Increasing the Discrimination Power in a Voting System

Abdolah Hadi-Vencheh

Department of Mathematics, Islamic Azad University, Khorasgan Branch, Isfahan, Iran

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ABSTRACT

Ranking DMUs based on individual preferences is an interesting and useful part of decision-making problems. Comparing the weighted sum of the selected number of rank votes, after determining the weights in a selected rank, can be regarded as a common approach to compute the total ranking of alternatives. In actual applications, making the weight of a certain rank zero means that we throw away the corresponding part of the obtained rank voting data. This paper proposes a new model to assess the non-zero weights for each position.

1 Introduction

In preferential voting systems, each voter selects m candidates from among n candidates $(n \geq m)$ and ranks them from the most to the least preferred. Each candidate may receive some votes in different ranking places. The total score of each candidate is the weighted sum of the votes he/she receives in different places. The winner is the one with the biggest total score. So, the key issue of the preference aggregation in a preferential voting system is how to determine the weights associated with different ranking places [6-13,16]. Borda's method is perhaps the most widely used procedure for determining the weights. By the Borda's method, the first place is given a weight or mark of m, the second place is given a weight or mark of $m-1$, followed by $m-2, ..., 2$ and the last place is given a weight or mark of one. Because of its computational simplicity, the Borda's method is very popular. But the determination of the weights is somewhat subjective. It is worth noting that the Borda rule has interesting properties in relation to other scoring rules. According to Brams and Fishburn [1]: "Among ranked positional scoring procedures to elect one candidate, Borda's method is superior in many respects, including susceptibility to strategic manipulation, propensity to elect Condorcet candidates, and ability to minimize paradoxical possibilities".

To avoid the subjectivity in determining the weights, Cook and Kress [2] suggest using data envelopment analysis (DEA) to determine the most favourable weights for each candidate. Different candidates utilize different sets of weights to calculate their total scores, which are referred to as the best relative total scores and are all restricted to be less than or equal to one. The candidate with the biggest relative total score of one is said to be DEA efficient and may be considered as a winner. This approach proves to be effective, but very often leads to more than one candidate to be DEA efficient. To choose a winner from among the DEA-efficient candidates, Cook and Kress [2] suggest maximizing the gap between the weights so that only one candidate is left DEA efficient. This has been found equivalent to imposing a common set of weights on all the candidates and equivalent to the Borda's method in a specific discrimination intensity function.

* Corresponding Author's E-mail: ahadi@khuisf.ac.ir

Green et al. [3] suggest using the cross-efficiency evaluation technique in DEA to choose the winner. Noguchi et al. [14] also utilize cross-efficiency evaluation technique to select the winner, but present a strong ordering constraint condition on the weights. Hashimoto [6] proposes the use of the DEA exclusion model (i.e. super-efficiency model) to identify the winner. Obata and Ishii [8] suggest excluding non-DEA-efficient candidates and using normalized weights to discriminate the DEA-efficient candidates. Their method is subsequently extended to rank non-DEA-efficient candidates by Foroughi and Tamiz [4] (see also Foroughi et al [5]). More recently Wang et al [10] proposed three models for preference voting and aggregation. Two of them are linear programming models which determine a common set of weights for all the candidates considered and the other is a nonlinear programming model that determines the most favourable weights for each candidate. But, Wang et al. [10] have not taken care of about making the weight of a certain rank zero means that we throw away the corresponding part of the obtained data. In actual applications, making the weight of a certain rank zero means that we throw away the corresponding part of the obtained rank voting data. Their incorrect model also used in [9]. To avoid possible more misapplications or spread in the future, we present in this paper an improved DEA-model to determine the weights of ranking places. The proposed model is simple and can lead to a stable full ranking for all the candidates considered. This will be illustrated with two numerical examples.

The rest of the paper is organized as follows. In the next section, we develop the model for preference aggregation to assess the weights associated with different ranking places. We then examine two numerical examples using the proposed model to illustrate its applications and show their capabilities of identifying the winner and producing a stable full ranking for all the candidates considered. Finally, we conclude the paper.

2 Model

Let w_j be the relative importance weight attached to the jth ranking place $(j = 1, \cdots, m)$ and v_{ij} be the vote of candidate i being ranked in the jth place. The total score of each candidate is defined as

$$Z_i = \sum_{j=1}^{m} v_{ij} w_j, \quad i = 1, 2, ..., n \tag{2.1}$$

which is a linear function of the relative importance weights. Once the weights are given or determined, candidates can be ranked in terms of their total scores.

To determine the score of each candidate, Wang et al [10] suggest the following DEA model, which maximize the minimum of the total scores of the n candidates and determine a common set of weights for all the candidates:

$$\begin{aligned} & max \;\; \alpha \\ s.t. \quad & \alpha \le Z_i = \sum_{j=1}^{m} v_{ij} w_j \le 1, \quad i = 1, \cdots, n \\ & w_1 \ge 2w_2 \ge \cdots \ge m w_m \ge 0, \end{aligned} \tag{2.2}$$

where $w_1 \ge 2w_2 \ge \cdots \ge m w_m \ge 0$ is the strong ordering constraint on decision variables.

