



Fuzzy Farthest Points and Fuzzy Best Approximation Points in Fuzzy Normed Spaces

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Abstract

In this paper we define fuzzy farthest points, fuzzy best approximation points and farthest orthogonality in fuzzy normed spaces and we will find some results. We prove some existence theorems, also we consider fuzzy Hilbert and show every nonempty closed and convex subset of a fuzzy Hilbert space has an unique fuzzy best approximation.

Key words: Normed fuzzy space, Fuzzy farthest orthogonality, Fuzzy best approximation points, Fuzzy farthest points.

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1 Introduction

It is well known that the conception of fuzzy sets, firstly defined by Zadeh in 1965. Fuzzy set theory provides us with a framework which is wider than that of classical set theory. Various mathematical structures, whose features emphasize the effects of ordered structure, can be developed on the theory. The theory of fuzzy sets has become an area of active research for the last forty years. On the other hand, the notion of fuzzyness has a wide application in many areas of science and engineering, chaos control, nonlinear dynamical systems, etc. In physics, for example, the fuzzy structure of space time is followed by the fact that in strong quantum gravity regime space time points are determined in a fuzzy manner and therefore the impossibility of determining position of particles gives a fuzzy structure.

In 1984, Kataras [16] defined a fuzzy norm on a linear space and at the same year Wu and Fang[30]also introduced a notion of fuzzy normed space and gave the generalization of the Kolomogroff normalized theorem for a fuzzy topological linear space. In [5], Biswas defined and studied fuzzy inner product space in linear space. Since then some mathematicians have defined fuzzy metrics and norms on a linear space from various points of view [6,9, 27, 29]. In 1994, Cheng and Mordeson introduced a definition of fuzzy norm on a linear space in such a manner that the corresponding induced fuzzy metric is of Kramosil and Michalek type [17]. In 2003, Bag and Samanta [3] modified the definition of Cheng and Mordeson [7] by removing a regular condition. They also established a decomposition theorem of fuzzy norm into a family of crisp norms and investigated some properties of fuzzy norms (see [4]). Following [3], we give the following notion of a fuzzy norm.

Definition 1.1. Let X be a real linear space. A function $N : X \times \mathbf{R} \rightarrow [0, 1]$ (the so-called fuzzy subset) is said fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbf{R}$:

$$(F_{N1}) \quad N(x, c) = 0 \text{ for } c \leq 0$$

$$(F_{N2}) \quad N(x, c) = 1 \text{ if and only if } x = 0 \text{ for every } c \in \mathbf{R}^+,$$

$$(F_{N3}) \quad N(cx, t) = N(x, \frac{t}{|c|}) \text{ for every } c \neq 0 \text{ and } t \in \mathbf{R}^+,$$

(F_{N4}) $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$ for every $s, t \in \mathbf{R}^+$,

(F_{N5}) $N(x, \cdot)$ is non-decreasing on \mathbf{R} , and $\lim_{t \rightarrow \infty} N(x, t) = 1$.

(F_{N6}) For $x \neq 0$, $N(x, \cdot)$ is (upper semi)continuous on \mathbf{R} .

The pair (X, N) is called a fuzzy normed linear space. One may regard $N(x, t)$ as the truth value of the statement thwe norm of x is less than or equal to thereal number t

Example 1.1 Let $(X, \|\cdot\|)$ be a normed linear space. Then:

$$N(x, t) = \begin{cases} \frac{t}{t + \|x\|}, & t > 0, x \in X, \\ 0 & t \leq 0, x \in X \end{cases}$$

is a fuzzy norm on X .

Example 1.2 Let $(X, \|\cdot\|)$ be a normed linear space. Then:

$$N(x, t) = \begin{cases} 0 & t \leq 0, \\ \frac{t}{\|x\|}, & 0 < t \leq \|x\|, \\ 1. & t \geq \|x\| \end{cases}$$

is a fuzzy norm on X .

Lemma 1.1 Let (X, N) be a norm fuzzy space. Define the function $N' : X \times \mathbf{R} \rightarrow [0, 1]$ as follows:

$$N'(x, t) = \begin{cases} \vee\{\alpha \in (0, 1) : \|x\|_\alpha \leq t\} & (x, t) \neq (0, 0) \\ 0 & (x, t) = (0, 0) \end{cases}$$

Then

a) N' is a norm fuzzy on X .

b) $N = N'$.

