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Some New Fixed Point Theorems for expansive map on S-metric spaces

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Abstract

In this note, we define the expansive map on S-metric space, also we study behavior of expansive map defined on a complete S-metric space and offer some new ways to proving fixed point type theorems and survey it by some illustrative examples.

1 Introduction

In 2006, a new structure of generalized metric space was introduced by [4] as an appropriate notion of generalized metric space called G-metric space, for applications of this structure see [3, 5]. Recently [8, 2, 1] simplify properties of G-metric space and introduced the notion of S-metric space. In this note we will examine some fixed point theorems and behavior of expansive map on S-metric space.

2 Basic Concepts

We briefly give some basic definitions of concepts which serve a background to this work.

Definition 2.1. Let X be a nonempty set. An S-**metric** on X is a function $S : X^3 \to [0, \infty)$ which satisfies the following conditions for each $x, y, z, a \in X$
(i) $S(x, y, z) \geq 0$,
(ii) $S(x, y, z) = 0$ if and only if $x = y = z$,
(iii) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.
The set X with an S-metric is called an S-metric space.

The standard examples of S-metric spaces are:
(a) Let X be any normed space, then $S(x, y, z) = \| y + z - 2x \| + \| y - z \|$ is an S-metric on X.
(b) Let (X, d) be a metric space, then $S_d(x, y, z) = d(x, z) + d(y, z)$ is an S-metric on X. This S-metric is called the *usual* S-metric on X.
(c) Another S-metric on (X, d) is $S'_d(x, y, z) = d(x, y) + d(x, z) + d(y, z)$ which is symmetric with respect to the argument.
(d) $S(x, y, z) = \max\{d(x, z), d(y, z)\}$ is another S-metric on X

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Example 2.1. Let $X = \mathbb{R}^+$. Define

$$S(x,y,z) = \begin{cases} 0 & \text{if } x = y = z, \\ \max\{x,y,z\} & \text{otherwise.} \end{cases}$$

Then S is an S-metric. To show $S(x,y,z) \le S(x,x,a) + S(y,y,a) + S(z,z,a)$, assume $S(x,y,z) = x$. We have $x \le max\{x,a\} \le max\{x,a\} + max\{y,a\} + max\{z,a\}$. That is, $S(x,y,z) \le S(x,x,a) + S(y,y,a) + S(z,z,a)$.

Example 2.2. (See[10]). Let $X = \{1,2,3\}$, define $S : X \times X \times X \to [0,\infty]$ as follows:
$S(1,1,2) = S(2,2,1) = 5$,
$S(2,2,3) = S(3,3,2) = S(1,1,3) = S(3,3,1) = 2$,
For $x = y = z$, $S(x,y,z) = 0$, otherwise $S(x,y,z) = 1$.
S is an S-metric on X.

Example 2.3. (See[6]). Let $X = \mathbb{R}^+$. Define $S(x,y,z) = |\ln \frac{x}{y}| + |\ln \frac{xy}{z^2}|$. Then S is an S-metric on X. We have $S(x,y,z) = 0 \Leftrightarrow \ln \frac{x}{y}, \ln \frac{xy}{z^2} = 0 \Leftrightarrow x = y, xy = z^2 \Leftrightarrow x = y = z$.

To show $S(x,y,z) \le S(x,x,a) + S(y,y,a) + S(z,z,a)$, we have:

$$\begin{aligned} S(x,y,z) &= |\ln x - \ln y| + |\ln x + \ln y - 2lnx| \le |\ln x - \ln a| + |\ln a - \ln y| + |\ln x - \ln z| + |\ln z - \ln y| \\ &\le 2|\ln \tfrac{x}{a}| + 2|\ln \tfrac{y}{a}| + 2\ln |\tfrac{z}{a}| = S(x,x,a) + S(y,y,a) + S(z,z,a). \end{aligned}$$

In this note we will often use the following important fact.

Lemma 2.1. (See[8]). In any S-metric space (X,S), we have $S(x,x,y) = S(y,y,x)$ for $x,y \in X$.

