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Redefined (anti) fuzzy BM-algebras

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Abstract

In this paper by using the notion of *anti fuzzy points* and its *besideness* to and *non-quasi-coincidence* with a fuzzy set the concepts of an anti fuzzy subalgebras in BM-algebras are generalized and their inter-relations and related properties are investigated.

Keywords: non-quasi coincident, $(\alpha, \beta)^*$-fuzzy subalgebra, BM-algebras.

1 Introduction

Y. Imai and K. Iseḱi introduced two classes of abstract algebras: BCK-algebras and BCI-algebras [6, 7]. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [4, 5] Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. J. Neggers and H. S. Kim [13] introduced the notion of $d-$algebras which is another generalization of BCK-algebras, and also they introduced the notion of B-algebras [14, 15]. Moreover, Y. B. Jun, E. H. Roh and H. S. Kim [11] introduced a new notion, called a BH-algebra, which is a generalization of $BCH/BCI/BCK$-algebras. Walendziak obtained the another equivalent axioms for B-algebra [18]. H. S. Kim, Y. H. Kim and J. Neggers [9] introduced the notion a (pre-) Coxeter algebra and showed that a Coxeter algebra is equivalent to an abelian group all of whose elements have order 2, i.e., a Boolean group. C. B. Kim and H. S. Kim [8] introduced the notion of a BM-algebra which is a specialization of B-algebras.

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The concept of a fuzzy set was introduced in [19] by L. A. Zadeh. Since then it has become a vigorous area of research in engineering, medical science, social science, physics, statistics, graph theory, etc. In this paper, we introduce the concept of an anti fuzzy subalgebra of BM-algebras by using the notion of *anti fuzzy points* and its *besideness* to and *non-quasi-coincidence* with a fuzzy set, and investigate their inter-relations and related properties.

2 Preliminaries

Definition 2.1. [8] A BM-*algebra* is a non-empty set X with a consonant 0 and a binary operation $*$ satisfying the following axioms:
(I) $x * 0 = x$,
(II) $(z * x) * (z * y) = y * x$,
for all $x, y, z \in X$.

In X we can define a binary relation by $x \leq y$ if and only if $x * y = 0$.

Proposition 2.2. [8] Let X be a BM-algebra. Then for any x, y and z in X, the following hold:
(a) $x * x = 0$,
(b) $0 * (0 * x) = x$,
(c) $0 * (x * y) = y * x$,
(d) $(x * z) * (y * z) = x * y$,
(e) $x * y = 0$ if and only if $y * x = 0$,
(f) $(x * y) * z = (x * z) * y$.

Definition 2.3. A non-empty subset S of a BM-algebra X is called a *subalgebra* of X if $x * y \in S$ for any $x, y \in S$.

A mapping $f : X \longrightarrow Y$ of BM-algebras is called a BM-*homomorphism* if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$.

We now review some fuzzy logic concept (see [19]).

Let X be a set. A fuzzy set A in X is characterized by a membership function $\mu_A : X \longrightarrow [0, 1]$. Let f be a mapping from the set X to the set Y and let B be a fuzzy set in Y with membership function μ_B.

The inverse image of B, denoted $f^{-1}(B)$, is the fuzzy set in X with membership function $\mu_{f^{-1}(B)}$ defined by $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$ for all $x \in X$.

Conversely, let A be a fuzzy set in X with membership function μ_A. Then the image of A, denoted by $f(A)$, is the fuzzy set in Y such that:

$$\mu_{f(A)}(y) = \begin{cases} \sup\limits_{z \in f^{-1}(y)} \mu_A(z) & \text{if } f^{-1}(y) = \{x : f(x) = y\} \neq \emptyset, \\ 0 & \text{otherwise} \end{cases}$$

A fuzzy set $\mathcal{A}$ in X of the form

$$\mathcal{A}(y) := \begin{cases} t \in [0,1) & if\ \ y = x, \\ 1 & if\ \ y \neq x \end{cases}$$

is called an *anti fuzzy point* with support x and value t and is denoted by x_t. A fuzzy set $\mathcal{A}$ in X is said to be *non-unit* if there exists $x \in X$ such that $\mathcal{A}(x) < 1$.

A fuzzy set $\mathcal{A}$ in a BM-algebra X is called an *anti-fuzzy subalgebra* of X if it satisfies [3]

$$(\forall x, y \in X)\,(\mathcal{A}(x * y) \leq \max\{\mathcal{A}(x), \mathcal{A}(y)\}). \tag{2.1}$$

3 Redefined (anti) fuzzy subalgebras

From now $(X, *, 0)$ or simply X is a BM-algebra.

Definition 3.1. An anti-fuzzy point x_t is said to *beside to* (resp. *be non-quasi coincident with*) a fuzzy set $\mathcal{A}$, denoted by $x_t \lessdot \mathcal{A}$ (resp. $x_t \Upsilon \mathcal{A}$), if $\mathcal{A}(x) \leq t$ (resp. $\mathcal{A}(x) + t < 1$). We say that $\lessdot$ (resp. Υ) is a *beside to relation* (resp. *non-quasi coincident with relation*) between anti-fuzzy points and fuzzy sets.

If $x_t \lessdot \mathcal{A}$ or $x_t \Upsilon \mathcal{A}$ (resp. $x_t \lessdot \mathcal{A}$ and $x_t \Upsilon \mathcal{A}$), we say that $x_t \lessdot \vee \Upsilon\, \mathcal{A}$ (resp. $x_t \lessdot \wedge \Upsilon\, \mathcal{A}$).

Proposition 3.2. Let $\mathcal{A}$ be a fuzzy set in a BM-algebra X. Then $\mathcal{A}$ satisfies the condition (2.1) if and only if it satisfies the following condition.

$$(\forall x, y \in X)\,(\forall t_1, t_2 \in [0,1))\,(x_{t_1}, y_{t_2} \lessdot \mathcal{A} \;\Rightarrow\; (x * y)_{\max\{t_1, t_2\}} \lessdot \mathcal{A}). \tag{3.1}$$

Proof. Assume that $\mathcal{A}$ satisfies the condition (2.1). Let $x, y \in X$ and $t_1, t_2 \in [0,1)$ satisfy $x_{t_1}, y_{t_2} \lessdot \mathcal{A}$. Then $\mathcal{A}(x) \leq t_1$ and $\mathcal{A}(y) \leq t_2$. Using (2.1) induces that

$$\mathcal{A}(x * y) \leq \max\{\mathcal{A}(x), \mathcal{A}(y)\} \leq \max\{t_1, t_2\}.$$

Hence $(x * y)_{\max\{t_1, t_2\}} \lessdot \mathcal{A}$.

Conversely, suppose that the condition (3.1) is valid. Since $x_{\mathcal{A}(x)} \lessdot \mathcal{A}$ and $y_{\mathcal{A}(y)} \lessdot \mathcal{A}$ for all $x, y \in X$, it follows from (3.1) that

$$(x * y)_{\max\{\mathcal{A}(x), \mathcal{A}(y)\}} \lessdot \mathcal{A}$$

so that $\mathcal{A}(x * y) \leq \max\{\mathcal{A}(x), \mathcal{A}(y)\}$. This completes the proof.

Note that if $\mathcal{A}$ is a fuzzy set in X such that $\mathcal{A}(x) \geq 0.5$ for all $x \in X$, then the set $\{x_t \mid x_t \lessdot \wedge \Upsilon\, \mathcal{A}\}$ is empty. In what follows let α and β denote any one of $\lessdot$, Υ, $\lessdot \vee \Upsilon$, or $\lessdot \wedge \Upsilon$ unless otherwise specified. To say that $x_t \overline{\alpha} \mathcal{A}$ means that $x_t \alpha \mathcal{A}$ does not hold.

