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Upper Bounds for 2-Restrained Domination Number of $GP(n,2)$

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ABSTRACT

Ghanbari and Mojdeh [3] initiated the concept of restrained 2-rainbow domination in graphs. In this paper is given upper bounds for 2-restrained domination number of a particular case of generalized Petersen graphs.

1 Introduction and Preliminary

Throughout this paper, we consider G as a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. We use cf. [5] as a reference for terminology and notation which are not explicitly defined here.

For a graph $G=(V(G),E(G))$, a set $S\subseteq V(G)$ is called a *dominating set* if every vertex not in S has a neighbor in S cf. [4] .The *domination number* $\gamma(G)$ of G is the minimum cardinality among all dominating sets of G. Let G be a graph and $v\in V(G)$. The open neighborhood of v is the set $N(v)=\{u\in V(G)|uv\in E(G)\}$, and its closed neighborhood is the set $N[v]=N(v)\cup v$. Let f be a function that assigns to each vertex a set of colors chosen from the set $\{1,...,k\}$; that is, $f:V(G)\rightarrow P(\{1,...,k\})$. If for each vertex $v\in V(G)$ such that $f(v)=\emptyset$ we have $\cup_{u\in N(v)}f(u)=\{1,...,k\}$, then f is called a *k-rainbow dominating function* (kRDF) of G cf. [1] and [2]. The weight, $\omega(f)$, of a function f is defined as $\omega(f)=\Sigma_{v\in V(G)}|f(v)|$. Given a graph G, the minimum weight of a kRDF is called the *k-rainbow domination number of G*, which we denote by $\gamma_{rk}(G)$. Ghanbari and Mojdeh [3] initiated the concept of *restrained 2- rainbow domination in graphs*.

Let f be a function that assigns to each vertex a set of colors chosen from the set $\{1,2\}$; that is, $f:V(G)\rightarrow P(\{1,2\})$. If for each vertex $v\in V(G)$, such that $f(v)=\emptyset$ we have $\cup_{u\in N(v)}f(u)=\{1,2\}$, and v is adjacent to a vertex $w\in V(G)$ such that $f(w)=\emptyset$ then f is called a *restrained 2-rainbow dominating function* (R2RDF) of G. The weight, $\omega(f)$, of a function f is defined as $\omega(f)=\Sigma_{v\in V(G)}|f(v)|$. Given a graph G, the minimum weight of a R2RDF is called the *restrained 2-rainbow domination number of G*, which we denote by $\gamma_{rr2}(G)$.

2 Main Result

Let $n\geq 3$ and k be relatively prime natural numbers and $k<n$. The generalized Petersen graph $GP(n,k)$ is defined as follows. Let C_n, C'_n be two disjoint cycles of length n. Let the vertices of C_n be $u_1,...,u_n$ and edges u_iu_{i+1} for $i=1,...,n-1$ and u_nu_1.

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Let the vertices of C'_n be $v_1, ..., v_n$ and edges v_iv_{i+k} for $i = 1, ..., n$, the sum $i+k$ being taken modulo n (throughout this section). The graph $GP(n, k)$ is obtained from the union of C_n and C'_n by adding the edges u_iv_i for $i = 1, ..., n$. Its obvious that $GP(n, k) = GP(n, n-k)$. The graph $GP(5, 2)$ or $GP(5, 3)$ is the well-known Petersen graph.

Theorem 2.1. *For* $n \geq 5$
(a) If $n \equiv 0(mod5)$*, the inequality* $\gamma_{rr2}(GP(n, 2)) = \gamma_{rr2}(GP(n, n-2)) \leq \frac{4n}{5} + 2$ *is satisfied.*
(b) If $n \equiv 1(mod5)$*, the inequality* $\gamma_{rr2}(GP(n, 2)) = \gamma_{rr2}(GP(n, n-2)) \leq 4\lfloor\frac{n}{5}\rfloor + 2$ *is satisfied.*
(c) If $n \equiv 2(mod5)$*, the inequality* $\gamma_{rr2}(GP(n, 2)) = \gamma_{rr2}(GP(n, n-2)) \leq 4(\lfloor\frac{n}{5}\rfloor + 1)$ *is satisfied.*
(d) If $n \equiv 3(mod5)$*, the inequality* $\gamma_{rr2}(GP(n, 2)) = \gamma_{rr2}(GP(n, n-2)) \leq 4(\lfloor\frac{n}{5}\rfloor + \frac{3}{2})$ *is satisfied.*
(e) If $n \equiv 4(mod5)$*, the inequality* $\gamma_{rr2}(GP(n, 2)) = \gamma_{rr2}(GP(n, n-2)) \leq 4(\lfloor\frac{n}{5}\rfloor + \frac{3}{2})$ *is satisfied.*