Definition 2.2. A sequence $\{x_n\}$ in X converges to x if $S(x_n,x_n,x) \to 0$ as $n \to \infty$ and we denote this by $\mathbf{lim_{n\to\infty} x_n = x}$ or $\mathbf{x_n \to x}$.

Lemma 2.2. (See[8]). Let (X,S) be an S-metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty} S(x_n,x_n,y_n) = S(x,x,y)$.

Remark 2.1. Let $X = \mathbb{R}^+$ and $S(x,y,z) = |x - z| + |x + z - 2y|$. S is an S-metric on X but it is not generated by any metric. To show this, assume S is generated by a metric d. So we have, $S(x,y,z) = d(x,z) + d(y,z)$. For $y = z$, we have $S(x,z,z) = 2|x - z| = d(x,z)$. For $y = x$ we have $S(x,x,z) = 2|x - z| = 2d(x,z)$. That is for every $x,z \in X$, $2d(x,z) = 2|x - z| = d(x,z)$, which is a contradiction.

There exists a natural topology on an S-metric space. At first let us define the notion of (open) ball.

Definition 2.3. Let (X,S) be an S-metric space. For $r > 0$ and $x \in X$ we define an open ball with center x and radius r as follows:

$$B_s(x,r) = \{y \in X : S(y,y,x) < r\}.$$

This is a quite different concept of the ball in the usual metric space. We have:

Example 2.4. Let (X,d) be a metric space and let $S_d(x,y,z) = d(x,z) + d(y,z)$ be the usual S-metric on X. Then:

$$\begin{aligned} B_s(x_0,2) = \{y \in X : S(y,y,x_0) < 2\} &= \{y \in \mathbb{R} : 2d(y,x_0) < 2\} \\ &= \{y \in \mathbb{R} : d(y,x_0) < 1\} = B_d(x_0,1). \end{aligned}$$

By using the notion of open ball we can introduce the standard topology on an S-metric space such that its basis is the open balls.

Definition 2.4. The sequence $\{x_n\}$ in an S-metric space (X, S) is called **Cauchy sequence** if $\lim_{n,m\to\infty} S(x_n, x_n, x_m) = 0$.

Definition 2.5. An S-metric space (X, S) is said to be **complete** if every Cauchy sequence converges.

We prove the following result:

Lemma 2.3. Any S-metric space is Hausdorff.

Proof. Let (X, S) be an S-metric space. Suppose $x \neq y$ and put $r = \frac{1}{3}S(x, x, y)$. We have $B_S(x, r) \cap B_S(y, r) = \emptyset$, for $x, y \in X$. Otherwise there exists $z \in X$ such that $z \in B_S(x, r) \cap B_S(y, r)$, therefore by definition of open ball we have $S(z, z, x) < r$ and $S(z, z, y) < r$. By Lemma 2.1 and (iii), we get

$$3r = S(x, x, y) \leq 2S(z, z, x) + S(z, z, y) = 2S(x, x, z) + S(y, y, z) < 3r,$$

which is a contradiction. □

Remark 2.2. We have:
$x_n \to x$ in (X, d) if and only if $d(x_n, x) \to 0$, if and only if $S_d(x_n, x_n, x) = 2d(x_n, x) \to 0$, that is, $x_n \to x$ in (X, S_d).

Definition 2.6. Let (X, S_1) and (Y, S_2) be S-metric spaces. A map $f : X \to Y$ is called **continuous** at $x \in X$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$S_1(x, x, y) < \epsilon \Rightarrow S_2(f(x), f(x), f(y)) < \delta,$$

$$\text{or} \qquad f(B_{s_1}(x, \delta)) \subset B_{s_2}(f(x), \epsilon).$$

Lemma 2.4. (See[7]). Let (X, S_1) and (Y, S_2) be S-metric spaces. Then $f : X \to Y$ is continuous at $x \in X$ if and only if $f(x_n) \to f(x)$ whenever $x_n \to x$.

