Nonlinear Viscosity Algorithm with Perturbation for Nonexpansive Multi-Valued Mappings

Hamid Reza Sahebi

Department of Mathematics, Ashtian Branch, Islamic Azad University, Ashtian, Iran

Article Info

Keywords

Nonexpansive multi-valed
strongly positive linear bounded operator
fixed point
Hilbert space.

Article History

Received: 2019 June 15

Accepted: 2020 February 11

Abstract

In this paper, based on viscosity technique with perturbation, we introduce a new nonlinear viscosity algorithm for finding a element of the set of fixed points of nonexpansive multi-valued mappings in a Hilbert space. We derive a strong convergence theorem for this new algorithm under appropriate assumptions. Moreover, in support of our results, some numerical examples (using Matlab software) are also presented.

1 Introduction

Throughout the paper unless otherwise stated, H denotes a real Hilbert space, we denote the norm and inner product of H by $\langle .,. \rangle$ and $\|.\|$, respectively. The set C (C be a nonempty closed convex subset of H) is called proximinal if for each $x \in H$, there exists an element $y \in C$ such that $\|x - y\| = d(x, C)$, where $d(x, C) = \inf\{\|x - z\| : z \in C\}$. Let $CB(D)$, $K(C)$ and $P(C)$ be the families of nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximinal bounded subsets of C, respectively. The Hausdorff metric on $CB(C)$ is defined by

$$H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\}, \quad A, B \in CB(C).$$

A multi-valued mapping $T : C \to CB(C)$ is said to be nonexpansive if $H(Tx, Ty) \leq \|x - y\|$ for all $x, y \in C$. An element $p \in C$ is called a fixed point of $T : C \to CB(C)$ if $p \in Tp$. The fixed points set of T is denoted by $\mathrm{Fix}(T)$.

A mapping $M : C \to H$ is said to be monotone, if

$$\langle Mx - My, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

M is called α-inverse-strongly-monotone if there exist a positive real number α such that

$$\langle Mx - My, x - y\rangle \geq \alpha\|Mx - My\|^2, \quad \forall x, y \in C.$$

* Corresponding Author's E-mail: sahebi.aiau.ac.ir@gmail.com

It is obvious that any α-inverse strongly monotone mapping M is monotone and Lipschitz continuous.

Ceng et.al [5], introduced the following generalized mixed equilibrium problem with perturbation: Find $x^* \in C$ such that

$$f(x^*, y) + \langle (A+B)x^*, y - x^* \rangle \geq 0, \ \ \forall y \in C. \tag{1.1}$$

where $A, B : C \to H$ are nonlinear mappings, $\phi : C \to R$ is a function and $f : C \times C \to R$ is a bifunction. The problem (1.1)is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problems in noncooperative games and others [2, 3, 4, 15, 18]

Very recently, Azhini and Taherian [16], motivated by [5, 19], proposed the following iteration process for finding a common element of the set of solutions of variational inequality (1.1) and the set of common fixed points of infinitely many nonexpansive mappings $\{S_n\}$ of C into itself and proved the strong convergence of the sequence generated by this iteration process to an element of $F(P_C S) = \bigcap_{n=1}^{\infty} F(P_C S_n)$.

$$\begin{cases} F(u_n, y) + \langle (M+N)x_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, & \forall y \in C \\ x_{n+1} = \beta_n P_C f(x_n) + \gamma_n x_n + \lambda_n P_C S_n[\alpha_n z + (1-\alpha_n)u_n], & \forall n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $\beta_n + \gamma_n + \lambda_n = 1$.

Motivated and inspired by Azhini and Taherian [16], Ceng et.al [5] and Takahashi [19] we introduce the iterative algorithm for finding a common element of the set of fixed point of a nonexpansive set-valued mapping in a real Hilbert space. Some strong convergence theorems and lemmas of the proposed algorithm are proven under new techniques and some mild assumption on the control conditions. Finally, some numerical examples that show the efficiency and implementation of our algorithm are presented.

The paper is structured as follows. In Section 2, we collect some lemmas, which are essential to prove our main results. In Section 3, we introduce a new algorithm for finding a common element of the set of fixed point of a nonexpansive set-valued mapping in a real Hilbert space. Then, we establish and prove the strong convergence theorem under some proper conditions. In Section 4, we also give some numerical examples to support our main theorem.

