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Comparison Between Fuzzy Number Sequences via Interactive Arithmetic J_0 and Standard Arithmetic

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(This paper is dedicated to Professor "John N. Mordeson" on the occasion of his 91st birthday.)

Abstract. The focus of this work is to study sequences of interactive fuzzy numbers. The interactivity relation is associated with the concept of joint possibility distribution. In this case, the type of interactivity studied is linked to a family of joint possibility distributions (J_γ), in which the parameter γ intrinsically models levels of interactivity between the fuzzy numbers involved. Each element of the sequence of interactive fuzzy numbers is obtained through a discrete equation, and the arithmetic operations present in the equation are extended to this type of fuzzy number. Some simulations are performed to illustrate the behavior of the sequences, called interactive, and to compare them with the sequences obtained by other fuzzy arithmetic operations.

AMS Subject Classification 2020: MSC 39A26; MSC 03B52; MSC 94D05
Keywords and Phrases: Interactive fuzzy numbers, Fuzzy number sequence, Fuzzy discrete equations, Sup-J extension principle.

1 Introduction

Real number sequences have been a subject of study within Mathematics, particularly in the field of Analysis. In this context, the domain of the function that generates the sequence is the set of natural numbers ($\mathbb{N}$) and the image is defined as the set of real numbers ($\mathbb{R}$). In addition, one can define a sequence of real numbers by writing the current value in terms of its predecessors. This type of sequence is also known as a recursive sequence, which must be started from one or more initial conditions, as occurs, for instance, in the Fibonacci and plant growth sequences [1].

There are several well-known real sequences, such as the Lucas sequence, in which the real sequence is the same as in the Fibonacci sequence, but the initial values differ. Also, there is the arithmetic sequence (each term is the sum of the previous term and a constant difference), geometric sequence (each term is the product of the previous term and a constant ratio), triangular number sequence (each term can be arranged in an equilateral triangle), and many others.

This work focuses on the study of an extension of recursive sequences, in the following sense: the domain of the function that generates the sequence remains the set of natural numbers, but its values lie in the set of fuzzy numbers ($\mathbb{R}_\mathcal{F}$). Such sequences are known as fuzzy sequences, and the motivation for working with this approach is based on the uncertainty in determining an exact value for the initial conditions of a recursive sequence, as seen in population dynamics [2].

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In this particular work, the fuzzy sequences, considered here, are given in the form of

$$X_n = f(X_0, X_1, \dots, X_{n-1}),$$

where $f : \mathbb{R}^n_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ is a linear fuzzy function. An example of this type of fuzzy sequence is given by $X_n = 3X_{n-1} + 2X_{n-2}$. An example of a fuzzy sequence that is not of this type is given by $X_n = X_{n-1} * X_{n-2}$.

For this purpose, the initial conditions of the recursive sequence must be given by fuzzy numbers, consequently, the operations involved in obtaining the n-th value of the sequence, in terms of the previous $n-1$ values, must be appropriated for fuzzy numbers. In the literature, there are various arithmetics for fuzzy numbers. This work will explore only two: the standard arithmetic and the interactive arithmetic.

The standard arithmetic is considered because it is the most common arithmetic operation used in the literature. Moreover, several properties of this arithmetic are well known. For example, it is always possible to compute the standard sum between fuzzy numbers; it is a commutative and associative operation, but it does not satisfy the opposite element; it always produces a fuzzy number with a bigger width than each width of the operands; and so on [2].

On the other hand, the choice of interactive arithmetic arises from the fact that the n-th term of the sequence depends on its predecessors. This dependence is intrinsically modeled by the concept of interactivity [3]. Interactivity is a fuzzy relation that emerges from a joint possibility distribution between fuzzy numbers. This relation is similar, but not equivalent, to the concept of dependence for random variables.

In the context of interactivity, there are several arithmetic operations proposed in the literature, all of which incorporate this relation. Carlsson and Fuller [4] proposed an addition (subtraction) for fuzzy numbers that assumes a linear correlation between the fuzzy numbers. Barros and Santo Pedro [5] explored these operations by proposing a fuzzy derivative. Wasques et al. [6] showed that Hukuara difference and its generalizations incorporate the relation of interactivity, which means that several papers in the literature use the relation of interactivity implicitly or explicitly since these fuzzy differences are widely considered in the fuzzy set theory.

This work addresses fuzzy number sequences that incorporate the interactivity relation, illustrating their advantages over using usual arithmetic for fuzzy numbers. The paper is organized as follows. Section 2 provides the mathematical background for the fuzzy sets theory and the construction of the interactive sum J_0. Section 3 explores fuzzy number sequences with different types of arithmetic operations. Section 4 presents the conclusion of the paper.