But, Wang et al [10] have not taken care of about making the weight of a certain rank zero. In actual applications, making the weight of a certain rank zero means that we throw away the corresponding part of the obtained rank voting data. Here, using an example we show this assertion. Consider the example in which 20 voters are asked to rank two out of four candidates A-D on a ballot. The votes each candidate receives are shown in Table 1.

Table 1. Votes received by four candidates.

Candidate	First place	Second place
A	7	7
B	7	8
C	3	3
D	3	2

If the model (2.2) is employed to solve the example then we get $\alpha^* = 0.4286,\ w_1^* = 0.1429$ and $w_2 = 0.0000$ (this is the unique solution of the model). As we see the weight of second place is zero, which means that the second place vote does not have any meaning. In actual applications, making the weight of a certain place vote zero means that we throw away the corresponding part of the obtained data.

In what follows, we present our models, which avoid producing a zero weight for a certain place vote and make full use of all the data. Consider the following model

$$\begin{aligned} & max\ \ \alpha \\ s.t.\ \ & \alpha \le Z_i = \sum_{j=1}^{m} v_{ij} w_j \le 1, \quad i = 1, ..., n \\ & w_1 \ge 2w_2 \ge \cdots \ge m w_m, \\ & w_m \ge \varepsilon \end{aligned} \tag{2.3}$$

As a theoretical construct, ε provides a lower bound for scoring of grades to keep them away from zero. Hence, the following LP is proposed to determine the ε.

$$\begin{aligned} & \varepsilon^* = max\ \ \varepsilon \\ s.t.\ \ & \sum_{j=1}^{m} v_{ij} w_j \le 1, \quad i = 1, ..., n \\ & w_1 \ge 2w_2 \ge \cdots \ge m w_m, \\ & w_m - \varepsilon \ge 0 \end{aligned} \tag{2.4}$$

It is clear that $\varepsilon = 0, \forall j : w_j = 0$ is a feasible solution to the model (2.4). The optimal value of model (2.4) is greater than zero, that is $\varepsilon^* > 0$.

Proof. The dual of model (2.4) is as follows:

$$\begin{aligned} & min\ \ \sum_{i=1}^{n} \theta_i \\ s.t.\ \ & \sum_{i=1}^{n} v_{i1} \theta_i - \delta_1 = 0 \\ & \sum_{i=1}^{n} v_{ij} \theta_i + j\delta_{j-1} - j\delta_j = 0, \quad j = 2, ..., m-1 \\ & \sum_{i=1}^{n} v_{im} \theta_i + m\delta_{m-1} - \delta_m = 0 \\ & \delta_m = 1 \\ & \theta_i, \delta_j \ge 0,\ i = 1, ..., n;\ j = 1, ..., m \end{aligned} \tag{2.5}$$

By contradiction assume that $\varepsilon^* = 0$. Hence, $\theta^* = \sum_{i=1}^{n} \theta_i^* = 0$. Therefore according to the constraints of model (2.5), for all $j = 1, ..., m$, we have $\delta_j = 0$ which contradicts to the last constraint of Model (2.5), So $\varepsilon^* = \theta^* > 0$. □

$$\varepsilon^* \leq \min_{1 \leq i \leq n} \left\{ \frac{1}{m \sum_{j=1}^{m} \frac{v_{ij}}{j}} \right\}.$$

Proof. From $w_1 \geq 2w_2 \geq \cdots \geq mw_m$, we have $w_j \geq \frac{mw_m}{j}$. Thus

$$Z_i = \sum_{j=1}^{m} v_{ij} w_j \geq \sum_{j=1}^{m} \frac{mw_m}{j} v_{ij} = mw_m \sum_{j=1}^{m} \frac{v_{ij}}{j}$$

But according to the constraints of model (2.3) for each $i = 1, ..., n$ we have $Zi \leq 1$. Therefore $mw_m \sum_{j=1}^{m} \frac{v_{ij}}{j} \leq 1$, or $w_m \leq \frac{1}{m \sum_{j=1}^{m} \frac{v_{ij}}{j}}$, $i = 1, ..., n$. On the other hand according to the last constraint of model (2.3), $w_m \geq \varepsilon$. Hence

$\varepsilon \leq \frac{1}{m \sum_{j=1}^{m} \frac{v_{ij}}{j}}$, $i = 1, ..., n$. So, $\varepsilon^* \leq \min_{1 \leq i \leq n} \left\{ \frac{1}{m \sum_{j=1}^{m} \frac{v_{ij}}{j}} \right\}$. □

The optimal value of model (2.4) is greater than zero and bounded.