2 Preliminaries

Let H be a Hilbert space and C be a nonempty closed and convex subset of H.For each point $x \in H$, there exists a unique nearest point of C, denote by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. P_C is called the metric projection of H onto C. It is well known that P_C is nonexpansive mapping. Recall that a mapping $T : H \to H$ is said to be firmly nonexpansive if

$$\langle Tx - Ty, x - y \rangle \geq \|Tx - Ty\|^2, \qquad \forall x, y \in H.$$

It is also known that H satisfies Opial's condition [13], i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \tag{2.1}$$

holds for every $y \in H$ with $y \neq x$.

The following lemmas will be used for proving the convergence result of this paper in the sequel.

Lemma 2.1. *[1] Let C be a nonempty and weakly compact subset of a Banach space E with the Opial condition and $T : C \to K(E)$ a nonexpansive mapping. Then $I - T$ is demiclosed.*

Lemma 2.2. *[6] The following inequality holds in real space H:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H.$$

Lemma 2.3. *[7] Let C be a closed and convex subset of a real Hilbert space H and $T : C \to CB(C)$ be a nonexpansive multi-valued map with Fix$(T) \neq \emptyset$, and $Tp = \{p\}$ for each $p \in$ Fix(T). Then Fix(T) is a closed and convex subset of C.*

Lemma 2.4. *[9] Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 2 and let T_r^F be defined as in Lemma 2.5, for $r > 0$. Let $x, y \in H$ and $t, s > 0$. Then,*

$$\|T_s^F y - T_t^F x\| \leq \|x - y\| + |\frac{s - t}{s}|\|T_s^F y - y\|.$$

Lemma 2.5. *[10] Let C be a nonempty, closed convex subset of H and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 2. Then for $r > 0$ and $x \in H$, there exists $z \in C$ such that $F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C$.*
Further define

$$T_r^F x = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0\}, \ \ \forall y \in C$$

for all $r > 0$ and $x \in H$. Then, the following hold:

$(i) T_r^F \ is \ single-valued.$

$(ii) T_r^F \ is \ firmly \ nonexpansive, i.e.,$

$$\|T_r^F(x) - T_r^F(y)\|^2 \leq \langle T_r^F(x) - T_r^F(y), x - y\rangle, \ \ \forall x, y \in H.$$

(iii)***Fix***$(T_r^F) =$ ***EP***(F).

(iv)***EP***(F) $is \ compact \ and \ convex.$

Lemma 2.6. *[11] Assume that B is a strong positive linear bounded self adjoint operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|B\|^{-1}$.*

Lemma 2.7. *[12, 17] Let C be a closed and convex subset of a real Hilbert space H and let P_C be the metric projection from H onto C. Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if*

$$\langle x - z, y - z\rangle \leq 0, \qquad \forall y \in C.$$

Lemma 2.8. *[14] Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$, for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.9. *[20] Let $\{a_n\}$ be a sequence of nonnegative real numbers such that $a_{n+1} \leq (1-\alpha_n)a_n + \delta_n,\ \ n \geq 0$ where α_n is a sequence in $(0,1)$ and δ_n is a sequence in $\mathbb{R}$ such that*

(i) $\Sigma_{n=1}^{\infty}\alpha_n = \infty$; (ii) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ or $\Sigma_{n=1}^{\infty}\delta_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.10. *[19] Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 2 and let T_r^F be defined as in Lemma 2.5, for $r > 0$. Let $x \in H$ and $s, t > 0$. Then,*

$$\|T_s^F x - T_t^F x\|^2 \leq \frac{s-t}{s}\langle T_s^F(x) - T_t^F(x), T_s^F(x) - x\rangle.$$

Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the following assumptions:

1. $F(x,x) \geq 0,\ \ \forall x \in C$,

2. F is monotone, i.e., $F(x,y) + F(y,x) \leq 0,\ \ \forall x \in C$,

3. F is upper hemicontinuouse, i.e., for each $x, y, z \in C$,

   $\limsup_{t\to 0} F(tz + (1-t)x, y) \leq F(x,y)$,

4. For each $x \in C$ fixed, the function $x \to F(x,y)$ is convex and lower semicontinuous;

3 A Nonlinear Iterative Algorithm

Let C be a nonempty closed convex subset of real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 2. Let M, N be two $\bar{\alpha}$-inverse strongly monotone and $\bar{\beta}$-inverse strongly monotone mappings from C into H, respectively. Let T be a nonexpansive multi-valued mapping on C into $K(H)$ such that $\Theta = \mathrm{Fix}(T) \cap \mathrm{GEPP} \neq \emptyset$. Also $f : C \to H$ be a α-contraction mapping and A, B be a strongly positive bounded linear self adjoint operators on H with coefficient $\bar{\gamma}_1 > 0$ and $\bar{\gamma}_2 > 0$ respectively such that $0 < \gamma < \frac{\bar{\gamma}_1}{\alpha} < \gamma + \frac{1}{\alpha}$, $\bar{\gamma}_1 \leq \|A\| \leq 1$ and $\|B\| = \bar{\gamma}_2$. For given $x_0 \in C$ arbitrary, let the sequence $\{x_n\}$ be generated by:

$$\begin{cases} u_n = T_{r_n}^F(x_n - r_n(M+N)x_n); \\ \\ x_{n+1} = \alpha_n\gamma f(x_n) + \beta_n Bx_n + ((1-\epsilon_n)I - \beta_n B - \alpha_n A)z_n, \end{cases} \tag{3.1}$$

where $z_n \in Tu_n$ such that $\|z_{n+1} - z_n\| \leq H(Tu_{n+1}, Tu_n)$.
Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\epsilon_n\}$ are sequences in $(0,1)$, $\{r_n\} \subset [r, \infty)$ with $r > 0$ satisfied the following conditions:

$(C1)\lim_{n\to\infty}\alpha_n=0,\ \ \Sigma_{n=1}^{\infty}\alpha_n=\infty;$

$(C2)\limsup_{n\to\infty}\beta_n\neq 1;$

$(C3)\lim_{n\to\infty}|r_{n+1}-r_n|=0,\ \ \liminf_{n\to\infty}r_n>0,\ \ 0<b<r_n<a<2min\{\bar{\alpha},\bar{\beta}\}.$

Lemma 3.1. *Let $p\in\Theta=Fix(T)\cap GEPP$. Then the sequence $\{x_n\}$ generated by Algorithm 3 is bounded.*

proof: We may assume without loss of generality that $\alpha_n\leq(1-\epsilon_n-\beta_n\|B\|)\|A\|^{-1}$. Since A and B are linear bounded self adjoint operators, we have

$$\|A\|=\sup\{|\langle Ax,x\rangle|:x\in H,\|x\|=1\},$$
$$\|B\|=\sup\{|\langle Bx,x\rangle|:x\in H,\|x\|=1\}$$

observe that

$$\begin{aligned}\langle((1-\epsilon_n)I-\beta_nB-\alpha_nA)x,x\rangle &=(1-\epsilon_n)\langle x,x\rangle-\beta_n\langle Bx,x\rangle-\alpha_n\langle Ax,x\rangle\\&\geq 1-\epsilon_n-\beta_n\|B\|-\alpha_n\|A\|\\&\geq 0.\end{aligned}$$

Therefore, $(1-\epsilon_n)I-\beta_nB-\alpha_nA$ is positive. Then, by strong positivity of A and B, we get

$$\begin{aligned}\|(1-\epsilon_n)I-\beta_nB-\alpha_nA\| &=\sup\{\langle((1-\epsilon_n)I-\beta_nB-\alpha_nA)x,x\rangle\ x\in H,\|x\|=1\}\\&=\sup\{(1-\epsilon_n)\langle x,x\rangle-\beta_n\langle Bx,x\rangle-\alpha_n\langle Ax,x\rangle:x\in H,\|x\|=1\}\\&\leq 1-\epsilon_n-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1\\&\leq 1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1.\end{aligned}\tag{3.2}$$

Let $p\in\Theta:=\text{Fix}(T)\cap\text{GEPP}$. Since $p\in\text{GEPP}$, from Theorem 3.1 [16] we have

$$\begin{aligned}\|u_n-p\|^2 &\leq\|x_n-p\|^2+r_n(r_n-2\bar{\alpha})\|Mx_n-Mp\|^2+r_n(r_n-2\bar{\beta})\|Nx_n-Np\|^2\\&\leq\|x_n-p\|^2.\end{aligned}\tag{3.3}$$

