Application of semi-analytic method to compute the moments for solution of logistic model

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Abstract

The population growth, is increase in the number of individuals in population and it depends on some random environment effects. There are several different mathematical models for population growth. These models are suitable tool to predict future population growth. One of these models is logistic model. In this paper, by using Feynman-Kac formula, the Adomian decomposition method is applied to compute the moments for the solution of logistic stochastic differential equation.

1 Introduction

The population growth is the increase in number of individual over time. Let assume that a population contains $N(t)$ individuals at time t. In deterministic exponential growth model [2] (suggested by Thomas Robert Malthus (1766-1834)) it is supposed that the number of births and deaths are fractions of total population. Let the number of births and deaths in time interval $[t, t+1]$ are α and β respectively. Therefore, we can write the increase in population for time interval $[t, t+1]$ in terms of $N(t)$, α, and β as follows

$$\begin{aligned} N(t+1) - N(t) &= \alpha N(t) - \beta N(t) \\ &= N(t)(\alpha - \beta). \end{aligned} \tag{1.1}$$

Now by using given initial condition $N(0) = N_0$, we get

$$N(t) = N_0(1 + \alpha - \beta)^t = N_0 R^t, \quad R = 1 + \alpha - \beta. \tag{1.2}$$

Above relation can be written in differential equation as

$$\begin{cases} \dfrac{dN(t)}{dt} = rN(t), \quad r = \ln R, \\ \\ N(0) = N_0, \end{cases} \tag{1.3}$$

with exact solution

$$N(t) = N_0 e^{rt}. \tag{1.4}$$

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In deterministic logistic model (suggested by Pierre Francois Verhust in nineteen century (1804-1849)), the population will not increase forever (for example because of insufficient resourses) and has environment limiting N (i.e. $\lim_{t\to\infty} N(t) = N$) and given by

$$\frac{dN(t)}{dt} = rN(t)\left(1 - \frac{N(t)}{N}\right). \tag{1.5}$$

In this model the rate of growth decrease as the limiting population (i.e. N) is approached and for $N(t) << N$, deterministic logistic model is similar to exponential growth model. The model can be solved by separation of variables as

$$N(t) = \frac{N}{1 + \left(\frac{N-N_0}{N_0}\right)e^{-kt}}. \tag{1.6}$$

There are a great number of different deterministic population model in the literature (for instance, Von Bertalanffy growth model, Richard growth model, Blumberg growth model, Gompertz growth model, Generic growth model, generalized logistic model). In general, population growth models have the following form

$$\frac{dN(t)}{dt} = N(t)F(t, N(t)). \tag{1.7}$$

Since the growth rate at time t is not exactly definite, aforesaid models can be improved by considering some random environment effects [3]. Therefore population models can be improved by considering both deterministic and stochastic terms as

$$dN(t) = f(t, N(t))dt + g(t, N(t))\frac{dW(t)}{dt}dt, \tag{1.8}$$

where $dW(t)$ is the differential of Brownian motion and $\frac{dW(t)}{dt}$ called withe noise.

Feynman-Kac formula named after Richard Feynman and Mark Kac, expresses a close connection between the expectations for solutions of SDEs and partial differential equations (PDEs) [4,5]. Let $\{X(t)\}_{t\geqslant 0}$ be a solution of the SDE

$$dX(t) = a(t, X(t))dt + \sigma(t, X(t))dW(t). \tag{1.9}$$

Assume that f and ρ be given functions. Fix a final time $T > 0$ and define a new function $V(t,x)$ for $t \in [0,T]$ by

$$V(t,x) = e^{-\int_t^T \rho(u)du}\mathbb{E}\left[f(X(T)|X(t) = x\right]. \tag{1.10}$$

Assume that $V(t,x) < \infty$ for all (t,x). Then $V = V(t,x)$ solves the following boundary value problem

$$\begin{cases} \dfrac{\partial V(t,x)}{\partial t} + \dfrac{\sigma^2(t,x)\partial^2 V(t,x)}{2\quad\partial x^2} + a(t,x)\dfrac{\partial V(t,x)}{\partial x} = \rho(t)V, \\ V(T,x) = f(x). \end{cases} \tag{1.11}$$

Consequently, by solving the boundary value problem (11), the expectations for solutions of SDEs can be easily computed.

The paper is organized as follows. In section 2, some preliminary in stochastic calculus are reviewed. In section 3, for convenience of the reader, a short review of the ADM is presented. In section 4, the moments for solution of the logistic stochastic differential equation are obtained. In section 5, two examples are presented.

