Combination of a modified fuzzy network DEA approach with ML algorithms and its application in the automobile manufacturing industry

Abstract

This paper proposes, for the first time, the integration of a new network Data Envelopment Analysis (DEA) approach with Artificial Intelligence (AI) techniques to predict the efficiency score of multi-process production and service systems with imprecise data. In many cases, accurate and sufficient information is not available, and the data is imprecise. Structural complexity and imprecise data lead to a large volume of variables and constraints in models that assess the performance of systems. The use of AI techniques, such as Machine Learning (ML) algorithms, is impressive for accurately predicting the performance scores of network systems in fuzzy data environments. The findings indicate that of the three AI algorithms examined- Logistic Regression (LR), Random Forest (RF), and Decision Tree (DT)- the RF algorithm has almost the highest accuracy in predicting interval efficiency scores and is utilized to predict the efficiency score of new DMUs.

Keywords: Data envelopment analysis (DEA), Performance evaluation, Fuzzy Network DEA, Machin Learning (ML) algorithms, After-sales services.

Subject classification code: C44, C61

1. Introduction

Data Envelopment Analysis (DEA) is a robust performance evaluation method that employs mathematical programming to assess the efficiency of homogenous Decision-Making Units (DMUs) (Charnes et al. 1978). Production and service systems, in many businesses, have a network multi-stage structure, where the outputs of one sub-division are the inputs of another stage. Performance evaluation of multi-stage DMUs, as a network system, involves considering the intermediate activities between sub-processes. Taking into account the inner relationships between components in a network system provide managers with useful information to make effective decisions. Evaluating a multi-stage DMU by considering the interrelation between sub-divisions reveals that the inefficiency of which sub-division causes the overall inefficiency. In many network DEA models, constraints related to intermediate activities are considered as inequalities. In this paper, we propose modified approaches to deal with intermediate activities. Free links between the stages are assumed in this study.

In addition to the structural complexity of DMUs, dealing with imprecise data is another issue in the performance evaluation process. In many real-world situations, there are no clear boundaries for information. To address such vague concepts, fuzzy techniques are employed, through which the quantitative concepts can be incorporated into mathematical models and made applicable for managerial policies. In this paper, we propose fuzzy network DEA models to evaluate the overall efficiency of network systems in imprecise environment.

The complexity of the structure and imprecise data introduce numerous variables and constraints to the mathematical model used for calculating the efficiency score of network systems. Furthermore, to evaluate the performance of a new DMU, the relative efficiency scores of all DMUs must be recalculated. Additionally, in many scenarios, such as medical resource allocations during epidemics, it is crucial to predict the required resources and performance for the future. AI can be employed in conjunction with performance evaluation approaches to tackle the computational burdens and make accurate predictions. Utilizing ML techniques enables more efficient handling of large amounts of data compared to conventional methods (Batta, 2020). By implementing ML methods, the efficiency values of new DMUs can be found without re-measuring the whole DMUs (Zhang et al., 2022).

In this paper, we utilize modified fuzzy network DEA models to calculate efficiency scores, which are then used as training and testing data sets for ML algorithms. Subsequently, the best-performing algorithm is identified and used to predict the efficiency score of new DMUs. We apply these approaches to assess the performance of after-sales services departments in an automobile company. A company offers post-purchase assistance to its customers, which encompasses transportation, installation, and commissioning services, as well as maintenance, supply and distribution of spare parts, instructional documentation for product usage, warranty provisions, and more (Rebelo et al. 2021). High quality after-sales services result in satisfied and loyal customers. Increasing customer retention enhances the company's brand reputation and leads to additional business from word-of-mouth sales, recommendations and customer referrals. The after-sales services department is a fundamental part of a company. A notable proportion of the revenue in the manufacturing industry is related to after-sales service activities. Optimizing the performance of the after-sales service department in a manufacturing business involves providing customers with perfect support, such as on time delivery and warranty (obligations undertaken by the seller), repair or replacement, etc. This results in a high level of customer satisfaction benefiting the seller. The production of goods and the provision of services involve multi-process operations. Evaluating production operations and services to optimally allocate resources, reduce costs, ensure customer satisfaction, and achieve similar objectives is fundamental in management and decision-making. In this study, we consider the level of customer satisfaction and the degree of obligation fulfilment as fuzzy data.

The contributions of this paper are as follows:

1) Instead of considering a single-stage black box structure, which results in missing information from inner processes, here, a network DEA approach has been applied to investigate the effect of each sub-process performance on the overall efficiency score. 2) In the previous network DEA and traditional DEA models for the evaluation of DMUs, the objective is to maximize input contraction or output expansion. Here, our model optimizes the performance of the overall system by both input contraction and output expansion. 3) In many network DEA models, constraints related to intermediate activities are considered as inequalities (Fukuyama, H. & Mirdehghan, 2012, Fukuyama & Matousek, 2017, Shafiei Nikabadi et al., 2017, and Niknafs et al., 2020) which, as will be shown in this paper, may lead to contradictions in optimality. In this paper, we present modified approaches to deal with intermediate activities. 4) We treat the level of customer satisfaction and the grade of obligation fulfilment as fuzzy numbers. By employing fuzzy methods, qualitative information can be incorporated in to the mathematical model allowing for more accurate simulations of real-world situations. This helps managers design effective strategies when dealing with inexact information. In addition, Through the combination of our network approach with the fuzzy method, we obtain a crisp interval efficiency score by which we are able to analyze and interpret the performance of a complex network system with vague and imprecise data. 5) This paper, for the first time, presents a combination of a modified fuzzy network DEA approach with AI techniques to predict the performance of multi-process production and service systems. 6) The network DEA approach proposed in this paper is applicable for performance evaluation of DMUs with interval data.

The next section reviews research studies conducted in network DEA, fuzzy DEA and fuzzy network DEA. Section 3 contains notations and preliminaries necessary to present the fuzzy network DEA method and ML algorithms. Section 4 covers a modified fuzzy network DEA model to evaluate the crisp interval efficiency scores of two-stage DMUs with fuzzy data. The application of the proposed approach in evaluating the performance of 21 after-sales service representatives of an Iranian auto-making company is provided in Section 5. Furthermore, an integrated approach including the combination of proposed fuzzy network DEA models with ML algorithms is developed to predict the efficiency scores of 60 after-sales service representatives.

2. Literature review

As the studies that took into account the internal structure of network systems, (Koushki, 2017) proposed a dynamic DEA network approach to evaluate two-stage structure DMUs where the activity and performance of DMU in one period affected its efficiency in the next. According to the results of the proposed dynamic model, the inefficiencies of DMU's improve considerably. (Galagedera et al., 2018) developed a network DEA model to assess the overall and stage-level performance of the fund management function as a three-stage production process. The stage-level processes operate under environmental conditions and levels of risk exposure, which are treated as conditions imposed on the intermediate measures (products). (Koushki, 2018) presented DEA approaches to achieve the most productivity in two-stage DMUs. (Shahbazifar et al., 2021) introduced network DEA models to assess and rank the groups of production systems utilizing average and minimum performance criteria. (Kooshki and Mashayekhi Nezam Abadi, 2018) proposed a two-stage network DEA model to identify the Most Productive Scale Size (MPSS) pattern in supplier-manufacturer supply chains. (Fukuyama et al., 2023) proposed a dynamic network DEA model to evaluate three-stage DMUs incorporating dual-role data, serving as both the final outputs in one period and the carryover inputs in the next period. They utilized a transformation function to characterize the production technology. (Koushki, 2023) introduced a modified network DEA approach and extended it to centralized method for evaluating units with a multi-stage structure. (Wanke et al., 2023) derived a deterministic linear programming model from a stochastic two-stage network DEA model in the presence of stochastic ratio data. (Emami et al., 2024) explored the issue of constant cost distribution within a particular two-stage system. They conducted a series of allocations based on common weights and size, implementing a min-max strategy to minimize the disparity between efficient allocations and those based on size. (Zheng et al., 2024) applied network DEA to evaluate a two-sided platform operation system by dividing it into two sub-processes: Marketing and Service (MS) sub-process and Value-Creating (VC) sub-process. (Xiao et al., 2024) developed a single-stage hierarchical additive self-evaluation model by integrating the well-established cross-efficiency method. They proposed a combination of a max-min secondary goal model and the Criteria Importance Through Inter-Criteria Correlation (CRITIC) method to expand the basic hierarchical self-evaluation model. (Gerami, 2025) evaluated bank branches as two-stage DMUs using a network DEA approach.

