Some notes concerning the convergence control parameter in homotopy analysis method

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Abstract

Homotopy analysis method (HAM) is a promising method for handling functional equations. Recent publications proved the effectiveness of HAM in solving wide variety of problems in different fields. HAM has a unique property which makes it superior to other analytic methods, this property is its ability to control the convergence region of the solution series. In this work, we clarified the advantages and effects of convergence-control parameter through an example.

Keywords: Homotopy analysis method, Convergence-control parameter, Zeroth order deformation equation.

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1   Introduction

The basic idea of Homotopy analysis method (HAM) was first proposed by Liao, in this Ph.D. thesis, in 1992 [9], but nowadays what is known, among scientists, is a more developed version of it. The first applications of the method is restricted to some papers of Liao and his colleagues and these are papers where the fundamental and basic concepts are presented [10, 11, 12]. Then in 2003, Liao published a book "

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Beyond perturbation: An introduction to homotopy analysis method ", and he gave a general introduction of HAM [13]. In this book, Liao proves its superiority over other classic semi-analytic methods and uses the method to solve a variety of problems and equations.

It was actually after Liao's book that the method attracted so many attention. Scientists and researchers, from various fields, found the method a promising tool for handling nonlinear equations. The method has been applied to solve so many problems, among them we can mention equations related to viscous flows of non-Newtonian fluids [5, 6, 7], KdV type equations [3, 19, 25], nonlinear equations in heat transfer [1, 2], financial problems [23, 24], differential-difference equations [22], Laplace equation with Neumann and Dirichlet boundary condition [8], and many other problems. A simple search in databases will show us that how much the method is popular. Other than the known applications of HAM in solving functional equation, there have been some interesting innovations, for example Prof. Abbasbandy and Liao have proposed a new *homotopy-Newton method* which is a universalization of the famous Newton-Raphson method [4]. Also Liao has recently introduced a transform, called *homotopy-transform*, which he proves that Euler transform is a special case of it [14]. Especially, HAM is employed to solve a kind of differential equations, that neither other analytic methods nor numerical techniques have been able to solve them [15, 16]. This proves the great potential of the method. In the next section we will give basic ideas of the method according to latest accepted terminology in the community.

2   Basic idea of HAM

Let us consider the following nonlinear equation in a general form:

$$N[u(r,t)] = 0, \tag{2.1}$$

where N is a nonlinear operator, $u(r,t)$ is an unknown function and r and t denote spatial and temporal independent variables, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated without any difficulties. The objective is to find the unknown function $u(r,t)$, for this Liao chooses an initial guess of the solution, $u_0(x,t)$ and defines a homotopy between this initial guess and the exact solution. An auxiliary operator $\mathcal{L}$ is needed to construct the homotopy and using the embedding parameter q, $q \in [0,1]$, (which he renames it to *homotopy parameter* [17]) the homotopy would be:

$$\mathcal{H}[\phi(r,t;q); u_0(r,t), q] = (1-q)\mathcal{L}[\phi(r,t;q) - u_0(r,t)] - qN[\phi(r,t;q)]. \tag{2.2}$$

This was the first homotopy proposed by Liao [9], later in order to make HAM more effective, he added an auxiliary function $H(r,t)$ and an auxiliary parameter c_0 to the homotopy equation and defined the new homotopy

$$\mathcal{H}[\phi(r,t;q); u_0(r,t), H, c_0, q] = (1-q)\mathcal{L}[\phi(r,t;q) - u_0(r,t)] - qc_0HN[\phi(r,t;q)]. \tag{2.3}$$

We will later explain that $H(r, t)$ helps to represent the solution in a special format and c_0 is actually a *convergence-control parameter* (This parameter was erstwhile denoted by $\hbar$, but as this symbol is also used to represent Plank's constant in quantum mechanics, Liao himself in [18] has changed it to c_0). Enforcing the homotopy to be zero, we have the so called *zeroth order deformation equation* (ZODE):

$$(1 - q)\mathcal{L}[\phi(r, t; q) - u_0(r, t)] = qc_0 H(r, t)N[\phi(r, t; q)]. \tag{2.4}$$

