Spectral Method for Solving Fuzzy Volterra Integral Equations of Second kind

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ABSTRACT

This paper, about the solution of fuzzy Volterra integral equation of fuzzy Volterra integral equation of second kind (F-VIE2) using spectral method is discussed. The parametric form of fuzzy driving term is applied for F-VIE2. Then three cases for (F-VIE2) are searched to solve them. This classifications are considered based on the sign of interval. The Gauss-Legendre points and Legendre weights for arithmetics in spectral method are used to solve (F-VIE2). Finally two examples are got to illustrate more.

1 Introduction

The integral equations method are used for solving many problems in mathematics, the solutions of Integral equations are important for applied science for example, mechanics or physics. The numerical methods are used widely in recent years. Many Integral equations have fuzzy parameters and a few method expanded to fuzzy integral equations.

The concepts of fuzzy numbers were first introduced by Zadeh, [10] but Friedman et.al introduced the numerical solution of fuzzy integral equation by embedding method. Sufficient conditions for convergence of their proposed method was given, [13]. There are some research over existence and unique of integral equations, [14],[15],[17] and [25]. Special Park and Jeong search exist and uniqueness of Fredholm-Volterra integral equation (F-VIE), [19]. Abbasbandyet. al, used a parametric fuzzy number to convert a linear fuzzy Fredholm integral equation of second kind such as the Nystrom approximation, [20]. Molabahrami et.al, by using the parametric form of fuzzy number converted a linear fuzzy F-VIE to two linear system of the second kind of integral equation in crisp case. They used the homotopy analysis method to find the approximate solution of these systems, [17]. Ghanbari et.al, used the Block-pulse functions to approximate the numerical solution of fuzzy F-IE, [23]. Sadeghi goghari et. al, present two method which exploit hybrid Legendre and Block Pulse functions and Legendre wavelets to find the approximate solution for a system of linear fuzzy F-IE of the second kind with two variables, [21]. Babolian et.al converted a linear fuzzy F-VIE of the second kind to a linear system of integral equation in the crisp case by Adomian method, [24].

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In this paper the numerical method for solving the F-VIE2 equations in the form

$$\tilde{u}(x) + \int_a^x K(x,s)\tilde{u}(s)ds = \tilde{g}(x), \quad x \in [a,b] \tag{1.1}$$

are discussed. $\tilde{u}$ is a notation for the fuzzy set of u. This numerical method is restricted to spectral method. In section 2, we review briefly some needed concepts. In section 3, we proposed our method to find solution by spectral method. In section 4, we classify the F-VIE2, we consider four cases in behavior of F-VIE2. Finally we get two example to illustrate more our method. In whole of paper $\tilde{a}$ is a notation for the fuzzy set of a.

2 Basic concepts

The basic definitions of a fuzzy number are given in [15, 25, 26, 11] as follows:

Definition 2.1. *A fuzzy number is a fuzzy set like $u : \mathbb{R} \to [0,1]$ which satisfies:*

1. *u is an upper semi-continuous function,*
2. *$u(x) = 0$ outside some interval [a,d],*
3. *There are real numbers b, c such as $a \leq b \leq c \leq d$ and*
   - *3.1 $u(x)$ is a monotonic increasing function on [a, b],*
   - *3.2 $u(x)$ is a monotonic decreasing function on [c, d],*
   - *3.3 $u(x) = 1$ for all $x \in [b,c]$.*

Definition 2.2. *A fuzzy number u in parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r)$, $\overline{u}(r)$, $0 \leq r \leq 1$, which satisfy the following requirements:*

1. *$\underline{u}(r)$ is a bounded non-decreasing left continuous function in $(0,1]$, and right continuous at 0,*
2. *$\overline{u}(r)$ is a bounded non-increasing left continuous function in $(0,1]$, and right continuous at 0,*
3. *$\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.*

Definition 2.3. *For arbitrary $\tilde{u} = (\underline{u}(r), \overline{u}(r))$ and $\tilde{v} = (\underline{v}(r), \overline{v}(r))$, $0 \leq r \leq 1$, and scalar k, we define addition, subtraction, scalar product by k and multiplication are respectively as following:*

$addition: \quad \underline{u+v}(r) = \underline{u}(r) + \underline{v}(r), \quad \overline{u+v}(r) = \overline{u}(r) + \overline{v}(r)$

