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A Hybrid Method for Numerical Solution of Fuzzy Mixed Delay Volterra-Fredholm Integral Equations System

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(This paper is dedicated to Professor "John N. Mordeson" on the occasion of his 91st birthday.)

Abstract. A hybrid method for the numerical solution of the system of delayed linear fuzzy mixed Volterra-Fredholm integral equations (FMDVFIES) is introduced. Using the hybrid of Bernstein polynomials and block-pulse functions (HBBFs), an approximate solution for the equations system is provided. Firstly, the HBBFs and their operational matrices are introduced, and some of their characteristics are described. Then by applying the operational matrices on FMDVFIES convert it to the algebraic equations system. The numerical solution is obtained by solving this algebraic system. Then the convergence is investigated and some numerical examples are presented to show the effectiveness of the method.

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Keywords and Phrases: Fuzzy integral equation, Block pulse function, Bernstein polynomials.

1 Introduction

Some decisions are uncertain or imprecise because they are based on imprecise data. And the dynamics governing these data cannot be stated definitively. This area of imprecise logic was first described by Zadeh in [1]. For the role of fuzzy concepts in real life, I refer the reader to the text of Lotfizadeh's letter in [2]: "The first significant real life applications of fuzzy set theory and fuzzy logic began to appear in the late seventies and early eighties. Among such applications were fuzzy logic controlled cement kilns and production of steel. The first consumer product was Matsushitas shower head, 1986. Soon, many others followed, among them home appliances, photo-graphic equipment, and automobile transmissions. A major real life application was Sendais fuzzy logic control system which began to operate in 1987 and was and is a striking success. In the realm of medical instrumentation, a notable real life application is Omrons fuzzy logic based and widely used blood pressure meter."

This concept of fuzzy quickly spread in most fields of science and engineering. Especially the role of fuzzy mathematics in this expansion has been very significant. It can be claimed that it is used in all branches of classical mathematics. Including mathematical analysis, which has a wide expansion in all its concepts such as derivatives [3, 4] differential equations, [5, 6, 7, 8], the concept of fuzzy integral [9, 10]. Differential and integral equations [11, 12], and various exact and approximate methods for solving them have been

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presented. Abbasbandy et al. [13] applied Rung-Kutta method for fuzzy differential equations, Araghi et al. [14] introduced the Lagrange interpolation based on the extension principle for fuzzy Fredholm integral equations, Ezzati et al. [15] presented numerical solution of two-dimensional fuzzy Fredholm integral equation of the second kind using fuzzy bivariate Bernstein polynomials, Shafiee et al. [16] applied predictor corrector method for nonlinear fuzzy Volterra integral equations, and Amin et al. [17] used Haar wavelet for solution of delay Volterra-Fredholm integral equations.

Many researchers have demonstrated the efficiency and error reduction of the combined Bernstein and Block Pulse methods for various fuzzy and non-fuzzy problems, such as [18] and [19] for fuzzy Fredholm integral equations, and [20] for fractional differential equations, and [21] for a system of linear Fredholm integral equations.

Delayed integral equations are a very important area in mathematics, where many phenomena in physics, biology and economics are modeled by such equations. Therefore, finding an exact or approximate solution for them is very important. Considering that many parameters in these models can have an uncertain nature. Therefore, their solution can be considered based on fuzzy concepts.

A numerical approximation method is proposed using the combination of Bernstein and block-pulse functions (HBBF) to FMDVFIES.

$$\mathbf{y}(t) = \mathbf{f}(t) \oplus \sum_{j=1}^{\sigma}{}^{*} \mathbf{A}_j \odot \mathbf{y}(t-\tau_j) \oplus \int_0^t \int_0^1 \mathbf{k}(s,t) \odot \mathbf{y}(t) dtds, \;\; \tau_j, t \in [0,1], \tag{1}$$

where $0 \leq \tau_j \leq 1$, $\mathbf{A}_j \in M_{p\times p}$, the set of real $p \times p$ matrices, for $j = 1, ..., \sigma$, and $\mathbf{y}(t) = \mathbf{y}_0(t)$, $t \leq 0$, and

$$\mathbf{y}(t) = [y_1(t), y_2(t), ..., y_p(t)]^T$$

is unknown function for every $t \in (0, 1]$. While $\mathbf{f}(t)$ and $\mathbf{k}(s,t)$ are known vector and matrix functions respectively,

$$\mathbf{f}(t) = [f_1(t), f_2(t), ..., f_p(t)]^T,$$

and

$$\mathbf{k}(s,t) = [k_{ij}(s,t)], \quad i, j = 1, 2, ..., p.$$

The main outlines of the hybrid method to FMDVFIES can be expressed as follows:

• The non-zero coefficients of Bernstein polynomials are natural numbers. Therefore, there is no coefficient error in the computations, a property that some polynomials, such as the Legendre and Bernoulli polynomials, do not have it.
• Presenting the transformation matrix of Bernstein polynomials to block pulse functions.
• Determined operational matrices.
• By substituting these matrices into the fuzzy integral equations system with time delay, we arrive at a system of algebraic equations.
• By solving this system of linear equations, we obtain a numerical solution to the problem.

The structure of the article is as follows: In Section 2, some basic results from Bernstein polynomial, hybrid functions and an overview of fuzzy concepts are given. The main idea are presented in Section 3. In Section 4, uniqueness of the solution and convergence analysis are investigated. The proposed method is tested through two numerical examples in Section 5. The conclusions are given in the last section.

2 Preliminaries

2.1 Bernstein polynomials

The M order of Bernstein polynomials on $[0, 1]$ are defined as [22]:

$$\mathcal{B}_{m,M}(t) = \binom{M}{m} t^m (1-t)^{M-m}, \quad m = 0, 1, ..., M. \tag{2}$$

Hybrid functions $\psi_{nm}(t)$, for $n = 1, 2, ..., N$ and $m = 0, 1, 2, ..., M-1$ on $[0, 1]$ are defined as

$$\psi_{nm}(t) = \begin{cases} \mathcal{B}_{m,M-1}(Nt - n + 1), & \frac{n-1}{N} \le t < \frac{n}{N} \\ 0, & otherwise \end{cases} \tag{3}$$

where n and M are the number of BPFs and the order of Bernstein polynomials respectively.
A function $f \in L^2[0, 1]$ can be expanded in terms of HBBFs as follows:

$$f(t) \simeq \sum_{n=1}^{N} \sum_{m=0}^{M-1} c_{nm} \psi_{nm}(t) = C^T \Psi(t), \tag{4}$$

where

$$C = [c_{10}, \ \dots \ , c_{1M-1}, c_{20}, \ \dots \ , c_{2M-1}, c_{N0}, \ \dots \ , c_{NM-1},]^T$$

$$\Psi = [\psi_{10}, \ \dots \ , \psi_{1M-1}, \psi_{20}, \ \dots \ , \psi_{2M-1}, \psi_{N0}, \ \dots \ , \psi_{NM-1}]^T,$$

and $c_{nm} = \frac{\langle f(t), \psi_{nm}(t) \rangle}{\langle \psi_{nm}(t), \psi_{nm}(t) \rangle}$ where $\langle \cdot, \cdot \rangle$ denote the inner product on $L^2[0, 1]$.