Then

$$\|u_n-p\|\leq\|x_n-p\|.$$

We obtain

$$\begin{aligned}
\|x_{n+1}-p\| &= \|\alpha_n\gamma f(x_n)+\beta_n Bx_n+((1-\epsilon_n)I-\beta_n B-\alpha_n A)z_n-p\| \\
&\le \alpha_n\|\gamma f(x_n)-Ap\|+\beta_n\|Bx_n-Bp\|+\epsilon_n\|p\| \\
&\quad +\|((1-\epsilon_n)I-\beta_n B-\alpha_n A)\|\|z_n-p\| \\
&\le \alpha_n(\|\gamma f(x_n)-\gamma f(p)\|+\|\gamma f(p)-Ap\|)+\beta_n\|Bx_n-Bp\|+\epsilon_n\|p\| \\
&\quad +(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)d(z_n,Tp) \\
&\le \alpha_n\gamma\alpha\|x_n-p\|+\alpha_n\|\gamma f(p)-Ap\|+\beta_n\bar{\gamma}_2\|x_n-p\|+\alpha_n\|p\| \\
&\quad +(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)H(Tu_n,Tp) \\
&\le \alpha_n\gamma\alpha\|x_n-p\|+\alpha_n\|\gamma f(p)-Ap\|+\beta_n\bar{\gamma}_2\|x_n-p\|+\alpha_n\|p\| \\
&\quad +(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\|u_n-p\| \\
&\le (1-(\bar{\gamma}_1-\gamma\alpha)\alpha_n)\|x_n-p\|+\alpha_n(\|p\|+\|\gamma f(p)-Ap\|) \\
&\le \max\{\|x_n-p\|,\tfrac{\|\gamma f(p)-Ap\|+\|p\|}{\bar{\gamma}_1-\gamma\alpha}\} \\
&\ \vdots \\
&\le \max\{\|x_0-p\|,\tfrac{\|\gamma f(p)-Ap\|+\|p\|}{\bar{\gamma}_1-\gamma\alpha}\}.
\end{aligned} \tag{3.4}$$

Hence $\{x_n\}$ is bounded.This implies that the sequences $\{u_n\}$, $\{z_n\}$ and $\{f(x_n)\}$ are bounded.

Lemma 3.2. *The following properties are satisfying for the Algorithm 3*

$P1.$ $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.$

$P2.$ $\lim_{n\to\infty}\|x_n-t_n\|=0.$

$P3.$ $\lim_{n\to\infty}\|Mx_n-Mp\|=0 \quad and \quad \lim_{n\to\infty}\|Nx_n-Np\|=0.$

$P4.$ $\lim_{n\to\infty}\|x_n-u_n\|=0.$

proof: P1:We have

$$\begin{aligned}
\|u_{n+1}-u_n\| &= \|T_{r_{n+1}}(x_{n+1}-r_{n+1}(M+N)x_{n+1})-T_{r_n}(x_n-r_n(M+N)x_n)\| \\
&\leq \|(x_{n+1}-r_{n+1}(M+N)x_{n+1})-(x_n-r_n(M+N)x_n)\| \\
&\quad +\tfrac{|r_{n+1}-r_n|}{r_{n+1}}\|T_{r_{n+1}}(x_{n+1}-r_{n+1}(M+N)x_{n+1})-(x_{n+1}-r_{n+1}(M+N)x_{n+1})\| \\
&\leq \|x_{n+1}-x_n\|+|r_{n+1}-r_n|\|(M+N)(x_{n+1}-x_n)\|+\tfrac{|r_{n+1}-r_n|}{r_{n+1}}\sigma_{n+1}
\end{aligned} \tag{3.5}$$

where

$\sigma_{n+1}=\sup_{n\in\mathbb{N}}\|T_{r_{n+1}}(x_{n+1}-r_{n+1}(M+N)x_{n+1})-(x_{n+1}-r_{n+1}(M+N)x_{n+1})\|$.

