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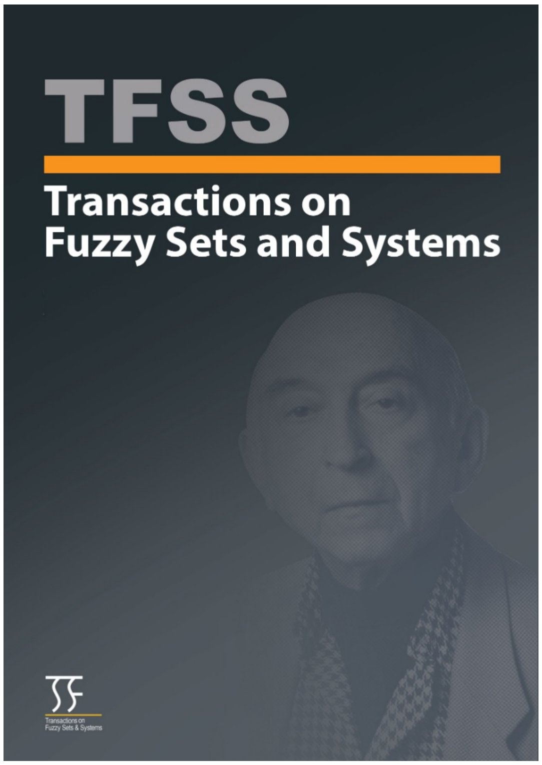

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A Novel Generalization of Hesitant Fuzzy Model with Application in Sustainable Supply Chain Optimization

Wajid Ali*, Tanzeela Shaheen, Iftikhar Ul Haq, Mohammad Mehedi Hassan

(This paper is dedicated to Professor "John N. Mordeson" on the occasion of his 91st birthday.)

Abstract. The n,m-rung orthopair fuzzy set theory is a robust model for managing uncertainty, particularly in multi-attribute decision-making. Meanwhile, the hesitant fuzzy model is a well-established tool in decision-making processes. Recognizing the similarities between these models, we propose a new framework called "c,d-rung orthopair hesitant fuzzy sets," which integrates both approaches. We examine key operations such as union, intersection, complement, subset, and equality, and introduce aggregation operators like the c,d-RHFPA, c,d-RHFWA, c,d-RHFPG, and c,d-RHFWPG operators. Additionally, an algorithm for multi-attribute decision-making is developed, which is applied to determine optimal business strategies for sustainable supply chain management. A comparative analysis with existing methods demonstrates the model's effectiveness, offering insights into its strengths and limitations. This paper introduces a novel approach to decision-making, outlining its real-world application and future research directions.

AMS Subject Classification 2020: 03E72; 94D05
Keywords and Phrases: Hesitant fuzzy sets, c,d-rung orthopair fuzzy sets, Decision making, MADM, Sustainable supply chain.

1 Introduction

1.1 Sustainable supply chain (SSC)

A sustainable supply chain, often referred to as an environmentally friendly or eco-conscious supply chain, is a business strategy that significantly focuses on incorporating environmentally and socially responsible practices throughout all phases of the supply chain process. This model has gained prominence in recent years due to growing concerns about climate change, natural resource depletion, social responsibility, and the need to mitigate the environmental and social impacts of business operations. Various re-searchers have explored this field from different perspectives. In 2015, Eskandarpour et al. [1] developed a supply chain network, and [2] outlined multi-objective optimization for sustainable supply chain. Linton et al. [3], in 2007, introduced the standard model for a sustainable supply chain, as well as decision models for its design and management in [4]. Resat et al. [5] developed an innovative model for multi-objective optimization approaches to sustainable supply chain management. Zhao et al. [6] applied a supply chain optimization model to continuous process industries with sustainability considerations. Eskandari [7] formulated and optimized a sustainable supply chain network for a blood platelet bank under conditions of uncertainty. Zhang et al. [8] introduced a novel

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multi-objective optimization model for sustainable supply chain network design, considering multiple distribution channels. Yu et al. [9] proposed two distinct frameworks for managing supply chain uncertainty by integrating a fuzzy structure with supply chain network optimization. Kazancoglu et al. [10] focused on leveraging emerging technologies to enhance the sustainability and resilience of supply chains in a fuzzy environment, particularly in the context of the COVID-19 pandemic. Goodarzian et al. [11] defined a new bi-objective green medicine supply chain network design under a fuzzy environment.

Liu et al. [12] demonstrated the application of a supply chain system in agriculture with a Crop Harvest Time Prediction Model for Better Sustainability, Integrating Feature Selection and Artificial Intelligence Methods. Yadav et al. [13] developed a sustainable supply chain model for multi-stage manufacturing with partial backlogging under a fuzzy environment, considering the effect of learning in the screening process. In 2020, Poujavad [14] designed a hybrid model for analyzing the risks of green supply chains in a fuzzy environment. Alsaed et al. [15] established a sustainable green A Novel Generalization of Hesitant Fuzzy Model with Application in Sustainable Supply Chain Optimization [16, 17] worked on a green Supply Chain Member Selection Method Considering Green Innovation Capability in a Hesitant Fuzzy Environment. Mistarihi et al. [18] developed a Strategic Framework for Disruption Management under a Fuzzy Environment. Liu [19] utilized q-rung interval-valued orthopair fuzzy data in a large-scale green supplier selection approach. Chang et al. [20] introduced a fuzzy optimization model for decision-making in supply chain management. Rehman et al. [21] constructed the application of a supply chain model in enhancing healthcare supply chain resilience by fuzzy decision-making.

1.2 Fuzzy Sets and their Generalizations

In 1965, Zadeh introduced the concept of Fuzzy sets [22] as a tool to address uncertainty. Fuzzy sets are ordered pairs where elements from a universal set are assigned member-ship values ranging from 0 to 1. Dubois et al. [23] authored a book on the fundamentals of fuzzy sets, discussing applications in detail. Attansove et al. [24] designed an extension of fuzzy sets called intuitionistic fuzzy sets, and Fermatean fuzzy sets were introduced by Senapati et al. [25]. Picture fuzzy sets and generalized orthopair fuzzy sets were introduced in [26, 27]. Torra [28] extended the FS model into a Hesitant fuzzy structure and discussed the generalized membership grade. Numerous researchers have contributed to the field of fuzzy sets and its generalizations [29-35]. Recently, Shahzadi et al. [36] introduced the latest extension of q-rung orthopair fuzzy sets, known as p,q-rung orthopair fuzzy sets, applied in multi-criteria decision-making. Ibrahim et al. [37] defined a topological approach for n, m-Rung orthopair fuzzy sets with applications to the diagnosis of learning disabilities. Continuously, Ibrahim et al. [38] combined two fuzzy frame-works—bipolar fuzzy sets and n,m-rung orthopair fuzzy sets—and defined an approach for multi-attribute group decision-making based on bipolar n, m-rung orthopair fuzzy sets. Furthermore, Ibrahim et al. [39] worked on an innovative method for group decision-making using n, m-rung orthopair fuzzy soft expert set knowledge. Mahmood at al [40] combined intuitionistic fuzzy sets and hesitant fuzzy sets and called intuitionistic hesitant fuzzy sets with their application in decision-making. Qahtan et al. [41] used Pythagorean hesitant fuzzy sets for supply chain systems and multiple-attribute decision-making in [42]. Krisci et al. [43] developed Fermatean hesitant fuzzy sets with medical decision-making applications. Liu et al. [44] constructed q-rung hesitant fuzzy sets and their application in multi-criteria decision-making. Sarwar et al. [45] established a decision-making model for failure modes and effects analysis based on rough fuzzy integrated clouds. Punnam et al. [46] explored a Linear Diophantine Fuzzy Soft Set-Based Decision-Making Approach Using a Revised Max-Min Average Composition Method. Recently, some novel extensions and generalizations of fuzzy models have been developed with their applications [47-49]. Aggregation operators play
a crucial role in information calculation, leading to the development of several aggregation operators in literature. Yager [50] introduced power average operators in 2001, and Xu et al. [51] developed pow-er geometric operators and their application in group decision-making. Yager and Ronald [52] designed generalized OWA aggregation operators. Dhankhar et al. [53] discussed multi-attribute decision-making based on the q-rung orthopair fuzzy Yager power weighted geometric aggregation operator of q-rung orthopair fuzzy values. Ali et al. [54] developed an Innovative Decision Model Utilizing Intuitionistic Hesitant Fuzzy Aczel-Alsina Aggregation Operators and Its Application. Haq et al. [55] designed a novel Fermatean Fuzzy Aczel–Alsina Model for Investment Strategy Selection.

