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Best Proximity Points Results for Cone Generalized Semi-Cyclic $\varphi-$Contraction Maps

Marzieh Ahmadi Baseri[a,*],Hamid Mazaheri Tehrani[a], T. D. Nrang[b]

[a]*Deparment of Mathematics, Yazd University, Yazd, Iran.*

[b]*Deparment of Mathematics,Guru Nanak Dev University, Amristar, India.*

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Abstract

In this paper, we introduce a cone generalized semi-cyclic $\varphi-$contraction map and prove best proximity points theorems for such maps in cone metric spaces. Also, we study existence and convergence results of best proximity points of such maps in normal cone metric spaces. Our results generalize some results on the topic.

1 Introduction

Huang and Zhang [8] introduced cone metric spaces as a generalization of metric spaces. The study of existence of best proximity points in cone metric spaces is very interesting. More precisely, for two given nonempty subsets A and B of a cone metric space (X,d), a point $x \in A$ is called a best proximity point of map $T : A \to B$ if $d(x,Tx) = d(A,B)$, where $d(A,B) = \inf\{d(a,b),\ a \in A, b \in B\}$. Now, let us consider S and T be given mappings from A to B, a common best proximity point is $x \in A$ such that $d(x,Tx) = d(x,Sx) = d(A,B)$. In this area, in 2014, Lee [10] presented cone metric version of existence and convergence for best proximity points. In 2016, best proximity point results for cone generalized cyclic contraction maps elicited in [2] on cone metric spaces. One could find more results about best proximity points on cone metric spaces in [11, 5] On the other hand, in 2011, Gabeleh and Abkar [7] proved best proximity point theorems for a semi-cyclic contraction pair on metric spaces.

In 2014, Thakur and Sharma [13] investigated best proximity point theorems for semi-cyclic $\varphi-$contraction pair in metric space.

In 2016, best proximity point results for generalized semi-cyclic $\varphi-$contractions in metric spaces proved in [1]. Also, there are some works about best proximity points of some cyclic contraction maps (for example, [6, 3, 9, 4]).

In this paper, we prove best proximity point results for a new class of semi-cyclic contraction pair (S,T), in cone metric spaces, is called cone generalized semi-cyclic φ-contraction pair. Our results generalize results in [1, 10]. Since cone metric spaces generalized metric spaces, our results extend the corresponding results in the literature. To prove our results in the next section we recall some definitions and facts. In the present paper E stands for a real Banach space.

* Corresponding Author's E-mail: m.ahmadi@stu.yazd.ac.ir(M.Ahmadi-Baseri)

A subset P of E is called a cone if and only if

(P1) P is closed, nonempty and $P \neq \{0\}$;
(P2) $a, b \geq 0$ and $x, y \in P$ implies $ax + by \in P$;
(P3) $x \in P$ and $-x \in P$ implies $x = 0$.

We define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$, for $x, y \in P$. $x \prec y$ will stand for $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of P.

A cone P is said to be normal if there is a number $M > 0$ such that for all $x, y \in E$

$$0 \preceq x \preceq y \quad implies \quad \|x\| \leq M\|y\|.$$

The least positive number M satisfying the above inequality is called the normal constant of P.

A map $f : P \to P$ is said to be increasing (strictly increasing) whenever $x \preceq y$ implies that $f(x) \preceq f(y)$ ($x \prec y$ implies that $f(x) \prec f(y)$).

Definition 1.1 *[8] Let X be a nonempty set. Suppose that a mapping $d : X \times X \to E$ satisfies:*

(d1) $0 \preceq d(x, y)$ for every $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(d2) $d(x, y) = d(y, x)$ for every $x, y \in X$;
(d3) $d(x, y) \preceq d(x, z) + d(z, y)$ for every $x, y, z \in X$.

Then d is called a cone metric and (X, d) is called a cone metric space.

