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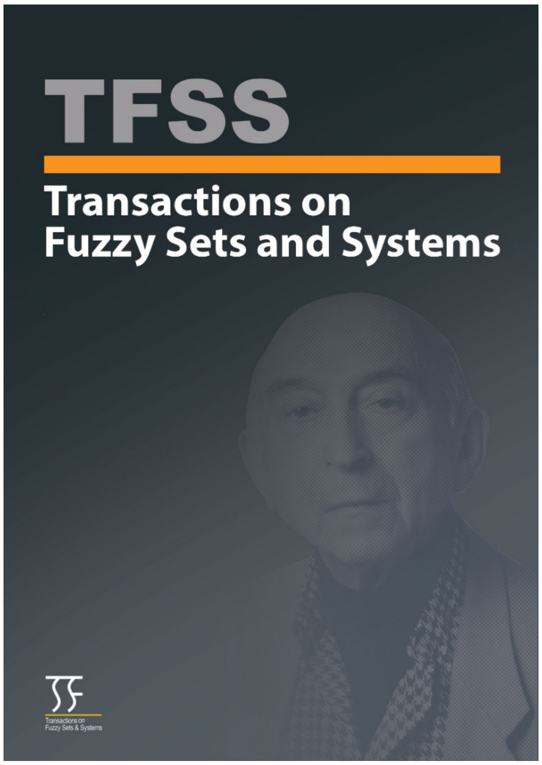


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# A Note on the Fuzzy Leonardo Numbers

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# A Note on the Fuzzy Leonardo Numbers

**Elen Viviani Pereira Spreafico*** , **Eudes Antonio Costa** , **Paula Maria Machado Cruz Catarino**

(This paper is dedicated to Professor "John N. Mordeson" on the occasion of his 91st birthday.)

**Abstract.** In this work, we define a new sequence denominated by fuzzy Leonardo numbers. Some algebraic properties of this new sequence are studied and several identities are established. Moreover, the relations between the fuzzy Fibonacci and fuzzy Lucas numbers are explored, and several results are given. In addition, some sums involving fuzzy Leonardo numbers are provided.

**AMS Subject Classification 2020:** 03E72; 11B39
**Keywords and Phrases:** Triangular fuzzy numbers, Fuzzy Fibonacci numbers, Fuzzy Lucas numbers, Leonardo numbers, Algebraic properties, Identities, Sum identities.

## 1 Introduction

Recently, several researchers have worked enthusiastically with numerical sequences. Their studies cover a wide range of fascinating aspects, including exploring unique properties, revealing previously known identities, and even unlocking the secrets behind generating functions and matrices. One of these interesting sequences is the Fibonacci sequence of numbers. The sequence of Fibonacci $\{F_n\}_{n\geq 0}$ is defined by a recurrence relation of order two, given by

$$F_n = F_{n-1} + F_{n-2},\ (n \geq 2), \tag{1}$$

with initial conditions $F_0 = 0$ and $F_1 = 1$. Other classical sequence is the sequence of Lucas numbers $\{L_n\}_{n\geq 0}$, defined by the same recurrence relation of Fibonacci sequence,

$$L_n = L_{n-1} + L_{n-2},\ (n \geq 2), \tag{2}$$

but with different initial conditions, $L_0 = 2$ and $L_1 = 1$. The Fibonacci sequence has motivated the study of many other numerical sequences. We can find not only properties of the sequences of Fibonacci but also the correlated sequences such as Lucas, Pell, and Pell-Lucas and their applications in the following works [1], [2] and [3].

One of these correlated sequences is the sequence of Leonardo, introduced by Catarino and Borges in [4], and defined by the recurrence relation

$$Le_n = Le_{n-1} + Le_{n-2} + 1,\ (n \geq 2), \tag{3}$$

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with initial conditions $Le_0 = Le_1 = 1$. This recurrence relation can have the equivalent form

$$Le_{n+1} = 2Le_n - Le_{n-2}, \ (n \geq 2).$$

The relation between Leonardo and Fibonacci numbers is given by

$$Le_n = 2F_{n+1} - 1, \tag{4}$$

according to Proposition 2.2 in [4].

The Leonardo sequence has given rise to many related research studies, which are, for example, those of Alp and Koçer in [5], Alves and Vieira in [6], Catarino and Borges in [7], Gokbas in [8], Kara and Yilmaz in [9], Kuhapatanakul and Chobsorn in [10], and Tan and Leung in [11], among others.

On the other hand, since fuzzy set theory has a lot of applications in real life, the interest in workings and researching has increased in recent years [12, 13, 14]. To face the challenges of ambiguity in various areas, Zadeh, in the article [15], introduced the fuzzy set theory. The fuzzy set theory is based on the fuzzy membership function. Given a set $A$, the membership function denoted by $\mu_A$ is a function that associates an element of a set $A$ to an element in the interval $[0, 1]$. A fuzzy set $A$ is described by its membership function $\mu_A$, and by the fuzzy membership function, we can determine the membership grade of an element concerning a set (see more details in [16, 17, 18, 19]). Following Duman, in [20], there are many fuzzy membership function types, which most commonly used are the triangular, trapezoidal, Gaussian, and generalized Bell. Fuzzy operations on fuzzy sets are defined as crisp operations performed on crisp sets. Operations on fuzzy sets are done using fuzzy membership functions. Operations such as addition, subtraction, multiplication, and division are defined in a fuzzy set, [16, 21]. When fuzzy set operations are applied to a set, the result is a fuzzy set. But these sets need to be converted to a real number, that is, an inference must be made. This process is called defuzzyfication, which means inversion of fuzzyfication [22].

Recently, a bridge between fuzzy sets and number theory was built when fuzzy Fibonacci and Lucas number sequences were defined using the triangular membership function by [23], and also several identities were provided. In addition, other properties are investigated in [20].

We aim to introduce the fuzzy Leonardo numbers using the triangular membership function and give some new properties of this new sequence. The article is organized as follows. In Section 2, we present the triangular fuzzy numbers with their operations. Also, the definitions of fuzzy Fibonacci numbers and fuzzy Lucas numbers are given as identities related to these sequences, which will be useful for the next sections. Section 3 introduces the fuzzy Leonardo numbers and establishes some properties and identities of this new set of numbers. In Section 4, some sums involving fuzzy Leonardo numbers are provided. Finally, some conclusions are stated.