As the studies conducted on fuzzy DEA, (Singh and Pant, 2020) introduced a fuzzy DEA method that integrates fuzzy weights within the objective function and employs alpha cuts to establish the fuzzy interval. (Ebrahimnejad and Amani, 2021) developed two virtual fuzzy DMUs, namely the fuzzy ideal DMU (FIDMU) and the fuzzy anti-ideal DMU (FADMU), while addressing the issue of undesirable outputs. They also put forward a fuzzy ranking algorithm designed to facilitate the comparison and ranking of the fuzzy efficiencies of DMUs. In a separate study, (Lu, 2021) developed a fuzzy network DEA approach for selecting advanced manufacturing technology (AMT) alternatives, taking into account multiple decision-makers (DMs) and weight restrictions. (Yang et al., 2022) extended a model utilizing triangular intuitionistic fuzzy numbers to assess the efficiency scores associated with solid waste recycling. Their approach integrated the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) method alongside the entropy weight method to ascertain the proportion of solid waste recycling. (Zadmirzaei et al., 2024) proposed a fuzzy undesirable slacks-based DEA method aimed at assessing environmental efficiency through the application of the directional distance function and the concept of weak disposability. They also explored various combined methodologies, including Artificial Neural Networks (ANN), ANN enhanced with particle swarm optimization (PSO), and the artificial immune system (AIS), to address inefficiencies. Additionally, (Sahil and Danish Lohani, 2024) advanced the network two-stage DEA framework by integrating undesirable outputs and shared resources within intuitionistic fuzzy contexts, utilizing parabolic intuitionistic fuzzy numbers. (Gholami Golsefid et al., 2024) evaluated the cost and revenue efficiency of two-stage systems under fuzzy data assumption. (Yang and Yang, 2025) developed models for the supplier selection problem in multi-stage supply chains under uncertain conditions.

In research focused on the integration of DEA methods with ML techniques, (Kuo et al., 2010) proposed a model for selecting green suppliers that combines ANN and two multi-attribute decision analysis (MADA) approaches: DEA and analytic network process (ANP). (Vlahogianni et al., 2016) applied DEA alongside neural network regression to evaluate the efficiency of bus depots and routes in the Athens bus system in Greece. (Nandy and Singh, 2020) applied a combined fuzzy DEA-ML method in the agricultural production. (Jomthanachai et al., 2021) applied a combination of the DEA and ML approaches for risk management. They utilized the DEA cross-efficiency method to assess a group of risk factors derived from the Failure Mode and Effects Analysis (FMEA). (Shi and Zhao, 2023) proposed an integrated ML-DEA approach to predict the performance of DMUs. Efficiency scores are measured and labeled as the good, the acceptable, and the underperforming classes utilizing RF, support vector machine, and LR classifiers. Then, Synthetic Minority Oversampling Technique (SMOTE) with Manhattan distance metric is used to solve class imbalance in the labeled, high-dimensional dataset. (Rezaee et al., 2024) employed a combined DEA-ML method to project DMUs onto a benchmark efficiency level by adjusting actionable and feasible inputs and outputs.

Few studies have been conducted on integrating network DEA with ML algorithms. To bridge this gap, this study introduces a combination of a modified fuzzy network DEA with ML algorithms to evaluate the sub divisions and calculate the overall efficiency of DMUs with a network structure.

3. Notations and preliminaries

3.1. Series Two-stage DMU

In a DMU with a two-stage series structure, there are intermediate activities between the stages denoted by vector Z . Stage 1 may have outputs that leave the system, and Stage 2 may have inputs except Z . Many production and service systems have a series structure. The models proposed in this paper can be used to evaluate series two-stage DMUs in general. Here, we will consider an after-sales service department in detail. The after-sales service department is viewed as a two-stage system consisting of reception (Stage 1) and repair (Stage 2) centers. The personnel costs and the degree of obligations fulfilled by the after-sales service team are inputs of stage1. Income from selling spare parts are the outputs of Stage 1. Other outputs of stage1 are cars in need of repairs. The transfer of these cars from Stage 1 to Stage 2 is considered an intermediate activity between the two stages. Cars in need of repairs and labor are inputs of Stage 2, while general income and customer satisfaction are the outputs of Stage 2.

We denote an after-sales service department by DMU . Consider DMU_j $(j=1,..,n)$ with a two-stage structure. In Stage 1, let X_j (costs and obligations fulfilment) be the vector of inputs , Z_j (cars needing repairs) and V_j be the vectors of the outputs (income from selling the spare parts). Z_j and R_j (labor) are the vectors of the inputs of Stage 2, and Y_j (general income and customer satisfaction) is the output vector. A network with two stages connected in a series structure is depicted in Figure1.

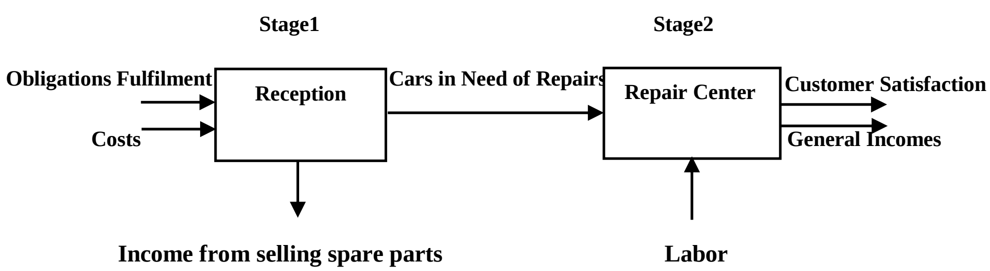

Fig.1. DMU with two-stage structure: after-sales service department

3.2. Fuzzy arithmetic

We treat the level of customer satisfaction and the degree of fulfilment of obligations by the after-sales service department as fuzzy numbers. Fuzzy arithmetic is a suitable method to deal with inexact qualitative data (Zadeh, 1965). Let U represents universe of discourse and $A \subseteq U$. function $\mu_A : U \rightarrow [0,1]$ is called membership function and $\mu_A(x)$ is the grade of membership of x in A. The fuzzy subset of A is defined as $\{(x, \mu_A(x)) \mid x \in U\}$. The $\alpha-$cut set of A is defined as $A^{\alpha} = \{x \mid \mu_A(x) \geq \alpha, \alpha \in [0,1]\}$. The support set of A is defined as $S(A) = \{x \mid \mu_A(x) > 0\}$. A triangular fuzzy number represented as $A = (a,,b,c)$, is a fuzzy number with a membership function as follows:

$$\mu_A(x) = \begin{cases} \dfrac{x-a}{b-a}, & a \leq x \leq b \\ \dfrac{c-x}{c-b}, & b \leq x \leq c \\ 0, & therwise. \end{cases}$$

