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Generalized composition operators from logarithmic Bloch type spaces to $\mathcal{Q}_K$ type spaces

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Abstract

In this paper boundedness and compactness of generalized composition operators from logarithmic Bloch type spaces to $\mathcal{Q}_K$ type spaces are investigated.

Key words: Generalized composition operator; Logarithmic Bloch type space; $\mathcal{Q}_K$ type space.

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1 Introduction

Let $\mathcal{H}(\mathbb{D})$ be the space of all analytic functions on the open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$ and $\alpha \in (0,\infty)$. The Bloch type space $\mathcal{B}^\alpha = \mathcal{B}^\alpha(\mathbb{D})$ is the space of all $f \in \mathcal{H}(\mathbb{D})$ satisfying

$$b_\alpha(f) = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)| < \infty.$$

The little Bloch type space $\mathcal{B}_0^\alpha$ consists of those functions $f \in \mathcal{B}^\alpha$ for which

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha|f'(z)| = 0.$$

For $\beta \in [0,\infty)$, the logarithmic Bloch type space $\mathcal{B}^\alpha_{\log^\beta} = \mathcal{B}^\alpha_{\log^\beta}(\mathbb{D})$ introduced by Stevic in [9], is the space of all $f \in \mathcal{H}(\mathbb{D})$ satisfying

$$b_{\alpha,\beta}(f) = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha(\ln\frac{e^{\frac{\beta}{\alpha}}}{1-|z|^2})^\beta|f'(z)| < \infty.$$

The little logarithmic Bloch type space $\mathcal{B}^\alpha_{\log^\beta,0}$ consists of those functions $f \in \mathcal{B}^\alpha_{\log^\beta}$ for which

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha(\ln\frac{e^{\frac{\beta}{\alpha}}}{1-|z|^2})^\beta|f'(z)| = 0.$$

In some papers (see [9]), the definitions of this kind of spaces are based on the coefficient $1-|z|$, instead of $1-|z|^2$. We first show that these are equivalent.

Obviously, $1-|z| < 1-|z|^2 < 2(1-|z|)$ and $\ln\frac{e^{\frac{\beta}{\alpha}}}{1-|z|^2} < \ln\frac{e^{\frac{\beta}{\alpha}}}{1-|z|}$ for all $z \in \mathbb{D}$. On the other hand,

$$\begin{aligned}\ln\tfrac{e^{\frac{\beta}{\alpha}}}{1-|z|} &= \ln\tfrac{e^{\frac{\beta}{\alpha}}}{1-|z|^2} + \ln(1+|z|)\\ &< \ln\tfrac{e^{\frac{\beta}{\alpha}}}{1-|z|^2} + \ln\tfrac{e}{1-|z|^2}\\ &= (1+\tfrac{\alpha}{\beta})\ln e^{\frac{\beta}{\alpha}} + 2\ln\tfrac{1}{1-|z|^2}\\ &\le \max\{1+\tfrac{\alpha}{\beta},2\}\ln\tfrac{e^{\frac{\beta}{\alpha}}}{1-|z|^2}.\end{aligned}$$

Therefore, we can replace $1 - |z|^2$ by $1 - |z|$ in the definitions of Bloch type spaces and logarithmic Bloch type spaces.

The space $\mathcal{B}^{\alpha}_{\log^{\beta}}$ is a Banach space with the norm $\|f\| := |f(0)| + b_{\alpha,\beta}(f)$, and $\mathcal{B}^{\alpha}_{\log^{\beta},0}$ is a closed subspace of $\mathcal{B}^{\alpha}_{\log^{\beta}}$. If $\beta = 0$, then $\mathcal{B}^{\alpha}_{\log^{\beta}}$ ($\mathcal{B}^{\alpha}_{\log^{\beta},0}$) coincides with the Bloch type space $\mathcal{B}^{\alpha}$ (little Bloch type space $\mathcal{B}^{\alpha}_0$). For $\beta = 1$, the space $\mathcal{B}^{\alpha}_{\log^{\beta}}$ is the generally weighted Bloch space (see [5]). When $\alpha = \beta = 1$, the space $\mathcal{B}^{\alpha}_{\log^{\beta}}$ is just the weighted Bloch space $\mathcal{B}_{\log}$.

