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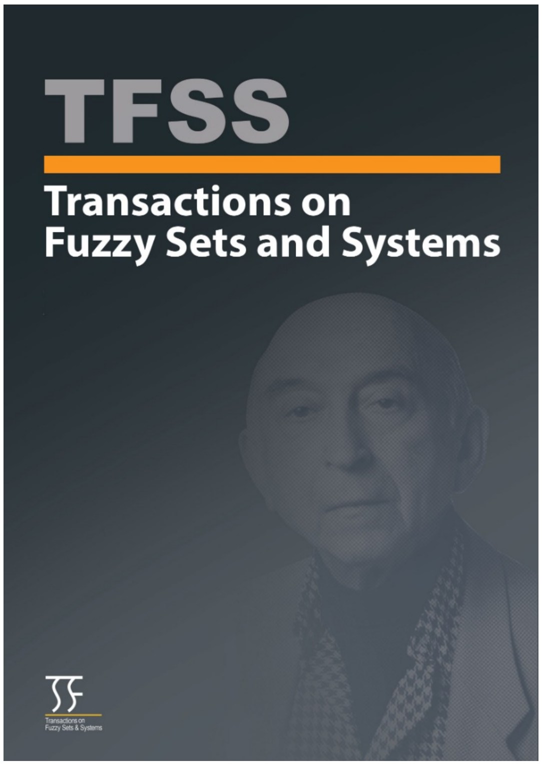

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A Journey from Traditional to Fuzzy Methods of Decision-Making

Michael Gr. Voskoglou*

Abstract. Decision-Making (DM) is one of the most important components of human cognition. Starting with a review of the traditional criteria for DM, this work presents also a method for the verification of a decision, a step of the DM process which, due to its special interest, is usually examined separately from its other steps. Frequently in everyday life, however, the data of a DM problem are vague and characterized by uncertainty. In such cases the traditional techniques for DM, which are based on principles of the bivalent logic (yes-no), cannot help effectively in making the right decision. The first who introduced principles of the fuzzy sets theory in DM were Bellman and Zadeh in 1970 and an example is given here illustrating their fuzzy criterion for DM. Also, among the several fuzzy methods proposed later by other researchers for a more effective DM, a hybrid method is developed here for parametric multiple-criteria DM using soft sets and grey numbers (or intuitionistic fuzzy sets, or neutrosophic sets) as tools, which improves an earlier method proposed by Maji et al. in 2002. All the DM approaches presented in this paper are illustrated with everyday practical examples.

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1 Introduction

Decision-making (DM), one of the most important components of human cognition, is the process of choosing a solution between two or more alternatives for the purpose of achieving the optimal result for a given problem. Obviously DM has sense if, and only if, there exists more than one feasible solution, together with one or more suitable criteria helping the decision maker to choose the best among these solutions. We recall that a solution is characterized as feasible, if it satisfies all the restrictions imposed onto the real system by the statement of the problem as well as all the natural restrictions imposed onto the problem by the real system; e.g. if x denotes the quantity of stock of a product, we must have $x \geq 0$. The choice of the suitable criterion, especially when the results of DM are affected by random events, depends upon the desired goals of the decision maker; e.g. optimistic or conservative criterion, etc.

The rapid technological progress, the impressive development of transportation means, the globalization of human society, the continuous changes appearing in the local and international economies, and other related reasons, led during the last 60-70 years to a continuously increasing complexity of the problems of our everyday life. As a result the DM process became in many cases a very difficult task, which is impossible to be based on the decision makers experience, intuition and skills only, as it usually happened in the past. Thus, from the beginning of 1950 a progressive development started of a systematic methodology for the DM

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process, termed *Statistical Decision Theory*, which is based on principles of Probability Theory, Statistics, Economics, Psychology and of other related scientific sectors [1].

The DM process involves the following steps:

• $\mathbf{d_1}$: *Analysis* of the decision problem, i.e. understanding, simplifying and reformulating the problem in a form permitting the application of the standard DM techniques it.

• $\mathbf{d_2}$: *Collection* and *interpretation* of all the necessary information related to the problem.

• $\mathbf{d_3}$: *Determination* of all the feasible solutions.

• $\mathbf{d_4}$: *Choice* of the best solution in terms of the suitable, according to the decision-makers goals, criterion (-ia).

One could add one more step to the DM process, the *verification* of the chosen decision according to the results obtained by applying it in practice. However, this step is extended to areas which, due to their depth and importance, have become autonomous. Therefore, it is usually examined separately from the other steps of the DM process.

Note that the first three steps of the DM process are continuous, in the sense that the completion of each one of them usually needs some time, during which the decision- maker's reasoning is characterized by transitions between hierarchically neighbouring steps. In other words, the DM process, the flow diagram of which is represented in Figure 1, cannot be characterized as a linear process.

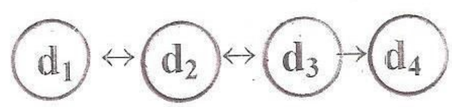

Figure 1: The flow diagram of the DM process

For facilitating the DM process, at the step of analysis a decision problem is usually represented by a *decision matrix*, otherwise termed as the *matrix of the pay-offs*. Each row of this matrix corresponds to an event and each column of it corresponds to a decision. The events are all the possible outcomes of the corresponding DM problem, whereas the entries of the matrix correspond to the results of each decision (pay-offs). Mathematically speaking, in a DM problem with n events and m possible decisions the decision matrix is an n x m matrix of the form $[a_{ij}]$, where a_{ij} denotes the pay-off corresponding to the event E_i and the decision D_j. Table 1, for example, represents the decision matrix of the classical DM problem of the judge.