1.3 Motivation and Contribution

In our comprehensive literature review, we have identified key areas of interest and gaps in current research. This review underscores the significance of fuzzy sets and their extensions, as well as highlighting the burgeoning field of 'Sustainable Supply Chain with Multi-Objective Decision-Making' and its applications. Addressing a notable gap in existing literature, our research primarily focuses on bridging the disconnect between the advanced "n,m-rung orthopair fuzzy model" and its application in sustainable supply chain systems. We have recognized the absence of methodologies based on "n,m-rung orthopair hesitant fuzzy sets" that facilitate multiple-attribute decision-making within sustainable supply chain contexts. In this paper, we have made several groundbreaking contributions to address these existing gaps as follows:

i. We successfully designed and developed an innovative combination of n,m-rung orthopair fuzzy sets and hesitant fuzzy sets, which we have named c,d-rung orthopair hesitant fuzzy sets. This pioneering work utilizes the synergies between the n,m and c,d models to propel the field forward.

ii. We have introduced and validated a comprehensive series of theorems and properties specific to our proposed model. This effort has significantly strengthened the theoretical underpinnings of our research.

iii. We have completed the development of an extensive series of aggregation operators for the c,d-rung orthopair hesitant fuzzy sets. This series includes the c,d-rung orthopair hesitant fuzzy power averaging (c,d-RHFPA) operator, the c,d-rung orthopair hesitant fuzzy power weighted averaging (c,d-RHFPWA) operator, the c,d-rung orthopair hesitant fuzzy power geometric (c,d-RHFPG) operator, and the c,d-rung orthopair hesitant fuzzy power weighted geometric (c,d-RHFWPG) operator.

iv. We have established a detailed algorithm for multiple criteria decision-making using c,d-RHF information. This robust framework is tailored for navigating com-plex decision processes efficiently.

v. Our research has successfully applied the developed multiple criteria decision-making model to identify optimal strategies for maintaining sustainable supply chain systems under c,d-rung orthopair hesitant fuzzy information.

vi. We conducted a thorough comparison of our model with existing techniques, demonstrating its consistency and superiority in the field.

vii. Finally, we have clearly articulated the benefits and advantages of our proposed model, emphasizing its significant impact and practical applications in the realm of sustainable supply chain management.

The article is structured as follows: Section 2 introduces the fundamental concepts relevant to our proposed approach. Section 3 develops the novel concept of "c,d-rung orthopair hesitant fuzzy Sets," including their operations and properties. Section 4 details the creation of aggregation operators for c,d-rung orthopair hesitant fuzzy sets, along with essential results and proofs. Section 5 elucidates the MCDM algorithm using c,d-rung orthopair hesitant fuzzy power averaging and geometric operators. Section 6 applies these concepts to a sustainable supply chain (SSC) model, providing a comprehensive exploration of MCDM. Section 7 presents a comparative analysis with existing techniques, highlighting the strengths and limitations of our approach. Finally, Section 8 concludes the paper and outlines future research directions. Figure 1 illustrates the manuscript's workflow.

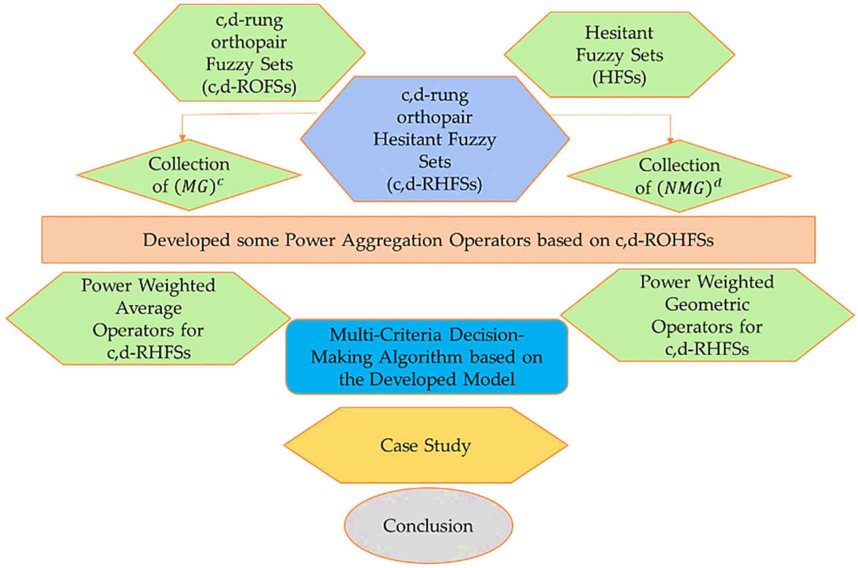

Figure 1: Flow chart of the sequence of research

2 Preliminaries

In this section, we recall the basic concepts such as intuitionistic hesitant fuzzy sets, Pythagorean hesitant fuzzy sets, Q-ROHFSs and c,d-rung rung orthopair fuzzy sets. Table 1 shows the symbols, and their descriptions used in the article.

Definition 2.1. let ∂ be a universal set. Then, $L = \{u, \pi_L(u), \psi_L(u) : u \in \partial\}$ is called.

1. an intuitionistic hesitant fuzzy set (IHFS) [40] if $0 \leq \max(\pi_L(u)) + \max(\psi_L(u)) \leq 1$ where $\pi_L(u)$ and $\psi_L(u)$ is a collection of distinct elements from [0,1]
2. a Pythagorean hesitant fuzzy set (PHFS) [42] if $0 \leq \max(\pi_L(u))^2 + \max(\psi_L(u))^2 \leq 1$ where $\pi_L(u)$ and $\psi_L(u)$ is a collection of distinct elements from [0,1]
3. a Fermatean hesitant fuzzy set (FHFS) [43] if $0 \leq \max(\pi_L(u))^3 + \max(\psi_L(u))^3 \leq 1$ where $\pi_L(u)$ and $\psi_L(u)$ is a collection of distinct elements from [0,1].
4. a Q-ROFS [44] if $0 \leq \max(\pi_L(u))^q + \max(\psi_L(u))^q \leq 1$, for $q \geq 1$. where $\pi_L(u)$ and $\psi_L(u)$ is a collection of distinct elements from [0,1].

Where $\pi_L(u), \psi_L(u) : \partial \to [0,1]$ are MG and NMG, respectively.

Definition 2.2. [38] let ∂ be a universal set. Then, $L = \{u, \pi_L(u), \psi_L(u) : u \in \partial\}$ is called a c,d-rung orthopair fuzzy set if $0 \leq (\pi_L(u))^c + (\psi_L(u))^d \leq 1$ such that $c, d \in N$. The degree of indeterminacy for $u \in \partial$ to L is given as,

$$\gamma_L(u) = \sqrt[c+d]{1 - \left[(\pi_L(u))^c + (\psi_L(u))^d\right]}, \quad \gamma_L(u) \in [0,1]$$

Definition 2.3. For a c,d-rung orthopair fuzzy set $L = (\pi_L(u), \psi_L(u))$, The score (SF) and accuracy functions (AF) are defined as

$$S(L) = (\pi_L(u))^c - (\psi_L(u))^d, \quad A(L) = (\pi_L(u))^c + (\psi_L(u))^d$$

Wherever, $S(L) \in [-1,1]$ and $A(L) \in [0,1]$.

3 An idea of c,d-rung orthopair hesitant fuzzy sets (c,d RHFSs)

The concept of the novel model called c,d-RHFSs with their basic properties and operators are briefly discussed in this section.

Table 1: Symbols and their descriptions

Symbols	Descriptions	Symbols	Descriptions
Q-ROFSS	q-rung orthopair fuzzy sets	c,d RHFSs	c,d-rung hesitant fuzzy sets
MG	Membership grade	c,d RHFPA	c,d-rung hesitant fuzzy power average
NMG	Non-membership grade	c,d RHFPG	c,d-rung hesitant fuzzy power geometric
SSC	Sustainable supply chain	MCDM	Multi criteria decision making
SF	Score function	AF	Accuracy function

Definition 3.1. let ∂ be a universal set. A c,d rung orthopair hesitant fuzzy sets(c,d-RHFSs) L in ∂ is stated as, $L = \{u, \pi_L(u), \psi_L(u): u \in \partial\}$ where $\pi_L(u)$ and $\psi_L(u)$ is a set of elements from the [0,1] and $0 \leq max\left(\pi_L(u)\right)^c + max\left(\psi_L(u)\right)^d \leq 1$ such that $c, d \in N$. The degree of indeterminacy for $u \in \partial$ to L is given as,

$$\gamma_L(u) = \bigcup_{\substack{e \in \pi_L(u) \\ f \in \psi_L(u)}} \sqrt[c+d]{1 - \left[\left(e_L(u)\right)^c + \left(f_L(u)\right)^d\right]}$$

and $\gamma_L(u) \in [0,1]$

Throughout the paper, for our easiness a c,d-RHFS is represented as $L = (\pi_L, \psi_L)$.

Remark 3.2. If c=d for a c,d-RHFS $L = (\pi_L, \psi_L)$, then we call $L = (\pi_L, \psi_L)$ is a Q-RHFS where q=c=d.

