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# Constuction of solitary solutions for nonlinear differential-difference equations via Adomain decomposition method

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**Abstract**

Here, Adomian decomposition method has been used for finding approximate and numerical solutions of nonlinear differential difference equations arising in mathematical physics. Two models of special interest in physics, namely, the Hybrid nonlinear differential difference equation and Relativistic Toda coupled nonlinear differential-difference equation are chosen to illustrate the validity and the great potential of the proposed method. Comparisons are made between the results of the proposed method and exact solutions. The results show that the Adomian Decomposition Method is an attractive method in solving the nonlinear differential difference equations. It is worthwhile to mention that the Adomian decomposition method is also easy to be applied to other nonlinear differential difference equation arising in physics.

***Keywords***: Hybrid nonlinear difference equation, Relativistic Toda coupled nonlinear difference equation, Adomian Decomposition method.

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## 1 Introduction

The nonlinear differential-difference equations (DDEs) have been the focus of many nonlinear studies. The DDEs play an important role in modeling complicated physical

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phenomena such as particle vibrations in lattices, currents row in electrical networks, and pulses in biological chains. The solutions of these DDEs can provide numerical simulations of nonlinear partial differential equations, queuing problems, and discretizations in solid state and quantum physics. Since the works of Fermi et al. in the 1950s [1], there were quite a number of research works developed during the last decades on DDEs. For instances, Levi and his coworkers analyzed the condition for existence of higher symmetries for a class of DDEs [2], [3], Yamilov and his coworkers [4], [5] outstanding contribution to the classification of DDEs, integrability tests and connections between integrable PDEs and DDEs $[6-23]$. The Adomian Decomposition Method (ADM) $[24-30]$ has been used to solve effectively, easily, and accurately a large class of linear and nonlinear equations, solutions partial, deterministic or stochastic differential equations with approximates which converge rapidly. The paper has been organized as follows. In Section 2, we extended the ADM to solving nonlinear differential-difference equations with initial condition, an brief outline of the methodology is presented. In Section 3, two models arising in physics are chosen to illustrate the validity of the Adomian decomposition method in solving NDDE(s). Finally, discussions and conclusions are presented in Section 4.

## 2 Methods and its Applications

For a given the nonlinear differential-difference equation as

$$L(u_n) + R(u_n, u_{n+1}, u_{n-1} + ..........) + N(u_n, u_{n+1}, u_{n-1}, .......) = g, \qquad (2.1)$$

with prescribed conditions, where $u_n$ is the unknown function, $L$ is the highest order derivative which is assumed to be $n$ easily invertible, $R$ is a linear differential operator of less order than $L$, $N(u_n, u_{n+1}, u_{n-1}, ...)$ represents the nonlinear term and $g_n$ is the source term. Assuming the inverse operator $L^{-1}$ exists and it can be taken as the define integral with respect to from $t_0$ to $t$,i.e.

$$L^{-1} = \int_{t_0}^{t} (.)dt \qquad (2.2)$$

Applying $L^{-1}$ to both sides of (2.1) with initial conditions,we have

$$u_n = f(x) - L^{-1}[R(u_n, u_{n+1}, u_{n-1} + ...) + N(u_n, u_{n+1}, u_{n-1}, ...)], \qquad (2.3)$$

where the function $f(x)$ represents the term arising from integrating the source term $g$ and from using the given initial or boundary conditions,all are assumed to be prescribed.

The Adomian decomposition,assumes a series that the unknown function $u_n(t)$ can be expressed by an infinite series as

$$u_n(t) = \sum_{m=0}^{\infty} u_{m,n}(t) \qquad (2.4)$$

The nonlinear operator $N(u_n, u_{n+1}, u_{n-1}, ...)$ can be decomposed by an infinite series of polynomials given by

$$N(u_n, u_{n+1}, u_{n-1}, ...) = \sum_{m=0}^{\infty} A_{m,n}, \tag{2.5}$$

where $A_{m,n}$ are the Adomian's polynomials. To determine the Adomian polynomials, we introduce a parameter $\lambda$ and (2.5) becomes

$$N(\sum_{m=0}^{\infty} u_{m,n}(t)\lambda^m, \sum_{m=0}^{\infty} u_{m,n+1}(t)\lambda^m, \sum_{m=0}^{\infty} u_{m,n-1}(t)\lambda^m, ...) = \sum_{m=0}^{\infty} A_{m,n}\lambda^m, \tag{2.6}$$

$$A_{m,n} = \frac{1}{m!}[\frac{d^m}{d\lambda^m} N(\sum_{m=0}^{\infty} u_{m,n}(t)\lambda^m, \sum_{m=0}^{\infty} u_{m,n+1}(t)\lambda^m, \sum_{m=0}^{\infty} u_{m,n-1}(t)\lambda^m, ...)]_{\lambda=0}, \tag{2.7}$$

