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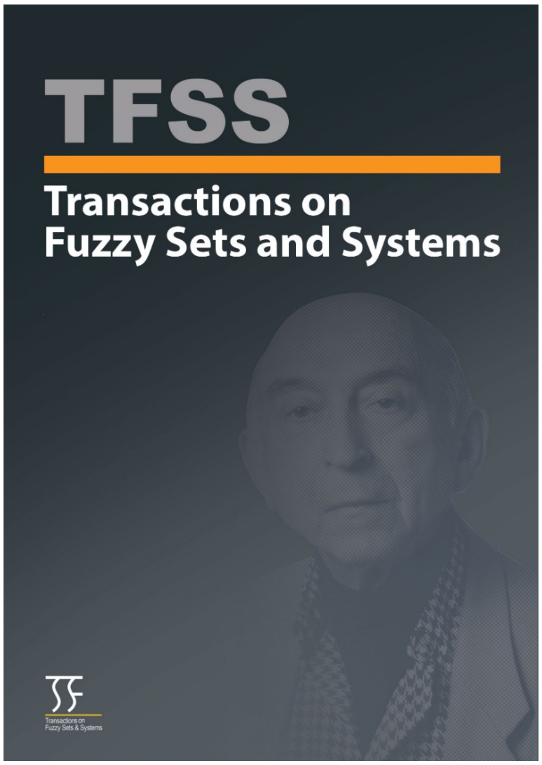

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Fuzzy Metric Spaces and Corresponding Fixed Point Theorems for Fuzzy Type Contraction

Musa Adeku Ibrahim, **Muhammed Raji***, **Kamilu Rauf**

Abstract. In this paper, we present innovative concepts of fuzzy type contractions and leverage them to establish fixed point theorems for fuzzy mappings within the framework of fuzzy metric spaces. The results of this article are applied to multivalued mappings and fuzzy mappings for contractive fuzzy type mappings. Through illustrative examples, we showcase the practical applicability of our proposed notions and results, demonstrating their effectiveness in real-world scenarios.

AMS Subject Classification 2020: 47H10; 54H25; 46S40
Keywords and Phrases: Fixed point, Fuzzy fixed point, Contractive type mapping, Hausdorff fuzzy metric space, fuzzy mapping.

1 Introduction

Throughout the years, Banach's fixed point theorems for contraction mappings have emerged as pivotal findings in the realm of mathematical analysis. These results, particularly Banach's contraction principle [1], have greatly contributed to the evolution of metric fixed point theory. By offering a reliable framework, this principle and its variations serve as invaluable tools in ensuring both the existence and uniqueness of solutions to nonlinear problems, including integral equations, differential equations, variational inequalities, and optimization problems. Numerous mathematicians have dedicated extensive efforts to refine and broaden this principle from various angles. Below, we delve into some of these noteworthy contributions. By reducing the triangle inequality constraint of the standard metric spaces, Czerwik [2] established the idea of b-metric spaces. The fixed-point properties of set-valued operators in b-metric spaces were then examined by Boriceanu [3], who also gave some specific instances of b-metric spaces. The notion of dislocated b-metric space, which is a generalization of b-metric spaces, was further developed by Hussain et al. [4]. They also proved certain fixed-point findings for four mappings that meet the generalized weak contractive conditions in a partially ordered dislocated b-metric space. The idea of fuzzy sets was first introduced by Zadeh [5], who also laid the groundwork for further studies in fuzzy mathematics. Weiss [6] explored fuzzy mappings and obtained multiple fixed point findings, expanding on Zadeh's work. Heilpern [7] introduced the idea of fuzzy contraction mappings, which is a further development of fuzzy mappings. Similar to Nadler's [8] fixed point theorem for multivalued mappings, he established a fixed point theorem for fuzzy contraction mappings. Later, in order to establish some common fixed point results for fuzzy mappings obeying a new rational F-contraction of Ciric type, Shahzad et al. [9] introduced the concept of F-contractions. The presence of fuzzy fixed

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points of set-valued fuzzy mappings in metric and fuzzy metric spaces that meet the Ciric type contraction in complete metric spaces was recently studied by Kanwal et al. [10]. In this direction several authors obtained further results, some of which can be found in [11, 12, 13, 14, 15, 16] . Considering the insights, we aim to introduce novel concepts of fuzzy type contractions and subsequently establish fixed point results for fuzzy mappings within the framework of fuzzy metric spaces. To bolster our findings, we offer illustrative examples demonstrating the practical application of the presented results and concepts. In addition, we present applications of our main results to multivalued mappings and fuzzy mappings.
Throughout our discourse, we let $CB(X)$ denote the family of all closed and bounded subsets of the metric space (X, d).

2 Preliminaries

In this section, we will introduce some definitions and lemmas that will be used in the rest of this work.

Definition 2.1. [17, 18] A function with X as its domain and the interval $[0, 1]$ as its range is called a fuzzy set in X. $F(X)$ represents the set of all fuzzy sets in X. The degree of membership of x in A is denoted by the value $A(x)$, given a fuzzy set A and a point x in X. A fuzzy set A's α-level set is represented by $[A]_\alpha$ and has the following definition:

$$[A]_\alpha = \{x : A(x) \geq \alpha\} \;\; where\ \alpha \in (0,1), [A]_0 = \{x : A(x) > 0\}$$

Definition 2.2. [19, 20] Let Y be a metric space and X a nonempty set. If a mapping T is a mapping from X into $F(Y)$, the set of all fuzzy sets on Y, then it is referred to as a fuzzy mapping. The degree to which y is a member of $T(x)$ is the membership function of a fuzzy mapping T, represented as $T(x)(y)$. Stated differently, $T(x)(y)$ represents y's degree of membership in the fuzzy set $T(x)$. Instead of using $[T(x)]_\alpha$ to denote the α-level set of $T(x)$, we will simply use $[Tx]_\alpha$.

Definition 2.3. [21, 22, 23] A fuzzy fixed point of a fuzzy mapping $T : X \to F(X)$ is defined as a point $x \in X$ where $\alpha \in (0, 1]$ and $x \in [Tx]_\alpha$.

Definition 2.4. [24] Let (X, d) be a metric space. Hausdorff metric H on $CB(X)$ induced by d is defined as

$$H(A,B) = max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\}, forall A,B \in CB(X),$$

where $d(a, B) = \inf\{d(a, b) : b \in B\}$.