Table 1: Decision matrix of the DM problem of the judge

Events	**Decisions of the judge**	
	INNOCENT	GUILTY
INNOCENT	An innocent is decided to be innocent	An innocent is decided to be guilty
GUILTY	A guilty is decided to be innocent	A guilty is decided to be guilty

An alternative way to represent a DM problem is the use of a *decision tree*, which has the form of a logical diagram. The decision tree of the DM problem of the judge, for example, is shown in Figure 2.

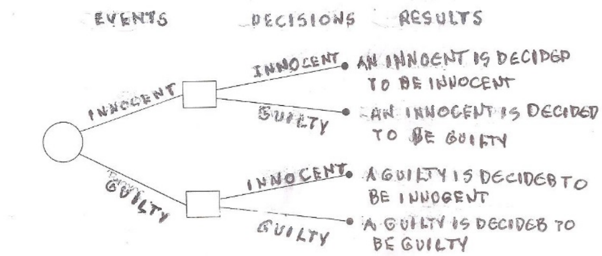

Figure 2: The decision tree of the DM problem of the judge

The use of a decision tree is usually preferred in the case of composite and complicated DM problems.

In this review paper, starting from the traditional criteria for DM, based on principles of the bivalent logic [2, 3], we also present a method for studying the verification of a decision, based on the calculation of the GPA index. Next the criterion of Bellman and Zadeh is presented for DM under fuzzy conditions [4] and a parametric method for multiple criteria DM is developed [5–8], which improves an earlier DM of Maji et al. [9] using soft sets as tools.

2 Traditional Criteria for Decision-Making

According to the existing information, a decision is made under conditions of *certainty, risk, uncertainty* or *complete ignorance*. In the first case the DM is obviously an easy task, whereas the complete lack of information is something that happens very seldom. Uncertainty in the field of Management is understood to be a situation in which all the possible outcomes of future action are known, but not the probabilities of the appearance of each outcome. On the contrary, in a situation of risk both the outcomes of an action and the probabilities of them to happen are known. The turn of a coin, for example, is a situation of risk, whereas the color of the first car that will pass in front of an observer is a situation of uncertainty.

As already mentioned in the previous section, a necessary condition for the DM is the existence of at least one suitable criterion helping the decision-maker to make the right decision. When the pay-offs are numerical quantities, the most commonly used decision criteria among those reported in the literature [2, 3], are the following:

- ***Maximization of the minimal pay-offs (maxi min pay-offs)***

Using this criterion, the decision-maker considers the minimal pay-offs corresponding to each possible decision and chooses the maximal among them. This criterion, otherwise known as the *criterion of Wald*, is based on the law of Murphy, according to which the worst that could happen will happen. It is, therefore, a conservative criterion, which is frequently used when the decision-maker knows that he/she has no chance to make a wrong estimation. On the other end, the *maximization of the maximal pay-offs (maxi max pay-offs)* is a super optimistic criterion, which is used very rarely, because it involves a great risk.

- ***Minimization of the maximal lost opportunities (mini max lost opportunities)***

The *lost opportunity* x_{ij} is defined to be the difference of the maximal pay-off corresponding to the event E_i, minus the pay-off a_{ij}, for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. To apply this criterion, one forms the $n\ x\ m$ *matrix of the lost opportunities* $\{x_{ij}\}$ and chooses the column (and therefore the decision) corresponding to the minimal among the maximal lost opportunities. This criterion, also known as the *regret criterion* because of the decision-makers disappointment with the lost opportunities, is more optimistic than the criterion of Wald.

- ***Maximization of the expected pay-offs***

Let p_i be the probability of appearance of the event E_i, $i = 1, 2, \ldots, n$, then the *expected pay-off* a_j corresponding to the possible decision j, $j = 1, 2, \ldots, m$, is defined by

$$a_j = \sum_{i=1}^{n} p_i a_{ij}. \tag{1}$$

According to this criterion, which obviously can be applied when the decision is made under conditions of risk, the right decision corresponds to the $\max(a_1, a_2, \ldots, a_m)$. For applying this criterion under conditions of uncertainty, i.e. when the probabilities p_i are not known, one may assume that all of them are equal to each other, a simplification which is not always true in practice. In this case equation (1) takes the form

$$a_j = \frac{1}{n} \sum_{i=1}^{n} a_{ij}. \tag{2}$$

In this form the criterion is known as the *criterion of Laplace*.

- ***Minimization of the expected lost opportunities***

Under conditions of risk, if p_i denotes the probability of realization of the event E_i, $i = 1, 2, \ldots, n$, the expected lost opportunity x_j corresponding to the possible decision D_j, $j = 1, 2, \ldots, m$, is defined by

$$x_j = \sum_{i=1}^{n} p_i x_{ij}. \tag{3}$$

According to this criterion, the right decision corresponds to the $\min(x_1, x_2, \ldots, x_m)$. In case of uncertainty one may set again $p_i = \frac{1}{n}$.

Remark 2.1. On the basis of the definition of the lost opportunities it becomes evident that the criteria of the maximization of the expected pay-offs and of the minimization of the expected lost opportunities are equivalent, leading always to the same decision, e.g. see below the case (iii) of Example 2.2.