Definition 2.7. (See[8]). Let (X, S) be an S-metric space. A map $T : X \to X$ is said to be a **contraction** if there exists a constant $0 \leq k < 1$ such that

$$S(Tx, Tx, Ty) \leq kS(x, x, y), \quad \text{for all} \quad x, y \in X.$$

Theorem 2.1. (See[8]). Let (X, S) be a complete S-metric space and $T : X \to X$ be a contraction. Then T has a unique fixed point.

Definition 2.8. (See[3]). Let (X,S) be an S-metric space and T be a self-map on X. Then T is called an **expansive map** if there exists a constant $a > 1$ such that for all $x, y \in X$, we have

$$S(Tx, Tx, Ty) \geq aS(x, x, y).$$

The constant a is called the **expansion coefficient**.

Expansive map on S-metric space need not to be continuous, consider to following example:

Example 2.5. Let $T : (\mathbb{R}, S) \to (\mathbb{R}, S)$ be defined by

$$Tx = \begin{cases} 4x & \text{if } x \leq 2, \\ 4x + 3 & \text{if } x > 2, \end{cases}$$

where $S(x, y, z) = \max\{|x - z|, |y - z|\}$. Then $(\mathbb{R}, S)$ is a complete S-metric space and T is an expansive map with expansion coefficient $a = 2$.

3 Main Result

We state our main result:

Theorem 3.1. Let (X, S) be a complete S-metric space and let $T : X \to X$ be a surjective and expansive map with the expansion coefficient a. Then T has a unique fixed point.

Proof. Assume $Tx = Ty$, then $0 = S(Tx, Tx, Ty) \geq aS(x, x, y)$, which implies that $S(x, x, y) = 0$, hence $x = y$. So, T is injective and invertible. Let H be the inverse map of T. Then

$$S(x, x, y) = S(T(Hx), T(Hx), T(Hy)) \geq aS(Hx, Hx, Hy).$$

Thus, for all $x, y \in X$, we have $S(Hx, Hx, Hy) \leq kS(x, x, y)$, where $k = \frac{1}{a} < 1$. Applying Theorem 2.1, we conclude that H has a unique fixed point $u \in X$; $H(u) = u$. But, $u = T(H(u)) = T(u)$, so u is also a fixed point of T. Suppose there exists $v \neq u$ such that $Tv = v$, then $Tv = v = H(Tv)$, so Tv is another fixed point for H. By uniqueness we conclude that $u = Tv = v$. □

Corollary 3.1. Let (X, S) be a complete S-metric space and let $T : X \to X$ be a surjective map satisfying the following condition, for all $x, y, z \in X$

$$S(T(x), T(y), T(z)) \geq k\{S(x, x, Tx) + S(y, y, Tx) + S(z, z, Tx)\} \tag{1}$$

where $k > 1$. Then T has a unique fixed point.

Proof. From (iii) we have $S(x, x, Tx) + S(y, y, Tx) + S(z, z, Tx) \geq S(x, y, z)$, then by inequality (1) we have $S(T(x), T(y), T(z)) \geq kS(x, y, z)$ for all $x, y, z \in X$, by putting $x = y$, the proof follows from Theorem 3.1. □

Example 3.1. Accomplish $X = \mathbb{R}$ with the S-metric $S(x, y, z) = \max\{|x - z|, |y - z|\}$ for all $x, y, z \in X$. Define $T : \mathbb{R} \to \mathbb{R}$ by

$$Tx = \begin{cases} 4x & \text{if } x \leq 2, \\ 2x + 4 & \text{if } x > 2, \end{cases}$$

Obviously T is a surjective map on X. Now

$$S(Tx, Tx, Ty) = \begin{cases} 4|x - y| & \textit{if } x, y \leq 2, \\ 2|x - y| & \textit{if } x, y > 2, \\ |4x - 2y - 4| & \textit{if } y > 2, x \leq 2, \\ |2x - 4y + y| & \textit{if } x > 2, y \leq 2, \end{cases}$$

$$\geq 2|x - y| = 2S(x, x, y).$$

So for $a = 2$ all conditions of the Theorem 3.1 are satisfied. Therefore, T has unique fixed point zero.