Definition 1.2 *[14] A nonempty subset A of (X, d), is said to be bounded above if there exists $c \in intP$ such that $c - d(x, y) \in P$ for all $x, y \in A$ and is said to be bounded if $\delta(A) = \sup\{d(x, y) : \ x, y \in A\}$ exists in E.*

Let $\{x_n\}$ be a sequence in a cone metric space (X, d) and $x \in X$. If for every $c \in intP$, there exists a natural number N such that for every $n > N$, $c - d(x_n, x) \in intP$, then $\{x_n\}$ converges to x with respect to P and is denoted as $\lim_{n\to\infty} x_n = x$.

Lemma 1.1 *[8] Let (X, d) be a cone metric space, P a normal cone, $\{x_n\}$ and $\{y_n\}$ be sequences in X. Then*

(i) x_n converges to x with respect to P if and only if $d(x_n, x) \to 0$ as $n \to \infty$,

(ii) If $x_n \to x$ and $y_n \to y$ as $n \to \infty$ with respect to P, then $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$,

(iii) If $x_n \to x$ and $y_n \to y$ as $n \to \infty$ with respect to P and $y_n - x_n \in P$ for every $n \in \mathbb{N}$, then $y - x \in P$.

2 Main results

Throughout this section, A and B are nonempty subsets of a cone metric space (X, d).

Definition 2.1 *Let two maps $S, T : A \cup B \to A \cup B$ be such that $S(A) \subseteq B$ and $T(B) \subseteq A$. Then the pair (S, T) is said to be cone generalized semi-cyclic φ-contraction pair, if $\varphi : E \to E$ is a strictly increasing map and*

$$\begin{aligned} d(Sx, Ty) &\preceq (1/3)\{d(x, y) + d(Sx, x) + d(Ty, y)\} \\ &- \varphi(d(x, y) + d(Sx, x) + d(Ty, y)) + \varphi(3d(A, B)), \end{aligned} \tag{2.1}$$

for all $x \in A$ and $y \in B$.

Example 2.1 *Let* $E=\mathbb{R}^2$*,* $P=\{(x,y)\in E:\ x,y\geq 0\}\subset\mathbb{R}^2$ *,* $X=\mathbb{R}$ *and* $d:X\times X\to E$ *be such that* $d(x,y)=(|x-y|,\lambda|x-y|)$*, where* $\lambda\geq 0$ *is a constant. Let* $A=[0,1],\ B=[-1,0]$*. Define* $S,T:A\cup B\to A\cup B$ *by*

$$S(x)=\begin{cases}\frac{-x}{6}, & x\in A\\ \frac{x}{6}, & x\in B,\end{cases}\qquad T(x)=\begin{cases}\frac{x}{6}, & x\in A\\ \frac{-x}{6}, & x\in B\end{cases}$$

and $\varphi(t_1,t_2)=(\frac{t_1}{6},\frac{t_2}{6})$ *for* $(t_1,t_2)\in\mathbb{R}^2$*, then*

$$\begin{aligned}&(1/3)\{d(x,y)+d(Sx,x)+d(Ty,y)\}-\varphi(d(x,y)+d(Sx,x)+d(Ty,y))\\&+\varphi(3d(A,B))-d(Sx,Ty)\\&=(1/6)\{d(x,y)+d(Sx,x)+d(Ty,y)\}-d(Sx,Ty)\\&=(1/6)\{(|x-y|,\lambda|x-y|)+(7/6)(x,\lambda x)-(7/6)(y,\lambda y)\}-(1/6)(|x-y|,\lambda|x-y|)\\&=(7/36)(x-y,\lambda(x-y))\in P,\ for\ x\in A\ and\ y\in B.\end{aligned}$$

Hence

$$(1/3)\{d(x,y)+d(Sx,x)+d(Ty,y)\}-\varphi(d(x,y)+d(Sx,x)+d(Ty,y))$$
$$+\varphi(3d(A,B))-d(Sx,Ty)\in P,$$

which implies that

$$d(Sx,Ty)\preceq(1/3)\{d(x,y)+d(Sx,x)+d(Ty,y)\}$$
$$-\varphi(d(x,y)+d(Sx,x)+d(Ty,y))+\varphi(3d(A,B)).$$

