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Logical Entropy of Partitions for Interval-Valued Intuitionistic Fuzzy Sets

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Abstract. In this study, we present the ideas of logical entropy and logical conditional entropy for partitions in interval-valued intuitionistic fuzzy sets, and we establish their fundamental properties. First, we establish the definitions of logical entropy and logical conditional entropy, demonstrating their key characteristics and relationships. We then define logical mutual information and explore its properties, providing a comprehensive understanding of its behavior within the context of interval-valued intuitionistic fuzzy sets. Additionally, we propose the concept of logical divergence of states defined on interval-valued intuitionistic fuzzy sets and examine its properties in detail, including its application and implications for understanding state transitions within these fuzzy sets. Finally, we extend our study to dynamical systems, introducing the logical entropy of such systems when modeled with interval-valued intuitionistic fuzzy sets. We present several results related to this extension, highlighting the applicability and relevance of logical entropy in analyzing and understanding the behavior of dynamical systems. Overall, this paper offers a thorough exploration of logical entropy, mutual information, and divergence within the framework of interval-valued intuitionistic fuzzy sets, providing new insights and potential applications in various fields.

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Keywords and Phrases: Interval-valued intuitionistic fuzzy sets, Logical entropy.

1 Introduction

Entropy is a crucial concept in numerous scientific disciplines, Including fields like physics, computer science, systems theory, information theory, statistics, sociology, and various others. It was initially introduced in the dynamical systems theory by Kolmogorov in 1958 [1]. Sinai later extended this concept by defining entropy for a dynamical system with a probability space as the state space [2]. Shannon conceptualized entropy in information theory [3], and more recently, Ellerman introduced logical entropy based on logical partitioning [4]. Several authors have recently defined entropy and logical entropy for dynamical systems with an algebraic structure [5, 6, 7, 8, 9, 10, 11].
Fuzzy generalizations of dynamical systems and their Shannon entropy have also been studied [6, 7, 8]. In 1975, Zadeh introduced interval-valued fuzzy sets ($IVFS$) as an extension of fuzzy sets [12]. Subsequently, in 1989, Atanassov and Gargov proposed the concept of interval-valued intuitionistic fuzzy sets ($IVIFS(X)$) as an extension of interval-valued fuzzy sets [13].
This paper aims to explore logical entropy for interval-valued intuitionistic fuzzy sets in a non-empty set X and to introduce the logical entropy of dynamical systems in $IVIFS(X)$. The structure of the paper is as follows:

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Section 2 presents the concepts of logical entropy for partitions in $IVIFS(X)$ and its logical conditional entropy, along with an investigation of their properties. In Section 3 delves into the concepts of logical mutual information and their properties. Then in Section 4 defines the logical divergence of states on $IVIFS(X)$ moreover examines its properties. Finally, Section 5 focuses on the study of logical entropy of dynamical systems in $IVIFS(X)$.

The subsequent sections offer essential information that will be used throughout the paper. The concepts mentioned below are all derived from references [13, 10]. For further details, please refer to [13, 10].

Consider $C[0,1]$ as the set comprising all closed subintervals of $[0,1]$. An interval-valued intuitionistic fuzzy set I within a universe X is described as $I = \{\langle x, \alpha_I(x), \beta_I(x)\rangle \mid x \in X\}$, where $\alpha_I : X \to C[0,1]$ and $\beta_I : X \to C[0,1]$, and these functions satisfy the condition $\alpha_{IU}(x) + \beta_{IU}(x) \leq 1$ for every $x \in X$. The intervals $\alpha_I(x)$ and $\beta_I(x)$indicate the degrees of membership and non-membership of an element x in I, respectively. Expressly, $\alpha_{IL}(x)$ and $\alpha_{IU}(x)$ represent the minimum and maximum levels of membership of x in I, respectively, and these values satisfy $0 \leq \alpha_{IL}(x) \leq \alpha_{IU}(x) \leq 1$.

For convenience, the collection of all interval-valued intuitionistic fuzzy sets over X is denoted as $IVIFS(X)$. In this discussion, we will write

$$I = \langle[\alpha_{IL}, \alpha_{IU}], [\beta_{IL}, \beta_{IU}]\rangle,$$

instead of

$$\{\langle x, [\alpha_{IL}(x), \alpha_{IU}(x)], [\beta_{IL}(x), \beta_{IU}(x)] \mid x \in X\rangle\}.$$

Two partial binary operations $\oplus$ and $\cdot$ on $IVIFS(X)$ are defined as follows: for any $I = \langle[\alpha_{IL}, \alpha_{IU}], [\beta_{IL}, \beta_{IU}]\rangle$ and $J = \langle[\alpha_{JL}, \alpha_{JU}], [\beta_{JL}, \beta_{JU}]\rangle \in IVIFS(X)$,

$$I \oplus J = \langle[\alpha_{IL} + \alpha_{JL}, \alpha_{IU} + \alpha_{JU}], [\beta_{IL} + \beta_{JL} - 1, \beta_{IU} + \beta_{JU} - 1]\rangle,$$

whenever $\alpha_{IL} + \alpha_{JL} \leq 1, \alpha_{IU} + \alpha_{JU} \leq 1, \beta_{IL} + \beta_{JL} \geq 1$, and $\beta_{IU} + \beta_{JU} \geq 1$, and

$$I \cdot J = \langle[\alpha_{IL} \cdot \alpha_{JL}, \alpha_{IU} \cdot \alpha_{JU}], [\beta_{IL} + \beta_{JL} - \beta_{IL} \cdot \beta_{JL}, \beta_{IU} + \beta_{JU} - \beta_{IU} \cdot \beta_{JU}]\rangle.$$

See the properties of these partial binary operations in [10].

A function $\mathbf{m} : IVIFS(X) \to [0,1]$ is termed a state on $IVIFS(X)$ if it satisfies certain conditions that allow it to measure or evaluate the interval-valued intuitionistic fuzzy sets over X within the range from 0 to 1.