## 2 Preliminaries concepts

In this section, we will present the definition of triangular fuzzy numbers, such as their arithmetic operations of the $\alpha$-cut, $\alpha \in [0, 1]$. In addition, the definitions of fuzzy Fibonacci and fuzzy Lucas numbers are given, and some properties of these numbers are presented.

First, consider the definition of the triangular fuzzy number given by Irmak and Demirtas in [23]. A triangular fuzzy number, denoted by $\tilde{A} = (a_1, a_2, a_3)$ is represented by three points, two of which are left and right of the interval, such that $a_1, a_2, a_3$ are real numbers. The triangular membership function with $\tilde{A} = (a_1, a_2, a_3)$ is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \leq a_1 \\ \frac{x-a_1}{a_2-a_1}, & a_1 < x \leq a_2 \\ \frac{a_3-x}{a_3-a_2}, & a_2 < x \leq a_3 \\ 0, & x \geq a_3 \end{cases}.$$

A triangular fuzzy number can be represented by $\alpha$-cut operation, which denotes $A^\alpha$. To convert a triangular fuzzy number to $\alpha-$cut interval, we follow that

$$A^\alpha = [a_1^\alpha, a_3^\alpha] = [a_1 + \alpha(a_2 - a_1), a_3 - \alpha(a_3 - a_2)], \tag{5}$$

where $\alpha \in [0,1]$ and $a_j$ for $j = 1, 2, 3$ are real numbers.

Let $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1, b_2, b_3)$ be the triangular fuzzy numbers and $A^\alpha = [a_1^\alpha, a_3^\alpha]$ and $B^\alpha = [b_1^\alpha, b_3^\alpha]$ be the $\alpha-$cut obtained from these numbers. The arithmetic operations of the $\alpha-$cut are given in [23] as follows

$$\begin{aligned}
A^\alpha + B^\alpha &= [a_1^\alpha + b_1^\alpha, a_3^\alpha + b_3^\alpha], && (6)\\
A^\alpha - B^\alpha &= [a_1^\alpha - b_1^\alpha, a_3^\alpha - b_3^\alpha], \\
A^\alpha B^\alpha &= [min\{a_1^\alpha b_1^\alpha, a_3^\alpha b_1^\alpha, a_1^\alpha b_3^\alpha, a_3^\alpha b_3^\alpha\}, max\{a_1^\alpha b_1^\alpha, a_3^\alpha b_1^\alpha, a_1^\alpha b_3^\alpha, a_3^\alpha b_3^\alpha\}], && (7)\\
A^\alpha / B^\alpha &= [min\{a_1^\alpha/b_1^\alpha, a_3^\alpha/b_1^\alpha, a_1^\alpha/b_3^\alpha, a_3^\alpha/b_3^\alpha\}, max\{a_1^\alpha/b_1^\alpha, a_3^\alpha/b_1^\alpha, a_1^\alpha/b_3^\alpha, a_3^\alpha/b_3^\alpha\}], \\
kA^\alpha &= [min\{ka_1^\alpha, ka_3^\alpha\}, max\{ka_1^\alpha, ka_3^\alpha\}],
\end{aligned}$$

with $k$ real number. Note that, if $a_1, b_1, a_2, b_2, a_3$ and $b_3$ are positive real numbers with $a_1 \leq a_2 \leq a_3$, and $b_1 \leq b_2 \leq b_3$, then $A^\alpha B^\alpha = [a_1^\alpha b_1^\alpha, a_3^\alpha b_3^\alpha]$. Moreover, if $k$ is a positive real number then we have $kA^\alpha = [min\{ka_1^\alpha, ka_3^\alpha\}, max\{ka_1^\alpha, ka_3^\alpha\}] = [ka_1^\alpha, ka_3^\alpha]$, (see more details in [16, 21]).

In [23], the author introduced the fuzzy Fibonacci numbers and the fuzzy Lucas numbers, which will be very useful for this article. Let $\{F_n\}_{n\geq 0}$ be the Fibonacci sequence (1). The triangular fuzzy number of Fibonacci is given by $\tilde{F_n} = (F_{n-1}, F_n, F_{n+1})$. Then, we have the following definition.

**Definition 2.1.** *Let $\{F_n\}_{n\geq 0}$ be the Fibonacci sequence (1). The fuzzy Fibonacci numbers are defined by the expression*

$$F_n^\alpha = [F_{n-1}^\alpha, F_{n+1}^\alpha] = [F_{n-1} + \alpha F_{n-2}, F_{n+1} - \alpha F_{n-1}], \tag{8}$$

*for $n \geq 2$, where $\alpha \in [0,1]$ and initial conditions are $F_0^\alpha = [1-\alpha, 1+\alpha]$ and $F_1^\alpha = [\alpha, 1]$.*

Similarly, the definition of fuzzy Lucas numbers, as proposed by Irmak and Demirtas in [23], is as follows:

**Definition 2.2.** *Let $\{L_n\}_{n\geq 0}$ be the Lucas sequence (2). The fuzzy Lucas numbers are defined by the expression*

$$L_n^\alpha = [L_{n-1}^\alpha, L_{n+1}^\alpha] = [L_{n-1} + \alpha L_{n-2}, L_{n+1} - \alpha L_{n-1}], \tag{9}$$

*for $n \geq 2$, where $\alpha \in [0,1]$ and initial conditions are $L_0^\alpha = [-1-3\alpha, 1+\alpha]$, and $L_1^\alpha = [2-\alpha, 3-2\alpha]$.*