Definition 3.3. A fuzzy set $\mathcal{A}$ in a BM-algebra X is called an $(\alpha, \beta)^*$*-fuzzy subalgebra* of X, where $\alpha \neq \lessdot \wedge \Upsilon$, if it satisfies the following implication:

$$(\forall x, y \in X)\,(\forall t_1, t_2 \in [0,1))\,(x_{t_1}\alpha\mathcal{A}, y_{t_2}\alpha\mathcal{A} \;\Rightarrow\; (x * y)_{\max\{t_1,t_2\}}\beta\mathcal{A}). \qquad (3.2)$$

Example 3.4. [3] Let $X = \{0, 1, 2\}$ be a set with the following table:

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Then $(X, *, 0)$ is a BM-algebra. Let $\mathcal{A}$ be a fuzzy set in X defined by $\mathcal{A}(0) = 0.4$, $\mathcal{A}(1) = 0.3$, and $\mathcal{A}(2) = 0.7$. It is routine to verify that $\mathcal{A}$ is a $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X.

Theorem 3.5. In a BM-algebra, every $(\lessdot \vee \Upsilon, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra is a $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra.

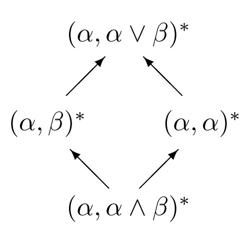
Proof. Let $\mathcal{A}$ be a $(\lessdot \vee \Upsilon, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of a BM-algebra X. Let $x, y \in X$ and $t_1, t_2 \in [0, 1)$ satisfy $x_{t_1} \lessdot \mathcal{A}$ and $y_{t_2} \lessdot \mathcal{A}$. Then $x_{t_1} \lessdot \vee \Upsilon\, \mathcal{A}$ and $y_{t_2} \lessdot \vee \Upsilon\, \mathcal{A}$, which imply that $(x * y)_{\max\{t_1,t_2\}} \lessdot \vee \Upsilon\, \mathcal{A}$. Hence $\mathcal{A}$ is a $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X.

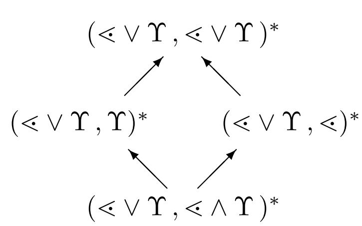
The converse of Theorem 3.5 is not true in general. For example, the $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra $\mathcal{A}$ of X in Example 3.4 is not a $(\lessdot \vee \Upsilon, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X since $1_{0.5} \lessdot \vee \Upsilon\, \mathcal{A}$ and $0_{0.4} \lessdot \vee \Upsilon\, \mathcal{A}$, but $(0 * 1)_{\max\{0.5,0.4\}} = 2_{0.5} \overline{\lessdot \vee \Upsilon}\, \mathcal{A}$.

Obviously any $(\lessdot, \lessdot)^*$-fuzzy subalgebra is a $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra, but the converse is not true. For example, the $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra $\mathcal{A}$ of X in Example 3.4 is not a $(\lessdot, \lessdot)^*$-fuzzy subalgebra of X since $1_{0.38} \lessdot \mathcal{A}$ and $1_{0.34} \lessdot \mathcal{A}$, but $(1 * 1)_{\max\{0.34,0.38\}} = 0_{0.38} \overline{\lessdot} \mathcal{A}$.

Also a $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra $\mathcal{A}$ of X may not be a $(\Upsilon, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra. For example, the $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra $\mathcal{A}$ of X in Example 3.4 is not a $(\Upsilon, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X since $1_{0.6}\Upsilon\mathcal{A}$ and $2_{0.1}\Upsilon\mathcal{A}$ but $(1 * 2)_{max\{0.6,0.1\}} = 2_{0.6} \overline{\lessdot \vee \Upsilon}\, \mathcal{A}$.

Theorem 3.6. Let $\mathcal{A}$ be a fuzzy set in a BM-algebra X. Then the left diagram shows the relationship between $(\alpha, \beta)^*$-fuzzy subalgebras of X, where α, β are one of $\lessdot$ and Υ. Also we have the right diagram.





Proposition 3.7. Let $\mathcal{A}$ be a fuzzy set in a BM-algebra X which is non-unit. If $\mathcal{A}$ is an $(\alpha, \beta)^*$-fuzzy subalgebra of X, then $\mathcal{A}(0) < 1$.

Proof. Assume that $\mathcal{A}(0) = 1$. Since $\mathcal{A}$ is non-unit, there exists $x \in X$ such that $\mathcal{A}(x) = t < 1$. If $\alpha = \lessdot$ or $\alpha = \lessdot \vee \Upsilon$, then $x_t \alpha \mathcal{A}$, but $(x * x)_{\max\{t,t\}} = 0_t \overline{\beta} \mathcal{A}$. This is a contradiction. If $\alpha = \Upsilon$, then $x_0 \alpha \mathcal{A}$ because $\mathcal{A}(x) + 0 = t + 0 = t < 1$. But $(x * x)_{\max\{0,0\}} = 0_0 \overline{\beta} \mathcal{A}$, which is a contradiction. Hence $\mathcal{A}(0) < 1$.

Proposition 3.8. Let $\mathcal{A}$ be a fuzzy set in a BM-algebra X. If $\mathcal{A}$ is a $(\lessdot, \lessdot)^*$-fuzzy subalgebra of X, then $\mathcal{A}(0) \leq \mathcal{A}(x)$, for all $x \in X$.

Proof. Since $x * x = 0$, for all $x \in X$. Then we get that $\mathcal{A}(0) = \mathcal{A}(x * x) \leq \max(\mathcal{A}(x), \mathcal{A}(x)) = \mathcal{A}(x)$.

For a fuzzy set $\mathcal{A}$ in a BM-algebra X, we denote

$$X^* := \{x \in X \mid \mathcal{A}(x) < 1\}.$$

Theorem 3.9. Let $\mathcal{A}$ be a fuzzy set in a BM-algebra X which is non-unit. If $\mathcal{A}$ is an $(\alpha, \beta)^*$-fuzzy subalgebra of X where (α, β) is one of the following:

- $(\lessdot, \lessdot)$,
- $(\lessdot, \Upsilon)$,
- $(\Upsilon, \lessdot)$,
- (Υ, Υ),

then the set X^* is a subalgebra of X.

Proof. (i) Assume that $\mathcal{A}$ is a $(\lessdot, \lessdot)^*$-fuzzy subalgebra of X. Let $x, y \in X^*$. Then $\mathcal{A}(x) < 1$ and $\mathcal{A}(y) < 1$. Assume that $\mathcal{A}(x * y) = 1$. Note that $x_{\mathcal{A}(x)} \lessdot \mathcal{A}$ and $y_{\mathcal{A}(y)} \lessdot \mathcal{A}$. But, since $\mathcal{A}(x * y) = 1 > \max\{\mathcal{A}(x), \mathcal{A}(y)\}$, we get $(x * y)_{\{\mathcal{A}(x), \mathcal{A}(y)\}} \overline{\lessdot} \mathcal{A}$. This is a contradiction, and so $\mathcal{A}(x * y) < 1$ which shows that $x * y \in X^*$. Hence X^* is a subalgebra of X.

(ii) Assume that $\mathcal{A}$ is a $(\lessdot, \Upsilon)^*$-fuzzy subalgebra of X. Let $x, y \in X^*$. Then $\mathcal{A}(x) < 1$ and $\mathcal{A}(y) < 1$. If $\mathcal{A}(x * y) = 1$, then

$$\mathcal{A}(x * y) + \max\{\mathcal{A}(x), \mathcal{A}(y)\} \geq 1.$$

Hence $(x * y)_{\max\{\mathcal{A}(x), \mathcal{A}(y)\}} \overline{\Upsilon} \mathcal{A}$, which is a contradiction since $x_{\mathcal{A}(x)} \lessdot \mathcal{A}$ and $y_{\mathcal{A}(y)} \lessdot \mathcal{A}$. Thus $\mathcal{A}(x * y) < 1$, and so $x * y \in X^*$. Therefore X^* is a subalgebra of X.