To be specific, the function $\mathbf{m}$ must meet the following criteria:

1. Normalization: The state assigns the value 1 to the specific fuzzy set $\langle[1,1],[0,0]\rangle$. This particular fuzzy set represents an element that is entirely a member (with a membership degree interval of $[1,1]$) and not at all a non-member (with a non-membership degree interval of $[0,0]$). Mathematically, this is expressed as:

$$\mathbf{m}(\langle[1,1],[0,0]\rangle) = 1.$$

2. Additivity: The state is additive for the operation $\oplus$, which is a binary operation defined for combining two interval-valued intuitionistic fuzzy sets. For any $I, J \in IVIFS(X)$, if $I \oplus J$ is defined, the state of the combined set $I \oplus J$ is equal to the sum of the states of the individual sets I and J. Formally, this is written as:

$$\mathbf{m}(I \oplus J) = \mathbf{m}(I) + \mathbf{m}(J).$$

A finite collection $\mathcal{F} = \{I_1, ..., I_n\}$ of elements of $IVIFS(X)$ is said to be a partition if

$$\oplus_{i=1}^{n} I_i = \langle[1,1],[0,0]\rangle.$$

Thus, the relation between a state $\mathbf{m}$ and a partition $\mathcal{F} = \{I_1, ..., I_n\}$ is

$$\mathbf{m}(\oplus_{i=1}^{n} I_i) = \sum_{i=1}^{n} \mathbf{m}(I_i).$$

Let $\mathcal{F}_1 = \{I_1, ..., I_n\}$ and $\mathcal{F}_2 = \{J_1, ..., J_m\}$. The partition $\mathcal{F}_2 = \{J_1, ..., J_m\}$ is called a refinement of $\mathcal{F}_1 = \{I_1, ..., I_n\}$, written as $\mathcal{F}_1 \preceq \mathcal{F}_2$, if there exists a partition $k(1), ..., k(n)$ of the set $\{1, ..., m\}$ such that

$$\mathbf{m}(I_i) = \sum_{h \in k(i)} \mathbf{m}(J_h),$$

for every $i = 1, ..., n$. The collection

$$\mathcal{F}_1 \vee \mathcal{F}_2 = \{I_i \cdot J_j : i = 1, ..., n, j = 1, ..., m\},$$

which is a partition.

2 Logical Entropy of Partitions in Interval-Valued Intuitionistic Fuzzy Sets

Logical entropy and logical conditional entropy provide more refined tools for measuring and managing uncertainty in $IVIFS$, where both fuzziness and hesitation need to be considered. These concepts extend classical entropy ideas to work better within the richer structure of $IVIFS$. For this purpose, in this section, we introduce the concepts of logical entropy and logical conditional entropy for partitions within $IVIFS$ and explore their properties.

Definition 2.1. Let $\mathcal{F} = \{I_1, \cdots, I_n\}$ be a partition in $IVIFS(X)$,, and let $\mathbf{m} : IVIFS(X) \to [0, 1]$ be a state. The logical entropy of $\mathcal{F}$ for state $\mathbf{m}$ is defined as follows:

$$H^l_{\mathbf{m}}(\mathcal{F}) = \sum_{i=1}^{n} \mathbf{m}(I_i)(1 - \mathbf{m}(I_i)). \tag{1}$$

Remark 2.2. The logical entropy $H^l_{\mathbf{m}}(A)$ is always non-negative. Given that $\sum_{i=1}^{n} \mathbf{m}(I_i) = \mathbf{m}(\oplus_{i=1}^{n} I_i) = \mathbf{m}(\langle [1, 1], [0, 0] \rangle) = 1$, equation (1) can also be expressed in the form shown below:

$$H^l_{\mathbf{m}}(\mathcal{F}) = 1 - \sum_{i=1}^{n} (\mathbf{m}(I_i))^2. \tag{2}$$

Example 2.3. $M = \{\langle [1, 1], [0, 0] \rangle\}$ represents a partition of $IVIFS(X)$, and for every partition B of $IVIFS(X)$, it holds that $B \preceq M$. If we set $M = \{\langle [1, 1], [0, 0] \rangle\}$, then $H^l_{\mathbf{m}}(M) = 0$.

Example 2.4. Suppose that $\langle [\alpha_{IL}, \alpha_{IU}], [\beta_{IL}, \beta_{IU}] \rangle \in IVIFS(X)$. Then, $\mathcal{F} = \{I_1 = \langle [\alpha_{IL}, \alpha_{IU}], [\beta_{IL}, \beta_{IU}] \rangle, I_2 = \langle [1 - \alpha_{IL}, 1 - \alpha_{IU}], [1 - \beta_{IL}, 1 - \beta_{IU}] \rangle\}$ forms a partition of $IVIFS(X)$.

If $\mathbf{m}$ is a state such that $\mathbf{m}(\langle [\alpha_{IL}, \alpha_{IU}], [\beta_{IL}, \beta_{IU}] \rangle) = s < 1$ and $\mathbf{m}(\langle [1 - \alpha_{IL}, 1 - \alpha_{IU}], [1 - \beta_{IL}, 1 - \beta_{IU}] \rangle) = 1 - s$, then $H^l_{\mathbf{m}}(\mathcal{F}) = 2s(1 - s)$.

Definition 2.5. Consider $\mathcal{F}_1 = \{I_1, \ldots, I_n\}$ and $\mathcal{F}_2 = \{J_1, \ldots, J_m\}$ as two partitions of $IVIFS(X)$. The logical conditional entropy of $\mathcal{F}_1$ given $\mathcal{F}_2$ is defined as follows:

$$H^l_{\mathbf{m}}(\mathcal{F}_1|\mathcal{F}_2) = \sum_{i=1}^{n}\sum_{j=1}^{m} \mathbf{m}(I_i \cdot J_j)\left(\mathbf{m}(J_j) - \mathbf{m}(I_i \cdot J_j)\right). \tag{3}$$

Proposition 2.6. *([10]) Consider $\mathcal{F} = \{I_1, ..., I_n\}$ as a partition of $IVIFS(X)$, and let $K \in IVIFS(X)$. then*

$$\mathbf{m}(K) = \sum_{i=1}^{n} \mathbf{m}(I_i \cdot K).$$

Remark 2.7. According to Proposition 2.6, we have $\sum_{i=1}^{n} \mathbf{m}(I_i \cdot I_j) = \mathbf{m}(I_j)$. Therefore, equation (3) can be rewritten in the following form:

$$H^l_{\mathbf{m}}(\mathcal{F}_1|\mathcal{F}_2) = \sum_{j=1}^{m}(J_j)^2 - \sum_{i=1}^{n}\sum_{j=1}^{m}(\mathbf{m}(I_i \cdot J_j))^2. \tag{4}$$

Remark 2.8. Since $\mathbf{m}(I_i \cdot J_j) \leq \mathbf{m}(J_j)$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$, the logical conditional entropy $H^l_{\mathbf{m}}(\mathcal{F}_1 \mid \mathcal{F}_2)$ is always nonnegative. Suppose $M = \{\langle [1,1], [0,0] \rangle\}$. It is straightforward to verify that $H^l_{\mathbf{m}}(\mathcal{F} \mid M) = H^l_{\mathbf{m}}(\mathcal{F})$ for any partition $\mathcal{F}$ of $IVIFS(X)$.