3.3. An overview of ML

In this paper, we utilize modified fuzzy network DEA models to calculate efficiency scores, which are then used as training and testing data sets for ML algorithms. Subsequently, the best-performing algorithm is identified and used to predict the efficiency score of new DMUs. ML is a robust method for data analysis that combines elements of mathematics, statistics, artificial intelligence, and computer science to extract insights from input data and autonomously generate predictions. This predictive functionality is achieved by examining input data to yield specific outputs through two main learning approaches: supervised learning and unsupervised learning (Sarker, 2021). Supervised learning is especially adept at handling classification and regression tasks, utilizing a range of algorithms, including DT, RF, and LR. DT is a supervised machine learning model that functions as a predictor applicable to both regression and classification tasks, as noted by (Guggari et al., 2018). Typically, decision trees comprise two categories of nodes: internal nodes and leaf nodes. Internal nodes signify decisions made based on specific features, whereas leaf nodes indicate the predicted outcome or class label. The journey from the root node to a leaf node encompasses a series of decisions that culminate in the final prediction (Koulinas et al., 2020). The primary challenges associated with

DTs involve identifying the most effective feature for data partitioning and determining which features should be chosen at each stage throughout the decision tree's operation (Ramos et al., 2022). RF is a supervised ML model known for its ability to deliver highly accurate predictions while minimizing the risk of overfitting. This technique, as indicated by its name, involves the random creation of a collection of decision trees, which are subsequently utilized in conjunction to generate predictions. Each decision tree within the random forest is trained on a randomly selected subset of the training data, as well as a random selection of features. The final output of the random forest is achieved by aggregating the predictions made by all the individual decision trees (Lee and Keck, 2022).LR is powerful supervised ML algorithm frequently employed for predictive tasks or for categorizing data into distinct classes. In its fundamental form, LR utilizes a logistic function, as defined below, to model a binary dependent variable; however, it can be adapted to accommodate multiple classes. This function effectively transforms any real number into a value that lies between 0 and 1. The logistic function can be mathematically expressed as $p(value) = \frac{1}{1+e^{-(w.value+b)}}$,where $value$ represents the input variable, $p(value)$ indicates the predicted probability, w denotes coefficients (weights) vector, and b shows the bias term, commonly known as intercept (Hosmer and Lemeshow, 2000).

4. Performance evaluation

4.1. Network DEA

To measure the efficiency score of DMU_o based on the contraction of input values or the expansion of output values (but not both) without considering intermediate products Z_j , the input-oriented and output-oriented CCR models (Charnes et al., 1978) are defined as models (1) and (2) respectively.

$$
\begin{aligned}
&\theta^* = \min \theta \\
&s.t. \sum_{j=1}^{n} \lambda_j X_j \le \theta X_o \\
&\quad \sum_{j=1}^{n} \lambda_j Y_j \ge Y_o \\
&\quad \lambda \ge 0 \qquad\qquad (1)
\end{aligned}
$$

$$
\begin{aligned}
&\phi^* = \max \phi \\
&s.t. \sum_{j=1}^{n} \lambda_j X_j \le X_o \\
&\quad \sum_{j=1}^{n} \lambda_j Y_j \ge \phi Y_o \\
&\quad \lambda \ge 0 \qquad\qquad (2)
\end{aligned}
$$

These models only consider the overall input and output X_j and Y_j of the system. However, the network structure of DMU, involving the series relationships and connectivity between the stages, should be considered. The intermediate vector Z_j is the output of stage 1 and also is the input of Stage 2. Constraint related to vectors $Z_j \; , j = 1,...,n$ as the outputs of Stage 1 is $\sum_{j=1}^{n} \lambda_j^1 Z_j \ge Z_o$ and as the inputs of Stage 2 is $\sum_{j=1}^{n} \lambda_j^2 Z_j \le Z_o$. Therefore, the connectivity between the stages implies that $\sum_{j=1}^{n} \lambda_j^1 Z_j = \sum_{j=1}^{n} \lambda_j^2 Z_j$ (3).

Many existing models to evaluate DMUs with network structure considered inequalities $\sum_{j=1}^{n} \lambda_j^1 Z_j \ge Z_o$, $\sum_{j=1}^{n} \lambda_j^2 Z_j \le Z_o$ related to the intermediate activities. However, according to these inequalities, in optimality we will have $\sum_{j=1}^{n}\lambda_j^{1^*}Z_j - S^{*+} = Z_o$, $\sum_{j=1}^{n}\lambda_j^{2^*}Z_j + S^{*-} = Z_o$ (4),

where S^{*+}, $S^{*-} \geq 0$. Thus, in optimality if at least one slack variable is non-zero, then equalities

$$\sum_{j=1}^{n}\lambda_j^{1^*}Z_j = Z_o + S^{*+} , \sum_{j=1}^{n}\lambda_j^{2^*}Z_j = Z_o - S^{*-} \quad (5)$$

result in $\sum_{j=1}^{n}\lambda_j^{1}Z_j \neq \sum_{j=1}^{n}\lambda_j^{2}Z_j$ which contradicts the series relation between two stages. The constraint (3) can be rewritten by using free variables as

$$\sum_{j=1}^{n}\lambda_j^{1}Z_j = Z_o + S , \sum_{j=1}^{n}\lambda_j^{2}Z_j = Z_o + S \quad (6),$$

where S is a free in sign vector and the intermediate values are assumed as free links. S represents the reduction or increment of Z_o value needed to obtain the optimal intermediate products. In the previous network DEA and traditional DEA models for evaluation of DMUs, the objective is to maximize input contraction or output expanding. Here, our model optimizes the performance of the overall system by both input-contraction and output-expansion. Assume that the input reduction and output increment factors of Stage 1 are denoted by α_1 and α_2, respectively. Thus, the constraints related to inputs and outputs will be as

$$\sum_{j=1}^{n}\lambda_j^{1}X_j \leq \alpha_1 X_o , \sum_{j=1}^{n}\lambda_j^{1}V_j \geq \alpha_2 V_o \quad (7).$$

Similar assumptions can be considered for the inputs and outputs of Stage 2.

According to the above discussion, our proposed network DEA model to calculate the overall efficiency of a two-stage DMU as the mean of the efficiency scores of the stages, considering intermediate activities within the system, and constraints (6) and (7), is as follows:

$$\begin{aligned}
&\min \ \frac{1}{2}\left(\frac{\alpha_1}{\alpha_2}+\frac{\beta_1}{\beta_2}\right)\\
&s.t. \sum_{j=1}^{n}\lambda_j^{1}X_j \leq \alpha_1 X_o\\
&\sum_{j=1}^{n}\lambda_j^{1}V_j \geq \alpha_2 V_o\\
&\sum_{j=1}^{n}\lambda_j^{1}Z_j = Z_o + S\\
&\sum_{j=1}^{n}\lambda_j^{2}Z_j = Z_o + S\\
&\sum_{j=1}^{n}\lambda_j^{2}R_j \leq \beta_1 R_o\\
&\sum_{j=1}^{n}\lambda_j^{2}Y_j \geq \beta_2 Y_o\\
&\alpha_1 \leq 1, \alpha_2 \geq 1\\
&\beta_1 \leq 1, \beta_2 \geq 1\\
&\lambda_j^1, \lambda_j^2 \geq 0 \qquad j=1,\dots,n \qquad (8)
\end{aligned}$$

A feasible solution for model (8) is given by considering

$$\begin{Bmatrix} \lambda_o^l = 1, \lambda_j^l = 0, l=1,2, j=1,\dots,n, j\neq o, \\ S=0, \alpha_1=\alpha_2=\beta_1=\beta_2=1 \end{Bmatrix}.$$

Thus, in optimality we have $\frac{\alpha_1^*}{\alpha_2^*} \leq 1, \frac{\beta_1^*}{\beta_2^*} \leq 1$.