This formula is easy to compute by using Maple software or by setting a computer code to get as many polynomials as we need in the calculation of the numerical as well as explicit solutions. Knowing the zeros component $u_n(0)$, the remaining components where $n > 1$ can be determined by using recurrence relation

$$u_0(n) = f(x), \tag{2.8}$$

$$u_m(n+1, t) = -L^{-1}[R(u_{m,n}, u_{m,n+1}, u_{m,n-1}, ...) + A_{m,n}], m > 0 \tag{2.9}$$

For the numerical computation,we consider the expression as follows:

$$\phi_{n,k} = \sum_{m=0}^{k} u_{m,n}, \tag{2.10}$$

denotes the $n$-term approximation to $u_n(t)$.

## 3 Applications

To illustrate the effectiveness and the advantages of the proposed method. Here, we consider two models of nonlinear differential-difference equations of special interest physically, namely, the Hybrid nonlinear differential difference equation and Relativistic Toda coupled nonlinear differential-difference equation.

## 3.1 The Hybrid nonlinear differential difference equation

**Example 1.**Let us first consider the Hybrid nonlinear differential difference equation [20]

$$\frac{\partial u_n}{\partial t} = (\alpha - u_n^2)(u_{n+1} - u_{n-1}), \tag{3.11}$$

subject to the initial condition

$$u_n(0) = \sqrt{-\alpha} tanh(d/2)(1 + cosh(d))sech(dn - 2), \tag{3.12}$$

where $d$ is constant. The Hybrid nonlinear difference (3.11) describe the discretization of the KdV and modified KdV equations. Applying ADM, (3.11) can be rewritten in the operator form.

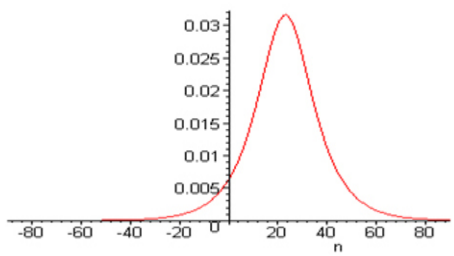


**Fig.[1.a]**: Approximate solution for 5-order approximation of $u_n(t)$ obtained by Adomian decomposition method with a fixed values of $d = 0.1$, $t = 15$ and $\alpha = -0.1$ for a different values of $n$.

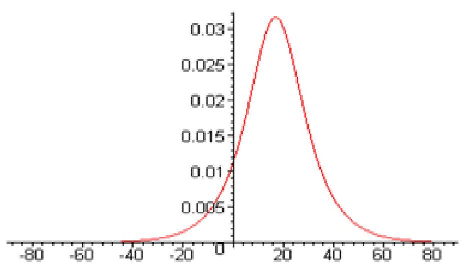


**Fig.[1.b]**:Exact solution of $u_n(t)$, (3.18) with a fixed values of $d = 0.1$, $t = 15$ and $\alpha = -0.1$ for a different values of $n$.

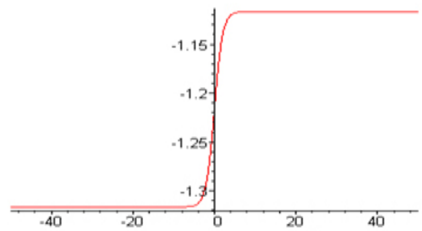


**Fig.[2.a]**: Approximate solution for 5-order approximation of $u_n(t)$ obtained by Adomian decomposition method with a fixed values of $d = 0.5$, $t = .1$ and $c = -0.1$ for a different values of $n$.