Setting $x_{n+1}=\epsilon_n x_n+(1-\epsilon_n)e_n$, then we have

$$\begin{aligned}
e_{n+1}-e_n &= \tfrac{\alpha_{n+1}\gamma f(x_{n+1})+\beta_{n+1}Bx_{n+1}+((1-\epsilon_{n+1})I-\beta_{n+1}B-\alpha_{n+1}A)z_{n+1}-\epsilon_{n+1}x_{n+1}}{1-\epsilon_{n+1}} \\
&\quad -\tfrac{\alpha_n\gamma f(x_n)+\beta_nBx_n+((1-\epsilon_n)I-\beta_nB-\alpha_nA)z_n-\epsilon_nx_n}{1-\epsilon_n} \\
&= \tfrac{\alpha_{n+1}}{1-\epsilon_{n+1}}(\gamma f(x_{n+1})-Az_{n+1})+\tfrac{\alpha_n}{1-\epsilon_n}(Az_n-\gamma f(x_n)) \\
&\quad +(\tfrac{\beta_{n+1}}{1-\epsilon_{n+1}}-\tfrac{\beta_n}{1-\epsilon_n})B(x_{n+1}-x_n)+(z_{n+1}-z_n) \\
&\quad +(\tfrac{\beta_n}{1-\epsilon_n}-\tfrac{\beta_{n+1}}{1-\epsilon_{n+1}})B(z_{n+1}-z_n)+(\tfrac{\epsilon_n}{1-\epsilon_n}-\tfrac{\epsilon_{n+1}}{1-\epsilon_{n+1}})(x_n-x_{n+1}).
\end{aligned}$$

Using (3.5), we have

$$\begin{aligned}
&\|e_{n+1}-e_n\| \\
&\leq \tfrac{\alpha_{n+1}}{1-\epsilon_{n+1}}\|\gamma f(x_{n+1})-Az_{n+1}\|+\tfrac{\alpha_n}{1-\epsilon_n}\|\gamma f(x_n)-Az_n\|+|\tfrac{\beta_{n+1}}{1-\epsilon_{n+1}}-\tfrac{\beta_n}{1-\epsilon_n}|\|B\|\|x_{n+1}-x_n\| \\
&\quad+\|z_{n+1}-z_n\|+|\tfrac{\beta_n}{1-\epsilon_n}-\tfrac{\beta_{n+1}}{1-\epsilon_{n+1}}|\|B\|\|z_{n+1}-z_n\|+|\tfrac{\epsilon_{n+1}}{1-\epsilon_{n+1}}-\tfrac{\epsilon_n}{1-\epsilon_n}|\|x_{n+1}-x_n\| \\
&\leq \tfrac{\alpha_{n+1}}{1-\epsilon_{n+1}}\|\gamma f(x_{n+1})-Az_{n+1}\|+\tfrac{\alpha_n}{1-\epsilon_n}\|\gamma f(x_n)-Az_n\|+|\tfrac{\beta_{n+1}}{1-\epsilon_{n+1}}-\tfrac{\beta_n}{1-\epsilon_n}|\|B\|\|x_{n+1}-x_n\| \\
&\quad+H(Tu_{n+1},Tu_n)+|\tfrac{\beta_n}{1-\epsilon_n}-\tfrac{\beta_{n+1}}{1-\epsilon_{n+1}}|\|B\|H(Tu_{n+1},Tu_n)+|\tfrac{\epsilon_{n+1}}{1-\epsilon_{n+1}}-\tfrac{\epsilon_n}{1-\epsilon_n}|\|x_{n+1}-x_n\| \\
&\leq \tfrac{\alpha_{n+1}}{1-\epsilon_{n+1}}\|\gamma f(x_{n+1})-Az_{n+1}\|+\tfrac{\alpha_n}{1-\epsilon_n}\|\gamma f(x_n)-Az_n\|+|\tfrac{\beta_{n+1}}{1-\epsilon_{n+1}}-\tfrac{\beta_n}{1-\epsilon_n}|\|B\|\|x_{n+1}-x_n\| \\
&\quad+\|u_{n+1}-u_n\|+|\tfrac{\beta_n}{1-\epsilon_n}-\tfrac{\beta_{n+1}}{1-\epsilon_{n+1}}|\|B\|\|u_{n+1}-u_n\|+|\tfrac{\epsilon_{n+1}}{1-\epsilon_{n+1}}-\tfrac{\epsilon_n}{1-\epsilon_n}|\|x_{n+1}-x_n\| \\
&\leq \tfrac{\alpha_{n+1}}{1-\epsilon_{n+1}}\|\gamma f(x_{n+1})-Az_{n+1}\|+\tfrac{\alpha_n}{1-\epsilon_n}\|\gamma f(x_n)-Az_n\|+|\tfrac{\beta_{n+1}}{1-\epsilon_{n+1}}-\tfrac{\beta_n}{1-\epsilon_n}|\|x_{n+1}-x_n\| \\
&\quad+\|x_{n+1}-x_n\|+|r_{n+1}-r_n|\|(M+N)(x_{n+1}-x_n)\|+\tfrac{|r_{n+1}-r_n|}{r_{n+1}}\sigma_{n+1} \\
&\quad+|\tfrac{\beta_n}{1-\epsilon_n}-\tfrac{\beta_{n+1}}{1-\epsilon_{n+1}}|\bar{\gamma}_2(\|x_{n+1}-x_n\|+|r_{n+1}-r_n|\|(M+N)(x_{n+1}-x_n)\|+\tfrac{|r_{n+1}-r_n|}{r_{n+1}}\sigma_{n+1}) \\
&\quad+|\tfrac{\epsilon_{n+1}}{1-\epsilon_{n+1}}-\tfrac{\epsilon_n}{1-\epsilon_n}|\|x_{n+1}-x_n\|,
\end{aligned}$$