Definition 1. A DMU_o is efficient if and only if we have $\alpha_1^* = 1, \alpha_2^* = 1, \beta_1^* = 1,$ and $\beta_2^* = 1$ in the optimal solution of model (8).

Theorem1. Let $\{\lambda_j^{l^*}, l=1,2, j=1,\dots,n, S^*, \alpha_1^*, \alpha_2^*, \beta_1^*, \beta_2^*\}$ be an optimal solution of model (8). Any DMU with the data vectors as

$(\beta_1^* X_o, \beta_1^* R_o, Z_o + S^*, \beta_2^* V_o, \beta_2^* Y_o)$ is efficient.

Proof. Assume $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_1$ and $\hat{\beta}_2$ as the optimal values of $\alpha_1, \alpha_2, \beta_1$ and β_2 after solving the following model. We should prove that $\hat{\alpha}_1 = \hat{\alpha}_2 = \hat{\beta}_1 = \hat{\beta}_2 = 1$.

$$
\begin{aligned}
&\min \ \frac{1}{2}\left(\frac{\alpha_1}{\alpha_2} + \frac{\beta_1}{\beta_2}\right) \\
&s.t. \sum_{j=1}^{n} \lambda_j^1 X_j \le \alpha_1 (\alpha_1^* X_o) \\
&\sum_{j=1}^{n} \lambda_j^1 V_j \ge \alpha_2 (\alpha_2^* V_o) \\
&\sum_{j=1}^{n} \lambda_j^1 Z_j = (Z_o + S^*) + S \\
&\sum_{j=1}^{n} \lambda_j^2 Z_j = (Z_o + S^*) + S \\
&\sum_{j=1}^{n} \lambda_j^2 R_j \le \beta_1 (\beta_1^* R_o) \\
&\sum_{j=1}^{n} \lambda_j^2 Y_j \ge \beta_2 (\beta_2^* Y_o) \\
&\alpha_1 \le 1, \alpha_2 \ge 1 \\
&\beta_1 \le 1, \beta_2 \ge 1 \\
&\lambda_j^1, \lambda_j^2 \ge 0 \qquad j = 1,\dots,n \qquad (9)
\end{aligned}
$$

If $\hat{\alpha}_1 < 1$, then from $\hat{\alpha}_2 \ge 1, \hat{\beta}_2 \ge 1, \hat{\beta}_1 \le 1$, we have $\frac{\hat{\alpha}_1 \alpha_1^*}{\hat{\alpha}_2 \alpha_2^*} + \frac{\hat{\beta}_1 \beta_1^*}{\hat{\beta}_2 \beta_2^*} < \frac{\alpha_1^*}{\alpha_2^*} + \frac{\beta_1^*}{\beta_2^*}$, which contradicts the optimality of $\alpha_1^*, \alpha_2^*, \beta_1^*$ and β_2^*. Assuming $\hat{\alpha}_2 > 1$ or $\hat{\beta}_2 > 1$ or $\hat{\beta}_1 < 1$, results in a similar contradiction.■

To obtain the Linear form of model (8), divide the constraints containing λ_j^1 by α_2, and the constraints containing λ_j^2 by β_2. Then, let $\eta_1 = \frac{1}{\alpha_2}$ and $\eta_2 = \frac{1}{\beta_2}$. Therefore, the objective function will be equal to $\frac{1}{2}(\gamma_1 + \gamma_2)$, where $\gamma_1 = \alpha_1 \eta_1$ and $\gamma_2 = \beta_1 \eta_2$. Consequently, the linear form of model (8) is obtained as follows:

$$
\begin{aligned}
&\min \ \frac{1}{2}(\gamma_1 + \gamma_2) \\
&s.t. \sum_{j=1}^{n} \bar{\lambda}_j^1 X_j \le \gamma_1 X_o \\
&\sum_{j=1}^{n} \bar{\lambda}_j^1 V_j \ge V_o \\
&\sum_{j=1}^{n} \bar{\lambda}_j^1 Z_j = \eta_1 Z_o + \bar{S}_1 \\
&\sum_{j=1}^{n} \bar{\lambda}_j^2 Z_j = \eta_2 Z_o + \bar{S}_2 \\
&\sum_{j=1}^{n} \bar{\lambda}_j^2 R_j \le \gamma_2 R_o \\
&\sum_{j=1}^{n} \bar{\lambda}_j^2 Y_j \ge Y_o \\
&\gamma_1 \le \eta_1 \le 1, \gamma_2 \le \eta_2 \le 1 \\
&\bar{\lambda}_j^1, \bar{\lambda}_j^2 \ge 0 \qquad j = 1,\dots,n \qquad (10)
\end{aligned}
$$

Where

$\bar{\lambda}_j^1 = \frac{1}{\alpha_2} \lambda_j^1, \bar{\lambda}_j^2 = \frac{1}{\beta_2} \lambda_j^2, \bar{S}_1 = \frac{1}{\alpha_2} S, \bar{S}_2 = \frac{1}{\beta_2} S, \ j = 1,\dots,n.$

Note that the constraints $\gamma_1 \le \eta_1$ and $\gamma_2 \le \eta_2$ are obtained from $\gamma_1 = \alpha_1 \eta_1$, $\alpha_1 \le 1, \gamma_2 = \beta_1 \eta_2$, and $\beta_1 \le 1$.

According to Definition 1, DMU_o is efficient iff in optimality we have $\gamma_1^* = 1, \eta_1^* = 1, \gamma_2^* = 1$ and $\eta_2^* = 1$.

4.2. Fuzzy network DEA

Assume that the inputs, outputs and intermediate values are triangular fuzzy numbers. I, Q, K, M

and P are the dimensions of the vectors X, Y, Z, V and R, respectively. For $i = 1,...,I$, assume that $\tilde{X}_{ij}$ is the set of i-th input of DMU_j $(j = 1,...,n)$ and consider similar assumptions for $\tilde{Y}_{qj}, \tilde{Z}_{kj}, \tilde{V}_{mj}$ and $\tilde{R}_{pj}$ where $q = 1,...,Q$, $k = 1,...,K$, $m = 1,...,M$ and $p = 1,...,P$. For $\alpha \in (0,1]$, let $(\tilde{X}_{ij})_\alpha^L = \min\{x_{ij} \in S(\tilde{X}_{ij}) \mid \mu_{\tilde{X}_{ij}}(x_{ij}) \geq \alpha\}$ and $(\tilde{X}_{ij})_\alpha^U = \max\{x_{ij} \in S(\tilde{X}_{ij}) \mid \mu_{\tilde{X}_{ij}}(x_{ij}) \geq \alpha\}$. Hence, we obtain a crisp interval $[(\tilde{X}_{ij})_\alpha^L, (\tilde{X}_{ij})_\alpha^U] =$
$[\alpha X_{ij}^m + (1-\alpha) X_{ij}^l, \alpha X_{ij}^m + (1-\alpha) X_{ij}^u]$

for a fuzzy input value. Crisp intervals related to $\tilde{Y}_{qj}, \tilde{Z}_{kj}, \tilde{V}_{mj}$ and $\tilde{R}_{pj}$ can be obtained by similar methods. According to crisp intervals obtained for fuzzy data, the pessimistic and optimistic values of efficiency scores of DMU_j $(j = 1,...,n)$ can be calculated.