2.2 An overview of fuzzy concepts

A pair $y = (\underline{y}(r), \overline{y}(r))$ for $r \in [0, 1]$ is called a parametric form of y if
1. $\underline{y}(r)$ is a bounded left continuous monotonic increasing function on $[0, 1]$,
2. $\overline{y}(r)$ is a bounded left continuous monotonic decreasing function on $[0, 1]$,
3. $\forall r \in [0, 1], \ \underline{y}(r) \le \overline{y}(r)$.
A number $a \in \mathbb{R}$ can be represented as $\underline{y}(r) = \overline{y}(r) = a, \ \forall r \in [0, 1]$.
Suppose E^1 be the set of all upper semi-continuous normal convex fuzzy numbers with bounded $r-$level intervals. It means that if $v \in E^1$ then the $r-$level set

$$[v]_r = \{s | v(s) \ge r\}, \quad 0 < r \le 1,$$

is a closed bounded interval which is denoted by $[v]_r = [v_1(r), v_2(r)]$.

Lemma 2.1. Let $v, w \in E^1$ and s be scalar. Then for $r \in (0, 1]$

$$[v + w]_r = [v_1(r) + w_1(r), v_2(r) + w_2(r)],$$

$$[v - w]_r = [v_1(r) - w_2(r), v_2(r) - w_1(r)],$$

$$[v \cdot w]_r = [\min\{v_1(r) \cdot w_1(r), v_1(r) \cdot w_2(r), v_2(r) \cdot w_1(r), v_2(r) \cdot w_2(r)\}, \\ \max\{v_1(r) \cdot w_1(r), v_1(r) \cdot w_2(r), v_2(r) \cdot w_1(r), v_2(r) \cdot w_2(r)\}],$$

$$[sv]_r = s[v]_r.$$

So, the set of all fuzzy numbers E^1 with addition and multiplication which is a convex cone and can be embedded into the Banach space $B = \overline{C}[0, 1] \times \overline{C}[0, 1], \ (B, \| \, . \, \|)$ where

$$\| \, (u, v) \, \| = \sup\{ \max_{0 \le r \le 1} |u(r)|, \quad \max_{0 \le r \le 1} |v(r)| \}. \tag{5}$$

The distance between u and v can be denoted as:

$$D(u,v) = \sup_{0\leq r\leq 1} \{\max[|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|]\}, \tag{6}$$

If $\widetilde{f}(t)$ is continuous in the metric D, then its definite integral exists [23], and

$$\underline{\int_a^b \widetilde{f}(t;r)dt} = \int_a^b \underline{f}(t;r)dt, \quad \overline{\int_a^b \widetilde{f}(t;r)dt} = \int_a^b \overline{f}(t;r)dt.$$

2.3 Block Pulse Functions and transformation matrix

The block-pulse functions (BPFs) and some well-known properties are introduced.

$$\mathfrak{b}_i(t) = \begin{cases} 1, & \frac{(i-1)T}{n} \leq t < \frac{iT}{n} \\ 0, & otherwise \end{cases} \tag{7}$$

for $i = 1, 2, \cdots n$ are defined as a set of BPFs that have the following properties:

$$\mathfrak{b}_i(t)\mathfrak{b}_j(t) = \begin{cases} \mathfrak{b}_i(t), & i = j, \\ 0, & i \neq j, \end{cases} \tag{8}$$

$$\int_0^T \mathfrak{b}_i(t)\mathfrak{b}_j(t)dt = \begin{cases} \frac{T}{n}, & i = j, \\ 0, & i \neq j. \end{cases} \tag{9}$$

The set of BPFs is complete.

The BPFs expansion:

The expansion of $f \in L[0,T)$, with respect to BPFs $\mathfrak{B}(t) = (\mathfrak{b}_1(t), \mathfrak{b}_2(t), ..., \mathfrak{b}_n(t))^T$ is defined as [24]:

$$f(t) \simeq (f_1, f_2, ..., f_n)\mathfrak{B}(t) = F^T\mathfrak{B}(t) = \mathfrak{B}^T(t)F,$$

where $F = (f_1, f_2, ..., f_n)^T$ is given by $F = \frac{1}{h}\int_0^T f(t)\mathfrak{B}(t)dt$ and f_i is the block pulse coefficient with respect to $\mathfrak{b}_i(t)$ for $i = 1, 2, \cdots, n$.
Now, assume that $K(s,\tau)$ belongs to $L^2([0,T] \times [0,T])$ we can write

$$K(s,\tau) \simeq \mathfrak{B}^T(s)K\mathfrak{B}(\tau),\ with\ K = \frac{1}{h^2}\int_0^T\int_0^T \mathfrak{B}^T(s)K(s,\tau)\mathfrak{B}(\tau)d\tau ds,$$

and $h = \frac{T}{n}$.
And also, from [24], can be found that

$$\int_0^T \mathfrak{B}(t)\mathfrak{B}^T(t)dt = hI, \tag{10}$$

and $\int_0^t \mathfrak{B}(t)dt \simeq \mathcal{P}\mathfrak{B}(t)$ where

$$\mathcal{P} = \frac{h}{2}\begin{bmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 2 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Therefore,

$$\int_0^T f(t)dt \simeq \int_0^T F^T \mathfrak{B}(t)dt \simeq F^T \mathcal{P} \mathfrak{B}(t). \tag{11}$$

For time delay $\tau = qh$ with a non-negative integer q, we have

$$\mathfrak{B}(t-\tau) = H^q \mathfrak{B}(t), \tag{12}$$

where

$$\begin{array}{c}\text{(q+1)th-column}\\ \downarrow\end{array}$$

$$H^q = \begin{bmatrix} 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & ... & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 1 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

2.4 Transformation matrix

The HBBFs can be expanded into $NM-$terms of BPFs [25] as

$$\Psi_{NM\times 1}(t) = \Phi_{NM\times NM} \mathfrak{B}_{NM\times 1}(t) \tag{13}$$

where

$$\Phi = \begin{bmatrix} A & O & O & \cdots & O \\ 0 & A & O & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & O & A \end{bmatrix}$$

and $A = (a_{m+1,i})_{M\times M}$ whit

$$\begin{aligned} a_{m+1,i} = M \sum_{k=0}^{M-1-m} (-1)^k \begin{pmatrix} M-1 \\ m \end{pmatrix} \begin{pmatrix} M-1-m \\ k \end{pmatrix} \frac{1}{k+m+1} \\ \left[\left(\frac{i+1}{M} \right)^{k+m+1} - \left(\frac{i}{M} \right)^{k+m+1} \right], \\ i = 0, 1, \cdots, NM-1. \end{aligned} \tag{14}$$