which implies

$$\begin{aligned}
&\|e_{n+1}-e_n\|-\|x_{n+1}-x_n\| \\
&\leq \tfrac{\alpha_{n+1}}{1-\epsilon_{n+1}}\|\gamma f(x_{n+1})-Az_{n+1}\|+\tfrac{\alpha_n}{1-\epsilon_n}\|\gamma f(x_n)-Az_n\|+|\tfrac{\beta_{n+1}}{1-\epsilon_{n+1}}-\tfrac{\beta_n}{1-\epsilon_n}|\|x_{n+1}-x_n\| \\
&\quad+|r_{n+1}-r_n|\|(M+N)(x_{n+1}-x_n)\|+\tfrac{|r_{n+1}-r_n|}{r_{n+1}}\sigma_{n+1} \\
&\quad+|\tfrac{\beta_n}{1-\epsilon_n}-\tfrac{\beta_{n+1}}{1-\epsilon_{n+1}}|\bar{\gamma}_2(\|x_{n+1}-x_n\|+|r_{n+1}-r_n|\|(M+N)(x_{n+1}-x_n)\|+\tfrac{|r_{n+1}-r_n|}{r_{n+1}}\sigma_{n+1}) \\
&\quad+|\tfrac{\epsilon_{n+1}}{1-\epsilon_{n+1}}-\tfrac{\epsilon_n}{1-\epsilon_n}|\|x_{n+1}-x_n\|.
\end{aligned}$$

Hence, it follows by conditions $(C1)-(C4)$ that

$$\limsup_{n\to\infty}(\|e_{n+1}-e_n\|-\|x_{n+1}-x_n\|)\leq 0. \tag{3.6}$$

From (3.6) and Lemma 2.8, we get $\lim_{n\to\infty}\|e_n-x_n\|=0$, and then

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\epsilon_n)\|e_n-x_n\| = 0. \tag{3.7}$$

P2: We can write

$$\begin{aligned}
\|x_n-z_n\| &\le \|x_{n+1}-x_n\| + \|\alpha_n\gamma f(x_n)+\beta_n Bx_n+((1-\epsilon_n)I-\beta_n B-\alpha_n A)z_n - z_n\| \\
&\le \|x_{n+1}-x_n\| + \alpha_n\|\gamma f(x_n)-Az_n\| + \beta_n\|Bx_n-Bz_n\| + \epsilon_n\|z_n\| \\
&\le \|x_{n+1}-x_n\| + \alpha_n\|\gamma f(x_n)-Az_n\| + \beta_n\bar{\gamma}_2\|x_n-z_n\| + \epsilon_n\|z_n\|.
\end{aligned}$$

Then

$$(1-\beta_n\bar{\gamma}_2)\|x_n-z_n\| \le \|x_{n+1}-x_n\| + \alpha_n\|\gamma f(x_n)-Az_n\| + \epsilon_n\|z_n\|.$$

Therefore

$$\begin{aligned}
\|x_n-z_n\| &\le \tfrac{1}{1-\beta_n\bar{\gamma}_2}\|x_{n+1}-x_n\| + \tfrac{\alpha_n}{1-\beta_n\bar{\beta}}\|\gamma f(x_n)-At_n\| + \tfrac{\epsilon_n}{1-\beta_n\bar{\gamma}_2}\|z_n\| \\
&\le \tfrac{1}{1-\beta_n\bar{\gamma}_2}\|x_{n+1}-x_n\| + \tfrac{\alpha_n}{1-\beta_n\bar{\gamma}_2}(\|\gamma f(x_n)-Az_n\| + \|z_n\|).
\end{aligned}$$

