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A Method for Finding LR Fuzzy Eigenvectors of Real Symmetric Matrix †

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Abstract. In this paper, the calculation methods of the real eigenvalues and LR fuzzy eigenvectors of clear real symmetry matrices are deeply considered. The original fuzzy feature problem is extended by using the arithmetic algorithm of LR fuzzy numbers into a simple feature problem with a high-order clear real symmetry matrix. We discuss two cases: (a) λ is a non-negative unknown eigenvalue; (b) λ is a negative unknown eigenvalue. We established two computational models and proposed an algorithm for finding the fuzzy eigenvectors of the true symmetry matrix. Some numerical examples are used to illustrate our proposed method.

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1 Introduction

Compared with the phenomenon of certainty, there are a large number of uncertain events exist and occur in real life. Fuzzy mathematics is born from this and happens to be one of the best tools to describe and analyze these uncertain phenomena. Some descriptive processes of motion and change often have uncertainty of all and part of the parameters. Sometime the uncertainty of the parameters is represented and computed by the fuzzy numbers. The concept of fuzzy numbers and their arithmetic operations were first coined and studied by Zadeh [1, 2], Dubois et al.[3] and Nahmias [4]. In the past half century, many scholars at home and abroad have paid more and more attention to a series of studies based on algebraic equations of fuzzy numbers. A new method is proposed to study fuzzy numbers and fuzzy number spatial structures by Puri and Ralescu [5] Goetschell et al.[6] and Wu Congxin et al.[7, 8].In the past two decades, more scholars have studied some more general and complex fuzzy linear systems based on Friedman et al. [9]'s 1998 embedding method to discuss a class of semi-fuzzy linear systems $A\widetilde{x} = \widetilde{b}$, such as dual fuzzy linear systems, generalized fuzzy linear systems, complex fuzzy linear systems, dual fully fuzzy linear systems, and general dual fuzzy linear systems, see [10, 11, 12, 13, 14].Through the joint efforts of many scholars, new theories and methods have been proposed in recent years, enriching fuzzy numbers and fuzzy linear systems [15, 16, 17]. The combination of fuzzy numbers and many mathematical problems has become a new research direction for many scholars. Guo et al. related linear matrix equations to fuzzy numbers and did some research [18, 19, 20, 14, 21, 22].

In recent years, the problem of fuzzy eigenvalues and eigenvectors has attracted the attention of many scholars. The reason is that the problem of finding the eigenvalues and eigenvectors of a matrix is widely used in problems in many fields such as engineering, management, physics and finance, but many of the

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parameters are uncertain. This uncertainty has been linked to fuzzy theory by scholars, and the methods of solving fuzzy eigenvalues and fuzzy eigenvectors have been used to solve the problem.The fuzzy eigenvalues and the generalized fuzzy eigenvalues of the form $\tilde{A}\tilde{x} = \tilde{\lambda}\tilde{B}\tilde{x}$ were studied by Buckley [23] and Chiao [24] using the same method, respectively. After that, Thoedorou al. [25] obtained a fuzzy eigenvector based on trigonometric fuzzy numbers by using a two-step method. In 2010, Tian founded the real matrix fuzzy eigenvector and studied the relationship between the real eigenvectors and the fuzzy eigenvectors [26], and he also studied the structure of the fuzzy eigenspace with some results. In 2013, Allahviranloo et al. [12] studied how to obtain the required price difference by deriving the maximum and minimum eigenvalues and general fuzzy eigenvalues under different conditions.

Most of the matrices involved in many studies today are symmetry matrices, and the role of symmetry matrices in many aspects is undoubtedly very large. We are not help guessing that which fuzzy vectors under symmetric linear transformation are simply just scaling This paper focuses on the study of the real eigenvalues and fuzzy eigenvectors of clear real matrices, which are based on the extension of LR fuzzy numbers and the universal application of symmetric matrices. Here's how this article is structured:

In Section 2, we will review some of the relevant definitions and arithmetic operations of LR fuzzy numbers, and in Section 3 we will deepen and extend the initial fuzzy vector eigenvalue problem to make it a simple eigenvalue problem for high-order clear real symmetry matrices, and we will propose an algorithm to solve the fuzzy eigenvectors of real symmetric matrices. In Section 4, we will give a few representative examples to illustrate. In Section 5, we draw some conclusions and conduct in-depth investigation and outlook.