Proof. The proof is clear using the above lemmas. □

The model (2.4) and the following model are equivalent:

$$\begin{aligned} & max \ \ w_m \\ s.t. \ \ & \sum_{j=1}^{m} v_{ij} w_j \leq 1, \quad i = 1, ..., n \\ & w_1 \geq 2w_2 \geq \cdots \geq mw_m \geq 0 \end{aligned} \tag{2.6}$$

Proof. From the last constraint of model (2.5) we have $\delta_m = 1$. Hence we can write the model (2.5) as follows

$$
\begin{aligned}
& min \quad \sum_{i=1}^{n} \theta_i \\
s.t. \quad & \sum_{i=1}^{n} v_{i1}\theta_i - \delta_1 = 0 \\
& \sum_{i=1}^{n} v_{ij}\theta_i + j\delta_{j-1} - j\delta_j = 0, \quad j = 2, ..., m-1 \\
& \sum_{i=1}^{n} v_{im}\theta_i + m\delta_{m-1} = 1 \\
& \theta_i, \delta_j \geq 0, \; i = 1, ..., n; \; j = 1, ..., m-1
\end{aligned}
\tag{2.7}
$$

Now the dual of model (2.7) is as follow

$$
\begin{aligned}
& max \;\; w_m \\
s.t. \quad & \sum_{j=1}^{m} v_{ij} w_j \leq 1, \quad i = 1, ..., n \\
& w_1 \geq 2w_2 \geq \cdots \geq m w_m \geq 0
\end{aligned}
\tag{2.8}
$$

But we now that the dual of the dual is primal, thus the above model is the same as model (2.4). □

By solving model (2.4) for data of Table 1 we have $\varepsilon^* = 0.04545$. If this ε^* is employed to solve the model (2.3) for example 1 we get $\alpha^* = 0.36364$, $w_1^* = 0.9091$ and $w_2^* = 0.04545$. Hence the ranking of the four candidates is as: $B \succ A \succ C \succ D$. So, candidate B is the winner.

3 Numerical examples

In this section, we examine two numerical examples using the proposed model to illustrate their applications and show its capabilities of choosing the winner and ranking candidates.
Consider the example investigated by Cook and Kress [2] and Wang et al [10] in which 20 voters are asked to rank four out of six candidates A-F on a ballot. The votes each candidate receives are shown in Table 2.

Table 2. Votes received by six candidates.

Candidate	First place	Second place	Third place	Fourth place
A	3	3	4	3
B	4	5	5	2
C	6	2	3	2
D	6	2	2	6
E	0	4	3	4
F	1	4	3	3

By solving (2.4), we get $\varepsilon^* = 0.02727$. Now the model (2.3) yields $\alpha^* = 0.43636$, $w_1^* = 0.10909$, $w_2^* = 0.05455$, $w_3^* = 0.03636$ and $w_4^* = 0.02727$. Solving the model (2.8), we have: $\alpha^* = 1.0000$, $w_1^* = 0.4800$, $w_2^* = 0.2400$, $w_3^* = 0.1600$ and $w_4^* = 0.1200$. The rankings of the six candidates produced by our model is shown in Table

3, from which it is clear that the model leads to the ranking: $D \succ B \succ C \succ A \succ F \succ E$. So, candidate D is the winner.

Table 3. Scores and rankings of the six candidates by proposed model.

Candidate	Score	Rank
A	0.71818	4
B	0.94546	2
C	0.92727	3
D	1.00000	1
E	0.43636	6
F	0.51818	5

Consider the example investigated by Obata and Ishii [14] and Foroughi and Tamiz [4], in which seven candidates A-G are ranked. Table 4 shows the votes each candidate receives in the first two places.

Table 4. Votes received by seven candidates.

Candidate	First place	Second place
A	32	10
B	28	20
C	13	36
D	20	27
E	27	19
F	30	8
G	0	30

Using the model (2.4), we have $\varepsilon^* = 0.01316$. Now by solving model (2.3) we get $\alpha^* = 0.39474,\ w_1^* = 0.02632$ and $w_2^* = 0.01316$. Table 5 shows the ranking of each candidate obtained by our model. As can be seen from Table 5, our model leads to the ranking: $B \succ A \succ E \succ F \succ D \succ C \succ G$. So, candidate B is the winner.

Table 5. Scores and rankings of the seven candidates by our model.

Candidate	Score	Rank
A	0.97369	2
B	1.00000	1
C	0.81579	6
D	0.88158	5
E	0.96053	3
F	0.89474	4
G	0.39474	7

4 Conclusion

We discussed applicability of the ranking method proposed by Wang et al. [10], and by using DEA, we determine the weights from rank voting data. Their model, gives rise to the case such that the data of some rank is ignored. Thus, we analyze the procedure to determine weights, and proposed an extended model for preference voting and aggregation. The contribution of this paper is to maintains the effects of all data in the final solution, an improvement over the model proposed by Wang et al. [10].

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