Definition 3.3. let $L = (\pi_L, \psi_L)$, $L_1 = \left(\pi_{L_1}, \psi_{L_1}\right)$ and $L_2 = \left(\pi_{L_2}, \psi_{L_2}\right)$ be three c,d-rung orthopair hesitant fuzzy sets then,

1. $L_1 \wedge L_2$

$$= \bigcup_{\substack{e_1 \in \pi_{L_1} \\ e_2 \in \pi_{L_2} \\ f_1 \in \psi_{L_1} \\ f_2 \in \psi_{L_2}}} (\min\{e_1, e_2\}, \max\{f_1, f_2\})$$

2. $L_1 \vee L_2$

$$= \bigcup_{\substack{e_1 \in \pi_{L_1} \\ e_2 \in \pi_{L_2} \\ f_1 \in \psi_{L_1} \\ f_2 \in \psi_{L_2}}} (\max\{e_1, e_2\}, \min\{f_1, f_2\})$$

3. L'

$$= \bigcup_{\substack{e \in \pi_L \\ f \in \psi_L}} \left(f^{\frac{d}{c}}, e^{\frac{c}{d}}\right)$$

Theorem 3.4. If $L = (\pi_L, \psi_L)$ is a c,d-rung orthopair hesitant fuzzy sets then L' is also (c,d-RHFSs) and $(L')' = L$.

Proof. Let $0 \leq \pi_L^c + \psi_L^d \leq 1$, then.

$0 \leq \left(f^{\frac{d}{c}}\right)^c + \left(e^{\frac{c}{d}}\right)^d = (e)^c + (\boldsymbol{f})^d \leq 1$ where $e \in \pi_L$, $f \in \psi_L$. Thus, L' is also (c,d-RHFS) and it is obvious

$$(L')' = \bigcup_{\substack{e\in\pi_L \\ f\in\psi_L}} \left(\left(f^{\frac{d}{c}}, e^{\frac{c}{d}}\right)\right)' = \bigcup_{\substack{e\in\pi_L \\ f\in\psi_L}} \left(\left(e^{\frac{c}{d}}\right)^{\frac{d}{c}}, \left(f^{\frac{d}{c}}\right)^{\frac{c}{d}}\right)$$

Which is again L. □

Remark 3.5. If $L_1 = (\pi_{L_1}, \psi_{L_1})$ and $L_2 = (\pi_{L_2}, \psi_{L_2})$are two c,d-rung orthopair hesitant fuzzy sets then $L_1 \wedge L_2$ and $L_1 \vee L_2$ are also (c,d-RHFSs).

Theorem 3.6. let $L_1 = (\pi_{L_1}, \psi_{L_1})$ and $L_2 = (\pi_{L_2}, \psi_{L_2})$ be two c,d-rung orthopair hesitant fuzzy sets then,

1. $L_1 \wedge L_2 = L_2 \wedge L_1$
2. $L_1 \vee L_2 = L_2 \vee L_1$
3. $(L_1 \wedge L_2) \vee L_2 = L_2$
4. $(L_1 \vee L_2) \wedge L_2 = L_2$

Proof. From Definition 3.3, we have:

1. $L_1 \wedge L_2$

$$= \bigcup_{\substack{e_1\in\pi_{L_1} \\ e_2\in\pi_{L_2} \\ f_1\in\psi_{L_1} \\ f_2\in\psi_{L_2}}} (\min\{e_1, e_2\}, \max\{f_1, f_2\}) = \bigcup_{\substack{e_1\in\pi_{L_1} \\ e_2\in\pi_{L_2} \\ f_1\in\psi_{L_1} \\ f_2\in\psi_{L_2}}} (\min\{e_2, e_1\}, \max\{f_2, f_1\}) = L_2 \wedge L_1$$

2. $L_1 \vee L_2$

$$= \bigcup_{\substack{e_1\in\pi_{L_1} \\ e_2\in\pi_{L_2} \\ f_1\in\psi_{L_1} \\ f_2\in\psi_{L_2}}} (\max\{e_1, e_2\}, \min\{f_1, f_2\}) = \bigcup_{\substack{e_1\in\pi_{L_1} \\ e_2\in\pi_{L_2} \\ f_1\in\psi_{L_1} \\ f_2\in\psi_{L_2}}} (\max\{e_2, e_1\}, \min\{f_2, f_1\}) = L_2 \vee L_1$$

3. $(L_1 \wedge L_2) \vee L_2$

$$= \bigcup_{\substack{e_1\in\pi_{L_1} \\ e_2\in\pi_{L_2} \\ f_1\in\psi_{L_1} \\ f_2\in\psi_{L_2}}} (\min\{e_1, e_2\}, \max\{f_1, f_2\}) \vee \left(\bigcup_{\substack{e_2\in\pi_{L_2} \\ f_2\in\psi_{L_2}}} (e_2, f_2)\right) = \bigcup_{\substack{e_1\in\pi_{L_1} \\ e_2\in\pi_{L_2} \\ f_1\in\psi_{L_1} \\ f_2\in\psi_{L_2}}} \left(\max(\min\{e_2, e_1\}, e_2), \min\left(\max\{f_2, f_1\}, f_2\right)\right) = L_2$$

4. $(L_1 \vee L_2) \wedge L_2$

$$= \bigcup_{\substack{e_1\in\pi_{L_1} \\ e_2\in\pi_{L_2} \\ f_1\in\psi_{L_1} \\ f_2\in\psi_{L_2}}} (\max\{e_1, e_2\}, \min\{f_1, f_2\}) \wedge \left(\bigcup_{\substack{e_2\in\pi_{L_2} \\ f_2\in\psi_{L_2}}} (e_2, f_2)\right) = \bigcup_{\substack{e_1\in\pi_{L_1} \\ e_2\in\pi_{L_2} \\ f_1\in\psi_{L_1} \\ f_2\in\psi_{L_2}}} (\min(\max\{e_2, e_1\}, e_2), \max(\min\{f_2, f_1\}, f_2)) = L_2$$

□

Theorem 3.7. let $L_1 = (\pi_{L_1}, \psi_{L_1})$ and $L_2 = (\pi_{L_2}, \psi_{L_2})$ be two c,d-rung orthopair hesitant fuzzy sets then,

1. $(L_1 \wedge L_2)' = L_1' \vee L_2'$
2. $(L_1 \vee L_2)' = L_1' \wedge L_2'$

Proof. For the c,d-RHFSs L_1 and L_2, we have:

1. $(L_1 \wedge L_2)'$

$$= \bigcup_{\substack{e_1 \in \pi_{L_1} \\ e_2 \in \pi_{L_2} \\ f_1 \in \psi_{L_1} \\ f_2 \in \psi_{L_2}}} (\min\{e_1, e_2\}, \max\{f_1, f_2\})' = \bigcup_{\substack{e_1 \in \pi_{L_1} \\ e_2 \in \pi_{L_2} \\ f_1 \in \psi_{L_1} \\ f_2 \in \psi_{L_2}}} \left(\max\left\{e_1^{\frac{d}{c}}, e_2^{\frac{d}{c}}\right\}, \min\left\{f_1^{\frac{c}{d}}, f_2^{\frac{c}{d}}\right\}\right) = \bigcup_{\substack{e_1 \in \pi_{L_1} \\ f_1 \in \psi_{L_1}}} \left(e_1^{\frac{d}{c}}, f_1^{\frac{c}{d}}\right) \vee \bigcup_{\substack{e_2 \in \pi_{L_2} \\ f_2 \in \psi_{L_2}}} \left(e_2^{\frac{d}{c}}, f_2^{\frac{c}{d}}\right)$$

$$= L_1' \vee L_2'$$

2. Similar to (1). □

Definition 3.8. let $L = (\pi_L, \psi_L)$, $L_1 = (\pi_{L_1}, \psi_{L_1})$ and $L_2 = (\pi_{L_2}, \psi_{L_2})$ be three c,d-rung orthopair hesitant fuzzy sets, and Δ is a positive real number ($\Delta > 0$),then

1. $L_1 \oplus L_2$

$$= \bigcup_{\substack{e_1 \in \pi_{L_1} \\ e_2 \in \pi_{L_2} \\ f_1 \in \psi_{L_1} \\ f_2 \in \psi_{L_2}}} \left(\sqrt[c]{e_1^c + e_2^c - e_1^c e_2^c}, f_1 f_2\right)$$

2. $L_1 \otimes L_2$

$$= \bigcup_{\substack{e_1 \in \pi_{L_1} \\ e_2 \in \pi_{L_2} \\ f_1 \in \psi_{L_1} \\ f_2 \in \psi_{L_2}}} \left(e_1 e_2, \sqrt[d]{f_1^d + f_2^d - f_1^d f_2^d}\right)$$

3. ΔL

$$= \bigcup_{\substack{e \in \pi_L \\ f \in \psi_L}} \left(\sqrt[c]{1 - (1 - e^c)^\Delta}, (f)^\Delta\right)$$