2 Preliminaries

The concept of fuzzy numbers can be defined in some ways(see [3, 4, 1]).

Definition 2.1. *Let X be a non-empty set. Let's put a fuzzy set $\widetilde{A}$ in X like this*

$$\mu_{\widetilde{A}} : X \to [0, 1],$$

each of these elements $x \in X$, is associated with a real number in the closed interval $[0, 1]$, where the value $\mu_{\widetilde{A}}$ represents the degree of membership of x in fuzzy set $\widetilde{A}$, the function $\mu_{\widetilde{A}} : X \to [0, 1]$ is called the membership function of $\widetilde{A}$. A fuzzy set $\widetilde{A}$ is represented by the set of ordered pairs of element x and grade $\mu_{\widetilde{A}}$ which can be written as

$$\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x))|x \in X\}.$$

Definition 2.2. *A fuzzy number is a fuzzy set like $t : R \to I = [0, 1]$ which satisfaction is as follows:*
(1) t is upper semi-continuous,
(2) t is fuzzy convex, i.e. $t(\lambda x + (1-\lambda)y) \geq \min\{t(x), t(y)\}$ for all $x, y \in R, \lambda \in [0, 1]$,
(3) t is normal, i.e. there exists $x_0 \in R$ such that $u(x_0) = 1$,
(4) suppt $= \{x \in R \mid u(x) > 0\}$ is the support of the t, and its closure cl(suppt) is compact.
Let E^1 be the set of all fuzzy numbers on R.

Definition 2.3. *If a fuzzy number $\widetilde{t}$ satisfies the following conditions, then $\widetilde{t}$ is called a LR fuzzy number:*

$$\mu_{\widetilde{t}}(x) = \begin{cases} L(\frac{t-x}{\alpha}), & x \leq t, \quad \alpha > 0, \\ R(\frac{t-m}{\beta}), & x \geq t, \quad \beta > 0, \end{cases}$$

where t, α and β are called the mean value, left and right spreads of $\widetilde{t}$, respectively. The left shape function $L(\cdot)$, satisfies:

(1) $L(x) = L(-x)$,
(2) $L(0) = 1$ and $L(1) = 0$,
(3) $L(x)$ is non increasing on $[0, \infty)$.
Under the similar conditions, the right shape function $R(\cdot)$, satisfies:
(1) $R(x) = R(-x)$,
(2) $R(0) = 1$ and $R(1) = 0$,
(3) $R(x)$ is non decreasing on $(-\infty, 0]$.

So we can get a situation like this, when two LR fuzzy numbers $\widetilde{t} = (t, \alpha, \beta)_{LR}$ and $\widetilde{u} = (u, \gamma, \delta)_{LR}$ are equal, if and only if $t = u, \alpha = \gamma$ and $\beta = \delta$.

Definition 2.4. *For arbitrary LR fuzzy numbers $\widetilde{t} = (t, \alpha, \beta)_{LR}$ and $\widetilde{u} = (u, \gamma, \delta)_{LR}$, we have*

(1) Addition

$$\widetilde{t} + \widetilde{u} = (t, \alpha, \beta)_{LR} + (u, \gamma, \delta)_{LR} = (t + u, \alpha + \gamma, \beta + \delta)_{LR}.$$

(2) Subtraction

$$\widetilde{t} - \widetilde{u} = (t, \alpha, \beta)_{LR} - (u, \gamma, \delta)_{LR} = (t - u, \alpha - \delta, \beta - \gamma)_{LR}.$$

(3) Scalar multiplication

$$\lambda\widetilde{t} = \lambda(t, \alpha, \beta)_{LR} \cong \begin{cases} (\lambda t, \lambda\alpha, \lambda\beta)_{LR}, & \lambda \geq 0, \\ (\lambda t, -\lambda\beta, -\lambda\alpha)_{RL}, & \lambda < 0. \end{cases}$$