2 Mathematical Background

A fuzzy subset A of a universe X is characterized by a membership function $\mu_A : X \to [0,1]$, where $\mu_A(x)$, or simply $A(x)$, indicates the degree to which $x \in X$ belongs to A. Every classical subset A of X is, in particular, a fuzzy set, as it can be described by the characteristic function $\chi_A : X \to \{0,1\}$, which is a particular case of a membership function. One way to handle fuzzy sets computationally is through α-cuts, defined by $[A]^\alpha = \{x \in X : A(x) \geq \alpha\}$ if $0 < \alpha \leq 1$ and $[A]^\alpha = \overline{\{x \in X : A(x) > 0\}}$ if $\alpha = 0$, where $\overline{Y}$ represents the closure of the set $Y \subseteq X$.

The set of fuzzy numbers, denoted by $\mathbb{R}_{\mathcal{F}}$, is formed by fuzzy subsets of $\mathbb{R}$ whose α-cuts are non-empty, bounded, closed, and nested intervals for all $\alpha \in [0,1]$. These α-cuts are denoted by $[A]^\alpha = [a^-_\alpha, a^+_\alpha]$, $\forall \alpha \in [0,1]$ [2]. The set of fuzzy numbers with continuous endpoints $a^-_{(\cdot)}, a^+_{(\cdot)} : [0,1] \to \mathbb{R}$ is denoted by $\mathbb{R}_{\mathcal{F}_C}$. An example of this type of fuzzy number is the triangular fuzzy number, denoted by triple $(a; b; c)$, with $a \leq b \leq c$, and characterized by the α-cuts $[a + \alpha(b-a), c + \alpha(b-c)]$. The width of a fuzzy number A is defined by $width(A) = |a^+_0 - a^-_0|$ [2].

Let A and B be fuzzy numbers. The Pompeiu-Hausdorff distance $D_\infty : \mathbb{R}_\mathcal{F} \times \mathbb{R}_\mathcal{F} \to [0, +\infty)$ is given by

$$D_\infty(A, B) = \sup_{\alpha \in [0,1]} (\max\{|a_\alpha^- - b_\alpha^-|, |a_\alpha^+ - b_\alpha^+|\}), \ \forall A, B \in \mathbb{R}_\mathcal{F}.$$

A sequence of fuzzy numbers is defined by a function $F : \mathbb{N} \to \mathbb{R}_\mathcal{F}$. This sequence is denoted by X_n, where X_n represents the value $F(n)$ and X_n is referred to as the n-th term of the sequence, that is, $F(n) = X_n$, for all $n \in \mathbb{N}$. A sequence X_n converges to X_p if for every $\epsilon > 0$, there exists n_0 such that $D_\infty(X_n, X_p) < \epsilon$, for all $n > n_0$.

A fuzzy relation $J \in \mathcal{F}(\mathbb{R}^2)$ is said to be a joint possibility distribution between the fuzzy numbers $A_1, A_2 \in \mathbb{R}_\mathcal{F}$ if

$$A_i(y) = \sup_{\{(x_1,x_2) \in \mathbb{R}^2 \, : \, x_i = y\}} J(x_1, x_2),$$

for all $y \in \mathbb{R}$ and $\forall i = 1, 2$.

This means that A_1 and A_2 can be obtained by the projection of J in x and y direction, respectively. The fuzzy numbers A_1 and A_2 are also called be the marginals of J.

Let $A_1, A_2 \in \mathbb{R}_\mathcal{F}$ and let J be a joint possibility distribution between them. The fuzzy numbers A_1 and A_2 are said to be non-interactive if

$$J(x_1, x_2) = J_{\min}(x_1, x_2) = \min\{A_1(x_1), A_2(x_2)\}, \quad \forall (x_1, x_2) \in \mathbb{R}^2.$$

Otherwise, that is, if $J \neq J_{\min}$, then A_1 and A_2 are said to be interactive fuzzy numbers.

The above definition states that the concept of interactivity between fuzzy numbers arises from the notion of joint possibility distribution. This idea is similar (but not equivalent) to the definition of dependence in the case of random variables, that is, the relation of dependence is similar to interactivity and independence is similar to non-interactivity.

