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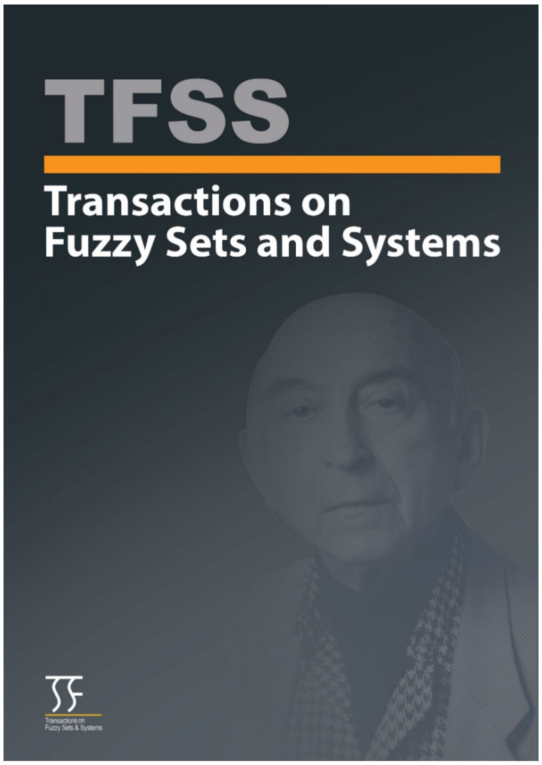

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Fuzzy Cone Metric Spaces and Fixed Point Theorems for Fuzzy Type Contraction

Muhammed Raji, **Laxmi Rathour***, **Lakshmi Narayan Mishra**, **Vishnu Narayan Mishra**

Abstract. The paper aims to introduce novel concepts of fuzzy type contractions and establish fixed point theorems for fuzzy mappings within the framework of fuzzy cone metric spaces. These contributions extend the existing literature on fuzzy mappings and fixed point theory. Through illustrative examples, we showcase the practical applicability of our proposed notions and results, demonstrating their effectiveness in real-world scenarios.

AMS Subject Classification 2020: 54H25; 47H10
Keywords and Phrases: Fuzzy cone metric spaces, Fixed point, Fuzzy mapping, Real Banach space.

1 Introduction

Banach's fixed point theorem for contraction mappings has been one of the most influential results in mathematical analysis. Banach's contraction principle [1] has been instrumental in the development of metric fixed point theory, and has been used to solve a wide range of problems, including differential equations, integral equations, optimization problems, and variational inequalities. Since its introduction, the Banach contraction mapping principle has been generalized and refined in numerous ways, leading to a wealth of articles dedicated to its improvement [2, 3, 4].
Guang and Xian [5] extended the notion of metric spaces by considering a real Banach space as the range set, thereby introducing the concept of cone metric spaces. Through their exploration of cone metric spaces, they uncovered significant properties that led to the derivation of several fixed point theorems, some of which can be found in [6, 7, 8].
Zadeh [9] pioneered the concept of fuzzy sets, laying the foundation for subsequent research in fuzzy mathematics. Building upon Zadeh's work, Weiss [10] delved into fuzzy mappings and derived numerous fixed point results. Heilpern [11] further expanded upon fuzzy mappings by introducing the concept of fuzzy contraction mappings. He established a fixed point theorem for fuzzy contraction mappings akin to Nadler's fixed point theorem for multivalued mappings. Moreover, Bag [12] introduced the innovative notion of fuzzy cone metric spaces, leveraging this framework to derive fixed point results for fuzzy T-Kannan contraction and fuzzy T-Chatterjea contraction mappings. Recently, Raji and Ibrahim [13] proved some fixed point results for fuzzy mappings in a complete dislocated b-metric space.
Based on the above insight, we introduce novel concepts of fuzzy type contractions and subsequently establish fixed point results for fuzzy mappings within the framework of fuzzy cone metric spaces. To bolster our findings, we offer illustrative examples demonstrating the practical application of the presented results and

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concepts.
Throughout our discourse, we denote by E a fuzzy real Banach space, by $\mathcal{F}$ a fuzzy cone in E with a non-empty interior, and by $\leq$ a partial ordering with respect to $\mathcal{F}$.

2 Preliminaries

A fuzzy cone metric space integrates concepts from fuzzy metric space and cone metric space, offering a broader and more adaptable approach to handle uncertainty and fuzziness in distance measurements. We begin this section with a few key definitions.

Definition 2.1. [14, 15] A function with X as its domain and the interval $[0, 1]$ as its range is called a fuzzy set in X. $\mathcal{F}(X)$ represents the set of all fuzzy sets in X. The degree of membership of x in A is denoted by the value $A(x)$, given a fuzzy set A and a point x in X. A fuzzy set A's α-level set is represented by $[A]_\alpha$ and has the following definition:

$$[A]_\alpha = \{x : A(x) \geq \alpha\} \text{ where } \alpha \in (0, 1), [A]_0 = \{x : A(x) > 0\}$$

Definition 2.2. [16, 17] Let Y be a metric space and X a nonempty set. If a mapping T is a mapping from X into $\mathcal{F}(Y)$, the set of all fuzzy sets on Y, then it is referred to as a fuzzy mapping. The degree to which y is a member of $T(x)$ is the membership function of a fuzzy mapping T, represented as $T(x)(y)$. Stated differently, $T(x)(y)$ represents y's degree of membership in the fuzzy set $T(x)$. Instead of using $[T(x)]_\alpha$ to denote the α-level set of $T(x)$, we will simply use $[Tx]_\alpha$.

Definition 2.3. [18, 19] A fuzzy fixed point of a fuzzy mapping $T : X \to \mathcal{F}(X)$ is defined as a point $x \in X$ where $\alpha \in (0, 1]$ and $x \in [Tx]_\alpha$.

Definition 2.4. [20] Consider the fuzzy real Banach space $(E, \|.\|)$, where $\|\square\| : E \to R(I)$. Use $E^*(I)$ to indicate the range of $\|.\|$,Thus, $E^*(I) \subset R^*(I)$.

Definition 2.5. [21] An interior point is defined as member $\eta \in A \subset R^*(I)$ if there exists $r > 0$ such that

$$S(\eta, r) = \{\delta \in R^*(I) : \delta \ominus \eta < \bar{r}\} \subset A$$

set of all interior points of A is called interior A.

Definition 2.6. [11] Fuzzy closed subset $\mathcal{F}$ of $E^*(I)$ is defined as follows: for each sequence $\{\eta_n\}$, such that

$$\lim_{n\to\infty} \eta_n = \eta \quad \text{implies} \quad \eta \in \mathcal{F}.$$

Definition 2.7. [22] A fuzzy cone is defined as a subset $\mathcal{F}$ of $E^*(I)$ if
i. $\mathcal{F}$ is fuzzy closed,nonempty and $\mathcal{F} \neq \{\bar{0}\}$,
ii. $a, b \in R, a, b \geq 0, \eta, \delta \in \mathcal{F} \implies a\eta \oplus b\delta \in \mathcal{F}$

Definition 2.8. [22] A mapping $x : R \mapsto [0, 1]$ over the set R of all real numbers is called a fuzzy real number

Definition 2.9. [22] A fuzzy real number x is convex if $x(t) \geq \wedge (x(s), x(r))$ where $s \leq t \leq r$.

Definition 2.10. [9] α-level set of fuzzy real number x is defined by $\{t \in R : x(t) \geq \alpha\}$ where $\alpha \in (0, 1]$. If there exists a $t_0 \in R$ such that $x(t_0) = 1$, then x called normal. For $0 < \alpha \leq 1$, α-level set of an upper semi continuous convex normal fuzzy real number η denoted by $[\eta]_\alpha$, serves as a closed interval $[a_\alpha, b_\alpha]$, where $a_\alpha = -\infty$ and $b_\alpha = +\infty$ are admissible.

Definition 2.11. [22] Given a fuzzy cone $\mathcal{F} \in E^*(I)$ define a partial odering $\leq$ with respect to $\mathcal{F}$ by $\eta \leq \delta$ iff $\delta \ominus \eta \in \mathcal{F}$ and $\eta < \delta$ indicates that $\eta \leq \delta$ but $\eta \neq \delta$ while $\eta << \delta$ will stand for $\delta \ominus \eta \in \ Int\mathcal{F}$ where $Int\ \mathcal{F}$ denote the interior of $\mathcal{F}$.

Definition 2.12. [22] The fuzzy cone $\mathcal{F}$ is called normal if there exists a number $K > 0$ such tha for all $x, y \in E$ with $\bar{0} \leq \|x\| \leq \|y\|$ implies $\|x\| \leq K\|y\|$. The least positive number satisfying above is called the normal constant of $\mathcal{F}$.