2.1 Fuzzy Eigenvector of Real Matrix

Definition 2.5. *If the mean value of a LR fuzzy number is 0 and the left and right spread values are α and β where $0 \leq \alpha, \beta < 1$, this fuzzy number is called LR zero fuzzy number and denoted by $\widetilde{0} = (0, \alpha, \beta)$.*

A fuzzy vector $\widetilde{x} = (\widetilde{x}_i), i = 1, \cdots, n$ is called a LR zero fuzzy vector, if each element $\widetilde{x}_i$ of $\widetilde{x}$ is a LR zero fuzzy number.

Definition 2.6. *Let A to be a $n \times n$ real matrix. If the real number λ and the non zero fuzzy vector $\widetilde{x}$ satisfies the following linear system*

$$A\widetilde{x} = \lambda\widetilde{x}, \tag{2.1}$$

i.e.,

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{12} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \\ \vdots \\ \widetilde{x}_n \end{pmatrix} = \lambda \begin{pmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \\ \vdots \\ \widetilde{x}_n \end{pmatrix}, \tag{2.2}$$

in this case, λ is a real eigenvalue of the real matrix A and the fuzzy eigenvector belonging to the real matrix A with the eigenvalue λ is $\widetilde{x}$.

3 Finding the Fuzzy Eigenvectors

We mainly study the problem of how to obtain fuzzy eigenvectors of real matrices by calculation in this paper. We first assume that A is a symmetric matrix, which makes the problem and calculation become simple and universal. On this basis, we consider the case of eigenvalues λ which are non-negative and negative respectively.

3.1 Extended Models and Its Solution

On the basis of the LR fuzzy number multiplication algorithm (by Dubois et al.), we can get the following conclusions.

(a) When a non-negative eigenvalue of matrix A is λ.

Theorem 3.1. *Suppose A a real matrix. In the case of $\lambda \geq 0$, the fuzzy feature problem* (2.1) *can be extended into a clear linear system as follows:*

$$\begin{cases} Ax = \lambda x, \\ \begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix} \begin{pmatrix} x^l \\ x^r \end{pmatrix} = \lambda \begin{pmatrix} x^l \\ x^r \end{pmatrix}, \end{cases} \tag{3.1}$$

where

$$\widetilde{x} = (x, x^l, x^r), A = A^+ + A^-. \tag{3.2}$$

And the elements a_{ij}^+ of matrix A^+ and a_{ij}^- of matrix A^- are determined by this way: if $a_{ij} \geq 0, a_{ij}^+ = a_{ij}$ else $a_{ij}^+ = 0, 1 \leq i,j \leq n$; if $a_{ij} < 0, a_{ij}^- = a_{ij}$ else $a_{ij}^- = 0, 1 \leq i,j \leq n$.

Proof. Let $A = A^+ + A^-, \widetilde{x} = (x, x^l, x^r)$. The elements a_{ij}^+ of matrix A^+ and a_{ij}^- of matrix A^- are determined by this way: if $a_{ij} \geq 0, a_{ij}^+ = a_{ij}$ else $a_{ij}^+ = 0, 1 \leq i,j \leq n$; if $a_{ij} < 0, a_{ij}^- = a_{ij}$ else $a_{ij}^- = 0, 1 \leq i,j \leq n$.

Firstly

$$\begin{aligned} A\widetilde{x} &= (A^+ + A^-)(x, x^l, x^r) = (A^+x, A^+x^l, A^+x^r) + (A^-x, -A^-x^r, -A^-x^l) \\ &= (A^+x + A^-x, A^+x^l - A^-x^r, A^+x^r - A^-x^l), \end{aligned} \tag{3.3}$$

On the other hand,

$$\lambda\widetilde{x} = (\lambda x, \lambda x^l, \lambda x^r), \lambda \geq 0. \tag{3.4}$$