When $q = 0$ this equation has the obvious solution $\phi(r, t; 0) = u_0(r, t)$, and if $q = 1$ then the ZODE is equivalent to the original equation (1). So the original equation is transformed to a system of equations. For every $q_0 \in [0, 1]$ we have an equation whose solution is denoted by $\phi(r, t; q_0)$. Since the homotopy is a continuous function, as q increases from 0 to 1, the solution $\phi(r, t; q)$ varies continuously from the initial guess $u_0(r, t)$ to the solution $u(r, t)$. What left is to evaluate $\phi(r, t; q)$ at $q = 1$. Expanding $\phi(r, t; q)$ in a Maclaurin series with respect to q, we have

$$\phi(r, t; q) = \phi(r, t; 0) + \sum_{k=1}^{\infty} D_k(\phi)q^k, \tag{2.5}$$

where $D_k(\phi)$ is the kth order *homotopy-derivative* of ϕ, which is defined as:
   Definition 2.1: Let ϕ be a function of the homotopy-parameter q, then

$$D_m(\phi) = \frac{1}{m}\frac{\partial^m \phi}{\partial q^m}\Big|_{q=0}, \qquad m \geq 0, \tag{2.6}$$

is called the mth-order homotopy-derivative of ϕ, where $m \geq 0$ is an integer.
   The properties of this type of derivative is fairly discussed in [17]. For simplicity we express (5) as

$$\phi(r, t; q) = u_0(r, t) + \sum_{k=1}^{\infty} u_k(r, t)q^k, \tag{2.7}$$

where $u_k = D_k(\phi)$. This series is called *homotopy series*. If the homotopy series is convergent in $q = 1$ then the solution to the original equation would be

$$u(r, t) = u_0(r, t) + \sum_{k=1}^{\infty} u_k(r, t) = \sum_{k=0}^{\infty} u_k(r, t). \tag{2.8}$$

So what remains is just to find $u_k(r, t)$, for $k = 1, 2, \cdots$. Liao takes mth order homotopy-derivative from ZODE and constructs the so called *high order deformation equations* (HODEs):

$$\mathcal{L}[u_m(r, t) - \chi_m u_{m-1}(r, t)] = c_0 H(r, t)D_m(N[\phi(r, t; q)]), \tag{2.9}$$

where

$$\chi_m = \begin{cases} 0 & m \leq 1 \\ 1 & m > 1 \end{cases}, \tag{2.10}$$

especially for an specific m, the equation is called mth order deformation equation.

Directly substituting series (7) into the ZODE, equating coefficients of the like-powers of the embedding parameter q, one can get the same HODEs, as proved in [21]. Equations (9) are linear ones, so we transformed a nonlinear equation to a system of linear ones which can be easily solved using an iterative procedure, this is the main consequence of HAM. After solving equations (9), we can substitute $u_m(r, t)$ in (8) and obtain an approximation of arbitrary order.

3   Validity of HAM

Liao in [13], states the following theorem which clarifies the challenges we have in HAM:

Theorem 3.1: *If the homotopy series (8), obtained from the HODEs (9), is convergent then the series must converge to the exact solution of (1).*

So what remains is just to choose the auxiliary elements in a way that insures the convergence of the resulted series. To do so we have the four auxiliary elements in the ZODE, namely $\mathcal{L}$, v_0, H and c_0. Here we, briefly, mention the role and duties of these auxiliary elements:

> $\mathcal{L}$: The auxiliary (linear) operator, it should have a simple inverse because we expect the HODEs to be easy-to-solve equations.

> v_0: The initial guess, It should have a reasonable relation to our desired solution, so must be in a form logically related to the main problem.

> H: The auxiliary function. we know that one of our expectations from a semi-analytic method is that it should have the potential to represent the solution in different base functions. The auxiliary function H helps us to have such representations.

> c_0: Convergence-control parameter, undoubtedly it is the most important element in HAM. Whenever we have a HAM series solution, which is dependent upon c_0, we can use it to control the convergence region of the series.

The early HAM, which is proposed in 1992, didn't have any convergence-control parameter or auxiliary function, but in 1997, Liao added these elements to the homotopy equation and it was start of a revolution in analytic techniques. From the literature it seems that, in most problems, we can choose the 4 auxiliary elements of HAM, in such a way that leads the method to convergent series, although we still don't have any proof for this assertion.

Liao's conjecture: *If a nonlinear equation has at least one solution, then there is at least one zeroth order deformation equation for which the homotopy solution series is convergent to the exact solution.*

Liao asserts that we usually can choose the four elements $\mathcal{L}$, H, v_0 and c_0, in a suitable way that makes the solution series, obtained from HAM, converge to the

exact solution. It is not proved yet but we confess that rejecting and giving a counterexample is a hard work because there are many choices for $\mathcal{L}$, H and v_0. Moreover we can't talk about the convergence problem separately for each auxiliary element, the role of the four elements are dependent upon each other in this problem.

