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Fixed point type theorem in *S*-metric spaces

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Abstract

A variant of fixed point theorem is proved in the setting of S-metric spaces.

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1 Introduction.

There are different type of generalization of metric spaces in several ways. For example, concepts of 2-metric spaces and D-metric spaces introduced by [2] and [3], respectively. The idea of partial metric space was introduced by [5] or the notion of G-metric spaces announced by [6]. Some authors have proved fixed point type theorems in these spaces (see, e.g.

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[9,10]). Impression of D^*-metric space and also S-metric spaces was initiated by Sedghi, [8,7].

In this paper,we find some new results on S-metric spaces and prove fixed point type theorem for k-contraction condition on S-metric space and offer some examples.

2 Basic Concepts of S-metric spaces

In this section we offer some concepts introduced S. Sedghi et al. ([7]) and results (see, e.g. [4,**?**]). We modify them for our purposes and present some new considerations.

Definition 2.1 *Let X be a nonempty set. We call S-metric on X is a function $S : X^3 \to [0, \infty)$ which satisfies the following conditions for each $x, y, z, a \in X$*
(i) $S(x, y, z) \geq 0$,
(ii) $S(x, y, z) = 0$ if and only if $x = y = z$,
(iii) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.
The set X in which S-metric is defined is called S-metric space.

The standard examples of such S-metric spaces are:
(a) Let X be any normed space, then $S(x, y, z) = \parallel y + z - 2x \parallel + \parallel y - z \parallel$ is a S-metric on X.
(b) Let (X, d) be a metric space, then $S(x, y, z) = d(x, z) + d(y, z)$ is a S-metric on X. This S-metric is called the *usual* S-metric on X.
(c) Another S-metric on (X, d) is $S(x, y, z) = d(x, y) + d(x, z) + d(y, z)$ which is symmetric with respect to the argument.
In the paper we will often use a following important relation.

Lemma 2.1 (See[7]). *In a S-metric space $S(x, x, y) = S(y, y, x)$ for $x, y \in X$.*

Lemma 2.2 *Let (X, S) be a S-metric space. If there exists sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x, x, y)$.*

There exists a natural topology on a S-metric spaces. At first let us remind a notion of (open) ball.

Definition 2.2 *Let (X, S) be a S-metric space. For $r > 0$ and $x \in X$ we define a ball with the center x and radius r as follows:*

$$B_s(x, r) = \{y \in X : S(y, y, x) < r\}.$$

This is quite different concept of ball in a usual metric space which shows the following example:

Example 2.1 *Let $X = \mathbb{R}$. Let $S(x, y, z)$ be a usual S-metric on $\mathbb{R}$ for all $x, y, z \in \mathbb{R}$. Therefore*

$$\begin{aligned} B_s(x_0, 2) = \{y \in X : S(y, y, x_0) < 2\} &= \{y \in \mathbb{R} : 2d(y, x_0) < 2\} \\ &= \{y \in \mathbb{R} : d(y, x_0) < 1\} = B_d(x_0, 1). \end{aligned}$$

By using the notion of ball we can introduce the standard topology on S-metric space.

Remark 2.1 *Any ball is open set in this topology and $x_n \to x$ means that $S(x_n, x_n, x) \to 0$ and $\{x_n\}$ is cauchy sequence if for every $\epsilon > 0$ there exsits a positive integer N, if $n, m > N$ then $x_n \in B_d(x_m, \epsilon)$(which is the same as $x_m \in B_d(x_n, \epsilon)$).*

We prove the following very important result:

Lemma 2.3 *Any S-metric space is a Hausdorff space.*

Proof. Let (X, S) be a S-metric space. Suppose $x \neq y$ and put $r = \frac{1}{3}S(x, x, y)$. Let us show that $B_S(x, r) \cap B_S(y, r) = \emptyset$, for $x, y \in X$. Suppose this is not true then there exists $z \in X$ such that $z \in B_S(x, r) \cap B_S(y, r)$,therefor by definition of ball we have $S(z, z, x) < r$ and $S(z, z, y) < r$. By Lemma 2.1 and (iii), we get

$$3r = S(x, x, y) \leq 2S(z, z, x) + S(z, z, y) = 2S(x, x, z) + S(y, y, z) < 3r,$$

which is a contradiction. $\square$

The following concepts which will be used in our consideration was introduced in [1,4].

