$ for some $\mu > 0$. Our objective is to demonstrate the existence of a fixed point for a constructed operator $\mathcal{Q}$ from (24) or (25) within the subset $\mathrm{P}_\mu$:

$$
\mathrm{P}_\mu = \{u \in \mathcal{T} \mid d(u,\widehat{0}) \le \mu\}.
$$

To prove the existence of a solution to (24), we require the following assumptions:

($\mathcal{H}_1$) $\mathfrak{F} : J \times \mathcal{T} \to \mathcal{T}$ is continuous, and there exists a constant $\xi_1 > 0$ such that

$$
d\left(\mathfrak{F}(t,u(t)),\mathfrak{F}(t,v(t))\right) \le \xi_1 d(u(t),v(t)),
$$

for all $t \in J$ and $u,v \in \mathcal{T}$.

Additionally, for every $t \in J$,

$$
\mathfrak{F}(t,\widehat{0}) = z_0 > 0.
$$

($\mathcal{H}_2$) $\Lambda : J \times \mathcal{T} \to \mathcal{T}$ is continuous, and there exists a constant $\Lambda_1 > 0$ such that

$$
d\left(\Lambda(t,u(t)),\Lambda(t,v(t))\right) \le \Lambda_1 d(u(t),v(t)),
$$

for all $t \in J$ and $u,v \in \mathcal{T}$.

Additionally, for every $t \in J$,

$$
\Lambda(t,\widehat{0}) = \widehat{0}.
$$

($\mathcal{H}_3$) There exists a positive number $\mu$ such that

$$
\frac{T\,(\xi_1\mu + z_0)\,\Lambda_1}{\Gamma(\alpha+1)} < 1.
$$

**Theorem 4.2.** *Under the assumptions ($\mathcal{H}_1$)-($\mathcal{H}_3$), the problem (23) has at least one solution of type 1 in $\mathcal{C}(J,\mathcal{T})$.*

**Proof.** Consider the operator $\mathcal{Q}:\mathcal{C}(J,\mathcal{T})\to\mathcal{C}(J,\mathcal{T})$ defined as

$$(\mathcal{Q}u)(t)=\frac{\mathfrak{F}(t,u(t))}{\Gamma(\alpha)}\int_0^t(t-\eta)^{\alpha-1}\Lambda(\eta,u(\eta))\,d\eta.$$

**Step (1):** We show that $\mathcal{Q}$ maps $\mathrm{P}_\mu$ into $\mathrm{P}_\mu$. Let $u\in\mathrm{P}_\mu$. Then,

$$\begin{aligned}
d\left((\mathcal{Q}u)(t),\widehat{0}\right)&\le d\left(\frac{\mathfrak{F}(t,u(t))}{\Gamma(\alpha)}\int_0^t(t-\eta)^{\alpha-1}\Lambda(\eta,u(\eta))\,d\eta,\widehat{0}\right)\\
&\le\frac{1}{\Gamma(\alpha)}d\left(\mathfrak{F}(t,u(t)),\widehat{0}\right)\int_0^t(t-\eta)^{\alpha-1}d\left(\Lambda(\eta,u(\eta)),\widehat{0}\right)\,d\eta\\
&\le\frac{1}{\Gamma(\alpha)}\left(d\left(\mathfrak{F}(t,u(t)),\mathfrak{F}(t,\widehat{0})\right)+d\left(\widehat{0},\mathfrak{F}(t,\widehat{0})\right)\right)\int_0^t(t-\eta)^{\alpha-1}d\left(\Lambda(\eta,u(\eta)),\Lambda(\eta,\widehat{0})\right)\,d\eta\\
&\le\frac{1}{\Gamma(\alpha)}\left(\xi_1 d(u,\widehat{0})+z_0\right)\int_0^t(t-\eta)^{\alpha-1}\Lambda_1 d(u,\widehat{0})\,d\eta\\
&\le\frac{1}{\Gamma(\alpha)}\left(\xi_1\mu+z_0\right)\Lambda_1\mu\int_0^t(t-\eta)^{\alpha-1}\,d\eta\\
&\le\frac{1}{\Gamma(\alpha+1)}\left(\xi_1\mu+z_0\right)t^\alpha\Lambda_1\mu\\
&\le\frac{T^\alpha\left(\xi_1\mu+z_0\right)\Lambda_1}{\Gamma(\alpha+1)}\mu.
\end{aligned}$$