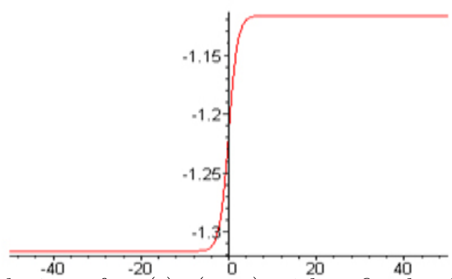


**Fig.[2.b]**: Exact solution of $u_n(t)$, (3.32) with a fixed values of $d = 0.5$, $t = .1$ and $c = -0.1$ for a different values of $n$.

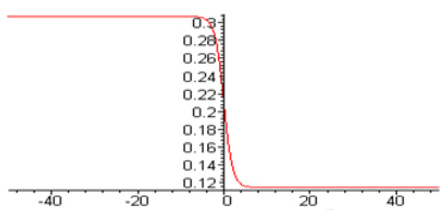


**Fig.[2.c]**: Approximate solution for 5-order approximation of $v_n(t)$ obtained by Adomian decomposition method with a fixed values of $d = 0.5$, $t = .1$ and $c = -0.1$ for a different values of $n$.

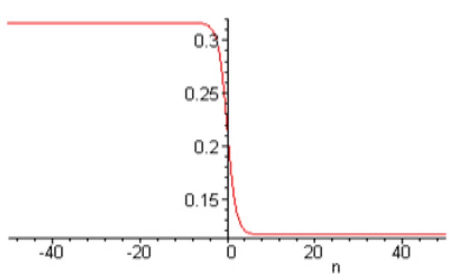

**Fig.[2.d]**: Exact solution of $v_n(t)$, (3.33) with a fixed values of $d = 0.5$, $t = .1$ and $c = -0.1$ for a different values of $n$.

$$L[u_n(t)] = [R(u) + N(u)], \tag{3.13}$$

where $L = \frac{\partial}{\partial t}$ is a linear operator $R(u) = \alpha[u(n+1) - u(n-1)]$ is the remainder of the linear operator.The nonlinear term is given by $N(u) = -u^2(n)(u(n+1) - u(n-1))$.Opeating with $L^{-1}$ on to both sides of (3.13) with initial condition (3.12) gives

$$u_n(t) = u_n(0) + L^{-1}[R(u) + N(u)] \tag{3.14}$$

The Adomian decomposition method assumes an infinite series solutions of the unknown function $u_n(t)$ in the form

$$u_n(t) = \sum_{m=0}^{\infty} u_m(n,t) \tag{3.15}$$

It is to be noted that the subscript m does not stand for the m-th lattice any more, it means the m-th element is the decomposition series. Similarly, the nonlinear operator $N(u)$ can be decomposed as

$$N(u) = \sum_{m=0}^{\infty} A_m(u_0, u_1, u_2, ...., u_m), \tag{3.16}$$

where $A_m$ are the appropriate Adomian polynomials, which is defined as [24, 25]. The first four components of $A_m$ are given by [21].

According to (3.14), (3.15) and (3.16), the first few components of the decomposition series $u_m(n,t)$ are readily found to be

$$u_0(n) = \sqrt{-\alpha} tanh(d/2)(1 + cosh(d))sech(dn - 2),$$

$$u_1(n,t) = L^{-1}[\alpha[u_0(n+1,t) - u_0(n-1,t)] + A_0],$$

$$u_2(n,t) = L^{-1}[\alpha[u_1(n+1,t) - u_1(n-1,t)] + A_1],$$

$$u_3(n,t) = L^{-1}[\alpha[u_2(n+1,t) - u_2(n-1,t)] + A_2],$$

$$u_{m+1}(n,t) = L^{-1}[\alpha[u_m(n+1,t) - u_m(n-1,t)] + A_m], \tag{3.17}$$

and so on. The rest of the components of $u_m(n,t)$ can be directly evaluated via (3.17) in a similar way. It is worth noting that the accuracy of the Adomian approach is significantly enhanced by calculation as many more terms as we like. This may be made but only at the expense of a considerable increase in the complexity of analysis. The Explicit forms for $u_1$,$u_2$,$u_3$, ..... are written in **Appendix A**.