Definition 2.13. [22] If every growing sequence that is bounded from above is convergent, the fuzzy cone $\mathcal{F}$ is said to be regular. That is $\{x_n\}$ is a sequence in E such that
$\|x_1\| \leq \|x_2\| \leq \cdots \leq \|y\|$ for some $y \in E$, then there exists $x \in E$ such that $\|x_n - x\| \to \bar{0}$ as $n \to \infty$

Definition 2.14. [22] Let X be a nonempty set. Suppose the mapping $d : X \times X \mapsto E^*(I)$ satisfies
$(fd1)$ $d(x, y)\ \geq \bar{0}$ and $d(x, y) = \bar{0}$ iff $x = y$;
$(fd2)$ $d(x, y) = d(y, x)$;
$(fd3)$ $d(x, y) \leq d(x, z) \oplus d(z, y)$ for all $x, y, z \in X$.
Then, the d is called a fuzzy cone metric and the pair (X, d) is called a fuzzy cone metric space.

Definition 2.15. [22] Let (X, d) be a fuzzy cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $\bar{0} << \|c\|$, there is a positive integer N such that for all $n > N$, $d(x_n, x) << \|c\|$, then, $\{x_n\}$ is said to be convergent and converges to x and x is called the limit of $\{x_n\}$. Denoted by

$$\lim_{n\to\infty} x_n = x$$

Definition 2.16. [22] Let $\{x_n\}$ be a sequence in X and (X, d) be a fuzzy cone metric space. $\{x_n\}$ is referred to as a Cauchy sequence in X if, for any $c \in E$ with $\bar{0} << \|c\|$, there exists a natural integer N such that, for any $m, n > N, d(x_n, x_m) << \|c\|$.

Definition 2.17. [22] Let (X, d) be a metric space with fuzzy cones. X is referred to as a complete fuzzy cone metric space if every Cauchy sequence is convergent in it.

Definition 2.18. [22] Let $\{x_n\}$ be a sequence in X and (X, d) be a fuzzy cone metric space with normal fuzzy cone. Then
i. $\{x_n\}$ converges to x if and only if $d(x_n, x) \to \bar{0}$ as $n \to \infty$
ii. $\{x_n\}$ is a Cauchy sequence if and only if $d(x_m, x_n) \to \bar{0}$ as $m, n \to \infty$

3 Main Results

We start this section with the definitions that follow.

Definition 3.1. Suppose $(X,\ d)$ is a fuzzy cone metric space. Let $T, S : X \to X$ be two functions. Then S is said to be fuzzy cone T-type I contraction if for all $x, y \in X, Tx \neq Ty$ and $a_1, a_2, a_3, a_4 \geq 0$ with $2a_1 + a_2 + a_3 + a_4 < 1$ satisfying the following condition:

$$d(TSx, TSy) \leq a_1 [d(Tx, TSx) \oplus d(Ty, TSy)] \oplus a_2 \frac{d(Tx, TSx)d(Ty, TSy)}{d(Tx, Ty)}$$
$$\oplus a_3 \frac{d(Tx, TSx)d(Ty, TSy)}{d(Tx, Ty) \oplus d(Tx, TSy) \oplus d(Ty, TSx)} \oplus a_4 \frac{d(Tx, TSx)d(Tx, TSy) \oplus d(Ty, TSx)d(Ty, TSy)}{d(Tx, TSy) \oplus d(Ty, TSx)} \tag{3.1}$$

Definition 3.2. Suppose $(X,\ d)$ is a fuzzy cone metric space. Let $T, S : X \to X$ be two functions. Then S is said to be fuzzy cone T-type II contraction if for all $x, y \in X, Tx \neq Ty$ and $a_1, a_2, a_3, a_4 \geq 0$ with $2a_1 + a_2 + a_3 + a_4 < 1$ satisfying the following condition:

$$\begin{aligned} d(TSx, TSy) \leq a_1 [d(Tx, TSy) \oplus d(Ty, TSx)] \oplus a_2 \frac{d(Tx, TSx)\, d(Ty, TSy)}{d(Tx, Ty)} \\ \oplus a_3 \frac{d(Tx, TSx)\, d(Ty, TSy)}{d(Tx, Ty) \oplus d(Tx, TSy) \oplus d(Ty, TSx)} \\ \oplus a_4 \frac{d(Tx, TSx)\, d(Tx, TSy) \oplus d(Ty, TSx)\, d(Ty, TSy)}{d(Tx, TSy) \oplus d(Ty, TSx)} \end{aligned} \tag{3.2}$$

Theorem 3.3. *Suppose $(X,\ d)$ is a complete fuzzy cone metric space, $\mathcal{F}$ be a normal fuzzy cone with normal constant K. Let $T : X \to X$ be a one-one continuous function and $S : X \to X$ be a fuzzy cone T-type I contraction mapping. Then, the following conditions are satisfied:*
i. for every $x_0 \in X$, $\lim_{n\to\infty} d(TS^n x_0, TS^{n+1} x_0) = \bar{0}$;
ii. there exists $v \in X$ such that $\lim_{n\to\infty} TS^n x_0 = v$;
iii. if T is sequentially convergent, then $\{S^n x_0\}$ has a convergent subsequence;
iv. there is a unique $u \in X$ such that $Su = u$;
v. if T is sequentially convergent, then for each $x_0 \in X$ the iterate sequence $\{S^n x_0\}$ converges to u.

Proof. Let $x_0 \in X$ be any arbitrary point in X. Define the iterate sequence $\{x_n\}$ by $x_{n+1} = Sx_n = S^n x_0$, Now, by using (3.1), we get

$$\begin{aligned} (Tx_n, Tx_{n+1}) &= d(TSx_{n-1}, TSx_n) \\ &\leq a_1 [d(Tx_{n-1}, TSx_{n-1}) \oplus d(Tx_n, TSx_n)] \\ &\oplus a_2 \frac{d(Tx_{n-1}, TSx_{n-1})\, d(Tx_n, TSx_n)}{d(Tx_{n-1}, Tx_n)} \\ &\oplus a_3 \frac{d(Tx_{n-1}, TSx_{n-1})\, d(Tx_n, TSx_n)}{d(Tx_{n-1}, Tx_n) \oplus d(Tx_{n-1}, TSx_n) \oplus d(Tx_n, TSx_{n-1})} \\ &\oplus a_4 \frac{d(Tx_{n-1}, TSx_{n-1})\, d(Tx_{n-1}, TSx_n) \oplus d(Tx_n, TSx_{n-1})\, d(Tx_n, TSx_n)}{d(Tx_{n-1}, TSx_n) \oplus d(Tx_n, TSx_{n-1})} \\ &= a_1 [d(Tx_{n-1}, Tx_n) \oplus d(Tx_n, Tx_{n+1})] \oplus a_2 \frac{d(Tx_{n-1}, Tx_n)\, d(Tx_n, Tx_{n+1})}{d(Tx_{n-1}, Tx_n)} \\ &\oplus a_3 \frac{d(Tx_{n-1}, Tx_n)\, d(Tx_n, Tx_{n+1})}{d(Tx_{n-1}, Tx_n) \oplus d(Tx_{n-1}, Tx_{n+1}) \oplus d(Tx_n, Tx_n)} \\ &\oplus a_4 \frac{d(Tx_{n-1}, Tx_n)\, d(Tx_{n-1}, Tx_{n+1}) \oplus d(Tx_n, Tx_n)\, d(Tx_n, Tx_{n+1})}{d(Tx_{n-1}, Tx_{n+1}) \oplus d(Tx_n, Tx_n)} \end{aligned}$$

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &\leq a_1 d(Tx_{n-1}, Tx_n) \oplus a_1 d(Tx_n, Tx_{n+1}) \oplus a_2 d(Tx_n, Tx_{n+1}) \\ &\oplus a_3 d(Tx_{n-1}, Tx_n) \oplus a_4 d(Tx_{n-1}, Tx_n) \\ d(Tx_n, Tx_{n+1}) &\leq \frac{a_1 + a_3 + a_4}{1 - (a_1 + a_2)} d(Tx_{n-1}, Tx_n) \end{aligned} \tag{3.3}$$

Let $\lambda = \frac{a_1+a_3+a_4}{1-(a_1+a_2)}$. Since $2a_1 + a_2 + a_3 + a_4 < 1$ implies that $\frac{a_1+a_3+a_4}{1-(a_1+a_2)} < 1$. Hence,

$$d\left(Tx_n, Tx_{n+1}\right) \leq \lambda d\left(Tx_{n-1}, Tx_n\right) \ \forall\ n \in \mathbb{N} \tag{3.4}$$

Then, by repeated application of (3.4), we have

$$d\left(TS^n x_0, TS^{n+1} x_0\right) \leq \lambda^n d\left(Tx_0, TSx_0\right) \ \forall\ n \in \mathbb{N} \tag{3.5}$$