There are different types of interactivity associated with various joint possibility distributions, such as interactivity via J_L [4, 5, 7]. This joint possibility distribution establishes a linear correlation between the membership functions of the involved fuzzy numbers, which restricts the applicability of J_L [8, 9]. For example, the joint possibility distribution J_L can not be applied for the pair of fuzzy numbers A_1 and A_2, where A_1 is a triangular symmetric fuzzy number (for example $A_1 = (1; 2; 3)$) and A_2 is a triangular non-symmetric fuzzy number (for example $A_1 = (1; 2; 4)$).

The following joint possibility distribution does not have such restrictions. Specifically, it can be applied to any pair of fuzzy numbers in $\mathbb{R}_{\mathcal{F}C}$. Given $A_1, A_2 \in \mathbb{R}_{\mathcal{F}C}$, for each $z \in \mathbb{R}$ and $\alpha \in [0, 1]$, consider the functions [10]:

$$g_1(z, \alpha) = \min_{w \in [A_2]^\alpha} |w + z|, \qquad \text{and} \quad g_2(z, \alpha) = \max_{w \in [A_1]^\alpha} |w + z|. \tag{1}$$

Also consider the sets R_α^i and $L^i(z, \alpha)$ defined as follows:

$$R_\alpha^i = \begin{cases} \{a_{i_\alpha}^-, a_{i_\alpha}^+\} & \text{if } \alpha \in [0, 1) \\ [A_i]^1 & \text{if } \alpha = 1 \end{cases},$$

and $L^i(z, \alpha) = [A_{3-i}]^\alpha \cap [-g_i(z, \alpha) - z, g_i(z, \alpha) - z]$, with $i = \{1, 2\}$.

The joint possibility distribution J_0 is defined by the following membership function [10]

$$J_0(x_1, x_2) = \begin{cases} \min\{A_1(x_1), A_2(x_2)\}, & \text{if } (x_1, x_2) \in P \\ 0 \quad , & \text{otherwise} \end{cases}, \tag{2}$$

where

$$P = \bigcup_{i=1}^{2} \bigcup_{\alpha \in [0,1]} P^i(\alpha) \quad \text{with} \quad P^i(\alpha) = \{(x_1, x_2) : x_i \in R^i_\alpha \text{ e } x_{3-i} \in L^i(x_i, \alpha)\}.$$

The following definition is a generalization of Zadeh's extension principle [11], which aims to extend real functions to fuzzy functions. Let $J \in \mathcal{F}(\mathbb{R}^n)$ be a joint possibility distribution of $A_1, ..., A_n \in \mathbb{R}_{\mathcal{F}}$ and $f : \mathbb{R}^n \to \mathbb{R}$. The sup-$J$ extension of the function f applied to $(A_1, ..., A_n)$ is defined by

$$f_J(A_1, ..., A_n)(y) = \sup_{(x_1,...,x_n) \in f^{-1}(y)} J(x_1, ..., x_n),$$

where $f^{-1}(y) = \{(x_1, ..., x_n) \in \mathbb{R}^n : f(x_1, ..., x_n) = y\}$.

Through the sup-J extension principle, the arithmetic between interactive fuzzy numbers is obtained. For example, the interactive sum and difference between A_1 and A_2 is defined as follows:

$$(A_1 +_J A_2)(y) = \sup_{x_1+x_2=y} J(x_1, x_2) \quad \text{and} \quad (A_1 -_J A_2)(y) = \sup_{x_1-x_2=y} J(x_1, x_2),$$

where J is an arbitrary joint possibility distribution.

Definition 2.1. [6] Let $A, B \in \mathbb{R}_{\mathcal{F}_C}$. The interactive fuzzy sum defined by

$$(A_1 +_0 A_2)(y) = \sup_{x_1+x_2=y} J_0(x_1, x_2) \tag{3}$$

is called the J_0-sum.

The J_0-sum for triangular fuzzy numbers can be easily computed according to the following theorem.

Theorem 2.2. *[12] Let $A = (a; b; c)$ and $B = (d; e; f)$ be triangular fuzzy numbers. Let J_0 be the joint possibility distribution between A and B, given by (2). Thus*

$$A +_0 B = \begin{cases} ((a+f) \wedge (b+e); b+e; (b+e) \vee (c+d)), & \text{if } width(A) \geq width(B) \\ ((c+d) \wedge (b+e); b+e; (b+e) \vee (a+f)), & \text{if } width(A) \leq width(B) \end{cases}. \tag{4}$$

For example, the J_0-sum between $A = (1; 2; 3)$ and $B = (0; 2; 4)$ is equal to

$$A +_0 B = (\min\{3+0, 2+2\}; 2+2; \max\{1+4, 2+2\}) = (3; 4; 5).$$

On the other hand, the usual sum is given by

$$A + B = (1+0; 2+2; 3+4) = (1; 4; 7),$$

which has a bigger width than $(3; 4; 5)$.