$subtraction: \quad \underline{u-v}(r) = \underline{u}(r) - \overline{v}(r), \quad \overline{u-v}(r) = \overline{u}(r) - \underline{v}(r)$

scalar product: $k\tilde{u} = \begin{cases} (k\underline{u}(r), k\overline{u}(r)), & k \geq 0 \\ (k\overline{u}(r), k\underline{u}(r)), & k < 0 \end{cases}$ $multiplication: \quad \underline{uv}(r) = max\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\overline{v}(r), \overline{u}(r)\underline{v}(r)\overline{u}(r)\overline{v}(r)\}$

$\overline{uv}(r) = min\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\overline{v}(r), \overline{u}(r)\underline{v}(r)\overline{u}(r)\overline{v}(r)\}$

Definition 2.4. *The metric structure is given by Hausdorff distance*

$D : \mathbb{R}_F \times \mathbb{R}_F \longrightarrow \mathbb{R}_+ \cup 0$

$D(u(r), v(r)) = Max\{sup|\underline{u} - \underline{v}|, sup|\overline{u} - \overline{v}|\}$

$(\mathbb{R}_F, D)$ *is a complete metric space and following properties are well known:*

$D(u+w,v+w)=D(u,v), \quad \forall u,v,w\in\mathbb{R}_F$
$D(ku,kv)=|k|D(u,v), \quad \forall u,v\in\mathbb{R}_F, \quad \forall k\in\mathbb{R}$
$D(u+v,w+e)\leq D(u,w)+D(v,e), \quad \forall u,v,w,e\in\mathbb{R}_F$

If the fuzzy function $f(x)$ is continuous in the metric D, its definite integral exists. Furthermore

$$\underline{(\int_a^b f(x)dx)}=(\int_a^b \underline{f}(x)dx)$$

$$\overline{(\int_a^b f(x)dx)}=(\int_a^b \overline{f}(x)dx)$$

For arithmetic in overall $s\in[a,b]$ for the following equation

$$\tilde{u}(x)=\tilde{f}(x)+\int_a^x K(x,s)\tilde{u}(s)ds,$$

can be transform to two equations:

$$\underline{u}(x)=\underline{f}(x)+\int_a^x v_1(x,s,\underline{u}(s),\overline{u}(s))ds$$

$$\overline{u}(x)=\overline{f}(x)+\int_a^x v_2(x,s,\underline{u}(s),\overline{u}(s))ds$$

which

$$v_1(s,t,\underline{u}(s,r),\overline{u}(s,r))=\begin{cases} k(x,s)\underline{u}(s), & k(x,s)\geq 0\\ k(x,s)\overline{u}(s), & k(x,s)<0\end{cases}$$

and

$$v_2(s,t,\underline{u}(s,r),\overline{u}(s,r))=\begin{cases} k(x,s)\overline{u}(s), & k(x,s)\geq 0\\ k(x,s)\underline{u}(s), & k(x,s)<0\end{cases}$$

Definition 2.5. *For simplify in arithmetics over parametric form of fuzzy number we define: Let* $u(r)=[\underline{u}(r),\overline{u}(r)], 0\leq r\leq 1$ *be a fuzzy number we take*

$$u^c=\frac{\underline{u}(r)+\overline{u}(r)}{2}$$

$$u^d=\frac{\overline{u}(r)-\underline{u}(r)}{2}$$

It is clear that $u^d(r)\geq 0$ *and* $\underline{u}(r)=u^c(r)-u^d(r)$ *and* $\overline{u}(r)=u^c(r)+u^d(r)$

Definition 2.6. *Let* $u(r)=[\underline{u}(r),\overline{u}(r)], v(r)=[\underline{v}(r),\overline{v}(r)], 0\leq r\leq 1$ *are two fuzzy numbers and also* k,s *are two arbitrary real numbers. If* $w=ku+sv$ *then by using definition (2.5)*

$$w^c(r)=ku^c(r)+sv^c(r)$$

$$w^d(r)=|k|u^d(r)+|s|v^d(r)$$

3 Fuzzy Legendre-collocation method

We consider the second-kind Volterra fuzzy integral equation is of the form (1). If the collocation points are chosen as the the set of $(N+1)$ Gauss-Legendre $\{x_i\}_{i=0}^{N}$. Then Eq. (1) holds that

$$\tilde{u}(x_i)+\int_{-1}^{x_i}K(x_i,s)\tilde{u}(s)ds=\tilde{g}(x_i),\quad 0\leq i\leq N \tag{3.1}$$