For example, with $N = 2$ and $M = 3$,

$$\Psi(t) = [\psi_{10}(t), \psi_{11}(t), \psi_{12}(t), \psi_{20}(t), \psi_{21}(t), \psi_{22}(t)]^T$$

and

$$\mathfrak{B} = [\mathfrak{b}_1(t), \mathfrak{b}_2(t), ..., \mathfrak{b}_6(t)]^T.$$

Such that

$$\left.\begin{array}{l}\psi_{10}(t) = 4t^2 - 4t + 1\\ \psi_{11}(t) = -8t^2 + 4t\\ \psi_{12}(t) = 4t^2\end{array}\right\}, \quad when\ 0 \leq t < \frac{1}{2}, \tag{15}$$

$$\left.\begin{array}{l}\psi_{20}(t) = 4t^2 - 8t + 4\\ \psi_{21}(t) = -8t^2 + 12t - 4\\ \psi_{22}(t) = 4t^2 - 4t + 1\end{array}\right\}, \quad when\ \frac{1}{2} \leq t < 1, \tag{16}$$

and

$$\Phi_{6\times 6} = \begin{bmatrix} A & O \\ 0 & A \end{bmatrix},$$

with

$$A = \begin{pmatrix} \frac{19}{27} & \frac{7}{27} & \frac{1}{27} \\ \frac{7}{27} & \frac{13}{27} & \frac{7}{27} \\ \frac{1}{27} & \frac{7}{27} & \frac{19}{27} \end{pmatrix}.$$

For more details, see [25].

3 The Main idea

Consider the parametric form of LFMDVFIES as follows:

$$\widetilde{\mathbf{y}}(t) = \widetilde{\mathbf{f}}(t) \oplus \sum_{j=1}^{\sigma}{}^{*} \mathbf{A}_j \odot \widetilde{\mathbf{y}}(t-\tau_j) \oplus \int_0^t \int_0^1 \mathbf{k}(s,t) \odot \widetilde{\mathbf{y}}(t) dt ds, \ \tau_j, t \in [0,1], \tag{17}$$

where $\widetilde{\mathbf{y}}(t)$ and $\widetilde{\mathbf{f}}(t)$ are parametric form of $\mathbf{y}(t)$ and $\mathbf{f}(t)$ in Eq. (1),

$$\widetilde{y_i}(t) = (\underline{y_i}(t,r), \overline{y_i}(t,r), \ \widetilde{y_i}(t-\tau_j) = (\underline{y_i}(t-\tau_j,r), \overline{y_i}(t-\tau_j,r)),$$
$$\widetilde{f_i}(t) = (\underline{f_i}(t,r), \overline{f_i}(t,r)), \ i = 1,2,...,p, \ j = 1,2,...,\sigma.$$

The expansion of functions $\widetilde{y_i}(t), \widetilde{f_i}(t)$ and $k_{ij}(s,\tau)$ can be written as follows:

$$\widetilde{y_i}(t) \simeq Y_i^T \odot \Psi(t), \quad \widetilde{f_i}(t) \simeq F_i^T \odot \Psi(t), \quad k_{ij}(s,\tau) \simeq \Psi^T(s) K_{i,j} \Psi(\tau), \tag{18}$$

where Y_i, F_i are vectors of $NM \times 1$ and K_{ij} is a matrix of $NM \times NM$.
$h = 1/(NM), \tau_j = q_j h, \ j = 1,2,...,\sigma.$

$$\begin{aligned} y_i(t-\tau_j) &\simeq Y_i^T \odot \Psi(t-\tau_j) = Y_i^T \odot \Phi\mathfrak{B}(t-\tau_j) \\ &= Y_i^T \odot \Phi H^{q_j}\mathfrak{B}(t) = Y_i^T \odot \Phi H^{q_j}\Phi^{-1}\Psi(t), \end{aligned}$$

$HB^{q_j} = \Phi H^{q_j} \Phi^{-1}$, and also, we have

$$\begin{aligned} &y_i(t-\tau_j) \simeq Y_i^T HB^{q_j} \odot \Psi(t), \ if \ t-\tau_j > 0, \\ &y_i(t-\tau_j) = y_{0i}(t), \ if \ t-\tau_j \leq 0, \end{aligned} \tag{19}$$

where $y_{0i}(t)$ denotes the $i-$th element of the $\mathbf{y}_0(t)$. The integration of vector $\Psi(t)$ can be approximated as

$$(I\Psi)(t) \simeq P\Psi(t), \tag{20}$$

where the $NM \times NM$ matrix P is called the HBBfs operational matrix of integration.

$$(I\Psi)(t) \simeq (I\Phi\mathfrak{B})(t) = \Phi(I\mathfrak{B})(t) \simeq \Phi\mathcal{P}\mathfrak{B}(t) = \Phi\mathcal{P}\Phi^{-1}\Psi(t), \tag{21}$$

so

$$P = \Phi\mathcal{P}\Phi^{-1}.$$

And also

$$\int_0^T \Psi(t)\Psi^T dt = \int_0^T \Phi\mathfrak{B}(t)\mathfrak{B}^T(t)\Phi^T dt = h\Phi\Phi^T = D_h,$$

hence

$$D_h = \frac{T}{NM}\begin{bmatrix} AA^T & O & O & \cdots & O \\ 0 & AA^T & O & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & O & AA^T \end{bmatrix}.$$

By substituting Eqs (18)–(21) into (17), we have

$$\begin{aligned} Y^T \odot \Psi = F^T \odot \Psi \oplus \sum_{j=1}^{\sigma} A^{(j)} \odot Y^T \odot HB^{q_j} \odot \Psi \\ \oplus \int_0^t \int_0^1 \Psi^T K \Psi \odot \psi^T Y dtds. \end{aligned} \tag{22}$$

And hence

$$Y^T = F^T \oplus \sum_{j=1}^{\sigma} Y^T \odot D_{A^{(j)}} \odot HB^{q_j} \oplus Y^T \odot D_h^T K^T P,$$

where

$$D_{A^{(j)}} = \begin{bmatrix} A^{(j)} & O & O & \cdots & O \\ 0 & A^{(j)} & O & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & O & A^{(j)} \end{bmatrix},$$

then

$$Y^T \odot \left(I \ominus \sum_{j=1}^{\sigma} D_{A^{(j)}} \odot HB^{q_j} \ominus D_h^T K^T P \right) = F^T.$$

The approximate solution of Eq.(17) can be found by solving these linear equations and taking $\widetilde{y}(t) = Y^T \odot \Psi(t)$.