Also, the subtraction operator can be defined in a similar way.

Definition 2.3. Let $A, B \in \mathbb{R}_{\mathcal{F}}$. The usual fuzzy difference is defined by

$$(A_1 - A_2)(y) = \sup_{x_1-x_2=y} \min\{A_1(x_1), A(x_2)\}. \tag{5}$$

Definition 2.4. [6] Let $A, B \in \mathbb{R}_{\mathcal{F}_C}$. The interactive fuzzy difference defined by

$$(A_1 -_I A_2)(y) = \sup_{x_1-x_2=y} J_0(x_1, x_2) \tag{6}$$

is called the I-difference.

For example, the I-difference between $A = (1; 3; 4)$ and $B = (1; 2; 3)$ is equal to

$$A -_I B = A +_0 (-B) = (\min\{1 + (-1), 3 + (-2)\}; 3 + (-2); \max\{4 + (-3), 3 + (-2)\}) = (0; 1; 1).$$

On the other hand, the usual sum is given by

$$A - B = A + (-B) = (1 - 3; 3 - 2; 4 - 1) = (-2; 1; 3),$$

which has bigger width than $(0; 1; 1)$. Also, note that

$$A -_I A = A +_0 (-A) = (a; b; c) +_0 (-c; -b; -a) = (0; 0; 0),$$

for all triangular fuzzy numbers A. Indeed, this result holds for any fuzzy number, that is, $A -_I A = 0$ for all $A \in \mathbb{R}_{\mathcal{F}_\mathcal{C}}$ [6].

The next section discusses sequences that are obtained through a discrete equation, where the arithmetic operations involved in the equation are given by interactive arithmetic operations.

3 Fuzzy Number Sequence

The sequences that will be considered here are obtained recurrently, that is, each term $x_n \in \mathbb{R}$ of the sequence is given as a function of the previous terms $x_1, \ldots, x_{n-1}$ from one or more initial conditions. For example, the sequence defined by the Equation (7)

$$x_n = x_{n-1} - r x_{n-2}, \tag{7}$$

where $r \in \mathbb{R}$, with initial conditions x_1 and x_2.

Taking the value of $r = 0.25$ and initial conditions $x_1 = x_2 = 1$, this sequence assumes the following values $\{1; 1; 0.75; 0.5; 0.3125; 0.1875; \ldots\}$, converging to 0.

Considering that the initial conditions are uncertain and given by fuzzy numbers, the sequence given in (7) is extended by the following fuzzy numbers sequence

$$X_n = X_{n-1} \ominus r X_{n-2}, \tag{8}$$

where $r \in \mathbb{R}$, with X_1 and X_2 being fuzzy numbers, and the operation $\ominus$ is a difference between fuzzy numbers.

Two cases will be analyzed here. The first one is when the fuzzy initial conditions are non-interactive, in this case, the usual difference must be considered. In the second case the fuzzy initial conditions are interactive, and thus, an interactive difference must be taken into account.

3.1 Usual Arithmetic Sequence

For the usual difference, we have the following sequence

$$X_n = X_{n-1} - r X_{n-2}. \tag{9}$$

Taking the initial conditions $X_1 = X_2 = (0; 1; 2)$ and $r = 0.25$, we obtain the following sequence of fuzzy numbers represented in Figure 1. Figure 2 shows the 16-th term X_{16} computed in this sequence.

Each element of the sequence X_n given in (9) can be found in Table 1. Note that the width of X_n, that is, the size of the 0-cut of X_n, is increasing with n. This implies that the uncertainty about the elements increases as n increases, this behavior is connected to the usual arithmetic.