Theorem 2.9. *For any arbitrary partitions $\mathcal{F}_1$ and $\mathcal{F}_2$ of $IVIFS(X)$, the following property hold.*

$$H^l_{\mathbf{m}}(\mathcal{F}_1 \vee \mathcal{F}_2) = H^l_{\mathbf{m}}(\mathcal{F}_1) + H^l_{\mathbf{m}}(\mathcal{F}_2 \mid \mathcal{F}_1). \tag{5}$$

Proof. Suppose that $\mathcal{F}_1 = \{I_1, \cdots, I_n\}$ and $\mathcal{F}_2 = \{J_1, \cdots, J_m\}$. Then By equations (2) and (3) we derive:

$$\begin{aligned}
H^l_{\mathbf{m}}(\mathcal{F}_1) + H^l_{\mathbf{m}}(\mathcal{F}_2 \mid \mathcal{F}_1) &= 1 - \sum_{i=1}^{n}(\mathbf{m}(I_i)^2 + \sum_{i=1}^{n}(\mathbf{m}(I_i)^2 \\
&- \sum_{i=1}^{n}\sum_{j=1}^{m}(\mathbf{m}(I_i \cdot J_j))^2 \\
&= 1 - \sum_{i=1}^{n}\sum_{j=1}^{m}(\mathbf{m}(I_i \cdot J_j)) \\
&= H^l_{\mathbf{m}}(\mathcal{F}_1 \vee \mathcal{F}_2).
\end{aligned}$$

□

Remark 2.10. Let $\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_n$ be partitions of $IVIFS(X)$. Using induction, we obtain the following generalization of equation (5):

$$\begin{aligned}
H^l_{\mathbf{m}}(\mathcal{F}_1 \vee \mathcal{F}_2 \vee \cdots \vee \mathcal{F}_n) &= H^l_{\mathbf{m}}(P_1) \\
&+ \sum_{i=2}^{n} H^l_{\mathbf{m}}(\mathcal{F}_i \mid \mathcal{F}_1 \vee \cdots \vee \mathcal{F}_{i-1} \vee \mathcal{F}_{i+1} \cdots \vee \mathcal{F}_n).
\end{aligned} \tag{6}$$

Remark 2.11. For any arbitrary partitions $\mathcal{F}_1$ and $\mathcal{F}_2$ of $IVIFS(X)$, the following relationship is easily obtained:

$$H^l_{\mathbf{m}}(\mathcal{F}_1 \vee \mathcal{F}_2) = H^l_{\mathbf{m}}(\mathcal{F}_1) + H^l_{\mathbf{m}}(\mathcal{F}_2 \mid \mathcal{F}_1) = H^l_{\mathbf{m}}(\mathcal{F}_2) + H^l_{\mathbf{m}}(\mathcal{F}_1 \mid \mathcal{F}_2). \tag{7}$$

Theorem 2.12. *For any partitions $\mathcal{F}_1$ and $\mathcal{F}_2$ of $IVIFS(X)$, the following assertions hold:*

(i) $H^l_{\mathbf{m}}(\mathcal{F}_1 \mid \mathcal{F}_2) \leq H^l_{\mathbf{m}}(\mathcal{F}_1)$.

(ii) $H^l_{\mathbf{m}}(\mathcal{F}_1 \vee \mathcal{F}_2) \leq H^l_{\mathbf{m}}(\mathcal{F}_1) + H^l_{\mathbf{m}}(\mathcal{F}_2)$.

Proof. Assume that $\mathcal{F}_1 = \{I_1, \ldots, I_n\}$ and $\mathcal{F}_2 = \{J_1, \ldots, J_m\}$. (i) Given that proposition 2.6 establishes that $\sum_{j=1}^{m} \mathbf{m}(I_i \cdot J_j) = \mathbf{m}(I_i)$, it follows that:

$$\begin{aligned}
\sum_{j=1}^{m} \mathbf{m}(I_i \cdot J_j)(\mathbf{m}(J_j) - \mathbf{m}(I_i \cdot J_j)) &\leq (\sum_{j=1}^{m} \mathbf{m}(I_i \cdot J_j))(\sum_{j=1}^{m}(\mathbf{m}(J_j) - \mathbf{m}(I_i \cdot J_j)) \\
&= \mathbf{m}(I_i)(\sum_{j=1}^{m} \mathbf{m}(J_j) - \sum_{j=1}^{m} \mathbf{m}(I_i \cdot J_j) \\
&= \mathbf{m}(I_i)(1 - \mathbf{m}(I_i)).
\end{aligned}$$

So

$$\begin{aligned}
H^l_{\mathbf{m}}(\mathcal{F}_1 \mid \mathcal{F}_2) &= \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbf{m}(I_i \cdot J_j)(\mathbf{m}(J_j) - \mathbf{m}(I_i \cdot J_j)) \\
&\leq \sum_{i=1}^{n} \mathbf{m}(I_i)(1 - \mathbf{m}(I_i)) \\
&= H^l_{\mathbf{m}}(\mathcal{F}_1).
\end{aligned}$$

(ii) Based on equation (5) and property (i), property (ii) is derived. □

Proposition 2.13. *([10]) If $\mathcal{F}_1 = \{I_1, \ldots, I_n\}$ and $\mathcal{F}_2 = \{J_1, \ldots, J_m\}$ are both partitions of $IVIFS(X)$, then $\mathcal{F}_1 \vee \mathcal{F}_2$ also constitutes a partition. Furthermore $\mathcal{F}_1 \preceq \mathcal{F}_1 \vee \mathcal{F}_2$.*

Theorem 2.14. *For any partitions $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}_3$ of $IVIFS(X)$, the following properties apply:*

(i) $\mathcal{F}_1 \preceq \mathcal{F}_2$ *implies* $H^l_{\mathbf{m}}(\mathcal{F}_1) \leq H^l_{\mathbf{m}}(\mathcal{F}_2)$;

(ii) $H^l_{\mathbf{m}}(\mathcal{F}_1 \vee \mathcal{F}_2) \geq max[H^l_{\mathbf{m}}(\mathcal{F}_1), H^l_{\mathbf{m}}(\mathcal{F}_2)]$.

(iii) $\mathcal{F}_1 \preceq \mathcal{F}_2$ *implies* $H^l_{\mathbf{m}}(\mathcal{F}_1 \mid \mathcal{F}_3) \leq H^l_{\mathbf{m}}(\mathcal{F}_2 \mid \mathcal{F}_3)$.