In the optimistic viewpoint: 1) The inputs of under-evaluation, DMU_o, have the lower bound values, while the input values of other DMUs are in upper bound. 2) The outputs of under-evaluation, DMU_o, have the upper bound values while, the output values of other DMUs are in the lower bound.

In the pessimistic viewpoint, the levels of inputs and outputs are now adjusted unfavorably for the under-evaluation DMU and in favor of the other units. It is notable that since intermediate products are produced by one stage and used by another stage, they should not be considered as inputs or outputs. Therefore, in both pessimistic and optimistic situations, we consider the same values, such as the middle value, for intermediate products. The crisp upper and lower bounds of the interval efficiency of DMU_o, based on model (10), are determined from the pessimistic and optimistic viewpoints as follows:

$$
\begin{aligned}
\gamma_\alpha^L = \min \ & \frac{1}{2}(\gamma_{1\alpha}^L + \gamma_{2\alpha}^L) \\
s.t. & \sum_{j \neq o} \tilde{\lambda}_j^1 (\tilde{X}_{ij})_\alpha^L + \tilde{\lambda}_o^1 (\tilde{X}_{io})_\alpha^U \leq \gamma_{1\alpha}^L (\tilde{X}_{io})_\alpha^U \qquad i = 1,...,I \\
& \sum_{j \neq o} \tilde{\lambda}_j^1 (\tilde{V}_{mj})_\alpha^U + \tilde{\lambda}_o^1 (\tilde{V}_{mo})_\alpha^L \geq (\tilde{V}_{mo})^L \qquad m = 1,...,M \\
& \sum_{j=1}^{n} \tilde{\lambda}_j^1 \tilde{Z}_j = \eta_1 \tilde{Z}_o + \tilde{S}_1 \\
& \sum_{j=1}^{n} \tilde{\lambda}_j^2 \tilde{Z}_j = \eta_2 \tilde{Z}_o + \tilde{S}_2 \\
& \sum_{j \neq o} \tilde{\lambda}_j^2 (\tilde{R}_{pj})_\alpha^L + \tilde{\lambda}_j^2 (\tilde{R}_{po})_\alpha^U \leq \gamma_{2\alpha}^L (\tilde{R}_{po})_\alpha^U \qquad p = 1,...,P \\
& \sum_{j \neq 0} \tilde{\lambda}_j^2 (\tilde{Y}_{qj})_\alpha^U + \tilde{\lambda}_j^2 (\tilde{Y}_{qo})_\alpha^L \geq (\tilde{Y}_{qo})_\alpha^L \qquad q = 1,...,Q \\
& \tilde{\lambda}_j^1, \tilde{\lambda}_j^2 \geq 0 \qquad j = 1,...,n \\
& \gamma_{1\alpha}^L \leq \eta_1 \leq 1, \ \gamma_{2\alpha}^L \leq \eta_2 \leq 1
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
\gamma_\alpha^U = \min \ & \frac{1}{2}(\gamma_{1\alpha}^U + \gamma_{2\alpha}^U) \\
s.t. & \sum_{j \neq o} \hat{\lambda}_j^1 (\tilde{X}_{ij})_\alpha^U + \hat{\lambda}_o^1 (\tilde{X}_{io})_\alpha^L \leq \gamma_{1\alpha}^U (\tilde{X}_{io})_\alpha^L \qquad i = 1,...,I \\
& \sum_{j \neq o} \hat{\lambda}_j^1 (\tilde{V}_{mj})_\alpha^L + \hat{\lambda}_o^1 (\tilde{V}_{mo})_\alpha^U \geq (\tilde{V}_{mo})^U \qquad m = 1,...,M \\
& \sum_{j=1}^{n} \hat{\lambda}_j^1 \tilde{Z}_j = \eta_3 \tilde{Z}_o + \hat{S}_1 \\
& \sum_{j=1}^{n} \hat{\lambda}_j^2 \tilde{Z}_j = \eta_4 \tilde{Z}_o + \hat{S}_2 \\
& \sum_{j \neq o} \hat{\lambda}_j^2 (\tilde{R}_{pj})_\alpha^U + \hat{\lambda}_j^2 (\tilde{R}_{po})_\alpha^L \leq \gamma_{2\alpha}^U (\tilde{R}_{po})_\alpha^L \qquad p = 1,...,P \\
& \sum_{j \neq 0} \hat{\lambda}_j^2 (\tilde{Y}_{qj})_\alpha^L + \hat{\lambda}_j^2 (\tilde{Y}_{qo})_\alpha^U \geq (\tilde{Y}_{qo})_\alpha^U \qquad q = 1,...,Q \\
& \hat{\lambda}_j^1, \hat{\lambda}_j^2 \geq 0 \qquad j = 1,...,n \\
& \gamma_{1\alpha}^U \leq \eta_3 \leq 1, \ \gamma_{2\alpha}^U \leq \eta_4 \leq 1
\end{aligned}
\tag{12}
$$

Where

$$\tilde{z}_{kj} = \frac{(\tilde{Z}_{kj})^L_\alpha + (\tilde{Z}_{kj})^U_\alpha}{2} \quad k = 1,\ldots,K \ , \ j = 1,\ldots,n \quad (13)$$

.

The symbols L and U represent the lower and upper bounds of interval data, respectively. After solving models (11) and (12) the interval efficiency of DMU_o is obtained as $[\gamma^L_\alpha, \gamma^U_\alpha]$. A feasible solution for model (11) is given by considering $\left\{\begin{matrix} \tilde{\lambda}^l_o = 1, \tilde{\lambda}^l_j = 0, l = 1,2, j = 1,\ldots,n, j \neq o, \\ \tilde{S}_1 = \tilde{S}_2 = 0, \gamma_{1\alpha}^{\ L} = \gamma_{2\alpha}^{\ L} = 1, \eta_1 = \eta_2 = 1 \end{matrix}\right\}$.

A feasible solution for model (12) is obtained by similar consideration.

According to the data interval formulation, the $\alpha -$cut method provides the efficiency score range of a DMU for individual possibility levels separately as the data intervals change for different α levels. Specifically, the minimum possible value of efficiency scores is provided at $\alpha = 0$ when $\gamma^* = \min \{\gamma^L_{\alpha\, o} \mid o = 1,\ldots,n\}$, where $\gamma^L_{\alpha\, o}$ is the optimal efficiency score of DMU_o in pessimistic viewpoint. This means that for any $\alpha \in [0,1]$, the efficiency score of no DMU in pessimistic viewpoint falls below γ^*. Models (11) and (12) can be used to calculate interval efficiency when the data is not fuzzy numbers but is interval.

5. Numerical results and interpretations

5.1. Analysis of efficiency scores measured using the novel DEA approach and comparison with traditional DEA results

We apply our approaches to assess 21 after-sales service departments located in a province in Iran in the year 2023. These departments are associated with an Iranian automobile factory. Each department is considered a two-stage DMU. In $DMU_j \ \ (j = 1,\ldots,n)$, inputs of Stage 1, are denoted by x_{1j} and x_{2j}, where x_{1j} represents the total costs. The degree of obligation fulfillment undertaken by the department, denoted by x_{2j}, is considered a triangular fuzzy number. The vectors V_j, Z_j and R_j are defined in Section 3.1. General income and level of customer satisfaction (considered as triangular fuzzy number) are the outputs of stage2 denoted by y_{1j} and y_{2j}, respectively. The costs and income values are in millions of dollars.