From $A\widetilde{x} = \lambda\widetilde{x}$, we have

$$\begin{cases} A^+x + A^-x = \lambda x, \\ A^+x^l - A^-x^r = \lambda x^l, \\ A^+x^r - A^-x^l = \lambda x^r. \end{cases} \tag{3.5}$$

By the matrix multiplication, the Eqs.(3.5) can be written as

$$\begin{cases} Ax = \lambda x, \\ \begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix} \begin{pmatrix} x^l \\ x^r \end{pmatrix} = \lambda \begin{pmatrix} x^l \\ x^r \end{pmatrix}, \end{cases}$$

where

$$\widetilde{x} = (x, x^l, x^r), A = A^+ + A^-.$$

The proof was completed. □

(b) When a negative eigenvalue of matrix A is λ.

Theorem 3.2. *Suppose A a real matrix. In the case of $\lambda < 0$, the fuzzy feature problem (*2.1*) can be extended into a clear linear system as follows:*

$$\begin{cases} Ax = \lambda x, \\ \begin{pmatrix} -A^- & A^+ \\ A^+ & -A^- \end{pmatrix} \begin{pmatrix} x^r \\ x^l \end{pmatrix} = \lambda \begin{pmatrix} x^r \\ x^l \end{pmatrix}, \end{cases} \tag{3.6}$$

where

$$\widetilde{x} = (x, x^l, x^r), A = A^+ + A^-. \tag{3.7}$$

And the elements a_{ij}^+ of matrix A^+ and a_{ij}^- of matrix A^- are determined by this way: if $a_{ij} \geq 0, a_{ij}^+ = a_{ij}$ else $a_{ij}^+ = 0, 1 \leq i, j \leq n$; if $a_{ij} < 0, a_{ij}^- = a_{ij}$ else $a_{ij}^- = 0, 1 \leq i, j \leq n$.

Proof. Let $A = A^+ + A^-, \widetilde{x} = (x, x^l, x^r)$. The elements a_{ij}^+ of matrix A^+ and a_{ij}^- of matrix A^- are determined by this way: if $a_{ij} \geq 0, a_{ij}^+ = a_{ij}$ else $a_{ij}^+ = 0, 1 \leq i, j \leq n$; if $a_{ij} < 0, ba_{ij}^- = a_{ij}$ else $a_{ij}^- = 0, 1 \leq i, j \leq n$.

Firstly

$$\begin{aligned} A\widetilde{x} &= (A^+ + A^-)(x, x^l, x^r) = (A^+x, A^+x^l, A^+x^r) + (A^-x, -A^-x^r, -A^-x^l) \\ &= (A^+x + A^-x, A^+x^l - A^-x^r, A^+x^r - A^-x^l), \end{aligned} \tag{3.8}$$

On the other hand,

$$\lambda\widetilde{x} = (\lambda x, -\lambda x^r, -\lambda x^l), \lambda < 0. \tag{3.9}$$

From $A\widetilde{x} = \lambda\widetilde{x}$, we have

$$\begin{cases} A^+x + A^-x = \lambda x, \\ A^+x^l - A^-x^r = -\lambda x^r, \\ A^+x^r - A^-x^l = -\lambda x^l. \end{cases} \tag{3.10}$$

By the matrix multiplication, the Eqs.(3.10) can be written as

$$\left\{ \begin{array}{c} Ax = \lambda x, \\ \begin{pmatrix} -A^- & A^+ \\ A^+ & -A^- \end{pmatrix} \begin{pmatrix} x^r \\ x^l \end{pmatrix} = -\lambda \begin{pmatrix} x^r \\ x^l \end{pmatrix}, \end{array} \right.$$

where

$$\widetilde{x} = (x, x^l, x^r), A = A^+ + A^-.$$

The proof was completed. □

3.2 Solving the Extended Model

With the above preparations, let's consider the calculation of the model (3.1) and (3.9).

In the first step, under the condition of $Ax = \lambda x$, i.e., we need to compute all the eigenvalues and eigenvectors of the real matrix $A = A^+ + A^-$.