Proof. (i) Assume $\mathcal{F}_1 = \{I_1, \ldots, I_n\}$ and $\mathcal{F}_2 = \{J_1, \ldots, J_m\}$. Under the premise $\mathcal{F}_1 \preceq \mathcal{F}_2$, a partition $\{k(1), \ldots, k(n)\}$ of the set $\{1, 2, \ldots, m\}$ exists such that $I_i = \sum_{j \in k(i)} J_j$ for all $i = 1, \ldots, n$. Therefore,

$$\begin{aligned}
H^l_{\mathbf{m}}(\mathcal{F}_1) &= 1 - \sum_{i=1}^{n} (\mathbf{m}(I_i))^2 \\
&= 1 - \sum_{i=1}^{n} (\mathbf{m}(\sum_{j \in k(i)} J_j))^2 \\
&= 1 - \sum_{i=1}^{n} (\sum_{j \in k(i)} \mathbf{m}(J_j))^2 \\
&\leq 1 - \sum_{i=1}^{n} \sum_{j \in k(i)} (\mathbf{m}(J_j))^2 \\
&= 1 - \sum_{j=1}^{m} (\mathbf{m}(J_j))^2 \\
&= H^l_{\mathbf{m}}(\mathcal{F}_2).
\end{aligned}$$

The inequality mentioned previously arises from the inequality $(a_1 + a_2 + \cdots + a_n)^2 \geq a_1^2 + a_2^2 + \cdots + a_n^2$ which holds for all nonnegative real numbers $a_1, \ldots, a_n$.
(ii) Since $\mathcal{F}_1 \preceq \mathcal{F}_1 \vee \mathcal{F}_2$ and $\mathcal{F}_2 \preceq \mathcal{F}_1 \vee \mathcal{F}_2$, property (ii) follows as a result of property (i).
(iii) Assuming $\mathcal{F}_1 \preceq \mathcal{F}_2$, Proposition 2.13 indicates that $\mathcal{F}_1 \vee \mathcal{F}_3 \preceq \mathcal{F}_2 \vee \mathcal{F}_3$. Consequently, using equation (5) and property (i), we can deduce that:

$$H^l_{\mathbf{m}}(\mathcal{F}_1 \mid \mathcal{F}_3) = H^l_{\mathbf{m}}(\mathcal{F}_1 \vee \mathcal{F}_3) - H^l_{\mathbf{m}}(\mathcal{F}_3) \leq H^l_{\mathbf{m}}(\mathcal{F}_2 \vee \mathcal{F}_3) - H^l_{\mathbf{m}}(\mathcal{F}_3) = H^l_{\mathbf{m}}(\mathcal{F}_2 \mid \mathcal{F}_3).$$

□

The set of all states defined on $IVIFS(X)$ is represented by $\mathbf{M}(IVIFS(X))$. In the subsequent theorem, we demonstrate that $\mathbf{M}(IVIFS(X))$ forms a convex set.

Theorem 2.15. *If $\mathbf{m}_1, \mathbf{m}_2 \in \mathbf{M}(IVIFS(X))$, then for any t within the interval $[0, 1]$, the combination $t\mathbf{m}_1 + (1-t)\mathbf{m}_2$ belongs to $\mathbf{M}(IVIFS(X))$.*

Proof. This proof is straightforward. □

The theorem below establishes that logical entropy is a convex function on $\mathbf{M}(IVIFS(X))$.

Theorem 2.16. *Given a partition $\mathcal{F}$ of $IVIFS(X)$, it is true that for any $\mathbf{m}_1, \mathbf{m}_2 \in \mathbf{M}(IVIFS(X))$ and for any t within the interval $[0, 1]$, the following holds:*

$$tH^l_{\mathbf{m}_1}(\mathcal{F}) + (1-t)H^l_{\mathbf{m}_2}(\mathcal{F}) \leq H^l_{t\mathbf{m}_1 + (1-t)\mathbf{m}_2}(\mathcal{F}).$$

Proof. Assume $\mathcal{F} = \{I_1, \ldots, I_n\}$. Given that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is convex for all

$x \in \mathbb{R}$, it follows that for any t in the interval $[0,1]$, we derive:

$$\begin{aligned} tH^l_{\mathbf{m}1}(\mathcal{F}) + (1-t)H^l_{\mathbf{m}_2}(\mathcal{F}) &= t\left[1 - \sum_{i=1}^{n}(\mathbf{m}_1(I_i))^2\right] \\ &+ (1-t)\left[1 - \sum_{i=1}^{n}(\mathbf{m}_2(I_i))^2\right] \\ &= 1 - t\sum_{i=1}^{n}(\mathbf{m}_1(I_i))^2 - (1-t)\sum_{i=1}^{n}(\mathbf{m}_2(I_i))^2 \\ &\leq 1 - \sum_{i=1}^{n}((t\mathbf{m}_1(I_i) + (1-t)\mathbf{m}_2(I_i))^2 \\ &= 1 - \sum_{i=1}^{n}(t\mathbf{m}_1 + (1-t)\mathbf{m}_2)(I_i))^2 \\ &= H^l_{t\mathbf{m}_1+(1-t)\mathbf{m}_2}(\mathcal{F}). \end{aligned}$$

□

3 Logical Mutual Information in Interval-Valued Intuitionistic Fuzzy Sets

In this section, the concept of logical mutual information for partitions in $IVIFS$ is introduced. The introduction of logical mutual information for partitions in $IVIFS$ is aimed at quantifying the interdependence between different fuzzy partitions in a way that takes into account both fuzziness and hesitation due to intuitionistic uncertainty. This measure provides a nuanced way to understand how one fuzzy concept can reduce uncertainty about another in complex, real-world decision-making and data analysis tasks where uncertainty is an inherent challenge.

Definition 3.1. The logical mutual information of partitions $\mathcal{F}_1$ and $\mathcal{F}_2$ in $IVIFS(X)$ is defined as follows:

$$\mathcal{I}^l_{\mathbf{m}}(\mathcal{F}_1, \mathcal{F}_2) = H^l_{\mathbf{m}}(\mathcal{F}_1) - H^l_{\mathbf{m}}(\mathcal{F}_1 \mid \mathcal{F}_2). \tag{8}$$

Remark 3.2. Since $H^l_{\mathbf{m}}(\mathcal{F}_1 \mid \mathcal{F}_2) \leq H^l_{\mathbf{m}}(\mathcal{F}_1)$, this implies that the logical mutual information $I^l_{\mathbf{m}}(\mathcal{F}_1, \mathcal{F}_2)$ is consistently nonnegative.