4. L^Δ

$$= \bigcup_{\substack{e \in \pi_L \\ f \in \psi_L}} \left(e^\Delta, \sqrt[d]{1 - (1 - f^d)^\Delta}\right)$$

Theorem 3.9. let $L_1 = (\pi_{L_1}, \psi_{L_1})$, $L_2 = (\pi_{L_2}, \psi_{L_2})$ and $L_3 = (\pi_{L_3}, \psi_{L_3})$ be three c,d-RHFSs, and Δ is a positive real number ($\Delta > 0$),then

1. $L_1 \oplus L_2 = L_2 \oplus L_1$
2. $L_1 \otimes L_2 = L_2 \otimes L_1$
3. $L_1 \oplus L_2 \oplus L_3 = L_1 \oplus L_3 \oplus L_2$
4. $L_1 \otimes L_2 \otimes L_3 = L_1 \otimes L_3 \otimes L_2$

Proof. From Definition 3.8 we have:

1. $L_1 \oplus L_2$

$$= \bigcup_{\substack{e_1 \in \pi_{L_1} \\ e_2 \in \pi_{L_2} \\ f_1 \in \psi_{L_1} \\ f_2 \in \psi_{L_2}}} \left(\sqrt[c]{e_1^c + e_2^c - e_1^c e_2^c}, f_1 f_2 \right) = \bigcup_{\substack{e_1 \in \pi_{L_1} \\ e_2 \in \pi_{L_2} \\ f_1 \in \psi_{L_1} \\ f_2 \in \psi_{L_2}}} \left(\sqrt[c]{e_2^c + e_1^c - e_2^c e_1^c}, f_2 f_1 \right) = L_2 \oplus L_1$$

2. $L_1 \otimes L_2$

$$= \bigcup_{\substack{e_1 \in \pi_{L_1} \\ e_2 \in \pi_{L_2} \\ f_1 \in \psi_{L_1} \\ f_2 \in \psi_{L_2}}} \left(e_1 e_2, \sqrt[d]{f_1^d + f_2^d - f_1^d f_2^d} \right) = \bigcup_{\substack{e_1 \in \pi_{L_1} \\ e_2 \in \pi_{L_2} \\ f_1 \in \psi_{L_1} \\ f_2 \in \psi_{L_2}}} \left(e_2 e_1, \sqrt[d]{f_2^d + f_1^d - f_2^d f_1^d} \right) = L_2 \otimes L_1$$

3. It is similar to 1.

4. It is similar to 2. □

Theorem 3.10. let $L = (\pi_L, \psi_L)$, $L_1 = (\pi_{L_1}, \psi_{L_1})$ and $L_2 = (\pi_{L_2}, \psi_{L_2})$ be three c,d-rung orthopair hesitant fuzzy sets, and Δ is a positive real number $(\Delta > 0)$,then

1. $(L_1 \oplus L_2)' = L_1' \otimes L_2'$
2. $(L_1 \otimes L_2)' = L_1' \oplus L_2'$
3. $(L')^{\Delta} = (\Delta L)'$
4. $\Delta(L)' = (L^{\Delta})'$

Proof. For the c,d-RHFSs L, L_1 and L_3, we have

1. $(L_1 \oplus L_2)'$

$$= \bigcup_{\substack{e_1 \in \pi_{L_1} \\ e_2 \in \pi_{L_2} \\ f_1 \in \psi_{L_1} \\ f_2 \in \psi_{L_2}}} \left(\sqrt[c]{e_1^c + e_2^c - e_1^c e_2^c}, f_1 f_2 \right)' = \bigcup_{\substack{e_1 \in \pi_{L_1} \\ e_2 \in \pi_{L_2} \\ f_1 \in \psi_{L_1} \\ f_2 \in \psi_{L_2}}} \left((f_1 f_2)^{\frac{d}{c}}, \left(\sqrt[c]{e_1^c + e_2^c - e_1^c e_2^c} \right)^{\frac{c}{d}} \right)$$

$$= \bigcup_{\substack{e_1 \in \pi_{L_1} \\ f_1 \in \psi_{L_1}}} \left(f_1^{\frac{d}{c}}, e_1^{\frac{c}{d}} \right) \otimes \bigcup_{\substack{e_2 \in \pi_{L_2} \\ f_2 \in \psi_{L_2}}} \left(f_2^{\frac{d}{c}}, e_2^{\frac{c}{d}} \right) = L_1' \otimes L_2'$$

2. It is similar to 1.

3. $(L')^{\Delta}$

$$= \bigcup_{\substack{e\in\pi_L\\ f\in\psi_L}} \left(f^{\frac{d}{c}}, e^{\frac{c}{d}}\right)^{\Delta} = \bigcup_{\substack{e\in\pi_L\\ f\in\psi_L}} \left(\left(f^{\frac{d}{c}}\right)^{\Delta}, \left(1-\left(1-\left(e^{\frac{c}{d}}\right)^{d}\right)^{\Delta}\right)^{\frac{1}{d}}\right) = \bigcup_{\substack{e\in\pi_L\\ f\in\psi_L}} \left((f^{\Delta})^{\frac{d}{c}}, \left(1-(1-(e^{c})^{\Delta})^{\frac{1}{d}}\right)^{\frac{c}{d}}\right)$$

$$= \bigcup_{\substack{e\in\pi_L\\ f\in\psi_L}} \left((1-(1-e^{c})^{\Delta})^{\frac{1}{c}} \quad, f^{\Delta}\right) = (\Delta L)'$$

4. $\Delta(L)'$

$$= \Delta \bigcup_{\substack{e\in\pi_L\\ f\in\psi_L}} \left(f^{\frac{d}{c}}, e^{\frac{c}{d}}\right) = \bigcup_{\substack{e\in\pi_L\\ f\in\psi_L}} \left(\left((1-\left(1-\left(f^{\frac{d}{c}}\right)^{c}\right)^{\Delta}\right)^{\frac{1}{c}}, \left((e)^{\frac{c}{d}}\right)^{\Delta}\right)$$

$$= \bigcup_{\substack{e\in\pi_L\\ f\in\psi_L}} \left(e^{\Delta}, \left((1-\left(1-(f^{d})\right)^{\Delta}\right)^{\frac{1}{d}}\right) = (L^{\Delta})' \qquad \square$$

4 c,d-rung orthopair hesitant fuzzy aggregation operators

Here, a series of average and geometric operators for c,d-RHFSs is briefly discussed. Moreover, their basic properties are also explained.

Definition 4.1. let $L_i = \left(\pi_{L_i}, \psi_{L_i}\right), (i = 1,2,\dots,k)$ be a set of c,d-RHFNs and $\tau = (\tau_i)^T$ be weight vector of L_i with $\tau_i > 0$ such that $\sum_{i=1}^{k} \tau_i = 1$ then, the

1. c,d-rung orthopair hesitant fuzzy weighted averaging (c,d-RHFWA) operator is a mapping $c,d-RHFWA: L^k \to L$ such that $c,d-RHFWA(L_1, L_2, \dots, L_k) = \oplus_{i=1}^{k} \tau_i L_i = \tau_1 L_1 \oplus \tau_2 L_2 \dots . \oplus \tau_k L_k$
2. c,d-rung orthopair hesitant fuzzy weighted geometric (c,d-RHFWG) operator is a mapping $c,d-RHFWG: L^k \to L$ such that $c,d-RHFWA(L_1, L_2, \dots, L_k) = \otimes_{i=1}^{k} L_i^{\tau_i} = L_1^{\tau_1} \otimes L_2^{\tau_2} \dots . \otimes L_k^{\tau_k}$.

Theorem 4.2. let $L_i = \left(\pi_{L_i}, \psi_{L_i}\right), (i = 1,2,\dots,k)$ be a set of c,d-RHFNs and $\tau = (\tau_i)^T$ be weight vector of L_i with $\tau_i > 0$ such that $\sum_{i=1}^{k} \tau_i = 1$ then,

1. The aggregation value of c,d-RHFNs $L_i (i = 1,2,\dots,k)$ by using $c,d-RHFWA$ operator is also c,d-RHFN. And

$$c,d-RHFWA(L_1, L_2, \dots, L_k) = \bigcup_{\substack{e_{L_i}\in\pi_{L_i}\\ f_{L_i}\in\psi_{L_i}}} \langle \left(1-\prod_{i=1}^{k}\left(1-\left(e_{L_i}\right)^{c}\right)^{\tau_i}\right)^{\frac{1}{c}}, \prod_{i=1}^{k}\left(f_{L_i}\right)^{\tau_i} \rangle$$

2. The aggregation value of c,d-RHFNs $L_i (i = 1,2,\dots,k)$ by using $c,d-RHFWG$ operator is also c,d-RHFN. And

$$c,d-RHFWG(L_1, L_2, \dots, L_k) = \bigcup_{\substack{e_{L_i}\in\pi_{L_i}\\ f_{L_i}\in\psi_{L_i}}} \langle \prod_{i=1}^{k}\left(e_{L_i}\right)^{\tau_i}, \left(1-\prod_{i=1}^{k}\left(1-\left(f_{L_i}\right)^{c}\right)^{\tau_i}\right)^{\frac{1}{c}} \rangle$$

Proof.