Thus (S,T) *is a cone generalized semi-cyclic* φ*-contraction pair.*

Example 2.2 *Let* $E=l^1$*,* $P=\{(x_n)\in E:\ x_n\geq 0\ for\ all\ n\}$*,* $X=\mathbb{R}$ *with the usual metric* ρ *and* $A=[0,1],\ B=[-1,0]$*. Define* $d:X\times X\to E$ *by* $d(x,y)=\{\frac{\rho(x,y)}{2^n}\}_{n\geq 1}$*,* $\varphi((t_n))=(\frac{t_n}{6})_{n\geq 1}$ *for* $(t_n)_{n\geq 1}\in E$ *and the maps* $S,\ T$ *be defined similar as in Example 2.1. Then* (S,T) *is a generalized cone semi-cyclic* φ*-contraction pair.*

Remark 2.1 *(i) A cone generalized semi-cyclic contraction map is a cone generalized semi-cyclic* φ*-contraction with* $\varphi(x)=(1-k)(x/3)$ *for* $x\in E$*, where* $k\in(0,1)$ *is a constant. In this case* (S,T) *satisfies in (2), for some* $k\in(0,1)$*,*

$$d(Sx,Ty)\preceq(k/3)\{d(x,y)+d(Sx,x)+d(Ty,y)\}+(1-k)d(A,B),\tag{2.2}$$

for all $x\in A$ *and* $y\in B$*.*
(ii) when in cone generalized semi-cyclic φ*-contraction* $S=T,\ T$ *is called cone generalized cyclic* φ*-contraction. A cone generalized cyclic contraction map is a cone generalized cyclic* φ*-contraction with* $\varphi(x)=(1-k)(x/3)$ *for* $x\in E$*, where* $k\in(0,1)$ *is a constant. In this case* T *satisfies in (3), for some* $k\in(0,1)$*,*

$$d(Tx,Ty)\preceq(k/3)\{d(x,y)+d(Tx,x)+d(Ty,y)\}+(1-k)d(A,B),\tag{2.3}$$

for all $x\in A$ *and* $y\in B$*.*

Let pair (S,T) be a cone generalized semi-cyclic φ-contraction. Consider $x_0 \in A$, then $Sx_0 \in B$, so there exists $y_0 \in B$ such that $y_0 = Sx_0$. Now $Ty_0 \in A$, so there exists $x_1 \in A$ such that $x_1 = Ty_0$. Inductively, we define sequences $\{x_n\}$ and $\{y_n\}$ in A and B, respectively by

$$x_{n+1} = Ty_n, \quad y_n = Sx_n. \tag{2.4}$$

Lemma 2.1 *Let $S,T : A \cup B \to A \cup B$ be cone generalized semi-cyclic φ-contraction maps. For $x_0 \in A$, if the sequences $\{x_n\}$ and $\{y_n\}$ are generated by (4) then for all $x \in A,\ y \in B$, and $n \geq 1$, we have*

(a) $\varphi(3d(A,B)) \preceq \varphi(d(x,y) + d(x,Sx) + d(y,Ty))$,

(b) $d(Sx,Ty) \preceq (1/3)\{d(x,y) + d(x,Sx) + d(y,Ty)\}$,

(c) $d(x_n,Sx_n) \preceq d(x_{n-1},Sx_{n-1})$,

(d) $d(x_{n+1},y_n) \preceq d(y_n,Ty_{n-1})$.

Proof We have $3d(A,B) \preceq d(x,y) + d(Tx,x) + d(Ty,y)$. Since φ is a strictly increasing map, (a) and (b) are obtained. Hence

$$d(x_n,Sx_n) \preceq (1/3)\{d(y_{n-1},x_n) + d(x_n,Sx_n) + d(y_{n-1},x_n)\},$$

$$d(x_n,Sx_n) \preceq d(y_{n-1},x_n). \tag{2.5}$$

Also, since

$$d(y_{n-1},x_n) \preceq (1/3)\{d(y_{n-1},x_{n-1}) + d(x_{n-1},Sx_{n-1}) + d(y_{n-1},x_n)\},$$

$$d(y_{n-1},x_n) \preceq d(x_{n-1},Sx_{n-1}). \tag{2.6}$$

The relations (5) and (6) implies (c). Since

$$d(x_{n+1},y_n) \preceq (1/3)\{d(x_n,y_n) + d(x_{n+1},y_n) + d(x_n,y_n)\},$$

so

$$d(x_{n+1},y_n) \preceq d(y_n,Ty_{n-1}),$$

that is inequality (d).