Then we solve the roots to the equation about λ

$$f(\lambda) = det(\lambda I - A) = 0, \tag{3.11}$$

and the nonzero solution of homogeneous group of linear equations

$$(\lambda I - A)x = O, \tag{3.12}$$

The above method can help us solve all the eigenvalues and eigenvectors of the real symmetry matrix A. Since the eigenvalues of the symmetry matrix A are all real numbers, we can sort these eigenvalues by magnitude. The assumptions are shown below

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{j-1} \geq 0 \geq \lambda_{j+1} \geq \cdots \geq \lambda_n, \tag{3.13}$$

and all eigenvectors belonging to the real symmetry matrix A are

$$x_1, x_2, \cdots, x_n, \tag{3.14}$$

each x_i is an eigenvector of the eigenvector of the real matrix A that belongs to the eigenvalue λ_i.

In the second step, we have obtained our eigenvalues, and we naturally start to solve the eigenvectors of the real matrix S.

We solve homogeneous group nonzero solutions of linear equations for every $\lambda_i, i = 1, 2, \cdots, j$ as follows:

$$(\lambda_i I - S) \begin{pmatrix} x^l \\ x^r \end{pmatrix} = O, \tag{3.15}$$

where

$$S = \begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix}.$$

We solve homogeneous group nonzero solutions of linear equations for every $\lambda_i, i = j+1, \cdots, n$ as follows:

$$(\lambda_i I + S) \begin{pmatrix} x^r \\ x^l \end{pmatrix} = O, \tag{3.16}$$

where

$$S = \begin{pmatrix} -A^- & A^+ \\ A^+ & -A^- \end{pmatrix}.$$

In the last step, we have the solution to the models (3.1) and (3.9) is as follows:

$$eigenvalues : \lambda_1, \lambda_2, \cdots, \lambda_n,$$

$$eigenvectors : \widehat{x}_1, \widehat{x}_2, \cdots, \widehat{x}_n, \tag{3.17}$$

where $\widehat{x}_i = (x_i, x_i^l, x_i^r), i = 1, \cdots, n$.

Remark 3.3. *We know that the eigenvalues of symmetric positive-definite matrices are positive, then when matrix A is a symmetric positive-definite matrix*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0,$$

and the eigenvectors

$$\widehat{x}_1, \widehat{x}_2, \cdots, \widehat{x}_n$$

of real matrix S are determined by the model (3.1).

When matrix A is symmetric negative definite matrix, its eigenvalues are all negative.

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n < 0,$$

and the eigenvectors

$$\widehat{x}_1, \widehat{x}_2, \cdots, \widehat{x}_n$$

of real matrix S are determined by the model (3.9).

3.3 Fuzzy Eigenvector

But we find that the LR fuzzy number solution vector $\widehat{x}$ obtained from the above may still be inappropriate, except for $x^l \geq 0, x^r \geq 0$. Therefore, we give the definition of the LR fuzzy feature vector to the problem (2.1) as follows:

Definition 3.4. *Let $\widehat{x} = (x, x^l, x^r)$. If (x, x^l, x^r) is the minimal solution of the Eqs.(3.1) or (3.9),such that $x^l \geq 0, x^r \geq 0$, we define $\widetilde{x} = \widehat{x} = (x, x^l, x^r)$ is a strong LR fuzzy eigenvector of fuzzy eigen problem (2.1) . Otherwise, the $\widetilde{x} = (x, x^l, x^r)$ is defined as a weak LR fuzzy eigenvector of fuzzy eigen problem (2.1) given by*

$$\widetilde{x} = \widetilde{x}_i,$$

where

$$\widetilde{x}_i = \begin{cases} (x_i, x_i^l, x_i^r), & x_i^l > 0, \quad x_i^r > 0, \\ (x_i, 0, \max\{-x_i^l, x_i^r\}), & x_i^l < 0, \quad x_i^r > 0, \\ (x_i, \max\{x_i^l, -x_i^r\}, 0,), & x_i^l > 0, \quad x_i^r < 0, \\ (x_i, -x_i^l, -x_i^r), & x_i^l < 0, \quad x_i^r < 0. \end{cases} \quad i = 1, \ldots, n. \tag{3.18}$$

The solution matrix can be a strong LR fuzzy solution only if $\widetilde{x} = (x, x^l, x^r)$ is an LR fuzzy vector, that is, every element in $\widetilde{x} = (x, x^l, x^r)$ is an LR fuzzy number.