To verify numerically whether the proposed methodology lead to high accuracy, we evaluate the numerical solutions using the fourth-term approximation and compared it with the exact analytical solution. So, we could get fourth-order approximation $\phi_5(n) = \sum_{m=0}^{5} u_m(n,t)$. Figs.[1.(a,b)] shows comparisons between the numerical approximate solutions obtained by ADM with a fixed values of $d$, $t$ for a different values of $n$ and exact solutions [20], which proofs the two solutions are quite good. We could say that solution to the Hybrid nonlinear differential difference equation obtained by ADM agree well with the exact solutions [20]

$$u_n(t) = \sqrt{-\alpha} tanh(d/2)(1 + cosh(d))sech[dn - 2\alpha tanh(d/2)(1 + coshd)t - 2]. \quad (3.18)$$

## 3.2 Relativistic Toda coupled nonlinear differential-difference equation

**Example 2.** A second instructive model is the Relativistic Toda coupled nonlinear differential difference equation [20]

$$\frac{\partial u_n(t)}{\partial t} = (1 + \alpha u_n(t))(v_n(t) - v_{n-1}(t)), \quad (3.19)$$

$$\frac{\partial v_n(t)}{\partial t} = v_n(t)(u_{n+1}(t) - u_n(t) + \alpha v_{n+1}(t) - \alpha v_{n-1}(t)), \quad (3.20)$$

subject to the initial conditions

$$u_n(0) = -1 - c_1 coth(d_1) + c_1 tanh[d_1 n], \quad (3.21)$$

$$v_n(0) = c_1 coth(d_1) - c_1 tanh[d_1 n], \quad (3.22)$$

where $c_1$ and $d_1$ are constants. The Toda lattice difference (3.19) and (3.20) describe vibrations in mass spring lattices with an exponential interation force. Applying ADM, Eqs. (3.19) and (3.20) can be written in an operator form

$$L[u_n(t)] = [v(n) - v(n-1)] + \alpha M[u(n), v(n), v(n-1)], \quad (3.23)$$

$$L[v_n(t)] = N[u(n), u(n+1), v(n), v(n+1), v(n-1)], \quad (3.24)$$

Operating with $L^{-1}$ on to both sides of (3.23) and (3.24) with initial conditions (3.21) and (3.22) gives

$$u_n(t) = u_n(0) + L^{-1}[v(n) - v(n+1) + \alpha M[(u(n), v(n), v(n-1)], \quad (3.25)$$

$$v_n(t) = v_n(0) + L^{-1}[N(u(n), u(n+1), v(n), u(n+1))] \quad (3.26)$$

The nonlinear operators $M[u(n),v(n),v(n-1)] = u(n)(v(n)-v(n-1))$ and $N[(u(n),u(n+1),v(n),u(n+1)] = v(n)[u(n+1)-u(n)+\alpha v(n+1)-\alpha v(n-1)]$ are given by the infinite series of Adomian polynomial

$$M(u_n, v_n, v_{n-1}) = \sum_{m=0}^{\infty} A_m, \tag{3.27}$$

$$N(u_n, u_{n+1}, v_n, v_{n+1}, v_{n-1}) = \sum_{m=0}^{\infty} B_m, \tag{3.28}$$

where $A_m$ and $B_m$ are the approximate Adomian polynomials which are generated according to algorithms determined. The first four components $A_m$ and $B_m$ of Adomain polynomials are written in **Appendix B**.

The Adomian decomposition method assumes an infinite series solutions of the unknown functions $u_n(t)$ and $v_n(t)$ in the form

$$u_n(t) = \sum_{m=0}^{\infty} u_m(n,t), \tag{3.29}$$

$$v_n(t) = \sum_{m=0}^{\infty} v_m(n,t) \tag{3.30}$$

Inserting (3.29) and (3.30) into (3.25) and (3.26) gives

$$u_0(n) = -1 - c_1 coth(d_1) + c_1 tanh[d_1 n],$$

$$v_0(n) = c_1 coth(d_1) - c_1 tanh[d_1 n],$$

$$u_{m+1}(n,t) = L^{-1}[v_m(n,t) - v_m(n+1,t) + A_m], m > 0$$

$$v_{m+1}(n,t) = L^{-1}[B_m], m > 0 \tag{3.31}$$

With the aid of the zeroth components of $u_{n,0}$ and $v_{n,0}$, by all terms arise from the initial conditions (3.21) and (3.22),and as a result, the remaining components $u_m(n,t)$ and $v_m(n,t)$, $m > 0$ can be determined. The Explicit forms for $u_1$,$u_2$,$u_3$ and $v_1$,$v_1$ and $v_1$ are written in **Appendix C**.