Theorem 3.3. *The logical mutual information of partitions $\mathcal{F}_1$ and $\mathcal{F}_2$ in $IVIFS(X)$ exhibits the following properties:*

(i) $\mathcal{I}^l_{\mathbf{m}}(\mathcal{F}_1, \mathcal{F}_2) = H^l_{\mathbf{m}}(\mathcal{F}_1) + H^l_{\mathbf{m}}(\mathcal{F}_2) - H^l_{\mathbf{m}}(\mathcal{F}_1 \vee \mathcal{F}_2)$;

(ii) $\mathcal{I}^l_{\mathbf{m}}(\mathcal{F}_1, \mathcal{F}_2) = \mathcal{I}^l_{\mathbf{m}}(\mathcal{F}_2, \mathcal{F}_1)$;

(iii) $\mathcal{I}^l_{\mathbf{m}}(\mathcal{F}_1, \mathcal{F}_2) \leq min\left[H^l_{\mathbf{m}}(\mathcal{F}_1), H^l_{\mathbf{m}}(\mathcal{F}_2)\right]$.

Proof. (i) Based on equation 5, we have $H^l_{\mathbf{m}}(\mathcal{F}_1 \mid \mathcal{F}_2) = H^l_{\mathbf{m}}(\mathcal{F}_1) - H^l_{\mathbf{m}}(\mathcal{F}_1 \vee \mathcal{F}_2)$. Consequently, utilizing equation (8), the following identities are established:

$$\mathcal{I}^l_{\mathbf{m}}(\mathcal{F}_1, \mathcal{F}_2) = H^l_{\mathbf{m}}(\mathcal{F}_1) + H^l_{\mathbf{m}}(\mathcal{F}_2) - H^l_{\mathbf{m}}(\mathcal{F}_1 \vee \mathcal{F}_2). \tag{9}$$

(ii) This property is derived from equation (9).
(iii) According to part (iii) of Theorem 2.14, $H^l_{\mathbf{m}}(\mathcal{F}_1) \leq H^l_{\mathbf{m}}(\mathcal{F}_1 \vee \mathcal{F}_2)$, which implies that $\mathcal{I}^l_{\mathbf{m}}(\mathcal{F}_1, \mathcal{F}_2) \leq min\left[H^l_{\mathbf{m}}(\mathcal{F}_1), H^l_{\mathbf{m}}(\mathcal{F}_2)\right]$. □

Theorem 3.4. *If partitions $\mathcal{F}_1$ and $\mathcal{F}_2$ are statistically independent, then:*

(i) $\mathcal{I}^l_{\mathbf{m}}(\mathcal{F}_1, \mathcal{F}_2) = H^l_{\mathbf{m}}(\mathcal{F}_1).H^l_{\mathbf{m}}(\mathcal{F}_2)$*;*

(ii) $1 - H^l_{\mathbf{m}}(\mathcal{F}_1 \vee \mathcal{F}_2) = \left(1 - H^l_{\mathbf{m}}(\mathcal{F}_1)\right) . \left(1 - H^l_{\mathbf{m}}(\mathcal{F}_2)\right)$.

Proof. (i) Assume $\mathcal{F}_1 = \{I_1, \dots, I_n\}$ and $\mathcal{F}_2 = \{J_1, \dots, J_m\}$. Based on equations (2) and (9), we derive:

$$\begin{aligned}
\mathcal{I}^l_{\mathbf{m}}(\mathcal{F}_1, \mathcal{F}_2) &= 1 - \sum_{i=1}^{n} (\mathbf{m}(I_i))^2 + 1 - \sum_{j=1}^{m} (\mathbf{m}(J_j)^2 - 1 \\
&+ \sum_{i=1}^{n} \sum_{j=1}^{m} (\mathbf{m}(I_i \cdot J_j))^2 \\
&= \left(1 - \sum_{i=1}^{n} (\mathbf{m}(I_i))^2\right) . \left(1 - \sum_{j=1}^{m} (\mathbf{m}(J_j))^2\right) \\
&= H^l_{\mathbf{m}}(\mathcal{F}_1).H^l_{\mathbf{m}}(\mathcal{F}_2).
\end{aligned}$$

(ii) Utilizing item (i) and equation (9), we arrive at:

$$\begin{aligned}
(1 - H^l_{\mathbf{m}}(\mathcal{F}_1)).(1 - H^l_{\mathbf{m}}(\mathcal{F}_2)) &= 1 - H^l_{\mathbf{m}}(\mathcal{F}_1) - H^l_{\mathbf{m}}(\mathcal{F}_2) + H^l_{\mathbf{m}}(\mathcal{F}_1).H^l_{\mathbf{m}}(\mathcal{F}_2) \\
&= 1 - H^l_{\mathbf{m}}(\mathcal{F}_1) - H^l_{\mathbf{m}}(\mathcal{F}_2) + I^l_{\mathbf{m}}(\mathcal{F}_1, \mathcal{F}_2) \\
&= 1 - H^l_{\mathbf{m}}(\mathcal{F}_1) - H^l_{\mathbf{m}}(\mathcal{F}_2) + H^l_{\mathbf{m}}(\mathcal{F}_1) + H^l_{\mathbf{m}}(\mathcal{F}_2) \\
&- H^l_{\mathbf{m}}(\mathcal{F}_1 \vee \mathcal{F}_2) \\
&= 1 - H^l_{\mathbf{m}}(\mathcal{F}_1 \vee \mathcal{F}_2).
\end{aligned}$$

□

4 Logical Divergence in Interval-Valued Intuitionistic Fuzzy Sets

In this section, we introduce the concept of logical divergence entropy within $IVIFS$. The introduction of this concept is motivated by the need to measure the divergence or difference between fuzzy partitions that include both fuzziness and intuitionistic hesitation (due to interval-valued uncertainty). This measure captures not only the imprecision in membership but also the hesitation in decision-making processes. It is useful for various applications, such as decision-making, pattern recognition, and data analysis, providing a more nuanced way to compare fuzzy sets in uncertain environments.

Definition 4.1. Assume $\mathcal{F} = \{I_1, \dots, I_n\}$ is a partition of $IVIFS(X)$ and $\mathbf{m}_1, \mathbf{m}_2 \in \mathbf{M}(IVIFS(X))$. The logical divergence of states $\mathbf{m}_1$ and $\mathbf{m}_2$ for $\mathcal{F}$ is defined as follows:

$$\mathcal{D}^l_{\mathcal{F}}(\mathbf{m}_1 \| \mathbf{m}_2) = \frac{1}{2} \sum_{i=1}^{n} (\mathbf{m}_1(I_i) - \mathbf{m}_2(I_i))^2.$$

Theorem 4.2. *Assume $\mathcal{F} = \{I_1, \dots, I_n\}$ is a partition of $IVIFS(X)$ and $\mathbf{m}_1, \mathbf{m}_2 \in \mathbf{M}(IVIFS(X))$. The logical divergence of states $\mathbf{m}_1$ and $\mathbf{m}_2$ for $\mathcal{F}$ fulfills the following conditions:*

(i) $\mathcal{D}^l_{\mathcal{F}}(\mathbf{m}_1 \| \mathbf{m}_2) = \mathcal{D}^l_{\mathcal{F}}(\mathbf{m}_2 \| \mathbf{m}_1)$.