Proposition 2.1 *Let $S,T : A\cup B \to A\cup B$ be cone generalized semi-cyclic φ-contraction maps. For $x_0 \in A$, the sequences $\{x_n\}$ and $\{y_n\}$ are generated by (4). Then two sequences $\{d(x_n,Sx_n)\}$ and $\{d(y_{n-1},Ty_n)\}$ converge to $d(A,B)$ in E.*

Proof Let $d_n = d(x_n,Sx_n)$, then by Lemma 2.1, $\{d_n\}$ is decreasing and bounded below. So $\lim_{n\to\infty} d_n$ exists. Put $t_0 := \lim_{n\to\infty} d_n$, hence for every $c \in intP$, there is a natural number N such that for every $n > N$,

$$c - d(d_n,t_0) \in intP. \tag{2.7}$$

Assume that $t_0 \succ d(A,B)$, then there exists $(x_0,y_0) \in A\times B$ such that

$$c - d(d_n,t_0) + d(A,B) - d(x_0,y_0) \in intP. \tag{2.8}$$

From (7) and (8), $d(A,B) - d(x_0,y_0) \in (intP - intP) \subset intP$, which is a contraction. Therefore $t_0 = d(A,B)$.

Remark 2.2 *Proposition 2.1 for $S = T$ is generalization of Proposition 6 of [9] in metric spaces.*

Proposition 2.2 *Let $S, T : A \cup B \to A \cup B$ be cone generalized semi-cyclic φ-contraction maps. For $x_0 \in A$, the sequences $\{x_n\}$ and $\{y_n\}$ are generated by (4). Then two sequences $\{x_n\}$ and $\{y_n\}$ are bounded.*

Proof By Proposition 2.1, we have $d(x_n, Sx_n) \to d(A, B)$ as $n \to \infty$. It is sufficient to show that $\{Sx_n\}$ is bounded. For the unbounded map φ, take $M \in E$ such that

$$\varphi(M) \succ (4/3)d(x_0, Sx_0) + \varphi(3d(A, B)).$$

If $\{Sx_n\}$ is not bounded, then there exists a natural number $N \in \mathbb{N}$, such that

$$d(x_1, Sx_N) \succ M, \qquad d(x_1, Sx_{N-1}) \prec M.$$

Then

$$\begin{aligned}
M &\prec d(x_1, Sx_N)\\
&\preceq d(y_0, x_N)\\
&\preceq (1/3)\{d(x_0, y_{N-1}) + d(x_0, Sx_0) + d(y_{N-1}, Ty_{N-1})\}\\
&-\varphi(d(x_0, y_{N-1}) + d(x_0, Sx_0) + d(y_{N-1}, Ty_{N-1})) + \varphi(3d(A, B))\\
&\preceq (1/3)\{d(x_0, x_1) + d(x_1, y_{N-1}) + d(x_0, Sx_0) + d(x_{N-1}, y_{N-1})\}\\
&-\varphi(d(x_0, y_{N-1})) + \varphi(3d(A, B))\\
&\preceq (1/3)\{d(x_0, y_0) + d(y_0, x_1) + M + d(x_0, Sx_0) + d(x_{N-1}, y_{N-2})\}\\
&-\varphi(d(x_0, y_{N-1})) + \varphi(3d(A, B))\\
&\preceq (1/3)\{3d(x_0, Sx_0) + M + d(x_{N-2}, y_{N-2})\}\\
&-\varphi(d(x_0, y_{N-1})) + \varphi(3d(A, B))\\
&\prec (4/3)d(x_0, Sx_0) + M - \varphi(d(x_0, y_{N-1})) + \varphi(3d(A, B)).
\end{aligned}$$

Hence

$$\varphi(d(x_0, y_{N-1})) \prec (4/3)d(x_0, Sx_0) + \varphi(3d(A, B)).$$

Therefore

$$\begin{aligned}
&\varphi(M) \prec \varphi(d(x_1, Sx_N)) \preceq \varphi(d(y_0, x_N)) \preceq \varphi(d(x_0, y_{N-1}))\\
&\prec (4/3)d(x_0, Sx_0) + \varphi(3d(A, B)),
\end{aligned}$$

which is a contradiction. Hence $\{Sx_n\}$ is bounded, therefore $\{x_n\}$ is bounded.