Since $\mathcal{F}$ is a normal cone with constant K, we have from (3.5),

$$d\left(TS^n x_0, TS^{n+1} x_0\right) \leq \lambda^n K d\left(Tx_0, TSx_0\right) \ \forall\ n \in \mathbb{N} \tag{3.6}$$

Implies

$$d\left(TS^n x_0, TS^{n+1} x_0\right) \leq \lambda^n K d_\alpha^i\left(Tx_0, TSx_0\right) \ for\ i = 1, 2 \tag{3.7}$$

On taking the limit in (3.7), we have

$$\lim_{n\to\infty} d_\alpha^i\left(TS^n x_0, TS^{n+1} x_0\right) = 0 \text{ for } i = 1, 2, \alpha \in (0, 1] \ \left(\text{since } \frac{a_1 + a_3 + a_4}{1 - (a_1 + a_2)} < 1\right)$$

Hence,

$$\lim_{n\to\infty} d\left(TS^n x_0, TS^{n+1} x_0\right) = 0 \tag{3.8}$$

For any $m > n$ where $m, n \in \mathbb{N}$, we have,

$$\begin{aligned} d\left(Tx_n, Tx_m\right) &\leq d\left(Tx_n, Tx_{n+1}\right) \oplus d\left(Tx_{n+1}, Tx_{n+2}\right) \oplus \cdots \oplus d\left(Tx_{m-1}, Tx_m\right) \\ &\leq \left[\left(\frac{a_1 + a_3 + a_4}{1 - (a_1 + a_2)}\right)^n + \left(\frac{a_1 + a_3 + a_4}{1 - (a_1 + a_2)}\right)^{n+1} + \cdots + \left(\frac{a_1 + a_3 + a_4}{1 - (a_1 + a_2)}\right)^{m-1}\right] d\left(Tx_0, TSx_0\right) \end{aligned}$$

$$\begin{aligned} &\leq \left[\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}\right] d\left(Tx_0, TSx_0\right) \\ &\leq \lambda^n \frac{1}{1-\lambda} d\left(Tx_0, TSx_0\right) \end{aligned} \tag{3.9}$$

So

$$d\left(TS^n x_0, TS^m x_0\right) \leq \lambda^n \frac{1}{1-\lambda} d\left(Tx_0, TSx_0\right) \tag{3.10}$$

Since $\mathcal{F}$ is normal, we get

$$d\left(TS^n x_0, TS^m x_0\right) \leq \lambda^n \frac{k}{1-\lambda} d\left(Tx_0, TSx_0\right) \tag{3.11}$$

Taking the limit as $m, n \to \infty$, we get

$$\lim_{m,n\to\infty} d\left(TS^n x_0, TS^m x_0\right) = \bar{0} \ \left(\text{since } \frac{a_1 + a_2 + a_3 + a_4}{1 - (a_2 + a_3)} < 1\right) \tag{3.12}$$

This proves that $\{(TS^n x_0\}$ is Cauchy sequence in X. Since X is a complete metric space, there exists $v \in X$ such that

$$\lim_{n\to\infty} (TS^n x_0 = v). \tag{3.13}$$

Now if T is subsequentially convergent, $\{S^n x_0\}$ has a convergent subsequence.So there exists $u \in X$ and $\{n_i\}$ such that

$$\lim_{i\to\infty} S^{n_i} x_0 = u. \tag{3.14}$$

Since T is continuous by (3.13), we get

$$\lim_{i\to\infty} TS^{n_i} x_0 = Tu. \tag{3.15}$$

Considering (3.14) and (3.15), we get $Tu = u$.
Now

$$d(TSu, Tu) \leq d(TSu, TS^{n_i}(x_0)) \oplus d\left(TS^{n_i}(x_0), TS^{n_i+1}(x_0)\right) \oplus d\left(TS^{n_i+1}(x_0), Tu\right).$$

$$\begin{aligned} d(TSu, Tu) \leq{} & a_1\left[d(Tu, TSu) \oplus d\left(TS^{n_i-1}(x_0), TS^{n_i}(x_0)\right)\right] \oplus a_2 \frac{d(Tu, TSu)\, d\left(TS^{n_i-1}(x_0), TS^{n_i}(x_0)\right)}{d(Tu, TS^{n_i-1}(x_0))} \\ & \oplus a_3 \frac{d(Tu, TSu)\, d\left(TS^{n_i-1}(x_0), TS^{n_i}(x_0)\right)}{d(Tu, TS^{n_i-1}(x_0)) \oplus d(Tu, TS^{n_i}(x_0)) \oplus d(TS^{n_i-1}(x_0), TSu)} \\ & \oplus a_4 \frac{d(Tu, TSu)\, d(Tu, TS^{n_i}(x_0)) \oplus d\left(TS^{n_i-1}(x_0), TSu\right) d\left(TS^{n_i-1}(x_0), TS^{n_i}(x_0)\right)}{d(Tu, TS^{n_i-1}(x_0)) \oplus d(TS^{n_i-1}(x_0), TSu)} \\ & \oplus \lambda^{n_i} d(Tx_0, TSx_0) \oplus d\left(TS^{n_i+1}(x_0), Tu\right) \end{aligned}$$

So

$$(TSu, Tu) \leq \lambda d\left(TS^{n_i-1}(x_0), TS^{n_i}(x_0)\right) \oplus \frac{1}{1-\lambda}\lambda^n d(Tx_0, TSx_0) \oplus \frac{1}{1-\lambda} d\left(TS^{n_i+1}(x_0), Tu\right)$$

Since $\mathcal{F}$ is normal cone with normal constant K, we have

$$(TSu, Tu) \leq \lambda K d\left(TS^{n_i-1}(x_0), TS^{n_i}(x_0)\right) \oplus \frac{k}{1-\lambda}\lambda^n d(Tx_0, TSx_0) \oplus \frac{k}{1-\lambda} d\left(TS^{n_i+1}(x_0), Tu\right)$$

Taking the llimit $i \to \infty$, using (3.15) and $\lambda < 1$, we get
$d^i_\alpha(TSu, Tu) = 0$ for all $\alpha \in (0, 1]$ and $i = 1, 2$,
Hence,

$$d(TSu, Tu) = \bar{0} \tag{3.17}$$

So that $TSu = Tu$.
Since T is one-one, we get $Su = u$. So S has a fixed point.
if v is another fixed point of S, then $Sv = v$. Since S is type I contraction, we obtain

$$\begin{aligned} d(TSu, TSv) \leq{} & a_1[d(Tu, TSu) \oplus d(Tv, TSv)] \oplus a_2 \frac{d(Tu, TSu)\, d(Tv, TSv)}{d(Tu, Tv)} \oplus \\ & a_3 \frac{d(Tu, TSu)\, d(Tv, TSv)}{d(Tu, Tv) \oplus d(Tu, TSv) \oplus d(Tv, TSu)} \oplus a_4 \frac{d(Tu, TSu)\, d(Tu, TSv) \oplus d(Tv, TSu)\, d(Tv, TSv)}{d(Tu, TSv) \oplus d(Tv, TSu)} \end{aligned} \tag{3.18}$$

$$\begin{aligned} ={} & a_1[d(Tu, Tu) \oplus d(Tv, Tv)] \oplus a_2 \frac{d(Tu, Tu)\, d(Tv, Tv)}{d(Tu, Tv)} \oplus \\ & a_3 \frac{d(Tu, Tu)\, d(Tv, Tv)}{d(Tu, Tv) \oplus d(Tu, Tv) \oplus d(Tv, Tu)} \oplus a_4 \frac{d(Tu, Tu)\, d(Tu, Tv) \oplus d(Tv, Tu)\, d(Tv, Tv)}{d(Tu, Tv) \oplus d(Tv, Tu)} \end{aligned}$$

Implies

$$d\left(TSu, TSv\right) = \bar{0}$$

So that $TSu = TSv$. Since S is injective, we get $u = v$. Thus,S has a unique fixed point.
Lastly, if T is sequentially convergent, by replacing n for n_i, we get that

$$\lim_{n\to\infty} S^n x_0 = u.$$

Thus, $\{S^n x_0\}$ is convergent to the fixed point u.
□

Theorem 3.4. *Suppose $(X,\ d)$ is a complete fuzzy cone metric space, $\mathcal{F}$ be a normal fuzzy cone with normal constant K. Let $T : X \to X$ be a one-one continuous function and $S : X \to X$ be a fuzzy cone T -type II contraction mapping. Then, the following conditions are satisfied:*
i. for every $x_0 \in X$, $\lim_{n\to\infty} d(TS^n x_0, TS^{n+1} x_0) = \bar{0}$;
ii. there exists $v \in X$ such that $\lim_{n\to\infty} TS^n x_0 = v$;
iii. if T is sequentially convergent, then $\{S^n x_0\}$ has a convergent subsequence;
iv. there is a unique $u \in X$ such that $Su = u$;
v. if T is sequentially convergent, then for each $x_0 \in X$ the iterate sequence $\{S^n x_0\}$ converges to u.