(iii) Assume that $\mathcal{A}$ is a $(\Upsilon, \lessdot)^*$-fuzzy subalgebra of X. Let $x, y \in X^*$. Then $\mathcal{A}(x) < 1$ and $\mathcal{A}(y) < 1$. Thus $x_0 \Upsilon \mathcal{A}$ and $y_0 \Upsilon \mathcal{A}$. If $\mathcal{A}(x * y) = 1$, then $\mathcal{A}(x * y) = 1 > 0 = \max\{0, 0\}$. Therefore $(x * y)_{\max\{0,0\}} \overline{\lessdot} \mathcal{A}$, which is a contradiction. Hence $\mathcal{A}(x * y) < 1$, and so $x * y \in X^*$.

(iv) Assume that $\mathcal{A}$ is a $(\Upsilon, \Upsilon)^*$-fuzzy subalgebra of X. Let $x, y \in X^*$. Then $\mathcal{A}(x) < 1$ and $\mathcal{A}(y) < 1$. If $\mathcal{A}(x * y) = 1$, then $\mathcal{A}(x * y) + \max\{0, 0\} = 1$ and so $(x * y)_{\max\{0,0\}} \overline{\Upsilon} \mathcal{A}$. This is impossible, and hence $\mathcal{A}(x * y) < 1$, i.e., $x * y \in X^*$. This completes the proof.

Corollary 3.10. Let $\mathcal{A}$ be a fuzzy set in a BM-algebra X which is non-unit. If $\mathcal{A}$ is an $(\alpha,\beta)^*$-fuzzy subalgebra of X where (α,β) is one of the following:

- $(\lessdot, \lessdot \wedge \Upsilon)$,
- $(\lessdot, \lessdot \vee \Upsilon)$,
- $(\Upsilon, \lessdot \wedge \Upsilon)$,
- $(\Upsilon, \lessdot \vee \Upsilon)$,
- $(\lessdot \vee \Upsilon, \lessdot \vee \Upsilon)$,
- $(\lessdot \vee \Upsilon, \lessdot \wedge \Upsilon)$,

then the set X^* is a subalgebra of X.

Proof. By Theorem 3.6, it is enough to prove for the cases:

$$\text{(i) } (\lessdot, \lessdot \vee \Upsilon) \quad \text{and} \quad \text{(ii) } (\Upsilon, \lessdot \vee \Upsilon).$$

(i) Let $x, y \in X^*$. Then $\mathcal{A}(x) < 1$ and $\mathcal{A}(y) < 1$, and so $\mathcal{A}(x) = t_1$ and $\mathcal{A}(y) = t_2$ for some $t_1, t_2 \in [0,1)$. It follows that $x_{t_1} \lessdot \mathcal{A}$ and $y_{t_2} \lessdot \mathcal{A}$ so that $(x*y)_{\max\{t_1,t_2\}} \lessdot \vee \Upsilon\, \mathcal{A}$, i.e., $(x*y)_{\max\{t_1,t_2\}} \lessdot \mathcal{A}$ or $(x*y)_{\max\{t_1,t_2\}}\Upsilon\mathcal{A}$. If $(x*y)_{\max\{t_1,t_2\}} \lessdot \mathcal{A}$, then $\mathcal{A}(x*y) \leq \max\{t_1,t_2\} < 1$ and thus $x*y \in X^*$. If $(x*y)_{\max\{t_1,t_2\}}\Upsilon\mathcal{A}$, then $\mathcal{A}(x*y) \leq \mathcal{A}(x*y) + \max\{t_1,t_2\} < 1$. Hence $x*y \in X^*$. For the case (ii), let $x, y \in X^*$. Then $\mathcal{A}(x) < 1$ and $\mathcal{A}(y) < 1$, which imply that $x_0\Upsilon\mathcal{A}$ and $y_0\Upsilon\mathcal{A}$. Since $\mathcal{A}$ is a $(\Upsilon, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra, $(x*y)_0 = (x*y)_{\max\{0,0\}} \lessdot \vee\Upsilon\, \mathcal{A}$, i.e., $(x*y)_0 \lessdot \mathcal{A}$ or $(x*y)_0\Upsilon\mathcal{A}$. If $(x*y)_0 \lessdot \mathcal{A}$, then $\mathcal{A}(x*y) = 0 < 1$. If $(x*y)_0\Upsilon\mathcal{A}$, then $\mathcal{A}(x*y) = \mathcal{A}(x*y) + 0 < 1$. Therefore $x*y \in X^*$. This completes the proof.

Theorem 3.11. Let $\mathcal{A}$ be a fuzzy set in a BM-algebra X which is non-unit. Then every $(\Upsilon,\Upsilon)^*$-fuzzy subalgebra of X is a constant on X^*.

Proof. Let $\mathcal{A}$ be a $(\Upsilon,\Upsilon)^*$-fuzzy subalgebra of X which is non-unit. Assume that $\mathcal{A}$ is not constant on X^*. Then there exists $y \in X^*$ such that $t_y = \mathcal{A}(y) \neq \mathcal{A}(0) = t_0$. Then either $t_y > t_0$ or $t_y < t_0$. If $t_y < t_0$, then $\mathcal{A}(y) + (1-t_0) = t_y + 1 - t_0 < 1$ and so $y_{1-t_0}\Upsilon\mathcal{A}$. Since

$$\mathcal{A}(y*y) + (1-t_0) = \mathcal{A}(0) + 1 - t_0 = t_0 + 1 - t_0 = 1,$$

we have $(y*y)_{\max\{1-t_0,1-t_0\}}\overline{\Upsilon}\mathcal{A}$. This is a contradiction. Now assume that $t_y > t_0$. Choose $t_1, t_2 \in [0,1)$ such that $t_1 < 1 - t_y < t_2 < 1 - t_0$. Then $\mathcal{A}(0) + t_2 = t_0 + t_2 < 1$ and $\mathcal{A}(y) + t_1 = t_y + t_1 < 1$. Thus $0_{t_2}\Upsilon\mathcal{A}$ and $y_{t_1}\Upsilon\mathcal{A}$. Since

$$\mathcal{A}(y*0) + \max\{t_1,t_2\} = \mathcal{A}(y) + t_2 = t_y + t_2 > 1,$$

we get $(y*0)_{\max\{t_1,t_2\}}\overline{\Upsilon}\mathcal{A}$, which is a contradiction. Therefore $\mathcal{A}$ is a constant on X^*.

Theorem 3.12. Let $\mathcal{A}$ be a fuzzy set in a BM-algebra X. Then $\mathcal{A}$ is a non-unit $(\Upsilon,\Upsilon)^*$-fuzzy subalgebra of X if and only if there exists a subalgebra S of X such that

$$\mathcal{A}(x) := \begin{cases} t \in [0,1) & if\ \ x \in S, \\ 1 & otherwise \end{cases} \tag{3.3}$$

Proof. Let $\mathcal{A}$ be a non-unit $(\Upsilon, \Upsilon)^*$-fuzzy subalgebra of X. Then by Proposition 3.7 and Theorems 3.11 and 3.9 we get that $\mathcal{A}(x) < 1$, for all $x \in X$ and X^* is a subalgebra of X, and

$$\mathcal{A}(x) := \begin{cases} \mathcal{A}(0) & if\ x \in X^*, \\ 1 & otherwise \end{cases}$$

Conversely, let S be a subalgebra of X which satisfy (3.3). Assume that $x_s \Upsilon \mathcal{A}$ and $y_r \Upsilon \mathcal{A}$ for some $s, r \in [0, 1)$. Then $\mathcal{A}(x) + s < 1$ and $\mathcal{A}(y) + r < 1$, and so $\mathcal{A}(x) \neq 1$ and $\mathcal{A}(y) \neq 1$. Thus $x, y \in S$ and so $x * y \in S$. It follows that $\mathcal{A}(x * y) + \max\{s, r\} = t + \max\{s, r\} < 1$ so that $(x * y)_{\max\{s,r\}} \Upsilon \mathcal{A}$. Therefore $\mathcal{A}$ is a non-unit $(\Upsilon, \Upsilon)^*$-fuzzy subalgebra of X.