- ***Criterion of optimism - pessimism***

In this criterion an *optimism* index q_j is assigned to the maximal pay-off, say t_j, of each decision D_j, $j = 1, 2, \ldots, m$. Also the *pessimism index* $1q_j$ is assigned to the minimal pay-off, say s_j, of the same decision. The index q_j either depends on the personal goals of the decision-maker, or it is determined with the help of existing statistical data. Then the expected pay-off a_j of the possible decision j, $j = 1, \ldots, m$, is calculated by the formula

$$a_j = q_j t_j + (1 - q_j) s_j, \tag{4}$$

and the right decision corresponds to the $\max(a_1, a_2, \ldots, a_m)$. This criterion is also referred to as the criterion of Hurwicz.

Example 2.2. The management of an industry must choose the optimal among three methods, say A_1, A_2, A_3, for the production of a good, which will be put on sale at a price of 100 euros per unit. The application of A_1 requires an initial capital of one million euros for buying and setting the necessary equipment, plus 50 euros per unit for the production expenses. The corresponding amounts of money are 1.6 million, 40 euros for A_2 and 3 million, 30 euros for A_3 respectively. The markets research has shown that the probability for a low demand of the good (25000 units) is 10%, for a mediocre demand (100000 units) is 70% and for a high demand (150000 units) is 20%. Further, the optimistic indices for each method of production were estimated to be $q_1 = q_2 = 0.6$ and $q_3 = 0.8$ respectively.

Find which the optimal choice for the industry is by applying the criteria:

i) Maxi min pay-offs,

ii) Mini max lost opportunities,

iii) Maximization of the expected pay-offs or minimization of the expected lost opportunities.

iv) Optimism pessimism.

Solution: Denote by E_1, E_2 and E_3 the events of low, mediocre and high demand of the good respectively.

i) The pay-offs a_{ij} are equal to the revenue from the sale of the good minus the initial capital and the expenses for the production of the good. In case of the event E_1 and the method A_2, for example, one finds that $a_{12} = 25000.100 - (1600000 + 25000.40) = -100000$ euros. The matrix of pay-offs (in thousands of euros) is the following:

$$\begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \end{array} \begin{array}{c} \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ \begin{bmatrix} 250 & -100 & -1250 \\ 4000 & 4400 & 4000 \\ 6500 & 7400 & 7500 \end{bmatrix} \end{array}$$

The minimal pay-offs corresponding to each method of production are 250, -100 and -1250 respectively and the maximal pay-off among them is 250. Therefore, the industry must choose the method A_1.

ii) With the help of the matrix of pay-offs one calculates the lost opportunities; for example, $x_{32} = 7.500 - 7.400 = 100$, $x_{33} = 7.500 - 7.500 = 0$, The matrix of the lost opportunities is, therefore, the following:

$$\begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \end{array} \begin{array}{c} \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ \begin{bmatrix} 0 & 350 & 1500 \\ 400 & 0 & 400 \\ 1000 & 100 & 0 \end{bmatrix} \end{array}$$

The maximal lost opportunities for each method of production are 1000, 350 and 1500, with $\min(1.000, 350, 1500) = 350$. Therefore, the industry must choose the method A_2. This decision is more optimistic than the decision made with the help of the previous criterion, since it corresponds to a maximum possible pay-off of 7.400.000 euros, in comparison to the 6.500.000 euros corresponding to the previous decision.

iii) From the problems data it turns out that $p_1 = 0.1$, $p_2 = 0.7$ and $p_3 = 0.2$. Therefore, equation (1) gives that the expected payoffs for each decision are $a_1 = (0.1).250 + (0.7).4000 + (0.2).6500 = 4125$ and similarly $a_2 = 4450$, $a_3 = 4175$. Therefore, since $\max(4125, 4450, 4175) = 4450$, the industry must choose the method A_2.

Also, with the help of equation (3) one finds that the expected lost opportunities for each method of production are $x_1 = 0.(0.1) + 400.(0.7) + 1000.(0.2) = 480$ and similarly $x_2 = 55$ and $x_3 = 430$. Therefore, since $\min(x_1, x_2, x_3) = 55$, the industry must choose again the method A_2 (see Remark 2.1).

iv) The maximal pay-off of the method A_1 is 6500 and the minimal is 250. Therefore, equation (4) gives that $a_1 = (0.6).6500 + (1 - 0.6).250 = 4000$ and similarly $a_2 = 4400$, $a_3 = 3500$. Therefore, since $\max(a_1, a_2, a_3) = 4400$, the industry must choose the method A_2.

3 Verification of a Decision

As it was already mentioned, the *verification* of a decision is a step of the DM process, which is usually examined separately from its other steps. A method will be presented here for investigating this important step of the DM process by using the *Grade Point Average (GPA)* index.

It is recalled that the GPA index is a weighted mean which is frequently used for assessing a groups quality performance (since greater coefficients are assigned to the higher grades) during a certain activity. For this, consider the qualitative grades $A =$ excellent, $B =$ very good, $C =$ good, $D =$ satisfactory and $F =$ unsatisfactory (failed). Then the GPA index is calculated by the formula

$$\text{GPA} = \frac{0n_F + n_D + 2n_C + 3n_B + 4n_A}{n}. \tag{5}$$

In formula (5) n denotes the total number of the groups members and nA, nB, nC, nD and nF denote the numbers of the groups members that demonstrated excellent, very good, good, satisfactory and unsatisfactory performance respectively [10, Chapter 6]. In case of the worst performance ($nF = n$), formula (5) gives that GPA $= 0$, whereas in case of the ideal performance ($nA = n$) it gives that GPA $= 4$. Therefore, we have in general that $0 \leq$ GPA ≤ 4.