Table 1. Data

DMU	x_{j1}	x_{j2}	R_j	V_j	y_{j1}	y_{j2}	Z_j
1	1.21	(3,5,7)	56	3.28	1.33	(2,5,9)	2100
2	1.05	(3,5,8)	41	3.60	.99	(5,6,7)	1954
3	.63	(4,5,8)	37	1.40	.54	(4,5,8)	1258
4	.83	(5,7,9)	45	1.69	.57	(4,6,9)	1823
5	.32	(3,7,9)	20	.85	.44	(3,5,8)	984
6	.94	(3,5,8)	34	1.90	.61	(3,4,6)	1205
7	.71	(5,8,9)	37	1.33	.51	(4,7,9)	1360
8	.19	(4,7,9)	18	.44	.19	(2,4,8)	871
9	.95	(3,4,8)	51	1.05	.67	(3,5,7)	1900
10	1.39	(3,5,7)	70	3.89	1.54	(3,6,9)	2410
11	.30	(3,4,7)	22	.78	.43	(3,7,9)	934
12	.55	(3,8,9)	24	.92	.68	(3,5,9)	895
13	.31	(3,6,9)	30	.82	.57	(2,4,8)	1100
14	.60	(3,6,9)	38	1.21	.82	(3,4,7)	1273
15	.29	(4,6,9)	19	.73	.34	(5,6,7)	876
16	.75	(3,6,8)	45	1.56	.97	(6,7,9)	1723
17	.79	(5,6,9)	55	1.93	1.10	(5,7,8)	1983
18	.74	(4,5,7)	41	1.51	.32	(2,4,7)	1830
19	.88	(4,5,7)	36	1.50	.55	(3,5,9)	1698
20	.43	(3,5,9)	25	.98	.23	(2,5,6)	950
21	.48	(3,4,7)	26	1.17	.79	(3,6,8)	823

We solve models (11) and (12) without considering the intermediate products which results in the interval $[\theta_\alpha^L, \theta_\alpha^U]$ for the efficiency scores shown in Table 2. Models (11) and (12) without constraints related to the intermediate products are the traditional black box DEA models. Then, these models are solved to calculate interval efficiency scores $[\gamma_\alpha^L, \gamma_\alpha^U]$. By using the traditional black box DEA method, efficiency scores of DMUs are calculated based on inputs and outputs of the overall system, while the intermediate activities within the system are neglected.

Table 2. Results of solving models (11) and (12) in classic and network cases for $\alpha = .6$

DMU	$[\theta_\alpha^L, \theta_\alpha^U]$	$[\gamma_\alpha^L, \gamma_\alpha^U]$	$[\gamma_{1\alpha}^L, \gamma_{1\alpha}^U]$	$[\gamma_{2\alpha}^L, \gamma_{2\alpha}^U]$
1	[.9246,1]	[.7861,.8908]	[.7906,1]	[.7816,.7816]
2	[1,1]	[.8973,.9077]	[1,1]	[.7946,.8154]
3	[.7016,1]	[.5683,.6554]	[.6563,.6563]	[.4803,.6546]
4	[.6382,1]	[.5121,.7974]	[.5948,.5948]	[.4293,1]
5	[.9471,1]	[.7901,.9083]	[.8167,.8167]	[.7635,1]
6	[.7134,1]	[.5948,.6879]	[.5992,.7142]	[.5904,.6615]
7	[.6114,1]	[.5408,.6447]	[.5551,.5551]	[.5266,.7344]
8	[.8617.,1]	[.6500,.9048]	[.7862,.8097]	[.5138,1]
9	[.5357,1]	[.4840,.6703]	[.5357,.8333]	[.4323,.5072]
10	[.9787,1]	[.8395,.9314]	[.8162,1]	[.8629,.8629]
11	[.9684,1]	[.8122,.9166]	[.8333,.8333]	[.7011,1]
12	[.9324,1]	[.7306,.7643]	[.5287,.5287]	[.9324,1]
13	[1,1]	[.7358,.8145]	[.8462,.8463]	[.6253,.7827]
14	[.8111,1]	[.6646,.6739]	[.6190,.6525]	[.7101,.7228]
15	[.9089,1]	[.8394,.9104]	[.8206,.8208]	[.8583,1]
16	[.8061,1]	[.6672,.7140]	[.6250,.6250]	[.7094,.8030]
17	[.9151,1]	[.7016,.7054]	[.7236,.7238]	[.6797,.6870]
18	[.6146,1]	[.4461,.5422]	[.6146,.6521]	[.2776,.4323]
19	[.5895,1]	[.5100,.6775]	[.5172,.6521]	[.5028,.7028]
20	[.7150,1]	[.5786,.7239]	[.7150,.7150]	[.4422,.7328]
21	[1,1]	[.8753,.9166]	[.7506,.8333]	[1,1]

According to the results shown in Table 2, it can be inferred that: 1) The classic traditional DEA approach concludes that in the optimistic point of view, all departments are efficient and it cannot recognize any difference between the 21 system performances. 2) The efficiency scores obtained by applying our network method have improved compared to those obtained by using the traditional DEA method. 3) The results obtained by using the classic black box DEA approach show that Departments No. 2,13, and 21 are efficient and Department No. 10 is the closest in ranking. However, the results obtained by solving our network model imply that no department is efficient in the both stages. 4) Through the combination of our network approach with the fuzzy method, we obtain a crisp interval efficiency score by which we are able to analyze and interpret the performance of a complex network system with vague and imprecise data. 5) In the pessimistic point of view, the reception part of Department No. 19 and the repair center of Department No. 18 have the lowest efficiency scores and require the most improvement in input values compared to other inefficient DMUs. Tables 3 and 4 contain the values of costs, labor, and income required to achieve the optimal efficiency score in both of pessimistic and optimistic scenarios.

Table 3. Optimal solutions of the fuzzy network models for $\alpha = .6$ in pessimistic viewpoint

DMU	x_{j1}	x_{j2}	R_j	V_j	y_{j1}	y_{j2}
1	.95	3.82	47	3.28	1.33	9
2	1.05	6.2	32	3.60	.99	8.52
3	.42	3	17	1.40	.54	4.64
4	.50	3	19	1.69	.57	5.20
5	.26	3	15	.85	.44	4.20
6	.56	3	20	1.90	.61	5.25
7	.40	3	19	1.33	.51	5.80
8	.14	3	9	.44	.19	3.20
9	.71	3	25	2.01	.67	5.80
10	1.13	4.53	60	3.89	1.54	9
11	.25	3	17	.78	.43	5.40
12	.29	3	22	.92	.68	5.85
13	.26	3	18	.82	.57	4.90
14	.37	3	27	1.21	.82	7.05
15	.23	3	16	.73	.34	5.6
16	.47	3	31	1.56	.97	8.34
17	.57	3	37	1.93	1.10	9
18	.45	3	11	1.51	.32	3.20
19	.45	3	18	1.50	.55	4.73
20	.30	3	11	.98	.23	3.80
21	.36	3	26	1.17	.79	4.8

Table 4. Optimal solutions of the fuzzy network models for $\alpha = .6$ in optimistic viewpoint

DMU	x_{j1}	x_{j2}	R_j	V_j	y_{j1}	y_{j2}
1	1.21	4.20	43	3.28	1.33	8.08
2	1.05	4.20	33	3.60	.99	6.4
3	.41	3	24	1.40	.54	6.20
4	.54	4.05	27	1.69	.57	7.20
5	.26	3	20	.85	.44	6.20
6	.67	3	22	1.95	.61	4.80
7	.39	3	27	1.33	.51	7.80
8	.15	3	18	.44	.19	5.60
9	.77	3	42	1.17	1.16	9
10	1.39	4.20	70	3.89	1.90	9
11	.25	3	22	.79	.43	7.80
12	.28	3	24	.92	.68	6.60
13	.25	3	23	.82	.57	5.60
14	.37	3	27	1.26	.82	5.20
15	.22	3	19	.73	.34	6.4
16	.46	3	36	1.59	.97	7.80
17	.56	3.32	37	1.93	1.10	7.40
18	.48	3	17	1.64	.32	5.20
19	.57	3	25	1.82	.55	6.60
20	.30	3	18	1.01	.32	5.40
21	.40	3	26	1.35	.79	6.8

In the pessimistic viewpoint, the greatest reduction occurs in the cost value of Department No. 19 and the labor value of Department No. 18 compared to the cost and labor values of other inefficient departments. These results are consistent with the efficiency scores of each stage as shown in Table 2. Furthermore, the income from selling spare parts (output of stage 1) has significantly increased in department No. 9.