If we can guess, in a way, that the solution series (8), which is dependent upon c_0, is convergent for some values of c_0 then, according to Liao, it is possible to find the convergence region. Liao proposes the *technique of c_0-curves* for finding this region. His theorization is as follows: If the series $u(r,t)$ from (8) comes to be convergent for some values of c_0 then, according to Theorem 2.3, for all of these values the series must converge to the exact solution. Moreover convergence of $u(r,t)$, from (8), results in the convergence of some dependent quantities like $\dfrac{\partial u}{\partial t}(r_0,t_0)$ or $\dfrac{\partial^2 u}{\partial t^2}(r_0,t_0)$, where (r_0,t_0) is some point in the domain. So if the curves of these quantities are plotted with respect to c_0, the curve must contain a horizontal line segment. According to Liao, for c_0s in this interval, the solution series converges to the exact solution [13].

However it is not always possible to have the exact solution series in a closed form, so a truncated version of the homotopy series (a suitable mth order approximation) is used to plot the curves.

Although the technique of c_0-curves is a smart method, but there is no powerful mathematical governing theory for this technique. So there still exist some challenging questions about the convergence-control parameter.

Is there any guarantee for existence of such c_0s?

Is there a best c_0? and if so, how can we find it?

Homotopy series is obtained by solving the high order deformation equations, these equations are derived from the zeroth order deformation equation, which is dependent upon $\mathcal{L}$, v_0 and H, so the convergence of the homotopy series, for some values of c_0, is directly related to the behavior of these elements. The only key for this problem is Liao's conjecture, which states that there are always a combination of the four elements which leads us to the exact solution. So the question of "existence of a suitable c_0" is equivalent to Liao's conjecture.

For the question of the best c_0, although the technique of c_0-curves lead us to suitable c_0s, but they are unable to decide on the best c_0. Recently Liao have proposed a modification of HAM called *optimal homotopy analysis approach* where instead of just one convergence-control parameter, he uses a number of them, simultaneously, and gets good result [18].

4   The role of CCP in HAM

The HAM has a unique property which makes it superior to other analytic methods, this property is the ability to control the convergence region of the solution series. HAM does this by adding a convergence-control parameter, c_0, to the homotopy

equation. This parameter doesn't change the problem under study and the final series would depend upon this parameter. Here we clarify the advantages and effects of CCP through an example.

Example 4.1 Consider the diffusion equation $u_t = (uu_x)_x$, with initial condition $u(x,0) = x^2$. The solution is $u(x,t) = \dfrac{x^2}{1-6t}$. Choosing the auxiliary linear operator $\mathcal{L} = \dfrac{\partial}{\partial t}$ and $u_0 = 0$, we have the homotopy equation

$$(1-q)\phi_t = c_0 q(\phi_t - (\phi_x\phi)_x),$$

now substituting $\phi(x,t;q) = u_0(x,t) + u_1(x,t)q + \cdots$ in the homotopy equation and equating the like powers of q, we will obtain the HODEs. Solving HODEs, considering the initial condition, we have

$$
\begin{aligned}
u_0(x,t) &= x^2, \\
u_1(x,t) &= -6c_0tx^2, \\
u_2(x,t) &= -6c_0tx^2(-6c_0t + (1+c_0)), \\
u_3(x,t) &= -6c_0tx^2(-6c_0t + (1+c_0))^2, \\
&\vdots \\
u_n(x,t) &= -6c_0tx^2(-6c_0t + (1+c_0))^{n-1}, \quad n \geq 1.
\end{aligned}
$$

So the mth order approximation reads

$$app_m(x,t) = u_0(x,t) + u_1(x,t) + \cdots + u_m(x,t) = x^2 + \sum_{n=1}^{m} -6c_0tx^2(-6c_0t + (1+c_0))^{n-1}.$$

$$(4.1)$$

First we consider the $c_0 = -1$ case, which is a special case known to be the homotopy perturbation method,

$$
\begin{aligned}
u_0(x,t) &= x^2, \\
u_1(x,t) &= 6tx^2, \\
u_n(x,t) &= (6t)^n x^2, \quad n \geq 2 \quad.
\end{aligned}
$$

So we have

$$app_m(x,t) = u_0(x,t) + u_1(x,t) + \cdots + u_m(x,t) = x^2(1 + 6t + \cdots + (6t)^m)$$

which results in

$$\lim_{m\to\infty} app_m(x,t) = x^2(1 + 6t + (6t)^2 + \cdots).$$

As far as t satisfies $\dfrac{-1}{6} < t < \dfrac{1}{6}$, this series converges to $u(x,t) = \dfrac{x^2}{1-6t}$, which is the exact solution we look for. So in the special case $c_0 = -1$, HAM gives a series which converges to the exact solution for all x and $\dfrac{-1}{6} < t < \dfrac{1}{6}$. However this region

is relatively a small one.