Our method is illustrated with the help of the following example:

Example 3.1. The car industry circulates a new car in the market in two different types, the luxury (L) Class and the regular (R) Class. Six months after the purchase with their cars, the customers were asked to complete a written questionnaire concerning the degree of their satisfaction for their cars. Their answers were divided by the industrys marketing department into the following five categories: $A =$ Fully satisfied customers, $B =$ Very well satisfied customers, $C =$ Satisfied customers, $D =$ Rather satisfied customers and $E =$ Unsatisfied customers. The data collected from the customers answers are depicted in Table 2. What is the general conclusion obtained by the car industry concerning the degree of satisfaction of its customers for their new cars?

Table 2: Questionnaires data

Customers Categories	L Class	R Class
A	60	60
B	30	90
C	30	45
D	30	45
E	20	15
Total	170	255

Solution: Replacing the data of Table 2 to formula (5) one finds that the GPA index concerning the degree of satisfaction of the owners of the L Class and the R Class is equal to $\frac{42}{17} \approx 2.47$ and $\frac{43}{17} \approx 2.529$ respectively. Taking into account that $0 \leq$ GPA ≤ 4, this means that the owners were satisfied with their cars at a percentage of $\frac{2.47x100}{4} \approx 61.75\%$ for the L Class and $\frac{2.529x100}{4} \approx 63.22\%$ for the R Class.

4 Criterion of Bellman and Zadeh for Decision-Making under Fuzzy Conditions

Frequently in everyday life the data of a DM problem are fuzzy; e.g. when a company wants to employ as a sales manager a well-experienced person whose residence is not very far from the companys place. In such cases the traditional techniques of DM, which are based on principles of bivalent logic (yes-no), cannot help effectively in making the right decision. On the contrary, *fuzzy sets (FSs)* and their extensions, due to their nature of including multiple values, offer a rich field of resources for this purpose; e.g. see [4, 11–18], etc.

It is recalled that Zadeh in 1965 extended the concept of the crisp set to that of a FS by replacing the characteristic with the membership function as follows [19]:

Definition 4.1. A FS, say A, in the universal set of the discourse U is of the form $A = \{(x, m(x)) : x \in U\}$, where $m : U \to [0, 1]$ is its membership function. The value $m(x)$ is called the membership degree of x in A, for all x in U. The closer $m(x)$ to 1, the better x satisfies the characteristic property of A.

For example, if A is the FS of the high mountains and $m(x) = 0.7$, then x is a rather high mountain, if $m(x) = 0.4$, then x is a rather low mountain, etc.
Bellman and Zadeh were the first, in 1970, who applied principles of FS theory to DM, their method being known as *Criterion of Bellman and Zadeh for DM* [4].

A DM problem under fuzzy conditions is characterized by its *fuzzy goal* (G) and by the *fuzzy constraints* C_i, $i = 1, 2, \ldots, n$, where n is a positive integer. The steps of the method of Bellman and Zadeh are the following:

- *Choice of the universal set of the discourse U*
- *Fuzzification of the decision problems data*

In this step the fuzzy goal G and the fuzzy constraints C_i are expressed as fuzzy sets (FSs) in U by defining properly the corresponding membership functions m_G and m_{C_i}.

- *Evaluation of the fuzzy data*

The *fuzzy decision* F, expressed as a fuzzy set in U, is equal to the intersection of the FSs G and C_i of U Therefore, the membership function m_F of F is defined by

$$m_F(x) = m_G \cap m_{C_1} \cap \ldots \cap m_{C_i}(x) = \min\{m_G(x), m_{C_1}(x), \ldots, m_{C_2}(x)\}, \tag{6}$$

for all x in U.

- *Defuzzification*

The solution of the problem corresponds to the element x of U having the highest membership degree in F.

The following example illustrates the DM model of Bellman and Zadeh in practice:

Example 4.2. A company is willing to employ as a sales manager the candidate with the best qualifications (G), provided that his/her salary demand is not very high (C_1) and that his/her residence is in a close distance from the companys central offices (C_2). There are four candidates for this position, say , , C and D, with annual salary demands of 29050, 25000, 14050, and 6250 euros respectively. Who of them is the best choice for the company under the fuzzy constraints C_1 and C_2?

Solution: In this problem the universal set of the discourse is the set $U = \{A, B, C, D\}$ of the four candidates. In order to express the fuzzy goal and the fuzzy constraints as FSs in U, one must properly define the corresponding membership functions.

For example, having in mind that there is not any general criterion available for the definition the membership functions, the membership function $m_{C_1} : U \to [0, 1]$ of the fuzzy constraint C_1, may be defined by $m_{C_1} = 1$ for $s(x) < 6000$, $m_{C_1}(x) = 1 - 2x10^{-5}xs(x)$ for $6000 \leq s(x) \leq 30000$ and $m_{C_1}(x) = 0$ for $s(x) > 30000$, where $s(x)$ denotes the salary of the candidate x, for all x in U. Then $m_{C_1}(A) = 1 - 2x0.2905 = 0.419$ and similary $m_{C_1}(B) = 0.5$, $m_{C_1}(C) = 0.719$ and $m_{C_1}(D) = 0.875$. Consequently, the constraint C_1 can be written as a FS in U in the form $C_1 = \{(A, 0.419), (B, 0.5), (C, 0.719), (D, 0.875)\}$.

Assume further that in an analogous way the fuzzy goal G and the fuzzy constraint C_2 were expressed as fuzzy sets in U in the form $G = \{(A, 0.9), (B, 0.6), (C, 0.8), (D, 0.6)\}$ and $C_2 = \{(A, 0.1), (B, 0.9), (C, 0.7), (D, 1)\}$ respectively. Therefore, with the help of equation (6) it is straightforward to check that F can be written as a FS in U in the form $F = \{(A, 0.1), (B, 0.5), (C, 0.7), (D, 0.6)\}$. The highest membership degree in F is 0.7 and corresponds to C. Therefore the candidate C is the best choice for the company.