Definition 2.3 (See[1]). *An element $(x, y) \in X \times X$ is called a* ***coupled fixed point****(c.f.p) of a mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.*

Remark 2.2 *An element (x, y) is a coupled coincidence point of $F : X \times X \to X$ if and only if it is usual fixed point for mapping $\tilde{F} : X \times X \to X \times X$ given by $\tilde{F}(x, y) = (F(x, y), F(y, x))$.*

Definition 2.4 (See[4]). *An element $(x, y) \in X \times X$ is called a* ***coupled coincidence point****(c.c.p) of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = gx$ and $F(y, x) = gy$.*

Definition 2.5 *Let X be a nonempty set. We say the mappings $F : X \times X \to X$ and $g : X \to X$ satisfy the L-condition if $gF(x, y) = F(gx, gy)$,for all $x, y \in X$.*

The next notion is modification of usual contraction condition.

Definition 2.6 *Let (X, S) be a S-metric space. We say the mappings $F : X \times X \to X$ and $g : X \to X$ satisfy the k-contraction if*

$$S(F(x, y), F(x, y), F(z, w)) \leq k(S(gx, gx, gz) + S(gy, gy, gw)), \quad (2.1)$$

for all $x, y, z, w, u, v \in X$.

As in classical case this condition is quite important for our results.

3 Main Result

The following crucial lemma help us to prove c.c.p theorem on S-metric space . The results such kind can be found e.g. in [10].

Lemma 3.1 *Let (X, S) be a S-metric space and $F : X \times X \to X$ and $g : X \to X$ be two mappings satisfying k-contraction for $k \in (0, \frac{1}{2})$. If (x, y)*

is a c.c.p of the mappings F and g, then $F(x,y) = gx = gy = F(y,x)$.

Proof. Since (x,y) is a c.c.p of the mappings F and g, we have $gx = F(x,y)$ and $gy = F(y,x)$. Suppose $gx \neq gy$. Then by (2.1), and Lemma 2.1, we get

$$\begin{aligned} S(gx,gx,gy) &= S(F(x,y),F(x,y),F(y,x)) \\ &\leq k(S(gx,gx,gy) + S(gy,gy,gx)) \\ &= 2kS(gx,gx,gy). \end{aligned}$$

Since $gx \neq gy$ by (ii) we have $S(gx,gx,gy) \neq 0$. Hence $2k \geq 1$ which is a contradiction. So $gx = gy$, and therefore $F(x,y) = gx = gy = F(y,x)$. $\square$

Theorem 3.1 *Let (X,S) be a S-metric space and $F : X \times X \to X$ and $g : X \to X$ be two mappings satisfying k-contraction for $k \in (0, \frac{1}{2})$ and L-condition. If $g(X)$ is continuous with closed range such that $F(X \times X) \subseteq g(X)$, then there is a unique x in X such that $gx = F(x,x) = x$.*

Proof. Let $x_0, y_0 \in X$. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $gx_1 = F(x_0,y_0)$ and $gy_1 = F(y_0,x_0)$. Then starting from the pair (x_1,y_1), we can choose $x_2, y_2 \in X$ such that $gx_2 = F(x_1,y_1)$ and $gy_2 = F(y_1,x_1)$. Then there exists sequences $\{x_n\}$ and $\{y_n\}$ in X such that $gx_{n+1} = F(x_n,y_n)$ and $gy_{n+1} = F(y_n,x_n)$. For $n \in \mathbb{N}$, from k-contraction condition, we have

$$S(gx_n,gx_n,gx_{n+1}) \leq k(S(gx_{n-1},gx_{n-1},gx_n) + S(gy_{n-1},gy_{n-1},gy_n)).$$