If we apply the following linear transformation

$$s(x,\theta)=\frac{x+1}{2}\theta+\frac{x-1}{2},\quad -1\leq\theta\leq 1 \tag{3.2}$$

then Eq. (3) becomes

$$\tilde{u}(x_i)+\frac{1+x_i}{2}\int_{-1}^{1}K(x_i,s(x_i,\theta))\tilde{u}(s(x_i,\theta))d\theta=\tilde{g}(x_i),\quad 0\leq i\leq N \tag{3.3}$$

Using $(N+1)-$point Gauss-Legendre quadrature formula relative to the Legendre weights $\{\omega_k\}$ gives

$$\tilde{u}(x_i)+\frac{1+x_i}{2}\sum_{j=0}^{N}K(x_i,s(x_i,\theta))\tilde{u}(s(x_i,\theta))\omega_j=\tilde{g}(x_i),\quad 0\leq i\leq N \tag{3.4}$$

we estimate $\tilde{u}$ using Lagrange interpolation polynomials

$$\tilde{u}(\sigma)\approx\sum_{j=0}^{N}\tilde{u}_j l_j(\sigma) \tag{3.5}$$

where the l_j is j-th Lagrange basic function. Combination Eqs. (6) and (7) yields

$$\tilde{u}_i+\frac{1+x_i}{2}\sum_{j=0}^{N}\tilde{u}_j(\sum_{p=0}^{N}K(x_i,s(x_i,\theta))l_p(s(x_i,\theta))\omega_p)=\tilde{g}(x_i),\quad 0\leq i\leq N \tag{3.6}$$

If $K(x,s)$ is continuous over $[-1,1]$ respect to the sign of $K(x,s)$ -that is positive or negative over $[-1,1]$ -we have three cases, that we peruse them in the next section .

4 The classification of F-VIE2

By using sign of $K(x,s)$ over $[-1,1]$ we have three cases, that we peruse them. The parametric form of equation (1) can be write in following:

$$[\overline{u}(x),\underline{u}(x)]+\int_{-1}^{x}K(x,s)[\overline{u}(x),\underline{u}(s)]ds=[\overline{g}(x),\underline{g}(x)],\quad x\in[-1,1] \tag{4.1}$$

$\underline{u}(r)$ and $\overline{u}(r)$ for all $0 \le r \le 1$ and for $x \in [-1,1]$ by Lagrange polynomials can be consider in following terms:

$$\underline{u}(x) \approx \sum_{j=0}^{N} \underline{u}_j(r) l_j(x)$$

$$\overline{u}(x) \approx \sum_{j=0}^{N} \overline{u}_j(r) l_j(x)$$

4.1 Case (1)

If $K(x,s)$ is positive over $[-1,1]$ Eq. (1) is transformed to

$$\underline{u}(x) + \int_{-1}^{x} K(x,s)\underline{u}(s)ds = \underline{g}(x), \quad x \in [-1,1] \tag{4.2}$$

and

$$\overline{u}(x) + \int_{-1}^{x} K(x,s)\overline{u}(s)ds = \overline{g}(x), \quad x \in [-1,1] \tag{4.3}$$

Now by using Eq. (4) we can consider Eqs. (9) and (10) in following respectively:

$$\underline{u}(x_i)(r) + \frac{1+x_i}{2}\int_{-1}^{1} K(x_i, s(x_i,\theta))\underline{u}(s(x_i,\theta))(r)d\theta = \underline{g}(x_i)(r), \quad 0 \le i \le N \tag{4.4}$$

$$\overline{u}(x_i)(r) + \frac{1+x_i}{2}\int_{-1}^{1} K(x_i, s(x_i,\theta))\overline{u}(s(x_i,\theta))(r)d\theta = \overline{g}(x_i)(r), \quad 0 \le i \le N \tag{4.5}$$

by using set of $(N+1)$ Gauss-Legendre $\{x_i\}_{i=0}^{N}$, the Eqs. (11) and (12) transform to the following equations respectively:

$$\underline{u}_i(r) + \frac{1+x_i}{2}\sum_{j=0}^{N} \underline{u}_j(r)(\sum_{p=0}^{N} K(x_i, s(x_i,\theta))l_p(s(x_i,\theta))\omega_p) = \underline{g}(x_i)(r), \quad 0 \le i \le N \tag{4.6}$$

$$\overline{u}_i(r) + \frac{1+x_i}{2}\sum_{j=0}^{N} \overline{u}_j(r)(\sum_{p=0}^{N} K(x_i, s(x_i,\theta))l_p(s(x_i,\theta))\omega_p) = \overline{g}(x_i)(r), \quad 0 \le i \le N \tag{4.7}$$

Now we must solve two crisp equations of the matrix form (13) and (14) and we can obtain $\underline{u}(r)$ and $\overline{u}(r)$ for all $0 \le r \le 1$ and for $x \in [-1,1]$.