(ii) $\mathcal{D}^l_{\mathcal{F}}(\mathbf{m}_1 \| \mathbf{m}_2) \geq 0$*, where equality holds if and only if the states $\mathbf{m}_1$ and $\mathbf{m}_2$ are identical over $\mathcal{F}$.*

Proof. Based on the definition provided above, this proof is straightforward. □ In the example below, it is demonstrated that logical divergence does not qualify as a distance metric because it does not fulfill the triangle inequality.

Example 4.3. In Example 2.4, assume $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$ are three states on $IVIFS(X)$ where $\mathbf{m}_1(I_1) = s_1$, $\mathbf{m}_2(I_1) = s_2$, and $\mathbf{m}_3(I_1) = s_3$, with s_1, s_2, s_3 each in the interval (0,1). Consequently, $\mathbf{m}_1(I_2) = 1 - s_1$, $\mathbf{m}_2(I_2) = 1 - s_2$, and $\mathbf{m}_3(I_2) = 1 - s_3$. Thus, we derive:

$$\mathcal{D}^l_{\mathcal{F}}(\mathbf{m}_1 \| \mathbf{m}_2) = \frac{1}{2}(\mathbf{m}_1(I_1) - \mathbf{m}_2(I_1))^2 + \frac{1}{2}(\mathbf{m}_1(I_2) - \mathbf{m}_2(I_2))^2 = (s_1 - s_2)^2.$$

Similarly, we have:

$$\mathcal{D}^l_{\mathcal{F}}(\mathbf{m}_1 \| \mathbf{m}_3) = (s_1 - s_3)^2, \text{ and } \mathcal{D}^l_{\mathcal{F}}(\mathbf{m}_2 \| \mathbf{m}_3) = (s_2 - s_3)^2.$$

Set $s_1 = \frac{1}{3}$, $s_2 = \frac{1}{4}$, $s_3 = \frac{1}{5}$. Clearly,

$$\mathcal{D}^l_{\mathcal{F}}(\mathbf{m}_1 \| \mathbf{m}_3) \geq \mathcal{D}^l_{\mathcal{F}}(\mathbf{m}_1 \| \mathbf{m}_2) + \mathcal{D}^l_{\mathcal{F}}(\mathbf{m}_2 \| \mathbf{m}_3).$$

This outcome indicates that the triangle inequality does not generally hold for logical divergence in $IVIFS(X)$.

Theorem 4.4. *Assume $\mathcal{F} = \{I_1, \ldots, I_n\}$ is a partition of $IVIFS(X)$. Then, for every pair of states $\mathbf{m}_1$ and $\mathbf{m}_2$ defined on $IVIFS(X)$, the following is true:*

$$\mathcal{D}^l_{\mathcal{F}}(\mathbf{m}_1 \| \mathbf{m}_2) = \left(\sum_{i=1}^{n} \mathbf{m}_1(I_i)(1 - \mathbf{m}_2(I_i)) \right) - \left[\frac{1}{2}(H^l_{\mathbf{m}_1}(\mathcal{F}) + H^l_{\mathbf{m}_2}(\mathcal{F})) \right].$$

Proof. Assume $\mathcal{F} = \{I_1, \ldots, I_n\}$. Let's proceed with the calculation:

$$\left(\sum_{i=1}^{n} \mathbf{m}_1(I_i)(1 - \mathbf{m}_2(I_i)) \right) - \left[\frac{1}{2}(H^l_{\mathbf{m}_1}(\mathcal{F}) + H^l_{\mathbf{m}_2}(\mathcal{F})) \right]$$

$$= 1 - \sum_{i=1}^{n} \mathbf{m}_1(I_i)\mathbf{m}_2(I_i) - \frac{1}{2}(1 - \sum_{i=1}^{n}(\mathbf{m}_1(I_i))^2) - \frac{1}{2}(1 - \sum_{i=1}^{n}(\mathbf{m}_2(I_i))^2)$$

$$= \frac{1}{2}(1 - \sum_{i=1}^{n}(\mathbf{m}_1(I_i) - \mathbf{m}_2(I_i))^2) = \mathcal{D}^l_{\mathcal{F}}(\mathbf{m}_1 \| \mathbf{m}_2).$$

□

5 The Logical Entropy of Dynamical System in $IVIFS(X)$

The concept of logical entropy for a dynamical system within the framework of IVIFS(X) is introduced to measure and track the evolving uncertainty and distinctions in systems characterized by both fuzziness and intuitionistic uncertainty. Logical entropy allows for a more nuanced analysis of dynamical systems where both imprecision and hesitation are present, providing deeper insights into the complexity and unpredictability of the systems behavior over time.

Definition 5.1. [10] A dynamical system in $IVIFS(X)$ consists of the triple $(IVIFS(X), \mathbf{m}, \psi)$, where $\mathbf{m} : IVIFS(X) \to [0, 1]$ is a state function on $IVIFS(X)$ and $\psi : IVIFS(X) \to IVIFS(X)$ is a mapping that meets the following criteria:

1. If $I \cdot J = \langle [0,0],[1,1] \rangle$, then $\psi(I) \cdot \psi(J) = \langle [1,1],[0,0] \rangle$ and $\psi(I \oplus J) = \psi(I) \oplus \psi(J)$, for any $I, J \in IVIFS(X)$.
2. $\psi(\langle [1,1],[0,0] \rangle) = \langle [1,1],[0,0] \rangle$;
3. $s(\psi(I)) = \mathbf{m}(I)$ for any $I \in IVIFS(X)$.

Theorem 5.2. *Consider $(IVIFS(X), \mathbf{m}, \psi)$ as a dynamical system in $IVIFS(X)$, with $\mathcal{F}_1$ and $\mathcal{F}_2$ as partitions within $IVIFS(X)$. The following assertions hold:*

(i) $\psi(\mathcal{F}_1 \vee \mathcal{F}_2) = \psi(\mathcal{F}_1) \vee \psi(\mathcal{F}_2)$.

(ii) $\mathcal{F}_1 \preceq \mathcal{F}_2$ *implies* $\psi(\mathcal{F}_1) \preceq \psi(\mathcal{F}_2)$.