Example 1.3 Let X be the Real or Complex vector space and let N define on $X \times \mathbb{R}$ as follows :

$$N(x, t) = \begin{cases} 1 & t > |x| \\ 0 & t \leq |x| \end{cases}$$

Definition 1.1 Let W be a nonempty subset of a fuzzy normed space (X, N) . For $x \in X$, $t > 0$, let

$$d(W, x, t) = \bigvee_{y \in W} N(y - x, t).$$

An element $y_0 \in W$ is said to be a fuzzy best approximation point of x from W if

$$N(y_0 - x, t) = d(W, x, t).$$

Let W be a nonempty set of a fuzzy normed space (X, N) . For $x \in X$, we shall denote the set of all elements of fuzzy best approximation points of x from W by

$$P_W^F(x) = \{y \in W : d(W, x, t) = N(y - x, t) \text{ for every } t \in \mathbb{R}\}.$$

If each $x \in X$ has at least (respectively exactly) one fuzzy best approximation in W , then W is called a fuzzy proximal (respectively fuzzy Chebyshev) set.

Suppose W be a nonempty subset of a fuzzy normed space (X, N) . For $\omega_0 \in W$, we define

$$\begin{aligned} (P_W^F)^{-1}(\omega_0) &= \{x \in X : \omega_0 \in P_W^F(x)\} \\ &= \{x \in X : d(W, x, t) = N(\omega_0 - x, t) \text{ for every } t \in \mathbb{R}\}, \end{aligned}$$

which is called ω_0 -fuzzy farthest points set.

Definition 1.2 Let (X, N) be a fuzzy normed space. A non-empty subset W of X is called fuzzy bounded (F -bounded), if there exists a $t > 0$ and $0 < r < 1$ such that $N(x, t) > 1 - r$ for all $x \in W$.

Definition 1.3 Let W be a nonempty F -bounded subset of a fuzzy normed space (X, N) .

For $x \in X$, $t \in \mathbb{R}$, let

$$\delta(W, x, t) = \bigwedge_{y \in W} N(y - x, t).$$

An element $q_W^F(x) \in A$ is said to be a fuzzy farthest point of x from A if

$$N(q_W^F(x) - x, t) = \delta(W, x, t).$$

We shall denote the set of all elements of Fuzzy farthest points of x from W by $F_W^F(x)$; i.e.,

$$F_W^F(x) = \{y \in W : \delta(W, x, t) = N(y - x, t) \text{ for every } t \in \mathbb{R}\}.$$

If each $x \in X$ has at least one fuzzy farthest in A , then W is called a fuzzy remotal fuzzy set.

If each $x \in X$ has an unique fuzzy farthest in A , then W is called a fuzzy uniquely remotal fuzzy set.

Suppose W is a nonempty and F -bounded subset of a fuzzy normed space (X, N) . For $\omega_0 \in W$, we define

$$\begin{aligned} (F_W^F)^{-1}(\omega_0) &= \{x \in X : \omega_0 \in F_W^F(x)\} \\ &= \{x \in X : \delta(W, x, t) = N(\omega_0, x, t) \text{ for every } t \in \mathbb{R}\}, \end{aligned}$$

which is called ω_0 -fuzzy farthest points set.

2 ω_0 -Fuzzy best approximation Points and ω_0 -Fuzzy farthest Points in Fuzzy Normed Spaces

In this section we consider ω_0 -fuzzy best approximation points and ω_0 -fuzzy Farthest points in fuzzy normed spaces.

Example 2.1 Suppose $X = R$, $A = [0, 1]$ and $N(x, t) = \frac{t}{t+|x|}$. For arbitrary $t > 1$, put $r = \frac{1}{1+t}$. Then for every $x \in A$, $N(x, t) > 1 - r$, and A is F -bounded.