To verify numerically whether the proposed methodology lead to high accuracy, we evaluate the numerical solutions using the fourth-term approximation and compared it with the exact analytical solution. So, we could get fourth-order approximation $\phi_5(n) = \sum_{m=0}^{5} u_m(n,t)$. Figs. [2.(a-d)] shows comparisons between the numerical approximate solutions obtained by ADM and exact solutions [20], which proofs the two solutions are quite good. It is to be noted that solution to Relativistic Toda coupled nonlinear differential-difference equation obtained by ADM agree well with the exact solutions [20]

$$u_n(t) = -1 - c_1 coth(d_1) + c_1 tanh[d_1 n + c_1 t], \quad (3.32)$$

$$v_n(t) = c_1 coth(d_1) - c_1 tanh[d_1 n + c_1 t], \quad (3.33)$$

where $c_1$ and $d_1$ are constants to be determined later.

## 4 Summary and Discussion

In summary, we successfully apply the Adomian Decomposition method to solve nonlinear NDDEs. As an illustrative example, we employ DDEs-ADM to the Hybrid nonlinear differential difference equation and Relativistic Toda coupled nonlinear differential-difference equation. The comparison are made between the solutions obtained by Adomian Decomposition Method with exact solution. The comparison shows the validity of Adomian Decomposition Method applied to nonlinear differentialdifference equation. Since the numerous DDEs in many fields of scientific and the hardness of solving them, this DDEs-ADM will become a much more interesting method to solving nonlinear DDEs in science and engineering.

The results reported here provide further evidence of the usefulness of Adomain decomposition (AMD) method. The ADM was clearly very efficient and powerful technique in finding the solutions of the nonlinear differential difference equations. It is clear that this method avoids linearization and biologically unrealistic assumptions,and provides an efficient numerical solution.

A clear conclusion can be drawn from the numerical results that the ADM algorithm provides highly accurate numerical soultions without spatial discretization for nonlinear differential equations. It is also worth noting that the advantage of the decomposition methodology displays a fast convergence of the solutions.

Finally, we point out that the ADM confirm the correctness of those obtained by other methods. The method is straightforward and concise, and it can also be applied to other nonlinear differential difference equations in mathematical physics. This is our task in future work.

## Acknowledgements

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## Appendix A

$$u_{0,n} := \sqrt{-\alpha}\,\tanh\left(\frac{1}{2}d\right)(1+\cosh(d))\,\mathrm{sech}(d\,n-2)$$

$$u_{1,n} := -\sinh(d)\,t\,(\cosh(d)^2-1+\cosh(d\,n-2)^2)$$
$$(-\cosh(d\,n-d-2)+\cosh(d\,n+d-2))\sqrt{-\alpha}\,\alpha \Big/ (\cosh(d\,n-2)^2$$
$$\cosh(d\,n+d-2)\cosh(d\,n-d-2))$$