For $p \in (0, \infty)$ and $\alpha > -1$, the weighted Bergman space $\mathcal{A}^p_\alpha$ is the space of all $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|^p_{\mathcal{A}^p_\alpha} = \int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\alpha dA(z) < \infty,$$

where dA is the normalized area measure on $\mathbb{D}$. It is well known that $\mathcal{A}^p_\alpha$ is a Banach space for $p \geq 1$, and in the case that $0 < p < 1$, it is a complete metric space with the distance $d(f, g) = \|f - g\|^p_{\mathcal{A}^p_\alpha}$. In the special case when $p = 2$, $\mathcal{A}^2_\alpha$ is a Hilbert space. For a general background about weighted Bergman spaces we refer to [16].

For $p \in (0, \infty)$ and $\alpha > -1$, the weighted Dirichlet type space $\mathcal{D}^p_\alpha$ is the space of all $f \in \mathcal{H}(\mathbb{D})$ for which $f' \in \mathcal{A}^p_\alpha$. Note that $\mathcal{D}^p_\alpha$ is a Banach space with the norm $\|f\| := |f(0)| + \|f'\|_{\mathcal{A}^p_\alpha}$. When $\alpha = 0$, $\mathcal{D}^p_\alpha$ coincides with the Dirichlet space $\mathcal{D}^p$.

For $a \in \mathbb{D}$, $G(z, a) = \log \frac{1}{|\sigma_a(z)|}$ is the Green's function on $\mathbb{D}$, where $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$ is the Möbius transformation of $\mathbb{D}$. For $s \in (0, \infty)$, the space $\mathcal{Q}_s$ consists of all $f \in \mathcal{H}(\mathbb{D})$ for which

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 G^s(z, a) dA(z) < \infty,$$

and its closed subspace $\mathcal{Q}_{s,0}$ consists of those functions $f \in \mathcal{Q}_s$ such that

$$\lim_{|a| \to 1} \int_{\mathbb{D}} |f'(z)|^2 G^s(z, a) dA(z) = 0.$$

It is well known that $\mathcal{Q}_1 = BMOA$ ($\mathcal{Q}_{1,0} = VMOA$), the space of all analytic functions of bounded (vanishing) mean oscillation [1].

In [15], Zhao introduced a general family of analytic function spaces, called the $F(p,q,s)$-spaces with $p \in (1,\infty)$, $q \in (-2,\infty)$ and $s \in [0,\infty)$, consisting of all $f \in \mathcal{H}(\mathbb{D})$ such that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q G^s(z,a)\,dA(z) < \infty.$$

The closed subspace $F_0(p,q,s)$ of $F(p,q,s)$ consists of those functions $f \in F(p,q,s)$ such that

$$\lim_{|a|\to 1} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q G^s(z,a)\,dA(z) = 0.$$

If $q+s \leq -1$, $F(p,q,s)$ reduces to the space of constant functions. The interest in the $F(p,q,s)$-spaces arises from the fact that they cover a lot of well-known function spaces which are listed in the following.

- $F(2,0,s) = \mathcal{Q}_s, \quad F_0(2,0,s) = \mathcal{Q}_{s,0}$
- $F(2,0,s) = \mathcal{B}, \quad F_0(2,0,s) = \mathcal{B}_0$ $(s > 1)$
- $F(2,0,1) = BMOA, \quad F_0(2,0,1) = VMOA$
- $F(p,pq-2,s) = \mathcal{B}^q, \quad F_0(p,pq-2,s) = \mathcal{B}^q_0$ $(s > 1)$
- $F(p,pq-2,1) = BMOA^q_p$ (The BMOA type spaces)
- $F_0(p,pq-2,1) = VMOA^q_p$ (The VMOA type spaces)
- $F(p,q,0) = \mathcal{A}^p_{q-p}$ $(q-p > -1)$, $\quad F(p,q,0) = \mathcal{D}^p_q$ $(q > -1)$
- $F(2,1,0) = H^2$ (The Hardy space)

In [10] Wulan and Zhou introduced a new space, $\mathcal{Q}_K$ type space. For a right-continuous and nondecreasing function $K : [0,\infty) \to [0,\infty)$, and for $p \in (0,\infty)$, $q \in (-2,\infty)$, the $\mathcal{Q}_K$ type space denoted by $\mathcal{Q}_K(p,q)$ consists of $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|^p_{K,p,q} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q K(G(z,a))\,dA(z) < \infty.$$