For $x > 1$, we have

$$\delta(A, x, t) = \frac{t}{t+x} = N(x-0, t),$$

and

$$(F^F)^{-1}(0) = \{x|x > 1\}.$$

For $x < 0$, we have

$$\delta(A, x, t) = \frac{t}{t+x-1} = N(x-1, t),$$

and

$$(F^F)^{-1}(1) = \{x|x < 0\}.$$

Theorem 2.1 Let W be a fuzzy subspace of a fuzzy normed space $(X, N, *)$.

i) If $\omega_0 \in W$, then $(P_W^F)^{-1}(\omega_0) = \omega_0 + (P_W^F)^{-1}(0)$.

ii) If $w \in W$, then $d(W, w, t) = 1$.

iii) If $w \in W$, then $\delta(W, w, t) \leq d(W, w, t)$, also $\delta(W, w, t) \leq 1$.

Proof. i) $x \in \omega_0 + (P_W^F)^{-1}(0)$ if and only if $x - \omega_0 \in (P_W^F)^{-1}(0)$. Then

$$\begin{aligned} N(x - \omega_0, t) &= d(W, x - \omega_0, t) \\ &= \bigvee_{g \in W} N(g - x + \omega_0, t) \\ &= \bigvee_{w \in W} N(w - x, t) \\ &= d(W, x, t). \end{aligned}$$

If and only if $x \in (P_W^F)^{-1}(0)$.

ii) We know that for $y \in W$, $N(y - w, t) \leq 1$. Therefore $d(W, w, t) \leq 1$. Also $N(w - w, t) = 1$ and so $d(W, w, t) \geq 1$. Then $d(W, w, t) = 1$.

iii) It is trivial.

Theorem 2.2 *Let W be a subspace of fuzzy normed linear space X and $\omega_0 \in W$. Then W is fuzzy proximal if and only if $X = W + (P_W^F)^{-1}(\omega_0)$.*

Proof. Suppose W is fuzzy proximal. Then for every $x \in X$, there exists a $\omega \in P_W^F(x)$. Therefore $x - \omega \in (P_W^F)^{-1}(0)$. From Theorem 3.2(i), $x - \omega \in (P_W^F)^{-1}(\omega_0) - \omega_0$. Then $x - \omega + \omega_0 \in (P_W^F)^{-1}(\omega_0)$, and so $x \in W + (P_W^F)^{-1}(\omega_0)$.

Conversely, If $X = W + (P_W^F)^{-1}(\omega_0)$ and $x \in X$, then for some $\omega \in W$ and $u_0 \in (P_W^F)^{-1}(\omega_0)$, we have $x = \omega + u_0$. Since $x - \omega - \omega_0 \in (P_W^F)^{-1}(0)$. Therefore $\omega + \omega_0 \in P_W^F(x)$, and W is fuzzy proximal.

Theorem 2.3 *Let W be a fuzzy subspace of fuzzy normed linear space X and $\omega_0 \in W$. Then W is fuzzy Chebyshev if and only if $X = W \oplus (P_W^F)^{-1}(0)$ (the main representation sum two element is unique).*

Proof. Suppose W is fuzzy Chebyshev, then W is fuzzy proximal, therefore $X = W + (P_W^F)^{-1}(0)$. If $x = \omega_1 + x_1 = \omega_2 + x_2$ for $\omega_1, \omega_2 \in W$ and $x_1, x_2 \in (P_W^F)^{-1}(0)$. Then $\omega_1 - \omega_2 = x_1 - x_2 \in W$. From Theorem 3.2(ii), $W \cap (P_W^F)^{-1}(0) = \{0\}$. We have $x_1 = x_2$ and $\omega_1 = \omega_2$.

Conversely, If $X = W \oplus (P_W^F)^{-1}(0)$, then W is fuzzy proximal. For $x \in X$, if $\omega_1, \omega_2 \in P_W^F(x)$. Then $\widehat{\omega}_1 = x - \omega_1, \widehat{\omega}_2 = x - \omega_2 \in (P_W^F)^{-1}(0)$. Therefore $x = \omega_1 + \widehat{\omega}_1 = \omega_2 + \widehat{\omega}_2$, $\omega_1 = \omega_2$ and $\widehat{\omega}_1 = \widehat{\omega}_2$. Therefore W is fuzzy Chebyshev.

