



A Legendre-spectral scheme for solution of nonlinear system of Volterra-Fredholm integral equations

L. Hooshangian ^{a,*} D. Mirzaei ^b

^a*Department of Mathematics, Dezful Branch, Islamic Azad University, Dezful, Iran.*

^b*Department of Mathematics, University of Isfahan, Isfahan, Iran.*

Received 10 March 2011; accepted 11 January 2012

Abstract

This paper gives an efficient numerical method for solving the nonlinear system of Volterra-Fredholm integral equations. A Legendre-spectral method based on the Legendre integration Gauss points and Lagrange interpolation is proposed to convert the nonlinear integral equations to a nonlinear system of equations where the solution leads to the values of unknown functions at collocation points.

Key words: Volterra-Fredholm integral equations, Nonlinear integral equations, Legendre-spectral method, Gauss Legendre points, Lagrange interpolation.

2010 AMS Mathematics Subject Classification : 45B05; 45D05; 45G15.

* Corresponding author's E-mail: l_hooshangian@yahoo.com(L. Hooshangian)

1 Introduction

A nonlinear Volterra-Fredholm integral equation can be considered as the following:

$$u(x, t) = g(x, t) + \int_0^t \int_{\Omega} f(x, t, \xi, \tau, u(\xi, \tau)) d\xi d\tau, \quad (x, t) \in [0, T] \times \Omega, \quad (1.1)$$

where $u(x, t)$ is unknown function and $g(x, t)$ and $f(x, t, \xi, \tau, u(\xi, \tau))$ are analytical functions on $D = [0, T] \times \Omega$ and $S \times \mathbb{R}^n$ where $S = \{(x, t, \xi, \tau) : 0 \leq \tau \leq t \leq T; (\xi, \tau) \in \Omega \times \Omega\}$, respectively, Ω is a close subset of \mathbb{R}^n , with convenient $\|\cdot\|$, and are such that (1.1) possesses a unique solution $u(x, t) \in C(D)$. Existence and results for (1.1) maybe found in [1,2,3,4].

Equations of this type arise in the theory of parabolic boundary value problems, the mathematical modelling of the spatio-temporal development of an epidemic and various physical and biological problems. Detailed descriptions and analysis of these models may be found in [5,6] and the references therein.

Some numerical methods for (1.1) are known. For the linear case, some projection methods for numerical treatment of (1.1) are given in [3,4,7]. The results of [4] have been extended to nonlinear Volterra-Fredholm integral equations by Brunner [8]. The trapezoidal Nyström method considered in [9]. The authors of [10] introduced the Adomian decomposition method for this equation. In [11] the numerical approximation of (1.1) studied by discrete time collocation method.

A nonlinear system of Volterra-Fredholm integral equations can be written as the following [12]:

$$U(x, t) = G(x, t) + \int_0^t \int_{\Omega} F(x, t, \xi, \tau, U(\xi, \tau)) d\xi d\tau, \quad (x, t) \in [0, T] \times \Omega, \quad (1.2)$$

such that

$$\begin{aligned}
 U(x, t) &= [u_1(x, t), u_2(x, t), \dots, u_n(x, t)], \\
 G(x, t) &= [g_1(x, t), g_2(x, t), \dots, g_n(x, t)], \\
 F(x, t, \xi, \tau, U(\xi, \tau)) &= [f_1(x, t, \xi, \tau, U(\xi, \tau)), f_2(x, t, \xi, \tau, U(\xi, \tau)), \\
 &\dots, f_n(x, t, \xi, \tau, U(\xi, \tau))].
 \end{aligned}$$

In [12] a decomposition method was applied to solve this system.

Due to the high accuracy and efficiency, spectral methods have been used for some classes of integral equations in recent years. For example you can see [13,14,15,16] and the references therein. There are also many papers concerning the numerical solutions of the other types of integral equations via different schemes [17,18,19,20].