Since $d(u,\widehat{0})<\mu$, it follows from assumption $(\mathcal{H}_3)$ that

$$d\left((\mathcal{Q}u)(t),\widehat{0}\right)<\mu.$$

Thus, $\mathcal{Q}$ maps $\mathrm{P}_\mu$ into $\mathrm{P}_\mu$.

**Step (2):** We establish the continuity of $\mathcal{Q}$ on $\mathrm{P}_\mu$. Let $\varepsilon>0$ and $u,v\in\mathrm{P}_\mu$ such that $d(u,v)<\varepsilon$. Then,

$$\begin{aligned}
d\left((\mathcal{Q}u)(t),(\mathcal{Q}v)(t)\right)&\le d\left(\frac{\mathfrak{F}(t,u(t))}{\Gamma(\alpha)}\int_0^t(t-\eta)^{\alpha-1}\Lambda(\eta,u(\eta))\,d\eta,\frac{\mathfrak{F}(t,v(t))}{\Gamma(\alpha)}\int_0^t(t-\eta)^{\alpha-1}\Lambda(\eta,v(\eta))\,d\eta\right)\\
&\le\frac{1}{\Gamma(\alpha)}d\left(\mathfrak{F}(t,u(t)),\mathfrak{F}(t,v(t))\right)\int_0^t(t-\eta)^{\alpha-1}d\left(\Lambda(\eta,u(\eta)),\Lambda(\eta,v(\eta))\right)\,d\eta\\
&\le\frac{1}{\Gamma(\alpha)}\xi_1 d(u,v)\int_0^t(t-\eta)^{\alpha-1}\Lambda_1 d(u,v)\,d\eta\\
&\le\frac{1}{\Gamma(\alpha+1)}\xi_1\Lambda_1 t^\alpha d(u,v)\\
&\le\frac{1}{\Gamma(\alpha+1)}\xi_1\Lambda_1 T^\alpha\varepsilon.
\end{aligned}$$

As $\varepsilon\to0$, $d\left((\mathcal{Q}u)(t),(\mathcal{Q}v)(t)\right)\to0$. Therefore, $\mathcal{Q}$ is continuous on $\mathrm{P}_\mu$.

**Step (3):** Let $\Omega(\neq\emptyset)\subseteq\mathrm{P}_\mu$. Let $\varpi>0$ be arbitrary, and take $u\in\Omega$ and $t_1,t_2\in J$ such that $d(t_2,t_1)\le\varpi$

with $t_2 \geq t_1$. Then,

$$\begin{aligned}
&d\left((\mathcal{Q}u)(t_1),(\mathcal{Q}u)(t_2)\right)\\
&\leq d\left(\frac{1}{\Gamma(\alpha)}\mathfrak{F}(t_1,u(t_1))\int_0^{t_1}(t_1-\eta)^{\alpha-1}\Lambda(\eta,u(\eta))\,d\eta,\frac{1}{\Gamma(\alpha)}\mathfrak{F}(t_2,u(t_2))\int_0^{t_2}(t_2-\eta)^{\alpha-1}\Lambda(\eta,u(\eta))\,d\eta\right)\\
&\leq \frac{1}{\Gamma(\alpha)}d\left(\mathfrak{F}(t_1,u(t_1)),\mathfrak{F}(t_2,u(t_2))\right)\int_0^{t_1}(t_1-\eta)^{\alpha-1}d\left(\Lambda(\eta,u(\eta)),\widehat{0}\right)\,d\eta\\
&+\frac{1}{\Gamma(\alpha)}d\left(\mathfrak{F}(t_1,u(t_2)),\widehat{0}\right)\int_{t_1}^{t_2}(t_2-\eta)^{\alpha-1}d\left(\Lambda(\eta,u(\eta)),\widehat{0}\right)\,d\eta\\
&\leq \frac{1}{\Gamma(\alpha+1)}\xi_1 d(u(t_1),u(t_2))T^\alpha\Lambda_1\mu+\frac{1}{\Gamma(\alpha+1)}d\left(\mathfrak{F}(t_1,u(t_2)),\widehat{0}\right)\varpi^\alpha\Lambda_1\mu\\
&+\frac{1}{\Gamma(\alpha+1)}d\left(\mathfrak{F}(t_2,u(t_2)),\mathfrak{F}(t_1,u(t_2))\right)T^\alpha\Lambda_1\mu.
\end{aligned}$$