1. We can provide proof of the abovementioned results by using mathematical induction. Therefore, we follow as,

(i). For $i = 2$ since

$$\tau_1 L_1 = \bigcup_{\substack{e_{L_1}\in\pi_{L_1}\\ f_{L_1}\in\psi_{L_1}}} \langle \left(1-\left(1-\left(e_{L_1}\right)^{c}\right)^{\tau_1}\right)^{\frac{1}{c}}, \left(f_{L_1}\right)^{\tau_1} \rangle$$

and

$$\tau_2 L_2 = \bigcup_{\substack{e_{L_2}\in\pi_{L_2}\\ f_{L_2}\in\psi_{L_2}}} \langle \left(1-\left(1-\left(e_{L_2}\right)^c\right)^{\tau_2}\right)^{\frac{1}{c}}, \left(f_{L_2}\right)^{\tau_2}\rangle$$

then $c,d-RHFWA(L_1,L_2)=\tau_1 L_1 \oplus \tau_2 L_2$=

$$\bigcup_{\substack{e_{L_1}\in\pi_{L_1}\\ f_{L_1}\in\psi_{L_1}}} \langle \left(1-\left(1-\left(e_{L_1}\right)^c\right)^{\tau_1}\right)^{\frac{1}{c}}, \left(f_{L_1}\right)^{\tau_1}\rangle \oplus \bigcup_{\substack{e_{L_2}\in\pi_{L_2}\\ f_{L_2}\in\psi_{L_2}}} \langle \left(1-\left(1-\left(e_{L_2}\right)^c\right)^{\tau_2}\right)^{\frac{1}{c}}, \left(f_{L_2}\right)^{\tau_2}\rangle$$

$$= \bigcup_{\substack{e_{L_1}\in\pi_{L_1}\\ e_{L_2}\in\pi_{L_2}\\ f_{L_1}\in\psi_{L_1}\\ f_{L_2}\in\psi_{L_2}}} \langle \left(1-\left(1-\left(e_{L_1}\right)^c\right)^{\tau_1}+1-\left(1-\left(e_{L_2}\right)^c\right)^{\tau_2}\right)^{\frac{1}{c}}, \left(f_{L_1}\right)^{\tau_1}\left(f_{L_2}\right)^{\tau_2}\rangle$$

$$= \bigcup_{\substack{e_{L_1}\in\pi_{L_1}\\ e_{L_2}\in\pi_{L_2}\\ f_{L_1}\in\psi_{L_1}\\ f_{L_2}\in\psi_{L_2}}} \langle \left(1-\left(1-\left(e_{L_1}\right)^c\right)^{\tau_1}\left(1-\left(e_{L_2}\right)^c\right)^{\tau_2}\right)^{\frac{1}{c}}, \left(f_{L_1}\right)^{\tau_1}\left(f_{L_2}\right)^{\tau_2}\rangle$$

$$= \bigcup_{\substack{e_{L_1}\in\pi_{L_1}\\ e_{L_2}\in\pi_{L_2}\\ f_{L_1}\in\psi_{L_1}\\ f_{L_2}\in\psi_{L_2}}} \langle \left(1-\left(1-\left(e_{L_1}\right)^c\right)^{\tau_1}\left(1-\left(e_{L_2}\right)^c\right)^{\tau_2}\right)^{\frac{1}{c}}, \left(f_{L_1}\right)^{\tau_1}\left(f_{L_2}\right)^{\tau_2}\rangle$$

$$= \bigcup_{\substack{e_{L_1}\in\pi_{L_1}\\ e_{L_2}\in\pi_{L_2}\\ f_{L_1}\in\psi_{L_1}\\ f_{L_2}\in\psi_{L_2}}} \langle \left(1-\prod_{i=1}^{2}\left(1-\left(e_{L_i}\right)^c\right)^{\tau_i}\right)^{\frac{1}{c}}, \prod_{i=1}^{2}\left(f_{L_i}\right)^{\tau_i}\rangle$$

(ii). Suppose that this result is satisfied for $i=r$ which is,

$$c,d-RHFWA(L_1,L_2,\dots,L_r)=\tau_1 L_1 \oplus \tau_2 L_2 \dots .\oplus \tau_r L_r = \bigcup_{\substack{e_{L_i}\in\pi_{L_i}\\ f_{L_i}\in\psi_{L_i}}} \langle \left(1-\prod_{i=1}^{r}\left(1-\left(e_{L_i}\right)^c\right)^{\tau_i}\right)^{\frac{1}{c}}, \prod_{i=1}^{r}\left(f_{L_i}\right)^{\tau_i}\rangle$$

Now, we will prove that the result is true for $i=r+1$ by using (i) and (ii) we have.

$$c,d-RHFWA(L_1,L_2,\dots,L_{r+1})=\tau_1 L_1 \oplus \tau_2 L_2 \dots .\oplus \tau_{r+1} L_{r+1} =$$

$$\bigcup_{\substack{e_{L_i}\in\pi_{L_i}\\ f_{L_i}\in\psi_{L_i}}} \langle \left(1-\prod_{i=1}^{r}\left(1-\left(e_{L_i}\right)^c\right)^{\tau_i}\right)^{\frac{1}{c}}, \prod_{i=1}^{r}\left(f_{L_i}\right)^{\tau_i}\rangle \oplus \bigcup_{\substack{e_{L_{r+1}}\in\pi_{L_{r+1}}\\ f_{L_{r+1}}\in\psi_{L_{r+1}}}} \langle \left(1-\left(1-\left(e_{L_{r+1}}\right)^c\right)^{\tau_{r+1}}\right)^{\frac{1}{c}}, \left(f_{L_{r+1}}\right)^{\tau_{r+1}}\rangle$$

$$= \bigcup_{\substack{e_{L_i}\in\pi_{L_i}\\ f_{L_i}\in\psi_{L_i}}} \langle \left(1-\prod_{i=1}^{r}\left(1-\left(e_{L_i}\right)^c\right)^{\tau_i}+1-\left(1-\left(e_{L_{r+1}}\right)^c\right)^{\tau_{r+1}}-\left(1-\prod_{i=1}^{r}\left(1-\left(e_{L_i}\right)^c\right)^{\tau_i}\right)(1\right.$$

$$\left.-\left(1-\left(e_{L_{r+1}}\right)^c\right)^{\tau_{r+1}}\right)^{\frac{1}{c}}, \prod_{i=1}^{r}\left(f_{L_i}\right)^{\tau_i}\left(f_{L_{r+1}}\right)^{\tau_{r+1}}\rangle$$

$$= \bigcup_{\substack{e_{L_i}\in\pi_{L_i}\\ f_{L_i}\in\psi_{L_i}}} \langle \left(1-\prod_{i=1}^{r}\left(1-\left(e_{L_i}\right)^c\right)^{\tau_i}\left(1-\left(e_{L_{r+1}}\right)^c\right)^{\tau_{r+1}}\right)^{\frac{1}{c}}, \prod_{i=1}^{r}\left(f_{L_i}\right)^{\tau_i}\left(f_{L_{r+1}}\right)^{\tau_{r+1}}\rangle$$

$$= \bigcup_{\substack{e_{L_i}\in\pi_{L_i}\\ f_{L_i}\in\psi_{L_i}}} \langle \left(1-\prod_{i=1}^{r+1}\left(1-\left(e_{L_i}\right)^c\right)^{\tau_i}\right)^{\frac{1}{c}}, \prod_{i=1}^{r+1}\left(f_{L_i}\right)^{\tau_i} \rangle$$

The theorem is meeting for $i = r + 1$. Thus, theorem is fulfilled for whole i.

2. The proof is same as part 1. □

Theorem 4.3. (Idempotence) let $L_i = \left(\pi_{L_i}, \psi_{L_i}\right), (i = 1,2, \dots, k)$ be a set of c,d-rung orthopair hesitant fuzzy numbers and $\tau = (\tau_i)^T$ be weight vector of L_i with $\tau_i > 0$ such that $\sum_{i=1}^{k}\tau_i = 1$.If all of $L_i = \left(\pi_{L_i}, \psi_{L_i}\right), (i = 1,2, \dots, k)$ are identical to $L = (\pi_L, \psi_L)$ then

1. $c,d - RHFWA(L_1, L_2, \dots, L_k) = L$

2. $c,d - RHFWG(L_1, L_2, \dots, L_k) = L$

Proof.