4 Uniqueness of the solution and Convergence analysis

Consider the equation

$$\widetilde{\mathbf{y}}(t) = \widetilde{\mathbf{f}}(t) \oplus \widetilde{\mathbf{y}}(t-\tau) \oplus \int_0^t \int_0^1 \mathbf{k}(s,t) \odot \widetilde{\mathbf{y}}(t)dtds, \ \tau, t \in [0,1], \tag{23}$$

or of even more general form:

$$\widetilde{\mathbf{y}}(t) = \widetilde{\mathbf{f}}(t) \oplus \sum_{j=1}^{\sigma\ *} \mathbf{A}_j \odot \widetilde{\mathbf{y}}(t-\tau_j) \oplus \int_0^t \int_0^1 \mathbf{k}(s,t) \odot \widetilde{\mathbf{y}}(t)dtds, \ \tau_j, t \in [0,1]. \tag{24}$$

4.1 Uniqueness

For Eqs.(23) and (24), the delays $t-\tau$ and $t-\tau_j$ are bounded and $\widetilde{\mathbf{y}}(t)=\widetilde{\mathbf{y_0}}(t),\ t\leq 0$. Then $\widetilde{\mathbf{y}}(t-\tau)$ is a known function of t for $0\leq t\leq\tau$. We then show that Eq.(23) has a unique solution $\widetilde{\mathbf{y}}(t)$ for $-\tau\leq t\leq\tau$ and then can be compute the solution for $-\tau\leq t\leq 2\tau$ and so on. By continuing this process, the existence and uniqueness of the solution for all $-\tau\leq t\leq 1$ is obtained. For any $0\leq\tau\leq 1$, consider $C_0=C([-\tau,0],E^p)$. Suppose that $\widetilde{\mathbf{y}}\in C(J_0,E^p)$ where $J_0=[-\tau,\tau]$ and also $\widetilde{\mathbf{f}}\in C([0,1],E^p)$, and

$$\mathbf{k}=[k_{ij}]_{p\times p}, k_{ij}\in C([0,1]\times[0,1],\mathbb{R}),$$

such that

$$\max_{1\leq i,j\leq p}\max_{0\leq s,t\leq 1}\mid k_{ij}(s,t)\mid = K.$$

Theorem 4.1. Suppose that $\widetilde{\mathbf{y}}(t)=\widetilde{\mathbf{y_0}}(t),\ t\leq 0$, and $\widetilde{\mathbf{y_0}}(t)\in C_0$. If

$$\parallel\mathbf{k}\parallel_\infty=\max_{1\leq i\leq p}\sum_{j=1}^{p}\mid k_{ij}(s,t)\mid = pK<1,\ 0\leq s,t\leq 1$$

then Eq.(23) has a unique solution $\widetilde{\mathbf{y}}(t)$ on J_0.

Proof. Define the metric on $C(J_0,E^p)$ by

$$\mathbf{D}(\mathbf{u}(t),\mathbf{v}(t))=\begin{pmatrix}D(u_1(t),v_1(t))\\ D(u_2(t),v_2(t))\\ \vdots\\ D(u_p(t),v_p(t))\end{pmatrix}.$$

We define the operator T on $C(J_0,E^p)$ by

$$\begin{aligned}T\widetilde{\mathbf{u}}(t)&=\widetilde{\mathbf{y_0}}(t), \qquad -\tau\leq t\leq 0,\\ T\widetilde{\mathbf{u}}(t)&=\widetilde{\mathbf{f}}(t)\oplus\widetilde{\mathbf{y_0}}(t)\oplus\int_0^t\int_0^1\mathbf{k}(s,z)\odot\widetilde{\mathbf{u}}(z)dzds, t\in[0,\tau],\ \tau>0.\end{aligned}$$

We find that $\mathbf{D}(T\widetilde{\mathbf{u}}(t),T\widetilde{\mathbf{v}}(t))=0,\ -\tau\leq t\leq 0$, and for $0\leq t\leq\tau$,

$$\begin{aligned}\mathbf{D}(T\widetilde{\mathbf{u}}(t),T\widetilde{\mathbf{v}}(t))=&\\ &\mathbf{D}\Bigg(\widetilde{\mathbf{f}}(t)\oplus\widetilde{\mathbf{y_0}}(t)\oplus\int_0^t\int_0^1\mathbf{k}(s,z)\odot\widetilde{\mathbf{u}}(z)dzds,\\ &\widetilde{\mathbf{f}}(t)\oplus\widetilde{\mathbf{y_0}}(t)\oplus\int_0^t\int_0^1\mathbf{k}(s,z)\odot\widetilde{\mathbf{v}}(z)dzds,\Bigg)\\ &\leq\int_0^t\int_0^1 pK\mathbf{D}(\widetilde{\mathbf{u}}(z),\widetilde{\mathbf{v}}(z))dzds\end{aligned}$$

$$\parallel\mathbf{D}(T\widetilde{\mathbf{u}}(t),T\widetilde{\mathbf{v}}(t))\parallel_\infty\leq pK\int_0^t\int_0^1\parallel\mathbf{D}(\widetilde{\mathbf{u}}(z),\widetilde{\mathbf{v}}(z))\parallel_\infty dzds$$

Hence we have

$$\parallel\mathbf{D}(T\widetilde{\mathbf{u}}(t),T\widetilde{\mathbf{v}}(t))\parallel_\infty<\parallel\mathbf{D}(\widetilde{\mathbf{u}}(z),\widetilde{\mathbf{v}}(z))\parallel_\infty.$$

So, the operator T is a contraction on $C(J_0,E^p)$. Therefore T has a unique fixed point $\widetilde{\mathbf{y}}\in C(J_0,E^p)$, and consequently this $\widetilde{\mathbf{y}}=\widetilde{\mathbf{y}}(t)$ is the unique solution of Eq.(23) on J_0. $\square$

Theorem 4.2. Suppose that $\widetilde{y}_i(t)_{M,N}$ and $\widetilde{y}_i(t)$ are the approximate solution by HBBFs and exact solution of $i-th$ component of $\widetilde{\mathbf{y}}(t)$ in Eq. (17) respectively. If $k_{ij}(s,t)$ for all $s,t \in [0,1]$ is continuous and bounded, then $\widetilde{y}_i(t)_{M,N} \longrightarrow \widetilde{y}_i(t)$ as $M,N \longrightarrow \infty$, for any $i = 1,2,...,p$.