But recently Tang et. al [21] and Chen and Tang [22] presented a promising Legendre-spectral method for solving Volterra integral equations. Their methods based on the Legendre (or Jacobi) Gauss collocation points and Lagrange interpolation method. They proved that the numerical errors in the infinity norm will decay exponentially. In this paper we present this approach to solve the nonlinear system (1.2).

The remainder of the paper is organized as follows: In Section 2, the Legendre-spectral method is presented to nonlinear system (1.2). In Section 3, numerical results for some problems, are investigated and the corresponding tables and figures are presented. Finally in Section 4 the report ends with a brief conclusion.

2 Legendre-spectral method

Without loss of generality, suppose that $(x, t) \in [-1, 1] \times [-1, 1]$ for employing the coefficients and weights of Legendre-Gauss integration. Set the collocation points as the set of N Legendre-Gauss points $\{x_i\}_{i=1}^N$

and $\{t_j\}_{j=1}^N$. Assume that Eq. (1.2) holds at x_i and t_j on $[-1, 1]$:

$$U(x_i, t_j) = G(x_i, t_j) + \int_{-1}^{t_j} \int_{-1}^1 F(x_i, t_j, \xi, \tau, U(\xi, \tau)) d\xi d\tau, \quad (2.1)$$

where $1 \leq i \leq N$ and $1 \leq j \leq N$. The main difficulty to obtaining high rate of accuracy is to compute the first integral in (2.1). In fact for small values of t_j , there is a little information available for U [21]. To overcome this difficulty the integral interval $[-1, t_j]$ is transferred to a fix interval $[-1, 1]$. We first make the following simple linear transformations:

$$s(t, \theta) = \frac{t+1}{2}\theta + \frac{t-1}{2}. \quad (2.2)$$

Then (2.1) takes the form:

$$\begin{aligned} U(x_i, t_j) &= G(x_i, t_j) \\ &+ \frac{t_j+1}{2} \int_{-1}^1 \int_{-1}^1 F(x_i, t_j, \xi, s(t_j, \theta), U(\xi, s(t_j, \theta))) d\xi d\theta, \\ &1 \leq i \leq N, \quad 1 \leq j \leq N. \end{aligned}$$

Using a N -point Gauss quadrature rule $\{\theta_j\}$ related to the Legendre weights $\{w_j\}$ in $[-1, 1]$ gives:

$$\begin{aligned} U(x_i, t_j) &= G(x_i, t_j) \\ &+ \frac{t_j+1}{2} \sum_{p=1}^N \sum_{q=1}^N F(x_i, t_j, \theta_q, s(t_j, \theta_p), U(\theta_q, s(t_j, \theta_p))) w_q w_p \\ &1 \leq i \leq N, \quad 1 \leq j \leq N. \end{aligned}$$

where the set $\{\theta_j\}_{j=1}^N$ coincide with the collocation points $\{t_j\}_{j=1}^N$ and $\{x_j\}_{j=1}^N$. We now need to represent $U(\theta_q, s(t_j, \theta_p))$ in term of $U_{i,j}$ for $i, j = 1, 2, \dots, N$. To this end, we expand it using two dimensional Lagrange interpolation polynomials, i.e.

$$U(\sigma, \rho) \approx \sum_{k=1}^N \sum_{l=1}^N U_{k,l} \ell_k(\sigma) \ell_l(\rho), \quad (2.3)$$

where ℓ_k is the k -th Lagrange basis function. Combining Eq. (2) and (2.3) yields:

$$U_{i,j} = G(x_i, t_j) + \frac{t_j + 1}{2} \sum_{p=1}^N \sum_{q=1}^N F \left(x_i, t_j, \theta_q, s(t_j, \theta_p), \sum_{k=1}^N \sum_{l=1}^N U_{k,l} \ell_k(\theta_q) \ell_l(s(t_j, \theta_p)) \right) w_q w_p, \quad (2.4)$$

where $1 \leq i \leq N$ and $1 \leq j \leq N$. Eq. (2.4) can then be solved by some methods suitable for solving the non-linear systems. When the values of $U_{i,j}$ for $i, j = 1, 2 \dots N$ are resulted the numerical solution for $x, t \in [-1, 1]$ can be obtained by Lagrange interpolation as

$$U(x, t) \approx \sum_{i=1}^N \sum_{j=1}^N U_{i,j} \ell_i(x) \ell_j(t), \quad (x, t) \in [-1, 1]. \quad (2.5)$$