Proof. We use the following partition of $V(GP(n, 2))$:

$$V(GP(n, 2)) = \{U_{5k}, U_{5k-1}, U_{5k-2}, U_{5k-3}, U_{5k-4}, V_{5k}, V_{5k-1}, V_{5k-2}, V_{5k-3}, V_{5k-4}\}$$

such that
$U_{5k} = \{u_{5k}, k = 1, 2, \cdots\}$, $U_{5k-1} = \{u_{5k-1}, k = 1, 2, \cdots\}$, $U_{5k-2} = \{u_{5k-2}, k = 1, 2, \cdots\}$, $U_{5k-3} = \{u_{5k-3}, k = 1, 2, \cdots\}$, $U_{5k-4} = \{u_{5k-4}, k = 1, 2, \cdots\}$, $V_{5k} = \{u_{5k}, k = 1, 2, \cdots\}$, $V_{5k-1} = \{u_{5k-1}, k = 1, 2, \cdots\}$, $V_{5k-2} = \{u_{5k-2}, k = 1, 2, \cdots\}$, $V_{5k-3} = \{u_{5k-3}, k = 1, 2, \cdots\}$, $V_{5k-4} = \{u_{5k-4}, k = 1, 2, \cdots\}$ and all of indices are taken modulo n.
(a) If $n \equiv 0(mod5)$, we use the following algorithm and define the function f on $GP(n, 2)$:
step 1) $f(u_{5k}) = f(u_{5k-1}) = f(u_{5k-3}) = \emptyset, k = 1, 2, \cdots$.
step 2) $f(v_{5k-2}) = f(v_{5k-3}) = f(v_{5k-4}) = \emptyset, k = 1, 2, \cdots$.
step 3) If k ia an odd number, then $f(u_{5k-4}) = \{1\}, k = 1, 2, \cdots$ and $f(u_{5k-2}) = \{2\}, k = 1, 2, \cdots$ but $f(u_1) = \{1, 2\}$.
step 4) If k ia an even number, then $f(u_{5k-4}) = \{2\}, k = 1, 2, \cdots$ and $f(u_{5k-2}) = \{1\}, k = 1, 2, \cdots$.
step 5) If k ia an odd number, then $f(v_{5k-1}) = f(v_{5k}) = \{1\}, k = 1, 2, \cdots$ but $f(u_n) = \{1, 2\}$.
step 6) If k ia an even number, then $f(v_{5k-1}) = f(v_{5k}) = \{2\}, k = 1, 2, \cdots$.
Now we claim that f is a R2RDF on $GP(n, 2)$ and $\gamma_{rr2}(GP(n, 2)) = \gamma_{rr2}(GP(n, n-2)) \leq \frac{4n}{5} + 2$.
Firstly if there exists the vertex w of $GP(n, 2)$, such that $f(w) = \emptyset$, in according definition of f (steps 1 and 2), w is a member of $U_{5k} \bigcup U_{5k-1} \bigcup U_{5k-3} \bigcup V_{5k-2} \bigcup V_{5k-3} \bigcup V_{5k-4}$. In other hand u_{5k} is adjacent to u_{5k-1}, u_{5k-3} is adjacent to v_{5k-3} and v_{5k-2} is adjacent to v_{5k-4}. Therefore w is adjacent to a vertex z and $f(z) = \emptyset$.
Now if w is a vertex of $GP(n, 2)$ and $f(w) = \emptyset$, then the following cases has happened.
Case 1) There exist a positive integer k such that $w = u_{5k}$. If $w = u_n$, its obvious that w is adjacent to u_1 and $f(u_1) = \{1, 2\}$ otherwise w is adjacent to u_{5k-1}, $u_{5k+1} = u_{5(k+1)-4}$ and v_{5k}. If k is an odd number, according to step 1, step 4 and step 5, $f(u_{5k-1}) = \emptyset$, $f(u_{5k+1}) = \{2\}$ and $f(v_{5k}) = \{1\}$ respectively. If k is an even number, according to step 1, step 3 and step 6, $f(u_{5k-1}) = \emptyset$, $f(u_{5k+1}) = \{1\}$ and $f(v_{5k}) = \{2\}$ respectively. Note that $f(v_n) = \{1, 2\}$.
Case 2) There exist a positive integer k such that $w = u_{5k-1}$. Then w is adjacent to u_{5k-2}, v_{5k-1} and u_{5k}. If k is an odd number, according to step 1, step 3 and step 5, $f(u_{5k}) = \emptyset$, $f(u_{5k-2}) = \{2\}$ and $f(v_{5k-1}) = \{1\}$ respectively. If k is an even number, according to step 1, step 4 and step 6, $f(u_{5k}) = \emptyset$, $f(u_{5k-2}) = \{1\}$ and $f(v_{5k-1}) = \{2\}$ respectively.
Case 3) There exist a positive integer k such that $w = u_{5k-3}$. Then w is adjacent to u_{5k-4}, u_{5k-2} and v_{5k-3}. If k is an odd number, according to step 2 and step 3, $f(v_{5k-3}) = \emptyset$ and $f(u_{5k-2}) = \{2\}$ and $f(u_{5k-4}) = \{1\}$. If k is an