The space $\mathcal{Q}_K(p,q)$ is a Banach space with the norm $\|f\|_{\mathcal{Q}_K(p,q)} := |f(0)| + \|f\|_{K,p,q}$, when $p \geq 1$. The closed subspace $\mathcal{Q}_{K,0}(p,q)$ of $\mathcal{Q}_K(p,q)$ consists of those functions $f \in \mathcal{Q}_K(p,q)$ such that

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q K(G(z,a))\,dA(z) = 0.$$

If $q+2=p$, $\mathcal{Q}_K(p,q)$ is Möbius invariant i.e, $\|f\circ\sigma_a\|_{K,p,q}=\|f\|_{K,p,q}$ for all $a\in\mathbb{D}$. We say that the space $\mathcal{Q}_K(p,q)$ is trivial if it contains constant functions only. For example, if $\int_0^1(1-r^2)^qK(\log\frac{1}{r})rdr=\infty$, then $\mathcal{Q}_K(p,q)$ is trivial [10]. Also by [10, Theorem 3.1], if $K(1)>0$ then the kernel function K can be chosen as bounded. Throughout the paper, we assume $K(1)>0$ and

$$\int_0^1(1-r^2)^qK(\log\frac{1}{r})rdr<\infty.$$

By [10, Theorem 2.1] we have $\mathcal{Q}_K(p,q)\subseteq\mathcal{B}^{\frac{q+2}{p}}$ and for a fixed $r\in(0,1)$,

$$\|f\|_{K,p,q}^p\geq\pi r^2K(\log\frac{1}{r})b^p_{\frac{q+2}{p}}(f),$$

for all $f\in\mathcal{Q}_K(p,q)$. In the sequel, we use the inequality

$$b_{\frac{q+2}{p}}(f)\leq C\|f\|_{K,p,q}. \tag{1.1}$$

Also by [10, Theorem 2.1], we have $\mathcal{Q}_K(p,q)=\mathcal{B}^{\frac{q+2}{p}}$ if and only if

$$\int_0^1(1-r^2)^{-2}K(\log\frac{1}{r})rdr<\infty.$$

Now we recall some particular cases. If $p=2$, $q=0$, we have that $\mathcal{Q}_K(p,q)=\mathcal{Q}_K$. For more details on the spaces of $\mathcal{Q}$ classes we refer to [4,?,12]. For $s\in[0,\infty)$, if $K(t)=t^s$, then $\mathcal{Q}_K(p,q)=F(p,q,s)$.

Let φ be an analytic self-map of $\mathbb{D}$ and $g\in\mathcal{H}(\mathbb{D})$, the generalized composition operator C_φ^g is defined by

$$(C_\varphi^gf)(z)=\int_0^z f'(\varphi(\xi))g(\xi)d\xi,\quad f\in\mathcal{H}(\mathbb{D}),\ z\in\mathbb{D},$$

which is introduced in [6]. When $g=\varphi'$, this operator is essentially (up to a constant) the composition operator C_φ, which is defined by $C_\varphi f=f\circ\varphi$. Darus and Ibrahim has defined an integral operator on a class of analytic functions in the unit disk [3]. Zhang and Liu gave characterization of the compact generalized composition operators from Bloch type spaces to $\mathcal{Q}_K$ type spaces in terms of K-Carleson measure in [14]. Essential norm of generalized composition operators from weighted Dirichlet or Bloch type spaces to $\mathcal{Q}_K$ type spaces was studied in [8]. A characterization of boundedness and compactness of generalized composition and Volterra type operators between $\mathcal{Q}_K$ spaces was provided in [7]. In this paper, we determine conditions under which the generalized composition operator C_φ^g from logarithmic Bloch type spaces to $\mathcal{Q}_K$ type spaces is bounded or compact without using Carleson measure. In this paper constants are denoted by C, they are positive and not necessarily the same in each occurrence.

2 Main results

Note that, if $C_\varphi^g(\mathcal{B}^\alpha_{\log^\beta}) \subseteq \mathcal{Q}_K(p,q)$, then $C_\varphi^g : \mathcal{B}^\alpha_{\log^\beta} \to \mathcal{Q}_K(p,q)$ is bounded, by the closed graph theorem. We now give an equivalent condition for boundedness and compactness of this operator.