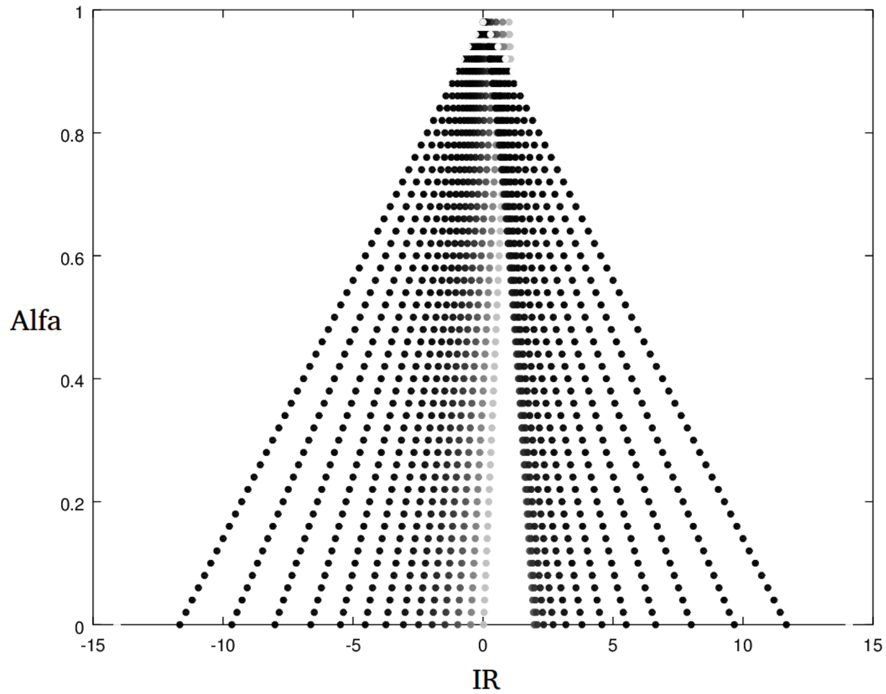

Figure 1: Fuzzy number sequence given by Equation (9) for $n = 16$. Each element of the sequence X_n is represented in shades of gray, with X_1 described by the lightest shade, and X_{16} by the darkest shade.

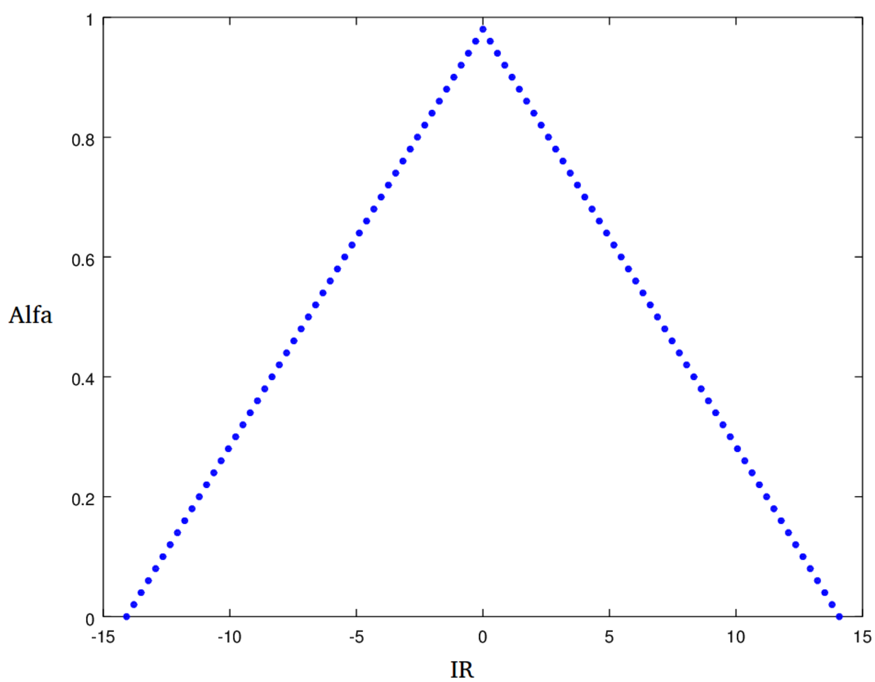

Figure 2: $X_{16} = (-14.080791; 0.00048828; 14.0818)$.

Note that, if the initial conditions are given by triangular fuzzy numbers, then the $(n-1)$-ary term of

Table 1: Sequence of fuzzy numbers obtained from Equation (8) from the initial conditions $X_1 = X_2 = (0; 1; 2)$, $r = 0.25$ and $n = 16$.

Usual Arithmetic	
n	X_n
1	$(0; 1; 2)$
2	$(0; 1; 2)$
3	$(-0.475; 0.75; 1.975)$
4	$(-0.97; 0.5; 1.97)$
5	$(-1.46375; 3.125; 2.0888)$
6	$(-1.95625; 0.1875; 2.3313)$
7	$(-2.478438; 0.10938; 2.6972)$
8	$(-3.06125; 0.0625; 3.1863)$
9	$(-3.735547; 0.035156; 3.8059)$
10	$(-4.532109; 0.019531; 4.5712)$
11	$(-5.483574; 0.010742; 5.5051)$
12	$(-6.626367; 0.0058594; 6.6381)$
13	$(-8.002632; 0.0031738; 8.009)$
14	$(-9.662153; 0.001709; 9.6656)$
15	$(-11.664398; 0.0091553; 11.6662)$
16	$(-14.080791; 0.00048828; 14.0818)$