Lemma 2.5. [25] *Assume that A and B are bounded, nonempty subsets of a metric space (X, d). If $a \in A$, then*

$$d(a,B) \leq H(A,B)$$

Lemma 2.6. [25] *Assume that A and B are bounded, nonempty subsets of a metric space (X, d) and $0 < \sigma \in R$. Then for $a \in A$, there exists $b \in B$ such that*

$$d(a,b) \leq H(A,B) + \sigma.$$

Lemma 2.7. [10] *If $A, B \in CB(X)$ with $H(A, B) < \varepsilon$, then for all $a \in A$, there exists $b \in B$ such that*

$$d(a,b) < \varepsilon.$$

Lemma 2.8. [10] *Let $\mu \in X$ and $A \in CB(X)$, $d(\mu, A) \leq d(\mu, v)$ for all $v \in A$.*

3 Main Results

Here are the definitions that we use to start this section.

Definition 3.1. Let $T : X \to F(X)$ be a fuzzy mapping and $(X,\ d)$ be a complete metric space. Assume that $\alpha(x) \in (0,1]$, and that the closed, bounded subsets of X are $[Tx]_{\alpha(x)}$ and $[Ty]_{\alpha(y)}$, respectively, non-empty. Then, T is considered fuzzy type I if it satisfies the following requirement for all x, y in X and $a_1, a_2, a_3, a_4 \geq 0$ with $a_1 + 2a_2 + a_3 + a_4 < 1$:

$$H([Tx]_{\alpha(x)}, [Ty]_{\alpha(y)}) \leq a_1 d(x,y) + a_2[d(x,[Tx]_{\alpha(x)}) + d(y,[Ty]_{\alpha(y)})] + a_3 \frac{d(x,[Tx]_{\alpha(x)})d(y,[Ty]_{\alpha(y)})}{d(x,y) + d(x,[Ty]_{\alpha(y)}) + d(y,[Tx]_{\alpha(x)})}$$
$$+ a_4 \frac{d(x,[Tx]_{\alpha(x)})d(x,[Ty]_{\alpha(y)}) + d(y,[Tx]_{\alpha(x)})d(y,[Ty]_{\alpha(y)})}{d(x,[Ty]_{\alpha(y)}) + d(y,[Tx]_{\alpha(x)})} \quad (3.1)$$

Definition 3.2. Let $T : X \to F(X)$ be a fuzzy mapping and $(X,\ d)$ be a complete metric space. Assume that $\alpha(x) \in (0,1]$, and that the closed, bounded subsets of X are $[Tx]_{\alpha(x)}$ and $[Ty]_{\alpha(y)}$, respectively, non-empty. Then, T is considered fuzzy type II if it satisfies the following requirement for all x, y in X and $a_1, a_2, a_3, a_4 \geq 0$ with $a_1 + 2a_2 + a_3 + a_4 < 1$:

$$H([Tx]_{\alpha(x)}, [Ty]_{\alpha(y)}) \leq a_1 d(x,y) + a_2[d(x,[Ty]_{\alpha(y)}) + d(y,[Tx]_{\alpha(x)})] + a_3 \frac{d(x,[Tx]_{\alpha(x)})d(y,[Ty]_{\alpha(y)})}{d(x,y) + d(x,[Ty]_{\alpha(y)}) + d(y,[Tx]_{\alpha(x)})}$$
$$+ a_4 \frac{d(x,[Tx]_{\alpha(x)})d(x,[Ty]_{\alpha(y)}) + d(y,[Tx]_{\alpha(x)})d(y,[Ty]_{\alpha(y)})}{d(x,[Ty]_{\alpha(y)}) + d(y,[Tx]_{\alpha(x)})} \quad (3.2)$$

Theorem 3.3. *Let $(X,\ d)$ be a complete metric space and $T : X \to F(X)$ be a fuzzy type I contraction mapping. Then, T has a fixed point in X.*

Proof. Let $x_0 \in X$ be any arbitrary point in X and $[Tx_0]_{\alpha(x_0)} \neq 0$ be a closed and bounded subsets of X. Let $x_1 \in [Tx_0]_{\alpha(x_0)}$. Since $[Tx_1]_{\alpha(x_1)} \neq 0$ is a closed and bounded subsets of X and by using Lemma 2.6, there exists $x_2 \in [Tx_1]_{\alpha(x_1)}$ such that

$$d(x_1, x_2) \leq H([Tx_0]_{\alpha(x_0)}, [Tx_1]_{\alpha(x_1)}) + \sigma \quad (3.3)$$

Again, $[Tx_0]_{\alpha(x_0)} \neq 0$ is a closed and bounded subsets of X and by using Lemma 2.6, there exists $x_3 \in [Tx_2]_{\alpha(x_2)}$ such that

$$d(x_2, x_3) \leq H([Tx_1]_{\alpha(x_1)}, [Tx_2]_{\alpha(x_2)}) + \sigma^2 \quad (3.4)$$

Continuing in this manner, we create a sequence x_n of points in X such that $x_n \in [Tx_{n-1}]_{\alpha(x_{n-1})}$, we can choose $x_{n+1} \in [Tx_{n-1}]_{\alpha(x_{n-1})}$ such that

$$d(x_n, x_{n+1}) \leq H([Tx_{n-1}]_{\alpha(x_{n-1})}, [Tx_n]_{\alpha(x_n)}) + \sigma n. \quad (3.5)$$

Now, from (3.3),

$$d(x_1, x_2) \leq H([Tx_0]_{\alpha(x_0)}, [Tx_1]_{\alpha(x_1)}) + \sigma,$$

using (3.1), we get

$$d(x_1, x_2) \leq H([Tx_0]_{\alpha(x_0)}, [Tx_1]_{\alpha(x_1)}) + \sigma,$$

$$\leq a_1 d(x_0, x_1) + a_2[d(x_0,[Tx_0]_{\alpha(x_0)}) + d(x_1,[Tx_1]_{\alpha(x_1)})] + a_3 \frac{d(x_0,[Tx_0]_{\alpha(x_0)})d(x_1,[Tx_1]_{\alpha(x_1)})}{d(x_0,x_1) + d(x_0,[Tx_1]_{\alpha(x_1)}) + d(x_1,[Tx_0]_{\alpha(x_0)})}$$

$$+ a_4 \frac{d(x_0,[Tx_0]_{\alpha(x_0)})d(x_0,[Tx_1]_{\alpha(x_1)}) + d(x_1,[Tx_0]_{\alpha(x_0)})d(x_1,[Tx_1]_{\alpha(x_1)})}{d(x_0,[Tx_1]_{\alpha(x_1)}) + d(x_1,[Tx_0]_{\alpha(x_0)})} + \sigma \tag{3.6}$$

$$d(x_1,x_2) \leq a_1 d(x_0,x_1) + a_2[d(x_0,x_1) + d(x_1,x_2)] + a_3 d(x_0,x_1) + a_4 d(x_0,x_1) + \sigma$$

$$d(x_1,x_2) \leq \frac{a_1+a_2+a_3+a_4}{1-a_2} d(x_0,x_1) + \frac{\sigma}{1-a_2}. \tag{3.7}$$