The convergence of the proposed method in consider for examples. Finally, in section 5, a short conclusion is expressed.

2 Preliminaries to stochastic calculus and stochastic logistic model

In this section, some preliminaries to stochastic calculus are presented. For more details see [4,5] and references therein.

Definition 2.1. *The normal distribution* $N(\mu, \sigma^2)$*, is the probability distribution defined by the following density function*

$$g(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right), \quad -\infty < x < \infty.$$

Remark 1 If x is $N(\mu, \sigma^2)$ then $\mathbf{E}[x] = \mu$ and $Var(x) = \sigma^2$.

Definition 2.2. *The lognormal distribution* $LN(\mu, \sigma^2)$ *is the distribution of* $y = e^x$*, where* x *is* $N(\mu, \sigma^2)$*. The probability distribution function of* y *is given by*

$$f(y; \mu, \sigma) = \frac{1}{y\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right).$$

Remark 2 If y is $LN(\mu, \sigma^2)$ then $\mathbf{E}[y] = e^{\mu + \frac{1}{2}\sigma^2}$ and $Var[y] = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right)$.

Definition 2.3. *Brownian motion is a stochastic process* $\{W(t) | t \in [0, \infty]\}$ *with the following properties:*
1. $W(0) = 0$.
2. It has a continuous path.
3. For all non-overlapping time intervals $[t_1, t_2]$*, and* $[t_3, t_4]$ *the random variables* $W(t_2) - W(t_1)$ *and* $W(t_4) - W(t_3)$ *are independent (i.e.* $W(t_2) - W(t_1) \perp W(t_4) - W(t_3)$*).*
4. The increment $W(t_2) - W(t_1)$ *is a normal variable, with zero mean and variance* $t_2 - t_1$ *(i.e.* $W(t_2) - W(t_1) \sim N(0, t_2 - t_1)$*).*

Theorem 2.1. *Let* $X(t)$ *be an Itô process given by*

$$dX(t) = udt + vdW(t). \tag{2.1}$$

Let $g(t, x) \in C^2([0, \infty) \times \mathbb{R})$*. Then* $Y(t) = g(t, X(t))$ *is again an Itô process, and*

$$dY(t) = \frac{\partial g(t, X(t))}{\partial t} dt + \frac{\partial g(t, X(t))}{\partial x} dX(t) + \frac{1}{2}\frac{\partial^2 g(t, X(t))}{\partial x^2} (dX(t))^2, \tag{2.2}$$

where $(dX(t))^2 = (dX(t)).(dX(t))$ *is computed according to the rules*
$dt.dt = dt.dW(t) = dW(t).dt = 0$*, and* $dW(t).dW(t) = dt$*.* ***Proof:*** *see [4,5].*

Remark 3 By using theorem 1, for $g(t, x) = \ln x$, we get

$$d\ln N(t) = \frac{1}{N(t)} dN(t) + \frac{1}{2}\frac{-1}{N^2(t)} (dN(t))^2. \tag{2.3}$$

On the other hand, using properties $dt.dW(t) = dW(t).dt = dt.dt = 0$, and $dW(t).dW(t) = dt$ then $(dN(t))^2$ can be written as

$$(dN(t))^2 = \beta^2 dt. \tag{2.4}$$

Now (14) becomes

$$d\ln N(t) = \frac{1}{N(t)}dN(t) - \frac{1}{2}\beta^2 dt. \tag{2.5}$$

Consequently, integrating both sides of (16) on $[0, t]$ gives us

$$\int_0^t \frac{dN(s)}{N(s)} = \ln N(t) - \ln N_0 + \frac{\beta^2}{2}t. \tag{2.6}$$

Definition 2.4. *The logistic model can be improved by considering some environment effects or a noise as*

$$dN(t) = rN(t)(K - N(t))dt + \beta N(t)dW(t). \tag{2.7}$$

Theorem 2.2. *Let*

$$\begin{cases} dN(t) = rN(t)(K - N(t))dt + \beta N(t)dW(t), \\ N(0) = N_0. \end{cases} \tag{2.8}$$

Then the exact solution to equation (19) is given by

$$N(t) = \left(\frac{1}{r}\right) \frac{N_0 \exp\left(\left(K - \frac{1}{2}\beta^2\right)t + \beta W(t)\right)}{1 + N_0 \int_0^t \exp\left(\left(K - \frac{1}{2}\beta^2\right)s + \beta W(s)\right)ds}. \tag{2.9}$$