Here we give a specific algorithm for how to find fuzzy eigenvectors of real symmetric matrices.

Algorithm 3.1

Step 1: Decomposing the matrix A with $A = A^+ + A^-$.

Step 2: By calculating the equation $Ax = \lambda x$,i.e, all the eigenvalues and eigenvectors of the real symmetry matrix A are obtained,

$$eigenvalues : \lambda_1, \lambda_2, \cdots, \lambda_n,$$

$$eigenvectors : x_1, x_2, \cdots, x_n.$$

Step 3: Solving the left and right spread values of fuzzy eigenvectors of real matrix A.

If $\lambda_i \geq 0$, then we can calculate the fuzzy eigen problem (2.1) by

$$\begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix} \begin{pmatrix} x^l \\ x^r \end{pmatrix} = \lambda_i \begin{pmatrix} x^l \\ x^r \end{pmatrix},$$

If $\lambda < 0$, then we can calculate the fuzzy eigen problem ((2.1) by

$$\begin{pmatrix} -A^- & A^+ \\ A^+ & -A^- \end{pmatrix} \begin{pmatrix} x^r \\ x^l \end{pmatrix} = \lambda_i \begin{pmatrix} x^r \\ x^l \end{pmatrix},$$

Step 4: Arrange the fuzzy eigenvectors of the real symmetric matrix A that we derive, i.e,

$$eigenvalues : \lambda_1, \lambda_2, \cdots, \lambda_n,$$

$$eigenvectors : \widetilde{x}_1, \widetilde{x}_2, \cdots, \widetilde{x}_n,$$

where $\widetilde{x}_i = (x_i, x_i^l, x_i^r), i = 1, \cdots, n.$

Step 5: Judge the strong LR fuzzy feature vector and take it as

$$\widetilde{x} = (x_i, x_i^l, x_i^r)$$

or a weak LR fuzzy eigenvector

$$\widetilde{x}_i = \begin{cases} (x_i, x_i^l, x_i^r), & x_i^l > 0, \quad x_i^r > 0, \\ (x_i, 0, \max\{-x_i^l, x_i^r\}), & x_i^l < 0, \quad x_i^r > 0, \\ (x_i, \max\{x_i^l, -x_i^r\}, 0,), & x_i^l > 0, \quad x_i^r < 0, \\ (x_i, -x_i^l, -x_i^r), & x_i^l < 0, \quad x_i^r < 0. \end{cases} \quad i = 1, \dots, n.$$

by the Definition 3.2.

Based on the above, we also need to discuss the existence conditions of strong fuzzy eigenvectors, so we re-analyze the equation (3.1) and (3.9).

Firstly, we can rewrite the equation (3.1) as

$$Sy = \lambda y, \tag{3.19}$$

where

$$S = \begin{pmatrix} A & O & O \\ O & A^+ & -A^- \\ O & -A^- & A^+ \end{pmatrix}, y = \begin{pmatrix} x \\ x^l \\ x^r \end{pmatrix}, \tag{3.20}$$

when $\lambda \geq 0$.

Also, we can rewrite the equation (3.9) as

$$Ty = (-\lambda)y, \tag{3.21}$$

where

$$T = \begin{pmatrix} -A & O & O \\ O & -A^- & A^+ \\ O & A^+ & -A^- \end{pmatrix}, y = \begin{pmatrix} x \\ x^r \\ x^l \end{pmatrix}, \tag{3.22}$$

when $\lambda < 0$.

Therefore, the matrix S is a higher order non-negative symmetric matrix if and only if matrix A is a non-negative symmetric matrix in equation (3.19). Similarly, matrix T is a higher order non-negative symmetric matrix if and only if A is an non-positive symmetric matrix in equation (3.21).