The fuzzy model of Bellman and Zadeh can be suitably modified to accommodate the relative importance that could exist for the goal and constraints by using *weighting coefficients*, whose sum is always equal to 1. The following example illustrates this case:

Example 4.3. Revisit Example 4.2 and assume that the management of the company, taking into account the existing companys budget and the results of the oral interviews of the four candidates, decided to attach weights $w = 0.5$, $w_2 = 0.2$ and $w_3 = 0.3$ to the goal G and to the constraints C_1 and C_2 respectively. Which will be the best companys choice under these new conditions?

Solution: In this case the membership function of the fuzzy decision F is defined as a linear combination of the weighted goal and constraints of the form

$$m_F(x) = w_1 x m_G(x) + w_2 x m_{C_1}(x) + w_3 x m_{C_2}(x), \tag{7}$$

where $m_G(x)$, $m_{C_1}(x)$, $m_{C_2}(x)$ are the membership degrees in G, C_1 and C_2 respectively of each x in U (see Example 4.2) and the coefficients w_1, w_2 and w_3 are the weights attached to the fuzzy goal and to the fuzzy constraints C_1 and C_2 respectively. Therefore, the membership degree of candidate A in the fuzzy decision F is equal to $m_F(A) = 0.5x0.9 + 0.2x0.419 + 0.3x0.1 = 0.638$. In the same way one finds that $m_F(B) = 0.67$, $m_F(C) = 0.7538$ and $m_F(D) = 0.775$. Therefore, candidate D is the companys best choice in this case.

5 MultipleCriteria Parametric Decision-Making

Following the criterion of Bellman and Zadeh, several other methods were proposed by other researchers for DM in fuzzy environments; e.g. [11–18], etc. Here we will present a hybrid, parametric, multiple-criteria DM method using soft sets, grey numbers and intuitionistic fuzzy sets as tools.

5.1 Decision-Making with Soft Sets

Molodstov introduced in 1999 the concept of *soft set (SS)* for tackling the uncertainty in a parametric manner, not needing, therefore, the definition of a membership function. Namely, a SS is defined as follows [20]:

Definition 5.1. Let E be a set of parameters, let A be a subset of E, and let f be a map from A into the power set $P(U)$ of the universe U. Then the SS (f, A) in U is defined as the set of the ordered pairs $(f, A) = \{(e, f(e)) : e \in A\}$. In other words, an SS is a parametric family of subsets of U. The term "soft" was introduced due to the fact that the form of (f, A) depends on the parameters of A. A FS in U with membership function $y = m(x)$ is a SS in U of the form $(f, [0, 1])$, where $f(a) = \{x \in U : m(x) \geq a\}$ is the corresponding a-cut of the FS, for each a in $[0, 1]$. Consequently the concept of SS is a generalization of the concept of FS. Most notions and operations defined on FSs are extended in a natural way to SSs.

Maji et al. [9] utilized the tabular form of a SS as a tool for DM in a parametric manner. Here this method is illustrated with the following example:

Example 5.2. Let $V = \{H_!, H_2, H_3, H_4, H_5, H_6\}$ be a set of houses and let $Q = \{e_1, e_2, e_3, e_4\}$ be the set of the parameters e_1 = beautiful, e_2 = wooden, e_3 = in the country and e_4 = cheap. Assume that H_1, H_2, H_6 are beautiful, H_2, H_3, H_5, H_6 are wooden, H_3, H_5 are the houses in the country and H_4 is the unique cheap house. Assume further that one is interested in buying a beautiful, wooden and cheap house in the country choosing among the previous six houses. Which is the best choice for the candidate buyer?

Solution: Consider the map $g : Q \to P(V)$ defined by $g(e_1) = \{H_1, H_2, H_6\}$, $g(e_2) = \{H_2, H_3, H_5, H_6\}$, $g(e_3) = \{H_3, H_5\}$ and $g(e_4) = \{H_4\}$ and the

$$\text{SS } (g, Q) = \{(e_1, \{H_1, H_2, H_6\}), (e_2, \{H_2, H_3, H_5, H_6\}), (e_3, \{H_3, H_5\}), (e_4, \{H_4\})\}.$$

The tabular representation of the SS (g, Q), which is shown in Table 3, is formed by assigning the binary elements 1, 0 to each of the houses having (not having) the property described by the corresponding parameter

The *choice value* of each house is calculated by adding the binary elements of the corresponding row of the tabular matrix containing it. The houses H_1 and H_4, therefore, have choice value 1 and all the other houses have choice value 2. Consequently, the buyer must choose one of the houses H_2, H_3, H_5 or H_6.

Table 3: Tabular form of the soft set (g, Q)

	e_1	e_2	e_3	e_4
H_1	1	0	0	0
H_2	1	1	0	0
H_3	0	1	1	0
H_4	0	0	0	1
H_5	0	1	1	0
H_6	1	1	0	0

5.2 Decision-Making Using Grey Numbers in the Decision Matrix

The decision of Example 5.2 was not very helpful for the candidate buyer, since it excluded only two among the six available for sale houses. This gave us the hint to modify the DM method of Maji et al. by using *grey numbers (GNs)* in the tabular form of the corresponding *SS* [6].