Proof.

$$D(\widetilde{y}_i(t), \widetilde{y}_i(t)_{M,N}) \leq \sum_{j=1}^{\sigma\ *}\sum_{l=1}^{p\ *} D(a_{il}^{(j)} \odot \widetilde{y}_l(t-\tau_j), a_{il}^{(j)} \odot \widetilde{y}_l(t-\tau_j)_{M,N}) +$$

$$\sum_{l=1}^{p\ *} D\left(\int_0^t\int_0^1 k_{il}(s,\tau)\widetilde{y}_l(\tau)d\tau ds, \int_0^t\int_0^1 k_{il}(s,\tau)\widetilde{y}_l(\tau)_{M,N}d\tau ds\right)$$

$$= \sum_{j=1}^{\sigma\ *}\sum_{l=1}^{p\ *} \mid a_{il}^{(j)} \mid D(\widetilde{y}_l(t-\tau_i), \widetilde{y}_l(t-\tau_i)_{M,N}) +$$

$$\sum_{l=1}^{p\ *} D\left(\int_0^t\int_0^1 k_{il}(s,\tau)\widetilde{y}_l(\tau)d\tau ds, \int_0^t\int_0^1 k_{il}(s,\tau)\sum_{n=1}^{N}\sum_{m=0}^{M-1} c_{l,nm}\psi_{n,m}(\tau)d\tau ds\right)$$

$$\leq \sum_{j=1}^{\sigma\ *}\sum_{l=1}^{p\ *} \mid a_{il}^{(j)} \mid D(\widetilde{y}_l(t-\tau_i), \widetilde{y}_l(t-\tau_i)_{M,N}) +$$

$$K\sum_{l=1}^{p\ *} D\left(\int_0^t\int_0^1 \widetilde{y}_l(\tau)d\tau ds, \int_0^t\int_0^1 \sum_{n=1}^{N}\sum_{m=0}^{M-1} c_{l,nm}\psi_{n,m}(\tau)d\tau ds\right),$$

where

$$\max_{1\leq i,j\leq p}\max_{0\leq s,t\leq 1} \mid k_{ij}(s,t) \mid = K.$$

$$\lim_{M,N\to\infty} D\left(\widetilde{y}_i(t), \widetilde{y}_i(t)_{M,N}\right)$$

$$\leq \lim_{M,N\to\infty}\sum_{j=1}^{\sigma\ *}\sum_{l=1}^{p\ *} \mid a_{il}^{(j)} \mid D(\widetilde{y}_l(t-\tau_j), \widetilde{y}_l(t-\tau_j)_{M,N}) +$$

$$K\lim_{M,N\to\infty}\sum_{l=1}^{p\ *} D\left(\int_0^t\int_0^1 \widetilde{y}_l(\tau)d\tau ds, \int_0^t\int_0^1 \sum_{n=1}^{N}\sum_{m=0}^{M-1} c_{l,nm}\psi_{n,m}(\tau)d\tau ds\right)$$

$$\leq \lim_{M,N\to\infty}\sum_{j=1}^{\sigma\ *}\sum_{l=1}^{p\ *} \mid a_{il}^{(j)} \mid D(\widetilde{y}_l(t-\tau_j), \widetilde{y}_l(t-\tau_j)_{M,N}) +$$

$$K\sum_{l=1}^{p\ *} D\left(\int_0^t\int_0^1 \widetilde{y}_l(\tau)d\tau ds, \int_0^t\int_0^1 \lim_{M,N\to\infty}\sum_{n=1}^{N}\sum_{m=0}^{M-1} c_{l,nm}\psi_{n,m}(\tau)d\tau ds\right)$$

$$\leq \lim_{M,N\to\infty}\sum_{j=1}^{\sigma\ *}\sum_{l=1}^{p\ *} \mid a_{il}^{(j)} \mid D(\widetilde{y}_l(t-\tau_j), \widetilde{y}_l(t-\tau_j)_{M,N}) +$$

$$K\int_0^t\int_0^1 \sum_{l=1}^{p\ *} D\left(\widetilde{y}_l(\tau), \lim_{M,N\to\infty}\sum_{n=1}^{N}\sum_{m=0}^{M-1} c_{l,nm}\psi_{n,m}(\tau)\right)d\tau ds.$$

Since

$$\widetilde{y}_l(t) = \lim_{M,N\to\infty}\sum_{n=1}^{N}\sum_{m=0}^{M-1} c_{l,nm}\psi_{n,m}(t),$$

and for any $t-\tau_j>0$

$$\lim_{M,N\to\infty}\widetilde{y}_l(t-\tau_j)_{M,N}=\lim_{M,N\to\infty}\sum_{n=1}^{N}\sum_{m=0}^{M-1}c_{l,nm}\psi_{n,m}(t-\tau_j)=\widetilde{y}_l(t-\tau_j),$$

and for $t-\tau_j\leq 0$

$$\widetilde{y}_l(t-\tau_j)_{M,N}=\phi_{0l}(t)=\widetilde{y}_l(t-\tau_j).$$

Hence,

$$\lim_{M,N\to\infty}\sum_{l=1}^{p}{}^{*}\mid a_{il}^{(j)}\mid\widetilde{y}_l(t-\tau_j)=\lim_{M,N\to\infty}\sum_{l=1}^{p}{}^{*}\mid a_{il}^{(j)}\mid\odot\widetilde{y}_l(t-\tau_j)_{M,N}.$$

Therefore, for every $i=1,2,...,p$,

$$\lim_{M,N\to\infty}D\left(\widetilde{y}_i(t),\lim_{M,N\to\infty}\sum_{n=1}^{N}\sum_{m=0}^{M-1}c_{i,nm}\psi_{n,m}(t)\right)=0.$$

□

5 Examples

Example 5.1. Consider the following FMDVFIES:

$$\widetilde{y_1}(t)=\widetilde{f_1}(t)\oplus(1/2)\widetilde{y_1}(t-1/3)\oplus\int_0^t\int_0^1 k_{11}\odot\widetilde{y_1}(t)dtds$$
$$\oplus\int_0^t\int_0^1 k_{12}\odot\widetilde{y_2}(t)dtds,\ \ t\in[0,1],$$

$$\widetilde{y_2}(t)=\widetilde{f_2}(t)\oplus\widetilde{y_1}(t-2/3)\oplus 2\widetilde{y_2}(t-1)\oplus\int_0^t\int_0^1 k_{21}\odot\widetilde{y_1}(t)dtds$$
$$\oplus\int_0^t\int_0^1 k_{22}\odot\widetilde{y_2}(t)dtds,$$

where $\widetilde{y_1}(t)=\widetilde{y_2}(t)=\widetilde{0},\ \ t\leq 0$, and

$$\underline{f_1}(t,r)=(\frac{r+1}{8})\left(e^{-t}-(2-5/e)(t-t^2/2)-(1/4)(t-t^3/3)\right.$$
$$\left.-(1/2)e^{-(t-1/3)}H(t-1/3)\right),$$

$$\overline{f_1}(t,r)=(\frac{3-r}{8})\left(e^{-t}-(2-5/e)(t-t^2/2)-(1/4)(t-t^3/3)\right.$$
$$\left.-(1/2)e^{-(t-1/3)}H(t-1/3)\right),$$

$$\underline{f_2}(t,r)=(\frac{r+1}{8})\left(t-(4/e-1)t^2/2-(1/5)(t-t^2/2)\right.$$
$$\left.-e^{-(t-2/3)}H(t-2/3)-(t-1)H(t-1)\right),$$

$$\overline{f_2}(t,r) = (\frac{3-r}{8})\left(t - (4/e-1)t^2/2 - (1/5)(t - t^2/2)\right.$$
$$\left. -e^{-(t-2/3)}H(t-2/3) - (t-1)H(t-1)\right),$$

H is the Heaviside function as:

$$H(t) = \begin{cases} 1, & t \geq 0, \\ 0, & otherwise, \end{cases}$$

and

$$k_{11} = t^2(1-s),\ k_{12} = t^2(1-s^2),$$
$$k_{21} = (1-t^2)s,\ k_{22} = t^3(1-s).$$

The exact solutions are as follows:

$$\underline{y_1}(t,r) = \frac{(1+r)e^{-t}}{8},\ \overline{y_1}(t,r) = \frac{(3-r)e^{-t}}{8},$$

$$\underline{y_2}(t,r) = \frac{(1+r)t}{8},\ \overline{y_2}(t,r) = \frac{(3-r)t}{8}.$$

The errors are shown in Figs. 1 and 2 and the numerical values and error values are given in Tables 1, 2, 3 and 4.