From

$$S(gx_{n-1},gx_{n-1},gx_n) \leq k(S(gx_{n-2},gx_{n-2},gx_{n-1}) + S(gy_{n-2},gy_{n-2},gy_{n-1})),$$

since the similar inequality is correct for $S(gy_{n-1},gy_{n-1},gy_n)$, we have

$$\begin{aligned} S(gx_{n-1},gx_{n-1},gx_n) + S(gy_{n-1},gy_{n-1},gy_n) \leq 2k(&S(gx_{n-2},gx_{n-2},gx_{n-1}) \\ &+ S(gy_{n-2},gy_{n-2},gy_{n-1})) \end{aligned}$$

holds for all $n \in \mathbb{N}$. By repeating this procedure enough time,we obtain for each $n \in \mathbb{N}$

$$S(gx_n, gx_n, gx_{n+1}) \leq \frac{1}{2}(2k)^n(S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)). \quad (3.1)$$

Let $m, n \in \mathbb{N}$ with $m > n+2$. By (iii)and Lemma 2.1, we have

$$\begin{aligned} S(gx_n, gx_n, gx_m) &\leq 2S(gx_n, gx_n, gx_{n+1}) + S(gx_m, gx_m, gx_{n+1}) \\ &= 2S(gx_n, gx_n, gx_{n+1}) + S(gx_{n+1}, gx_{n+1}, gx_m) \\ &\leq 2S(gx_n, gx_n, gx_{n+1}) + 2S(gx_{n+1}, gx_{n+1}, gx_{n+2}) \\ &+ S(gx_m, gx_m, gx_{n+2}) \\ &... \\ &\leq 2\textstyle\sum_{i=n}^{m-2} S(gx_i, gx_i, gx_{i+1}) + S(gx_{m-1}, gx_{m-1}, gx_m). \end{aligned}$$

By (3.1) we will have,

$$\begin{aligned} S(gx_n, gx_n, gx_m) &\leq 2\textstyle\sum_{i=n}^{m-2} S(gx_i, gx_i, gx_{i+1}) + S(gx_{m-1}, gx_{m-1}, gx_m) \\ &\leq 2\textstyle\sum_{i=n}^{m-2} \frac{1}{2}(2k)^i(S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)) \\ &+ \tfrac{1}{2}(2k)^{m-1}(S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)) \\ &\leq (2k)^n(S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)) \\ &\quad [1 + 2k + (2k)^2 + (2k)^3 + ...] \\ &\leq \tfrac{(2k)^n}{1-2k}(S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)). \end{aligned}$$

Letting $n, m \to \infty$, we have

$$\lim_{n,m\to\infty} S(gx_n, gx_n, gx_m) = 0.$$

Thus, $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Similarly, $\{gy_n\}$ is a Cauchy sequence. Since $g(X)$ is closed, $\{gx_n\}$ and $\{gy_n\}$ are convergent to some $x \in X$ and $y \in X$. Since g is continuous, $\{g(gx_n)\}$ is convergent to gx and $\{g(gy_n)\}$ is convergent to gy. Moreover, since F and g satisfy L-condition, we have $g(gx_{n+1}) = g(F(x_n, y_n)) = F(gx_n, gy_n)$, and

$g(gy_{n+1}) = g(F(y_n, x_n)) = F(gy_n, gx_n)$. Thus

$$S(g(gx_{n+1}), g(gx_{n+1}), F(x, y)) \leq k(S(g(gx_n), g(gx_n), gx) + S(g(gy_n), g(gy_n), gy)).$$