In real-life situations, the improvement of input and output values depends on financial, environmental, and social limitations and may not be fully achievable in reality. For example, the results show that in Department No. 18, labor should be reduced from 41 to 11 (30 workers). However, this would impose significant costs related to payoffs and settlements for the manager, who can only afford to make settlements for 7 workers. As another example, the results show that in Department No. 9, the income from selling spare parts should be increased significantly. This could be achieved by selling at a higher price, selling more products, or selling strictly for cash without any conditional sales.

5.2. Estimating the efficiency scores using ML algorithms

Data related to after-sales service departments during 60 periods is shown in Table 5. Table 6 contains efficiency intervals related to the data presented in Table 5, obtained by using the fuzzy network DEA method proposed in this paper (models (11) and (12) for $\alpha = .6$).

Table 5. Data related to 60 after-sales service departments

DMU	x_{j1}	x_{j2}	R_j	V_j	y_{j1}	y_{j2}	Z_j
1	1.21	(3,5,7)	56	3.28	1.33	(2,5,9)	2100
2	1.05	(3,5,8)	41	3.60	.99	(5,6,7)	1954
3	.63	(4,5,8)	37	1.40	.54	(4,5,8)	1258
4	.83	(5,7,9)	45	1.69	.57	(4,6,9)	1823
5	.32	(3,7,9)	20	.85	.44	(3,5,8)	984
6	.94	(3,5,8)	34	1.90	.61	(3,4,6)	1205
7	.71	(5,8,9)	37	1.33	.51	(4,7,9)	1360
8	.19	(4,7,9)	18	.44	.19	(2,4,8)	871
9	.95	(3,4,8)	51	1.05	.67	(3,5,7)	1900
10	1.39	(3,5,7)	70	3.89	1.54	(3,6,9)	2410
11	.30	(3,4,7)	22	.78	.43	(3,7,9)	934
12	.55	(3,8,9)	24	.92	.68	(3,5,9)	895
13	.31	(3,6,9)	30	.82	.57	(2,4,8)	1100
14	.60	(3,6,9)	38	1.21	.82	(3,4,7)	1273
15	.29	(4,6,9)	19	.73	.34	(5,6,7)	876
16	.75	(3,6,8)	45	1.56	.97	(6,7,9)	1723
17	.79	(5,6,9)	55	1.93	1.10	(5,7,8)	1983
18	.74	(4,5,7)	41	1.51	.32	(2,4,7)	1830
19	.88	(4,5,7)	36	1.50	.55	(3,5,9)	1698
20	.43	(3,5,9)	25	.98	.23	(2,5,6)	950
21	.48	(3,4,7)	26	1.17	.79	(3,6,8)	823
22	.21	(3,5,7)	18	.58	.33	(5,7,9)	920
23	1.45	(5,7,9)	71	4.60	1.87	(5,7,9)	2983
24	.74	(4,5,7)	41	1.65	.73	(4,5,8)	1370
25	.55	(3,5,7)	23	1.09	.57	(4,6,9)	1571
26	.41	(3,5,7)	20	1.07	.68	(4,6,7)	1120

DMU	x_{j1}	x_{j2}	R_j	V_j	y_{j1}	y_{j2}	Z_j
27	.22	(5,6,9)	19	.91	.51	(2,4,8)	1000
28	.42	(3,4,8)	29	1.26	.49	(5,7,9)	1190
29	.58	(3,6,9)	37	.93	.33	(3,5,7)	1346
30	.95	(4,5,7)	55	1.32	.88	(3,5,7)	1770
31	.39	(4,7,9)	33	.80	.29	(5,7,9)	910
32	.66	(4,8,9)	43	1.08	.57	(3,7,9)	1000
33	.19	(3,7,9)	22	.52	.20	(3,5,9)	895
34	.31	(3,4,8)	40	.82	.47	(5,7,9)	1100
35	.77	(5,7,9)	41	1.38	.79	(3,4,7)	1573
36	.90	(4,8,9)	50	1.03	.88	(5,6,7)	1890
37	.90	(4,6,8)	48	1.18	.79	(5,7,9)	1723
38	.47	(5,7,9)	22	.93	.51	(5,7,8)	1190
39	.93	(4,8,9)	41	1.11	.55	(2,4,7)	1460
40	.25	(4,5,7)	19	.77	.39	(3,5,9)	864
41	.33	(4,5,9)	25	.98	.45	(5,7,8)	950
42	.48	(3,4,7)	21	.97	.49	(4,6,8)	1370
43	.21	(4,8,9)	16	.58	.33	(2,5,9)	859
44	1.45	(3,8,9)	60	1.97	.89	(4,6,7)	2391
45	1.63	(4,5,8)	61	2.40	1.11	(4,5,7)	2680
46	.98	(5,7,9)	45	.99	.33	(2,5,9)	1791
47	.17	(3,7,9)	20	.61	.26	(3,5,7)	887
48	.29	(5,7,9)	34	.90	.56	(3,4,6)	1205
49	.38	(6,8,9)	28	.95	.51	(5,7,9)	1112
50	.47	(4,7,9)	22	1.09	.79	(2,4,8)	1086
51	1.78	(3,4,8)	65	3.05	1.71	(3,4,7)	2900
52	.70	(3,5,7)	37	1.33	.77	(4,7,8)	1379
53	1.30	(3,4,6)	49	2.78	1.22	(3,7,9)	2840
54	.65	(3,8,9)	34	.92	.55	(3,5,9)	1352
55	.49	(3,6,9)	29	.88	.45	(2,4,8)	1210
56	.46	(3,4,8)	38	1.21	.77	(3,4,7)	1320
57	.35	(4,6,9)	26	.73	.43	(5,6,7)	915
58	1.75	(3,5,6)	65	3.56	1.87	(3,7,9)	3000
59	1.79	(3,4,6)	71	3.2	1.89	(5,7,8)	3100

DMU	x_{j1}	x_{j2}	R_j	V_j	y_{j1}	y_{j2}	Z_j
60	.93	(4,5,7)	45	1.09	.71	(2,4,6)	2050

Table 6. Efficiency intervals related to the data presented in Table 5, obtained by solving models (11) and (12) for $\alpha = .6$