3 Numerical results

In this section, the method is applied to some numerical examples. All computations are performed by the Matlab *R2008a* software package. The numerical scheme (2.4) leads to a non-linear system for $\{U_{i,j}\}_{i,j=1}^N$, and a proper solver for the non-linear system should be used. To solve it, we use the robust routine *fsolve* from the optimization toolbox of Matlab. *fsolve* should be provided with an initial guess as a starting matrix. For different starting matrix we observed same convergence point with more or less iterations. In all examples the initial guess are chosen as following N by N constant matrix,

$$u_0 = \begin{bmatrix} 2 & 2 & \dots & 2 \\ 2 & 2 & \dots & 2 \\ \vdots & \vdots & & \vdots \\ 2 & 2 & \dots & 2 \end{bmatrix}_{N \times N}.$$

3.1 Example 1

Consider the following nonlinear Volterra-Fredholm integral equation which is given in [8,10,11]

$$u_1(x, t) = g_1(x, t) + \int_0^t \int_0^1 f_1(x, t, \xi, \tau)(1 - \exp(-u_1(\xi, \tau))) d\xi d\tau, \quad (3.1)$$

where $(x, t) \in [0, 1] \times [0, 1]$ and

$$f_1(x, t, \xi, \tau) = \frac{x(1 - \xi^2)}{(1 + t)(1 + \tau^2)},$$
$$g_1(x, t) = -\log\left(1 + \frac{xt}{1 + t^2}\right) + \frac{xt^2}{8(1 + t)(1 + t^2)}.$$

The exact solution is

$$u_1(x, t) = -\log\left(1 + \frac{xt}{1 + t^2}\right).$$

Computations are performed for different numbers of collocation points (N). Here we select N from 1 up to 9 and $\|e_1\|_\infty$ are presented in Table ?? and graphed in Figure 1. Infinity norm of error has the following meaning:

$$\|e\|_\infty = \max_{1 \leq i \leq N} \max_{1 \leq j \leq N} |u_{i,j} - u_{i,j}^N|,$$

where $u_{i,j}^N$ are the approximate solutions of $u_{i,j}$ using proposed method.

Comparison with those given in [8,10,11], shows the high accuracy of proposed scheme. The method of [11], using 32 collocation points leads to maximum error 9.29e-7, however by this method, more accurate result obtained using 6 collocation points.

N	$\ e_1\ _\infty$
1	1.77×10^{-2}
2	1.85×10^{-3}
3	1.25×10^{-4}
4	3.96×10^{-5}
5	8.56×10^{-6}
6	4.53×10^{-7}
7	2.25×10^{-7}
8	4.55×10^{-8}
9	2.12×10^{-9}

Table 1
Maximum absolute errors for Example 3.1.

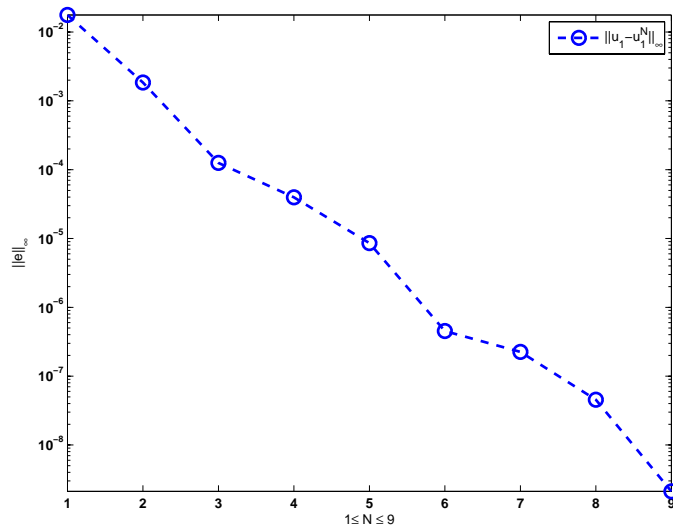


Fig. 1. Maximum errors at different N , Example 3.1.