Theorem 2.4 *Let W be a subset and F -bounded of fuzzy normed linear space X . Then W is fuzzy remotal if and only if $X = W + (F_W^F)^{-1}(0)$.*

Proof. Suppose W is fuzzy remotal, then for every $x \in X$ there exists a $\omega_0 \in F_W^F(x)$. Therefore $x - \omega_0 \in (F_W^F)^{-1}(0)$, and so $x \in \omega_0 + (F_W^F)^{-1}(0)$. Therefore $x \in W + (F_W^F)^{-1}(0)$ and $X = W + (F_W^F)^{-1}(0)$.

If $X = W + (F_W^F)^{-1}(0)$, it is clear that W is fuzzy remotal.

Theorem 2.5 *Let W be a fuzzy subspace of fuzzy normed linear space X . If W is fuzzy proximal and $(P_W^F)^{-1}(0)$ is singleton, then W is fuzzy Chebyshev.*

Proof. Suppose $(P_W^F)^{-1}(0) = \{x_0\}$ and $x \in X$. If $\omega_1, \omega_2 \in P_W^F(x)$. Then $x - \omega_1, x - \omega_2 \in (P_W^F)^{-1}(0)$, $x - \omega_1 = x - \omega_2 = x_0$, and so $\omega_1 = \omega_2$.

Theorem 2.6 *Let W be a subset and f -bounded of fuzzy normed linear space X . If W is fuzzy remotal and $(F_W^F)^{-1}(0)$ is singleton, then W is fuzzy uniquely remotal set.*

Proof. Suppose $(F_W^F)^{-1}(0) = \{x_0\}$ and $x \in X$. If $\omega_1, \omega_2 \in F_W^F(x)$. Then $x - \omega_1, x - \omega_2 \in (F_W^F)^{-1}(0)$ and $x - \omega_1 = x - \omega_2 = x_0$ and so $\omega_1 = \omega_2$.

Theorem 2.7 *Let W be a subset of fuzzy normed linear space X and $\omega_0 \in W$. If $(P_W^F)^{-1}(\omega_0) = X$, then W is a singleton set and $W = \{\omega_0\}$.*

Proof. Suppose there exists a $\omega_1 \in W$ and $\omega_1 \neq \omega_0$. Then $\omega_1 \in (P_W^F)^{-1}(\omega_0)$. Then

$$d(W, \omega_1, t) = N(\omega_1 - \omega_0, t).$$

Since $\omega_1 \in W$, from Theorem 3.2(ii), we have $d(W, \omega_1, t) = 1$, therefore $N(\omega_1 - \omega_0, t) = 1$, and so $\omega_1 = \omega_0$.

3 ω_0 -fuzzy best Approximation Points in Fuzzy Hilbert Spaces

In this section we consider ω_0 -fuzzy best approximation points in fuzzy Hilbert spaces. In the First step we give the new modified definition of a fuzzy inner product space and then we prove some interesting results which hold in any fuzzy inner product space. Throughout this paper we let

$$H(t) = \begin{cases} 1 & t > 0, \\ 0 & t \leq 0. \end{cases}$$

Definition 3.1 A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is said to be a continuous t -norm if $([0, 1], *)$ is a topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ ($a, b, c, d \in [0, 1]$).

Definition 3.2 A fuzzy inner product space (FIP-space) is a triplet $(X; F; *)$, where X is a real vector space, $*$ is a continuous t -norm, F is a fuzzy set on $X^2 \times \mathbb{R}$ and the following conditions hold for every $x; y; z \in X$ and $s; t; r \in \mathbb{R}$:

- (FI-1) $F(x; x; 0) = 0$ and $F(x; x; t) > 0$, for each $t > 0$
- (FI-2) $F(x; x; t) \neq H(t)$ for some $t \in \mathbb{R}$ if and only if $x \neq 0$
- (FI-3) $F(x; y; t) = F(y; x; t)$
- (FI-4) For any real number α , $F(\alpha x; y; t) = F(x; y; \frac{t}{\alpha})$ if $\alpha > 0$, $F(\alpha x; y; t) = H(t)$ if $\alpha = 0$ and $F(\alpha x; y; t) = 1 - F(x; y; \frac{t}{-\alpha})$ if $\alpha < 0$
- (FI-5) $F(x; x; t) * F(y; y; s) \leq F(x + y; x + y; t + s)$
- (FI-6) $\sup_{s+r=t} (F(x; z; s) * F(y; z; r)) = F(x + y; z; t)$
- (FI-7) $F(x; y; \cdot) : \mathbb{R} \rightarrow [0; 1]$ is continuous on $\mathbb{R} \setminus \{0\}$
- (FI-8) $\lim_{t \rightarrow +\infty} F(x; y; t) = 1$.