Motivated by the previous definitions, we will introduce the fuzzy Leonardo numbers and study some properties of this new fuzzy sequence of numbers in the next section. Moreover, this article will explore the connection between the fuzzy Leonardo numbers, the fuzzy Fibonacci numbers, and the fuzzy Lucas numbers by considering the following identities for non-negative integers $n$,

| | | |
|---|---|---|
| [20, Theorem 3.1] | $F_{n+4}^\alpha + F_n^\alpha = 3F_{n+2}^\alpha$ , | (10) |
| [20, Theorem 3.2] | $F_{n+10}^\alpha = 11F_{n+5}^\alpha + F_n^\alpha$ , | (11) |
| [20, Theorem 3.3] | $F_{n+2}^\alpha - F_{n+1}^\alpha = (-F_n, F_n, 2F_{n+1})$ , | (12) |
| [23, Theorem 3.1] | $F_{n+2}^\alpha - F_{n-2}^\alpha = L_n^\alpha$ , | (13) |
| [23, Theorem 3.2(a)] | $2F_{n+2}^\alpha - 3F_n^\alpha = L_n^\alpha$ , | (14) |
| [23, Theorem 3.2(d)] | $2F_{n+1}^\alpha - F_n^\alpha = L_n^\alpha$ , | (15) |
| [23, Theorem 3.2(g)] | $F_{n+1}^\alpha + F_{n-1}^\alpha = L_n^\alpha$ | (16) |
| [23, Theorem 3.2(b), (c) and (e)] | $5F_n^\alpha = 2L_{n+2}^\alpha - 3L_n^\alpha = L_{n+1}^\alpha + L_{n-1}^\alpha = 2L_{n+1}^\alpha - L_n^\alpha$ . | (17) |

For the reason to establish identities involving the fuzzy Leonardo numbers, in this article, we will consider the following classical identities for Leonardo numbers $\{Le_n\}_{n\geq 0}$ established in Proposition 2.3 [4],

$$Le_n = 2\left(\frac{L_n + L_{n+2}}{5}\right) - 1, \tag{18}$$

$$Le_{n+3} = \left(\frac{L_{n+1} + L_{n+7}}{5}\right) - 1, \tag{19}$$

$$Le_n = L_{n+2} - F_{n+2} - 1, \tag{20}$$

for all non-negative integer $n$, where $F_n$ is the $n$-th Fibonacci number given by (1) and $L_n$ is the $n$-th Lucas number given by (2).

# 3 The fuzzy Leonardo numbers, properties and identities

In this section, we will introduce the fuzzy Leonardo numbers and provide some properties of this new sequence. Moreover, some identities are established.

## 3.1 The fuzzy Leonardo numbers and properties

Let $\{Le_n\}_{n\geq 0}$ be the Leonardo sequence of numbers defined by Equation (3) and the triangular fuzzy number of Leonardo given by $\tilde{Le}_n = (Le_{n-1}, Le_n, Le_{n+1})$. Then, it is natural to consider the $\alpha-$cut of the triangular fuzzy numbers given in the following definition.

**Definition 3.1.** *Let $\{Le_n\}_{n\geq 0}$ be Leonardo sequence given by (3). The fuzzy Leonardo numbers are defined by the following expression*

$$Le_n^\alpha = [Le_{n-1}^\alpha, Le_{n+1}^\alpha] = [Le_{n-1} + \alpha(Le_{n-2} + 1), Le_{n+1} - \alpha(Le_{n-1} + 1)], \tag{21}$$

*for $n \geq 2$, where $\alpha \in [0,1]$ and initial conditions are $Le_0^\alpha = [1-\alpha, 1+\alpha]$, and $Le_1^\alpha = [\alpha, 1]$.*

By Definition 3.1, the elements of the sequence $\{Le_n^\alpha\}_{n\geq 0}$ are the $\alpha-$cut obtained from the triangular fuzzy number of Leonardo $\tilde{Le}_n$, and can be operated by using the $\alpha-$cut operations.

Observe that by considering the triangular fuzzy number $\tilde{1} = (1,1,1)$, and by applying the $\alpha-$cut, we obtain $I^\alpha = [1^\alpha, 1^\alpha] = [1 + \alpha(1-1), 1 - \alpha(1-1)] = [1,1]$.

Then, by using the rule of summation (6), we describe a recurrence relation for the fuzzy Leonardo numbers in the next proposition.

**Proposition 3.2.** *Consider $\alpha \in [0,1]$. Let $\{Le_n^\alpha\}_{n\geq 0}$ be the sequence of fuzzy Leonardo numbers. Then, it is verified*

$$Le_n^\alpha = Le_{n-1}^\alpha + Le_{n-2}^\alpha + I^\alpha, \tag{22}$$

*where $I^\alpha = [1,1]$.*

**Proof.** By considering the sum operation and Expression (21), we have

$$\begin{aligned}
Le_{n-1}^\alpha + Le_{n-2}^\alpha + I^\alpha &= [Le_{n-2}^\alpha, Le_n^\alpha] + [Le_{n-3}^\alpha, Le_{n-1}^\alpha] + [1^\alpha, 1^\alpha] \\
&= [Le_{n-2}^\alpha + Le_{n-3}^\alpha + 1^\alpha, Le_n^\alpha + Le_{n-1}^\alpha + 1^\alpha] \\
&= [Le_{n-2} + \alpha(Le_{n-3} + 1) + Le_{n-3} + \alpha(Le_{n-4} + 1) + 1, \\
&\quad Le_n - \alpha(Le_{n-2} + 1) + Le_{n-1} - \alpha(Le_{n-3} + 1) + 1] \\
&= [Le_{n-1} + \alpha(Le_{n-2} + 1), Le_{n+1} - \alpha(Le_{n-1} + 1)] \\
&= Le_n^\alpha,
\end{aligned}$$

which verifies the result. □

In addition, observe that the Leonardo sequence $\{Le_n\}_{n\geq 0}$ is an increasing sequence of positive integers, then it is verified the scalar operation $kA^\alpha = [ka_1^\alpha, ka_3^\alpha]$, for $k$ positive real number. Moreover, since it is verified the recurrence relation for the Leonardo numbers, $Le_{n+1} = 2Le_n - Le_{n-2}$, $(n \geq 2)$, with the same proceedings done in the proof of Proposition 22 and the scalar product, we can obtain a new equation for the fuzzy Leonardo numbers given by

$$Le_n^\alpha = 2Le_n^\alpha - Le_{n-2}^\alpha.$$

### 3.2 Some Identities

This subsection will provide some new identities for the fuzzy Leonardo numbers. In addition, we will establish new identities involving the fuzzy Leonardo numbers, the fuzzy Fibonacci numbers, and the fuzzy Lucas numbers.