Proof: *The equation (19) changes to the following form*

$$\frac{dN(t)}{N(t)} = r(K - N(t))dt + \beta dW(t). \tag{2.10}$$

Integrating both sides of (21) on $[0, t]$ *and using (17) gives us*

$$\begin{aligned} \ln\left(\frac{N(t)}{N_0}\right) + \frac{\beta^2}{2}t &= -r\int_0^t N(s)ds + \beta W(t) + rKt, \\ \ln\left(\frac{N(t)}{N_0}\right) + r\int_0^t N(s)ds &= \beta W(t) + rKt - \frac{\beta^2}{2}t, \\ \ln\left(\frac{N(t)}{N_0}\right) + r\int_0^t N(s)ds &= \beta W(t) + \left(rK - \frac{\beta^2}{2}\right)t, \\ \frac{N(t)}{N_0}\exp\left(r\int_0^t N(s)ds\right) &= \exp\left(\beta W(t) + \left(rK - \frac{\beta^2}{2}\right)t\right). \end{aligned} \tag{2.11}$$

Now integrating both sides of (22) on $[0,t]$ result in

$$\begin{aligned}
\int_0^t N(s)\exp\left(r\int_0^s N(u)du\right)ds &= N_0\int_0^t \exp\left(\beta W(s)+\left(rK-\frac{\beta^2}{2}\right)s\right)ds,\\
\int_0^t \exp\left(r\int_0^s N(u)du\right)d\left(\int_0^s N(u)du\right) &= N_0\int_0^t \exp\left(\beta W(s)+\left(rK-\frac{\beta^2}{2}\right)s\right)ds,\\
\frac{1}{r}\exp\left(r\int_0^s N(u)du\right)\Big|_0^t &= N_0\int_0^t \exp\left(\beta W(s)+\left(rK-\frac{\beta^2}{2}\right)s\right)ds,\\
\frac{1}{r}\left(\exp\left(r\int_0^t N(u)du\right)-1\right) &= N_0\int_0^t \exp\left(\beta W(s)+\left(rK-\frac{\beta^2}{2}\right)s\right)ds.
\end{aligned} \tag{2.12}$$

Therefore, we get

$$\begin{aligned}
\exp\left(r\int_0^t N(u)du\right) &= 1+rN_0\int_0^t \exp\left(\beta W(s)+\left(rK-\frac{\beta^2}{2}\right)s\right)ds,\\
\int_0^t N(u)du &= \frac{1}{r}\ln\left(1+rN_0\int_0^t \exp\left(\beta W(s)+\left(rK-\frac{\beta^2}{2}\right)s\right)ds\right).
\end{aligned} \tag{2.13}$$

Finally, by differentiating (24), (20) is obtained.

3 The Adomian Decomposition Method (ADM)

Consider the following differential equation

$$Lu+Ru+Nu=g, \tag{3.1}$$

where L is the highest order derivative which assumed to be easily invertible, R is a linear differential operator of less order than L, Nu represents the nonlinear terms, and g is source term. Applying L^{-1} to both side of the relation (25) and using initial conditions results in

$$u=f-L^{-1}(Ru)-L^{-1}(Nu), \tag{3.2}$$

where the function f represents the terms arising from integrating the source term g and from using the given conditions. Let $u=\sum_{n=0}^{\infty}u_n$. In this method, the components $u_0,u_1,u_2,\cdots$ are determined recursively as follows [1,6-14]

$$\begin{cases} u_0=f,\\ u_k=-L^{-1}(Ru_{k-1})-L^{-1}(Nu_{k-1}),\ k\in\mathbb{N}. \end{cases} \tag{3.3}$$

In the next section, by using the Feynman-Kac formula, the ADM is applied to obtained an explicit formula for the moments of the square-root diffusion process.