Theorem 3.2. (See[9]). Let (X, S) be a complete S-metric space and let $T : X \to X$ be a surjective map satisfying the following condition for all $x, y \in X$,

$$S(T(x), T(x), T(y)) \geq aS(x, x, y) + bS(x, x, Tx) + dS(y, y, Ty) \tag{2}$$

where a, b, c are non negative real numbers and $a + b + d > 1$ and $b < 1$. Then T has a fixed point.

Corollary 3.2. Let (X, S) be a complete S-metric space and let T be a surjective self-map on X satisfying the following condition for all $x, y \in X$

$$S(T(x), T(x), T(y)) \geq \alpha S(x, x, y) + \beta\{S(x, x, Tx) + S(y, y, Ty)\}, \tag{3}$$

where α, β are non negative real numbers and $\alpha + 2\beta > 1$ and $\beta < \frac{1}{2}$. Then T has a fixed point.

Proof. In Theorem 3.2, If we put $\alpha = a$, and $b = d = \beta$, then the condition (3) reduced to condition (2), so the proof follows from Theorem 3.2. □

Example 3.2. Accomplish $X = \mathbb{R}^+$ with the following S-metric

$$S(x, y, z) = \begin{cases} 0 & \text{if } x = y = z, \\ \max\{x, y, z\} & \text{otherwise.} \end{cases}$$

Define $T(x) = 2x$, suppose $x < y$. Then we should have:

$$\begin{aligned} S(Tx, Tx, Ty) &= S(2x, 2x, 2y) = 2y \\ &\geq aS(x, x, y) + bS(x, x, Tx) + cS(y, y, Ty) \\ &= ay + 2bx + 2cy, \end{aligned}$$

for $a = \frac{32}{31}$ and $b = c = \frac{1}{31}$ all conditions of Theorem 3.2 are satisfied and zero is the fixed point of T. For $y < x$ we have the same result.

Theorem 3.3. Let (X, S) be a complete S-metric space and $T : X \to X$ be an surjective map satisfying the following condition for all $x \in X$:

$$S(Tx, Tx, T^2x) \geq aS(x, x, Tx) \tag{4}$$

where $a > 1$. Then T has a fixed point.

Proof. Let $x_0 \in X$, since T is surjective, so there exists $x_1 \in T^{-1}(x_0)$. Successively we can pick up $x_n \in T^{-1}(x_{n-1})$ for $n = 2, 3, 4, 5, ...$ .If $x_m = x_{m-1}$ for some m, then x_m is a fixed point of T. Assume $x_n \neq x_{n-1}$, $T(x_n) = x_{n-1}$ for every n, then from (4) we have

$$S(x_n, x_n, x_{n-1}) \leq \frac{1}{\alpha} S(x_{n-1}, x_{n-1}, x_{n-2}). \tag{5}$$

Let $q = \frac{1}{a}$, then $q < 1$. By repeating (5) and using Lemma 2.1 we have

$$S(x_n, x_n, x_{n-1}) \leq q^{n-1} S(x_0, x_0, x_1). \tag{6}$$

Then by (iii) and Lemma 2.1 for all $n, m \in \mathbb{N}$; $n < m$ we have

$$\begin{aligned} S(x_n, x_n, x_m) &\leq 2S(x_n, x_n, x_{n+1}) + S(x_m, x_m, x_{n+1}) \\ &= 2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_m) \\ &\leq 2S(x_n, x_n, x_{n+1}) + 2S(x_{n+1}, x_{n+1}, x_{n+2}) + S(x_m, x_m, x_{n+2}) \\ &\dots \\ &\leq 2\sum_{i=n}^{m-2} S(x_i, x_i, x_{i+1}) + S(x_{m-1}, x_{m-1}, x_m). \end{aligned}$$