Recall the relation between the Leonardo and Fibonacci numbers given by (4), namely, $Le_n = 2F_{n+1} - 1$. Therefore, by using the scalar product, Definition 3.1, and Equation (4), we establish the following result.

**Proposition 3.3.** *Consider $\alpha \in [0,1]$. Let $\{Le_n^\alpha\}_{n\geq 0}$ be the sequence of fuzzy Leonardo numbers and $\{F_n^\alpha\}_{n\geq 0}$ be the sequence of fuzzy Fibonacci numbers. Then, it is verified*

$$Le_n^\alpha = 2F_{n+1}^\alpha - I^\alpha. \tag{23}$$

**Proof.** Equation (21) shows us that $Le_n^\alpha = [Le_{n-1} + \alpha(Le_{n-2}+1), Le_{n+1} - \alpha(Le_{n-1}+1)]$. Since it is verified $Le_n = 2F_{n+1} - 1$, then

$$\begin{aligned} Le_n^\alpha &= [Le_{n-1} + \alpha(Le_{n-2}+1), Le_{n+1} - \alpha(Le_{n-1}+1)] \\ &= [2F_n + 2\alpha(F_{n-1}) - 1, 2F_{n+2} - 2\alpha(F_n) - 1] \\ &= 2[F_n + \alpha F_{n-1}, F_{n+2} - \alpha F_n] - [1^\alpha, 1^\alpha] \\ &= 2F_{n+1}^\alpha - I^\alpha, \end{aligned}$$

by Equation (8). □

Similarly, recall the identities of the sequence of Leonardo numbers stated in Proposition 2.3 [4], and the operations in $\alpha$-cut. Then, the next proposition is stated.

**Proposition 3.4.** *For all non-negative integers $n$, the following identities hold:*

$$Le_n^\alpha = \frac{2}{5}\left(L_n^\alpha + L_{n+2}^\alpha\right) - I^\alpha, \tag{24}$$

$$Le_{n+3}^\alpha = \frac{1}{5}\left(L_{n+1}^\alpha + L_{n+7}^\alpha\right) - I^\alpha, \tag{25}$$

$$Le_n^\alpha = L_{n+2}^\alpha - F_{n+2}^\alpha - I^\alpha, \tag{26}$$

*where $I^\alpha = [1,1]$, $Le_n^\alpha$ is the $n$-th fuzzy Leonardo numbers, $F_n^\alpha$ is the $n$-th fuzzy Fibonacci number given by (8), and $L_n^\alpha$ is the $n$-th fuzzy Lucas number given by (9).*

**Proof.** By combining Definition 3.1 and Identity (18) we obtain

$$\begin{aligned}
& Le_n^\alpha \\
= \; & [Le_{n-1}+\alpha(Le_{n-2}+1), Le_{n+1}-\alpha(Le_{n-1}+1)] \\
= \; & 2\left[\left(\frac{L_{n-1}+L_{n+1}}{5}\right)-1+\alpha\left(\frac{L_{n-2}+L_n}{5}\right), \left(\frac{L_{n+1}+L_{n+3}}{5}\right)-1-\alpha\left(\frac{L_{n-1}+L_{n+1}}{5}\right)\right] \\
= \; & 2\left[\left(\frac{L_{n-1}+L_{n+1}}{5}\right)+\alpha\left(\frac{L_{n-2}+L_n}{5}\right), \left(\frac{L_{n+1}+L_{n+3}}{5}\right)-\alpha\left(\frac{L_{n-1}+L_{n+1}}{5}\right)\right]-1^\alpha \\
= \; & \frac{2}{5}\left[(L_{n-1}+L_{n+1})+\alpha\left(L_{n-2}+L_n\right), (L_{n+1}+L_{n+3})-\alpha\left(L_{n-1}+L_{n+1}\right)\right]-1^\alpha \\
= \; & \frac{2}{5}\left(L_n^\alpha+L_{n+2}^\alpha\right)-I^\alpha
\end{aligned}$$

Similarly, by using Definition 3.1 and Identity (19), we obtain Equation (25), as well as, by using Definition 3.1 and Identity (20) we obtain (26) □

Next, we will provide an identity related to the product of fuzzy Leonardo numbers. To do this, we need to observe the product rule (7) and the fact of the Leonardo sequence $\{Le_n\}_{n\geq 0}$ is an increasing sequence of positive integers. Then we have $Le_m^\alpha Le_k^\alpha = [Le_{m-1}^\alpha Le_{k-1}^\alpha, Le_{m+1}^\alpha Le_{k-1}^\alpha]$.

**Theorem 3.5.** *Consider $m$ and $n$ non-negative integers and let $Le_n^\alpha$ be the $n$-th fuzzy Leonardo numbers. Then*

$$\begin{aligned}
Le_m^\alpha Le_{n-m+1}^\alpha + Le_{m+1}^\alpha Le_{n-m}^\alpha = \; & [8(-1)^{n-m}(Le_{2m-n-1}+1)-Le_{m-1}-Le_{n-m}+Le_m+Le_{n-m-1} \\
& +\alpha(8(-1)^{n-m}(Le_{2m-n-2}+1)+8(-1)^{m-1}(Le_{n-2m-2}+1)+Le_{n-m}+2Le_{n-1}) \\
& +\alpha^2(8(-1)^{n-m-1}(Le_{2m-n-1}+1)+Le_{m-1}+Le_{n-m-2}), \\
& 8(-1)^{n-m+2}(Le_{2m-n}+1)-Le_{m+1}-Le_{n-m+2}+Le_{m+2}+Le_{n-m+1} \\
& -\alpha(12(-1)^{n-m+3}(Le_{2m-n-2}+1)-Le_{n-m+2} \\
& +12(-1)^{n-m+2}(Le_{2m-n}+1)+Le_{m+3}-1) \\
& +\alpha^2(8(-1)^{n-m-1}(Le_{2m-n-1}+1)+2Le_{n-m-2}+Le_{m-1}-Le_m+1)].
\end{aligned}$$