Theorem 2.1 *Let $\alpha, p \in (0,\infty)$, $\beta \in [0,\infty)$, $q \in (-2,\infty)$, $g \in \mathcal{H}(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then $C_\varphi^g(\mathcal{B}^\alpha_{\log^\beta}) \subseteq \mathcal{Q}_K(p,q)$ if and only if*

$$L := \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|g(z)|^p(1-|z|^2)^q K(G(z,a))}{(1-|\varphi(z)|^2)^{\alpha p}(\ln \frac{e^{\frac{\beta}{\alpha}}}{1-|\varphi(z)|^2})^{\beta p}} dA(z) < \infty. \tag{2.1}$$

Proof. Suppose that $C_\varphi^g(\mathcal{B}^\alpha_{\log^\beta}) \subseteq \mathcal{Q}_K(p,q)$. By [9, Theorem 3] there exist two functions $f_1, f_2 \in \mathcal{B}^\alpha_{\log^\beta}$ such that

$$\frac{C}{(1-|z|)^\alpha(\ln \frac{e^{\frac{\beta}{\alpha}}}{1-|z|})^\beta} \le |f_1'(z)| + |f_2'(z)|, \quad z \in \mathbb{D}.$$

Using (1), we get

$$\frac{C}{(1-|\varphi(z)|^2)^\alpha(\ln \frac{e^{\frac{\beta}{\alpha}}}{1-|\varphi(z)|^2})^\beta} \le |f_1'(\varphi(z))| + |f_2'(\varphi(z))|, \quad z \in \mathbb{D}.$$

It follows that

$$\frac{C|g(z)|^p(1-|z|^2)^qK(G(z,a))}{(1-|\varphi(z)|^2)^{\alpha p}(\ln\frac{e^{\frac{\beta}{\alpha}}}{1-|\varphi(z)|^2})^{\beta p}}$$
$$\leq 2^p(|f_1'(\varphi(z)|^p+|f_2'(\varphi(z)|^p)|g(z)|^p(1-|z|^2)^qK(G(z,a)).$$

Integrating with respect to z, we have

$$\int_{\mathbb{D}}\frac{|g(z)|^p(1-|z|^2)^qK(G(z,a))}{(1-|\varphi(z)|^2)^{\alpha p}(\ln\frac{e^{\frac{\beta}{\alpha}}}{1-|\varphi(z)|^2})^{\beta p}}dA(z)\leq C(\|C_\varphi^g(f_1)\|^p_{\mathcal{Q}_K(p,q)}+\|C_\varphi^g(f_2)\|^p_{\mathcal{Q}_K(p,q)}).$$

Since $C_\varphi^g(\mathcal{B}^\alpha_{\log^\beta})\subseteq\mathcal{Q}_K(p,q)$, the inequality (2.1) follows.

Conversely, for $f\in\mathcal{B}^\alpha_{\log^\beta}$ we have

$$\begin{aligned}\|C_\varphi^g(f)\|^p_{\mathcal{Q}_K(p,q)}&=\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(\varphi(z))|^p|g(z)|^p(1-|z|^2)^qK(G(z,a))dA(z)\\&\leq b^p_{\alpha,\beta}(f)\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{|g(z)|^p(1-|z|^2)^qK(G(z,a))}{(1-|\varphi(z)|^2)^{\alpha p}(\ln\frac{e^{\frac{\beta}{\alpha}}}{1-|\varphi(z)|^2})^{\beta p}}dA(z)\\&\leq L\|f\|^p_{\mathcal{B}^\alpha_{\log^\beta}},\end{aligned}$$

which implies that $C_\varphi^g(\mathcal{B}^\alpha_{\log^\beta})\subseteq\mathcal{Q}_K(p,q)$. $\square$
Moreover, the above argument shows that $C_\varphi^g:\mathcal{B}^\alpha_{\log^\beta}\to\mathcal{Q}_K(p,q)$ is, infact, bounded if and only if $L<\infty$.