even number, according to step 2 and step 4, $f(v_{5k-3})=\emptyset$, $f(u_{5k-2})=\{1\}$ and $f(u_{5k-4})=\{2\}$.
Case 4) There exist a positive integer k such that $w=v_{5k-2}$. Then w is adjacent to v_{5k-4}, u_{5k-2} and u_{5k}. If k is an odd number, according to step 2, step 3 and step 5, $f(v_{5k-4})=\emptyset$ and $f(u_{5k-2})=\{2\}$ and $f(v_{5k})=\{1\}$ respectively. If k is an even number, according to step 2, step 4 and step 6, $f(v_{5k-4})=\emptyset$ and $f(u_{5k-2})=\{1\}$ and $f(v_{5k})=\{2\}$ respectively.
Case 5) There exist a positive integer k such that $w=v_{5k-3}$. If $w=v_2$, its obvious that w is adjacent to v_n and $f(v_n)=\{1,2\}$ otherwise w is adjacent to u_{5k-3}, v_{5k-1} and $v_{5(k-1)}$. If $k>1$ is an odd number, according to step 1, step 5 and step 6, $f(u_{5k-3})=\emptyset$ and $f(v_{5(k-1)})=\{2\}$ and $f(v_{5k-1})=\{1\}$ respectively. If k is an even number, according to step 1, step 5 and step 6, $f(u_{5k-3})=\emptyset$ and $f(v_{5(k-1)})=\{1\}$ and $f(v_{5k-1})=\{2\}$ respectively.
Case 6) There exist a positive integer k such that $w=v_{5k-4}$. If $w=v_1$, its obvious that w is adjacent to u_1 and $f(u_1)=\{1,2\}$ otherwise w is adjacent to v_{5k-2}, $v_{5(k-1)-1}$ and u_{5k-4}. If $k>1$ is an odd number, according to step 2, step 6 and step 3, $f(v_{5k-2})=\emptyset$ and $f(v_{5(k-1)-1})=\{2\}$ and $f(u_{5k-4})=\{1\}$ respectively. If k is an even number, according to step 2, step 5 and step 4, $f(v_{5k-2})=\emptyset$ and $f(v_{5(k-1)-1})=\{1\}$ and $f(u_{5k-4})=\{2\}$ respectively.
Secondly since $n\equiv 0(mod5)$, then

$$|U_{5k}|=|U_{5k-1}|=|U_{5k-2}|=|U_{5k-3}|=|U_{5k-4}|=|V_{5k}|=|V_{5k-1}|=|V_{5k-2}|=|V_{5k-3}|=|V_{5k-4}|=\lfloor\frac{n}{5}\rfloor$$