Proof. Let $x_0 \in X$ be any arbitrary point in X. Define the iterate sequence x_n by $x_{(n+1)} = Sx_n = S^n x_0$. Now, by using (3.2), we get

$$\begin{aligned}
(Tx_n, Tx_{n+1}) &= d\left(TSx_{n-1}, TSx_n\right) \\
&\le a_1\left[d\left(Tx_{n-1}, TSx_n\right) \oplus d\left(Tx_n, TSx_{n-1}\right)\right] \\
&\oplus a_2 \frac{d\left(Tx_{n-1}, TSx_{n-1}\right) d\left(Tx_n, TSx_n\right)}{d\left(Tx_{n-1}, Tx_n\right)} \\
&\oplus a_3 \frac{d\left(Tx_{n-1}, TSx_{n-1}\right) d\left(Tx_n, TSx_n\right)}{d\left(Tx_{n-1}, Tx_n\right) \oplus d\left(Tx_{n-1}, TSx_n\right) \oplus d\left(Tx_n, TSx_{n-1}\right)} \\
&\oplus a_4 \frac{d\left(Tx_{n-1}, TSx_{n-1}\right) d\left(Tx_{n-1}, TSx_n\right) \oplus d\left(Tx_n, TSx_{n-1}\right) d\left(Tx_n, TSx_n\right)}{d\left(Tx_{n-1}, TSx_n\right) \oplus d\left(Tx_n, TSx_{n-1}\right)} \\
&= a_1\left[d\left(Tx_{n-1}, Tx_{n+1}\right) \oplus d\left(Tx_n, Tx_n\right)\right] \oplus a_2 \frac{d\left(Tx_{n-1}, Tx_n\right) d\left(Tx_n, Tx_{n+1}\right)}{d\left(Tx_{n-1}, Tx_n\right)} \\
&\oplus a_3 \frac{d\left(Tx_{n-1}, Tx_n\right) d\left(Tx_n, Tx_{n+1}\right)}{d\left(Tx_{n-1}, Tx_n\right) \oplus d\left(Tx_{n-1}, Tx_{n+1}\right) \oplus d\left(Tx_n, Tx_n\right)} \\
&\oplus a_4 \frac{d\left(Tx_{n-1}, Tx_n\right) d\left(Tx_{n-1}, Tx_{n+1}\right) \oplus d\left(Tx_n, Tx_n\right) d\left(Tx_n, Tx_{n+1}\right)}{d\left(Tx_{n-1}, Tx_{n+1}\right) \oplus d\left(Tx_n, Tx_n\right)}
\end{aligned}$$

$$\begin{aligned}
d\left(Tx_n, Tx_{n+1}\right) &\le a_1 d\left(Tx_{n-1}, Tx_n\right) \oplus a_1 d\left(Tx_n, Tx_{n+1}\right) \oplus a_2 d\left(Tx_n, Tx_{n+1}\right) \\
&\oplus a_3 d\left(Tx_{n-1}, Tx_n\right) \oplus a_4 d\left(Tx_{n-1}, Tx_n\right) \\
d\left(Tx_n, Tx_{n+1}\right) &\le \frac{a_1 + a_3 + a_4}{1 - (a_1 + a_2)} d\left(Tx_{n-1}, Tx_n\right) \qquad (3.20)
\end{aligned}$$

Let $\lambda = \frac{a_1+a_3+a_4}{1-(a_1+a_2)}$. Since $2a_1 + a_2 + a_3 + a_4 < 1$ implies that $\frac{a_1+a_3+a_4}{1-(a_1+a_2)} < 1$. Hence,

$$d\left(Tx_n, Tx_{n+1}\right) \le \lambda d\left(Tx_{n-1}, Tx_n\right)\ \ \forall\ n \in \mathbb{N} \qquad (3.21)$$

Then, by repeated application of (3.21), we have

$$d\left(TS^nx_0, TS^{n+1}x_0\right) \leq \lambda^n d\left(Tx_0, TSx_0\right) \ \forall\ n \in \mathbb{N} \tag{3.22}$$

Since $\mathcal{F}$ is a normal cone with constant K, we have from (3.22),

$$d\left(TS^nx_0, TS^{n+1}x_0\right) \leq \lambda^n K d\left(Tx_0, TSx_0\right) \ \forall\ n \in \mathbb{N} \tag{3.23}$$

Implies

$$d\left(TS^nx_0, TS^{n+1}x_0\right) \leq \lambda^n K d_\alpha^i\left(Tx_0, TSx_0\right) \ for\ i = 1, 2 \tag{3.24}$$

On taking the limit in (3.24), we have

$$\lim_{n\to\infty} d_\alpha^i\left(TS^nx_0, TS^{n+1}x_0\right) = 0 \text{ for } i = 1, 2, \alpha \in (0, 1] \ \left(\text{since } \frac{a_1 + a_3 + a_4}{1 - (a_1 + a_2)} < 1\right)$$

Hence,

$$\lim_{n\to\infty} d\left(TS^nx_0, TS^{n+1}x_0\right) = 0 \tag{3.25}$$

For any $m > n$ where $m, n \in \mathbb{N}$, we have,

$$\begin{aligned} d\left(Tx_n, Tx_m\right) &\leq d\left(Tx_n, Tx_{n+1}\right) \oplus d\left(Tx_{n+1}, Tx_{n+2}\right) \oplus \cdots \oplus d\left(Tx_{m-1}, Tx_m\right) \\ &\leq \left[\left(\frac{a_1 + a_3 + a_4}{1 - (a_1 + a_2)}\right)^n + \left(\frac{a_1 + a_3 + a_4}{1 - (a_1 + a_2)}\right)^{n+1} + \cdots + \left(\frac{a_1 + a_3 + a_4}{1 - (a_1 + a_2)}\right)^{m-1}\right] d\left(Tx_0, TSx_0\right) \\ &\leq \left[\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}\right] d\left(Tx_0, TSx_0\right) \\ &\leq \lambda^n \frac{1}{1 - \lambda} d\left(Tx_0, TSx_0\right) \end{aligned} \tag{3.26}$$

So

$$d\left(TS^nx_0, TS^mx_0\right) \leq \lambda^n \frac{1}{1 - \lambda} d\left(Tx_0, TSx_0\right) \tag{3.27}$$

Since $\mathcal{F}$ is normal, we get

$$d\left(TS^nx_0, TS^mx_0\right) \leq \lambda^n \frac{k}{1 - \lambda} d\left(Tx_0, TSx_0\right) \tag{3.28}$$

Taking the limit as $m, n \to \infty$, we get

$$\lim_{m,n\to\infty} d\left(TS^nx_0, TS^mx_0\right) = \bar{0} \ \left(\text{since } \frac{a_1 + a_2 + a_3 + a_4}{1 - (a_2 + a_3)} < 1\right) \tag{3.29}$$

This proves that $\{(TS^nx_0\}$ is Cauchy sequence in X. Since X is a complete metric space, there exists $v \in X$ such that

$$\lim_{n\to\infty} (TS^nx_0 = v). \tag{3.30}$$

Now if T is subsequentially convergent, $\{S^nx_0\}$ has a convergent subsequence.So there exists $u \in X$ and $\{n_i\}$ such that

$$\lim_{i\to\infty} S^{n_i}x_0 = u. \tag{3.31}$$

Since T is continuous by (3.31), we get

$$\lim_{i\to\infty} TS^{n_i}x_0 = Tu. \tag{3.32}$$

Considering (3.31) and (3.32), we get $Tu = u$.
Now

$$d(TSu, Tu) \leq d(TSu, TS^{n_i}(x_0)) \oplus d\left(TS^{n_i}(x_0), TS^{n_i+1}(x_0)\right) \oplus d\left(TS^{n_i+1}(x_0), Tu\right).$$

$$\begin{aligned} d(TSu, Tu) \leq a_1\left[d(Tu, TS^{n_i}(x_0)) \oplus d\left(TS^{n_i-1}(x_0), TSu\right)\right] \oplus a_2\frac{d(Tu, TSu)\, d\left(TS^{n_i-1}(x_0), TS^{n_i}(x_0)\right)}{d(Tu, TS^{n_i-1}(x_0))} \\ \oplus a_3\frac{d(Tu, TSu)\, d\left(TS^{n_i-1}(x_0), TS^{n_i}(x_0)\right)}{d(Tu, TS^{n_i-1}(x_0)) \oplus d(Tu, TS^{n_i}(x_0)) \oplus d(TS^{n_i-1}(x_0), TSu)} \\ \oplus a_4\frac{d(Tu, TSu)\, d(Tu, TS^{n_i}(x_0)) \oplus d\left(TS^{n_i-1}(x_0), TSu\right) d\left(TS^{n_i-1}(x_0), TS^{n_i}(x_0)\right)}{d(Tu, TS^{n_i-1}(x_0)) \oplus d(TS^{n_i-1}(x_0), TSu)} \\ \oplus\lambda^{n_i} d(Tx_0, TSx_0) \oplus d\left(TS^{n_i+1}(x_0), Tu\right) \end{aligned}$$