Let

$$\mathcal{M}(u,\varpi)=\sup\left\{d(u(t_1),u(t_2))\mid d(t_2,t_1)\leq\varpi;t_1,t_2\in J\right\}.$$

Since $\mathfrak{F}$ is continuous, $d\left(\mathfrak{F}(t_2,u(t_2)),\mathfrak{F}(t_1,u(t_2))\right)\to 0$ as $\varpi\to 0$. Therefore,

$$\begin{aligned}
\mathcal{M}(\mathcal{Q},\varpi)\leq \frac{1}{\Gamma(\alpha+1)}\xi_1 T^\alpha\Lambda_1\mu\mathcal{M}(u,\varpi)+\sup_{d(t_2,t_1)\leq\varpi}\frac{1}{\Gamma(\alpha+1)}d\left(\mathfrak{F}(t_1,u(t_2)),\widehat{0}\right)\varpi^\alpha\Lambda_1\mu\\
+\sup_{d(t_2,t_1)\leq\varpi}\frac{1}{\Gamma(\alpha+1)}d\left(\mathfrak{F}(t_2,u(t_2)),\mathfrak{F}(t_1,u(t_2))\right)T^\alpha\Lambda_1\mu.
\end{aligned}$$

As $\varpi\to 0$, taking the supremum over $u\in\Omega$, we obtain

$$\mathcal{M}_0(\Omega)\leq\frac{1}{\Gamma(\alpha+1)}\xi_1\Lambda_1 T^\alpha\mu\mathcal{M}_0(\Omega).$$

Hence, $\mathcal{M}(\Omega(t))\leq\mathcal{M}(\Omega(t))=0$, which implies that $\Omega(t)$ is relatively compact in $\mathrm{P}_\mu$. By the Ascoli-Arzela theorem, $\Omega$ is relatively compact in $\mathrm{P}_\mu$. Based on Theorem 2.13, there exists a fixed point $u$ of $\mathcal{Q}$ in $\Omega\subseteq\mathrm{P}_\mu$, i.e., Equation (23) has at least one solution in $\mathcal{C}(J,\mathcal{T})$. This completes the proof. □

**Remark 4.3.** *The same procedure can be applied to the other case* (25).

## 5 Examples

The following example demonstrates Theorem 3.2.

**Example 5.1.** Consider the fuzzy hybrid differential equation (FHDE):

$$\frac{d}{dt}\left[\frac{u(t)\left(1+t^2\right)}{3u(t)+1}\right]=\frac{t^3u(t)}{4+t^2},\quad \text{where } t\in J=[0,1] \text{ and } u(0)=\widehat{0}. \tag{26}$$

Here,

$$\mathfrak{F}(t,u)=\frac{3u+1}{1+t^2},$$

$$b=1,$$

and

$$\Lambda(t,u) = \frac{t^3 u}{4+t^2}.$$

It is clear that $\mathfrak{F}$ is continuous and satisfies

$$d\left(\mathfrak{F}(t,u(t)),\mathfrak{F}(t,v(t))\right) \leq \xi_1 d(u(t),v(t)),$$

for all $t \in J$ and $u,v \in \mathcal{T}$. Therefore, $\xi_1 = 3$ and $z_0 = \mathfrak{F}(t,0) = \frac{1}{1+t^2} \leq 1$.