4.2 Case (2)

In this case we hypothesis $K(x,s)$ be negative over $[-1,1]$. Then Eq. (5) is transformed to two following equations:

$$\underline{u}(x_i)(r) + \frac{1+x_i}{2}\int_{-1}^{1} K(x_i, s(x_i,\theta))\overline{u}(s(x_i,\theta))(r)d\theta = \underline{g}(x_i)(r), \quad 0 \le i \le N \tag{4.8}$$

$$\overline{u}(x_i)(r) + \frac{1+x_i}{2}\int_{-1}^{1} K(x_i, s(x_i,\theta))\underline{u}(s(x_i,\theta))(r)d\theta = \overline{g}(x_i)(r), \quad 0 \le i \le N \tag{4.9}$$

then if we use (4) in (15) and (16) we holds:

$$\underline{u}_i(r) + \frac{1+x_i}{2}\sum_{j=0}^{N}\overline{u}_j(r)(\sum_{p=0}^{N}K(x_i,s(x_i,\theta))l_p(s(x_i,\theta))\omega_p) = \underline{g}(x_i)(r), \quad 0 \le i \le N \tag{4.10}$$

$$\overline{u}_i(r) + \frac{1+x_i}{2}\sum_{j=0}^{N}\underline{u}_j(r)(\sum_{p=0}^{N}K(x_i,s(x_i,\theta))l_p(s(x_i,\theta))\omega_p) = \overline{g}(x_i)(r), \quad 0 \le i \le N \tag{4.11}$$

Now if we get

$$A_{i,j} = \frac{1+x_i}{2}\sum_{j=0}^{N}K(x_i,s(x_i,\theta))l_j(s(x_i,\theta))\omega_j, \quad 0 \le i \le N \tag{4.12}$$

then to find solution in (17) and (18) and by using definitions (2.5) and (2.6) for all $0 \le i \le N$ we have

$$u_i^c(r) + A_{ij}\sum_{p=0}^{N}u_p^c(r) = g^c(x_i) \tag{4.13}$$

$$u_i^d(r) + A_{ij}\sum_{p=0}^{N}u_p^d(r) = g^d(x_i) \tag{4.14}$$

Now we must solve two crisp equations of the matrix form (20) and (21). Eq. (20) and (21) can be solved by some suitable method for solving the linear systems. By using definition (2.6) we can obtain $\tilde{u}_i = [\underline{u}_i, \overline{u}_i]$ for $i = 0,1,...,N$. When the values of $\tilde{u}_i = [\underline{u}_i, \overline{u}_i]$ for $i = 0,1,...,N$, are resulted the numerical solution for $x \in [-1,1]$ can be obtained by Lagrange polynomials.

4.3 Case (3)

In case (3) we consider $K(x,s)$ be continuous in $-1 \le s \le 1$ and for fix x, $K(x,s)$ changes its sign in finite points as t_i, for example without loss generality $K(x,s)$ is positive over $[-1,t]$ and negative over $[t,x]$, we have

$$\tilde{u} + \int_{-1}^{t}K(x,s)\tilde{u}(s)ds + \int_{t}^{x}K(x,s)\tilde{u}(s)ds = \tilde{g}(x) \tag{4.15}$$

then by (N+1) Gauss-Legendre points we can write:

$$\tilde{u}(x_i) + \int_{-1}^{t}K(t,s)\tilde{u}(s)ds + \int_{t}^{x_i}K(x_i,s)\tilde{u}(s)ds = \tilde{g}(x_i) \tag{4.16}$$

Now we can transform interval $[t,1]$ to $[-1,1]$ by

$$s = \frac{x_i+t}{1-t}\eta + \frac{x-1}{1-t}, -1 \le \eta \le 1 \tag{4.17}$$

by (4) and (24) we have

$$\tilde{u}(x_i)+\frac{1+t}{2}\int_{-1}^{1}K(t,s(t,\theta))\tilde{u}(s)d\theta+\frac{x_i+t}{1-t}\int_{-1}^{1}K(x_i,s(t,\eta))\tilde{u}(s)d\eta=\tilde{g}(x_i) \tag{4.18}$$