For example, if $q=\beta=0$, $p=2$, $\alpha=1$, $K(t)=1$, $g(z)=1$ and $\varphi(z)=z$, then L is infinity. Note that in this case, by [13, Theorem 1.2.1] the lacunary series $f(z)=\sum_{k=0}^\infty z^{2^k}$ is in $\mathcal{B}^\alpha_{\log^\beta}=\mathcal{B}$ and by [11, Theorem 7], it is not in $\mathcal{Q}_K(p,q)=\mathcal{Q}_K$, hence $C_\varphi^g(\mathcal{B}^\alpha_{\log^\beta})\subseteq\mathcal{Q}_K(p,q)$ does not hold.

By [9, Lemma 3], we have the following estimates for the growth rate of the functions f in $\mathcal{B}^{\alpha}_{\log^{\beta}}$

$$|f(z)| \leq C \begin{cases} |f(0)| + \|f\|_{\mathcal{B}^{\alpha}_{\log^{\beta}}} & \alpha \in (0,1) \quad \text{or} \quad \alpha = 1, \beta > 1 \\ |f(0)| + \|f\|_{\mathcal{B}^{\alpha}_{\log^{\beta}}} \ln \ln \frac{e^{\frac{\beta}{\alpha}}}{1-|z|} & \alpha = \beta = 1 \\ |f(0)| + \|f\|_{\mathcal{B}^{\alpha}_{\log^{\beta}}} (\ln \frac{e^{\frac{\beta}{\alpha}}}{1-|z|})^{1-\beta} & \alpha = 1, \beta \in (0,1) \\ |f(0)| + \frac{\|f\|_{\mathcal{B}^{\alpha}_{\log^{\beta}}}}{(1-|z|)^{\alpha-1} (\ln \frac{e^{\frac{\beta}{\alpha}}}{1-|z|})^{\beta}} & \alpha > 1, \beta \geq 0, \end{cases} \tag{2.2}$$

for some $C > 0$ independent of f. By (1) we can replace $1-|z|$ by $1-|z|^2$ in (2.2).

Using (1.1), (2.2) and similar to the proof of [7, Lemma 2.1], we have the following Lemma.

Lemma 2.1 *Let $\alpha, p \in (0,\infty)$, $\beta \in [0,\infty)$, $q \in (-2,\infty)$, $g \in \mathcal{H}(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$ such that $C^g_{\varphi}(\mathcal{B}^{\alpha}_{\log^{\beta}}) \subseteq \mathcal{Q}_K(p,q)$. Then C^g_{φ} is Compact if and only if for any bounded sequence (f_n) in $\mathcal{B}^{\alpha}_{\log^{\beta}}$ which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$, we have $\|C^g_{\varphi}(f_n)\|_{\mathcal{Q}_K(p,q)} \to 0$ as $n \to \infty$.*

By Lemma 2.1 we prove the main result of this paper.

Theorem 2.2 *Let $0 \leq \beta < \alpha < \infty$, $p \in (0,\infty)$, $q \in (-2,\infty)$, $g \in \mathcal{H}(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$ such that $C^g_{\varphi}(\mathcal{B}^{\alpha}_{\log^{\beta}}) \subseteq \mathcal{Q}_K(p,q)$. Then C^g_{φ} is compact if and only if*

$$\lim_{r\to 1} \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} \frac{|g(z)|^p (1-|z|^2)^q K(G(z,a))}{(1-|\varphi(z)|^2)^{\alpha p} (\ln \frac{e^{\frac{\beta}{\alpha}}}{1-|\varphi(z)|^2})^{\beta p}} dA(z) = 0. \tag{2.3}$$

Proof. Let (2.3) hold and (f_n) be a sequence in the closed unit ball of $\mathcal{B}^{\alpha}_{\log^{\beta}}$ such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$.

By hypothesis for every $\varepsilon > 0$, there is $\delta \in (0,1)$ such that

$$\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>\delta}\frac{|g(z)|^p(1-|z|^2)^qK(G(z,a))}{(1-|\varphi(z)|^2)^{\alpha p}(\ln\frac{e^{\frac{\beta}{\alpha}}}{1-|\varphi(z)|^2})^{\beta p}}dA(z) < \varepsilon.$$