Now we turn to the general case, where we leave c_0 undetermined. According to (2.6.1), app_m is a geometric series which is convergent whenever $|1 + c_0 - 6c_0t| < 1$. With this condition app_m converges to

$$x^2 - 6c_0tx^2 \frac{1}{1 - (1 + c_0 - 6c_0t)} = \frac{x^2}{1 - 6t},$$

which is the exact solution.

Two separate cases could be considered:

1. If $c_0 > 0$ then $|1 + c_0 - 6c_0t| < 1$ results in $\frac{1}{6} < t < \frac{1}{6} + \frac{1}{3c_0}$, so the convergence region of the solution series, in $c_0 > 0$ case, would be

$$\begin{cases} x \in \mathbb{R} \\ \frac{1}{6} < t < \frac{1}{6} + \frac{1}{3c_0} \end{cases}$$

Here it is possible to extend the convergence region by choosing smaller values of c_0. However positive c_0s can not guarantee convergence of the solution series for $t < \frac{1}{6}$, for these values of t we refer to negative c_0s.

2. If $c_0 < 0$ then the condition $|1 + c_0 - 6c_0t| < 1$, would be equivalent to $\frac{1}{6} + \frac{1}{3c_0} < t < \frac{1}{6}$, so it is possible to satisfy convergence by suitable negative c_0s [20].

This analytic results also can be verified numerically. If we set $c_0 = 0.2$ then, according to the just mentioned results the convergence region for t must be $\frac{1}{6} < t < \frac{1}{6} + \frac{1}{3(0/2)} = \frac{11}{6}$. Using this value for c_0, we construct the 40th and 80th order approximation, the errors of these approximations for $x = 1$ and different values of t in $(\frac{1}{6}, \frac{11}{6})$ are reported in Table 4.1.

Table 4.1

Errors of the 40th and 80th order approximation when $c_0 = 0.2$

m	x	t=0.3	t= 0.5	t=0.8	t=1	t=1.2	t=1.5
40	1	0.002	2.01E-9	2.04E-25	0	1.88E-25	1.50E-9
80	1	1.20E-6	2.68E-18	3.00E-30	0	5.00E-30	2.01E-18
40	5	0.0526	5.01E-8	5.10E-24	0	4.69E-24	3.76E-8
80	5	4.92E-5	6.70E-17	8.00E-30	0	4.33E-28	5.02E-17

It is seen that as we approach boundary points the precision decreases. For extending the convergence region we have to choose smaller values for c_0. For example if $c_0 = 0.02$ is chosen then the convergence region would extend to $\dfrac{1}{6} < t < \dfrac{1}{6} + \dfrac{1}{3(0/02)}$, the errors of this case is also reported in Table 4.2.

Table 4.2
Errors of the 40th and 80th order approximation when $c_0 = 0.02$

m	t=0.5	t=1	t=5	t=8	t=12	t=15
40	0/29	0.017	8.80E-16	1.368E-49	8.63E-16	4.88E-5
80	0.057	2.62E-4	7.49E-31	1.82E-98	7.34E-31	2.36E-9

In the presented example we simply determined the valid region of c_0 by straightforward calculations. This success was mainly because of the general term we had for representing approximations of arbitrary order, but in other cases, where there is no general term for the approximations, how can one determine the valid region for CCP? The only strategy at hand is Liao's *technique of c_0-curves*. Here we briefly review Liao's reasoning. If the series obtained by HAM, which in most cases is dependent upon c_0, is convergent for some values of c_0, so are related quantities. For example if the mth order approximation app_m is convergent to u_e, the exact solution, we could conclude that other quantities, like $\dfrac{\partial}{\partial t} app_m$, would converge to $\dfrac{\partial}{\partial t} u_e$, similarly $\dfrac{\partial^2}{\partial x^2} app_m$ would also converge to $\dfrac{\partial^2}{\partial x^2} u_e$. But Liao makes a restriction on deciding these quantities, he confines himself to those quantities which have a physical explanation.