In the following we shall present a list of known lemmas and definitions which are needed in the proof of the main results.

Lemma 3.1 Let $(X, F, *)$ be a FIP-space. Then it is a fuzzy normed space. We can define

$$N(x, t) = \begin{cases} F(x, x, t^2) & t > 0, \\ 0 & t \leq 0. \end{cases}$$

Lemma 3.2 Suppose that $(X, F, *)$ be a FIP-space, where $*$ is a strong t -norm and for each $x, y \in X$, define $\langle \cdot, \cdot \rangle : X \times X \rightarrow \mathbb{R}$ by

$$\langle x, y \rangle = \sup\{t \in \mathbb{R} : F(x, y, t) < 1\}.$$

Then $(X, \langle \cdot, \cdot \rangle)$ is an inner product space.

Lemma 3.3 Let $(X, F, *)$ be a FIP-space, where $*$ is a strong t -norm and for each $x, y \in X$, $\sup\{t \in \mathbb{R} : F(x, y, t) < 1\} < \infty$. If we define $\|x\| = \langle x, y \rangle \frac{1}{2}$. Then $(X, \|\cdot\|)$ is a normed space.

Definition 3.3 (i) Let $(X, F, *)$ be a FIP-space. $u, v \in X$ is said to be fuzzy orthogonal if $F(u, v, t) = H(t)$ ($\forall t \in \mathbb{R}$).

(ii) Let $(X, F, *)$ be a FIP-space and M a subset of X . The set of vectors in X , fuzzy orthogonal to every vector in M , denoted by M^\perp , is called fuzzy orthogonal complement of M .

Definition 3.4 Let $(X, F, *)$ be a FIP-space, where $*$ is a strong t -norm and for each $x, y \in X$, $\sup\{t \in \mathbb{R} : F(x, y, t) < 1\} < \infty$. We say that $(X, F, *)$ is fuzzy Hilbert space and denoted by (FH, F, t) if $(X, \|\cdot\|)$ is a complete normed space.

Lemma 3.4 Let $(X, F, *)$ be a FIP-space. Then M^\perp is a closed subspace of X . Possibly consisting of just the zero vector. However, if $M \neq \{0\}$, then $M^\perp \neq X$.

We shall extend following lemma in this paper.

Lemma 3.5 If K is a nonempty, closed and convex subset of a τ_F -complete standard FIP-space $(X; F; \min)$, then there exists a unique vector in K of greatest norm.

Theorem 3.1 Let $(X, F, *)$ be a FIP-space, M is a subspace of X and $x \in X$. Then $g_0 \in P_M^F(x)$ if and only if $\|x - g_0\| \leq \|x - m\|$ for every $m \in M$. Thus if $g_0 \in P_M^F(x)$, then

$$\|x - g_0\| = \inf_{m \in M} \|x - m\| \quad (*)$$

Proof. We know that $\|x - g_0\| \leq \|x - m\|$ for every $m \in M$, if and only if $\langle x - g_0, x - g_0 \rangle \leq \langle x - m, x - m \rangle$ for every $m \in M$, if and only if $\sup\{t : F(x - g_0, x - g_0, t) < 1\} \leq \sup\{t : F(x - m, x - m, t) < 1\}$ for every $m \in M$. If and only if $F(x - g_0, x - g_0, s) \geq F(x - m, x - m, s)$ for every $s > 0$ and for every $m \in M$. If $s > 0$, there exist a $t \in \mathbb{R}$ such that $s = t^2$. If and only if $F(x - g_0, x - g_0, t^2) \geq F(x - m, x - m, t^2)$, that is $N(x - g_0, t) \geq N(x - m, t)$, therefore $g_0 \in P_M^t(x)$.

Theorem 3.2 If K is a nonempty, closed and convex subset of a fuzzy

Hilbert space $(FH, F, *)$. For $x \in X$, there exists a unique fuzzy best approximation $k_0 \in P_K^F(x)$.