Let $\sigma = \frac{a_1+a_2+a_3+a_4}{1-a_2}$. Since $a_1 + 2a_2 + a_3 + a_4 < 1$ implies that $\frac{a_1+a_2+a_3+a_4}{1-a_2} < 1$. Hence,

$$d(x_1,x_2) \leq \sigma\, d(x_0,x_1) + \frac{\sigma}{1-a_2} \;\; for \;\; all \;\; n \in N. \tag{3.8}$$

Now, from (3.4),

$$d(x_2,x_3) \leq H([Tx_1]_{\alpha(x_1)},[Tx_2]_{\alpha(x_2)}) + \sigma^2,$$

using (3.1), we get

$$d(x_2,x_3) \leq H([Tx_1]_{\alpha(x_1)},[Tx_2]_{\alpha(x_2)}) + \sigma^2,$$

$$\leq a_1 d(x_1,x_1) + a_2[d(x_1,[Tx_1]_{\alpha(x_1)}) + d(x_2,[Tx_2]_{\alpha(x_2)})] + a_3 \frac{d(x_1,[Tx_1]_{\alpha(x_1)})d(x_2,[Tx_2]_{\alpha(x_2)})}{d(x_1,x_2) + d(x_1,[Tx_2]_{\alpha(x_2)}) + d(x_2,[Tx_1]_{\alpha(x_1)})}$$

$$+ a_4 \frac{d(x_1,[Tx_1]_{\alpha(x_1)})d(x_1,[Tx_2]_{\alpha(x_2)}) + d(x_2,[Tx_1]_{\alpha(x_1)})d(x_2,[Tx_2]_{\alpha(x_2)})}{d(x_1,[Tx_2]_{\alpha(x_2)}) + d(x_2,[Tx_1]_{\alpha(x_1)})} + \sigma^2 \tag{3.9}$$

$$d(x_2,x_3) \leq a_1 d(x_1,x_2) + a_2[d(x_1,x_2) + d(x_2,x_3)] + a_3 d(x_1,x_2) + a_4 d(x_1,x_2) + \sigma^2$$

$$d(x_2,x_3) \leq \frac{a_1+a_2+a_3+a_4}{1-a_2} d(x_1,x_2) + \frac{\sigma^2}{1-a_2}. \tag{3.10}$$

Let $\sigma = \frac{a_1+a_2+a_3+a_4}{1-a_2}$. Since $a_1 + 2a_2 + a_3 + a_4 < 1$ implies that $\frac{a_1+a_2+a_3+a_4}{1-a_2} < 1$. Hence,

$$d(x_2,x_3) \leq \sigma\, d(x_1,x_2) + \frac{\sigma^2}{1-a_2} \;\; for \;\; all \;\; n \in N. \tag{3.11}$$

$$d(x_2,x_3) \leq \sigma\,[\sigma\, d(x_0,x_1) + \frac{\sigma}{1-a_2}\;] + \frac{\sigma^2}{1-a_2}$$

$$d(x_2,x_3) \leq \sigma^2 d(x_0,x_1) + \frac{\sigma^2}{1-a_2} + \frac{\sigma^2}{1-a_2}$$

$$d(x_2,x_3) \leq \sigma^2 d(x_0,x_1) + \frac{2\sigma^2}{1-a_2} \tag{3.12}$$

Now,

$$d(x_3,x_4) \leq H([Tx_2]_{\alpha(x_2)},[Tx_3]_{\alpha(x_3)}) + \sigma^3,$$

using (3.1), we get

$$d(x_3,x_4) \leq H([Tx_2]_{\alpha(x_2)},[Tx_3]_{\alpha(x_3)}) + \sigma^3,$$

$$\leq a_1 d(x_2,x_2) + a_2[d(x_2,[Tx_2]_{\alpha(x_2)}) + d(x_3,[Tx_3]_{\alpha(x_3)})] + a_3 \frac{d(x_2,[Tx_2]_{\alpha(x_2)})d(x_3,[Tx_3]_{\alpha(x_3)})}{d(x_2,x_3) + d(x_2,[Tx_3]_{\alpha(x_3)}) + d(x_3,[Tx_2]_{\alpha(x_2)})}$$

$$+ a_4 \frac{d(x_2,[Tx_2]_{\alpha(x_2)})d(x_2,[Tx_3]_{\alpha(x_3)}) + d(x_3,[Tx_2]_{\alpha(x_2)})d(x_3,[Tx_3]_{\alpha(x_3)})}{d(x_2,[Tx_3]_{\alpha(x_3)}) + d(x_3,[Tx_2]_{\alpha(x_2)})} + \sigma^3 \tag{3.13}$$

$$d(x_3, x_4) \leq a_1 d(x_2, x_3) + a_2[d(x_2, x_3) + d(x_3, x_4)] + a_3 d(x_2, x_3) + a_4 d(x_2, x_3) + \sigma^3$$

$$d(x_3, x_4) \leq \frac{a_1 + a_2 + a_3 + a_4}{1 - a_2} d(x_2, x_3) + \frac{\sigma^3}{1 - a_2}. \tag{3.14}$$

Let $\sigma = \frac{a_1+a_2+a_3+a_4}{1-a_2}$. Since $a_1 + 2a_2 + a_3 + a_4 < 1$ implies that $\frac{a_1+a_2+a_3+a_4}{1-a_2} < 1$. Hence,

$$d(x_3, x_4) \leq \sigma\, d(x_2, x_3) + \frac{\sigma^3}{1 - a_2} \;\; for \;\; all \;\; n \in N. \tag{3.15}$$

$$d(x_3, x_4) \leq \sigma\, [\sigma^2 d(x_0, x_1) + \frac{2\sigma^2}{1 - a_2} \;\,] + \frac{\sigma^3}{1 - a_2}$$

$$d(x_3, x_4) \leq \sigma^3 d(x_0, x_1) + \frac{2\sigma^3}{1 - a_2} + \frac{\sigma^3}{1 - a_2}$$

$$d(x_3, x_4) \leq \sigma^3 d(x_0, x_1) + \frac{3\sigma^3}{1 - a_2} \tag{3.16}$$

Again, continuing in this fashion, we have

$$d(x_n, x_{n+1}) \leq \sigma^n d(x_0, x_1) + \frac{n\sigma^n}{1 - a_2} \tag{3.17}$$