Example 5.2. Consider the following FMDVFIES:

$$\widetilde{y_1}(t) = \widetilde{f_1}(t) \oplus \widetilde{y_1}(t-1) \oplus \int_0^t \int_0^1 k_{11} \odot \widetilde{y_1}(t)dtds$$
$$\oplus \int_0^t \int_0^1 k_{12} \odot \widetilde{y_2}(t)dtds,$$

$$\widetilde{y_2}(t) = \widetilde{f_2}(t) \oplus (1/2)\widetilde{y_2}(t-5/6) \oplus \int_0^t \int_0^1 k_{21} \odot \widetilde{y_1}(t)dtds$$
$$\oplus \int_0^t \int_0^1 k_{22} \odot \widetilde{y_2}(t)dtds, \quad t \in [0,1],$$

where

$$\underline{f_1}(t,r) = (r/4)\left(e^{-t} - (6-16/e)(t - t^3/3) - (1/4)(t - t^4/4)\right.$$
$$\left. -e^{-(t-1)}H(t-1)\right),$$

$$\overline{f_1}(t,r) = (1/2 - r/4)\left(e^{-t} - (6-16/e)(t - t^3/3) - (1/4)(t - t^4/4)\right.$$
$$\left. -e^{-(t-1)}H(t-1)\right),$$

$$\underline{f_2}(t,r) = (r/4)\left(t - (1/4)(t + t^2/2) - (1/2)(t-5/6)H(t-5/6)\right)$$
$$+ (1/2 - r/4)t^3/e,$$

$$\begin{aligned}\overline{f_2}(t,r) =&(1/2-r/4)\left(t-(1/4)(t+t^2/2)-(1/2)(t-5/6)H(t-5/6)\right)\\&+(r/4)t^3/e.\end{aligned}$$

H is the Heaviside function, and

$$k_{11}=t^3(1-s^2),\ k_{12}=t^2(1-s^3),$$

$$k_{21}=(1+t^2)s^2,\ k_{22}=t^2(1+s).$$

The exact solutions are as follows:

$$\underline{y_1}(t,r)=\frac{re^{-t}}{4},\ \overline{y_1}(t,r)=\frac{(2-r)e^{-t}}{4}.$$

$$\underline{y_2}(t,r)=\frac{rt}{4},\ \overline{y_2}(t,r)=\frac{(2-r)t}{4}.$$

The errors are shown in Figs. 3 and 4 and the numerical values and error values are given in Tables 5, 6, 7 and 8.

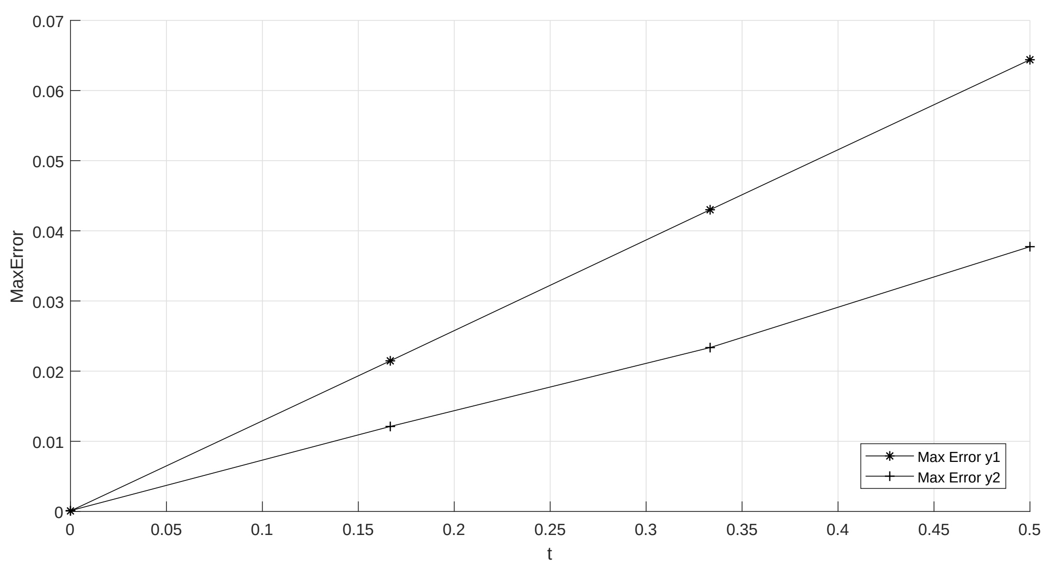

Figure 1: The errors for $M=3, N=2$ in Example 5.1.

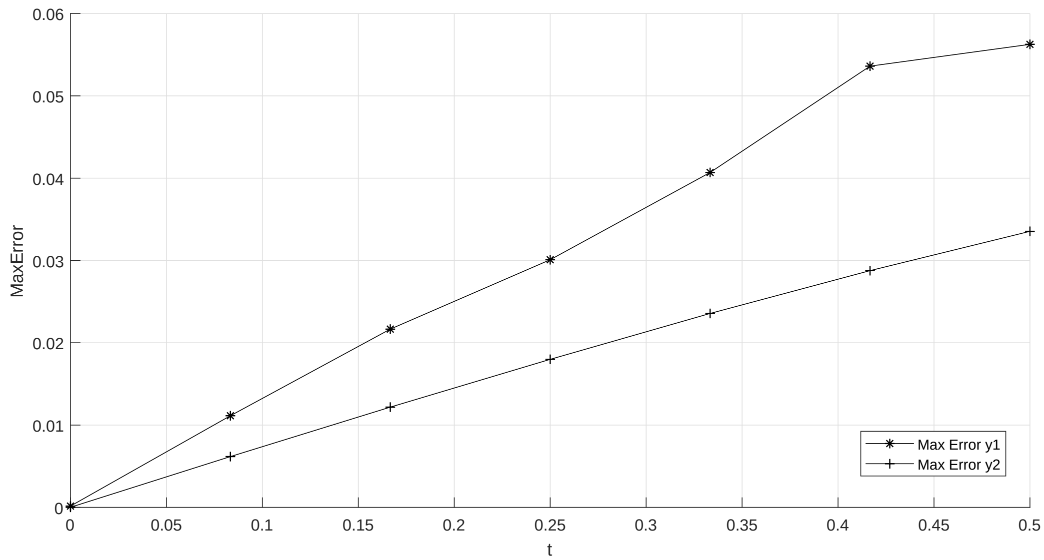

Figure 2: The errors for $M = 3, N = 4$ in Example 5.1.