Definition 5.3. A GN is understood to be a real number with known boundaries whose exact value is unknown. A GN, say G, is represented with the help of a closed real interval. Namely, we write $G \in [a, b]$, with a, b in the set $\mathbf{R}$ of real numbers. Frequently, however, G is accompanied by a *whitenization function* $g : [a, b] \to [0, 1]$, such that the closer $g(x)$ to 1, the more x approximates the exact value of G, for all x in $[a, b]$.

It is recalled that GNs are used as tools for performing all the necessary calculations in the theory of *grey systems* introduced by Deng in 1982 [21] as an alternative to Zadehs FSs for tackling the existing in real world uncertainty. The known arithmetic of the real intervals [22] is used for performing the arithmetic operations between GNs. Here we will make use of the *addition* of GNs and the *scalar multiplication* of a positive number with a GN, which are defined as follows:

Definition 5.4. Let $G_1 \in [a_1, b_1]$, $G_2 \in [a_2, b_2]$ be two given GNs and let k be a positive number. Then the sum $G_1 + G_2 \in [a_1 + a_2, b_1 + b_2]$ and the scalar product $kG_1 \in [ka_1, kb_1]$. When no whitenization function is assigned to $G \in [a, b]$, then the real number

$$W(G) = \frac{a + b}{2}, \tag{8}$$

is used for approximating the unknown exact value of G.

Revisiting now Example 5.2 one observes that the parameters e_1 and e_4 do have not a bivalent texture. In fact, how beautiful a house is depends on the subjective criteria of each observer, whereas its low or high price depends on the financial ability of the candidate buyer. For this reason, the characterization of the parameters e_1 and e_4 in Table 3 by using the binary elements 0, 1 is not the suitable one. One way to tackle this problem, is to replace the binary elements 0, 1 corresponding to the parameters e_1 and e_5 with GNs. This is illustrated with the following example.

Example 5.5. Revisit Example 5.2 and assume that the candidate buyer, after studying more carefully the existing information about the six available houses, decided to use Table 4 instead of Table 3 for making the final decision, where $G_1 \in [0.85, 1]$, $G_2 \in [0.6, 0.74]$, $G_3 \in [0.5, 0.59]$ and $G_4 \in [0, 0.49]$ are the GNs replacing the binary elements 0, 1 in the columns of e_1 and e_3. Which will be the optimal decision in this case?

Table 4: Revised tabular form of the soft set (g, Q) using grey numbers

	e_1	e_2	e_3	e_5
H_1	G_1	0	0	G_2
H_2	G_1	1	0	G_4
H_3	G_2	1	1	G_2
H_4	G_3	0	0	G_1
H_5	G_4	1	1	G_3
H_6	G_1	1	0	G_4

Solution: In Table 4 one calculates the choice value V_i of the house H_i, $i = 1, 2, 3, 4, 5, 6$ with the help of Definition 5.3 and equation (8) as follows: $V_1 = W(G_1 + G_2) = W([1.45, 1.74]) = \frac{1.45 + 1.74}{2} = 1.595$ and similarly $V_2 = 1 + W(G_1 + G_4) = 2.17$, $V_3 = 2 + W(G_2 + G_2) = 3.34$, $V_4 = W(G_3 + G_1) = 1.47$, $V_5 = 2 + W(G_4 + G_3) = 3.215$, $V_6 = 1 + W(G_1 + G_4) = 2.47$. Therefore, the optimal decision is to buy the house H_3. A second way for tackling this problem is to use *triangular fuzzy numbers (TFNs)* instead of GNs [5]. These two methods are equivalent, providing always the same outcomes.

5.3 Decision-Making Using Intuitionistic Fuzzy Pairs in the Decision Matrix

As we have seen in the previous example, the use of the GNs instead of the binary elements 0, 1 for characterizing the fuzzy parameters that exist in the tabular decision matrix, helps the decision-maker to make a better decision. DM situations, however, appear frequently in everyday life, in which the decision-maker is not sure about the accuracy of these characterizations. In such cases, one way to perform the DM process is to use intuitionistic fuzzy pairs instead of GNs in the tabular matrix of the corresponding soft set [8].

It is recalled that Atanassov in 1986, in order to tackle more effectively the existing in real life uncertainty, added to Zadehs membership degree the degree of non-membership and extended the concept of FS to the concept of *intuitionistic FS (IFS)* as follows [23]:

Definition 5.6. An IFS, say A, in the universe U is of the form $A = \{(x, m(x), n(x)) : x \in U, 0 \leq m(x) + n(x) \leq 1\}$, where $m : U \to [0, 1]$ and $n : U \to [0, 1]$ are its membership and non-membership functions of A respectively.

For example, if A is the set of the high mountains and $m(x) = 0.6$, $n(x) = 0.2$, then there is a 60% belief that x is a high mountain, but at the same time there is a 20% belief that I is not a high mountain. For brevity an IFS is denoted here by $A = \langle n, m \rangle$ and its elements are written in the form of *intuitionistic fuzzy pairs (IFPs)* (m, n), with $m + n \leq 1$. For the needs of the present work we define the addition of $IFPs$ and the scalar multiplication of a positive number with an IFP in the same way as for the ordinary ordered pairs, i.e. as follows:

Definition 5.7. Let $A = \langle m, n \rangle$ be an IFS, let (m_1, n_1), (m_2, n_2) be elements of A and let k be a positive number. Then:

i) The sum $(m_1, n_1) + (m_2, n_2) = (m_1 + m_2, n_1 + n_2)$

ii) ii) The scalar product $k(m_1, n_1) = (km_1, kn_1)$.

It becomes evident that the above defined sum and the scalar product are not closed operations in A, since it can be either $(m_1 + m_2) + (n_1 + n_2) > 1$ or (and) $km_1 + kn_1 > 1$.