Now, we prove a best proximity theorem in a normal cone metric space for a cone generalized semi-cyclic φ-contraction pair.

Theorem 2.1 *Let A and B be nonempty subsets of a normal cone metric space (X, d) and $S, T : A \cup B \to A \cup B$ be cone generalized semi-cyclic φ-contraction maps. For $x_0 \in A$, the sequences $\{x_n\}$ and $\{y_n\}$ are generated by (4).*
(i) If $\{y_n\}$ has a convergent subsequence in B, then there exists $y \in B$ such that $d(y, Ty) = d(A, B)$. Moerever, if $d(A, B) = 0$, then y is unique.
(ii) If $\{x_n\}$ has a convergent subsequence in A, then there exists $x \in A$ such that $d(x, Sx) = d(A, B)$. Moerever, if $d(A, B) = 0$, then x is unique.

Proof (i) Let $\{y_{n_k}\}$ be a subsequence of $\{y_n\}$ such that $\lim_{k\to\infty} y_{n_k} = y$. The relation

$$d(A,B) \preceq d(Ty_{n_k}, y) \preceq d(y_{n_k}, y) + d(y_{n_k}, Ty_{n_k})$$

holds for each $k \geq 1$. Letting $k \to \infty$, by Lemma 1.1, Lemma 2.1(d) and Proposition 2.1, we obtain

$$\lim_{k\to\infty} d(Ty_{n_k}, y) = d(A,B).$$

From Lemma 2.1(b),

$$\begin{aligned} d(Ty, y_{n_k}) &\preceq (1/3)\{d(y, x_{n_k}) + d(y, Ty) + d(x_{n_k}, Sx_{n_k})\} \\ &\preceq (1/3)\{d(y, y_{n_k}) + d(y_{n_k}, x_{n_k}) + d(y, y_{n_k}) + d(Ty, y_{n_k}) + d(x_{n_k}, Sx_{n_k})\}. \end{aligned}$$

Letting $k \to \infty$, by Lemma 1.1 and Proposition 2.1, we get

$$(2/3)d(A,B) \preceq (2/3) \lim_{k\to\infty} d(Ty, y_{n_k}) \preceq (2/3)d(A,B).$$

So $d(Ty, y) = d(A,B)$.
Now assume that $y, z \in B$, $y \neq z$ and $d(y,Ty) = d(z,Tz) = d(A,B) = 0$. then
$d(y,z) \preceq d(y,Ty) + d(Ty,z)$
$\preceq d(y,Ty) + d(Ty,Tz) + d(z,Tz)$
$= d(Ty,Tz)$
$\preceq (1/3)\{d(y,z) + d(y,Ty) + d(z,Tz)\}$
$-\varphi(d(y,z) + d(y,Ty) + d(z,Tz)) + \varphi(3d(A,B))$
$\prec d(y,z) - \varphi(d(y,z)) + \varphi(0)$.
Hence

$$\varphi(0) \prec \varphi(d(y,z)) \prec \varphi(0),$$

which is a contradiction.
(ii) It can be proved by the same method.

Definition 2.2 *A subset M of a cone metric space (X,d) is boundedly compact if each bounded sequence in M has a subsequence converging to a point in M.*

Theorem 2.2 *Let A and B be nonempty subsets of a normal cone metric space (X,d) and $S,T : A\cup B \to A\cup B$ be such that the pair (S,T) is generalized semi-cyclic φ-contraction. If A and B are boundedly compact then there exist $x \in A$ and $y \in B$ such that*

$$d(x,Sx) = d(A,B) = d(y,Ty).$$

Proof The proof follows from Theorem 2.1 and Proposition 2.2.

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