Theorem 3.13. Let S be a subalgebra of a BM-algebra X and let $\mathcal{A}$ be a fuzzy set in X such that

(i) $(\forall x \in X \setminus S)\ (\mathcal{A}(x) = 1)$,

(ii) $(\forall x \in S)\ (\mathcal{A}(x) \leq 0.5)$.

Then $\mathcal{A}$ is a $(\Upsilon, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X.

Proof. Let $x, y \in X$ and $t_1, t_2 \in [0, 1)$ be such that $x_{t_1} \Upsilon \mathcal{A}$ and $y_{t_2} \Upsilon \mathcal{A}$, that is, $\mathcal{A}(x) + t_1 < 1$ and $\mathcal{A}(y) + t_2 < 1$. If $x * y \notin S$, then $x \in X \setminus S$ or $y \in X \setminus S$, i.e., $\mathcal{A}(x) = 1$ or $\mathcal{A}(y) = 1$. It follows that $t_1 < 0$ or $t_2 < 0$. This is a contradiction, and so $x * y \in S$. Hence $\mathcal{A}(x * y) \leq 0.5$. If $\max\{t_1, t_2\} < 0.5$, then $\mathcal{A}(x * y) + \max\{t_1, t_2\} < 1$ and thus $(x * y)_{\max\{t_1,t_2\}} \Upsilon \mathcal{A}$. If $\max\{t_1, t_2\} \geq 0.5$, then $\mathcal{A}(x * y) \leq 0.5 \leq \max\{t_1, t_2\}$ and so $(x * y)_{\max\{t_1,t_2\}} \lessdot \mathcal{A}$. Therefore $(x * y)_{\max\{t_1,t_2\}} \lessdot \vee \Upsilon\, \mathcal{A}$. This completes the proof.

Theorem 3.14. Let $\mathcal{A}$ be a $(\Upsilon, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of a BM-algebra X such that $\mathcal{A}$ is not constant on X^*. Then there exists $x \in X$ such that $\mathcal{A}(x) \leq 0.5$. Moreover $\mathcal{A}(x) \leq 0.5$ for all $x \in X^*$.

Proof. Assume that $\mathcal{A}(x) > 0.5$ for all $x \in X$. Since $\mathcal{A}$ is not constant on X^*, there exists $x \in X^*$ such that $t_x = \mathcal{A}(x) \neq \mathcal{A}(0) = t_0$. Then either $t_0 > t_x$ or $t_0 < t_x$. For the first case, choose $\delta < 0.5$ such that $t_x + \delta < 1 < t_0 + \delta$. It follows that $x_\delta \Upsilon \mathcal{A}$,

$$\mathcal{A}(x * x) = \mathcal{A}(0) = t_0 > \delta = \max\{\delta, \delta\},$$

$$\mathcal{A}(x * x) + \max\{\delta, \delta\} = \mathcal{A}(0) + \delta = t_0 + \delta > 1$$

so that $(x * x)_{\max\{\delta,\delta\}} \overline{\lessdot \vee \Upsilon}\, \mathcal{A}$. This is a contradiction. For the second case, we can choose $\delta < 0.5$ such that $t_x + \delta > 1 > t_0 + \delta$. Then $0_\delta \Upsilon \mathcal{A}$ and $x_1 \Upsilon \mathcal{A}$, but $(x * 0)_{\max\{1,\delta\}} = x_1 \overline{\lessdot \vee \Upsilon}\, \mathcal{A}$ since $\mathcal{A}(x) > 0.5 > \delta$ and $\mathcal{A}(x) + \delta = t_x + \delta > 1$. This leads to a contradiction. Therefore $\mathcal{A}(x) \leq 0.5$ for some $x \in X$. We now show that $\mathcal{A}(0) \leq 0.5$. Assume that $\mathcal{A}(0) = t_0 > 0.5$. Since there exists $x \in X$ such that $\mathcal{A}(x) = t_x \leq 0.5$, we have $t_0 > t_x$. Choose $t_1 < t_0$ such that $t_x + t_1 < 1 < t_0 + t_1$. Then $\mathcal{A}(x) + t_1 = t_x + t_1 < 1$, and so $x_{t_1} \Upsilon \mathcal{A}$. Now we get

$$\mathcal{A}(x * x) + \max\{t_1, t_1\} = \mathcal{A}(0) + t_1 = t_0 + t_1 > 1,$$

$$\mathcal{A}(x*x)=\mathcal{A}(0)=t_0>t_1=\max\{t_1,t_1\}.$$

Hence $(x*x)_{\max\{t_1,t_1\}}\overline{\lessdot\vee\Upsilon}\,\mathcal{A}$, a contradiction. Therefore $\mathcal{A}(0)\leq 0.5$. Finally suppose that $t_x=\mathcal{A}(x)>0.5$ for some $x\in X^*$. Let t be such that $0<t<0.5$ and $t_x>0.5+t$. Therefore $\mathcal{A}(x)+0<1$ and $\mathcal{A}(0)+(0.5-t)<1$ which imply that $x_0\Upsilon\mathcal{A}$ and $0_{(0.5-t)}\Upsilon\mathcal{A}$. But $(x*0)_{\max(0,0.5-t)}=x_{(0.5-t)}$ and so $\mathcal{A}(x)>0.5-t$ and $\mathcal{A}(x)+0.5-t>1$, thus $(x*0)_{0,0.5-t)}\overline{\lessdot\vee\Upsilon}\,\mathcal{A}$, which is a contradiction. Hence $\mathcal{A}(x)\leq 0.5$.

We give a characterization of a $(\lessdot,\lessdot\vee\Upsilon)^*$-fuzzy subalgebra.

Theorem 3.15. Let $\mathcal{A}$ be a fuzzy set in a BM-algebra X. Then $\mathcal{A}$ is a $(\lessdot,\lessdot\vee\Upsilon)^*$-fuzzy subalgebra of X if and only if it satisfies the following inequality.

$$(\forall x,y\in X)\,(\mathcal{A}(x*y)\leq\max\{\mathcal{A}(x),\mathcal{A}(y),0.5\}). \tag{3.4}$$

Proof. Assume that $\mathcal{A}$ is a $(\lessdot,\lessdot\vee\Upsilon)^*$-fuzzy subalgebra of X. Let $x,y\in X$ be such that $\max\{\mathcal{A}(x),\mathcal{A}(y)\}>0.5$. Then $\mathcal{A}(x*y)\leq\max\{\mathcal{A}(x),\mathcal{A}(y)\}$. If it is not true, then $\mathcal{A}(x*y)<t<\max\{\mathcal{A}(x),\mathcal{A}(y)\}$ for some $t\in(0.5,1)$. It follows that $x_t\lessdot\mathcal{A}$ and $y_t\lessdot\mathcal{A}$, but $(x*y)_{\max\{t,t\}}=(x*y)_t\overline{\lessdot\vee\Upsilon}\,\mathcal{A}$ which is a contradiction. Hence $\mathcal{A}(x*y)\leq\max\{\mathcal{A}(x),\mathcal{A}(y)\}$ whenever $\max\{\mathcal{A}(x),\mathcal{A}(y)\}>0.5$. If $\max\{\mathcal{A}(x),\mathcal{A}(y)\}\leq 0.5$, then $x_{0.5}\lessdot\mathcal{A}$ and $y_{0.5}\lessdot\mathcal{A}$ which imply that $(x*y)_{0.5}=(x*y)_{\max\{0.5,0.5\}}\lessdot\vee\Upsilon\,\mathcal{A}$. Therefore $\mathcal{A}(x*y)\leq 0.5$ because if $\mathcal{A}(x*y)>0.5$, then $\mathcal{A}(x*y)+0.5>0.5+0.5=1$, a contradiction. Hence $\mathcal{A}(x*y)\leq\max\{\mathcal{A}(x),\mathcal{A}(y),0.5\}$ for all $x,y\in X$.