DMU	*Efficiecy interval*	DMU	*Efficiency interval*	DMU	*Efficiency interval*
1	[.66,.83]	**21**	[.71,.91]	**41**	[.73,.98]
2	[.80,.89]	**22**	[.82,1]	**42**	[.67,1]
3	[.46,.67]	**23**	[1,1]	**43**	[.68,.95]
4	[.39,.60]	**24**	[.60,.67]	**44**	[.42,.48]
5	[.52,.96]	**25**	[.66,1]	**45**	[.51,.65]
6	[.52,.65]	**26**	[.85,1]	**46**	[.32,.49]
7	[.40,.69]	**27**	[.94,1]	**47**	[.72,1]
8	[.42,1]	**28**	[.73,1]	**48**	[.63,.74]
9	[.45,.63]	**29**	[.40,.56]	**49**	[.63,.88]
10	[.73,.82]	**30**	[.48,.56]	**50**	[.82,.90]
11	[.63,.82]	**31**	[.49,.85]	**51**	[.78,1]
12	[.57,.80]	**32**	[.43,.64]	**52**	[.59,.79]
13	[.47,.82]	**33**	[.57,1]	**53**	[.75,1]
14	[.51,.68]	**34**	[.57,1]	**54**	[.43,.66]
15	[.58,.94]	**35**	[.53,.59]	**55**	[.48,.69]
16	[.52,.77]	**36**	[.46,.55]	**56**	[.66,.78]
17	[.49,.71]	**37**	[.46,.63]	**57**	[.58,.86]
18	[.42,.55]	**38**	[.68.92]	**58**	[1,1]
19	[.48,.63]	**39**	[.37,.46]	**59**	[.88,.88]
20	[.44,.68]	**40**	[.73,1]	**60**	[.48,.55]

We consider 80% of the dataset for training and allocate the remaining 20% for testing across three ML algorithms: LR, RF, and DT. To prepare the data for ML algorithms, it is transformed into crisp intervals, as outlined in Section 4.2 for $\alpha = .6$. During the training phase, the specific combination of input, output, and intermediate values that contributes to the efficiency score of the DMU is identified. following this, the trained ML model is utilized to predict efficiency scores for the testing data. The efficiency scores for the testing data are calculated using the fuzzy network DEA method, and the predicted values produced by the ML algorithms are displayed in Table 7.

Table 7. Fuzzy network DEA efficiency intervals as the test data and their predicted values by combined DEA-ML algorithms

DMU	DEA	$DEA-LR$	$DEA-DT$	$DEA-RF$
1	[.52,.96]	[.43,.79]	[.48,.89]	[.49,.81]

DMU	DEA	$DEA-LR$	$DEA-DT$	$DEA-RF$
2	[.42,1]	[.34,.96]	[.41,.96]	[.38,.88]
3	[.47,.82]	[.45,.70]	[.44,.69]	[.45,.79]
4	[.66,1]	[.64,.92]	[.62,.94]	[.64,.89]
5	[.73,1]	[.61,.96]	[.69,1]	[.70,.94]
6	[.57,1]	[.49,.89]	[.49,.92]	[.54,.93]
7	[.57,1]	[.44,.91]	[.53,.98]	[.52,.94]
8	[.73,1]	[.69,.89]	[.72,.95]	[.69,.93]
9	[.67,1]	[.62,.97]	[.65,.93]	[.64,.86]
10	[.68,.95]	[.63,.87]	[.63,.94]	[.66,.90]
11	[.72,1]	[.56,.98]	[.69,.98]	[.72,.97]
12	[.58,.86]	[.56,.82]	[.58,.84]	[.56,.86]

The values of Mean Squared Error (MSE), Root Mean Squared Error (RMSE), Absolute Error (MAE), and accuracy of three DEA-ML algorithms in predicting pessimistic and optimistic ranges of efficiency scores related to the testing data are presented in Figure4.

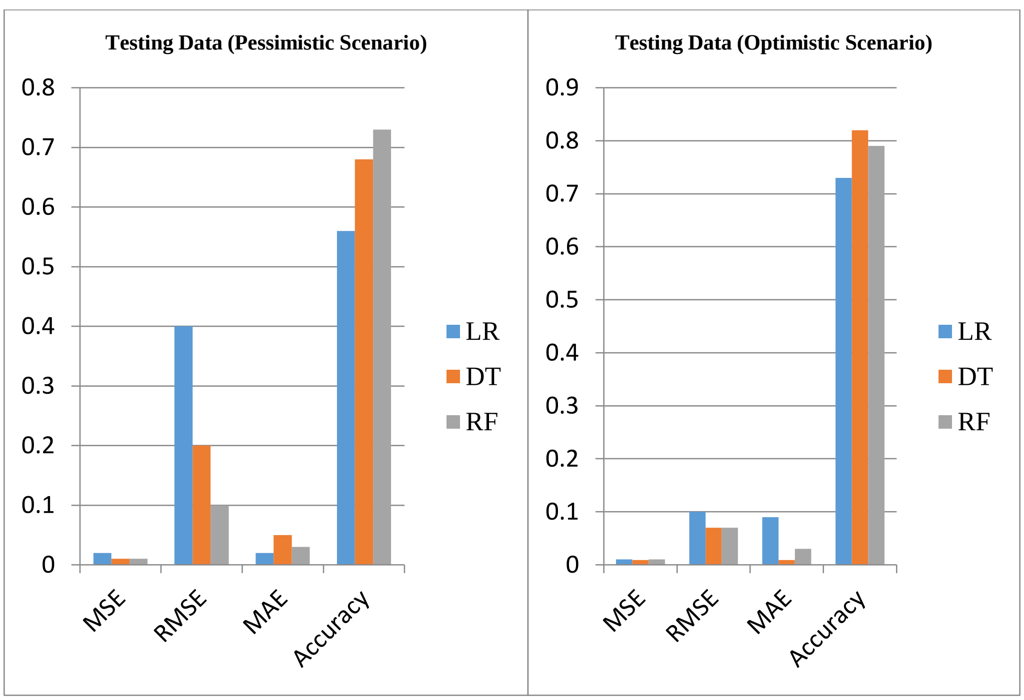

Fig.4. Comparison of three DEA-ML algorithms in predicting efficiency score ranges

According to the results depicted in Figure4, the DEA-RF algorithm has the highest accuracy in predicting efficiency scores under pessimistic scenario (model (11)), while the DEA-DT algorithm is the best in prediction of scores under optimistic scenario (model (12)), with a slightly higher accuracy value than the DEA-RF algorithm. Therefore, the DEA-RF algorithm has nearly the best performance in both pessimistic and optimistic scenarios and is used to predict the efficiency score of new DMUs.

6. Conclusion and directions for future research

In recent years, various types of DEA methods have been widely developed due to the different structures and data types of DMUs. Efficiency evaluation of DMUs with complex network structures involves network DEA approaches that consider the inner relationships between sub-processes. We have presented a modified network DEA model in which the efficiency score of overall system have been optimized by using different input reduction and output increment factors simultaneously. Furthermore, our network DEA model has been wisely designed so that the interactions between the stages are reflected truly. In addition to the structural complexity, data (i.e. inputs, outputs and intermediate products) come in various types in real world situations. Many processes involve imprecise, inexact, vague and qualitative information. Decision making in the presence of uncertain information is one of the main concerns in managerial issues. Fuzzy approaches make it possible to incorporate qualitative information into a mathematical formulation, empowering decision makers to adopt managerial policies. We have proposed a fuzzy network model to assess the after-sales service section in an auto-making company, where the levels of obligation fulfilment and customer satisfaction are considered as fuzzy data. Numerical results show that by employing our network method, the efficiency scores of after-sales service departments have improved compared to the efficiency measures calculated by solving the traditional DEA model. To deal with the computational complexity, we have integrated our fuzzy network approach with ML algorithms to predict the efficiency score ranges of DMUs from pessimistic and optimistic perspectives. The algorithm with the best performance is used to predict the performance of new DMUs. Applying fuzzy approaches in dealing with qualitative information helps in designing effective strategies for organizational management. Analyzing the performance of after-sale services departments over multiple periods is a major managerial concern. This analysis involves using the dynamic network DEA method, where the outputs in one period are carried over as the inputs to the next period. This topic warrants consideration for future studies.

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