Interactive Arithmetic	
n	X_n
1	$(0; 1; 2)$
2	$(0; 1; 2)$
3	$(-0.15; 0.75; 1.485)$
4	$(-0.01; 0.5; 0.99)$
5	$(-0.00625; 0.3125; 0.61875)$
6	$(-0.00375; 0.1875; 0.37125)$
7	$(-0.0021875; 0.10938; 0.21656)$
8	$(-0.00125; 0.0625; 0.12375)$
9	$(-0.00070312; 0.035156; 0.069609)$
10	$(-0.00039062; 0.019531; 0.038672)$
11	$(-0.00021484; 0.010742; 0.02127)$
12	$(-0.00011719; 0.0058594; 0.011602)$
13	$(-0.00063477; 0.0031738; 0.0062842)$
14	$(-0.00003418; 0.0017090; 0.0033838)$
15	$(-0.000018311; 0.00091553; 0.0018127)$
16	$(-0.000009765; 0.00048828; 0.0009668)$

this sequence is given by

$$X_{n-1} = (a_{n-2} - rc_{n-3}; b_{n-2} - rb_{n-3}; c_{n-2} - ra_{n-3}),$$

whose α-cuts are given by

$$\begin{aligned} [X_{n-1}]^\alpha &= [a_{n-2} - rc_{n-3} + \alpha(b_{n-2} - rb_{n-3} - (a_{n-2} - rc_{n-3})), \\ &\quad c_{n-2} - ra_{n-3} + \alpha(b_{n-2} - rb_{n-3} - (c_{n-2} - ra_{n-3}))] \\ &= [(a_{n-2} - rc_{n-3})(1-\alpha) + \alpha(b_{n-2} - rb_{n-3}), \\ &\quad (c_{n-2} - ra_{n-3})(1-\alpha) + \alpha(b_{n-2} - rb_{n-3})] \end{aligned}$$

and the n-ary term of this sequence is given by

$$X_n = (a_{n-1} - rc_{n-2}; b_{n-1} - rb_{n-2}; c_{n-1} - ra_{n-2}),$$

whose α-cuts are given by

$$\begin{aligned} [X_n]^\alpha &= [(a_{n-1} - rc_{n-2})(1-\alpha) + \alpha(b_{n-1} - rb_{n-2}), \\ &\quad (c_{n-1} - ra_{n-2})(1-\alpha) + \alpha(b_{n-1} - rb_{n-2})]. \end{aligned}$$

For all $r > 0$, it follows that

$$\begin{aligned} D_\infty(X_n, X_{n-1}) &= \sup_{\alpha\in[0,1]} (\max\{|a_\alpha^- - b_\alpha^-|, |a_\alpha^+ - b_\alpha^+|\}) \\ &= \max\{|a_{n-1} - rc_{n-2} - (a_{n-2} - rc_{n-3})|, |c_{n-1} - ra_{n-2} - (c_{n-2} - ra_{n-3})|\} \end{aligned}$$

or

$$\begin{aligned} D_\infty(X_n, X_{n-1}) &= \max\{|b_{n-1} - rb_{n-2} - (b_{n-2} - rb_{n-3})|, |b_{n-1} - rb_{n-2} - (b_{n-2} - rb_{n-3})|\} \\ &= |b_{n-1} - rb_{n-2} - (b_{n-2} - rb_{n-3})|. \end{aligned}$$

Since $width(X_{n-1}) \leq width(X_n)$, it follows that for $r > 1$ the above sequences do not converge. This comment gives raise to the following proposition.

Proposition 3.1. *Let be the fuzzy sequence given by*

$$X_n = X_{n-1} - rX_{n-2},$$

where the subtraction operation $-$ *is given by the usual difference for fuzzy numbers. Thus, the fuzzy sequence* X_n *diverges, for* $r > 1$.

3.2 Sequence via Interactive Arithmetic

For interactive arithmetic, several differences can be used, for example, gH-difference [13], L-difference [5] and I-difference [6]. In the simulations performed here, only the I-difference will be considered, since it exists for any pair of fuzzy numbers, in contrast to the gH-difference (which can not be computed for any triangular fuzzy numbers) and L-difference (which can not be computed for triangular fuzzy numbers with different shapes). For the I-difference, the following fuzzy sequence

$$X_n = X_{n-1} -_I rX_{n-2}, \tag{10}$$

is illustrated in Figure 3.