Proof. The proof of (i) is derived from condition (ii) of Definition 5.1.
Consider $\mathcal{F}_1 = \{I_1, \ldots, I_n\}$ and $\mathcal{F}_2 = \{J_1, \ldots, J_m\}$, with $\mathcal{F}_1 \preceq \mathcal{F}_2$. Consequently, there is a partition $\{k(1), \ldots, k(n)\}$ of the set $\{1, 2, \ldots, m\}$ such that $I_i = \sum_{j \in k(i)} J_j$ for each $i = 1, 2, \ldots, n$. Therefore, according to condition (i) of Definition 5.1, it follows:

$$\psi(I_i) = \psi(\sum_{j \in k(i)} J_j) = \sum_{j \in k(i)} \psi(J_j),$$

for $i = 1, 2, \ldots, n$. This implies that $\psi(\mathcal{F}_1) \preceq \psi(\mathcal{F}_2)$. □

Theorem 5.3. *Consider $(IVIFS(X), \mathbf{m}, \psi)$ as a dynamical system within $IVIFS(X)$, with $\mathcal{F}_1$ and $\mathcal{F}_2$ as partitions of $IVIFS(X)$. Then, for any non-integer n, the following assertions are valid:*

(i) $H^l_{\mathbf{m}}(\psi^n(\mathcal{F}_1)) = H^l_{\mathbf{m}}(\mathcal{F}_1)$;

(ii) $H^l_{\mathbf{m}}(\psi^n(\mathcal{F}_1) | \psi^n(\mathcal{F}_2)) = H^l_{\mathbf{m}}(\mathcal{F}_1 | \mathcal{F}_2)$.

Proof. Suppose that $\mathcal{F}_1 = \{I_1, \ldots, I_n\}$ and $\mathcal{F}_2 = \{J_1, \ldots, J_m\}$.

(i) Since for any non-negative integer n and for each $i = 1, \ldots, k$, it is true that $\mathbf{m}(\psi^n(I_i)) = \mathbf{m}(I_i)$, we conclude:

$$H^l_{\mathbf{m}}(\psi^n(\mathcal{F}_1)) = \sum_{i=1}^{n} \mathbf{m}(\psi^n(I_i) - \mathbf{m}(\psi^n(I_i))^2 = \sum_{i=1}^{n} \mathbf{m}(I_i) - \mathbf{m}(I_i)^2 = H^l_{\mathbf{m}}(\mathcal{F}_1).$$

(ii) Based on the same argument, we have:

$$H^l_{\mathbf{m}}(\psi^n(\mathcal{F}_1) | \psi^n(\mathcal{F}_2)) = \sum_{j=1}^{m} \mathbf{m}(\psi^n(J_i)^2 - \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbf{m}(\psi^n(I_i \cdot J_j))^2$$

$$= \sum_{j=1}^{m} \mathbf{m}(J_i)^2 - \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbf{m}(I_i \cdot J_j)^2 = H^l_{\mathbf{m}}(\mathcal{F}_1 | \mathcal{F}_2).$$

□

Theorem 5.4. *Take $(IVIFS(X), \mathbf{m}, \psi)$ as a dynamical system, where $\mathcal{F}$ is a partition of $IVIFS(X)$. Then,*

$$\lim_{n \longrightarrow \infty} \frac{1}{n} H^l_s(\bigvee_{i=0}^{n-1} \psi^i(\mathcal{F})).$$

Proof. Suppose $a_n = H^l_{\mathbf{m}}\left(\bigvee_{i=0}^{n-1} \psi^i(\mathcal{F})\right)$ for $n = 1, 2, \ldots$. Then the sequence $\{a_n\}_{n=1}^{\infty}$ consists of non-negative real numbers and satisfies the property $a_{s+r} \leq a_s + a_r$ for any natural numbers s and r. According to property (i) of Theorem 5.3 and using the sub-additivity of logical entropy, we have:

$$\begin{aligned}
a_{s+r} &= H^l_{\mathbf{m}}(\bigvee_{i=0}^{s+r-1} \psi^i(\mathcal{F})) \\
&\leq H^l_{\mathbf{m}}(\bigvee_{i=0}^{s-1} \psi^i(\mathcal{F}) + H^l_{\mathbf{m}}(\bigvee_{i=s}^{s+r-1} \psi^i(\mathcal{F})) \\
&= a_s + H^l_{\mathbf{m}}(\psi^s(\bigvee_{i=0}^{r-1} \psi^i(\mathcal{F})) \\
&= a_s + H^l_{\mathbf{m}}(\bigvee_{i=0}^{r-1} \psi^i(\mathcal{F}) = a_s + a_r.
\end{aligned}$$

Therefore, by Theorem 4.9 from [14], $\lim_{n\to\infty} \frac{1}{n} a_n$ exists. □

Definition 5.5. Consider $(IVIFS(X), \mathbf{m}, \psi)$ as a dynamical system, with $\mathcal{F}$ being a partition of $IVIFS(X)$. We then define the logical entropy of ψ relative to $\mathcal{F}$ as follows:

$$H^l_{\mathbf{m}}(\psi, \mathcal{F}) = \lim_{n\longrightarrow\infty} \frac{1}{n} H^l_{\mathbf{m}}(\bigvee_{i=0}^{n-1} \psi^i(\mathcal{F})).$$

Remark 5.6. Let $(IVIFS(X), \mathbf{m}, \psi)$ be a dynamical system in $IVIFS(X)$ and let $\mathcal{F} = \{\langle[1, 1], [0, 0]\rangle\}$. Then $\bigvee_{i=0}^{n-1} \psi^i(\mathcal{F}) = \mathcal{F}$, and

$$H^l_{\mathbf{m}}(\psi, \mathcal{F}) = \lim_{n\to\infty} \frac{1}{n} H^l_{\mathbf{m}}\left(\bigvee_{i=0}^{n-1} \psi^i(\mathcal{F})\right) = \lim_{n\to\infty} \frac{1}{n} H^l_{\mathbf{m}}(\mathcal{F}) = 0.$$

Theorem 5.7. *Consider $(IVIFS(X), \mathbf{m}, \psi)$ as a dynamical system, with $\mathcal{F}$ being a partition of $IVIFS(X)$. Then, for every non-negative integer k, the following holds:*

$$H^l_{\mathbf{m}}(\psi, \mathcal{F}) = H^l_{\mathbf{m}}(\psi, \bigvee_{i=0}^{k} \psi^i(\mathcal{F})).$$