So

$$\omega(f)=|U_{5k-2}|+|U_{5k-4}|+|V_{5k}|+|V_{5k-1}|+2=\frac{4n}{5}+2$$

(b) If $n\equiv 1(mod5)$, we use the following algorithm and define the function f on $GP(n,2)$:
step 1) $f(u_{5k})=f(u_{5k-1})=f(u_{5k-3})=\emptyset, k=1,2,\cdots$.
step 2) $f(v_{5k-2})=f(v_{5k-3})=f(v_{5k-4})=\emptyset, k=1,2,\cdots$.
step 3) If k ia an odd number, then $f(u_{5k-4})=\{1\}, k=1,2,\cdots$ and $f(u_{5k-2})=\{2\}, k=1,2,\cdots$.
step 4) If k ia an even number, then $f(u_{5k-4})=\{2\}, k=1,2,\cdots$ and $f(u_{5k-2})=\{1\}, k=1,2,\cdots$.
step 5) If k ia an odd number, then $f(v_{5k-1})=f(v_{5k})=\{1\}, k=1,2,\cdots$ but $f(v_4)=\{1,2\}$.
step 6) If k ia an even number, then $f(v_{5k-1})=f(v_{5k})=\{2\}, k=1,2,\cdots$.
Now similarly to proof of part (a) and a little changes, f is a R2RDF on $GP(n,2)$ and since $n\equiv 1(mod5)$, then

$$|U_{5k}|=|U_{5k-1}|=|U_{5k-2}|=|U_{5k-3}|=|V_{5k}|=|V_{5k-1}|=|V_{5k-2}|=|V_{5k-3}|=\lfloor\frac{n}{5}\rfloor$$

and $|U_{5k-4}|=|V_{5k-4}|=\lfloor\frac{n}{5}\rfloor+1$.
So

$$\omega(f)=|U_{5k-2}|+|U_{5k-4}|+|V_{5k}|+|V_{5k-1}|+1=4\lfloor\frac{n}{5}\rfloor+2$$

(c) If $n\equiv 2(mod5)$, we use the following algorithm and define the function f on $GP(n,2)$:
step 1) $f(u_{5k})=f(u_{5k-1})=f(u_{5k-3})=\emptyset, k=1,2,\cdots$.
step 2) $f(v_{5k-2})=f(v_{5k-3})=f(v_{5k-4})=\emptyset, k=1,2,\cdots$.
step 3) If k ia an odd number, then $f(u_{5k-4})=\{1\}, k=1,2,\cdots$ and $f(u_{5k-2})=\{2\}, k=1,2,\cdots$ but $f(u_1)=\{1,2\}$.
step 4) If k ia an even number, then $f(u_{5k-4})=\{2\}, k=1,2,\cdots$ and $f(u_{5k-2})=\{1\}, k=1,2,\cdots$.
step 5) If k ia an odd number, then $f(v_{5k-1})=f(v_{5k})=\{1\}, k=1,2,\cdots$ but $f(v_4)=\{1,2\}$ and $f(v_{n-2})=\{1,2\}$..
step 6) If k ia an even number, then $f(v_{5k-1})=f(v_{5k})=\{2\}, k=1,2,\cdots$.