**Proof.** Using the Definition 21 we obtain

$$\begin{aligned}
Le_m^\alpha Le_{n-m+1}^\alpha + Le_{m+1}^\alpha Le_{n-m}^\alpha = \; & [Le_{m-1}+\alpha(Le_{m-2}+1), Le_{m+1}-\alpha(Le_{m-1}+1)] \\
& \times[Le_{n-m}+\alpha(Le_{n-m-1}+1), Le_{n-m+2}-\alpha(Le_{n-m}+1)] \\
& +[Le_m+\alpha(Le_{m-1}+1), Le_{m+2}-\alpha(Le_m+1)] \\
& \times[Le_{n-m-1}+\alpha(Le_{n-m-2}+1), Le_{n-m+1}-\alpha(Le_{n-m-1}+1)] \\
= \; & [(Le_{m-1}+\alpha(Le_{m-2}+1))(Le_{n-m}+\alpha(Le_{n-m-1}+1)), \\
& (Le_{m+1}-\alpha(Le_{m-1}+1))(Le_{n-m+2}-\alpha(Le_{n-m}+1))] \\
& +[(Le_m+\alpha(Le_{m-1}+1))(Le_{n-m-1}+\alpha(Le_{n-m-2}+1)), \\
& (Le_{m+2}-\alpha(Le_m+1))(Le_{n-m+1}-\alpha(Le_{n-m-1}+1))] \\
= \; & [(Le_{m-1}+\alpha(Le_{m-2}+1))(Le_{n-m}+\alpha(Le_{n-m-1}+1)) \\
& +(Le_m+\alpha(Le_{m-1}+1))(Le_{n-m-1}+\alpha(Le_{n-m-2}+1)), \\
& (Le_{m+1}-\alpha(Le_{m-1}+1))(Le_{n-m+2}-\alpha(Le_{n-m}+1)) \\
& +(Le_{m+2}-\alpha(Le_m+1))(Le_{n-m+1}-\alpha(Le_{n-m-1}+1))].
\end{aligned}$$

Denote $A_n = Le_{n-1}+\alpha Le_{n-2}$ and $B_n = Le_{n+1}-\alpha Le_{n-1}$, then we have

$$\begin{aligned}&(Le_{m-1}+\alpha(Le_{m-2}+1))(Le_{n-m}+\alpha(Le_{n-m-1}+1)\\ &=(Le_{m-1}+\alpha Le_{m-2})(Le_{n-m}+\alpha Le_{n-m-1})+\alpha(Le_{m-1}+\alpha Le_{m-2}+Le_{n-m}+\alpha Le_{n-m-1})\\ &=A_mA_{n-m+1}+\alpha(Le_{m-1}+Le_{n-m})+\alpha^2(Le_{m-2}+Le_{n-m-1}),\end{aligned} \tag{27}$$

$$\begin{aligned}&(Le_{m+1}-\alpha(Le_{m-1}+1))(Le_{n-m+2}-\alpha(Le_{n-m}+1))\\ &=(Le_{m+1}-\alpha Le_{m-1})(Le_{n-m+2}-\alpha Le_{n-m})-\alpha(Le_{m+1}-\alpha Le_{m-1}+Le_{n-m+2}-\alpha Le_{n-m})\\ &=B_mB_{n-m+1}-\alpha(Le_{m+1}+Le_{n-m+2})+\alpha^2(Le_{m-1}+Le_{n-m}),\end{aligned} \tag{28}$$

$$\begin{aligned}&(Le_{m}+\alpha(Le_{m-1}+1))(Le_{n-m-1}+\alpha(Le_{n-m-2}+1))\\ &=(Le_{m}+\alpha Le_{m-1})(Le_{n-m-1}+\alpha Le_{n-m-2})+\alpha(Le_{m}+\alpha Le_{m-1}+Le_{n-m-1}+\alpha Le_{n-m-2})\\ &=A_{m+1}A_{n-m}+\alpha(Le_{m}+Le_{n-m-1})+\alpha^2(Le_{m-1}+Le_{n-m-2}),\end{aligned} \tag{29}$$

and

$$\begin{aligned}&(Le_{m+2}-\alpha(Le_{m}+1))(Le_{n-m+1}-\alpha(Le_{n-m-1}+1))\\ &=(Le_{m+2}-\alpha Le_{m})(Le_{n-m+1}-\alpha Le_{n-m-1})+\alpha(Le_{m+2}-\alpha Le_{m}+Le_{n-m+1}-\alpha Le_{n-m-1})\\ &=B_{m+1}B_{n-m}-\alpha(Le_{m+2}+Le_{n-m+1})+\alpha^2(Le_{m}+Le_{n-m-1}).\end{aligned} \tag{30}$$

Now, since

$$\begin{aligned}A_mA_{n-m+1}&=Le_{m-1}Le_{n-m}+\alpha(Le_{n-m}Le_{m-2}+Le_{m-1}Le_{n-m-1})+\alpha^2Le_{m-2}Le_{n-m-1},\\ A_{m+1}A_{n-m}&=Le_{m}Le_{n-m-1}+\alpha(Le_{n-m-1}Le_{m-1}+Le_{m}Le_{n-m-2})+\alpha^2Le_{m-1}Le_{n-m-2},\\ B_mB_{n-m+1}&=Le_{m+1}Le_{n-m+2}-\alpha(Le_{m-1}Le_{n-m+2}+Le_{m+1}Le_{n-m})+\alpha^2Le_{m-1}Le_{n-m},\\ B_{m+1}B_{n-m}&=Le_{m+2}Le_{n-m+1}-\alpha(Le_{m}Le_{n-m+1}+Le_{m+2}Le_{n-m-1})+\alpha^2Le_{m}Le_{n-m-1},\end{aligned}$$

then, by summing Equations (27) and (29), we obtain the first component given by

$$\begin{aligned}&A_mA_{n-m+1}+\alpha(Le_{m-1}+Le_{n-m})+\alpha^2(Le_{m-2}+Le_{n-m-1})\\ &+A_{m+1}A_{n-m}+\alpha(Le_{m}+Le_{n-m-1})+\alpha^2(Le_{m-1}+Le_{n-m-2})\\ &=Le_{m-1}Le_{n-m}+Le_mLe_{n-m-1}\\ &+\alpha(Le_{n-m}Le_{m-2}+Le_{m-1}Le_{n-m-1}+Le_{n-m-1}Le_{m-1}+Le_mLe_{n-m-2}+Le_{m-1}+Le_{n-m}+Le_m+Le_{n-m-1})\\ &+\alpha^2(Le_{m-2}Le_{n-m-1}+Le_{m-1}Le_{n-m-2}+Le_{m-2}+Le_{n-m-1}).\end{aligned} \tag{31}$$