Proof. Suppose $\delta = \inf_{m \in K} \|x - m\|$. Then there is a sequence $\{y_n\}_{n \in \mathbb{N}}$ in K such that $\|y_n - x\| \rightarrow \delta$. By the parallelogram law,

$$\begin{aligned} \|y_n - y_m\|^2 &= 2\|y_n - x\|^2 + 2\|y_m - x\|^2 - \|(y_n + y_m) - 2x\|^2 \\ &\leq 2\|y_n - x\|^2 + 2\|y_m - x\|^2 - 4\delta^2 \\ &\rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Hence $\{y_n\}_{n \in \mathbb{N}}$ is a Cauchy sequences. But K is closed (hence complete) and so y_n converges to $k_0 \in K$. It follows, by the continuity of the norm,

$$\|x - k_0\| = \delta.$$

Suppose $y \in K$ is another fuzzy best approximation to x , i.e. $\|x - y\| = \delta$. Then

$$\begin{aligned} \|k_0 - y\| &= 2\|k_0 - x\|^2 + 2\|y - x\|^2 - \|(k_0 + y) - 2x\|^2 \\ &\leq 2\|k_0 - x\|^2 + 2\|y - x\|^2 - 4\delta^2 \\ &= 0. \end{aligned}$$

and so $k_0 = y$.

The following corollary is an extension of Lemma 4.7.

Theorem 3.3 *If K is a nonempty, closed and convex subset of a fuzzy Hilbert space $(FH, F, *)$. Then there exists a unique vector in K of greatest norm.*

Proof. Since $0 \in X$, from above theorem there exists a unique fuzzy best approximation $k_0 \in P_K^t(0)$ therefore

$$N(k_0, t) = \bigvee_{k \in K} N(k, t).$$

Theorem 3.4 *Let $(X, F, *)$ be a FIP-space. If $u, v \in X$ is fuzzy orthogonal, then $\langle u, v \rangle = 0$.*

Proof. Since $u \perp v$, then $F(u, v, t) = H(t)$. From definition $\langle u, v \rangle = \sup\{t \in \mathbb{R} : F(u, v, t) < 1\}$, it follows that $\langle u, v \rangle = 0$.

Theorem 3.5 *Let $(X, F, *)$ be a FIP-space, M is a subspace of X and $x \in X$. Then $g_0 \in P_M^t(x)$ if and only if $\langle x - g_0, m \rangle = 0$ for every $m \in M$.*

Proof. Suppose g_0 be a fuzzy best approximation to x in M . Then for any real s , arbitrary $m \in M$ and any scalar α the element $g_0 + s\alpha m$ belongs to M so that function

$$g(s) = \langle (g_0 + s\alpha m), (g_0 + s\alpha m) \rangle,$$

must have a minimum at $s = 0$. This implies that

$$g'(0) = \langle \alpha m, g_0 - x \rangle + \langle g_0 - x, \alpha m \rangle.$$

If we choose $\alpha \neq 0$ such that $\langle \alpha m, g_0 - x \rangle$ is real, then $\langle \alpha m, g_0 - x \rangle = 0$. Since m was arbitrary this implies $\langle m, x - g_0 \rangle = 0$ for any $m \in M$.

Conversely, suppose $\langle m, x - g_0 \rangle = 0$ for any $m \in M$. Let m be any element in M . Then

$$\begin{aligned} \|x - m\|^2 &= \|x - g_0 + (g_0 - m)\|^2 \\ &= \langle x - g_0 + (g_0 - m), x - g_0 + (g_0 - m) \rangle \\ &= \|x - g_0\|^2 + \|g_0 - m\|^2, \end{aligned}$$

because $g_0 - m \in M$ and thus is orthogonal to $x - g_0$. Hence

$$\|x - g_0\| \leq \|x - m\|.$$

Therefore from Theorem 4.8, $g_0 \in P_M^t(x)$.

Theorem 3.6 *Let M be a subspace of fuzzy inner product space X . Then $(P_M^t)^{-1}(\omega_0) = \omega_0 + M^\perp$.*

Proof. Suppose $x \in (P_M^t)^{-1}(\omega_0)$, then $\omega_0 \in P_M^t(x)$. Therefore $x - \omega_0 \in M^\perp$, and so $x \in M^\perp + \omega_0$.