Now by using Legendre weights, the Eq. (25) is transformed to

$$\tilde{u}_i+\frac{1+t}{2}\sum_{j=0}^{N}(\tilde{u}_j\sum_{j=0}^{N}K(t,s(t,\theta))l_j(s(t,\theta))\omega_j)+\frac{x_i+t}{1-t}\sum_{j=0}^{N}(\tilde{u}_j\sum_{j=0}^{N}K(x_i,s(t,\eta))l_j(s(x_i,\eta))\omega_j)=\tilde{g}(x_i) \tag{4.19}$$

then with using the sign of K on the intervals we have

$$\underline{u}_i(r)+\frac{1+t}{2}\sum_{j=0}^{N}(\underline{u}_j(r)\sum_{j=0}^{N}K(t,s(t,\theta))l_j(s(t,\theta))\omega_j)+\frac{x_i+t}{1-t}\sum_{j=0}^{N}(\overline{u}_j(r)\sum_{j=0}^{N}K(x_i,s(t,\eta))l_j(s(x_i,\eta))\omega_j)=\underline{g}(x_i)(r) \tag{4.20}$$

$$\overline{u}_i(r)+\frac{1+t}{2}\sum_{j=0}^{N}(\overline{u}_j(r)\sum_{j=0}^{N}K(t,s(t,\theta))l_j(s(t,\theta))\omega_j)+\frac{x_i+t}{1-t}\sum_{j=0}^{N}(\underline{u}_j(r)\sum_{j=0}^{N}K(x_i,s(t,\eta))l_j(s(x_i,\eta))\omega_j)=\overline{g}(x_i)(r) \tag{4.21}$$

then we can write following matrices:

$$u_i^c(r)+(B_j+C_{ij})\sum_{p=0}^{N}u_p^c(r)=g_i^c,\quad 0\le i\le N \tag{4.22}$$

$$u_i^d(r)+(B_j-C_{ij})\sum_{p=0}^{N}u_p^d(r)=g_i^d,\quad 0\le i\le N \tag{4.23}$$

which

$$B_j=\frac{1+t}{2}\sum_{j=0}^{N}K(t,s(t,\theta))l_j(s(t,\theta))\omega_j$$

$$C_{ij}=\frac{x_i+t}{1-t}\sum_{j=0}^{N}K(x_i,s(t,\eta))l_j(s(x_i,\eta))\omega_j$$

Eq. (29) and (30) can be solved by some suitable method for solving the linear systems. By using definition (2.5) we can obtain $\tilde{u}_i=[\underline{u}_i,\overline{u}_i]$ for $i=0,1,...,N$. When the values of $\tilde{u}_i=[\underline{u}_i,\overline{u}_i]$ for $i=0,1,...,N$, are the numerical solution for $x\in[-1,1]$ can be obtained by Lagrange polynomials.

Denoting $U_N=\{u_0,u_1,...,u_N\}^t$ and $g_N=\{g(x_0),g(x_1),...,g(x_N)\}^t$ we can obtain an equation of the matrix form:

$$U_N+AU_N=g_N \tag{4.24}$$

where matrix A is given by

$$A_{i,j}=\frac{1+x_i}{2}\sum_{p=0}^{N}K(x_i,s(x_i,\theta))l_j(s(x_i,\theta))\omega_p \tag{4.25}$$

we express $L_j(s)$ in Legendre functions

$$l_j(s) = \sum_{p=0}^{N} \alpha_{p,j} L_p(s) \tag{4.26}$$

where $\alpha_{p,j}$ is discrete polynomial coefficients. The inverse relation

$$\alpha_{p,j}(s) = \frac{1}{\gamma_p} \sum_{i=0}^{N} l_j(x_i) L_p(x_i)\omega_i = \frac{L_p(x_j)\omega_j}{\gamma_p} \tag{4.27}$$

where

$$\gamma_p = \sum_{i=0}^{N} L_p^2(x_i)\omega_i = (p+\frac{1}{2})^{-1}, p < N \tag{4.28}$$

and $\gamma_N = (N+\frac{1}{2})^{-1}$ for the Gauss formulas.

$$l_j(s) = \sum_{p=0}^{N} \frac{L_p(x_j)L_p(s)\omega_j}{\gamma_p} \tag{4.29}$$

This equations in this section be spoken for crisp equations. We can use them for $\underline{U}_N = \{\underline{u}_1, \underline{u}_2, ...\underline{u}_N\}$ and $\overline{U}_N = \{\overline{u}_1, \overline{u}_2, ...\overline{u}_N\}$ instead of U_N and $\underline{g}_N = \{\underline{g}(x_1), \underline{g}(x_2), ...\underline{g}(x_N)\}$ and $\overline{g}_N = \{\overline{g}(x_1), \overline{g}(x_2), ...\overline{g}(x_N)\}$ instead of g_N.