If $n > m$ and $n, m \in N$, then we have

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m). \tag{3.18}$$

Applying (3.17) in (3.18) , we get

$$d(x_n, x_m) \leq \sigma^n d(x_0, x_1) + \frac{n\sigma^n}{1 - a_2} + \sigma^{n+1} d(x_0, x_1) + \frac{(n+1)\sigma^{n+1}}{1 - a_2} + \cdots + \sigma^{m-1} d(x_0, x_1) + \frac{(m-1)\sigma^{m-1}}{1 - a_2}$$

$$d(x_n, x_m) \leq \sigma^n d(x_0, x_1)(1 + \sigma + \sigma^2 + \sigma^3 + \cdots + \sigma^{m-n-1}) + \sum_{i=n}^{m-1} \frac{i\sigma^i}{1 - a_2}$$

$$d(x_n, x_m) \leq \sigma^n d(x_0, x_1)(\frac{1 - \sigma^{m-n}}{1 - \sigma}) + \sum_{i=n}^{m-1} \frac{i\sigma^i}{1 - a_2} \tag{3.19}$$

On taking $m, n \to \infty$ in (3.19), we get

$$d(x_n, x_m) = 0. \tag{3.20}$$

This proves that the sequence $\{x_n\}$ is a Cauchy sequence in X. Since X is a complete metric space, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Now,

$$d(x^*, [Tx^*]_{\alpha(x^*)}) \leq [d(x^*, x_n) + d(x_n, [Tx^*]_{\alpha(x^*)})],$$

using (3.1), we get

$$d(x^*, [Tx^*]_{\alpha(x^*)}) \leq d(x^*, x_n) + a_1 d(x_{n-1}, x^*) + a_2[d(x_{n-1}, [Tx_{n-1}]_{\alpha(x_{n-1})}) + d(x^*, [Tx^*]_{\alpha(x^*)})] +$$
$$a_3 \frac{d(x_{n-1}, [Tx_{n-1}]_{\alpha(x_{n-1})}) d(x^*, [Tx^*]_{\alpha(x^*)})}{d(x_{n-1}, x^*) + d(x_{n-1}, [Tx^*]_{\alpha(x^*)}) + d(x^*, [Tx_{n-1}]_{\alpha(x_{n-1})})} +$$

$$a_4\frac{d(x_{n-1},[Tx_{n-1}]_{\alpha(x_{n-1})})d(x_{n-1},[Tx^*]_{\alpha(x^*)})+d(x^*,[Tx_{n-1}]_{\alpha(x_{n-1})})d(x^*,[Tx^*]_{\alpha(x^*)})}{d(x_{n-1},[Tx^*]_{\alpha(x^*)})+d(x^*,[Tx_{n-1}]_{\alpha(x_{n-1})})}$$

$$d(x^*,[Tx^*]_{\alpha(x^*)})\le d(x^*,x_n)+a_1d(x_{n-1},x^*)+a_2[d(x_{n-1},x_n)+d(x^*,[Tx^*]_{\alpha(x^*)})]+$$
$$a_3\frac{d(x_{n-1},x_n)d(x^*,[Tx^*]_{\alpha(x^*)})}{d(x_{n-1},x^*)+d(x_{n-1},[Tx^*]_{\alpha(x^*)})+d(x^*,x_n)}+$$
$$a_4\frac{d(x_{n-1},x_n)d(x_{n-1},[Tx^*]_{\alpha(x^*)})+d(x^*,x_n)d(x^*,[Tx^*]_{\alpha(x^*)})}{d(x_{n-1},[Tx^*]_{\alpha(x^*)})+d(x^*,x_n)} \tag{3.21}$$

On taking $n\to\infty$ in (3.21), we get

$$d(x^*,[Tx^*]_{\alpha(x^*)})\le d(x^*,x^*)+a_1d(x^*,x^*)+a_2[d(x^*,x^*)+d(x^*,[Tx^*]_{\alpha(x^*)})]+$$
$$a_3\frac{d(x^*,x^*)d(x^*,[Tx^*]_{\alpha(x^*)})}{d(x^*,x^*)+d(x^*,[Tx^*]_{\alpha(x^*)})+d(x^*,x^*)}+$$
$$a_4\frac{d(x^*,x^*)d(x^*,[Tx^*]_{\alpha(x^*)})+d(x^*,x^*)d(x^*,[Tx^*]_{\alpha(x^*)})}{d(x^*,[Tx^*]_{\alpha(x^*)})+d(x^*,x^*)}$$

implies

$$(1-a_2)d(x^*,[Tx^*]_{\alpha(x^*)})\le 0 \tag{3.22}$$

Since $a_1+2a_2+a_3+a_4<1$ implies $a_1+a_2+a_3+a_4<1-a_2$, that is, $1-a_2\neq 0$. Hence,

$$d(x^*,[Tx^*]_{\alpha(x^*)})=0.$$

Implies

$$x^*\in[Tx^*]_{\alpha(x^*)}.$$

Thus, $x^*\in X$ is the fixed point. □

Theorem 3.4. *Let $(X,\ d)$ be a complete metric space and $T:X\to F(X)$ be a fuzzy type II contraction mapping. Then, T has a fixed point in X.*

Proof. Let $x_0\in X$ be any arbitrary point in X and $[Tx_0]_{\alpha(x_0)}\neq 0$ be a closed and bounded subsets of X. Let $x_1\in[Tx_0]_{\alpha(x_0)}$. Since $[Tx_1]_{\alpha(x_1)}\neq 0$ is a closed and bounded subsets of X and by using Lemma 2.6, there exists $x_2\in[Tx_1]_{\alpha(x_1)}$ such that

$$d(x_1,x_2)\le H([Tx_0]_{\alpha(x_0)},[Tx_1]_{\alpha(x_1)})+\sigma. \tag{3.23}$$

Again, $[Tx_0]_{\alpha(x_0)}\neq 0$ is a closed and bounded subsets of X and by using Lemma 2.6, there exists $x_3\in[Tx_2]_{\alpha(x_2)}$ such that

$$d(x_2,x_3)\le H([Tx_1]_{\alpha(x_1)},[Tx_2]_{\alpha(x_2)})+\sigma^2 \tag{3.24}$$

Continuing in this manner, we create a sequence x_n of points in X such that $x_n\in[Tx_{n-1}]_{\alpha(x_{n-1})}$, we can choose $x_{n+1}\in[Tx_{n-1}]_{\alpha(x_{n-1})}$ such that

$$d(x_n,x_{n+1})\le H([Tx_{n-1}]_{\alpha(x_{n-1})},[Tx_n]_{\alpha(x_n)})+\sigma^n. \tag{3.25}$$