Similarly, by summing Equations (27) and (29), we obtain the second component given by

$$\begin{aligned}&B_mB_{n-m+1}-\alpha(Le_{m+1}+Le_{n-m+2})+\alpha^2(Le_{m-1}+Le_{n-m})\\ &+B_{m+1}B_{n-m}-\alpha(Le_{m+2}+Le_{n-m+1})+\alpha^2(Le_{m}+Le_{n-m-1})\\ &=Le_{m+1}Le_{n-m+2}+Le_{m+2}Le_{n-m+1}\\ &-\alpha(Le_{m-1}Le_{n-m+2}+Le_{m+1}Le_{n-m}+Le_mLe_{n-m+1}+Le_{m+2}Le_{n-m-1}+Le_{m+1}+Le_{n-m+2})\\ &+\alpha^2(Le_{m-1}Le_{n-m}+Le_mLe_{n-m-1}+Le_{m-1}+Le_{n-m}).\end{aligned} \tag{32}$$

Theorems 2.1 and 2.14 in [5] established

$$Le_{-n} = (-1)^n(Le_{n-2}+1)-1,$$
$$Le_kLe_m + Le_sLe_t = 4(-1)^m(Le_{k-s-1}+1)(Le_{k-t-1}+1) - Le_k - Le_m + Le_s + Le_t,$$

for positive integers $n, k, m, s$ and $t$ with $k+m=s+t$, then holds:

$$\begin{aligned} Le_{m-1}Le_{n-m} + Le_mLe_{n-m-1} &= 4(-1)^{n-m}(Le_{-2}+1)(Le_{2m-n-1}+1) - Le_{m-1} - Le_{n-m} + Le_m + Le_{n-m-1} \\ &= 8(-1)^{n-m}(Le_{2m-n-1}+1) - Le_{m-1} - Le_{n-m} + Le_m + Le_{n-m-1}, \end{aligned}$$

$$Le_{m-2}Le_{n-m} + Le_{m-1}Le_{n-m-1} = 8(-1)^{n-m}(Le_{2m-n-2}+1) - Le_{m-2} - Le_{n-m} + Le_{m-1} + Le_{n-m-1},$$

$$Le_{n-m-1}Le_{m-1} + Le_mLe_{n-m-2} = 8(-1)^{m-1}(Le_{n-2m-2}+1) - Le_{n-m-1} - Le_{m-1} + Le_m + Le_{n-m-2},$$

and

$$Le_{m-1}Le_{n-m} + Le_mLe_{n-m-1} = 8(-1)^{n-m}(Le_{2m-n-1}+1) - Le_{m-1} - Le_{n-m} + Le_m + Le_{n-m-1}.$$

Therefore, we can rewrite Equation (31) in the form

$$\begin{gathered} A_mA_{n-m+1} + \alpha(Le_{m-1}+Le_{n-m}) + \alpha^2(Le_{m-2}+Le_{n-m-1}) \\ +A_{m+1}A_{n-m} + \alpha(Le_m + Le_{n-m-1}) + \alpha^2(Le_{m-1}+Le_{n-m-2}) \\ = 8(-1)^{n-m}(Le_{2m-n-1}+1) - Le_{m-1} - Le_{n-m} + Le_m + Le_{n-m-1} \\ +\alpha(8(-1)^{n-m}(Le_{2m-n-2}+1) + 8(-1)^{m-1}(Le_{n-2m-2}+1) + Le_{n-m} + 2Le_{n-1}) \\ +\alpha^2(8(-1)^{n-m-1}(Le_{2m-n-1}+1) + Le_{m-1} + Le_{n-m-2}). \end{gathered} \tag{33}$$

Similarly, we have,

$$Le_{m+1}Le_{n-m+2} + Le_{m+2}Le_{n-m+1} = 8(-1)^{n-m+2}(Le_{2m-n}+1) - Le_{m+1} - Le_{n-m+2} + Le_{m+2} + Le_{n-m+1},$$

$$Le_{m-1}Le_{n-m+2} + Le_{m+1}Le_{n-m} = 12(-1)^{n-m+3}(Le_{2m-n-2}+1) - Le_{m-1} - Le_{n-m+2} + Le_{m+1} + Le_{n-m},$$

$$Le_mLe_{n-m+1} + Le_{m+2}Le_{n-m-1} = 12(-1)^{n-m+2}(Le_{2m-n}+1) - Le_m - Le_{n-m+1} + Le_{m+2} + Le_{n-m-1},$$

and

$$Le_{m-2}Le_{n-m-1} + Le_{m-1}Le_{n-m-2} = 8(-1)^{n-m-1}(Le_{2m-n-1}+1) - Le_{m-2} - Le_{n-m-1} + Le_{m-1} + Le_{n-m-2}.$$

Therefore

$$\begin{gathered}
B_m B_{n-m+1} - \alpha(Le_{m+1} + Le_{n-m+2}) + \alpha^2(Le_{m-1} + Le_{n-m}) \\
+B_{m+1}B_{n-m} - \alpha(Le_{m+2} + Le_{n-m+1}) + \alpha^2(Le_m + Le_{n-m-1}) \\
= 8(-1)^{n-m+2}(Le_{2m-n} + 1) - Le_{m+1} - Le_{n-m+2} + Le_{m+2} + Le_{n-m+1} \\
-\alpha(12(-1)^{n-m+3}(Le_{2m-n-2} + 1) - Le_{n-m+2} + 12(-1)^{n-m+2}(Le_{2m-n} + 1) + Le_{m+3} - 1) \\
+\alpha^2(8(-1)^{n-m-1}(Le_{2m-n-1} + 1) + 2Le_{n-m-2} + Le_{m-1} - Le_m + 1).
\end{gathered}$$

which proves the theorem. □

Next, we will provide identities involving the fuzzy Leonardo numbers and the fuzzy Fibonacci numbers.