Now, by (6) we have:

$$\begin{aligned} S(x_n, x_n, x_m) &\leq 2\sum_{i=n}^{m-2} S(x_i, x_i, x_{i+1}) + S(x_{m-1}, x_{m-1}, x_m) \\ &\leq 2\sum_{i=n}^{m-2} q^{n-1} S(x_0, x_0, x_1) + q^{m-1} S(x_0, x_0, x_1)) \\ &\leq 2q^{n-1} S(x_0, x_0, x_1)[1 + q + q^2 + \dots] \\ &\leq \frac{2q^{n-1}}{1-q} S(x_0, x_0, x_1). \end{aligned}$$

Hence, $\lim S(x_n, x_m, x_m) = 0$, as $n, m \to \infty$. So $\{x_n\}$ is a Cauchy sequence. By the completeness of (X, S), there exists $u \in X$ such that $\{x_n\}$ converges to u. Since T is surjective there exists $b \in X$ such that $T(b) = u$. Also, there exists $c \in X$ such that $T(c) = b$. Now for every $n \in \mathbb{N}$ we have:

$$\begin{aligned} S(x_n, x_n, u) &= S(Tx_{n+1}, Tx_{n+1}, T^2(c)) \\ &\geq \alpha S(x_{n+1}, x_{n+1}, T(c)). \end{aligned}$$

That is $\lim x_n = T(c)$. So $b = T(c) = u = T(b)$. □

Example 3.3. Accomplish $X = \mathbb{R}^+$ with the following S-metric,

$$S(x, y, z) = \begin{cases} 0 & \text{if } x = y = z, \\ \max\{x, y, z\} & \text{otherwise.} \end{cases}$$

Assume,

$$Tx = \begin{cases} 4x & \textit{if } x < 2, \\ 2x & \textit{if } x \geq 2. \end{cases}$$

We have:

$$T^2 x = \begin{cases} 16x & \textit{if } x < \frac{1}{2}, \\ 8x & \textit{if } \frac{1}{2} \leq x < 2 \\ 4x & \textit{if } 2 \leq x, \end{cases}$$

$$S(Tx,Tx,T^2x)=T^2x=\begin{cases}16x & \textit{if } x<\frac{1}{2},\\ 8x & \textit{if } \frac{1}{2}\leq x<2\\ 4x & \textit{if } 2\leq x.\end{cases}$$

$$S(x,x,Tx)=\begin{cases}4x & \textit{if } x<\frac{1}{2},\\ 4x & \textit{if } \frac{1}{2}\leq x<2\\ 2x & \textit{if } 2\leq x\end{cases}$$

So, $S(Tx,Tx,T^2x)\geq 2S(x,x,Tx)$. That is, all conditions of Theorem 3.3 are satisfied and T has fixed point zero. (T is continuous).

Example 3.4. Accomplish $X=\mathbb{R}^+$ with the following S-metric,

$$S(x,y,z)=\begin{cases}0 & \text{if } x=y=z,\\ \max\{|x|,|y|,|z|\} & \text{otherwise.}\end{cases}$$

Assume,

$$Tx=\begin{cases}\sqrt{3}x & \textit{if } x<0,\\ 2x & \textit{if } x\geq 0.\end{cases}$$

We have:

$$T^2x=\begin{cases}3x & \textit{if } x<0,\\ 4x & \textit{if } x\geq 0,\end{cases}$$

$$S(x,x,Tx)=Tx=\begin{cases}-\sqrt{3}x & \textit{if } x<0,\\ 2x & \textit{if } x\geq 0,\end{cases}$$

$$S(Tx,Tx,T^2x)=\begin{cases}-3x & \textit{if } x<0,\\ 4x & \textit{if } x\geq 0.\end{cases}$$

So, $S(Tx,Tx,T^2x)\geq \sqrt{3}S(x,x,Tx)$. That is, all conditions of Theorem 3.3 are satisfied and zero is the fixed point of T.

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