Conversely, assume that $\mathcal{A}$ satisfies (3.4). Let $x,y\in X$ and $t_1,t_2\in[0,1)$ be such that $x_{t_1}\lessdot\mathcal{A}$ and $y_{t_2}\lessdot\mathcal{A}$. Then $\mathcal{A}(x)\leq t_1$ and $\mathcal{A}(y)\leq t_2$. Suppose that $\mathcal{A}(x*y)>\max\{t_1,t_2\}$. If $\max\{\mathcal{A}(x),\mathcal{A}(y)\}>0.5$ then

$$\mathcal{A}(x*y)\leq\max\{\mathcal{A}(x),\mathcal{A}(y),0.5\}=\max\{\mathcal{A}(x),\mathcal{A}(y)\}\leq\max\{t_1,t_2\}.$$

This is a contradiction, and so $\max\{\mathcal{A}(x),\mathcal{A}(y)\}\leq 0.5$. It follows that

$$\mathcal{A}(x*y)+\max\{t_1,t_2\}<2\mathcal{A}(x*y)\leq 2\max\{\mathcal{A}(x),\mathcal{A}(y),0.5\}\leq 1$$

so that $(x*y)_{\max\{t_1,t_2\}}\Upsilon\mathcal{A}$. Hence $(x*y)_{\max\{t_1,t_2\}}\lessdot\vee\Upsilon\,\mathcal{A}$, and consequently $\mathcal{A}$ is a $(\lessdot,\lessdot\vee\Upsilon)^*$-fuzzy subalgebra of X.

Theorem 3.16. For any subset S of a BM-algebra X, let χ_S denote the characteristic function of S. Then the function $\chi_S^c:X\to[0,1]$ defined by $\chi_S^c(x)=1-\chi_S(x)$ for all $x\in X$ is a $(\lessdot,\lessdot\vee\Upsilon)^*$-fuzzy subalgebra of X if and only if S is a subalgebra of X.

Proof. Assume that χ_S^c is a $(\lessdot,\lessdot\vee\Upsilon)^*$-fuzzy subalgebra of X and let $x,y\in S$. Then $\chi_S^c(x)=1-\chi_S(x)=0$ and $\chi_S^c(y)=1-\chi_S(y)=0$. Hence $x_0\lessdot\chi_S^c$ and $y_0\lessdot\chi_S^c$, which imply that $(x*y)_0=(x*y)_{\max\{0,0\}}\lessdot\vee\Upsilon\,\chi_S^c$. Thus $\chi_S^c(x*y)\leq 0$ or $\chi_S^c(x*y)+0<1$. If $\chi_S^c(x*y)\leq 0$, then $1-\chi_S(x*y)=0$, i.e., $\chi_S(x*y)=1$. Hence $x*y\in S$. If $\chi_S^c(x*y)+0<1$, then $\chi_S(x*y)>0$. Thus $\chi_S(x*y)=1$, and so $x*y\in S$. Therefore S is a subalgebra of X.

Conversely, suppose that S is a subalgebra of X. Let $x,y\in X$. If $x,y\in S$, then $x*y\in S$, and thus

$$\chi_S^c(x*y)=\max\{\chi_S^c(x),\chi_S^c(y)\}\leq\max\{\chi_S^c(x),\chi_S^c(y),0.5\}.$$

If any one of x and y does not belong to S, then $\chi_S^c(x) = 1$ or $\chi_S^c(y) = 1$. Hence $\chi_S^c(x * y) \leq \max\{\chi_S^c(x), \chi_S^c(y)\} \leq \max\{\chi_S^c(x), \chi_S^c(y), 0.5\}$. Using Theorem 3.15, we know that χ_S^c is a $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X.

Theorem 3.17. A fuzzy set $\mathcal{A}$ in a BM-algebra X is a $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X if and only if the set

$$L(\mathcal{A}; t) := \{x \in X \mid \mathcal{A}(x) \leq t\},\ t \in [0.5, 1)$$

is a subalgebra of X.

Proof. Assume that $\mathcal{A}$ is a $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X and let $x, y \in L(\mathcal{A}; t)$. Then $\mathcal{A}(x) \leq t$ and $\mathcal{A}(y) \leq t$, and so $x_t \lessdot \mathcal{A}$ and $y_t \lessdot \mathcal{A}$. It follows from Theorem 3.15 that

$$\mathcal{A}(x * y) \leq \max\{\mathcal{A}(x), \mathcal{A}(y), 0.5\} \leq \max\{t, 0.5\} = t$$

so that $x * y \in L(\mathcal{A}; t)$. Hence $L(\mathcal{A}; t)$ is a subalgebra of X.

Conversely, let $\mathcal{A}$ be a fuzzy set in X such that the set $L(\mathcal{A}; t) := \{x \in X \mid \mathcal{A}(x) \leq t\}$ is a subalgebra of X for all $t \in [0.5, 1)$. If there exist $x, y \in X$ such that $\mathcal{A}(x * y) > \max\{\mathcal{A}(x), \mathcal{A}(y), 0.5\}$, then we can take $t \in (0, 1)$ such that

$$\max\{\mathcal{A}(x), \mathcal{A}(y), 0.5\} < t < \mathcal{A}(x * y).$$

Thus $x, y \in L(\mathcal{A}; t)$ and $t > 0.5$, and so $x * y \in L(\mathcal{A}; t)$, i.e., $\mathcal{A}(x * y) \leq t$. This is a contradiction. Therefore $\mathcal{A}(x * y) \leq \max\{\mathcal{A}(x), \mathcal{A}(y), 0.5\}$ for all $x, y \in X$. Using Theorem 3.15, we conclude that $\mathcal{A}$ is a $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X.

Proposition 3.18. Let $\mathcal{A}$ be a fuzzy set in a BM-algebra X. Then $\mathcal{A}$ is a $(\lessdot, \lessdot)^*$-fuzzy subalgebra of X if and only if for all $t \in [0, 1]$, the nonempty level set $L(\mathcal{A}; t)$ is a subalgebra of X.

Proof. The proof follows from Proposition 3.2.

Theorem 3.19. Let $\mathcal{A}$ be a fuzzy set in a BM-algebra X. Then $\mathcal{A}$ is a non-unit $(\Upsilon, \Upsilon)^*$-fuzzy subalgebra of X if and only if $L(\mathcal{A}; \mathcal{A}(0)) = X^*$ and for all $t \in [0, 1]$, the nonempty level set $L(\mathcal{A}; t)$ is a subalgebra of X.