5 Numerical Example

In this section two examples are presented and solved by Legendre-spectral method and tables and figures are presented that numerical and exact solutions are compared in those. Without lose of generality, we will only use the Legendre-Gauss-Lobatto points (i.e., the zeros of $L(N+1)(x)$) as the collocation points. Our numerical evidences show that the other two kinds of Legendre-Gauss points produce results with similar accuracy. For the Legendre- Gauss-Lobatto points, the corresponding weights are

$$w_j = \frac{2}{(1-x_j^2)[L'_{N+1}(x_j)]^2}, 0 \le j \le N.$$

The example (5.1) is solved by Homotopy perturbation method in [18] and a comparison between HPM and Legendre-spectral method is obtained.

Example 5.1. *Consider the following fuzzy integral equation:*

$$u(t) + \int_0^t sinht.u(s)ds = [(cosht + 1 - cosh^2t)(r^2 + r), (cosht + 1 - cosh^2t)(4 - r^3 - r)]$$

The exact solution is $u(t) = [cosht(r^2+r), cosht(4-r^3-r)]$. *Spectral scheme is used for it. The absolute errors for* $\underline{u}(t)$ *and* $\overline{u}(t)$ *are shown in Table (1.5) and figure (1.5). These results indicate that the desired spectral accuracy is obtained.*

r	$\lvert Error(\underline{u})\rvert, N = 4$	$\lvert Error(\underline{u})\rvert, N = 8$	$\lvert Error(\underline{u})\rvert, N = 12$	$\lvert Error(\underline{u})\rvert, N = 20$	$Error(\underline{u})(HPM)$	$\lvert Error(\overline{u})\rvert, N = 4$	$\lvert Error(\overline{u})\rvert, N = 8$	$\lvert Error(\overline{u})\rvert, N = 12$	$\lvert Error(\overline{u})\rvert, N = 20$	$Error(\overline{u})(HPM)$
0	0	0	0	5.233E(-11)	0	3.2E(-3)	1.3E(-5)	3.199E(-7)	3.128E(-11)	0.00016
0.1	2E(-3)	3.903E(-5)	4.302E(-7)	4.809E(-11)	0.000005	2.9E(-3)	3.323E(-5)	4.1E(-7)	2.158E(-11)	0.00016
0.2	2.1E(-2)	3.032E(-5)	1.981E(-7)	4.013E(-11)	0.000010	2.5E(-3)	3.763E(-5)	4.3E(-7)	2.299E(-11)	0.00016
0.3	2.2E(-2)	3.103E(-5)	4.433E(-7)	3.857E(-11)	0.000016	3.5E(-3)	4.332E(-5)	3.3E(-7)	3.802E(-11)	0.00015
0.4	2.3E(-2)	3.325E(-5)	1.176E(-7)	3.409E(-11)	0.000024	4.2E(-3)	3.913E(-5)	1.9E(-7)	4.192E(-11)	0.00015
0.5	1.3E(-2)	2.333E(-5)	2.23E(-7)	3.109E(-12)	0.000031	3.5E(-3)	3.673E(-5)	3.1E(-7)	3.918E(-11)	0.00014
0.6	2.45E(-2)	3.311E(-5)	3.0454E(-7)	2.347E(-12)	0.000040	4.1E(-3)	3.763E(-5)	4.2E(-7)	4.681E(-11)	0.00013
0.7	2.5E(-2)	4.223E(-5)	5.423E(-7)	2.653E(-12)	0.000050	3.1E(-3)	3.671E(-5)	3.4E(-7)	3.018E(-11)	0.00012
0.8	2.65E(-2)	3.235E(-5)	8.423E(-7)	2.143E(-12)	0.000060	3.3E(-3)	3.319E(-5)	2.3E(-7)	3.100E(-11)	0.00011
0.9	2.7E(-2)	3.2E(-5)	1.134E(-7)	1.834E(-12)	0.000070	2.4E(-3)	3.304E(-5)	3.3E(-7)	2.018E(-11)	0.0001
1	1.9E(-2)	2.9E(-5)	1.681E(-8)	1.634E(-12)	0.000080	1.7E(-3)	3.103E(-5)	1.75E(-7)	4.008E(-11)	0.00008