Theorem 3.5. [27] *We assume that $G \in R^{m\times m}$ is a non-negative matrix with the eigenvalue λ, and that there exists a non-negative vector $z \in R^m, z \geq 0$, which is subject to*

$$Gz = \lambda z.$$

Now, by using the generalized Perron theorem for non-negative matrices, we give a sufficient condition to prove that strong fuzzy eigenvectors of real symmetric matrices exist.

Theorem 3.6. *Suppose the crisp matrix S and matrix T, when matrix A is a non negative one in the Eqs.*(3.19) *or matrix A is a non positive one in the Eqs.(*3.21*), the strong LR fuzzy eigenvector must exist in the real symmetric matrix A.*

Proof. According to the structure of matrix S or T and the Theorem 3.3, the proof of Theorem 3.5 is straightforward. The proof is completed. □

4 Numerical Examples

Example 4.1. Consider the fuzzy eigenvector of the following real symmetric matrix

$$A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix}.$$

Let

$$A = A^+ + A^- = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & -4 \\ -2 & -4 & 0 \end{pmatrix}.$$

From $Ax = \lambda x$, i.e., for λ, we solve the root of the equation respect to λ

$$f(\lambda) = det(\lambda I - A) = 0,$$

and the nonzero solution of homogeneous group of linear equations

$$(\lambda I - A)x = O,$$

we can get

$$\lambda_1 = \lambda_2 = 1, \lambda_3 = 10,$$

$$x_1 = \begin{pmatrix} -0.2981 \\ -0.5963 \\ -0.7455 \end{pmatrix}, x_2 = \begin{pmatrix} 0.8944 \\ -0.4472 \\ 0.0000 \end{pmatrix}, x_3 = \begin{pmatrix} 0.3333 \\ 0.6667 \\ 0.0000 \end{pmatrix},$$

the above $x_i (i = 1, 2, 3)$ and $\lambda_i (i = 1, 2, 3)$ can be used as all the eigenvalues and eigenvectors of the real symmetric positive definite matrix A.

For $\lambda_i = 1 > 0, i = 1, 2$, we solve non zero solutions to the homogeneous systems of linear equation

$$(2I - S)\begin{pmatrix} x^l \\ x^r \end{pmatrix} = O, S = \begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix},$$

and obtain

$$\begin{pmatrix} x^l \\ x^r \end{pmatrix} = \begin{pmatrix} -0.4149 & 0.2981 \\ 0.2075 & 0.5963 \\ -0.5491 & 0,0000 \\ 0.6598 & 0.0000 \\ 0.2192 & 0.0000 \\ 0.0000 & -0.7454 \end{pmatrix}.$$

For $\lambda_3 = 10 > 0$, we solve non zero solutions to the homogeneous systems of linear equation

$$(10I - S)\begin{pmatrix} x^r \\ x^l \end{pmatrix} = O, S = \begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix},$$

and get

$$\begin{pmatrix} x^r \\ x^l \end{pmatrix} = \begin{pmatrix} 0.3333 \\ 0.66674 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ 0.6667 \end{pmatrix}.$$

Based on the above, we can get all the real eigenvalues and fuzzy eigenvectors of the real symmetric matrix A, i.e

$$\lambda_1 = \lambda_2 = 1, \lambda_3 = 10,$$

$$\widetilde{x}_1 = \begin{pmatrix} (-0.2981, 0.0000, 0.6598) \\ (-0.5963, 0.2075, 0.2192) \\ (-0.7455, 0.0000, 0.0000) \end{pmatrix}, \widetilde{x}_2 = \begin{pmatrix} (0.8944, 0.2981, 0.0000) \\ (0.4472, 0.5963, 0.0000) \\ (0.0000, 0.0000, 0.0000) \end{pmatrix},$$

$$\widetilde{x}_3 = \begin{pmatrix} (0.3333, 0.3333, 0.0000) \\ (0.6667, 0.6667, 0.0000) \\ (-0.6667, 0.0000, 0.6667) \end{pmatrix},$$

among them, the eigenvectors $\widetilde{x}_1$ and $\widetilde{x}_2$ corresponding to the eigenvalues $\lambda_{1,2}$ of the original real matrix A are two weak LR fuzzy eigenvectors, and the eigenvectors $\widetilde{x}_3$ corresponding to the eigenvalues λ_3 of the original real matrix A is a strong LR fuzzy eigenvector.