Proof. Let $\mathcal{A}$ be a non-unit $(\Upsilon, \Upsilon)^*$-fuzzy subalgebra of X. Then by Theorem 3.12 we have

$$\mathcal{A}(x) = \begin{cases} \mathcal{A}(0) & \text{if } x \in X^* \\ 1 & otherwise \end{cases}$$

So it is easy to check that $L(\mathcal{A}; \mathcal{A}(0)) = X^*$. Let $x, y \in L(\mathcal{A}; t)$, for $t \in [0, 1]$. Then $\mathcal{A}(x) \leq t$ and $\mathcal{A}(y) \leq t$. If $t = 1$, then it is clear that $x * y \in L(\mathcal{A}; 1)$. Now let $t \in [0, 1)$. Then $x, y \in X^*$ and so $x * y \in X^*$. Hence $\mathcal{A}(x * y) = \mathcal{A}(0) \leq t$. Therefore $L(\mathcal{A}; t)$ is a subalgebra of X.

Conversely, since $L(\mathcal{A}; \mathcal{A}(0)) = X^*$ and $0 \in L(\mathcal{A}; \mathcal{A}(0))$, X^* is a subalgebra of X

and $\mathcal{A}$ is non-unit. Now let $x \in X^*$. Then $\mathcal{A}(x) \geq \mathcal{A}(0)$ and $\mathcal{A}(x) > 0$. Since $L(\mathcal{A};\mathcal{A}(x)) \neq \emptyset$, so $L(\mathcal{A};\mathcal{A}(x))$ is a subalgebra of X. Then $0 \in L(\mathcal{A};\mathcal{A}(x))$ implies that $\mathcal{A}(0) \geq \mathcal{A}(x)$. Hence $\mathcal{A}(x) = \mathcal{A}(0)$, for all $x \in X^*$. Therefore

$$\mathcal{A}(x) = \begin{cases} \mathcal{A}(0) & \text{if } x \in X^* \\ 1 & otherwise \end{cases}$$

Hence by Theorem 3.12 $\mathcal{A}$ is a $(\Upsilon, \Upsilon)^*$-fuzzy subalgebra of X.

Theorem 3.20. Every $(\Upsilon, \Upsilon)^*$-fuzzy subalgebra is a $(\lessdot, \lessdot)^*$-fuzzy subalgebra.

Proof. The proof follows from Theorem 3.19 and Proposition 3.18.

Theorem 3.21. Let $\mathcal{A}$ be a non-unit $(\Upsilon, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X. Then the nonempty level set $L(\mathcal{A};t)$ is a subalgebra of X, for all $t \in [0.5, 1]$.

Proof. If $\mathcal{A}$ is a constant on X^*, then by Theorem 3.12, $\mathcal{A}$ is a $(\Upsilon, \Upsilon)^*$-fuzzy subalgebra. Thus by Theorem 3.19 we have the nonempty level set $L(\mathcal{A};t)$ is a subalgebra of X, for $t \in [0, 1]$. If $\mathcal{A}$ is not a constant on X^*, then by Theorem 3.12, we have

$$\mathcal{A}(x) = \begin{cases} \alpha & \text{if } x \in X^* \\ 1 & otherwise \end{cases}$$

where $\alpha \leq 0.5$. Now we show that the nonempty level set $L(\mathcal{A};t)$ is a subalgebra of X for $t \in [0.5, 1]$. If $t = 1$, then it is clear that $L(\mathcal{A};t)$ is a subalgebra of X. Now let $t \in [0.5, 1)$ and $x, y \in L(\mathcal{A};t)$. Then $\mathcal{A}(x), \mathcal{A}(y) \leq t < 1$ imply that $x, y \in X^*$. Thus $x * y \in X^*$ and so $\mathcal{A}(x * y) \leq 0.5 \leq t$. Therefore $x * y \in L(\mathcal{A};t)$.

Theorem 3.22. Let $\mathcal{A}$ be a non-unit fuzzy set of BM algebra X, $L(\mathcal{A};0.5) = X^*$ and the nonempty level set $L(\mathcal{A};t)$ is a subalgebra of X, for all $t \in [0, 1]$. Then $\mathcal{A}$ is a $(\Upsilon, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X.

Proof. Since $\mathcal{A} \neq 1$ we get that $X^* \neq \emptyset$. Thus by hypothesis we have $L(\mathcal{A};0.5) \neq \emptyset$ and so X^* is a subalgebra of X. Also $\mathcal{A}(x) \leq 0.5$, for all $x \in X^*$ and $\mathcal{A}(x) = 1$, if $x \notin X^*$. Therefore by Theorem 3.21, $\mathcal{A}$ is a $(\Upsilon, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X.

Theorem 3.23. Let $\mathcal{A}$ be an $(\Upsilon, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of BM algebra X. Then for all $t \in [0.5, 1]$, the nonempty level set $L(\mathcal{A};t)$ is a subalgebra of X. Conversely, if the nonempty level set $\mathcal{A}$ is a subalgebra of X, for all $t \in [0, 1]$, then $\mathcal{A}$ is an $(\Upsilon, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X.

Proof. Let $\mathcal{A}$ be an $(\Upsilon, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X. If $t = 1$, then $L(\mathcal{A};t)$ is a subalgebra of X. Now let $L(\mathcal{A};t) \neq \emptyset$, $0.5 \leq t < 1$ and $x, y \in L(\mathcal{A};t)$. Then $\mathcal{A}(x), \mathcal{A}(y) \leq t$. Thus by hypothesis we have $\mathcal{A}(x * y) \leq \max(\mathcal{A}(x), \mathcal{A}(y), 0.5) \leq \max(t, 0.5) \leq t$. Therefore $L(\mathcal{A};t)$ is a subalgebra of X.

Conversely, let $x, y \in X$. Then we have

$$\mathcal{A}(x), \mathcal{A}(y) \leq \max(\mathcal{A}(x), \mathcal{A}(y), 0.5) = t_0$$

Hence $x, y \in L(\mathcal{A};t_0)$, for $t_0 \in [0, 1]$ and so $x * y \in L(\mathcal{A};t_0)$. Therefore $\mathcal{A}(x * y) \leq t_0 = \max(\mathcal{A}(x), \mathcal{A}(y), 0.5)$, then $\mathcal{A}$ is a $(\Upsilon, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X.

For any fuzzy set $\mathcal{A}$ in X and $t \in [0,1)$, we denote

$$\mathcal{A}_t := \{x \in X \mid x_t \Upsilon \mathcal{A}\} \quad \text{and} \quad [\mathcal{A}]_t := \{x \in X \mid x_t \lessdot \vee \Upsilon\, \mathcal{A}\}.$$

Obviously $[\mathcal{A}]_t = L(\mathcal{A};t) \cup \mathcal{A}_t$.

Theorem 3.24. A fuzzy set $\mathcal{A}$ in a BM-algebra X is a $(\lessdot, \lessdot \vee \Upsilon\,)^*$-fuzzy subalgebra of X if and only if $[\mathcal{A}]_t$ is a subalgebra of X for all $t \in [0,1)$.

Proof. Let $\mathcal{A}$ be a $(\lessdot, \lessdot \vee \Upsilon\,)^*$-fuzzy subalgebra of X and let $x, y \in [\mathcal{A}]_t$ for $t \in [0,1)$. Then $x_t \lessdot \vee \Upsilon\, \mathcal{A}$ and $y_t \lessdot \vee \Upsilon\, \mathcal{A}$, that is, $\mathcal{A}(x) \leq t$ or $\mathcal{A}(x) + t > 1$, and $\mathcal{A}(y) \leq t$ or $\mathcal{A}(y) + t > 1$. Since $\mathcal{A}(x * y) \leq \max\{\mathcal{A}(x), \mathcal{A}(y), 0.5\}$ by Theorem 3.15, we have $\mathcal{A}(x*y) \leq \max\{t, 0.5\}$. If it is not true, then $x_t \overline{\lessdot \vee \Upsilon}\, \mathcal{A}$ or $y_t \overline{\lessdot \vee \Upsilon}\, \mathcal{A}$, a contradiction. If $t \geq 0.5$, then $\mathcal{A}(x*y) \leq \max\{t, 0.5\} = t$ and so $x*y \in L(\mathcal{A};t) \subseteq [\mathcal{A}]_t$. If $t < 0.5$, then $\mathcal{A}(x*y) \leq \max\{t, 0.5\} = 0.5$ and thus $\mathcal{A}(x*y) + t < 0.5 + 0.5 = 1$. Hence $(x*y)_t \Upsilon \mathcal{A}$, and so $x * y \in \mathcal{A}_t \subseteq [\mathcal{A}]_t$. Therefore $[\mathcal{A}]_t$ is a subalgebra of X.