So

$$(TSu, Tu) \leq \lambda d\left(TS^{n_i-1}(x_0), TS^{n_i}(x_0)\right) \oplus \frac{1}{1-\lambda}\lambda^n d(Tx_0, TSx_0) \oplus \frac{1}{1-\lambda} d\left(TS^{n_i+1}(x_0), Tu\right)$$

Since $\mathcal{F}$ is normal cone with normal constant K, we have

$$(TSu, Tu) \leq \lambda K d\left(TS^{n_i-1}(x_0), TS^{n_i}(x_0)\right) \oplus \frac{k}{1-\lambda}\lambda^n d(Tx_0, TSx_0) \oplus \frac{k}{1-\lambda} d\left(TS^{n_i+1}(x_0), Tu\right)$$

Taking the llimit $i \to \infty$, using (3.33) and $\lambda < 1$, we get
$d^i_\alpha(TSu, Tu) = 0$ for all $\alpha \in (0, 1]$ and $i = 1, 2$,
Hence,

$$d(TSu, Tu) = \bar{0} \tag{3.34}$$

So that $TSu = Tu$.
Since T is one-one, we get $Su = u$. So S has a fixed point.
if v is another fixed point of S, then $Sv = v$. Since S is type I contraction, we obtain

$$\begin{aligned} d(TSu, TSv) \leq a_1[d(Tu, TSv) \oplus d(Tv, TSu)] \oplus a_2\frac{d(Tu, TSu)\, d(Tv, TSv)}{d(Tu, Tv)} \oplus \\ a_3\frac{d(Tu, TSu)\, d(Tv, TSv)}{d(Tu, Tv) \oplus d(Tu, TSv) \oplus d(Tv, TSu)} \oplus a_4\frac{d(Tu, TSu)\, d(Tu, TSv) \oplus d(Tv, TSu)\, d(Tv, TSv)}{d(Tu, TSv) \oplus d(Tv, TSu)} \end{aligned} \tag{3.35}$$

$$\begin{aligned} = a_1[d(Tu, Tv) \oplus d(Tv, Tu)] \oplus a_2\frac{d(Tu, Tu)\, d(Tv, Tv)}{d(Tu, Tv)} \oplus \\ a_3\frac{d(Tu, Tu)\, d(Tv, Tv)}{d(Tu, Tv) \oplus d(Tu, Tv) \oplus d(Tv, Tu)} \oplus a_4\frac{d(Tu, Tu)\, d(Tu, Tv) \oplus d(Tv, Tu)\, d(Tv, Tv)}{d(Tu, Tv) \oplus d(Tv, Tu)} \end{aligned}$$

Implies

$$\begin{aligned} d(TSu, TSv) &\leq 2a_1 d(Tu, Tv) \\ &< d(Tu, Tv) \text{ as } 2a_1 < 1, \end{aligned}$$

this is a contradiction. So that $TSu = TSv$. Since S is injective, we get $u = v$. Thus,S has a unique fixed point.
Lastly, if T is sequentially convergent, by replacing n for n_i, we get that

$$\lim_{n\to\infty} S^n x_0 = u.$$

Thus, $\{S^n x_0\}$ is convergent to the fixed point u.
□

Example 3.5. Consider $E = C[0,1]$ and $\mathcal{F} = \{\eta \in E^*(I) : \eta \geq \bar{0}\}$ and $X = R$. Let $d : X \times X \mapsto E^*(I)$ be a fuzzy mapping define by

$$d(x,y)(t) = \begin{cases} \frac{|x-y|e^{k_0}}{t}, & \text{if } t \geq |x-y|e^{k_0} \\ 0, & \text{if } t < |x-y|e^{k_0} \end{cases}$$

Where k_0 is a fixed number in $[0,1]$. Now,

$$\frac{|x-y|e^{k_0}}{t} \geq \alpha \implies t \leq \frac{|x-y|e^{k_0}}{\alpha}$$

Thus, α-level set of $d(x,y)$ are given by

$$[d(x,y)]_\alpha = \left[|x-y|e^{k_0}.\frac{|x-y|e^{k_0}}{\alpha}\right],\ \alpha \in (0,1].$$

Choose the ordering " $\leq$ " as " $\preceq$ ", then it is easy to verify that,
$(Fd1) d(x,y) \succeq \bar{0}$ and $d(x,y) = \bar{0}$ iff $x = y$;
$(Fd2) d(x,y) = d(y,x)$;
$(Fd3) d(x,y) \preceq d(x,z) \oplus d(z,y)$ forall $x, y, z \in X$.
Then, the pair (X,d) is completely fuzzy cone metric space.
Now, show that (X,d) is a complete fuzzy cone metric space.
Let $\{x_n\}$ be a Cauchy sequence in (X,d). Then $(x_n, x_m) \to \bar{0}$ as $m,n \to \infty$, that is $d^1_\alpha(x_n,x_m) \to \bar{0}$ as $m,n \to \infty$ forall $\alpha \in (0,1]$. So $\{x_n\}$ is Cauchy sequence in $X(R)$. Since X is complete, there exists $x \in X$ such that

$$|x_n - x| \to \bar{0} \text{ as } n \to \infty.$$

Thus, (X,d) is complete. Since for any $\eta, \mu \in E^*(I), \eta \leq \mu \implies \eta \leq 1.\mu$, then,$\mathcal{F}$ is a fuzzy normal cone with normal constant 1.

Now consider the functions $T, S : X \mapsto X$ defined by $Tx = x^2$ and $Sx = \frac{1}{2}$. Let $a_1 = \frac{1}{50}$,$a_2 = \frac{1}{20}$,$a_3 = \frac{1}{30}$,$a_4 = \frac{1}{40}$. Then, we have.

$$d^1_\alpha(TSx, TSy) = |TSx - TSy|e^{k_0} = \left|\frac{x^2}{4} - \frac{y^2}{4}\right| e^{k_0}$$

$$d^1_\alpha(TSx, TSy) \leq \frac{1}{50}\left[|Tx - TSx| + |Ty - TSy|\right] e^{k_0} + \frac{1}{20}\frac{\left[|Tx - TSx||Ty - TSy|\right] e^{k_0}}{|Tx - Ty|e^{k_0}}$$

$$+\frac{1}{30}\frac{\left[|Tx - TSx||Ty - TSy|\right] e^{k_0}}{\left[|Tx - Ty| + |Tx - TSy| + |Ty - TSx|\right] e^{k_0}} + \frac{1}{40}\frac{\left[|Tx - TSx||Tx - TSy| + |Ty - TSx||Ty - TSy|\right] e^{k_0}}{\left[|Tx - TSy| + |Ty - TSx|\right] e^{k_0}}$$

$$d_\alpha^1\left(TSx,TSy\right) \leq \frac{1}{50}\left[d_\alpha^1(Tx,TSx)+d_\alpha^1(Ty,TSy)\right]+\frac{1}{20}\frac{d_\alpha^1(Tx,TSx)d_\alpha^1(Ty,TSy)}{d_\alpha^1(Tx,Ty)}$$
$$+\frac{1}{30}\frac{d_\alpha^1(Tx,TSx)d_\alpha^1(Ty,TSy)}{d_\alpha^1(Tx,Ty)+d_\alpha^1(Tx,TSy)+d_\alpha^1(Ty,TSx)}+\frac{1}{40}\frac{d_\alpha^1(Tx,TSx)d_\alpha^1(Tx,TSy)+d_\alpha^1(Ty,TSx)d_\alpha^1(Ty,TSy)}{d_\alpha^1(Tx,TSy)+d_\alpha^1(Ty,TSx)} \tag{3.36}$$