Figure 4 depicts the 16-th term X_{16} computed from the sequence (10). It is possible to observe that the output produced by this sequence is indeed a fuzzy number. Moreover, the operation $-_I$ preserves the shape of the triangular fuzzy number.

As in usual arithmetic, the elements of $[X_n]^1$ are the same as in the classical sequence. Now, due to the interactive arithmetic obtained by the set J_0, the width of each $X_n \in \mathbb{R}_{\mathcal{F}_C}$ is decreasing with n. Therefore, the uncertainty about such elements decreases over time.

The right tabular of Table 1 illustrates the values of each element of the sequence (10). Analyzing the table, it is possible to quantitatively compare each X_n given by (9) and (10). It can be observed that the width of the fuzzy numbers produced by the sequence (10) is smaller or equal than the width of the fuzzy numbers produced by the sequence (9), for all $n \in \mathbb{N}$. Consequently, the uncertainty about the fuzzy sequence given in (8) is smaller using the I-difference than the usual difference.

Moreover, if the initial conditions are given by triangular fuzzy numbers, then the $(n-1)$-ary term of this sequence is given by

$$X_{n-1} = (\min\{a_{n-2} - ra_{n-3}, b_{n-2} - rb_{n-3}\}; b_{n-2} - rb_{n-3}; \max\{b_{n-2} - rb_{n-3}, c_{n-2} - rc_{n-3}\}),$$

if $width(X_{n-2}) \geq width(rX_{n-3})$ or

$$X_{n-1} = (\min\{c_{n-2} - rc_{n-3}, b_{n-2} - rb_{n-3}\}; b_{n-2} - rb_{n-3}; \max\{b_{n-2} - rb_{n-3}, a_{n-2} - ra_{n-3}\}),$$

if $width(X_{n-2}) \leq width(rX_{n-3})$ and the n-ary term of this sequence is given by

$$X_n = (\min\{a_{n-1} - ra_{n-2}, b_{n-1} - rb_{n-2}\}; b_{n-1} - rb_{n-2}; \max\{b_{n-1} - rb_{n-2}, c_{n-1} - rc_{n-2}\}),$$

if $width(X_{n-1}) \leq width(rX_{n-2})$ or

$$X_n = (\min\{c_{n-1} - rc_{n-2}, b_{n-1} - rb_{n-2}\}; b_{n-1} - rb_{n-2}; \max\{b_{n-1} - rb_{n-2}, a_{n-1} - ra_{n-2}\}).$$

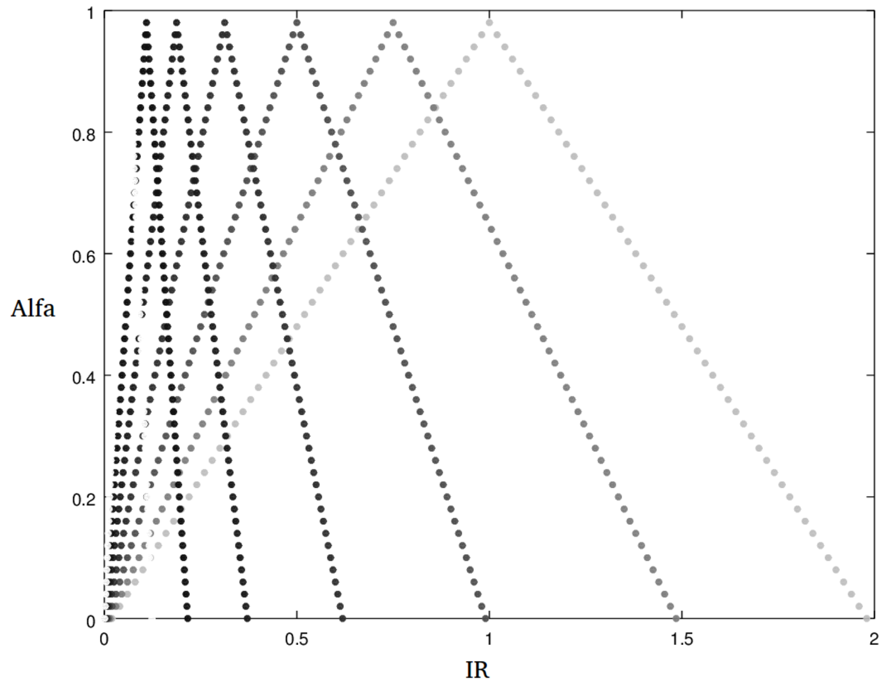

Figure 3: Fuzzy number sequence given by Equation (10) for $n = 16$. Each element of the sequence X_n is represented in shades of gray, with X_1 described by the lightest shade, and X_{16} by the darkest shade.