Table 1: Aproximate and exact solutions and errors for $M = 3$ and $N = 2$ in Example 5.1.

$t = 0.333$ r	App. $\underline{y_1}(t,r)$	Exact $\underline{y_1}(t,r)$	Error	App. $\overline{y_1}(t,r)$	Exact $\overline{y_1}(t,r)$	Error	Max Error
0.00000	0.01575	0.02986	0.01410	0.04656	0.08957	0.04300	0.04300
0.16667	0.01832	0.03483	0.01651	0.04400	0.08459	0.04060	0.04060
0.33333	0.02089	0.03981	0.01892	0.04143	0.07961	0.03819	0.03819
0.50000	0.02346	0.04478	0.02133	0.03886	0.07464	0.03578	0.03578
0.66667	0.02602	0.04976	0.02374	0.03629	0.06966	0.03337	0.03337
0.83333	0.02859	0.05474	0.02614	0.03373	0.06469	0.03096	0.03096
1.00000	0.03116	0.05971	0.02855	0.03116	0.05971	0.02855	0.02855

Table 2: Aproximate and exact solutions and errors for $M = 3$ and $N = 2$ in Example 5.1.

$t = 0.333$ r	App. $\underline{y_2}(t,r)$	Exact $\underline{y_2}(t,r)$	Error	App. $\overline{y_2}(t,r)$	Exact $\overline{y_2}(t,r)$	Error	Max Error
0.00000	0.03373	0.04167	0.00793	0.10164	0.12500	0.02336	0.02336
0.16667	0.03939	0.04861	0.00922	0.09599	0.11806	0.02207	0.02207
0.33333	0.04505	0.05556	0.01050	0.09033	0.11111	0.02078	0.02078
0.50000	0.05071	0.06250	0.01179	0.08467	0.10417	0.01950	0.01950
0.66667	0.05637	0.06944	0.01307	0.07901	0.09722	0.01821	0.01821
0.83333	0.06203	0.07639	0.01436	0.07335	0.09028	0.01693	0.01693
1.00000	0.06769	0.08333	0.01564	0.06769	0.08333	0.01564	0.01564

Table 3: Aproximate and exact solutions and errors for $M = 3$ and $N = 4$ in Example 5.1.

$t = 0.333$ r	App. $\underline{y_1}(t,r)$	Exact $\underline{y_1}(t,r)$	Error	App. $\overline{y_1}(t,r)$	Exact $\overline{y_1}(t,r)$	Error	Max Error
0.00000	0.01411	0.02986	0.01574	0.04888	0.08957	0.04069	0.04069
0.16667	0.01701	0.03483	0.01782	0.04598	0.08459	0.03861	0.03861
0.33333	0.01991	0.03981	0.01990	0.04308	0.07961	0.03653	0.03653
0.50000	0.02281	0.04478	0.02198	0.04019	0.07464	0.03445	0.03445
0.66667	0.02570	0.04976	0.02406	0.03729	0.06966	0.03237	0.03237
0.83333	0.02860	0.05474	0.02614	0.03439	0.06469	0.03029	0.03029
1.00000	0.03150	0.05971	0.02822	0.03150	0.05971	0.02822	0.02822

Table 4: Aproximate and exact solutions and errors for $M = 3$ and $N = 4$ in Example 5.1.

$t = 0.333$ r	App. $\underline{y_2}(t,r)$	Exact $\underline{y_2}(t,r)$	Error	App. $\overline{y_2}(t,r)$	Exact $\overline{y_2}(t,r)$	Error	Max Error
0.00000	0.03381	0.04167	0.00785	0.10144	0.12500	0.02356	0.02356
0.16667	0.03945	0.04861	0.00916	0.09580	0.11806	0.02225	0.02225
0.33333	0.04508	0.05556	0.01047	0.09017	0.11111	0.02094	0.02094
0.50000	0.05072	0.06250	0.01178	0.08453	0.10417	0.01964	0.01964
0.66667	0.05635	0.06944	0.01309	0.07890	0.09722	0.01833	0.01833
0.83333	0.06199	0.07639	0.01440	0.07326	0.09028	0.01702	0.01702
1.00000	0.06762	0.08333	0.01571	0.06762	0.08333	0.01571	0.01571

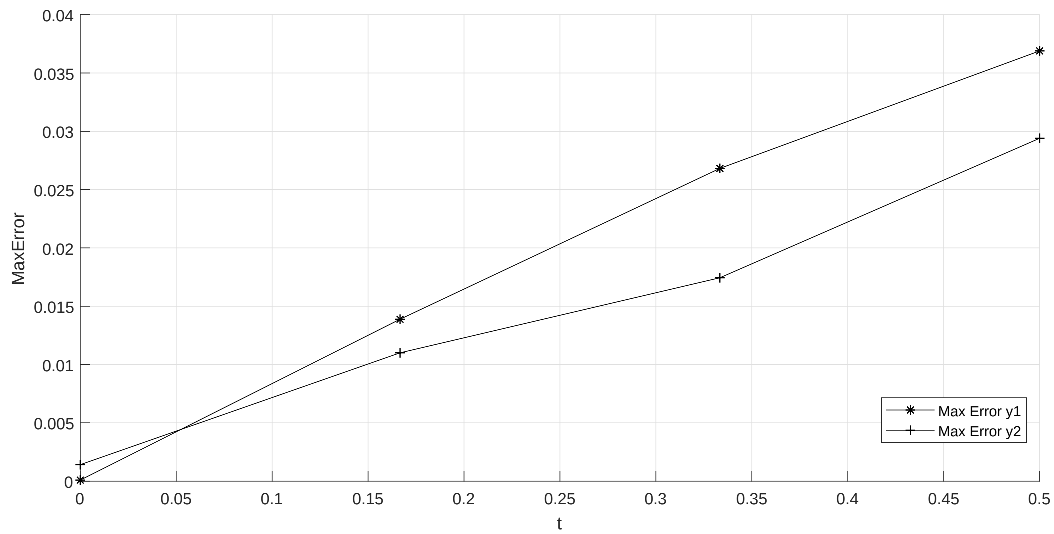

Figure 3: The errors for $M = 3, N = 2$ in Example 5.2.