Also,

$$d_\alpha^2\left(TSx,TSy\right) \leq \frac{1}{50}\left[\frac{|Tx-TSx}{\alpha}+\frac{|Ty-TSy}{\alpha}\right]e^{k_0}+\frac{1}{20}\frac{\left[\frac{|Tx-TSx|}{\alpha}+\frac{|Ty-TSy|}{\alpha}\right]e^{k_0}}{\frac{|Tx-TSx|}{\alpha}e^{k_0}}$$
$$+\frac{1}{30}\frac{\left[\frac{|Tx-TSx|}{\alpha}\frac{|Ty-TSy|}{\alpha}\right]e^{k_0}}{\left[\frac{|Tx-Ty|}{\alpha}+\frac{|Tx-TSy|}{\alpha}+\frac{|Ty-TSx|}{\alpha}\right]}+$$
$$\frac{1}{40}\frac{\left[\frac{|Tx-TSx|}{\alpha}\frac{|Tx-TSy|}{\alpha}+\frac{|Ty-TSx|}{\alpha}\frac{|Ty-TSy|}{\alpha}\right]e^{k_0}}{\left[\frac{|Tx-TSy|}{\alpha}+\frac{|Ty-TSx|}{\alpha}\right]e^{k_0}}$$

$$d_\alpha^2\left(TSx,TSy\right) \leq \frac{1}{50}\left[d_\alpha^2(Tx,TSx)+d_\alpha^2(Ty,TSy)\right]+\frac{1}{20}\frac{d_\alpha^2(Tx,TSx)d_\alpha^2(Ty,TSy)}{d_\alpha^2(Tx,Ty)}$$
$$+\frac{1}{30}\frac{d_\alpha^2(Tx,TSx)d_\alpha^2(Ty,TSy)}{d_\alpha^2(Tx,Ty)+d_\alpha^2(Tx,TSy)+d_\alpha^2(Ty,TSx)}+\frac{1}{40}\frac{d_\alpha^2(Tx,TSx)d_\alpha^2(Tx,TSy)+d_\alpha^2(Ty,TSx)d_\alpha^2(Ty,TSy)}{d_\alpha^2(Tx,TSy)+d_\alpha^2(Ty,TSx)} \tag{3.37}$$

From (3.36) and (3.37), we have

$$d\left(TSx,TSy\right) \leq \frac{1}{50}\left[d(Tx,TSx)+d(Ty,TSy)\right]+\frac{1}{20}\frac{d(Tx,TSx)d(Ty,TSy)}{d(Tx,Ty)}$$
$$+\frac{1}{30}\frac{d(Tx,TSx)d(Ty,TSy)}{d(Tx,Ty)+d(Tx,TSy)+d(Ty,TSx)}+\frac{1}{40}\frac{d(Tx,TSx)d(Tx,TSy)+d(Ty,TSx)d(Ty,TSy)}{d(Tx,TSy)+d(Ty,TSx)}$$

Thus, S is a fuzzy cone T-type I contraction for $2a_1+a_2+a_3+a_4<1$.
Now, to show that S is a fuzzy T-type II contraction, Let $a_1=\frac{1}{30}$,$a_2=\frac{1}{10}$,$a_3=\frac{1}{20}$,$a_4=\frac{1}{40}$.

Then, we have

$$d_\alpha^1(TSx,TSy) = |TSx - TSy|e^{k_0} = \left|\frac{x^2}{4} - \frac{y^2}{4}\right| e^{k_0}$$

$$\begin{aligned} d_\alpha^1(TSx,TSy) \leq & \frac{1}{30}\left[\left|\left(x^2-\frac{y^2}{4}\right)-\left(y^2-\frac{x^2}{4}\right)\right|\right]e^{k_0} + \frac{1}{10}\frac{\left[\left|\left(x^2-\frac{x^2}{4}\right)\left(y^2-\frac{y^2}{4}\right)\right|\right]e^{k_0}}{|x^2-y^2|e^{k_0}} \\ & + \frac{1}{20}\frac{\left[\left|\left(x^2-\frac{x^2}{4}\right)\left(y^2-\frac{y^2}{4}\right)\right|\right]e^{k_0}}{\left[|x^2-y^2| + \left|\left(x^2-\frac{y^2}{4}\right)-\left(y^2-\frac{x^2}{4}\right)\right|\right]e^{k_0}} \\ & + \frac{1}{40}\frac{\left[\left|\left(x^2-\frac{x^2}{4}\right)\left(x^2-\frac{y^2}{4}\right)-\left(y^2-\frac{x^2}{4}\right)\left(y^2-\frac{y^2}{4}\right)\right|\right]e^{k_0}}{\left[\left|\left(x^2-\frac{y^2}{4}\right)-\left(y^2-\frac{x^2}{4}\right)\right|\right]e^{k_0}} \end{aligned}$$

$$\begin{aligned} d_\alpha^1(TSx,TSy) \leq & \frac{1}{30}[|Tx-TSy| + |Ty-TSx|]e^{k_0} + \frac{1}{10}\frac{[|Tx-TSx||Ty-TSy|]e^{k_0}}{[|Tx-Ty|]e^{k_0}} \\ & + \frac{1}{20}\frac{[|Tx-TSx||Ty-TSy|]e^{k_0}}{[[|Tx-Ty|]e^{k_0} + |Tx-TSy| + |Ty-TSx|]\,e^{k_0}} \\ & + \frac{1}{40}\frac{[|Tx-TSx||Tx-TSy| + |Ty-TSx||Ty-TSy|]\,e^{k_0}}{[|Tx-TSy| + |Ty-TSx|]\,e^{k_0}} \end{aligned}$$

Implies

$$\begin{aligned} d_\alpha^1(TSx,TSy) \leq & \frac{1}{30}\left[d_\alpha^1(Tx,TSy) + d_\alpha^1(Ty,TSx)\right] + \frac{1}{10}\frac{d_\alpha^1(Tx,TSx)\,d_\alpha^1(Ty,TSy)}{d_\alpha^1(Tx,Ty)} \\ & + \frac{1}{20}\frac{d_\alpha^1(Tx,TSx)\,d_\alpha^1(Ty,TSy)}{d_\alpha^1(Tx,Ty) + d_\alpha^1(Tx,TSy) + d_\alpha^1(Ty,TSx)} \\ & + \frac{1}{40}\frac{d_\alpha^1(Tx,TSx)\,d_\alpha^1(Tx,TSy) + d_\alpha^1(Ty,TSx)\,d_\alpha^1(Ty,TSy)}{d_\alpha^1(Tx,TSy) + d_\alpha^1(Ty,TSx)} \end{aligned} \tag{3.38}$$

Also,

$$\begin{aligned} d_\alpha^2(TSx,TSy) \leq & \frac{1}{30}\left[d_\alpha^2(Tx,TSy) + d_\alpha^2(Ty,TSx)\right] + \frac{1}{10}\frac{d_\alpha^2(Tx,TSx)\,d_\alpha^2(Ty,TSy)}{d_\alpha^2(Tx,Ty)} \\ & + \frac{1}{20}\frac{d_\alpha^2(Tx,TSx)\,d_\alpha^2(Ty,TSy)}{d_\alpha^2(Tx,Ty) + d_\alpha^2(Tx,TSy) + d_\alpha^2(Ty,TSx)} \\ & + \frac{1}{40}\frac{d_\alpha^2(Tx,TSx)\,d_\alpha^2(Tx,TSy) + d_\alpha^2(Ty,TSx)\,d_\alpha^2(Ty,TSy)}{d_\alpha^2(Tx,TSy) + d_\alpha^2(Ty,TSx)} \end{aligned} \tag{3.39}$$

From (3.38) and (3.39), we have

$$d\left(TSx,TSy\right) \leq \frac{1}{30}\left[d\left(Tx,TSy\right)+d\left(Ty,TSx\right)\right]+\frac{1}{10}\frac{d\left(Tx,TSx\right)d\left(Ty,TSy\right)}{d\left(Tx,Ty\right)} +\frac{1}{20}\frac{d\left(Tx,TSx\right)d\left(Ty,TSy\right)}{d\left(Tx,Ty\right)+d\left(Tx,TSy\right)+d\left(Ty,TSx\right)} +\frac{1}{40}\frac{d\left(Tx,TSx\right)d\left(Tx,TSy\right)+d\left(Ty,TSx\right)d\left(Ty,TSy\right)}{d\left(Tx,TSy\right)+d\left(Ty,TSx\right)}$$

Thus, S is a fuzzy cone T-type II contraction for $2a_1+a_2+a_3+a_4<1$.

Example 3.6. Consider $E=C[0,1]$ and $\mathcal{F}=\{\eta\in E^*(I):\eta\geq\bar{0}\}\subset E^*(I)$, $X=R$. Let $d:X\times X\mapsto E^*(I)$ be a fuzzy mapping define by

$$d(x,y)=|x-y|e^{k_0}\ ,e^{k_0}\in E$$

Where k_0 is a fixed number in $[0,1]$ and the α-level set of $d(x,y)$ are given by

$$[d(x,y)]_\alpha=\left[|x-y|e^{k_0}.\frac{|x-y|e^{k_0}}{\alpha}\right],\ \alpha\in(0,1].$$

Then, the pair (X,d) is called a fuzzy cone metric space as in Example 3.5 and consider the functions $T,S:X\mapsto X$ defined by $Tx=x$ and $Sx=\frac{x}{2}$. Clearly, T is one-one and continuous. Then, by Theorem 3.4, $v=0$ is the unique fixed point of S in X.