3.2 Example 2

Consider the following system of nonlinear Volterra-Fredholm integral equations [12]:

$$\begin{cases} u_1(x, t) = g_1(x, t) + \int_0^t \int_0^1 f_1(x, t, \xi, \tau, u_1(\xi, \tau), u_2(\xi, \tau)) d\xi d\tau, \\ u_2(x, t) = g_2(x, t) + \int_0^t \int_0^1 f_2(x, t, \xi, \tau, u_1(\xi, \tau), u_2(\xi, \tau)) d\xi d\tau, \end{cases} \quad (3.2)$$

where

$$\begin{aligned} g_1(x, t) &= -2x \exp(t) - \frac{1}{4}x^2t^2 - \frac{t^4}{8} (2 \log(\cos 1) + 2 \tan 1 - 1), \\ g_2(x, t) &= \frac{2}{3}(x - t)(\exp(t) - 1) + t \tan x, \end{aligned}$$

and

$$\begin{aligned} f_1(x, t, \xi, \tau, u_1, u_2) &= \xi\tau(x^2 + u_2^2), \\ f_2(x, t, \xi, \tau, u_1, u_2) &= \xi(x - t)u_1. \end{aligned}$$

The exact solutions are

$$\begin{cases} u_1(x, t) = -2x \exp(t), \\ u_2(x, t) = t \tan x. \end{cases}$$

As before, numerical results are performed for different N and presented in Table 2 and Figure 2. Figure 2 depicts, numerical errors in the infinity norm decay very fast as N increases. Comparison with those given in [12], the method of this paper is more accurate.

N	$\ e_1\ _\infty$	$\ e_2\ _\infty$
2	3.39×10^{-3}	6.07×10^{-3}
3	5.17×10^{-4}	4.40×10^{-4}
4	5.37×10^{-5}	6.07×10^{-5}
5	4.58×10^{-6}	6.07×10^{-6}
6	3.58×10^{-7}	6.07×10^{-8}
7	2.66×10^{-8}	6.07×10^{-9}
8	1.91×10^{-9}	6.07×10^{-10}

Table 2
Maximum absolute errors for Example 3.2.

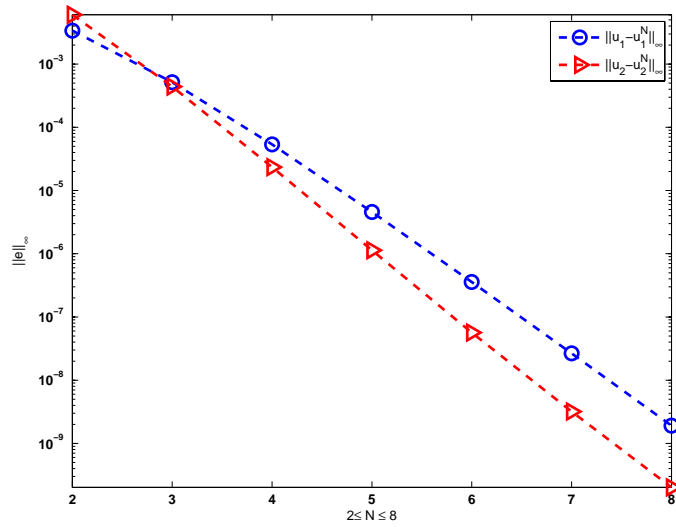


Fig. 2. Maximum errors at different N , Example 3.2.