$$u_{2,n} := \alpha\left(-\frac{1}{2}\sinh(d)\,t^2\,(\cosh(d)^2-1+\cosh(d\,(n+1)-2)^2)\right.$$
$$(-\cosh(d\,(n+1)-d-2)+\cosh(d\,(n+1)+d-2))\sqrt{-\alpha}\,\alpha \Big/ ($$
$$\cosh(d\,(n+1)-d-2)\cosh(d\,(n+1)+d-2)\cosh(d\,(n+1)-2)^2)+\frac{1}{2}\sinh(d)$$
$$t^2\,(\cosh(d)^2-1+\cosh(d\,(n-1)-2)^2)$$
$$(-\cosh(d\,(n-1)-d-2)+\cosh(d\,(n-1)+d-2))\sqrt{-\alpha}\,\alpha \Big/ ($$
$$\left.\cosh(d\,(n-1)-d-2)\cosh(d\,(n-1)+d-2)\cosh(d\,(n-1)-2)^2)\right)-\alpha^2$$
$$\tanh\left(\frac{1}{2}d\right)(1+\cosh(d))\,\mathrm{sech}(d\,n-2)\left(\right.$$
$$\sqrt{-\alpha}\,\tanh\left(\frac{1}{2}d\right)(1+\cosh(d))\,\mathrm{sech}(d\,(n+1)-2)$$
$$\left.-\sqrt{-\alpha}\,\tanh\left(\frac{1}{2}d\right)(1+\cosh(d))\,\mathrm{sech}(d\,(n-1)-2)\right)\sinh(d)\,t^2$$
$$(\cosh(d)^2-1+\cosh(d\,n-2)^2)(-\cosh(d\,n-d-2)+\cosh(d\,n+d-2)) \Big/ ($$
$$\cosh(d\,n-d-2)\cosh(d\,n+d-2)\cosh(d\,n-2)^2)+\alpha\tanh\left(\frac{1}{2}d\right)^2(1+\cosh(d))^2$$
$$\mathrm{sech}(d\,n-2)^2\left(-\frac{1}{2}\sinh(d)\,t^2\,(\cosh(d)^2-1+\cosh(d\,(n+1)-2)^2)\right.$$
$$(-\cosh(d\,(n+1)-d-2)+\cosh(d\,(n+1)+d-2))\sqrt{-\alpha}\,\alpha \Big/ ($$
$$\cosh(d\,(n+1)-d-2)\cosh(d\,(n+1)+d-2)\cosh(d\,(n+1)-2)^2)+\frac{1}{2}\sinh(d)$$
$$t^2\,(\cosh(d)^2-1+\cosh(d\,(n-1)-2)^2)$$

$$(-\cosh(d\,(n-1)-d-2)+\cosh(d\,(n-1)+d-2))\sqrt{-\alpha}\,\alpha \Big/ \Big(\cosh(d\,(n-1)-d-2)\cosh(d\,(n-1)+d-2)\cosh(d\,(n-1)-2)^2\Big)$$

## Appendix B

$$A_0 := u_{0,n}\,v_{0,n} - u_{0,n}\,v_{0,n-1} \qquad B_0 := -u_{0,n}\,v_{0,n} + v_{0,n}\,v_{0,n+1} - v_{0,n}\,v_{0,n-1} + v_{0,n}\,u_{0,n+1}$$

$$B_1 := -u_{0,n}\,v_{1,n} - u_{1,n}\,v_{0,n} + 2\,v_{0,n}\,v_{1,n+1} + v_{1,n}\,u_{0,n+1} - v_{0,n}\,v_{0,n-1} + v_{1,n}\,v_{0,n+1} - v_{1,n}\,v_{0,n-1}$$

$$A_1 := u_{0,n}\,v_{1,n} - u_{0,n}\,v_{1,n-1} + u_{1,n}\,v_{0,n} - u_{1,n}\,v_{0,n-1}$$

$$A_2 := u_{0,n}\,v_{2,n} - u_{0,n}\,v_{2,n-1} + u_{1,n}\,v_{1,n} - u_{1,n}\,v_{1,n-1} + u_{2,n}\,v_{0,n} - u_{2,n}\,v_{0,n-1}$$

$$B_2 := -u_{0,n}\,u_{2,n} - u_{1,n}\,v_{1,n} - u_{2,n}\,v_{0,n} + 2\,v_{0,n}\,v_{2,n+1} + v_{1,n}\,u_{1,n+1} + v_{2,n}\,u_{0,n+1} - v_{0,n}\,v_{2,n-1} + v_{1,n}\,v_{1,n+1} - 2\,v_{1,n}\,v_{1,n-1} + v_{2,n}\,v_{0,n+1}$$