Corollary 3.7. *Suppose $(X,\ d)$ is a complete fuzzy cone metric space, $\mathcal{F}$ be a normal fuzzy cone with normal constant K. Let $T:X\to X$ be a one-one continuous function and $S:X\to X$ be a fuzzy cone T -type I contraction mapping, that is, for all $x,y\in X$,$Tx\neq Ty$ and $a_1,a_2,a_3\geq 0$ with $2a_1+a_2+a_3<1$ satisfying the following*

$$d\left(TSx,TSy\right)\leq a_1\left[d(Tx,TSx)\oplus d(Ty,TSy)\right]\oplus a_2\frac{d(Tx,TSx)d(Ty,TSy)}{d(Tx,Ty)} \oplus a_3\frac{d(Tx,TSx)d(Ty,TSy)}{d(Tx,Ty)\oplus d(Tx,TSy)\oplus d(Ty,TSx)} \tag{3.40}$$

Then, the following conditions are satisfied:
i. for every $x_0\in X$, $\lim_{n\to\infty}d(TS^nx_0,TS^{n+1}x_0)=\bar{0}$;
ii. there exists $v\in X$ such that $\lim_{n\to\infty}TS^nx_0=v$;
iii. if T is sequentially convergent, then $\{S^nx_0\}$ has a convergent subsequence;
iv. there is a unique $u\in X$ such that $Su=u$;
v. if T is sequentially convergent, then for each $x_0\in X$ the iterate sequence $\{S^nx_0\}$ converges to u.

Proof. Let $a_4=0$ in Theorem 3.3, we get the result immediately. □

Corollary 3.8. *Suppose $(X,\ d)$ is a complete fuzzy cone metric space, $\mathcal{F}$ be a normal fuzzy cone with normal constant K. Let $T:X\to X$ be a one-one continuous function and $S:X\to X$ be a fuzzy cone T -type I contraction mapping, that is, for all $x,y\in X$,$Tx\neq Ty$ and $a_1,a_2,a_3\geq 0$ with $2a_1+a_2+a_3<1$ satisfying* *the following*

$$d(TSx,TSy) \leq a_1 [d(Tx,TSx) \oplus d(Ty,TSy)] \oplus a_2 \frac{d(Tx,TSx)d(Ty,TSy)}{d(Tx,Ty)} \oplus a_3 \frac{d(Tx,TSx)d(Tx,TSy) \oplus d(Ty,TSx)d(Ty,TSy)}{d(Tx,TSy) \oplus d(Ty,TSx)} \tag{3.41}$$

Then, the following conditions are satisfied:
i. for every $x_0 \in X$, $\lim_{n\to\infty} d(TS^n x_0, TS^{n+1}x_0) = \bar{0}$;
ii. there exists $v \in X$ *such that* $\lim_{n\to\infty} TS^n x_0 = v$;
iii. if T *is sequentially convergent, then* $\{S^n x_0\}$ *has a convergent subsequence;*
iv. there is a unique $u \in X$ *such that* $Su = u$;
v. if T *is sequentially convergent, then for each* $x_0 \in X$ *the iterate sequence* $\{S^n x_0\}$ *converges to* u.

Proof. Let $a_3 = 0$ in Theorem 3.3, we get the result immediately. □

Corollary 3.9. *Suppose* (X, d) *is a complete fuzzy cone metric space,* $\mathcal{F}$ *be a normal fuzzy cone with normal constant* K. *Let* $T : X \to X$ *be a one-one continuous function and* $S : X \to X$ *be a fuzzy cone* T *-type I contraction mapping, that is, for all* $x, y \in X$,$Tx \neq Ty$ *and* $a_1, a_2, a_3 \geq 0$ *with* $2a_1 + a_2 + a_3 < 1$ *satisfying the following*

$$d(TSx,TSy) \leq a_1 [d(Tx,TSx) \oplus d(Ty,TSy)] \oplus a_2 \frac{d(Tx,TSx)d(Ty,TSy)}{d(Tx,Ty) \oplus d(Tx,TSy) \oplus d(Ty,TSx)} \oplus a_3 \frac{d(Tx,TSx)d(Tx,TSy) \oplus d(Ty,TSx)d(Ty,TSy)}{d(Tx,TSy) \oplus d(Ty,TSx)} \tag{3.42}$$

Then, the following conditions are satisfied:
i. for every $x_0 \in X$, $\lim_{n\to\infty} d(TS^n x_0, TS^{n+1}x_0) = \bar{0}$;
ii. there exists $v \in X$ *such that* $\lim_{n\to\infty} TS^n x_0 = v$;
iii. if T *is sequentially convergent, then* $\{S^n x_0\}$ *has a convergent subsequence;*
iv. there is a unique $u \in X$ *such that* $Su = u$;
v. if T *is sequentially convergent, then for each* $x_0 \in X$ *the iterate sequence* $\{S^n x_0\}$ *converges to* u.

Proof. Let $a_2 = 0$ in Theorem 3.3, we get the result immediately. □

Corollary 3.10. *Suppose* (X, d) *is a complete fuzzy cone metric space,* $\mathcal{F}$ *be a normal fuzzy cone with normal constant* K. *Let* $T : X \to X$ *be a one-one continuous function and* $S : X \to X$ *be a fuzzy cone* T *-type I contraction mapping, that is, for all* $x, y \in X$,$Tx \neq Ty$ *and* $a_1, a_2, a_3 \geq 0$ *with* $2a_1 + a_2 + a_3 < 1$ *satisfying the following*

$$d(TSx,TSy) \leq a_1 \frac{d(Tx,TSx)d(Ty,TSy)}{d(Tx,Ty)} \oplus a_2 \frac{d(Tx,TSx)d(Ty,TSy)}{d(Tx,Ty) \oplus d(Tx,TSy) \oplus d(Ty,TSx)} \oplus a_3 \frac{d(Tx,TSx)d(Tx,TSy) \oplus d(Ty,TSx)d(Ty,TSy)}{d(Tx,TSy) \oplus d(Ty,TSx)} \tag{3.43}$$

Then, the following conditions are satisfied:
i. for every $x_0 \in X$, $\lim_{n\to\infty} d(TS^n x_0, TS^{n+1}x_0) = \bar{0}$;

ii. there exists $v \in X$ *such that* $\lim_{n\to\infty} TS^n x_0 = v$*;*
iii. if T *is sequentially convergent, then* $\{S^n x_0\}$ *has a convergent subsequence;*
iv. there is a unique $u \in X$ *such that* $Su = u$*;*
v. if T *is sequentially convergent, then for each* $x_0 \in X$ *the iterate sequence* $\{S^n x_0\}$ *converges to* u*.*

Proof. Let $a_1 = 0$ in Theorem 3.3, we get the result immediately. □

Corollary 3.11. *Suppose* $(X,\ d)$ *is a complete fuzzy cone metric space,* $\mathcal{F}$ *be a normal fuzzy cone with normal constant* K*. Let* $T : X \to X$ *be a one-one continuous function and* $S : X \to X$ *be a fuzzy cone* T *-type II contraction mapping, that is, for all* $x, y \in X$*,*$Tx \neq Ty$ *and* $a_1, a_2, a_3 \geq 0$ *with* $2a_1 + a_2 + a_3 < 1$ *satisfying the following*

$$d(TSx,TSy) \leq a_1 [d(Tx,TSy) \oplus d(Ty,TSx)] \oplus a_2 \frac{d(Tx,TSx)\, d(Ty,TSy)}{d(Tx,Ty)} \oplus a_3 \frac{d(Tx,TSx)\, d(Ty,TSy)}{d(Tx,Ty) \oplus d(Tx,TSy) \oplus d(Ty,TSx)} \tag{3.44}$$

Then, the following conditions are satisfied:
i. for every $x_0 \in X$*,* $\lim_{n\to\infty} d(TS^n x_0, TS^{n+1} x_0) = \bar{0}$*;*
ii. there exists $v \in X$ *such that* $\lim_{n\to\infty} TS^n x_0 = v$*;*
iii. if T *is sequentially convergent, then* $\{S^n x_0\}$ *has a convergent subsequence;*
iv. there is a unique $u \in X$ *such that* $Su = u$*;*
v. if T *is sequentially convergent, then for each* $x_0 \in X$ *the iterate sequence* $\{S^n x_0\}$ *converges to* u*.*