4 Main results

By using Feynman-Kac formula, to compute the n-moment for the solution of logistic stochstic differential equation, the following PDE is achieved

$$\begin{cases} \frac{\partial V(t,x)}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V(t,x)}{\partial x^2} + x(\alpha - \beta x)\frac{\partial V(t,x)}{\partial x} = 0, \\ V(T,x) = f(x) = x^n. \end{cases} \tag{4.1}$$

Let $L = \frac{\partial}{\partial t}$, $R = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + x(\alpha - \beta x)\frac{\partial}{\partial x}$, and $N = g = 0$. Integrating both sides of the relation (28) on $[t,T]$ results in

$$\int_t^T \frac{\partial V(s,x)}{\partial s} ds + \frac{1}{2}\sigma^2 x^2 \int_t^T \frac{\partial^2 V(s,x)}{\partial x^2} ds + x(\alpha - \beta x) \int_t^T \frac{\partial V(s,x)}{\partial x} ds = 0. \tag{4.2}$$

Therefore, by using initial condition $V(T,x) = f(x) = x^n$, the relation (29) simplifies as

$$V(t,x) = x^n + \frac{1}{2}\sigma^2 x^2 \int_t^T \frac{\partial^2 V(s,x)}{\partial x^2} ds + x(\alpha - \beta x) \int_t^T \frac{\partial V(s,x)}{\partial x} ds. \tag{4.3}$$

Let $V(t,x) = \sum_{n=0}^{\infty} V_n(t,x)$. According to the relation (27), the components $V_0(t,x), V_1(t,x)$, $V_2(t,x), \cdots$ are determined as follows

$$\begin{cases} V_0(t,x) = x^n, \\ V_k(t,x) = \frac{1}{2}\sigma^2 x^2 \int_t^T \frac{\partial^2 V_{k-1}(s,x)}{\partial x^2} ds + x(\alpha - \beta x) \int_t^T \frac{\partial V_{k-1}(s,x)}{\partial x} ds, \ k \in \mathbb{N}. \end{cases} \tag{4.4}$$

Subsequently, the n-term approximate can be used to approximate the solution.

The following program, written by Maple, generates all of the components , $V_1(t,x)$, $V_2(t,x), \cdots$ in the relation (31) for any given n, a, b, c, t and T. In this program, it is supposed that $alpha = 1$, $beta = 1$, $sigma = 1$, $T = 1$, $t = 0$, $n = 15$, and $V_0(t,x) = x^2$ which can easily changed by the user.

```
alpha:=1;
beta:=1;
sigma:=1;
T:=1;
t:=1;
n:=15;
v[0]:=x**2;
v[0]:=unapply(v[0],x);
for i from 1 to n do
v[i]:=simplify(sigma**2*x**2*diff(v[i-1](x),x,x)*((-1)**i*(T-t)**(i-1)/(i-1)!)
+(alpha-beta*x)*diff(v[i-1](x),x)*((-1)**i*(T-t)**(i-1)/(i-1)!));
v[i]:=unapply(v[i],x);
od;
approximate:=add(v[i](x),i=0..n);
```

5 Numerical examples

Example 5.1. *As a first example, consider the following stochastic differential equation*

$$\begin{cases} dN(t) = N(t)(1-N(t))dt + N(t)dW(t), \\ X(0) = x \geqslant 0 \end{cases} \tag{5.1}$$

According to the relation (28), to compute the second moment of $N(t)$, the following partial differential equation is considered

$$\begin{cases} \frac{\partial V(t,x)}{\partial t} + x\frac{\partial^2 V(t,x)}{\partial x^2} + (1-x)\frac{\partial V(t,x)}{\partial x} = 0, \\ V(T,x) = x^2 \end{cases} \tag{5.2}$$

Let $T = 1$. Thus, the first few components, calculated by the formula (31), are as follows

$$\begin{cases} V_0 = x^2, \\ V_1 = 2x^3 - 4x^2, \\ V_2 = -6x^4 + 26x^3 - 16x^2, \\ V_3 = -12x^5 + 87x^4 - 133x^3 + 32x^2, \\ V_4 = 10x^6 - 108x^5 + \frac{597}{2}x^4 - \frac{1261}{6}x^3 + \frac{64}{3}x^2, \\ V_5 = \frac{5}{2}x^7 - \frac{75}{2}x^6 + \frac{649}{4}x^5 - \frac{10813}{48}x^4 + \frac{11605}{144}x^3 - \frac{32}{9}x^2, \\ V_6 = \frac{-7}{48}x^8 + \frac{139}{48}x^7 - \frac{1729}{96}x^6 + \frac{1859}{45}x^5 - \frac{184613}{5760}x^4 + \frac{105469}{17280}x^3 - \frac{16}{135}x^2, \\ V_7 = \frac{-7}{4320}x^9 + \frac{1421}{34560}x^8 - \frac{5999}{17280}x^7 + \frac{12311}{10368}x^6 - \frac{61919}{38400}x^5 + \frac{1019759}{1382400}x^4 - \\ \frac{953317}{12441600}x^3 + \frac{4}{6075}x^2, \\ \vdots \end{cases} \tag{5.3}$$