Letting $n \to \infty$, and by Lemma 2.2, we get that $S(gx, gx, F(x, y)) \leq k(S(gx, gx, gx) + S(gy, gy, gy)) = 0$.
Hence $gx = F(x, y)$,and similarly, $gy = F(y, x)$. By Lemma 3.1, (x, y) is a c.c.p of the mappings F and g. So $gx = F(x, y) = F(y, x) = gy$. We have

$$\begin{aligned} S(gx_{n+1}, gx_{n+1}, gx) &= S(F(x_n, y_n), F(x_n, y_n), F(x, y)) \\ &\leq k(S(gx_n, gx_n, gx) + S(gy_n, gy_n, gy)). \end{aligned}$$

Letting $n \to \infty$, by Lemma 2.2, we get $S(x, x, gx) \leq k(S(x, x, gx) + S(y, y, gy))$. Similarly, $S(y, y, gy) \leq k(S(x, x, gx) + S(y, y, gy))$. Thus,

$$S(x, x, gx) + S(y, y, gy) \leq 2k(S(x, x, gx) + S(y, y, gy)). \tag{3.2}$$

Since $2k < 1$, inequality (3.2) occur only if $S(x, x, gx) = 0$ and $S(y, y, gy) = 0$. Hence $x = gx$ and $y = gy$. Thus, we get $gx = F(x, x) = x$. To prove the uniqueness, let $z \in X$ with $z \neq x$ such that $z = gz = F(z, z)$. Then

$$\begin{aligned} S(x, x, z) &\leq 2kS(gx, gx, gz) \\ &= 2kS(x, x, z). \end{aligned}$$

Since $2k < 1$ we get a contradiction. $\square$
The following result is immediate corollary from the previous theorem g being the identical mapping.

Theorem 3.2 *Let (X, S) be a complete S-metric space and $F : X \times X \to X$ be a mapping satisfying following contraction condition*

$$S(F(x, y), F(u, v), F(z, w)) \leq k(S(x, u, z) + S(y, v, w))$$

for all $x, y, u, v \in X$ and $k \in (0, \frac{1}{2})$. Then there is a unique $x \in X$ such that $F(x, x) = x$.

Now we present some examples.

Example 3.1 *Let $X = [0, 1]$. Suppose $S(x, y, z)$ be usual S-metric on X,for all $x, y, z \in X$. Then (X, S) is a complete S-metric space. Now we define a map $F : X \times X \to X$ by $F(x, y) = \frac{1}{6}xy$ for $x, y \in X$. Also, define $g : X \to X$ by $g(x) = x$ for $x \in X$. Since*

$$|xy - uv| \leq |x - u| + |y - v|$$

holds for all $x, y, u, v \in X$, we have

$$\begin{aligned} S(F(x,y), F(x,y), F(z,w)) &= 2|\frac{1}{6}xy - \frac{1}{6}zw| \\ &\leq \frac{1}{6}(2|x - z| + 2|y - w|) \\ &= \frac{1}{6}(S(gx, gu, gz) + S(gy, gv, gw)) \end{aligned}$$

holds for all $x, y, u, v, z, w \in X$. It's clear that F and g satisfy all the hypothesis of Theorem 3.1. Therefore F and g have a unique common fixed point. Here $F(0, 0) = g(0) = 0$.

Example 3.2 *Let $X = [0, 1]$. Suppose $S(x, y, z)$ be usual S-metric on X,for all $x, y, z \in X$. Then (X, S) is a complete S-metric space. Define a map $F : X \times X \to X$ by $F(x, y) = 1 - \frac{1}{6}(x + y)$ for $x, y \in X$. Also,*

$$\begin{aligned} S(F(x,y), F(u,v), F(z,w)) &= |F(x,y) - F(z,w)| + |F(u,v) - F(z,w)| \\ &= \frac{1}{6}|z - x + w - y| + \frac{1}{6}|z - u + v - w| \\ &\leq \frac{1}{6}(|x - z| + |u - z|) + \frac{1}{6}(|y - w| + |v - w|) \\ &= \frac{1}{6}(S(x, u, z) + S(y, v, w)). \end{aligned}$$

Then by Theorem 3.2, F has a unique fixed point. Here $x = \frac{3}{4}$ is the unique fixed point of F, that is $F(x, x) = x$.

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