Now, from (3.23),

$$d(x_1,x_2)\le H([Tx_0]_{\alpha(x_0)},[Tx_1]_{\alpha(x_1)})+\sigma,$$

using (3.2), we get

$$d(x_1,x_2) \leq H([Tx_0]_{\alpha(x_0)},[Tx_1]_{\alpha(x_1)})+\sigma,$$

$$\leq a_1 d(x_0,x_1)+a_2[d(x_0,[Tx_1]_{\alpha(x_1)})+d(x_1,[Tx_0]_{\alpha(x_0)})]+a_3\frac{d(x_0,[Tx_0]_{\alpha(x_0)})d(x_1,[Tx_1]_{\alpha(x_1)})}{d(x_0,x_1)+d(x_0,[Tx_1]_{\alpha(x_1)})+d(x_1,[Tx_0]_{\alpha(x_0)})}$$

$$+a_4\frac{d(x_0,[Tx_0]_{\alpha(x_0)})d(x_0,[Tx_1]_{\alpha(x_1)})+d(x_1,[Tx_0]_{\alpha(x_0)})d(x_1,[Tx_1]_{\alpha(x_1)})}{d(x_0,[Tx_1]_{\alpha(x_1)})+d(x_1,[Tx_0]_{\alpha(x_0)})}+\sigma \tag{3.26}$$

$$d(x_1,x_2) \leq a_1 d(x_0,x_1)+a_2[d(x_0,x_2)+d(x_1,x_1)]+a_3 d(x_0,x_1)+a_4 d(x_0,x_1)+\sigma.$$

By triangular inequality, we have

$$d(x_1,x_2) \leq \frac{a_1+a_2+a_3+a_4}{1-a_2}d(x_0,x_1)+\frac{\sigma}{1-a_2}. \tag{3.27}$$

Let $\sigma=\frac{a_1+a_2+a_3+a_4}{1-a_2}$. Since $a_1+2a_2+a_3+a_4<1$ implies that $\frac{a_1+a_2+a_3+a_4}{1-a_2}<1$. Hence,

$$d(x_1,x_2) \leq \sigma\, d(x_0,x_1)+\frac{\sigma}{1-a_2} \ \ for \ \ all \ \ n \in N. \tag{3.28}$$

Now, from (3.24),

$$d(x_2,x_3) \leq H([Tx_1]_{\alpha(x_1)},[Tx_2]_{\alpha(x_2)})+\sigma^2,$$

using (3.2), we get

$$d(x_2,x_3) \leq H([Tx_1]_{\alpha(x_1)},[Tx_2]_{\alpha(x_2)})+\sigma^2,$$

$$\leq a_1 d(x_1,x_1)+a_2[d(x_1,[Tx_2]_{\alpha(x_2)})+d(x_2,[Tx_1]_{\alpha(x_1)})]+a_3\frac{d(x_1,[Tx_1]_{\alpha(x_1)})d(x_2,[Tx_2]_{\alpha(x_2)})}{d(x_1,x_2)+d(x_1,[Tx_2]_{\alpha(x_2)})+d(x_2,[Tx_1]_{\alpha(x_1)})}$$

$$+a_4\frac{d(x_1,[Tx_1]_{\alpha(x_1)})d(x_1,[Tx_2]_{\alpha(x_2)})+d(x_2,[Tx_1]_{\alpha(x_1)})d(x_2,[Tx_2]_{\alpha(x_2)})}{d(x_1,[Tx_2]_{\alpha(x_2)})+d(x_2,[Tx_1]_{\alpha(x_1)})}+\sigma^2 \tag{3.29}$$

$$d(x_2,x_3) \leq a_1 d(x_1,x_2)+a_2[d(x_1,x_3)+d(x_2,x_2)]+a_3 d(x_1,x_2)+a_4 d(x_1,x_2)+\sigma^2$$

Again, by triangular inequality, we obtain

$$d(x_2,x_3) \leq \frac{a_1+a_2+a_3+a_4}{1-a_2}d(x_1,x_2)+\frac{\sigma^2}{1-a_2}. \tag{3.30}$$

Let $\sigma=\frac{a_1+a_2+a_3+a_4}{1-a_2}$. Since $a_1+2a_2+a_3+a_4<1$ implies that $\frac{a_1+a_2+a_3+a_4}{1-a_2}<1$. Hence,

$$d(x_2,x_3) \leq \sigma\, d(x_1,x_2)+\frac{\sigma^2}{1-a_2} \ \ for \ \ all \ \ n \in N. \tag{3.31}$$

$$d(x_2,x_3) \leq \sigma\,[\sigma\, d(x_0,x_1)+\frac{\sigma}{1-a_2}\ ]+\frac{\sigma^2}{1-a_2}$$

$$d(x_2,x_3) \leq \sigma^2 d(x_0,x_1)+\frac{\sigma^2}{1-a_2}+\frac{\sigma^2}{1-a_2}$$

$$d(x_2,x_3) \leq \sigma^2 d(x_0,x_1)+\frac{2\sigma^2}{1-a_2} \tag{3.32}$$

Now,

$$d(x_3,x_4) \leq H([Tx_2]_{\alpha(x_2)},[Tx_3]_{\alpha(x_3)}) + \sigma^3,$$

using (3.2), we get

$$d(x_3,x_4) \leq H([Tx_2]_{\alpha(x_2)},[Tx_3]_{\alpha(x_3)}) + \sigma^3,$$

$$\leq a_1 d(x_2,x_2) + a_2[d(x_2,[Tx_3]_{\alpha(x_3)}) + d(x_3,[Tx_2]_{\alpha(x_2)})] + a_3\frac{d(x_2,[Tx_2]_{\alpha(x_2)})d(x_3,[Tx_3]_{\alpha(x_3)})}{d(x_2,x_3)+d(x_2,[Tx_3]_{\alpha(x_3)})+d(x_3,[Tx_2]_{\alpha(x_2)})}$$

$$+ a_4\frac{d(x_2,[Tx_2]_{\alpha(x_2)})d(x_2,[Tx_3]_{\alpha(x_3)}) + d(x_3,[Tx_2]_{\alpha(x_2)})d(x_3,[Tx_3]_{\alpha(x_3)})}{d(x_2,[Tx_3]_{\alpha(x_3)})+d(x_3,[Tx_2]_{\alpha(x_2)})} + \sigma^3 \tag{3.33}$$

$$d(x_3,x_4) \leq a_1 d(x_2,x_3) + a_2[d(x_2,x_4)+d(x_3,x_3)] + a_3 d(x_2,x_3) + a_4 d(x_2,x_3) + \sigma^3$$