Proof. Using Definition 5.5, we derive:

$$\begin{aligned}
H^l_{\mathbf{m}}(\psi, \bigvee_{i=0}^{k} \psi^i(\mathcal{F})) &= \lim_{n\longrightarrow\infty} \frac{1}{n} H^l_{\mathbf{m}}(\bigvee_{j=0}^{n-1} \psi^j(\bigvee_{i=0}^{k} \psi^i(\mathcal{F})) \\
&= \lim_{n\longrightarrow\infty} \frac{k+n}{n} \frac{1}{k+n} H^l_{\mathbf{m}}(\bigvee_{j=0}^{k+n-1} \psi^j(\mathcal{F})) \\
&= \lim_{n\longrightarrow\infty} \frac{1}{k+n} H^l_{\mathbf{m}}(\bigvee_{j=0}^{k+n-1} \psi^j(\mathcal{F})) = H^l_{\mathbf{m}}(\psi, \mathcal{F}).
\end{aligned}$$

□

Theorem 5.8. *Consider $(IVIFS(X), \mathbf{m}, \psi)$ as a dynamical system, with $\mathcal{F}_1, \mathcal{F}_2$ being two partitions of $IVIFS(X)$ such that $\mathcal{F}_1 \preceq \mathcal{F}_2$. Then $H^l_{\mathbf{m}}(\psi, \mathcal{F}_1) \leq H^l_{\mathbf{m}}(\psi, \mathcal{F}_2)$.*

Proof. Suppose that $\mathcal{F}_1 \preceq \mathcal{F}_2$. By Theorems 5.2 and 5.3, we have $\bigvee_{i=0}^{n-1} \psi^i(\mathcal{F}_1) \preceq \bigvee_{i=0}^{n-1} \psi^i(\mathcal{F}_2)$ for $n = 1, 2, \ldots$. Therefore, by a property of logical entropy, we get:

$$H^l_{\mathbf{m}}(\bigvee_{i=0}^{n-1} \psi^i(\mathcal{F}_1)) \leq H^l_{\mathbf{m}}(\bigvee_{i=0}^{n-1} \psi^i(\mathcal{F}_2)).$$

Taking the limit as $n \to \infty$, we obtain $H^l_{\mathbf{m}}(\psi, \mathcal{F}_1) \leq H^l_{\mathbf{m}}(\psi, \mathcal{F}_2)$. □

Definition 5.9. Let $(IVIFS(X), \mathbf{m}, \psi)$ be a dynamical system in $IVIFS(X)$. The logical entropy of $(IVIFS(X), \mathbf{m}, \psi)$ is defined as:

$$H^l_{\mathbf{m}}(\psi) = \sup\{H^l_{\mathbf{m}}(\psi, \mathcal{F}) \mid \mathcal{F} \text{ is a partition of } IVIFS(X)\}.$$

Theorem 5.10. *Let $(IVIFS(X), \mathbf{m}, \psi)$ be a dynamical system in $IVIFS(X)$. Then, for every natural number n, $H^l_{\mathbf{m}}(\psi^n) = n \cdot H^l_{\mathbf{m}}(\psi)$.*

Proof. Suppose that P be a partition in $IVIFS(X)$. Then for every $n \in \mathbb{N}$, we have:

$$\begin{aligned}
H^l_{\mathbf{m}}(\psi^n, \bigvee_{i=0}^{n-1} \psi^i(\mathcal{F})) &= \lim_{k\to\infty} \frac{1}{k} H^l_{\mathbf{m}}(\bigvee_{j=0}^{k-1} (\psi^n(\mathcal{F}))^j)(\bigvee_{i=0}^{n-1} \psi^i(\mathcal{F})) \\
&= \lim_{k\to\infty} \frac{1}{k} H^l_{\mathbf{m}}(\bigvee_{j=0}^{k-1} \bigvee_{i=0}^{n-1} (\psi^{nj+i}(\mathcal{F})) \\
&= \lim_{k\to\infty} \frac{kn}{k} \frac{1}{kn} H^l_{\mathbf{m}}(\bigvee_{i=0}^{kn-1} \psi^i(\mathcal{F})) = n.H^l_{\mathbf{m}}(\psi, \mathcal{F}).
\end{aligned}$$

Therefore

$$\begin{aligned}
n.H^l_{\mathbf{m}}(\psi) &= n.\sup\{H^l_{\mathbf{m}}(\psi, \mathcal{F});\ \mathcal{F} \text{ is a partition } in\ IVIFS(X)\} \\
&= \sup\{H^l_{\mathbf{m}}(\psi^n, \bigvee_{i=0}^{n-1} \psi^i(\mathcal{F})); \mathcal{F} \text{ is a partition } in\ IVIFS(X)\} \\
&\leq \sup\{H^l_{\mathbf{m}}(\psi^n, \mathcal{G}); \mathcal{G} \text{ is a partition } in\ IVIFS(X)\} = H^l_{\mathbf{m}}(\psi^n).
\end{aligned}$$

On the other hand, Since $\mathcal{F} \preceq \bigvee_{i=0}^{n-1} \psi^i(\mathcal{F})$, by Theorem 5.8, we obtain:

$$H^l_{\mathbf{m}}(\psi^n, \mathcal{F}) \leq H^l_{\mathbf{m}}(\psi^n, \bigvee_{i=0}^{n-1} \psi^i(\mathcal{F})) = n.H^l_{\mathbf{m}}(\psi, \mathcal{F}).$$

Thus

$$\begin{aligned}
H^l_{\mathbf{m}}(\psi^n) &= \sup\{H^l_{\mathbf{m}}(\psi^n, \mathcal{F}); \mathcal{F} \text{ is a partition in } IVIFS(X)\} \\
&\leq n.\sup\{H^l_{\mathbf{m}}(\psi, \mathcal{F}); \mathcal{F} \text{ is a partition in } IVIFS(X)\} \\
&= n.H^l_{\mathbf{m}}(\psi).
\end{aligned}$$

□

6 Conclusion

This paper offers an in-depth examination of logical entropy and its associated measures in the context of interval-valued intuitionistic fuzzy sets (IVIFS). It begins by introducing the core concepts of logical entropy for partitions in IVIFS, alongside logical conditional entropy, and thoroughly explores their properties. The discussion then expands to cover logical mutual information, highlighting its importance in measuring shared information between fuzzy partitions. The concept of logical divergence is introduced next, providing a detailed analysis of state divergence in IVIFS and exploring the properties of these measures. The study concludes by applying logical entropy to dynamical systems in IVIFS, focusing on how evolving uncertainty and distinctions can be measured in such systems. Collectively, the paper presents a comprehensive theoretical framework for understanding and quantifying uncertainty in complex environments characterized by both fuzziness and intuitionistic hesitation, with potential applications in decision-making, control theory, and systems analysis.

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