We also define the *mean value* of a finite number of IFPS of A in the following way:

Definition 5.8. Let $A = \langle m, n\rangle$ be an IFS and let $(m_1, n_1), (m_2, n_2), \ldots, (m_k, n_k)$ be a finite number of elements of A. Then the mean value of these elements is defined to be $(m, n) = \frac{1}{k}[(m_1, n_1) + (m_2, n_2) + \ldots + (m_k, n_k)]$. It becomes evident that (m, n) is always an element of A.

The use of IFPs in the decision matrix will be illustrated with the following example:

Example 5.9. A company wants to employ a person among six candidates, say A_1, A_2, A_3, A_4, A_5 and A_6. The ideal qualifications for the new employee are to have satisfactory previous experience (p_1), to hold a university degree (p_2), to have a driving license (p_3) and to be young (p_4). Assume that A_2, A_3, A_5, A_6 are the holders of a university degree and that A_3, A_5 are the holders of a driving license. Assume further that the company has difficulty assigning accurate characterizations to the six candidates with respect to the fuzzy parameters p_1 and p_4. It was decided, therefore, to use $IFPs$ instead of the binary elements 0, 1 in the tabular decision matrix. For this, the analysts of the company considered the $IFSs$ of the candidates with satisfactory previous experience and of the young candidates, as well as the trivial IFSs of the holders of a university degree and of a driving license and represented their elements in the form of IFPs. As a result the tabular decision matrix took the form of Table 5.

Table 5: Tabular representation of the DM process using IFPs

	p_1	p_2	p_3	p_4
A_1	$(1,0)$	$(0,1)$	$(0,1)$	$(0.6,0.1)$
A_2	$(1,0)$	$(1,0)$	$(0,1)$	$(0.2,0.6)$
A_3	$(0.5,0.1)$	$(1,0)$	$(1,0)$	$(0.6,0.2)$
A_4	$(0.5,0.3)$	$(0,1)$	$(0,1)$	$(1,0)$
A_5	$(0.5,0.4)$	$(1,0)$	$(1,0)$	$(0.6,0.1)$
A_6	$(1,0)$	$(1,0)$	$(0,1)$	$(0.4,0.2)$

Which will be the best choice for the company?

Solution: In this case the choice value of each candidate A_i, $i = 1, 2, 3, 4, 5, 6$, is equal the *mean value* of the IFPs contained in the row of A_i. With the help of Definitions 5.7 and 5.8, therefore, one finds that the choice value of A_1 is equal to

$$\frac{1}{4}[(1,0) + 2(0,1) + (0.6,0.1)] = \frac{1}{4}(1.6, 2.1) = (0.4, 0.525).$$

In the same way the choice values of A_2, A_3, A_4, A_5 and A_6 can be find to be equal to $(0.55, 0.4)$, $(0.775, 0.075)$, $(0.375, 0.575)$, $(0.775, 0.125)$ and $(0.6, 0.3)$ respectively. The company now may use either an optimistic criterion by choosing the candidate with the greatest membership degree, or a conservative criterion by choosing the candidate with the lower non-membership degree, i.e. one of the candidates A_3 and A_5 in the first case, or the candidate A_3 in the second case. A combination of the two criteria leads finally to the choice of the candidate A_3.

5.4 Decision-Making Using Neutrosophic Triplets in the Decision Matrix

An alternative way for tackling the previous DM problem is to use *neutrosophic sets (NSs)* instead of $IFSs$ writing their elements in the form of *neutrosophic triplets (NTs)* in the tabular decision matrix [7]. In fact, Smarandache in 1995, inspired by the frequently appearing in the everyday life neutralities, like

$\langle tall, medium, short\rangle$, $\langle friend, neutral, enemy\rangle$,$\langle win, draw, defeat\rangle$, etc., introduced the degree of indeterminacy or neutrality and extended the notion of IFS to the notion of NS [24]. The simplest form a NS, known as a *single-valued NS (SVNS)* is defined in the following way [25]:

Definition 5.10. A SVNS, say A, in the universe U has the form

$$A = \{(x, m(x), i(x), n(x)) \; : \; x \in U, \; m(x), i(x), n(x) \in [0, 1], \; 0 \leq m(x) + i(x) + n(x) \leq 3\}.$$

In the SVNS A $m(x)$ is the degree of membership (or truth), $i(x)$ is the degree of indeterminacy (or neutrality) and $n(x)$ is the degree of non-membership (or falsity) of x in A, for all x in U. When $0 \leq m(x) + i(x) + n(x) \leq 1$, then the data about x in A are characterized by incomplete information, when $m(x)+i(x)+n(x) = 1$ by complete and when $m(x)+i(x)+n(x) > 1$ by inconsistent (contradiction relevant) information. A NS may contain simultaneously elements characterized by all these types of information. For brevity we write $A = \langle m, i, n\rangle$ and the elements of A as NTs in the form (m, i, n), with $0 \leq m + i + n \leq 3$. For example, if A is the NS of the high mountains and $(0.6, 0.3, 0.2) \in A$, then there exists a 60% belief that x is a high mountain, but at the same time a 30% belief that x is neither a high nor a low mountain and a 20% belief that it is a low mountain.
The *sum* of NTSs, the *scalar product* of a positive number cross a NT and the mean value of a finite number of NTs of a NS are defined similarly with the corresponding operations for $IFPs$ (see Definitions 5.7 and 5.8). The advantage of using NTs instead of IFPs in the decision matrix is that they enable one to handle data connected incomplete and/or inconsistent information. The following example illustrates this situation.