Now similarly to proof of part (a) and a little changes, f is a R2RDF on $GP(n,2)$ and since $n\equiv 2(mod5)$, then

$$|U_{5k}|=|U_{5k-1}|=|U_{5k-2}|=|V_{5k}|=|V_{5k-1}|=|V_{5k-2}|=\lfloor\frac{n}{5}\rfloor$$

and $|U_{5k-3}|=|U_{5k-4}|=|V_{5k-4}|=|V_{5k-3}|=\lfloor\frac{n}{5}\rfloor+1$.
So

$$\omega(f)=|U_{5k-2}|+|U_{5k-4}|+|V_{5k}|+|V_{5k-1}|+3=4\lfloor\frac{n}{5}\rfloor+4$$

(d) If $n\equiv 3(mod5)$, we use the following algorithm and define the function f on $GP(n,2)$:
step 1) $f(u_{5k})=f(u_{5k-1})=f(u_{5k-3})=\emptyset, k=1,2,\cdots$.
step 2) $f(v_{5k-2})=f(v_{5k-3})=f(v_{5k-4})=\emptyset, k=1,2,\cdots$.
step 3) If k ia an odd number, then $f(u_{5k-4})=\{1\}, k=1,2,\cdots$ and $f(u_{5k-2})=\{2\}, k=1,2,\cdots$ but $f(u_1)=\{1,2\}$ and $f(u_n)=\{1,2\}$.
step 4) If k ia an even number, then $f(u_{5k-4})=\{2\}, k=1,2,\cdots$ and $f(u_{5k-2})=\{1\}, k=1,2,\cdots$.
step 5) If k ia an odd number, then $f(v_{5k-1})=f(v_{5k})=\{1\}, k=1,2,\cdots$ but $f(v_4)=\{1,2\}$ and $f(v_{n-3})=\{1,2\}$.
step 6) If k ia an even number, then $f(v_{5k-1})=f(v_{5k})=\{2\}, k=1,2,\cdots$.
Now similarly to proof of part (a) and a little changes, f is a R2RDF on $GP(n,2)$ and since $n\equiv 3(mod5)$, then

$$|U_{5k}|=|U_{5k-1}|=|V_{5k}|=|V_{5k-1}|=\lfloor\frac{n}{5}\rfloor$$

and $|U_{5k-4}|=|U_{5k-3}|=|U_{5k-2}|=|V_{5k-4}|=|V_{5k-3}|=|V_{5k-2}|=\lfloor\frac{n}{5}\rfloor+1$.
So

$$\omega(f)=|U_{5k-2}|+|U_{5k-4}|+|V_{5k}|+|V_{5k-1}|+4=4\lfloor\frac{n}{5}\rfloor+6$$

(e) If $n\equiv 4(mod5)$, we use the following algorithm and define the function f on $GP(n,2)$:
step 1) $f(u_{5k})=f(u_{5k-1})=f(u_{5k-3})=\emptyset, k=1,2,\cdots$ but $f(u_n)=\{1\}$.
step 2) $f(v_{5k-2})=f(v_{5k-3})=f(v_{5k-4})=\emptyset, k=1,2,\cdots$.
step 3) If k ia an odd number, then $f(u_{5k-4})=\{1\}, k=1,2,\cdots$ and $f(u_{5k-2})=\{2\}, k=1,2,\cdots$ but $f(u_1)=\{1,2\}$.
step 4) If k ia an even number, then $f(u_{5k-4})=\{2\}, k=1,2,\cdots$ and $f(u_{5k-2})=\{1\}, k=1,2,\cdots$ but $f(u_{n-1})=\{1,2\}$.
step 5) If k ia an odd number, then $f(v_{5k-1})=f(v_{5k})=\{1\}, k=1,2,\cdots$.
step 6) If k ia an even number, then $f(v_{5k-1})=f(v_{5k})=\{2\}, k=1,2,\cdots$.
Now similarly to proof of part (a) and a little changes, f is a R2RDF on $GP(n,2)$ and since $n\equiv 4(mod5)$, then $|U_{5k}|=|V_{5k}|=\lfloor\frac{n}{5}\rfloor$ and

$$|U_{5k-4}|=|U_{5k-3}|=|U_{5k-2}|=|U_{5k-1}|=|V_{5k-4}|=|V_{5k-3}|=|V_{5k-2}|=|V_{5k-1}|=\lfloor\frac{n}{5}\rfloor+1.$$

So

$$\omega(f)=|U_{5k-2}|+|U_{5k-4}|+|V_{5k}|+|V_{5k-1}|+3=4\lfloor\frac{n}{5}\rfloor+6.$$

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