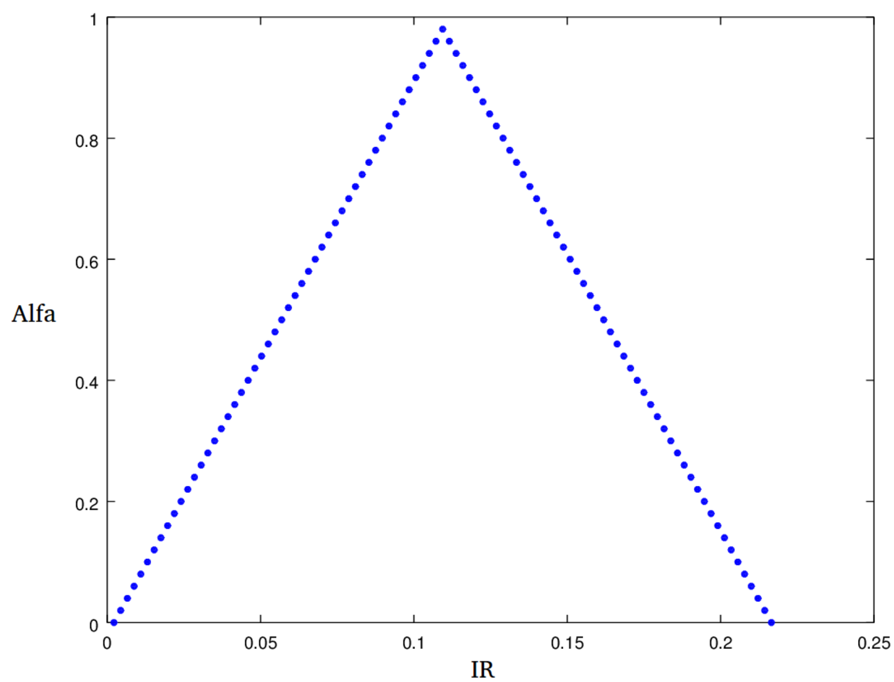

Figure 4: $X_{16} = (0.00009; 0.00488; 0.00966)$

If $0 < r < 1$, then $D_\infty(X_n, X_{n-1})$ is a lower bounded and decreasing function with respect to n, since $width(X_n) \leq width(X_{n-1})$, where $a_{n-2} - ra_{n-3}$, $b_{n-1} - rb_{n-2}$ and $c_{n-1} - rc_{n-2}$ are decreasing sequences.

Such reasoning is summarized in the following proposition.

Proposition 3.2. *Let be the fuzzy sequence given by*

$$X_n = X_{n-1} -_I rX_{n-2},$$

where the subtraction operation $-_I$ *is given by the interactive difference* (2.4). *Thus, the fuzzy sequence* X_n *converges, for any* $0 < r < 1$.

Similar results would be obtained using the gH-difference and the L-difference, if it were possible to calculate $X_{n-1} \ominus rX_{n-2}$ for each n. This comment is attributed to the fact that every arithmetic operation coming from a joint possibility distribution $J \neq J_{\min}$ produces fuzzy numbers with a smaller width than the usual arithmetic [14].

4 Conclusion

This work studied sequences of fuzzy numbers that assume values in $\mathbb{R}_{\mathcal{F}_C}$. Each element of this sequence is obtained by recurrence according to the equation (8), with fuzzy initial conditions.

Through some simulations, using the I-difference, it was noticed that the interactive arithmetic produces a sequence of elements with a smaller width than the width of the elements obtained by the usual arithmetic. This result is valid for all interactive arithmetic. It is worth mentioning that other interactive arithmetic could have been used, such as the differences gH and L, however, it is not always possible to compute them. The I-difference, on the other hand, does not have such restrictions.

From the point of view of applications, a smaller width implies less uncertainty about the elements of the sequence $\{X_n\}$. The sequence provided by usual arithmetic has an increasing width, and therefore, it propagates uncertainty over its elements. On the other hand, using the I-difference, the width of the sequence decreases, which is better for controlling uncertainty over time. This makes interactive arithmetic more suitable for modeling than the usual one.

It is worth noting that in several applications the usual sum and the gH-difference are used in the same equation. This is not consistent with joint possibility distributions, since the gH-difference is an interactive arithmetic operation [6], and the usual sum is not.

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