**Proposition 3.6.** *Consider $\alpha \in [0,1]$. Let $\{Le_n^\alpha\}_{n\geq 0}$ be the sequence of fuzzy Leonardo numbers and $\{F_n^\alpha\}_{n\geq 0}$ be the sequence of fuzzy Fibonacci numbers. Then, the following identities hold:*

$$Le_{n+9}^\alpha - Le_{n-1}^\alpha = 22F_{n+5}^\alpha,\ n \geq 1; \tag{34}$$

$$Le_{n+3}^\alpha + Le_{n-1}^\alpha + 2I^\alpha = 6F_{n+2}^\alpha,\ n \geq 1; \tag{35}$$

$$Le_{n+1}^\alpha - Le_n^\alpha = (-2F_n, 2F_n, 4F_{n+1}),\ n \geq 0; \tag{36}$$

$$Le_{n+1}^\alpha - Le_{n-3}^\alpha = 2L_n^\alpha,\ n \geq 3; \tag{37}$$

$$Le_{n+1}^\alpha = 3Le_{n-1}^\alpha + 2I^\alpha + 2L_n^\alpha,\ n \geq 1; \tag{38}$$

$$2Le_n^\alpha - Le_{n-1}^\alpha + I^\alpha = 2L_n^\alpha,\ n \geq 1; \tag{39}$$

$$Le_n^\alpha + Le_{n-2}^\alpha + 2I^\alpha = 2L_n^\alpha,\ n \geq 2, \tag{40}$$

$$5Le_{n-1}^\alpha + 5I^\alpha = 2(L_{n+1}^\alpha + L_{n-1}^\alpha) = 2(2L_{n+1}^\alpha - L_n^\alpha),\ n \geq 1, \tag{41}$$

*where $F_n^\alpha$ in the $n$-the fuzzy Fibonacci number, and where $L_n^\alpha$ in the $n$-the fuzzy Lucas number.*

**Proof.** First, by combining Equations (23) and (11), we have

$$\begin{aligned}
Le_{n+9}^\alpha &= 2F_{10}^\alpha - I^\alpha \\
&= 2(11F_{n+5}^\alpha + F_n^\alpha) - I^\alpha \\
&= 22F_{n+5}^\alpha + 2F_n^\alpha - I^\alpha \\
&= 22F_{n+5}^\alpha + Le_{n-1}^\alpha,
\end{aligned}$$

which proves Equation (34). Similarly, by combining Equations (23) and (10), we obtain Equation (35). By combining Equations (23) and (12) we get Equation (36). Finally, for to prove Equations (37), (38), (39), (40), and (41), we combine Equations (23) and (13), Equations (23) and (14), Equations (23) and (15), Equations (23) and (16), and using Equations (23) and (17), respectively. □

## 4 Some sums involving fuzzy Leonardo numbers

In this section, we will provide some identities involving the sums of fuzzy Leonardo numbers. First, recall $I^\alpha = [1^\alpha, 1^\alpha] = [1, 1]$. By definition of the fuzzy number, we obtain $A^\alpha I^\alpha = I^\alpha A^\alpha = A^\alpha$ for all $A^\alpha$. Therefore, we have the following lemma.

**Lemma 4.1.** *Consider the fuzzy number $I^\alpha = [1,1]$. Then*

$$\sum_{j=1}^{n} I^\alpha = \left[\frac{n(n+1)}{2}\right]^\alpha .$$

**Proof.** Note that, by summation rule (6),

$$\begin{aligned}\sum_{j=1}^{n} I^\alpha &= \sum_{j=1}^{n}[1^\alpha,1^\alpha] = \sum_{j=1}^{n}[1,1] = \left[\sum_{j=1}^{n}1, \sum_{j=1}^{n}1\right] \\ &= \left[\frac{n(n+1)}{2}, \frac{n(n+1)}{2}\right] = \left[\frac{n(n+1)}{2}\right]^\alpha ,\end{aligned}$$

as required. □

**Theorem 4.2.** *Let $\{Le_j^\alpha\}_{j\geq 0}$ be the sequence of fuzzy Leonardo numbers. Then the sum of the $n$ first terms of the sequence consisting of these fuzzy numbers is given by*

$$\sum_{j=0}^{n} Le_j^\alpha = 2(F_{n+1}^\alpha - F_1^\alpha) - \left[\frac{(n-1)n}{2}\right]^\alpha + [1-\alpha, 1+\alpha].$$

**Proof.** Combining Theorem 3.5 in [5], Lemma 4.1 and Proposition 3.3, we get

$$\begin{aligned}\sum_{j=0}^{n} Le_j^\alpha &= \sum_{j=1}^{n}\left(2F_{j-1}^\alpha - I^\alpha\right) + Le_0^\alpha \\ &= \left(\sum_{j=0}^{n-1} 2F_j^\alpha - \sum_{j=1}^{n} I^\alpha\right) + Le_0^\alpha \\ &= 2\sum_{j=0}^{n-1} F_j^\alpha - \sum_{j=1}^{n-1} I^\alpha - F_{-1}^\alpha \\ &= 2(F_{n+1}^\alpha - F_1^\alpha) - \left[\frac{(n-1)n}{2}\right]^\alpha + [1-\alpha, 1+\alpha],\end{aligned}$$

as required. □

**Proposition 4.3.** *Let $\{Le_j^\alpha\}_{j\geq 0}$ be the sequence of fuzzy Leonardo numbers. Then the sum of $n$ first even terms of the sequence is:*

$$\sum_{j=0}^{n} Le_{2j}^\alpha = 2(F_{2n}^\alpha - F_1^\alpha) - \left[\frac{(n-1)n}{2}\right]^\alpha + [1-\alpha, 1+\alpha].$$

**Proof.** Note that

$$\begin{aligned}\sum_{j=0}^{n} Le_{2j}^\alpha &= \sum_{j=1}^{n}\left(2F_{2j-1}^\alpha - I^\alpha\right) + Le_0^\alpha \\ &= \left(\sum_{j=0}^{n-1} 2F_{2j-1}^\alpha - \sum_{j=1}^{n} I^\alpha\right) + Le_0^\alpha \\ &= 2\sum_{j=0}^{n-1} F_{2j-1}^\alpha - \sum_{j=1}^{n-1} I^\alpha + [1-\alpha, 1+\alpha] .\end{aligned}$$