Example 4.2. Consider the fuzzy eigenvector of the following real matrix

$$A = \begin{pmatrix} 1 & -2 & 2 \\ -2 & -2 & 4 \\ 2 & 4 & -2 \end{pmatrix},$$

Let

$$A = A^+ + A^- = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 4 \\ 2 & 4 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

From $Ax = \lambda x$, i.e., we solve the roots of the equation respect to λ

$$f(\lambda) = det(\lambda I - A) = 0,$$

and the nonzero solution of homogeneous group of linear equations

$$(\lambda I - A)x = O,$$

and get

$$\lambda_1 = \lambda_2 = 2, \lambda_3 = -7,$$

$$x_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, x_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, x_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

For $\lambda_{1,2} = 2 > 0$, we solve non zero solutions to the homogeneous systems of linear equation

$$(2I - S) \begin{pmatrix} x^l \\ x^r \end{pmatrix} = O, S = \begin{pmatrix} A^+ & -A^- \\ -A^- & A^+ \end{pmatrix},$$

and obtain

$$\begin{pmatrix} x^l \\ x^r \end{pmatrix} = \begin{pmatrix} 0.6667 & 0.0002 \\ -0.6668 & 0.5000 \\ 0.1665 & 0,5000 \\ -0.6667 & -0.0002 \\ 0.6668 & -0.5000 \\ -0.1665 & -0,5000 \end{pmatrix}.$$

For $\lambda_3 = -7 < 0$, we solve non zero solutions to the homogeneous systems of linear equation

$$(-7I - S)\begin{pmatrix} x^r \\ x^l \end{pmatrix} = O, S = \begin{pmatrix} -A^- & A^+ \\ A^+ & -A^- \end{pmatrix},$$

and obtain

$$\begin{pmatrix} x^r \\ x^l \end{pmatrix} = \begin{pmatrix} 0.2357 \\ 0.4714 \\ -0.4714 \\ -0.2357 \\ -0.4714 \\ 0.4714 \end{pmatrix}.$$

Through the above operation, we can obtain all the real eigenvalues and fuzzy eigenvectors of the real symmetric matrix A, which are respectively

$$\lambda_1 = \lambda_2 = 2, \lambda_3 = -7,$$

$$\widetilde{x}_1 = \begin{pmatrix} (-2, 0.6667, 0.0000) \\ (1, 0.0000, 0.6668) \\ (0, 0.1665, 0.0000) \end{pmatrix}, \widetilde{x}_2 = \begin{pmatrix} (2, 0.0002, 0.0000) \\ (0, 0.5000, 0.0000) \\ (1, 0.5000, 0.0000) \end{pmatrix},$$

$$\widetilde{x}_3 = \begin{pmatrix} (1, 0.0000, 0.2357) \\ (1, 0.0000, 0.4714) \\ (-2, 0.4714, 0.0000) \end{pmatrix}.$$

According to the Definition 3.2.,we can draw the conclusion that the eigenvectors $\widetilde{x}_1$ and $\widetilde{x}_2$ corresponding to the eigenvalues $\lambda_{1,2}$ of the original real matrix A are two weak LR fuzzy eigenvectors, and the eigenvector $\widetilde{x}_3$ corresponding to the eigenvalue λ_3 of the original real matrix A is also a weak LR fuzzy eigenvector.

5 Conclusion

In this paper, we study the LR fuzzy eigenvector problem of fuzzy matrices, and propose two computational models and algorithms for real symmetric matrices, which can solve the non-negative or negative LR fuzzy eigenvectors. Clear and straightforward mathematical derivations are used in the proof process, which is easy to understand. The practical application value of the algorithm is illustrated by example. We can consider extending the algorithm to the complex number field to solve more complex problems, and we can also try to further explore other properties of fuzzy linear systems, such as stability, so as to better enrich the theory of fuzzy linear systems.

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