By triangular inequality, we get

$$d(x_3,x_4) \leq \frac{a_1+a_2+a_3+a_4}{1-a_2} d(x_2,x_3) + \frac{\sigma^3}{1-a_2}. \tag{3.34}$$

Let $\sigma = \frac{a_1+a_2+a_3+a_4}{1-a_2}$. Since $a_1+2a_2+a_3+a_4<1$ implies that $\frac{a_1+a_2+a_3+a_4}{1-a_2}<1$. Hence,

$$d(x_3,x_4) \leq \sigma\, d(x_2,x_3) + \frac{\sigma^3}{1-a_2} \;\; for \;\; all \;\; n \in N. \tag{3.35}$$

$$d(x_3,x_4) \leq \sigma\,[\sigma^2 d(x_0,x_1) + \frac{2\sigma^2}{1-a_2}\;] + \frac{\sigma^3}{1-a_2}$$

$$d(x_3,x_4) \leq \sigma^3 d(x_0,x_1) + \frac{2\sigma^3}{1-a_2} + \frac{\sigma^3}{1-a_2}$$

$$d(x_3,x_4) \leq \sigma^3 d(x_0,x_1) + \frac{3\sigma^3}{1-a_2} \tag{3.36}$$

Again, continuing in this fashion, we have

$$d(x_n,x_{n+1}) \leq \sigma^n d(x_0,x_1) + \frac{n\sigma^n}{1-a_2} \tag{3.37}$$

If $n>m$ and $n,m \in N$, then we have

$$d(x_n,x_m) \leq d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \cdots + d(x_{m-1},x_m) \tag{3.38}$$

Applying (3.37) in (3.38) , we get

$$d(x_n,x_m) \leq \sigma^n d(x_0,x_1) + \frac{n\sigma^n}{1-a_2} + \sigma^{n+1} d(x_0,x_1) + \frac{(n+1)\sigma^{n+1}}{1-a_2} + \cdots + \sigma^{m-1} d(x_0,x_1) + \frac{(m-1)\sigma^{m-1}}{1-a_2}$$

$$d(x_n,x_m) \leq \sigma^n d(x_0,x_1)(1+\sigma+\sigma^2+\sigma^3+\cdots+\sigma^{m-n-1}) + \sum_{i=n}^{m-1}\frac{i\sigma^i}{1-a_2}$$

$$d(x_n,x_m) \leq \sigma^n d(x_0,x_1)(\frac{1-\sigma^{m-n}}{1-\sigma}) + \sum_{i=n}^{m-1}\frac{i\sigma^i}{1-a_2} \tag{3.39}$$

On taking $m,n \to \infty$ in (3.19), we get

$$d(x_n,x_m) = 0. \tag{3.40}$$

This proves that the sequence $\{x_n\}$ is a Cauchy sequence in X. Since X is a complete metric space, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Now,

$$d(x^*, [Tx^*]_{\alpha(x^*)}) \leq [d(x^*, x_n) + (x_n, [Tx^*]_{\alpha(x^*)})],$$

Using (3.2), we get

$$d(x^*, [Tx^*]_{\alpha(x^*)}) \leq d(x^*, x_n) + a_1 d(x_{n-1}, x^*) + a_2[d(x_{n-1}, [Tx^*]_{\alpha(x^*)}) + d(x^*, [Tx_{n-1}]_{\alpha(x_{n-1})})] +$$

$$a_3 \frac{d(x_{n-1}, [Tx_{n-1}]_{\alpha(x_{n-1})}) d(x^*, [Tx^*]_{\alpha(x^*)})}{d(x_{n-1}, x^*) + d(x_{n-1}, [Tx^*]_{\alpha(x^*)}) + d(x^*, [Tx_{n-1}]_{\alpha(x_{n-1})})} +$$

$$a_4 \frac{d(x_{n-1}, [Tx_{n-1}]_{\alpha(x_{n-1})}) d(x_{n-1}, [Tx^*]_{\alpha(x^*)}) + d(x^*, [Tx_{n-1}]_{\alpha(x_{n-1})}) d(x^*, [Tx^*]_{\alpha(x^*)})}{d(x_{n-1}, [Tx^*]_{\alpha(x^*)}) + d(x^*, [Tx_{n-1}]_{\alpha(x_{n-1})})}$$

$$d(x^*, [Tx^*]_{\alpha(x^*)}) \leq d(x^*, x_n) + a_1 d(x_{n-1}, x^*) + a_2[d(x_{n-1}, [Tx^*]_{\alpha(x^*)}) + d(x^*, x_n)] +$$

$$a_3 \frac{d(x_{n-1}, x_n) d(x^*, [Tx^*]_{\alpha(x^*)})}{d(x_{n-1}, x^*) + d(x_{n-1}, [Tx^*]_{\alpha(x^*)}) + d(x^*, x_n)} +$$

$$a_4 \frac{d(x_{n-1}, x_n) d(x_{n-1}, [Tx^*]_{\alpha(x^*)}) + d(x^*, x_n) d(x^*, [Tx^*]_{\alpha(x^*)})}{d(x_{n-1}, [Tx^*]_{\alpha(x^*)}) + d(x^*, x_n)} \tag{3.41}$$

On taking $n \to \infty$ in (3.41), we get

$$d(x^*, [Tx^*]_{\alpha(x^*)}) \leq d(x^*, x^*) + a_1 d(x^*, x^*) + a_2[d(x^*, [Tx^*]_{\alpha(x^*)}) + d(x^*, x^*)] +$$

$$a_3 \frac{d(x^*, x^*) d(x^*, [Tx^*]_{\alpha(x^*)})}{d(x^*, x^*) + d(x^*, [Tx^*]_{\alpha(x^*)}) + d(x^*, x^*)} +$$

$$a_4 \frac{d(x^*, x^*) d(x^*, [Tx^*]_{\alpha(x^*)}) + d(x^*, x^*) d(x^*, [Tx^*]_{\alpha(x^*)})}{d(x^*, [Tx^*]_{\alpha(x^*)}) + d(x^*, x^*)}$$

implies

$$(1 - a_2) d(x^*, [Tx^*]_{\alpha(x^*)}) \leq 0 \tag{3.42}$$

Since $a_1 + 2a_2 + a_3 + a_4 < 1$ implies $a_1 + a_2 + a_3 + a_4 < 1 - a_2$, that is, $1 - a_2 \neq 0$. Hence,

$$d(x^*, [Tx^*]_{\alpha(x^*)}) = 0.$$

Implies

$$x^* \in [Tx^*]_{\alpha(x^*)}.$$

Thus, $x^* \in X$ is the fixed point. □

Example 3.5. Consider $X = [0, 2]$ the usual metric space which is complete and $T : X \to F(X)$ be a fuzzy type mapping such that $T(x) \in F(X)$, where $x \in X$ and $T(x) : X \to [0, 1]$ is a function defined by

$$T(x)(t) = \begin{cases} \frac{1}{2}, & 0 \leq t \leq \frac{1}{2} \\ \frac{1}{3}, & \frac{1}{2} < t < 1 \\ 0, & 1 \leq t \leq 2 \end{cases}$$