Conversely, let $\mathcal{A}$ be a fuzzy set in X and $t \in [0,1)$ be such that $[\mathcal{A}]_t$ is a subalgebra of X. Let $\max\{\mathcal{A}(x), \mathcal{A}(y), 0.5\} < t < \mathcal{A}(x*y)$ for some $t \in (0.5, 1)$. Then $x, y \in L(\mathcal{A};t) \subseteq [\mathcal{A}]_t$, which implies that $x*y \in [\mathcal{A}]_t$. Hence $\mathcal{A}(x*y) \leq t$ or $\mathcal{A}(x*y) + t < 1$, a contradiction. Therefore $\mathcal{A}(x*y) \leq \max\{\mathcal{A}(x), \mathcal{A}(y), 0.5\}$ for all $x, y \in X$. Using Theorem 3.15, we know that $\mathcal{A}$ is a $(\lessdot, \lessdot \vee \Upsilon\,)^*$-fuzzy subalgebra of X.

Theorem 3.25. Let $\{\mathcal{A}_i \mid i \in \Lambda\}$ be a family of $(\lessdot, \lessdot \vee \Upsilon\,)^*$-fuzzy subalgebras of a BM-algebra X. Then $\mathcal{A} := \bigcap_{i \in \Lambda} \mathcal{A}_i$ is a $(\lessdot, \lessdot \vee \Upsilon\,)^*$-fuzzy subalgebra of X.

Proof. By Theorem 3.15 we have $\mathcal{A}_i(x*y) \leq \max\{\mathcal{A}(x), \mathcal{A}(y), 0.5\}$, and so

$$\begin{aligned} \mathcal{A}(x*y) &= \inf_{i\in\Lambda} \mathcal{A}_i(x*y) \\ &\leq \inf_{i\in\Lambda} \max\{\mathcal{A}_i(x), \mathcal{A}_i(y), 0.5\} \\ &= \max\{\inf_{i\in\Lambda} \mathcal{A}_i(x), \inf_{i\in\Lambda} \mathcal{A}_i(y), 0.5\} \\ &= \max\{\mathcal{A}(x), \mathcal{A}(y), 0.5\}. \end{aligned}$$

By Theorem 3.15 we know that $\mathcal{A}$ is a $(\lessdot, \lessdot \vee \Upsilon\,)^*$-fuzzy subalgebra of X.

Theorem 3.26. Let $\{\mathcal{A}_i \mid i \in \Lambda\}$ be a family of $(\alpha, \beta)^*$-fuzzy subalgebras of X. Then $\mathcal{A} := \bigcap_{i\in\Lambda} \mathcal{A}_i$ is an $(\alpha, \beta)^*$-fuzzy subalgebra of X, where (α, β) is one of the following forms

(i) $(\lessdot, \Upsilon)$, (ii) $(\lessdot, \lessdot \wedge \Upsilon\,)$,
(iii) $(\Upsilon, \lessdot)$, (iv) $(\Upsilon, \lessdot \wedge \Upsilon\,)$,
(v) $(\lessdot \vee \Upsilon\,, \Upsilon)$, (vi) $(\lessdot \vee \Upsilon\,, \lessdot \wedge \Upsilon\,)$,
(vii) $(\lessdot \vee \Upsilon\,, \lessdot)$, (viii) $(\Upsilon, \lessdot \vee \Upsilon\,)$,
(ix) (Υ, Υ).

Proof. We prove theorem for an $(\Upsilon, \Upsilon)^*$-fuzzy subalgebra. The proof of the other cases is similar.
If there exists $i \in \Lambda$ such that $\mathcal{A}_i = 0$, then $\mathcal{A} = 0$. So $\mathcal{A}$ is a $(\Upsilon, \Upsilon)^*$-fuzzy subalgebra. Let $\mathcal{A}_i \neq 0$ for all $i \in \Lambda$. Then by Theorem 3.12 we have

$$\mathcal{A}_i(x) = \begin{cases} \mathcal{A}_i(0) & \text{if } x \in X_i^* \\ 1 & otherwise \end{cases}$$

for all $i \in \Lambda$. So it is clear that

$$\mathcal{A}(x) = \begin{cases} \mathcal{A}(0) & \text{if } x \in \bigcap\limits_{i\in\Lambda} X_i^* \\ 1 & otherwise \end{cases}$$

Since $\bigcap\limits_{i\in\Lambda} X_i^*$ is a subalgebra of X, then by Theorem 3.12 $\mathcal{A}$ is a $(\Upsilon, \Upsilon)^*$-fuzzy subalgebra of X.

Theorem 3.27. Let $\{\mathcal{A}_i \mid i \in \Lambda\}$ be a family of $(\lessdot, \lessdot)^*$-fuzzy subalgebras of a BM-algebra X. Then $\mathcal{A} := \bigcup\limits_{i\in\Lambda} \mathcal{A}_i$ is a $(\lessdot, \lessdot)^*$-fuzzy subalgebra of X.

Proof. Let $x_t \lessdot \mathcal{A}$ and $y_r \lessdot \mathcal{A}$, where $t, r \in [0, 1)$. Then $\mathcal{A}(x) \leq t$ and $\mathcal{A}(y) \leq r$. Thus for all $i \in \Lambda$, we have $\mathcal{A}_i(x) \leq t$ and $\mathcal{A}_i(y) \leq r$ and so $\mathcal{A}_i(x * y) \leq \max(t, r)$. Therefore $\mathcal{A}(x * y) \leq \max(t, r)$. Hence $(x * y)_{max(t,r)} \lessdot \mathcal{A}$.

The following is our question: Is the union of two $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebras of a BM-algebra X a $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X?

Lemma 3.28. Let $f : X \to Y$ be a BM-homomorphism and G be a fuzzy set of Y with membership function $\mathcal{A}_G$. Then $x_t \alpha \mathcal{A}_{f^{-1}(G)} \Leftrightarrow f(x)_t \alpha \mathcal{A}_G$, for all $\alpha \in \{\Upsilon, \lessdot, \lessdot \vee \Upsilon, \lessdot \wedge \Upsilon\}$.

Proof. Let $\alpha = \lessdot$. Then

$$x_t \alpha \mathcal{A}_{f^{-1}(G)} \Leftrightarrow \mathcal{A}_{f^{-1}(G)}(x) \leq t \Leftrightarrow \mathcal{A}_G(f(x)) \leq t \Leftrightarrow (f(x))_t \alpha \mathcal{A}_G$$

The proof of the other cases is similar to above argument.

Theorem 3.29. Let $f : X \to Y$ be a BM-homomorphism and G be a fuzzy set of Y with membership function $\mathcal{A}_G$.

(i) If G is an $(\alpha, \beta)^*$-fuzzy subalgebra of Y, then $f^{-1}(G)$ is an $(\alpha, \beta)^*$-fuzzy subalgebra of X,

(ii) Let f be epimorphism. If $f^{-1}(G)$ is an $(\alpha, \beta)^*$-fuzzy subalgebra of X, then G is an $(\alpha, \beta)^*$-fuzzy subalgebra of Y.