## Appendix C

$$v_{0,n} := c\coth(d) - c\tanh(d\,n)\,,\; u_{0,n} := -1 - c\coth(d) + c\tanh(d\,n)$$

$$v_{1,n} := -c^2\,(\cosh(d)\cosh(d\,n) - \sinh(d\,n)\sinh(d))\,(\cosh(d\,(n-1))\sinh(d\,n) - \cosh(d\,n)\sinh(d\,(n-1)))\,t \Big/ (\sinh(d)\cosh(d\,n)^2\cosh(d\,(n-1)))$$

$$u_{1,n} := (-c\tanh(d\,n) + c\tanh(d\,(n-1)) + (-1 - c\coth(d) + c\tanh(d\,n))\,(c\coth(d) - c\tanh(d\,n)) - (-1 - c\coth(d) + c\tanh(d\,n))\,(c\coth(d) - c\tanh(d\,(n-1))))\,t$$

$$u_{2,n} := -\frac{1}{2}c^2\,(\cosh(d)\cosh(d\,n) - \sinh(d\,n)\sinh(d))\,(\cosh(d\,(n-1))\sinh(d\,n) - \cosh(d\,n)\sinh(d\,(n-1)))\,t^2 \Big/ (\sinh(d)\cosh(d\,n)^2\cosh(d\,(n-1))) + \frac{1}{2}c^2\,(\cosh(d)\cosh(d\,(n-1)) - \sinh(d\,(n-1))\sinh(d))$$

$$(\cosh(d\,(n-2))\sinh(d\,(n-1)) - \cosh(d\,(n-1))\sinh(d\,(n-2)))\,t^2 \Big/ (\sinh(d)\cosh(d\,(n-1))^2\cosh(d\,(n-2))) - \frac{1}{2}(-1 - c\coth(d) + c\tanh(d\,n))\,c^2\,(\cosh(d)\cosh(d\,n) - \sinh(d\,n)\sinh(d))$$

$$(\cosh(d(n-1))\sinh(dn)-\cosh(dn)\sinh(d(n-1)))\,t^2 \big/ (\sinh(d)\cosh(dn)^2$$
$$\cosh(d(n-1)))+\frac{1}{2}(-1-c\coth(d)+c\tanh(dn))\,c^2$$
$$(\cosh(d)\cosh(d(n-1))-\sinh(d(n-1))\sinh(d))$$
$$(\cosh(d(n-2))\sinh(d(n-1))-\cosh(d(n-1))\sinh(d(n-2)))\,t^2 \big/ (\sinh(d)$$
$$\cosh(d(n-1))^2\cosh(d(n-2)))+\frac{1}{2}(-c\tanh(dn)+c\tanh(d(n-1))$$
$$+(-1-c\coth(d)+c\tanh(dn))(c\coth(d)-c\tanh(dn))$$
$$-(-1-c\coth(d)+c\tanh(dn))(c\coth(d)-c\tanh(d(n-1))))\,t^2$$
$$(c\coth(d)-c\tanh(dn))-\frac{1}{2}(-c\tanh(dn)+c\tanh(d(n-1))$$
$$+(-1-c\coth(d)+c\tanh(dn))(c\coth(d)-c\tanh(dn))$$
$$-(-1-c\coth(d)+c\tanh(dn))(c\coth(d)-c\tanh(d(n-1))))\,t^2$$
$$(c\coth(d)-c\tanh(d(n-1)))$$

$$v_{2,n} := \frac{1}{2}(-1-c\coth(d)+c\tanh(dn))\,c^2(\cosh(d)\cosh(dn)-\sinh(dn)\sinh(d))$$
$$(\cosh(d(n-1))\sinh(dn)-\cosh(dn)\sinh(d(n-1)))\,t^2 \big/ (\sinh(d)\cosh(dn)^2$$
$$\cosh(d(n-1)))-\frac{1}{2}(-c\tanh(dn)+c\tanh(d(n-1))$$
$$+(-1-c\coth(d)+c\tanh(dn))(c\coth(d)-c\tanh(dn))$$
$$-(-1-c\coth(d)+c\tanh(dn))(c\coth(d)-c\tanh(d(n-1))))\,t^2$$
$$(c\coth(d)-c\tanh(dn))-(c\coth(d)-c\tanh(dn))\,c^2$$
$$(\cosh(d)\cosh(d(n+1))-\sinh(d(n+1))\sinh(d))$$
$$(\cosh(dn)\sinh(d(n+1))-\cosh(d(n+1))\sinh(dn))\,t^2 \big/ (\sinh(d)$$
$$\cosh(d(n+1))^2\cosh(dn))-\frac{1}{2}c^2(\cosh(d)\cosh(dn)-\sinh(dn)\sinh(d))$$
$$(\cosh(d(n-1))\sinh(dn)-\cosh(dn)\sinh(d(n-1)))\,t^2$$