Now if $x \in M^\perp + \omega_0$, then $x - \omega_0 \in M^\perp$. Therefore $x \in (P_M^t)^{-1}(\omega_0)$.

4 Fuzzy farthest orthogonality

In this section we define the concept fuzzy farthest orthogonality in fuzzy normed linear spaces.

Definition 4.1 Let (X, N) be a fuzzy normed linear space and $x, y \in X$. We say that x is fuzzy farthest orthogonal to y and is denoted by $x \perp_{FF} y$ if and only if $N(x, t) \leq N(x - y, t)$. If W is a f -bounded subset of X and $x \in X$. We say that $x \perp_{FF} W$, if and only if $N(x, t) \leq \delta(W, x, t)$.

Theorem 4.1 Let (X, N) be a fuzzy normed linear space, W a F -bounded subset of X and $x \in X$. Then $x \perp_{FF} W$ if and only if for every $y \in W$, $x \perp_{FF} y$.

Proof. Suppose $x \perp_{FF} W$, then $N(x, t) \leq \delta(W, x, t) = \bigwedge_{y \in W} N(y - x, t)$ for every $t \in \mathbb{R}$. Therefore $N(x, t) \leq N(y - x, t)$ for every $y \in W$ and for every $t \in \mathbb{R}$. Conversely, if for every $y \in W$ and for every $t \in \mathbb{R}$, $N(x, t) \leq N(x - y, t)$. We get \inf on $y \in W$. Then $N(x, t) \leq \delta(W, x, t)$ for every $t \in \mathbb{R}$. \square

Definition 4.2 Let $(X, \|\cdot\|)$ be a normed linear space, W a bounded subset of X . The set

$$W^{\perp F} = \{x \in X : x \perp_F W\}.$$

is denoted by the farthest orthogonal complement with respect to W .

Theorem 4.2 Let (X, N) be a fuzzy normed linear space, W a f -bounded subset of X and $x \in X$. If $0 \in F_W^F(x)$, then $x \perp_{FF} W$.

Proof. If $0 \in F_W(x)$, then $N(x, t) = \delta(W, x, t)$ for every $t \in \mathbb{R}$. Therefore $x \perp_{FF} W$. \square

Proposition 1 Let (X, N) be a fuzzy normed linear space, W a f -bounded subset of X and $x \in X$. then: (i) For $x \in X$, $x \perp_{FF} x$.

(ii) If $x \in X$, then $x \perp_{FF} 0$.

(iv) If $x \perp_{FF} y$ and $N(x, t) = N(y, t)$ for every $t \in \mathbb{R}$, then $y \perp_F x$.

(iv) If $0 \perp_{FF} x$ for $x \in X$, then $x = 0$.

(v) For $\alpha \in \mathbb{C}$, $x \perp_{FF} y$ if and only if $\alpha x \perp_{FF} \alpha y$.

- (vi) If $x_n \rightarrow x$, $y_n \rightarrow y$ and $x_n \perp_{FF} y_n$, then $x \perp_{FF} y$.
(vii) For $x \neq 0$ and $\lambda, \mu \in \mathbb{C}$, $\lambda x \perp_{FF} \mu x$ if and only if $\lambda \perp_{FF} \mu$.

Proof. It is trivial. □

Theorem 4.3 Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space and $x, y \in X$.

- (i) If $\langle x, y \rangle = 0$ and $y \neq 0$, then x is not farthest orthogonal to y .
(ii) If $x \perp_{FF} y$ then $2\operatorname{Re} \langle x, y \rangle \geq \|y\|^2$.

Proof. It is trivial. □

Theorem 4.4 Let $(X, \|\cdot\|)$ be a normed linear space, $x, y \in X$. If $x \perp_{BY}$, then $x' \perp_{FY}$ for every $x' \in [x, y]$.

Proof. Since $x' \in [x, y]$, we have $\|x' - y\| = \|x' - x\| - \|x - y\|$. If $x \perp_{BY}$. It follows that $\|x\| \leq \|x - y\|$. Therefore

$$\begin{aligned} \|x'\| &\geq \|x' - x\| - \|x\| \\ &\geq \|x' - x\| - \|x - y\| \\ &= \|x' - y\|. \end{aligned}$$

Therefore, $x' \perp_{FY}$. □

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