Finally, the series have been obtained is convergent because the ratio of $||V_i||_\infty$ to $||V_{i-1}||_\infty$ for $i = 1, 2, 3, \cdots$ decrease to zero [14]. Below, these ratios for the first few are expressed

$$\begin{cases} \frac{||V_1||_\infty}{||V_0||_\infty} = 4.0, \\ \frac{||V_2||_\infty}{||V_1||_\infty} = 6.5, \\ \frac{||V_3||_\infty}{||V_2||_\infty} = 5.115384615, \\ \frac{||V_4||_\infty}{||V_3||_\infty} = 2.244360902, \\ \frac{||V_5||_\infty}{||V_4||_\infty} = 0.7546761586, \\ \frac{||V_6||_\infty}{||V_5||_\infty} = 0.1833841980, \\ \frac{||V_7||_\infty}{||V_6||_\infty} = 0.03903245192, \\ \quad\vdots \end{cases} \tag{5.4}$$

Example 5.2. *As a second example, consider the following stochastic differential equation*

$$\begin{cases} dN(t) = N(t)(10 - 2N(t))dt + 4N(t)dW(t), \\ X(0) = x \geqslant 0. \end{cases} \tag{5.5}$$

According to the relation (28), to compute the third moment of $X(t)$, the following partial differential equation *is considered*

$$\begin{cases} \frac{\partial V(t,x)}{\partial t} + 16x\frac{\partial^2 V(t,x)}{\partial x^2} + (10-2x)\frac{\partial V(t,x)}{\partial x} = 0, \\ V(T,x) = x^3. \end{cases} \tag{5.6}$$

Let $T = 2$. Thus, the first few components, calculated by the formula (31), are as follows

$$\begin{cases} V_0 = x^3, \\ V_1 = 6x^4 - 126x^3, \\ V_2 = -48x^5 + 2148x^4 - 1587x^3, \\ V_3 = -240x^6 + 17472x^5 - 296796x^4 + 1000188x^3, \\ V_4 = 480x^7 - 50720x^6 + 1473168x^5 - 12476300x^4, \\ V_5 = 280x^8 - 40200x^7 + 1755020x^6 - \frac{80610320}{3}x^5 + \frac{377565661}{3}x^4 - 110270727x^3, \\ V_6 = \frac{-112}{3}x^9 + \frac{20902}{3}x^8 - 424072x^7 + \frac{91230890}{9}x^6 - \frac{4105792961}{45}x^5 + \frac{44790053219}{180}x^4 \\ -\frac{2315685267}{20}x^3, \\ V_7 = \frac{-14}{15}x^{10} + \frac{29596}{135}x^9 - \frac{2388211}{135}x^8 + \frac{163613479}{270}x^7 - \frac{28738133261}{3240}x^6 + \frac{67030145917}{1350}x^5 \\ -\frac{5258169675613}{64800}x^4 + \frac{16209796869}{800}x^3, \\ \vdots \end{cases} \tag{5.7}$$

Finally, the series have been obtained is convergent because the ratio of $||V_i||_\infty$ to $||V_{i-1}||_\infty$ for $i = 1, 2, 3, \cdots$ decrease to zero [14]. Below, these ratios for the first few are expressed

$$\begin{cases} \frac{||V_1||_\infty}{||V_0||_\infty} = 126.0, \\ \frac{||V_2||_\infty}{||V_1||_\infty} = 126.0, \\ \frac{||V_3||_\infty}{||V_2||_\infty} = 63.0, \\ \frac{||V_4||_\infty}{||V_3||_\infty} = 21.0, \\ \frac{||V_5||_\infty}{||V_4||_\infty} = 5.991979238, \\ \frac{||V_6||_\infty}{||V_5||_\infty} = 1.977141896, \\ \frac{||V_7||_\infty}{||V_6||_\infty} = 0.3260997884, \\ \vdots \end{cases} \tag{5.8}$$

6 Conclusions

In this paper, by using the Feynman-Kac formula, the ADM is applied to compute the moments for solution of logistic stochastic differential equation. In this method, the solution is found in the form of a convergent series and usually converges to the exact solution. Moreover, the terms of the series can be computed easily. similarly, other semi-analytic methods such as homotopy perturbation method, and homotopy analysis method can be applied to compute the moments for solution of logistic differential equation.

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