3.3 Example 3

In this example, consider a 3×3 nonlinear system of equations as:

$$\begin{cases} u_1(x, t) = g_1(x, t) + \int_0^t \int_0^1 u_2(\xi, \tau) u_3(\xi, \tau) d\xi d\tau, \\ u_2(x, t) = g_2(x, t) + \int_0^t \int_0^1 u_1(\xi, \tau) u_3(\xi, \tau) d\xi d\tau, \\ u_3(x, t) = g_3(x, t) + \int_0^t \int_0^1 u_1(\xi, \tau) u_2(\xi, \tau) d\xi d\tau, \end{cases} \quad (3.3)$$

where

$$\begin{aligned} g_1(x, t) &= \frac{1}{t}(-\exp(t) - t \exp(t) + t^2 + 2t + 1) + \exp(-xt), \\ g_2(x, t) &= \frac{1}{t}(\exp(-t) + t \exp(-t) + t^2 - 1) + \exp(xt), \\ g_3(x, t) &= x, \end{aligned}$$

and the exact solutions are:

$$\begin{cases} u_1(x, t) = \exp(-xt), \\ u_2(x, t) = \exp(xt), \\ u_3(x, t) = x + t. \end{cases}$$

As before, Numerical results are performed using different number of integration points, N , and depicted in Table 3 and Figure 3 in terms of maximum norm of errors.

4 Conclusion

An efficient and accurate numerical scheme based on the Legendre-spectral method proposed for solving the nonlinear system of Volterra-Fredholm

N	$\ e_1\ _\infty$	$\ e_2\ _\infty$	$\ e_3\ _\infty$
1	3.22×10^{-2}	2.99×10^{-2}	8.54×10^{-2}
2	5.41×10^{-3}	4.22×10^{-3}	7.66×10^{-3}
3	3.28×10^{-4}	1.30×10^{-4}	7.80×10^{-5}
4	1.05×10^{-5}	4.75×10^{-6}	1.09×10^{-5}
5	4.13×10^{-7}	1.83×10^{-7}	1.26×10^{-7}
6	1.17×10^{-8}	4.77×10^{-9}	1.10×10^{-8}
7	4.80×10^{-10}	8.82×10^{-10}	8.45×10^{-10}

Table 3
Maximum absolute errors for Example 3.3.

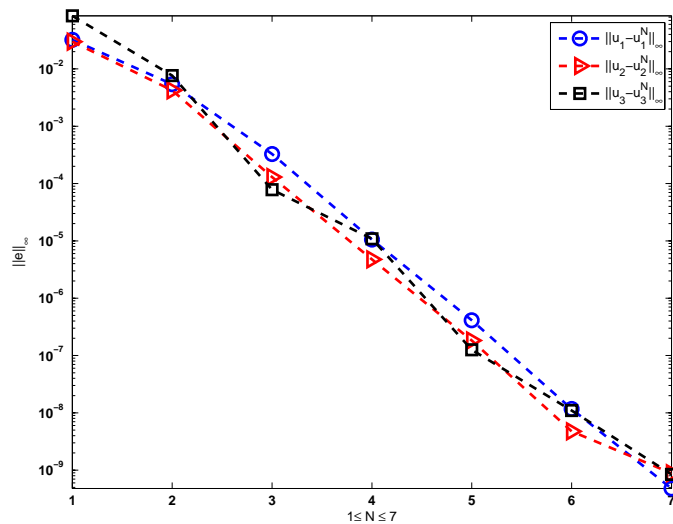


Fig. 3. Maximum errors at different N , Example 3.3.

integral equations. The Gaussian integration method with the Lagrange interpolation were employed to reduce the problem to the solution of nonlinear algebraic equations. Illustrative examples were given and compared with other references to demonstrate the validity and applicability of the method. As can be seen from the results reported in Section 3, by selecting few numbers of collocation points, excellent accurate results were produced.

Acknowledgement

The authors would like to thank the anonymous referees for their suggestions and remarks which improved the quality of paper.

This paper was extracted from a research project which has been done in Department of Mathematics, Dezful Branch, Islamic Azad University. The authors appreciate this university for all supports.