Let $\Delta = \{w \in \mathbb{D} : |w| \le \delta\}$. Then

$$\begin{aligned}
\|C_\varphi^g(f_n)\|^p_{\mathcal{Q}_K(p,q)} &= \sup\nolimits_{a\in\mathbb{D}}\textstyle\int_{\mathbb{D}}|f_n'(\varphi(z))|^p|g(z)|^p(1-|z|^2)^qK(G(z,a))dA(z)\\
&= \sup\nolimits_{a\in\mathbb{D}}[\textstyle\int_{|\varphi(z)|\le\delta}|f_n'(\varphi(z))|^p|g(z)|^p(1-|z|^2)^qK(G(z,a))dA(z)\\
&+ \textstyle\int_{\delta<|\varphi(z)|<1}|f_n'(\varphi(z))|^p|g(z)|^p(1-|z|^2)^qK(G(z,a))dA(z)]\\
&\le \sup\nolimits_{w\in\Delta}|f_n'(w)|^p\sup\nolimits_{a\in\mathbb{D}}\textstyle\int_{|\varphi(z)|\le\delta}|g(z)|^p(1-|z|^2)^q\\
&\times K(G(z,a))dA(z)\\
&+ b^p_{\alpha,\beta}(f_n)\sup\nolimits_{a\in\mathbb{D}}\textstyle\int_{\delta<|\varphi(z)|<1}\frac{|g(z)|^p(1-|z|^2)^qK(G(z,a))}{(1-|\varphi(z)|^2)^{\alpha p}(\ln\frac{e^{\frac{\beta}{\alpha}}}{1-|\varphi(z)|^2})^{\beta p}}dA(z)\\
&\le \sup\nolimits_{w\in\Delta}|f_n'(w)|^p\sup\nolimits_{a\in\mathbb{D}}\textstyle\int_{|\varphi(z)|\le\delta}|g(z)|^p(1-|z|^2)^q\\
&\times K(G(z,a))dA(z)+\varepsilon,
\end{aligned}$$

since $b^p_{\alpha,\beta}(f_n) \le \|f_n\|_{\mathcal{B}^\alpha_{\log^\beta}} \le 1$. By [2, VII,Theorem 2.1], the sequence (f_n') converges to zero uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$. In particular, $\sup_{w\in\Delta}|f_n'(w)|^p \to 0$ as $n \to \infty$. Hence the boundedness of the kernel function K and the boundedness of g on the compact subset $\{z : |\varphi(z)| \le \delta\}$ of $\mathbb{D}$ implies that $\|C_\varphi^g(f_n)\|_{\mathcal{Q}_K(p,q)} \to 0$ as $n \to \infty$. Therefore, by Lemma 2.1, $C_\varphi^g : \mathcal{B}^\alpha_{\log^\beta} \to \mathcal{Q}_K(p,q)$ is compact.

Conversely, let C_φ^g be compact. Since

$$\begin{aligned}
b_{\alpha,\beta}(\frac{z^n}{n^{1-\alpha+\beta}}) &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha(\ln\frac{e^{\frac{\beta}{\alpha}}}{1-|z|^2})^\beta n^{\alpha-\beta}|z|^{n-1}\\
&\le \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha(\frac{e^{\frac{\beta}{\alpha}}}{1-|z|^2})^\beta n^{\alpha-\beta}|z|^{n-1}\\
&\le C\sup_{z\in\mathbb{D}}(1-|z|)^{\alpha-\beta}n^{\alpha-\beta}|z|^{n-1},
\end{aligned}$$

and for $0 \leq \beta < \alpha < \infty$, $f(x) = n^{\alpha-\beta}x^{n-1}(1-x)^{\alpha-\beta}$ has a maximum in $\frac{n-1}{n-1+\alpha-\beta}$, the sequence $(\frac{z^n}{n^{1-\alpha+\beta}})$ is norm bounded in $\mathcal{B}^\alpha_{\log^\beta}$. It is well known that the series $\sum_{n=1}^\infty \frac{r^n}{n^{1-\alpha+\beta}}$ converges for any $r \in (0,1)$. Hence the sequence $(\frac{z^n}{n^{1-\alpha+\beta}})$ converges to zero uniformly on compact subsets of $\mathbb{D}$, using Lemma 2.1, we have $\|C^g_\varphi(\frac{z^n}{n^{1-\alpha+\beta}})\|_{\mathcal{Q}_K(p,q)} \to 0$ as $n \to \infty$. Whence for given $\varepsilon > 0$,

$$n^{(\alpha-\beta)p}\int_{\mathbb{D}} |\varphi(z)|^{p(n-1)}|g(z)|^p(1-|z|^2)^q K(G(z,a))dA(z) < \varepsilon,$$

for large enough n. Thus for each $r \in (0,1)$,

$$n^{(\alpha-\beta)p} r^{p(n-1)}\int_{|\varphi(z)|>r} |g(z)|^p(1-|z|^2)^q K(G(z,a))dA(z) < \varepsilon.$$