Table 5.1: Error results for $x = 0.2$ and $N = 4, 8, 12$ and $N = 20$ for $\underline{u}$ and $\overline{u}$ in Example (5.1)

r	$\lvert Error(\underline{u})\rvert, N = 4$	$\lvert Error(\underline{u})\rvert, N = 8$	$\lvert Error(\underline{u})\rvert, N = 12$	$\lvert Error(\underline{u})\rvert, N = 20$	$\lvert Error(\overline{u})\rvert, N = 4$	$\lvert Error(\overline{u})\rvert, N = 8$	$\lvert Error(\overline{u})\rvert, N = 12$	$\lvert Error(\overline{u})\rvert, N = 20$
0	2.245E(-3)	2.375E(-6)	0	5.3E(-11)	3.232E(-3)	1.3E(-6)	4.001E(-8)	3.438E(-14)
0.1	3.278E(-3)	1.163E(-6)	4.334E(-7)	4.812E(-13)	1.239E(-3)	5.345E(-6)	4.102E(-8)	2.348E(-14)
0.2	2.756E(-2)	1.263E(-6)	3.198E(-7)	4.543E(-14)	2.335E(-3)	5.123E(-6)	4.033E(-8)	2.945E(-14)
0.3	2.275E(-3)	1.327E(-6)	4.543E(-7)	3.876E(-14)	1.543E(-3)	5.373E(-6)	3.343E(-8)	3.128E(-14)
0.4	2.187E(-2)	1.735E(-6)	1.213E(-8)	3.344E(-14)	4.232E(-3)	5.233E(-6)	1.449E(-8)	4.218E(-14)
0.5	3.983E(-2)	1.318E(-6)	2.543E(-7)	3.1981E(-13)	3.523E(-3)	6.343E(-6)	3.761E(-8)	3.832E(-14)
0.6	3.345E(-2)	1.316E(-6)	3.543E(-7)	2.743E(-14)	4.221E(-3)	6.343E(-6)	4.262E(-8)	4.346E(-14)
0.7	3.185E(-3)	1.443E(-6)	5.129E(-7)	2.343E(-14)	3.271E(-3)	5.334E(-5)	3.465E(-8)	3.898E(-14)
0.8	2.551E(-3)	1.322E(-6)	8.324E(-7)	2.119E(-14)	3.193E(-3)	5.713E(-6)	2.334E(-8)	3.343E(-14)
0.9	2.347E(-2)	1.543E(-6)	2.154E(-7)	1.833E(-14)	2.443E(-3)	5.633E(-6)	3.233E(-8)	2.878E(-14)
1	3.342E(-3)	1.319E(-6)	1.643E(-8)	1.622E(-13)	1.287E(-3)	5.233E(-6)	1.751E(-8)	1.867E(-14)

Table 5.2: Error results for $x = 0.2$ and $N = 4, 8, 12$ and $N = 20$ for $\underline{u}$ and $\overline{u}$ in Example (5.2)

Example 5.2. *Consider the following fuzzy two-dimensional volterra integral equation:*

$$u(x) - \int_0^x 3xs^2 u(s)ds = [(x - x^2 s^3)(r-1), (x - x^2 s^3)(1-r)]$$

The exact solution is $u(x, y) = [x(r-1), x(1-r)]$. Spectral scheme is used for it. The absolute errors for $\underline{u}(t)$ and $\overline{u}(t)$ are shown in Table (5.2). These results indicate that the desired spectral accuracy is obtained.

6 Conclusion

In this work, one and two dimensional nonlinear fuzzy Volterra integral equations of second kind were studied. The Legendre-spectral scheme was successfully employed for solving them. This numerical method was based on the Legendre points and using Lagrange interpolation polynomials. The obtained results by this method was illustrated very near to exact solutions. Obtained data evince that the convergence rate is very fast, and lower approximations can accede high accuracy. A comparison between spectral method and Homotopy perturbation method is obtained in a example that is shown that the spectral method is more accurate. The computations in this paper were performed by using Maple 18.

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