If for all $x \in X$, there exist $\alpha(x) = \frac{1}{2}$ such that

$$[Tx]_{\frac{1}{2}} = \left\{ t : Tx(t) \geq \frac{1}{2} \right\},$$

$$[Tx]_{\frac{1}{2}} = \left[0, \frac{1}{2}\right] \, and \; [Ty]_{\frac{1}{2}} = \left[0, \frac{1}{2}\right].$$

Now,

$$H\left([Tx]_{\frac{1}{2}}, [Ty]_{\frac{1}{2}}\right) = \max\left\{ \sup_{x \in [Tx]_{\frac{1}{2}}} d\left(x, [Ty]_{\frac{1}{2}}\right), \sup_{y \in [Ty]_{\frac{1}{2}}} d\left(y, [Tx]_{\frac{1}{2}}\right) \right\},$$

$$H\left([Tx]_{\frac{1}{2}}, [Ty]_{\frac{1}{2}}\right) = 0.$$

$$H(x, y) = |x - y|.$$

$$H\left([Ty]_{\frac{1}{2}}, y\right) = \begin{cases} 0 \text{ if } & y \in [Ty]_{\frac{1}{2}} \\ \text{Otherwise nonzero} \end{cases}$$

$$H\left([Tx]_{\frac{1}{2}}, x\right) = \begin{cases} 0 \text{ if } & x \in [Tx]_{\frac{1}{2}} \\ \text{Otherwise nonzero} \end{cases}$$

$$H\left([Tx]_{\frac{1}{2}}, y\right) = \begin{cases} 0 \text{ if } & y \in [Tx]_{\frac{1}{2}} \\ \text{Otherwise nonzero} \end{cases}$$

$$H\left([Ty]_{\frac{1}{2}}, x\right) = \begin{cases} 0 \text{ if } & x \in [Ty]_{\frac{1}{2}} \\ \text{Otherwise nonzero} \end{cases}$$

let $a_1 = \frac{1}{50}, a_2 = \frac{1}{10}, a_3 = \frac{1}{20}, a_4 = \frac{1}{30}$. Then,

$$H([Tx]_{\frac{1}{2}}, [Ty]_{\frac{1}{2}}) \leq \frac{1}{50} d(x, y) + \frac{1}{10}[d(x, [Tx]_{\frac{1}{2}}) + d(y, [Ty]_{\frac{1}{2}})] +$$

$$\frac{1}{20} \frac{d(x, [Tx]_{\frac{1}{2}}) d(y, [Ty]_{\frac{1}{2}})}{d(x, y) + d(x, [Ty]_{\frac{1}{2}}) + d(y, [Tx]_{\frac{1}{2}})}$$

$$+ \frac{1}{30} \frac{d(x, [Tx]_{\frac{1}{2}}) d(x, [Ty]_{\frac{1}{2}}) + d(y, [Tx]_{\frac{1}{2}}) d(y, [Ty]_{\frac{1}{2}})}{d(x, [Ty]_{\frac{1}{2}}) + d(y, [Tx]_{\frac{1}{2}})}$$

$$0 \leq \frac{1}{50}|x - y| + \frac{1}{10}[d(x, [Tx]_{\frac{1}{2}}) + d(y, [Ty]_{\frac{1}{2}})] +$$

$$\frac{1}{20} \frac{d(x, [Tx]_{\frac{1}{2}}) d(y, [Ty]_{\frac{1}{2}})}{d(x, y) + d(x, [Ty]_{\frac{1}{2}}) + d(y, [Tx]_{\frac{1}{2}})}$$

$$+ \frac{1}{30} \frac{d(x, [Tx]_{\frac{1}{2}}) d(x, [Ty]_{\frac{1}{2}}) + d(y, [Tx]_{\frac{1}{2}}) d(y, [Ty]_{\frac{1}{2}})}{d(x, [Ty]_{\frac{1}{2}}) + d(y, [Tx]_{\frac{1}{2}})}$$

As all the conditions of Theorem 3.3 are satisfied, we can conclude that T has a fixed point in X. Similarly, for fuzzy type II contraction mapping, we have

$$H([Tx]_{\frac{1}{2}},[Ty]_{\frac{1}{2}}) \leq \frac{1}{50}d(x,y) + \frac{1}{10}[d(x,[Ty]_{\frac{1}{2}}) + d(y,[Tx]_{\frac{1}{2}})] +$$
$$\frac{1}{20}\frac{d(x,[Tx]_{\frac{1}{2}})d(y,[Ty]_{\frac{1}{2}})}{d(x,y)+d(x,[Ty]_{\frac{1}{2}})+d(y,[Tx]_{\frac{1}{2}})}$$
$$+\frac{1}{30}\frac{d(x,[Tx]_{\frac{1}{2}})d(x,[Ty]_{\frac{1}{2}})+d(y,[Tx]_{\frac{1}{2}})d(y,[Ty]_{\frac{1}{2}})}{d(x,[Ty]_{\frac{1}{2}})+d(y,[Tx]_{\frac{1}{2}})}$$
$$0 \leq \frac{1}{50}|x-y| + \frac{1}{10}[d(x,[Ty]_{\frac{1}{2}}) + d(y,[Tx]_{\frac{1}{2}})] +$$
$$\frac{1}{20}\frac{d(x,[Tx]_{\frac{1}{2}})d(y,[Ty]_{\frac{1}{2}})}{d(x,y)+d(x,[Ty]_{\frac{1}{2}})+d(y,[Tx]_{\frac{1}{2}})}$$
$$+\frac{1}{30}\frac{d(x,[Tx]_{\frac{1}{2}})d(x,[Ty]_{\frac{1}{2}})+d(y,[Tx]_{\frac{1}{2}})d(y,[Ty]_{\frac{1}{2}})}{d(x,[Ty]_{\frac{1}{2}})+d(y,[Tx]_{\frac{1}{2}})}$$

As all of the conditions of Theorem 3.4 are satisfied, we can conclude that T has a fixed point in X.