Taking $r \geq n^{\frac{\beta-\alpha}{n-1}}$, we obtain

$$\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r} |g(z)|^p(1-|z|^2)^q K(G(z,a))dA(z) < \varepsilon. \tag{2.4}$$

On the other hand, for any f in the closed unit ball $\mathbb{B}_{\mathcal{B}^\alpha_{\log^\beta}}$ of $\mathcal{B}^\alpha_{\log^\beta}$, if we set $f_t(z) = f(tz)$, then $f_t \to f$ uniformly on compact subsets of $\mathbb{D}$ as $t \to 1$. Since $C^g_\varphi : \mathcal{B}^\alpha_{\log^\beta} \to \mathcal{Q}_K(p,q)$ is compact, using Lemma 2.1, $\|C^g_\varphi(f_t - f)\|_{\mathcal{Q}_K(p,q)} \to 0$ as $t \to 1$. Let $\varepsilon > 0$ be given. Choose $t \in (0,1)$ such that

$$\int_{\mathbb{D}} |(C^g_\varphi f_t)'(z) - (C^g_\varphi f)'(z)|^p(1-|z|^2)^q K(G(z,a))dA(z) < \varepsilon.$$

Using this inequality along with (2.3), we have

$$\begin{aligned}
&\int_{|\varphi(z)|>r} |(C^g_\varphi f)'(z)|^p(1-|z|^2)^q K(G(z,a))dA(z)\\
&\leq C(\varepsilon + \int_{|\varphi(z)|>r} |(C^g_\varphi f_t)'(z)|^p(1-|z|^2)^q K(G(z,a))dA(z))\\
&\leq C\varepsilon(1+\sup_{z\in\mathbb{D}}|f_t'(z)|^p).
\end{aligned}$$

Thus for every $f \in \mathbb{B}_{\mathcal{B}^\alpha_{\log^\beta}}$ and every $\varepsilon > 0$, there exists a $\delta = \delta(f, \varepsilon)$ such that

$$\int_{|\varphi(z)|>r} |(C^g_\varphi f)'(z)|^p (1-|z|^2)^q K(G(z,a)) dA(z) < \varepsilon, \qquad (2.5)$$

for all $r \in [\delta, 1)$. As mentioned in the previous theorem, there are two functions $f_1, f_2 \in \mathcal{B}^\alpha_{\log^\beta}$ such that for each $z \in \mathbb{D}$,

$$\frac{C}{(1-|z|^2)^\alpha (\ln \frac{e^{\frac{\beta}{\alpha}}}{1-|z|^2})^\beta} \leq |f_1'(z)| + |f_2'(z)|.$$

Let $\delta = \max_{1\leq k\leq 2} \delta(\frac{f_k}{\|f_k\|}, \varepsilon)$ and using (2.5) then we have

$$\begin{aligned}
2\varepsilon >& \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} \frac{1}{\|f_1\|^p_{\mathcal{B}^\alpha_{\log^\beta}}} |(C^g_\varphi f_1)'(z)|^p (1-|z|^2)^q K(G(z,a)) dA(z) \\
&+ \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} \frac{1}{\|f_2\|^p_{\mathcal{B}^\alpha_{\log^\beta}}} |(C^g_\varphi f_2)'(z)|^p (1-|z|^2)^q K(G(z,a)) dA(z) \\
\geq& C \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} (|f_1'(\varphi(z))|^p + |f_2'(\varphi(z))|^p) |g(z)|^p (1-|z|^2)^q K(G(z,a)) dA(z) \\
\geq& C \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} \frac{|g(z)|^p (1-|z|^2)^q K(G(z,a))}{(1-|\varphi(z)|^2)^{\alpha p} (\ln \frac{e^{\frac{\beta}{\alpha}}}{1-|\varphi(z)|^2})^{\beta p}} dA(z),
\end{aligned}$$

for all $r \in [\delta, 1)$, which implies (2.3). □

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