Proof. (i) Let $x_t \alpha \mathcal{A}_{f^{-1}(G)}$ and $y_r \alpha \mathcal{A}_{f^{-1}(G)}$, for $t, r \in [0, 1)$. Then by Lemma 3.28, we get that $(f(x))_t \alpha \mathcal{A}_G$ and $(f(y))_r \alpha \mathcal{A}_G$. Hence by hypothesis $(f(x) * f(y))_{\max(t,r)} \beta \mathcal{A}_G$. Then $(f(x * y))_{\max(t,r)} \beta \mathcal{A}_G$ and so $(x * y)_{\max(t,r)} \beta \mathcal{A}_{f^{-1}(G)}$.

(ii) Let $x, y \in Y$. Then by hypothesis there exist $x^{'}, y^{'} \in X$ such that $f(x^{'}) = x$ and $f(y^{'}) = y$. Assume that $x_t \alpha \mathcal{A}_G$ and $y_r \alpha \mathcal{A}_G$, then $(f(x^{'}))_t \alpha \mathcal{A}_G$ and $(f(y^{'}))_r \alpha \mathcal{A}_G$. Thus $x^{'}_t \alpha \mathcal{A}_{f^{-1}(G)}$ and $y^{'}_r \alpha \mathcal{A}_{f^{-1}(G)}$ and therefore $(x^{'} * y^{'})_{\max(t,r)} \beta \mathcal{A}_{f^{-1}(G)}$. So

$$(f(x^{'} * y^{'}))_{\max(t,r)} \beta \mathcal{A}_G \Rightarrow (f(x^{'}) * f(y^{'}))_{\max(t,r)} \beta \mathcal{A}_G \Rightarrow (x * y)_{\max(t,r)} \beta \mathcal{A}_G.$$

Theorem 3.30. Let $f : X \to Y$ be a BM-homomorphism and H be a $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X with membership function $\mathcal{A}_H$. If $\mathcal{A}_H$ is f-invariant, then $f(H)$ is a $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of Y.

Proof. Let y_1 and $y_2 \in Y$. If $f^{-1}(y_1)$ or $f^{-1}(y_2) = \emptyset$, then $\mathcal{A}_{f(H)}(y_1 * y_2) \leq \max(\mathcal{A}_{f(H)}(y_1), \mathcal{A}_{f(H)}(y_2), 0.5)$. Now let $f^{-1}(y_1)$ and $f^{-1}(y_2) \neq \emptyset$. Then there exist $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Thus by hypothesis we have

$$\begin{aligned}
\mathcal{A}_{f(H)}(y_1 * y_2) &= \sup_{t \in f^{-1}(y_1 * y_2)} \mathcal{A}_H(t) \\
&= \sup_{t \in f^{-1}(f(x_1 * x_2))} \mathcal{A}_H(t) \\
&= \mathcal{A}_H(x_1 * x_2) \\
&\leq \max(\mathcal{A}_H(x_1), \mathcal{A}_H(x_2), 0.5) \\
&= \max(\sup_{t \in f^{-1}(y_1)} \mathcal{A}_H(t), \sup_{t \in f^{-1}(y_2)} \mathcal{A}_H(t), 0.5) \\
&= \max(\mathcal{A}_{f(H)}(y_1), \mathcal{A}_{f(H)}(y_2), 0.5).
\end{aligned}$$

So by Theorem 3.15, $f(H)$ is a $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of Y.

Lemma 3.31. Let $f : X \to Y$ be a BM-homomorphism.
(i) If S is a subalgebra of X, then $f(S)$ is a subalgebra of Y,
(ii) If $S^{'}$ is a subalgebra of Y, then $f^{-1}(S^{'})$ is a subalgebra of X.

Proof. The proof is easy.

Theorem 3.32. Let $f : X \to Y$ be a BM-homomorphism. If H is a non-unit $(\Upsilon, \Upsilon)^*$-fuzzy subalgebra of X with membership function $\mathcal{A}_H$, then $f(H)$ is a non-unit $(\Upsilon, \Upsilon)^*$-fuzzy subalgebra of Y.

Proof. Let H be a non-unit $(\Upsilon, \Upsilon)^*$-fuzzy subalgebra of X. Then by Theorem 3.12, we have

$$\mathcal{A}_H(x) = \begin{cases} \mathcal{A}_H(0) & \text{if } x \in X^* \\ 1 & \textit{otherwise} \end{cases}$$

Now we show that

$$\mathcal{A}_{f(H)}(y) = \begin{cases} \mathcal{A}_H(0) & \text{if } y \in f(X^*) \\ 1 & \textit{otherwise} \end{cases}$$

Let $y \in Y$. If $y \in f(X^*)$, then there exists $x \in X^*$ such that $f(x) = y$. Thus $\mathcal{A}_{f(H)}(y) = \sup_{t \in f^{-1}(y)} \mathcal{A}_H(t) = \mathcal{A}_H(0)$. If $y \notin f(X^*)$, then it is clear that $\mathcal{A}_{f(H)}(y) = 1$. Since X^* is a subalgebra of X, $f(X^*)$ is a subalgebra of Y. Therefore by Theorem 3.12, $f(H)$ is a non-unit $(\Upsilon, \Upsilon)^*$-fuzzy subalgebra of Y.

Theorem 3.33. Let $f : X \to Y$ be a BM-homomorphism. If H is an $(\alpha, \beta)^*$-fuzzy subalgebra of X with membership function $\mathcal{A}_H$, then $f(H)$ is an $(\alpha, \beta)^*$-fuzzy subalgebra of Y, where (α, β) is one of the following forms

(i) $(\lessdot, \Upsilon)$, (ii) $(\lessdot, \lessdot \wedge \Upsilon)$,
(iii) $(\Upsilon, \lessdot)$, (iv) $(\Upsilon, \lessdot \wedge \Upsilon)$,
(v) $(\lessdot \vee \Upsilon, \Upsilon)$, (vi) $(\lessdot \vee \Upsilon, \lessdot \wedge \Upsilon)$,
(vii) $(\lessdot \vee \Upsilon, \lessdot)$, (viii) $(\Upsilon, \lessdot \vee \Upsilon)$.

Theorem 3.34. Let $f : X \to Y$ be a BM-homomorphism and H be an $(\lessdot, \lessdot)^*$-fuzzy subalgebra of X with membership function $\mathcal{A}_H$. If $\mathcal{A}_H$ is an f-invariant, then $f(H)$ is an $(\lessdot, \lessdot)^*$-fuzzy subalgebra of Y.

Proof. Let $z_t \lessdot \mathcal{A}_{f(H)}$ and $y_r \lessdot \mathcal{A}_{f(H)}$, where $t, r \in [0, 1)$. Then $\mathcal{A}_{f(H)}(z) \leq t$ and $\mathcal{A}_{f(H)}(y) \leq r$. Thus $f^{-1}(z), f^{-1}(y) \neq \emptyset$ imply that there exist $x_1, x_2 \in X$ such that $f(x_1) = z$ and $f(x_2) = y$. Since $\mathcal{A}_H$ is f-invariant, then $\mathcal{A}_{f(H)}(z) \leq t$ and $\mathcal{A}_{f(H)}(y) \leq r$ imply that $\mathcal{A}_H(x_1) \leq t$ and $\mathcal{A}_H(x_2) \leq r$. So by hypothesis we have

$$\begin{aligned}
\mathcal{A}_{f(H)}(z * y) &= \sup_{t \in f^{-1}(z*y)} \mathcal{A}_H(t) \\
&= \sup_{t \in f^{-1}(f(x_1 * x_2))} \mathcal{A}_H(t) \\
&= \mathcal{A}_H(x_1 * x_2) \\
&\leq \max(t, r)
\end{aligned}$$

Therefore $(z * y)_{\max(t,r)} \in \mathcal{A}_{f(H)}$, and hence $f(H)$ is a $(\lessdot, \lessdot)^*$-fuzzy subalgebra of Y.

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