Proof. Let $a_4 = 0$ in Theorem 3.4, we get the result immediately. □

Corollary 3.12. *Suppose* $(X,\ d)$ *is a complete fuzzy cone metric space,* $\mathcal{F}$ *be a normal fuzzy cone with normal constant* K*. Let* $T : X \to X$ *be a one-one continuous function and* $S : X \to X$ *be a fuzzy cone* T *-type II contraction mapping, that is, for all* $x, y \in X$*,*$Tx \neq Ty$ *and* $a_1, a_2, a_3 \geq 0$ *with* $2a_1 + a_2 + a_3 < 1$ *satisfying the following*

$$d(TSx,TSy) \leq a_1 [d(Tx,TSy) \oplus d(Ty,TSx)] \oplus a_2 \frac{d(Tx,TSx)\, d(Ty,TSy)}{d(Tx,Ty)} \oplus a_3 \frac{d(Tx,TSx)\, d(Tx,TSy) \oplus d(Ty,TSx)\, d(Ty,TSy)}{d(Tx,TSy) \oplus d(Ty,TSx)} \tag{3.45}$$

Then, the following conditions are satisfied:
i. for every $x_0 \in X$*,* $\lim_{n\to\infty} d(TS^n x_0, TS^{n+1} x_0) = \bar{0}$*;*
ii. there exists $v \in X$ *such that* $\lim_{n\to\infty} TS^n x_0 = v$*;*
iii. if T *is sequentially convergent, then* $\{S^n x_0\}$ *has a convergent subsequence;*
iv. there is a unique $u \in X$ *such that* $Su = u$*;*
v. if T *is sequentially convergent, then for each* $x_0 \in X$ *the iterate sequence* $\{S^n x_0\}$ *converges to* u*.*

Proof. Let $a_3 = 0$ in Theorem 3.4, we get the result immediately. □

Corollary 3.13. *Suppose* $(X,\ d)$ *is a complete fuzzy cone metric space,* $\mathcal{F}$ *be a normal fuzzy cone with normal constant* K*. Let* $T : X \to X$ *be a one-one continuous function and* $S : X \to X$ *be a fuzzy cone* T

-type II contraction mapping, that is, for all $x, y \in X$,$Tx \neq Ty$ and $a_1, a_2, a_3 \geq 0$ with $2a_1 + a_2 + a_3 < 1$ satisfying the following

$$d(TSx, TSy) \leq a_1 [d(Tx, TSy) \oplus d(Ty, TSx)] \oplus a_2 \frac{d(Tx, TSx) d(Ty, TSy)}{d(Tx, Ty) \oplus d(Tx, TSy) \oplus d(Ty, TSx)} \oplus a_3 \frac{d(Tx, TSx) d(Tx, TSy) \oplus d(Ty, TSx) d(Ty, TSy)}{d(Tx, TSy) \oplus d(Ty, TSx)} \tag{3.46}$$

Then, the following conditions are satisfied:
i. for every $x_0 \in X$, $\lim_{n\to\infty} d(TS^n x_0, TS^{n+1} x_0) = \bar{0}$;
ii. there exists $v \in X$ such that $\lim_{n\to\infty} TS^n x_0 = v$;
iii. if T is sequentially convergent, then $\{S^n x_0\}$ has a convergent subsequence;
iv. there is a unique $u \in X$ such that $Su = u$;
v. if T is sequentially convergent, then for each $x_0 \in X$ the iterate sequence $\{S^n x_0\}$ converges to u.

Proof. Let $a_2 = 0$ in Theorem 3.4, we get the result immediately. □

4 Conclusion

The main findings of this study demonstrate applicability fuzzy cone metric spaces in establishing fixed point theorems for fuzzy mappings. This study provides significant advancements in the understanding of fuzzy cone metric spaces, with potential applications in differential equations and nonlinear Fredholm integral equation. Future work could also explore the extension of this results to other types of fuzzy mappings and their applications in real-world problems.

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References

[1] Banach S. Sur les operations dans les ensembles abstraits et leur application aux equations itegrales. *Fundamenta Mathematicae.* 1922; 3(1): 133-181. DOI: https://doi.org/10.4064/fm-3-1-133-181

[2] Bag T. Some results on D*-fuzzy metric spaces. *International Journal of Mathematics and Scientific Computing.* 2012; 2(1): 29-33.

[3] Ciric LB. A generalization of Banachs contraction principle. *Proceedings of the American Mathematical Society.* 1974; 45(2): 267-273. DOI: https://doi.org/10.2307/2040075

[4] Raji M. Generalized α-ψ contractive type mappings and related coincidence fixed point theorems with applications. *The Journal of Analysis.* 2023; 31(2): 1241-1256. DOI: https://doi.org/10.1007/s41478-022-00498-8

[5] Long-Guang H, Xian Z. Cone metric spaces and fixed point theorems for contractive mappings. *Journal of Mathematical Analysis and Applications.* 2007; 332(2): 1468-1476. DOI: http://doi.org/10.1016/j.jmaa.2005.03.087

[6] Ilic D, Rakocevic V. Common fixed point for maps on cone metric spaces. *Journal of Mathematical Analysis and Applications.* 2008; 341(2): 876-882. DOI: https://doi.org/10.1016/j.jmaa.2007.10.065

[7] Morales JR, Rajas E. Cone metric spaces and fixed point theorem of T-Kannan contractions mappings. *Int. J. Math. Anal.* 2010; 4(4): 175-184. DOI: https://doi.org/10.48550/arXiv.0907.3949

[8] Rezapour SH, Hamlbarani R. Some notes on the paper Cone metric spaces and fixed point theorems for contractive mappings. *Journal of Mathematical Analysis and Applications.* 2008; 345(2): 719-724. DOI: https://doi.org/10.1016/j.jmaa.2008.04.049

[9] Zadeh LA. Fuzzy sets. *Information and Control.* 1965; 8(3): 338-353.

[10] Weiss MD. Fixed points and induced fuzzy topologies for fuzzy sets. *J. Math. Anal. Appl.* 1975; 50: 142-150. DOI: https://doi.org/10.1016/0022-247X(75)90044-X

[11] Heilpern S. Fuzzy mappings and fixed point theorem. *Journal of Mathematical Analysis and Applications.* 1981; 83(2): 566-569. DOI: https://doi.org/10.1016/0022-247X(81)90141-4

[12] Bag T. Fuzzy cone metric spaces and fixed point theorems on fuzzy T-Kannan and fuzzy T-Chatterjea type contractive mappings. *Fuzzy Information and Engineering.* 2015; 7(3): 305-315. DOI: http://doi.org/10.1016/j.fiae.2015.09.004

[13] Raji M, Ibrahim MA. Fixed point theorems for fuzzy contractions mappings in a dislocated b-metric spaces with applications. *Annals of Mathematics and Computer Science.* 2024; 21: 1-13. DOI: https://doi.org/10.56947/amcs.v21.233

[14] Rashid, M, Shahzad A, Azam A. Fixed point theorems for L-fuzzy mappings in quasi-pseudo metric spaces. *Journal of Intelligent & Fuzzy Systems.* 2017; 32(1): 499-507. DOI: https://doi.org/10.3233/JIFS-152261

[15] Raji M, Ibrahim MA, Rauf K, Kehinde R. Common fixed point results for fuzzy F-contractive mappings in a dislocated metric spaces with application.*Qeios*. 2024. DOI: https://doi.org/10.32388/SV98CN

[16] Azam A. Fuzzy fixed points of fuzzy mappings via a rational inequality. *Hacet. J. Math. Stat.* 2011; 40(3): 421-431.

[17] Butnariu D. Fixed point for fuzzy mapping.*Fuzzy sets and Systems.* 1982; 7(2): 191-207. DOI: https://doi.org/10.1016/0165-0114(82)90049-5

[18] Phiangsungnoen S, Sintunavarat W, Kumam P. Common α-fuzzy fixed point theorems for fuzzy mappings via βF-admissible pair. *Journal of Intelligent & Fuzzy Systems.* 2014; 27(5): 2463-2472. DOI: http://dx.doi.org/10.22436/jnsa.008.01.07

[19] Shahzad A, Shoaib A, Khammahawong K, Kumam P. New Ciric Type Rational Fuzzy F-Contraction for Common Fixed Points. *In Beyond Traditional Probabilistic Methods in Economics 2.* 2019; 809: 215-229. DOI: https://doi.org/10.1007/978-3-030-04200-417

[20] Phiangsungnoen S, Kumam P. Fuzzy fixed point theorems for multivalued fuzzy contractions in b-metric spaces. *Journal of Nonlinear Science and Applications.* 2015; 8(1): 55-63. DOI: https://doi.org/10.1186/s40467-014-0020-6

[21] Shoaib A, Kumam P, Shahzad A, Phiangsungnoen S, Mahmood Q. Fixed point results for fuzzy mappings in a b-metric space. *Fixed point theory and Applications.* 2018; 2018: 1-12. DOI: https://doi.org/10.1186/s13663-017-0626-8

[22] Shahzad A, Shoaib A, Mahmood Q. Fixed point theorems for fuzzy mappings in b- metric space. *Ital. J. Pure Appl. Math.* 2017; 38: 419-42.

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