Suppose that the quantity at hand is $\dfrac{\partial^2}{\partial x^2} u_e$, then from convergence of $\dfrac{\partial^2}{\partial x^2} app_m$ to $\dfrac{\partial^2}{\partial x^2} u_e$, we conclude that for values r_0 and t_0, from domain of u_e, the series $\dfrac{\partial^2}{\partial t^2} app_m(r_0, t_0)$ (which is dependent upon c_0), must converge to $\dfrac{\partial^2}{\partial t^2} u_e(r_0, t_0)$. Now consider the values of c_0, where the quantity $\dfrac{\partial^2}{\partial t^2} app_m(r_0, t_0)$ converges, for this values the quantity converges to a unique value of $\dfrac{\partial^2}{\partial t^2} u_e(r_0, t_0)$, we call these values of c_0 the valid region of c_0 and denote it by R_{c_0}. So if we plot curve of $\dfrac{\partial^2}{\partial t^2} app_m(r_0, t_0)$ versus c_0, it must contain a horizontal line segment under this R_{c_0}. The curves of this related quantities versus c_0 are called c_0-curves.

Liao makes a reverse reasoning and concludes that for the values $c_0 \in R_{c_0}$, app_m must converge to the exact solution. Here we try to compare our analytic results (which has explicitly given us the R_{c_0}), with Liao's technique of c_0-curves.

We plot c_0-curves corresponding to $\dfrac{\partial}{\partial t} u_e$ and $\dfrac{\partial^2}{\partial x^2} u_e$ at $(x, t) = (1, 1)$ for approximations of order 40 and 80.
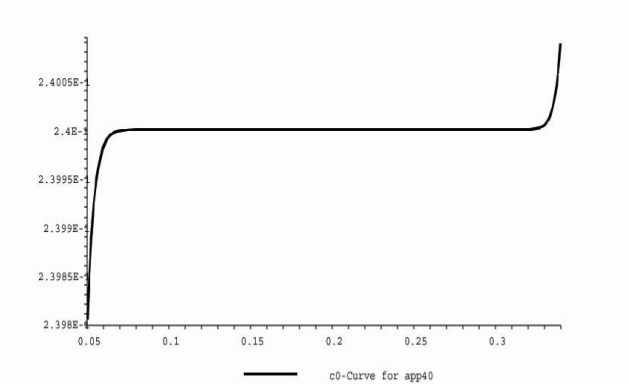
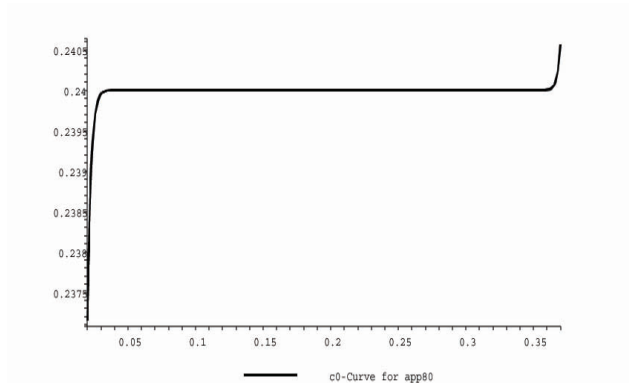Figure 4.1 : c_0-curve corresponding to $app_{40t}(1,1)$.

Figure 4.2 : c_0-curve corresponding to $app_{80t}(1,1)$.

For $\dfrac{\partial}{\partial t}u_e$ the exact value at $(1,1)$ is 0.24, by plotting the curves of $\dfrac{\partial}{\partial t}app_{40}(1,1)$ and $\dfrac{\partial}{\partial t}app_{80}(1,1)$ we have the corresponding R_{c_0} to be $[0.08, 0.31]$ and $[0.05, 0.35]$, respectively, see Figure 4.1 and 4.2.

We accept the valid region for c_0 to be the segment $[0.05, 0.35]$, but according to our analytic calculations we know that these c_0s can not satisfy convergence for all values of t, we know that when $c_0 \in [0.05, 0.35]$ the convergence is only guaranteed for $t \in [\dfrac{1}{6}, \dfrac{41}{6}]$. Although the technique of c_0-curves managed to give us a valid region for c_0 but in order to have complete region of convergence we have to do more calculations.

The results of the quantity $\dfrac{\partial^2}{\partial t^2}u_e(1,1)$ would be very same as the $\dfrac{\partial}{\partial t}u_e(1,1)$ case.

There is one more corollary for the presented example, it is mostly seen that when

we increase the order of approximation, in plotting c_0-curves, the valid region, R_{c_0}, gets smaller. But in our example it gets bigger which seems to be an interesting phenomena. However there is no change in the principle that increasing the order results in better convergence regions.

5   Conclusion

This work concerns with homotopy analysis method. Unlike other popular analytic methods, this one doesn't need any small parameters to be contained in the equation, instead the method itself introduces an auxiliary parameter $-c_0-$, by which it can control the convergence region of the solution series. In this work, we clarify the advantages and effects of convergence-control parameter through an example.

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