1. Since $L_i = L = (\pi_L, \psi_L)(i = 1,2, \dots, k)$ then$c,d - RHFWG(L_1, L_2, \dots, L_k)$

$$= \bigcup_{\substack{e_{L_i}\in\pi_{L_i}\\ f_{L_i}\in\psi_{L_i}}} \langle \left(1-\prod_{i=1}^{k}\left(1-\left(e_{L_i}\right)^c\right)^{\tau_i}\right)^{\frac{1}{c}}, \prod_{i=1}^{k}\left(f_{L_i}\right)^{\tau_i} \rangle$$

$$= \bigcup_{\substack{e_L\in\pi_L\\ f_L\in\psi_L}} \langle \left(1-\prod_{i=1}^{k}\left(1-(e_L)^c\right)^{\tau_i}\right)^{\frac{1}{c}}, \prod_{i=1}^{k}(f_L)^{\tau_i} \rangle$$

$$= \bigcup_{\substack{e_L\in\pi_L\\ f_L\in\psi_L}} \langle \left(1-(1-(e_L)^c)^{\sum_{i=1}^{k}\tau_i}\right)^{\frac{1}{c}}, \prod_{i=1}^{k}(f_L)^{\sum_{i=1}^{k}\tau_i} \rangle = \bigcup_{\substack{e_L\in\pi_L\\ f_L\in\psi_L}} \langle \left(1-(1-(e_L)^c)\right)^{\frac{1}{c}}, \prod_{i=1}^{k}(f_L) \rangle = L$$

2. The proof is same as 1. □

Theorem 4.4. (Boundedness) let $L_i = \left(\pi_{L_i}, \psi_{L_i}\right), (i = 1,2, \dots, k)$ be a set of c,d-rung orthopair hesitant fuzzy numbers and $\tau = (\tau_i)^T$ be weight vector of L_i with $\tau_i > 0$ such that $\sum_{i=1}^{k}\tau_i = 1$. Suppose that $e_L^- = \min_{1\le i\le k}\{e_i\}_{e_i\in\pi_{L_i}}$ and $e_L^+ = \max_{1\le i\le k}\{e_i\}_{e_i\in\pi_{L_i}}$ $f_L^+ = \max_{1\le i\le k}\{f_i\}_{f_i\in\psi_{L_i}}$, and $f_L^- = \min_{1\le i\le k}\{f_i\}_{f_i\in\psi_{L_i}}$. Then,

1. $(e_L^-, f_L^-) \le c,d - RHFWA(L_1, L_2, \dots, L_k) \le (e_L^+, f_L^+)$
2. $(e_L^-, f_L^-) \le c,d - RHFWG(L_1, L_2, \dots, L_k) \le (e_L^+, f_L^+)$

Proof.

1. For any $L_i = \left(\pi_{L_i}, \psi_{L_i}\right), (i = 1,2, \dots, k)$ we can get $e_L^- \le e_i \le e_L^+$ and $f_L^- \le f_i \le f_L^+$..Then we have

$$e_L^- = \left(1-(1-(e_L^-)^c)^{\sum_{i=1}^{k}\tau_i}\right)^{\frac{1}{c}} = \left(1-\prod_{i=1}^{k}\left(1-(e_L^-)^c\right)^{\tau_i}\right)^{\frac{1}{c}} \le \left(1-\prod_{i=1}^{k}\left(1-\left(e_{L_i}\right)^c\right)^{\tau_i}\right)^{\frac{1}{c}}$$

$$\le \left(1-\prod_{i=1}^{k}\left(1-(e_L^+)^c\right)^{\tau_i}\right)^{\frac{1}{c}} = \left(1-(1-(e_L^+)^c)^{\sum_{i=1}^{k}\tau_i}\right)^{\frac{1}{c}} = e_L^+$$

and

$$f_L^- = \left(1-(1-(f_L^-)^c)^{\sum_{i=1}^{k}\tau_i}\right)^{\frac{1}{c}} = \left(1-\prod_{i=1}^{k}\left(1-(f_L^-)^c\right)^{\tau_i}\right)^{\frac{1}{c}} \le \left(1-\prod_{i=1}^{k}\left(1-\left(f_{L_i}\right)^c\right)^{\tau_i}\right)^{\frac{1}{c}}$$

$$\le \left(1-\prod_{i=1}^{k}\left(1-(f_L^+)^c\right)^{\tau_i}\right)^{\frac{1}{c}} = \left(1-(1-(f_L^+)^c)^{\sum_{i=1}^{k}\tau_i}\right)^{\frac{1}{c}} = f_L^+$$

Therefore,

$$(e_L^-, f_L^-) \le c,d - RHFWA(L_1, L_2, \dots, L_k) \le (e_L^+, f_L^+)$$

2. The proof is same as part 1. □

Theorem 4.5. (Monotonicity) let $L_i = (\pi_{L_i}, \psi_{L_i})$and $M_i = (\pi_{M_i}, \psi_{M_i})(i = 1,2, \ldots, k)$ be two sets of c,d-rung orthopair hesitant fuzzy numbers. If $L_i \subseteq M_i, \forall i$,then

1. $c, d - RHFWA(L_1, L_2, \ldots, L_k) \leq c, d - RHFWA(M_1, M_2, \ldots, M_k)$
2. $c, d - RHFWG(L_1, L_2, \ldots, L_k) \leq c, d - RHFWG(M_1, M_2, \ldots, M_k)$

Proof.

1. Since for all i, we have $e_{L_i} \leq e_{M_i}, f_{L_i} \geq f_{M_i}$then

$$\bigcup_{\substack{e_{L_i} \in \pi_{L_i} \\ f_{L_i} \in \psi_{L_i}}} \langle \left(1 - \prod_{i=1}^{k} \left(1 - \left(e_{L_i}\right)^c\right)^{\tau_i}\right)^{\frac{1}{c}} \rangle \leq \bigcup_{\substack{e_{M_i} \in \pi_{M_i} \\ f_{M_i} \in \psi_{M_i}}} \langle \left(1 - \prod_{i=1}^{k} \left(1 - \left(e_{M_i}\right)^c\right)^{\tau_i}\right)^{\frac{1}{c}} \rangle , \prod_{i=1}^{k} \left(f_{L_i}\right)^{\tau_i} \leq \prod_{i=1}^{k} \left(f_{M_i}\right)^{\tau_i}$$

Therefore,

$$c, d - RHFWA(L_1, L_2, \ldots, L_k) = \bigcup_{\substack{e_{L_i} \in \pi_{L_i} \\ f_{L_i} \in \psi_{L_i}}} \langle \left(1 - \prod_{i=1}^{k} \left(1 - \left(e_{L_i}\right)^c\right)^{\tau_i}\right)^{\frac{1}{c}}, \prod_{i=1}^{k} \left(f_{L_i}\right)^{\tau_i} \rangle$$

$$\leq \bigcup_{\substack{e_{M_i} \in \pi_{M_i} \\ f_{m_i} \in \psi_{M_i}}} \langle \left(1 - \prod_{i=1}^{k} \left(1 - \left(e_{M_i}\right)^c\right)^{\tau_i}\right)^{\frac{1}{c}}, \prod_{i=1}^{k} \left(f_{M_i}\right)^{\tau_i} \rangle =\leq c, d - RHFWA(M_1, M_2, \ldots, M_k)$$

2. The proof is similar to (1). □

The important function for ranking two c,d-rung hesitant fuzzy sets is known as score function and accuracy function. Here, we will introduce these functions.

Definition 4.6. Let $L = (\pi_L, \psi_L)$ a c,d-rung orthopair hesitant fuzzy numbers. Then

1. SF of L is defined as follows:

$$H(L) = \frac{S(\pi_L) - S(\psi_L)}{2}$$

2. AF of L is defined as follows:

$$A(L) = \frac{S(\pi_L) + S(\psi_L)}{2}$$

Where

$$S(\pi_L) = \frac{\sum_{i=1}^{k} e_{L\,i}^{c}}{k}$$

and

$$S(\psi_L) = \frac{\sum_{i=1}^{k} f_{L\,i}^{c}}{k}$$

Remark 4.7. Let $L = (\pi_L, \psi_L)$ a c,d-rung orthopair hesitant fuzzy number. Then it is suggested that,

1. SF $H(L) \in [-1,1]$
2. AF $A(L) \in [0,1]$

Note 4.8. Let $L_1 = (\pi_{L_1}, \psi_{L_1})$and $L_2 = (\pi_{L_2}, \psi_{L_2})$ be two c,d-rung orthopair hesitant fuzzy numbers. Then comparison techniques supposed as,

1. If $H(L_1) < H(L_2)$, then $L_1 \prec L_2$,
2. If $H(L_1) > H(L_2)$, then $L_1 \succ L_2$,
3. If $H(L_1) = H(L_2)$, then

(a) If $A(L_1) < H(L_2)$, then $L_1 \prec L_2$,
(b) If $A(L_1) > A(L_2)$, then $L_1 \succ L_2$,
(c) If $A(L_1) = A(L_2)$, then $L_1 \approx L_2$.