Example 5.11. Revisiting Example 5.9 assume that the company, due to the existence of incomplete and inconsistent information for some candidates, decided to use NSs instead of IFSs for the formation of the decision matrix. Thus, considering the NSs of the candidates with satisfactory previous experience and of the young candidates, as well as the trivial NSs of the holders of a university degree and of the holders of a driving license and representing their elements in the form of NTs formed the decision matrix shown in Table 6. Which is the best choice for the company in this case?

Table 6: Tabular representation of the DM process using NTs

	p_1	p_2	p_3	p_4
A_1	$(1, 0, 0)$	$(0, 0, 1)$	$(0, 0, 1)$	$(0.6, 0.3, 0.1)$
A_2	$(1, 0, 0)$	$(1, 0, 0)$	$(0, 0, 1)$	$(0.2, 0.2, 0.7)$
A_3	$(0.5, 0.4, 0.2)$	$(1, 0, 0)$	$(1, 0, 0)$	$(0.6, 0.2, 0.1)$
A_4	$(0.5, 0.2, 0.2)$	$(0, 0, 1)$	$(0, 0, 1)$	$(1, 0, 0)$
A_5	$(0.5, 0.1, 0.4)$	$(1, 0, 0)$	$(1, 0, 0)$	$(0.6, 0.3, 0.1)$
A_6	$(1, 0, 0)$	$(1, 0, 0)$	$(0, 0, 1)$	$(0.4, 0.3, 0.2)$

Solution: In this case the choice value of the candidate A_1 is equal to $\frac{1}{4}[(1, 0, 0)+2(0, 0, 1)+(0.6, 0.3, 0.1)] = \frac{1}{4}(1.6, 0.3, 2.1) = (0.4, 0.075, 0.525)$ and in the same way the choice values of A_2, A_3, A_4, A_5 and A_6 are approximately equal to $(0.55, 0.07, 0.425)$, $(0.775, 0.15, 0.075)$, $(0.375, 0.05, 0.55)$, $(0.775, 0.13, 0.125)$ and $(0.6, 0.075, 0.3)$ respectively. Consequently, applying the optimistic criterion the company must choose one of the candidates A_3 or A_5, whereas applying the conservative criterion it must choose the candidate A_3. The final choice of the company, therefore, must be again the candidate A_3, although the indeterminacy degree of candidate A_5 is slightly smaller $(0.13 < 0.15)$.

5.5 Weighted Parametric Decision-Making

Cases appear frequently in DM in which the decision-makers goals are not equally important. In such cases, weight coefficients, whose sum is equal to 1, are assigned to each parameter. Assume, for instance, that the weight coefficients 0.4, 0.3, 0.2 and 0.1 have been assigned to the parameters p_1, p_2, p_3 and p_4 respectively of Example 5.9. Then the weighted choice value of the candidate A_1 is equal to

$$\frac{1}{4}[0.4(1,0)+0.3(0,1)+0.2(0,1)+0.1(0.6,0.1)]=\frac{1}{4}(0.46,0.51)=(0.115,0.1275).$$

In the same way one finds that the choice values of the candidates A_2, A_3, A_4, A_5 and A_6 are $(0.18, 0.065)$, $(0.19, 0.015)$, $(0.075, 0.115)$, $(0.19, 0.0425)$ and $(0.185, 0.055)$ respectively. The combination of the two criteria, therefore, shows again that the best decision for the company is to employ the candidate A_3.

Remark 5.12. (i) The parametric DM method presented in this work is of a general character, therefore it can be applied to all the analogous cases of multiple-criteria DM. Other examples that have been already presented in earlier works of the present author are related to decisions for buying a car [5], choosing a new player for a football team [7], etc.
(ii) There is no objective criterion for defining the membership function of a FS, its definition depends on the personal criteria of each observer. The same problem exists for all the extensions of FSs involving membership functions and in particular for the membership, non-membership and indeterminacy functions of the IFSs and of the NSs. As a result, the characterization of the fuzzy parameters p_1 and p_4 in Examples 5.9 and 5.11 using IFPs and NTs respectively was purely based on the companys analysts personal criteria. An analogous problem appears when using GNs (or TFNs) for characterizing the fuzzy parameters in the decision matrix (see Section 5.2), although no whitenization function was used for the corresponding GNs. This is, therefore, a general limitation of the parametric DM method presented in this work.

6 Conclusion

Frequently in everyday life the goal and/or the constraints of a DM problem are expressed in a vague way, characterized by uncertainty. The first who studied DM problems under fuzzy conditions were Bellman and Zadeh in 1970. Since then, several DM methods have been proposed by other researchers using FSs or their extensions as tools. In this work, starting from the traditional DM criteria of bivalent logic and continuing with the fuzzy criterion of Bellman and Zadeh, we also presented a hybrid model for multiple-criteria parametric DM in fuzzy environments. This model improves a DM method of Maji et al. using SSs as tools, by replacing the binary elements 0, 1 in the tabular matrix of the corresponding SS either with GNs (or TFNs), or by IFPs, or by NTs, depending on the form of the corresponding DM problem. In addition, a method was presented, based on the calculation of the GPA index, for the verification of a decision, a step of the DM process, which, due to its special importance, is usually examined separately from its other steps. All the DM methods presented in this work are illustrated by suitable examples, connected to everyday life situations. It seems that suitable combinations of two or more theories related to FS (e.g. SSs with GNs or with IFSs or with NSs in this work) provide better results than each one of these theories alone does. This is, therefore, a promising area for further research.

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