Corollary 3.6. *Let (X,d) be a complete metric space and $T : X \to CB(X)$ be a fuzzy mapping. Suppose there exists $\alpha(x) \in (0,1]$, with $[Tx]_{\alpha(x)}$ and $[Ty]_{\alpha(y)}$ a closed, bounded, non-empty subsets of X. Then T has a fixed point in X, if for all $x,y \in X$ and $a_1,a_2,a_3,a_4,a_5 \geq 0$ with $a_1+2a_2+2a_3+a_4+a_5 < 1$ satisfying the following condition:*

$$H([Tx]_{\alpha(x)},[Ty]_{\alpha(y)}) \leq a_1 d(x,y) + a_2[d(x,[Tx]_{\alpha(x)}) + d(y,[Ty]_{\alpha(y)})] +$$
$$a_3[d(x,[Ty]_{\alpha(y)}) + d(y,[Tx]_{\alpha(x)})] + a_4\frac{d(x,[Tx]_{\alpha(x)})d(y,[Ty]_{\alpha(y)})}{d(x,y)+d(x,[Ty]_{\alpha(y)})+d(y,[Tx]_{\alpha(x)})}$$
$$+ a_5\frac{d(x,[Tx]_{\alpha(x)})d(x,[Ty]_{\alpha(y)})+d(y,[Tx]_{\alpha(x)})d(y,[Ty]_{\alpha(y)})}{d(x,[Ty]_{\alpha(y)})+d(y,[Tx]_{\alpha(x)})} \tag{3.43}$$

4 Application

In this section, we explore a specific application of our results. We demonstrate how Theorems 3.3 and 3.4 can be applied to show the existence of fixed points for multivalued mappings in metric spaces. To begin, we start with the following definitions

Definition 4.1. Let $R : X \to CB(X)$ be a multivalued mapping and (X,d) be a complete metric space. Let us assume that $\alpha(x) \in (0,1]$, where $R(x)$ and $R(y)$ are closed, bounded, non-empty subsets of X. Then R is said to be fuzzy type I contraction if for all $x,y \in X$ and $a_1,a_2,a_3,a_4 \geq 0$ with $a_1+2a_2+a_3+a_4 < 1$ satisfying the following requirement:

$$H(R(x),R(y)) \leq a_1 d(x,y) + a_2[d(x,R(x)) + d(y,R(y))]$$
$$+ a_3\frac{d(x,R(x))d(y,R(y))}{(d(x,y)+d(x,R(y))+d(y,R(x)))} + a_4\frac{d(x,R(x))d(x,R(y))+d(y,R(x))d(y,R(y))}{(d(x,R(y))+d(y,R(x)))} \tag{4.1}$$

Definition 4.2. Let $R : X \to CB(X)$ be a multivalued mapping and (X, d) be a complete metric space. Let us assume that $\alpha(x) \in (0, 1]$, where $R(x)$ and $R(y)$ are closed, bounded, non-empty subsets of X. Then R is said to be fuzzy type II contraction if for all $x, y \in X$ and $a_1, a_2, a_3, a_4 \geq 0$ with $a_1 + 2a_2 + a_3 + a_4 < 1$ satisfying the following requirement:

$$H(R(x), R(y)) \leq a_1 d(x, y) + a_2[d(x, R(y)) + d(y, R(x))]$$

$$+ a_3 \frac{d(x, R(x))d(y, R(y))}{(d(x, y) + d(x, R(y)) + d(y, R(x)))} + a_4 \frac{d(x, R(x))d(x, R(y)) + d(y, R(x))d(y, R(y))}{(d(x, R(y)) + d(y, R(x)))} \tag{4.2}$$

Theorem 4.3. *Let (X, d) be a complete metric space and $R : X \to CB(X)$ be a fuzzy type I contraction mapping. Then, T has a fixed point in X.*

Proof. Let $\alpha : X \to (0, 1]$ be an arbitrary mapping. Consider a fuzzy mapping $R : X \to F(X)$ defined by

$$(Tx)(t) = \begin{cases} \alpha(x), & t \in Rx, \\ 0, & t \notin Rx. \end{cases}$$

We have that

$$[Tx]_{\alpha(x)} = \{t : Tx(t) \geq \alpha(X)\} = Rx.$$

Hence, condition (4.1) becomes condition (3.1) in Theorem 3.3. It implies that there exists $x^* \in X$ such that $x^* \in [Tx^*]_{\alpha(x^*)} = Rx^*$. □

Theorem 4.4. *Let (X, d) be a complete metric space and $R : X \to CB(X)$ be a fuzzy type II contraction mapping. Then, T has a fixed point in X.*

Proof. Let $\alpha : X \to (0, 1]$ be an arbitrary mapping. Consider a fuzzy mapping $R : X \to F(X)$ defined by

$$(Tx)(t) = \begin{cases} \alpha(x), & t \in Rx, \\ 0, & t \notin Rx. \end{cases}$$

We have that

$$[Tx]_{\alpha(x)} = \{t : Tx(t) \geq \alpha(x)\} = Rx.$$

Hence, condition (4.2) becomes condition (3.2) in in Theorem 3.4. It implies that there exists $x^* \in X$ such that $x^* \in [Tx^*]_{\alpha(x^*)} = Rx^*$. □

Corollary 4.5. *Let (X, d) be a complete metric space and $R : X \to CB(X)$ be a multivalued mapping. Suppose there exists $\alpha(x) \in (0, 1]$, with $R(x)$ and $R(y)$ a closed, bounded, non-empty subsets of X. Then R has a fixed point in X, if for all $x, y \in X$ and $a_1, a_2, a_3, a_4, a_5 \geq 0$ with $a_1 + 2a_2 + 2a_3 + a_4 + a_5 < 1$ satisfying the following condition:*

$$H(R(x), R(y)) \leq a_1 d(x, y) + a_2[d(x, R(x)) + d(y, R(y))] + a_3[d(x, R(y)) + d(y, R(x))]$$

$$+ a_4 \frac{d(x, R(x))d(y, R(y))}{(d(x, y) + d(x, R(y)) + d(y, R(x)))} + a_5 \frac{d(x, R(x))d(x, R(y)) + d(y, R(x))d(y, R(y))}{(d(x, R(y)) + d(y, R(x)))} \tag{4.3}$$

Proof. It follows from the logic of the proof of Theorem 4.3 and Theorem 4.4. □

5 Conclusion

The main findings of this study demonstrate applicability of fuzzy type contractions in establishing fixed point theorems for fuzzy mappings. This study provides significant advancements in the understanding of fuzzy metric spaces, with potential applications in differential equations and nonlinear Fredholm integral equation.

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