References

- [1] O. Diekmann, Thresholds and traveling waves for the geographical spread of infection, *J. Math. Biol.* 6 (1978) 109–130
- [2] B.G. Pachpatte, On mixed Volterra-Fredholm type integral equations, *Indian J. Pure Appl. Math.* 17 (1986), pp. 488–496.
- [3] L. Hacia, On approximate solution for integral equations of mixed type, *ZAMM. Z. Angew. Math. Mech.* 76 (1996) 415–416.
- [4] P.J. Kauthen, Continuous time collocation methods for Volterra-Fredholm integral equations, *Numer. Math.* 56, (1989) 409–424.
- [5] L. Hacia, On approximate solving of the Fourier problems, *Demonstratio Math.* 12 (1979) 913–922.
- [6] H.R. Thieme, A model for the spatio spread of an epidemic, *J. Math. Biol.* 4 (1977) 337–351.
- [7] H. Guoqiang and Z. Liqing, Asymptotic expansion for the trapezoidal Nyström method of linear Volterra- Fredholm equations, *J. Comput. Appl. Math.* 51 (3) (1994) 339–348.
- [8] H. Brunner, On the numerical solution of nonlinear Volterra-Fredholm integral equation by collocation methods, *SIAM J. Numer. Anal.* 27 (4)(1990) 987-1000.
- [9] H. Guoqiang, Asymptotic error expansion for the Nyström method for a nonlinear Volterra-Fredholm integral equations, *J. Comput. Appl. Math.* 59 (1995) 490–59.

- [10] K. Maleknejad and M. Hadizadeh, A new computational method for Volterra-Fredholm integral equations, *Comput. Math. Appl.* 37 (1999) 1-8.
- [11] M. Hadizadeh, Posteriori error estimates for the nonlinear Volterra-Fredholm integral equations, *Comput. Math. Appl.* 45 (2003) 677–687.
- [12] K. Maleknejad, M.R. Fadaei Yami, A computational method for system of Volterra-Fredholm integral equations, *Appl. Math. Comput.* 183 (2006) 589–595.
- [13] Gamal N. Elnagar, Mohammad A. Kazemi, A cell-averaging chebyshev spectral method for nonlinear fredholm-hammerstein integral equations, *Int. J. Comput. Math.* 60 (1996) 91–104.
- [14] Y.J.Jiang, On spectral methods for Volterra-type integro-differential equations, *J. Comput. Appl. Math.* 230 (2009) 333–340.
- [15] M. Lakestani, B. N. Saray, M. Dehghan, Numerical solution for the weakly singular Fredholm integro-differential equations using Legendre multiwavelets, *J. Comput. Appl. Math.* 235 (2011) 3291–3303.
- [16] F. Fakhari-Izadi, M. Dehghan, The spectral methods for parabolic Volterra integro-differential equations, *J. Comput. Appl. Math.* 235 (2011) 4032–4046.
- [17] M. Tavassoli Kajani, S. Mahdavi, Numerical solution of nonlinear integral equations by Galerkin methods with hybrid Legendre and Block-Pulse functions, *Mathematics Scientific Journal*, 7 (2011) 97–105.
- [18] Y. Liu, Solving abel integral equation by using Chebyshev wavelets, *Mathematics Scientific Journal*, 6 (2010) 51–57.
- [19] Y. Mahmoudi, Chebyshev collocation method for integral equations with singular kernels, *Mathematics Scientific Journal*, 5 (2010) 41–49.
- [20] A. Shamsavaran, Numerical solution of linear Volterra and Fredholm integro differential equations using Haar wavelets, *Mathematics Scientific Journal*, 6 (2010) 85–96.
- [21] T. Tang, Xiang Xu, Jin Cheng, On spectral methods for Volterra type integral equations and the convergence analysis, *J. Comput. Math.* 26 (2008) 825–837.

- [22] Y. Chen, T. Tang, Convergence analysis of the Jacobi spectral-collocation methods for Volterra integral equations with a weakly singular kernel, *Math. Comput.* 79 (2010) 147–167.