5 Decision making on c,d-rung orthopair hesitant fuzzy sets

This section includes the establishment of a model to use the proposed operators for MCDM under c,d-RHFNs. For a MCDM problem, assume that $L = \{L_1, L_2, \ldots, L_m\}$ is a finite set of alternatives and $Q = \{Q_1, Q_2, \ldots, Q_k\}$ is a set of criteria. Let $B = [L_{ij}] = [\pi_{L_{ij}}, \psi_{L_{ij}}]_{m \times k}$ be a decision matrix be provided by decision makers. A set of weight vector $\tau = (\tau_1, \tau_2, \dots, \tau_k)^T$ with $\tau_i > 0$ such that $\sum_{i=1}^{k} \tau_i = 1$ then, the model (Algorithm 1) of managing the MCDM troubles as follows:

Algorithm 1

1. We will establish a decision matrix based on c,d-rung orthopair hesitant fuzzy numbers $\boldsymbol{B} = [\boldsymbol{L_{ij}}]$ for MCDM.

2. Create a normalized c,d-rung orthopair hesitant fuzzy numbers decision matrix $\boldsymbol{B} = [\boldsymbol{L_{ij}}]$ from c,d-rung orthopair hesitant fuzzy numbers

3. Calculate the alternatives values $\boldsymbol{L_{ij}}$ by using the set of weight vector $\boldsymbol{\tau} = (\boldsymbol{\tau_1}, \boldsymbol{\tau_2}, \dots, \boldsymbol{\tau_k})^T$ and the averaging and geometric aggregation operaotrs discussed in Section 4.

$$\boldsymbol{L_j = c, d - RHFWA(L_{j1}, L_{j2}, \dots, L_{jk}) = \bigcup_{\substack{e_{L_i} \in \pi_{L_i} \\ f_{L_i} \in \psi_{L_i}}} \langle \left(1 - \prod_{i=1}^{k} \left(1 - (e_{L_i})^c\right)^{\tau_i}\right)^{\frac{1}{c}}, \prod_{i=1}^{k} (f_{L_i})^{\tau_i} \rangle}$$

or

$$\boldsymbol{L_j = c, d - RHFWG(L_{j1}, L_{j2}, \dots, L_{jk}) = \bigcup_{\substack{e_{L_i} \in \pi_{L_i} \\ f_{L_i} \in \psi_{L_i}}} \langle \prod_{i=1}^{k} (e_{L_i})^{\tau_i}, \left(1 - \prod_{i=1}^{k} \left(1 - (f_{L_i})^c\right)^{\tau_i}\right)^{\frac{1}{c}} \rangle}$$

For all $\boldsymbol{j = 1, 2, \dots, m}$.

4.Calculate the score results for all c,d-rung orthopair hesitant fuzzy numbers of $\boldsymbol{L_j}$ obtained from Step 3.

5. The best option can be found by obtaining from the comparing techniques using score values and accuracy values.

6 Case Study via c,d-rung orthopair hesitant aggregation operators

In a rapidly evolving global marketplace, a multinational manufacturing company, NB sons Ltd, is committed to enhancing its sustainability practices throughout its supply chain. The company recognizes that achieving sustainability goals requires making strategic decisions that balance economic, environmental, and social factors. NB Sons Ltd has adopted a c,d-rung orthopair hesitant Information System to evaluate potential solutions for optimizing its supply chain sustainability.

Background:

NB Sons Ltd operates in the electronics industry, producing consumer devices. Their supply chain consists of numerous suppliers, transportation networks, and manufacturing facilities distributed across various countries. They aim to reduce their environmental footprint, improve working conditions, and maintain cost-effectiveness. Four alternative strategies have been identified for supply chain optimization, each with four attributes:

Alternatives and Criteria for MCDM:

A set of alternatives $L = \{L_1, L_2, L_3, L_4\}$ and $Q = \{Q_1, Q_2, Q_3, Q_4\}$ are described in our scenario is, $\boldsymbol{L_1}$ **(Local Sourcing):** Emphasizing local suppliers and shortening transportation distances. $\boldsymbol{Q_1}$ Cost Efficiency: Lower transportation costs. $\boldsymbol{Q_2}$ Environmental Impact**:** Reduced carbon emissions due to shorter distances. $\boldsymbol{Q_3}$ Social Responsibility**:** Support for local economies and labor conditions. $\boldsymbol{Q_4}$ Product Quality: Product Quality should be attractive. $\boldsymbol{L_2}$ **(Global Sourcing):** Seeking suppliers from low-cost regions for cost savings.$\boldsymbol{Q_1}$ Cost Efficiency: Lower procurement costs. $\boldsymbol{Q_2}$ Environmental Impact: Increased transportation-related emissions. $\boldsymbol{Q_3}$Social Responsibility**:** Ethical concerns related to labor practices abroad. $\boldsymbol{Q_4}$ Product Quality: Product Quality should be attractive. $\boldsymbol{L_3}$ **(Green Logistics):** Investing in eco-friendly transportation and warehousing.$\boldsymbol{Q_1}$ Cost Efficiency: Higher initial investment but potential long-term savings. $\boldsymbol{Q_2}$Environmental Impact: Reduced emissions from sustainable logistics. $\boldsymbol{Q_3}$Social Responsibility**:** Improved supply chain sustainability practices. $\boldsymbol{Q_4}$ Product Quality: Product Quality should be attractive.

$\boldsymbol{L_4}$ **Supplier Collaboration:** Partnering closely with suppliers for sustainable practices. $\boldsymbol{Q_1}$Cost Efficiency: Potential for cost savings through collaborative efforts. $\boldsymbol{Q_2}$Environmental Impact: Reduction in the overall supply chain's carbon footprint. $\boldsymbol{Q_3}$Social Responsibility: Enhanced labor conditions and ethical sourcing. $\boldsymbol{Q_4}$ Product Quality: Product Quality should be attractive.

Objective: NB Sons Ltd aims to select the supply chain strategy that best aligns with its sustainability objectives. The decision-making process involves evaluating the four alternatives based on the three attributes: Cost Efficiency, Environmental Impact, and Social Responsibility. However, the decision-makers recognize that they have bipolar hesitant information, meaning they may have conflicting feelings or uncertainties regarding each attribute's importance and performance for each alternative.

The MCDM matrix is given in the form of Table 2 based on the c,d-rung orthopair hesitant information.

Table 2: c,d-RHF information

Alternatives	Q1	Q2	Q3	Q4
$\boldsymbol{L_1}$	$\{0.7,0.3\},\{0.6,0.5\}$	$\{0.5,0.4\},\{0.8,0.4\}$	$\{0.6,0.5\},\{0.7,0.4\}$	$\{0.2,0.1\},\{0.6,0.2\}$
$\boldsymbol{L_2}$	$\{0.5,0.2\},\{0.7,0.6\}$	$\{0.3,0.1\},\{0.6,0.4\}$	$\{0.6,0.3\},\{0.7,0.1\}$	$\{0.3,0.2\},\{0.8,0.3\}$
$\boldsymbol{L_3}$	$\{0.6,0.3\},\{0.6,0.4\}$	$\{0.7,0.1\},\{0.7,0.2\}$	$\{0.5,0.2\},\{0.8,0.3\}$	$\{0.4,0.4\},\{0.9,0.4\}$
$\boldsymbol{L_4}$	$\{0.8,0.5\},\{0.6,0.3\}$	$\{0.5,0.4\},\{0.8,0.5\}$	$\{0.4,0.1\},\{0.7,0.4\}$	$\{0.6,0.2\},\{0.5,0.3\}$

To aggregate the information given in Table 2, proposed aggregation operators are used where $c = 3\ and\ d = 1$ and weights for each attribute is taken as $\tau_1 = 0.1, \tau_1 = 0.2, \tau_1 = 0.2$ and $\tau_1 = 0.5$

$$c,d - RHFWA(L_{j1}, L_{j2}, \dots, L_{j4}) = \bigcup_{\substack{e_{L_i} \in \pi_{L_i} \\ f_{L_i} \in \psi_{L_i}}} \langle \left(1 - \prod_{i=1}^{4} \left(1 - \left(e_{L_i}\right)^c\right)^{\tau_i}\right)^{\frac{1}{c}}, \prod_{i=1}^{4} \left(f_{L_i}\right)^{\tau_i} \rangle$$

or

$$L_j = c,d - RHFWG(L_{j1}, L_{j2}, \dots, L_{j4}) = \bigcup_{\substack{e_{L_i} \in \pi_{L_i} \\ f_{L_i} \in \psi_{L_i}}} \langle \prod_{i=1}^{4} \left(e_{L_i}\right)^{\tau_i}, \left(1 - \prod_{i=1}^{4} \left(1 - \left(f_{L_i}\right)^c\right)^{\tau_i}\right)^{\frac{1}{c}} \rangle$$

After applying these aggregation operators, we obtain the calculated values shown in Table 3