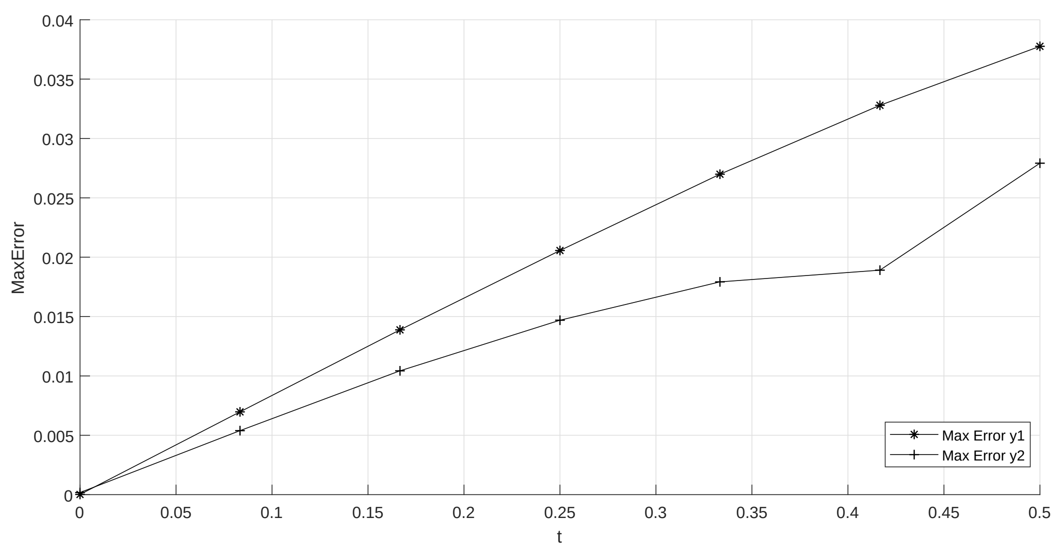

Figure 4: The errors for $M = 3, N = 4$ in Example 5.2.

Table 5: Aproximate and exact solutions and errors for $M = 3$ and $N = 2$ in Example 5.2.

$t = 0.333$ r	App. $\underline{y_1}(t,r)$	Exact $\underline{y_1}(t,r)$	Error	App. $\overline{y_1}(t,r)$	Exact $\overline{y_1}(t,r)$	Error	Max Error
0.00000	0.00000	0.00000	0.00000	0.33627	0.35827	0.02199	0.02199
0.16667	0.02539	0.02986	0.00447	0.30825	0.32841	0.02016	0.02016
0.33333	0.05077	0.05971	0.00894	0.28023	0.29855	0.01833	0.01833
0.50000	0.07616	0.08957	0.01341	0.25220	0.26870	0.01650	0.01650
0.66667	0.10154	0.11942	0.01788	0.22418	0.23884	0.01466	0.01788
0.83333	0.12693	0.14928	0.02235	0.19616	0.20899	0.01283	0.02235
1.00000	0.15231	0.17913	0.02682	0.16814	0.17913	0.01100	0.02682

Table 6: Aproximate and exact solutions and errors for $M = 3$ and $N = 2$ in Example 5.2.

$t = 0.333$ r	App. $\underline{y_2}(t,r)$	Exact $\underline{y_2}(t,r)$	Error	App. $\overline{y_2}(t,r)$	Exact $\overline{y_2}(t,r)$	Error	Max Error
0.00000	0.00747	0.00000	0.00747	0.16343	0.16667	0.00324	0.00747
0.16667	0.01721	0.01389	0.00332	0.15048	0.15278	0.00230	0.00332
0.33333	0.02694	0.02778	0.00083	0.13753	0.13889	0.00136	0.00136
0.50000	0.03668	0.04167	0.00499	0.12458	0.12500	0.00042	0.00499
0.66667	0.04642	0.05556	0.00914	0.11163	0.11111	0.00052	0.00914
0.83333	0.05616	0.06944	0.01329	0.09868	0.09722	0.00146	0.01329
1.00000	0.06589	0.08333	0.01744	0.08573	0.08333	0.00240	0.01744

Table 7: Aproximate and exact solutions and errors for $M = 3$ and $N = 4$ in Example 5.2.

$t = 0.333$ r	App. $\underline{y_1}(t,r)$	Exact $\underline{y_1}(t,r)$	Error	App. $\overline{y_1}(t,r)$	Exact $\overline{y_1}(t,r)$	Error	Max Error
0.00000	0.00000	0.00000	0.00000	0.33595	0.35827	0.02231	0.02231
0.16667	0.02536	0.02986	0.00450	0.30796	0.32841	0.02045	0.02045
0.33333	0.05071	0.05971	0.00900	0.27996	0.29855	0.01859	0.01859
0.50000	0.07607	0.08957	0.01349	0.25197	0.26870	0.01673	0.01673
0.66667	0.10143	0.11942	0.01799	0.22397	0.23884	0.01487	0.01799
0.83333	0.12679	0.14928	0.02249	0.19597	0.20899	0.01301	0.02249
1.00000	0.15214	0.17913	0.02699	0.16798	0.17913	0.01116	0.02699

Table 8: Aproximate and exact solutions and errors for $M = 3$ and $N = 4$ in Example 5.2.

$t = 0.333$ r	App. $\underline{y_2}(t,r)$	Exact $\underline{y_2}(t,r)$	Error	App. $\overline{y_2}(t,r)$	Exact $\overline{y_2}(t,r)$	Error	Max Error
0.00000	0.00817	0.00000	0.00817	0.16091	0.16667	0.00576	0.00817
0.16667	0.01771	0.01389	0.00382	0.14818	0.15278	0.00460	0.00460
0.33333	0.02725	0.02778	0.00052	0.13545	0.13889	0.00344	0.00344
0.50000	0.03679	0.04167	0.00487	0.12272	0.12500	0.00228	0.00487
0.66667	0.04633	0.05556	0.00922	0.11000	0.11111	0.00111	0.00922
0.83333	0.05587	0.06944	0.01357	0.09727	0.09722	0.00005	0.01357
1.00000	0.06541	0.08333	0.01792	0.08454	0.08333	0.00121	0.01792

6 Conclusion

First, the properties of Bernstein polynomials and their combination with block pulse functions are presented. Then, the important transformation matrix is introduced, which is one of the advantages of this work, because it can be generalized to other polynomials and its combination with block pulse functions, where the operational matrices are more easily calculated. We applied the method of combining functions to mixed fuzzy integral equations system with time delay. Then, by using the transformation matrix, we determined other operational, delay, and Fredholm and Volterra integrals matrices. By substituting these matrices into the fuzzy integral equations system with time delay, we arrive at a system of algebraic equations. By solving this system of linear equations, we obtain a solution to the problem. Then, we proved the uniqueness and convergence of the method. And some numerical examples are presented to show the effectiveness of the method. The results showed that the hybrid methods are very useful for these types of systems. For future research, this method can be used for such equations with nonlinear or nonlinear delay functions. And also, it can be applied to other polynomials and its combination with block pulse functions.

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