According Theorem 3.5 in [5], and Lemma 4.1, we have that

$$\sum_{j=0}^{n} Le_{2j}^{\alpha} \quad = \quad 2(F_{2n}^{\alpha} - F_{1}^{\alpha}) - \left[\frac{(n-1)n}{2}\right]^{\alpha} + [1-\alpha, 1+\alpha],$$

as required. □

**Proposition 4.4.** *Let $\{Le_j^{\alpha}\}_{j\geq 0}$ be the sequence of fuzzy Leonardo numbers. Then the sum of $n$ first odd terms of the sequence is:*

$$\sum_{j=0}^{n} Le_{2j+1}^{\alpha} = 2(F_{2n+1}^{\alpha} - F_{1}^{\alpha}) - \left[\frac{(n-1)n}{2}\right]^{\alpha} + [1-\alpha, 1+\alpha].$$

**Proof.** Observe that

$$\begin{aligned}
\sum_{j=0}^{n} Le_{2j+1}^{\alpha} &= \sum_{j=1}^{n} \left(2F_{2j}^{\alpha} - I^{\alpha}\right) + Le_{0}^{\alpha} \\
&= \left(\sum_{j=0}^{n-1} 2F_{2j}^{\alpha} - \sum_{j=1}^{n} I^{\alpha}\right) + Le_{0}^{\alpha} \\
&= 2\sum_{j=0}^{n} F_{2j}^{\alpha} - \sum_{j=1}^{n-1} I^{\alpha} + [1-\alpha, 1+\alpha]\ .
\end{aligned}$$

Therefore, by Theorem 3.5 in [5] we obtain

$$\sum_{j=0}^{n} Le_{2j+1}^{\alpha} \quad = \quad 2(F_{2n+1}^{\alpha} - F_{1}^{\alpha}) - \left[\frac{(n-1)n}{2}\right]^{\alpha} + [1-\alpha, 1+\alpha],$$

as desired. □

A direct and immediate consequence of Proposition 4.3 and Proposition 4.4 is the result we now present, which arises naturally from the established relationships and further reinforces the conclusions derived from the propositions.

**Proposition 4.5.** *Let $\{Le_n^{\alpha}\}_{jn\geq 0}$ be the sequence of fuzzy Leonardo numbers. For all non-negative integers $n$, we have the following formulas:*

$$\sum_{j=0}^{n} (-1)^k Le_k^{\alpha} = 2F_{2n}^{\alpha} - 2F_{2n+1}^{\alpha};$$

*if the last term is negative and*

$$\sum_{j=0}^{n} (-1)^k Le_k^{\alpha} = 2F_{2n+2}^{\alpha} - 2F_{2n+1}^{\alpha} + [2n+1]^{\alpha};$$

*if the last term is positive.*

**Proof.** First, consider that the last term is negative, then

$$\begin{aligned}
&\sum_{k=0}^{2n+1}(-1)^k Le_k^\alpha \\
= \;& Le_0^\alpha - Le_1^\alpha + Le_2^\alpha - Le_3^\alpha + \cdots + Le_{2n}^\alpha - Le_{2n+1}^\alpha \\
= \;& (Le_0^\alpha + Le_2^\alpha + \cdots + Le_{2n}^\alpha) - (Le_1^\alpha + Le_3^\alpha + \cdots + Le_{2n+1}^\alpha) \\
= \;& \sum_{k=0}^{n} Le_{2k}^\alpha - \sum_{k=0}^{n} Le_{(2k+1)}^\alpha \\
= \;& \left(2(F_{2n}^\alpha - F_1^\alpha) - \left[\frac{(n-1)n}{2}\right]^\alpha + [1-\alpha, 1+\alpha]\right) \\
& - \left(2(F_{2n+1}^\alpha - F_1^\alpha) - \left[\frac{(n-1)n}{2}\right]^\alpha + [1-\alpha, 1+\alpha]\right) \\
= \;& 2F_{2n}^\alpha - 2F_{2n+1}^\alpha.
\end{aligned}$$

In which case that last term is positive, then

$$\begin{aligned}
&\sum_{k=0}^{2(n+1)}(-1)^k Le_k^\alpha \\
= \;& Le_0^\alpha - Le_1^\alpha + Le_2^\alpha - Le_3^\alpha + \cdots + Le_{2n}^\alpha - Le_{2n+1}^\alpha + Le_{2n+2}^\alpha \\
= \;& \sum_{k=0}^{n+1} Le_{2k}^\alpha - \sum_{k=0}^{n} Le_{2k+1}^\alpha \\
= \;& \left(2(F_{2n+2}^\alpha - F_1^\alpha) - \left[\frac{(n+1)(n+2)}{2}\right]^\alpha + [1-\alpha, 1+\alpha]\right) \\
& - \left(2(F_{2n+1}^\alpha - F_1^\alpha) - \left[\frac{(n-1)n}{2}\right]^\alpha + [1-\alpha, 1+\alpha]\right) \\
= \;& 2F_{2n+2}^\alpha - 2F_{2n+1}^\alpha + [2n+1]^\alpha,
\end{aligned}$$

which verifies the result. □

## 5 Conclusion

In this study, we introduced a new sequence of fuzzy numbers, namely, the sequence of fuzzy Leonardo numbers. We established the recurrence relation to this new sequence, some properties, as well as some identities. In addition, we explored the relation between fuzzy Leonardo, fuzzy Fibonacci, and fuzzy Lucas numbers, and some identities were given. Moreover, we provided some sums identities for the fuzzy Leonardo numbers.

It seems to us that all results given here are new in the literature.

Number sequences, especially recurring ones, establish patterns in the real world and are therefore used as discrete growth models. Discrete models are easy to solve and, in some cases, can describe solutions with predictions that are as good as continuous models. On the other hand, in some real problems, we have a certain degree of uncertainty about the solution, and that is why we use a fuzzy number to give us flexibility in finding the best solution for that problem. The construction presented in this article, a priori, is simply the immersion of a recurring integer sequence over the fuzzy number structure. However, generally, the combination of both theories can be the premise for establishing discrete growth models that combine the flexibility of fuzzy logic with the structural properties of the discrete models, and then the models can be discussed closer to the real world.

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