Table 3: Aggregated results of c,d-RHF information

Alternatives	c,d-RFWA	c,d-RHFWG
$\boldsymbol{L_1}$	$\{0.0432,0.0106\},\{0.6454,0.2957\}$	$\{0.3444,0.1930\},\{0.2211,0.0610\}$
$\boldsymbol{L_2}$	$\{0.0225,0.0028\},\{0.7256,0.3270\}$	$\{0.3561,0.1813\},\{0.2758,0.0791\}$
$\boldsymbol{L_3}$	$\{0.0546,0.0129\},\{0.7799,0.3383\}$	$\{0.4961,0.2670\},\{0.3427,0.0651\}$
$\boldsymbol{L_4}$	$\{0.0860,0.0143\},\{0.5891,0.3419\}$	$\{0.5874,0.2574\},\{0.1981,0.0657\}$

By applying the score function for c,d-RHF information, we have results in Table 4

Table 4: Score values on different c,d parameters

Alternatives	Score (3,2-RFWA)	Score (3,2-RHFWG)	Score(1,1-RFWA)	Score(1,1-RFWG)
$\boldsymbol{L_1}$	-0.1259	-0.0011	-0.1771	0.0101
$\boldsymbol{L_2}$	-0.1583	-0.0078	-0.2154	-0.00542
$\boldsymbol{L_3}$	-0.1806	0.0048	-0.2098	0.0434
$\boldsymbol{L_4}$	-0.1158	0.0443	-0.1556	0.0893

Ranking results based on the score values presented in Table 4 are displayed in Table 5

Table 5: Ranking of alternatives derived from score values

Alternatives	Ranking	Best
3,2-RHFWA	$L4 > \mathrm{L1} > \mathrm{L3} > \mathrm{L2}$	$L4$
3,2-RHFWG	$L4 > \mathrm{L3} > \mathrm{L1} > \mathrm{L2}$	$L4$
1,1-RHFWA	$L4 > \mathrm{L1} > \mathrm{L3} > \mathrm{L2}$	$L4$
1,1-RHFWG	$L4 > \mathrm{L3} > \mathrm{L1} > \mathrm{L2}$	$L4$

This ranking shows that the alternative $\boldsymbol{L_4}$ **Supplier Collaboration:** Partnering closely with suppliers for sustainable practices is the best strategy identified for supply chain optimization.

7 Comparative analysis

In this section, we will compare the established approach with existing techniques and analyze the difference between these models. The comparison outcomes are presented in Table 6.

Table 6: Comparative analysis

Approaches	Alternatives	Ranking	Best
Proposed	2,3-RHFWA	$L4 > \mathrm{L2} > \mathrm{L1} > \mathrm{L3} > \mathrm{L5}$	$L4$
	3,2-RHFWA	$L4 > \mathrm{L2} > \mathrm{L1} > \mathrm{L3} > \mathrm{L5}$	$L4$
Ibrahim et al [38]	2,3-RFWA	$L4 > \mathrm{L2} > \mathrm{L1} > \mathrm{L3} > \mathrm{L5}$	$L4$
	3,2-RFWA	$L4 > \mathrm{L2} > \mathrm{L1} > \mathrm{L3} > \mathrm{L5}$	$L4$
Mahmood et al [40]	1,1-RFWA(IFWA)	$L4 > \mathrm{L2} > \mathrm{L1} > \mathrm{L3} > \mathrm{L5}$	$L4$
Khan et al [42]	2,2-RFWA(PFWA)	$L4 > \mathrm{L2} > \mathrm{L1} > \mathrm{L3} > \mathrm{L5}$	$L4$
Krisci [43]	3,3-RFWA(FFWA)	$L4 > \mathrm{L2} > \mathrm{L1} > \mathrm{L3} > \mathrm{L5}$	$L4$

It is evident that when we modified the c,d parameters of the proposed c,d-RHFS model, the results consistently aligned with those of existing approaches. The proposed model demonstrated compatibility with the outcomes achieved using n,m-rung orthopair fuzzy sets as presented by Ibrahim [38]. Similarly, our method exhibited promising results when compared to Intuitionistic hesitant fuzzy sets [40], Pythagorean hesitant fuzzy sets [42], and Fermatean hesitant fuzzy sets [43]. Through this comparison, we have discovered some significant characteristics of our suggested approach. In the subsequent section, we will examine in a detailed discussion the advantages of this methodology.

7.1 Benefits and limitations of the proposed technique

The benefits of the proposed approach are discussed as follows,

i. The proposed approach is a more generalized structure, Figure 2 shows this generic structure.
ii. By taking singleton elements in MG and NMG in c,d-RHFS then this is converted into n,m-rung orthopair fuzzy sets [38].
iii. By taking c=d, c,d-RHFS is converted into Q-RHFS [44].
iv. By taking c=d=3, c,d-RHFS is converted into Fermatean hesitant fuzzy set[43].
v. By taking c=d=2, c,d-RHFS is converted into Pythagorean hesitant fuzzy set[42].
vi. By taking c=d=1, c,d-RHFS is converted into intuitionistic hesitant fuzzy set[40].
vii. The proposed approach facilitates the selection process within a multi-attribute decision-making model.
viii. This approach can be expanded to accommodate other decision-making processes such as MULTIMORA, TOPSIS, and VIKOR models.

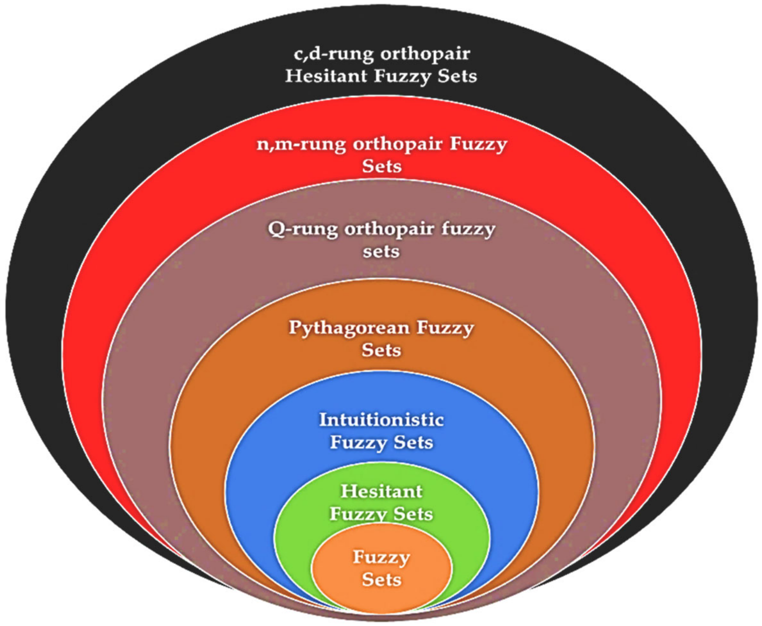

Figure 2: Generalizations of Fuzzy Sets

8 Conclusion

In this research article, we have explored the dimensions of Q-RHFSs, which encapsulate membership and non-membership grades within the [0,1] interval for each element in a given universe. The evolution of Q-RHFSs has led to the development of n,m-rung orthopair fuzzy sets, delineating a more expansive version of the original concept. Our research progresses this idea by amalgamating it with a hesitant fuzzy model, thereby forging the innovative concept of the c,d-rung orthopair hesitant fuzzy model. This innovative model is particularly suited for effectively managing scenarios laden with uncertainty.

Our study rigorously confirms that the proposed c,d-rung orthopair hesitant fuzzy model aligns seamlessly with the core principles and operational mechanisms fundamental to fuzzy set theory. We have innovatively designed a series of power averaging and geometric aggregation operators, offering an exhaustive elucidation of their roles in the calculation of fuzzy information. Further, we have applied this model to tackle a critical global challenge: the development of sustainable supply chain systems. This application focuses on the strategic selection process for corporations, considering a multitude of attributes. To facilitate this complex decision-making process, we have devised a tailored multiple-attribute decision-making model, which is attuned to the nuances of the c,d-rung orthopair hesitant fuzzy information.

Our contribution to the academic field is twofold. Firstly, we conduct a comprehensive comparative analysis with existing models, thereby underscoring the distinctive advantages of our innovative approach. Secondly, the integration of hesitant fuzzy modeling into the c,d-orthopair fuzzy sets framework markedly improves our capacity to make well-informed decisions in environments characterized by significant uncertainty.

Looking forward, we are poised to implement our methodology within the dynamic spheres of machine learning and artificial intelligence. We will further refine and elaborate on existing models by incorporating a variety of aggregation operators and undertaking comparative analyses. Additionally, we anticipate expanding our model to include the methodologies discussed in references [56-58]. This expansion will enable us to thoroughly assess the applicability and efficacy of our methods across an array of techniques and domains, thereby enriching the landscape of fuzzy set theory and its applications.

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