The function $\Lambda$ is also continuous and satisfies

$$d\left(\Lambda(t,u(t)),\Lambda(t,v(t))\right) \leq \frac{1}{4} d(u(t),v(t)),$$

for all $t \in J$ and $u,v \in \mathcal{T}$. Thus, $\Lambda(t,0) = 0$ and $\Lambda_1 = \frac{1}{4}$.

From the inequality in assumption $(\mathcal{H}_3)$, we have

$$b\left(\xi_1\mu + z_0\right)\Lambda_1 = 1 \cdot (3\mu + z_0) \cdot \frac{1}{4} \leq \frac{3\mu+1}{4} < 1.$$

This implies $\mu < 1$. The hypothesis $(\mathcal{H}_3)$ is satisfied for $\mu = \frac{1}{2}$.

Therefore, all the hypotheses $(\mathcal{H}_1)$-$(\mathcal{H}_3)$ of Theorem 3.2 are satisfied. Hence, Equation (26) has a solution in $\mathcal{C}(J,\mathcal{T})$.

The following example demonstrates Theorem 4.2.

**Example 5.2.** Consider the fuzzy fractional hybrid differential equation (FFHDE):

$$ {}^{C}_{gH}D_*^{\frac{1}{4}}\left[\frac{u(t)\left(1+t^2\right)}{3u(t)+1}\right] = \frac{t^3 u(t)}{4+t^2}, \quad \text{where } t \in J = [0,1] \text{ and } u(0) = \widehat{0}. \tag{27}$$

Here,

$$\mathfrak{F}(t,u) = \frac{3u+1}{1+t^2},$$

$$b = 1, \quad \alpha = \frac{1}{4},$$

and

$$\Lambda(t,u) = \frac{t^3 u}{4+t^2}.$$

The function $\mathfrak{F}$ is clearly continuous and satisfies

$$d\left(\mathfrak{F}(t,u(t)),\mathfrak{F}(t,v(t))\right) \leq \xi_1 d(u(t),v(t)),$$

for all $t \in J$ and $u,v \in \mathcal{T}$. Therefore, $\xi_1 = 3$ and $z_0 = \mathfrak{F}(t,0) = \frac{1}{1+t^2} \leq 1$.

The function $\Lambda$ is also continuous and satisfies

$$d\left(\Lambda(t,u(t)),\Lambda(t,v(t))\right) \leq \frac{1}{4} d(u(t),v(t)),$$

for all $t \in J$ and $u,v \in \mathcal{T}$. Thus, $\Lambda(t,0) = 0$ and $\Lambda_1 = \frac{1}{4}$.

From the inequality in assumption $(\mathcal{H}_3)$, we have

$$\frac{b\left(\xi_1\mu + z_0\right)\Lambda_1}{\Gamma(\alpha+1)} = \frac{1 \cdot (3\mu + z_0)}{4\Gamma(\alpha+1)} \leq \frac{3\mu+1}{4\Gamma\left(\frac{5}{4}\right)} < 1.$$

This implies $\mu < \frac{4\Gamma\left(\frac{5}{4}\right)-1}{3} \approx 0.87$. The hypothesis $(\mathcal{H}_3)$ is satisfied for $\mu = \frac{1}{2}$.

Therefore, all the hypotheses $(\mathcal{H}_1)$-$(\mathcal{H}_3)$ of Theorem 4.2 are satisfied. Hence, Equation (27) has a solution in $\mathcal{C}(J,\mathcal{T})$.

## 6 Conclusion

Fuzzy sets may exhibit two types of generalized differentiability. In fuzzy fractional hybrid differential equations (FFHDEs), a set can be either $(i)$-differentiable or $(ii)$-differentiable. This duality often introduces complexities when solving such equations.

In this study, we focus on hybrid differential equations and fractional hybrid differential equations. Initially, we establish existence theorems for fuzzy solutions under certain weaker constraints by employing the measure of non-compactness (m.n.c.) and Mnch's fixed-point theorem. The insights presented here extend and refine several previously established results. Finally, illustrative examples are provided to validate the findings.

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