Since $\alpha_n \to 0$, $\|x_{n+1}-x_n\| \to 0$ and $(C2)$ we obtain

$$\lim_{n\to\infty}\|x_n-z_n\| = 0. \tag{3.8}$$

P3: From (3.3), we have

$$
\begin{aligned}
&\|x_{n+1}-p\|^2\\
&=\|\alpha_n\gamma f(x_n)+\beta_nBx_n+((1-\epsilon_n)I-\beta_nB-\alpha_nA)z_n-p\|^2\\
&=\|\alpha_n(\gamma f(x_n)-Ap)+\beta_n(Bx_n-Bp)+((1-\epsilon_n)I-\beta_nB-\alpha_nA)(z_n-p)-\epsilon_np\|^2\\
&\leq\|((1-\epsilon_n)I-\beta_nB-\alpha_nA)(z_n-p)+\beta_n(Bx_n-Bp)-\epsilon_np\|^2\\
&\quad+2\langle\alpha_n(\gamma f(x_n)-Ap),x_{n+1}-p\rangle\\
&\leq((1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)d(z_n,Tp)+\beta_n\|B\|\|x_n-z_n\|+\epsilon_n\|p\|)^2+2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle\\
&\leq((1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)H(Tu_n,Tp)+\beta_n\|B\|\|x_n-z_n\|+\epsilon_n\|p\|)^2\\
&\quad+2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle\\
&\leq((1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\|u_n-p\|+\beta_n\|B\|\|x_n-z_n\|+\epsilon_n\|p\|)^2\\
&\quad+2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle\\
&=(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)^2\|u_n-p\|^2+(\beta_n)^2\|B\|^2\|x_n-z_n\|^2+(\epsilon_n)^2\|p\|^2\\
&\quad+2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\beta_n\|B\|\|u_n-p\|\|x_n-z_n\|\\
&\quad+2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\epsilon_n\|p\|\|u_n-p\|+2\beta_n\epsilon_n\|B\|\|p\|\|x_n-z_n\|\\
&\quad+2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle
\end{aligned}
\tag{3.9}
$$

$$\leq (1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)^2(\|x_n-p\|^2+r_n(r_n-2\bar{\alpha})\|Mx_n-Mp\|^2+r_n(r_n-2\bar{\beta})\|Nx_n-Np\|^2)$$

$$+(\beta_n)^2\|B\|^2\|x_n-z_n\|^2+(\epsilon_n)^2\|p\|^2+2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\beta_n\|B\|\|u_n-p\|\|x_n-z_n\|$$

$$+2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\epsilon_n\|p\|\|u_n-p\|+2\beta_n\epsilon_n\|B\|\|p\|\|x_n-z_n\|$$

$$+2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle$$

$$\leq \|x_n-p\|^2+(\beta_n\bar{\gamma}_2+\alpha_n\bar{\gamma}_1)^2\|x_n-p\|^2$$

$$+(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)^2(r_n(r_n-2\bar{\alpha})\|Mx_n-Mp\|^2+r_n(r_n-2\bar{\beta})\|Nx_n-Np\|^2)$$

$$+(\beta_n)^2\|B\|^2\|x_n-z_n\|^2+(\epsilon_n)^2\|p\|^2+2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\beta_n\|B\|\|u_n-p\|\|x_n-z_n\|$$

$$+2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\epsilon_n\|p\|\|u_n-p\|+2\beta_n\epsilon_n\|B\|\|p\|\|x_n-z_n\|$$

$$+2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle.$$

By $(C3)$, we can write

$$(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)^2(r_n(2\bar{\alpha}-r_n)\|Mx_n-Mp\|^2+r_n(2\bar{\beta}-r_n)\|Nx_n-Np\|^2)$$

$$\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+(\beta_n\bar{\gamma}_2+\alpha_n\bar{\gamma}_1)^2\|x_n-p\|^2+(\beta_n)^2\|B\|^2\|x_n-z_n\|^2+(\alpha_n)^2\|p\|^2$$

$$+2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\beta_n\|B\|\|u_n-p\|\|x_n-z_n\|+2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\alpha_n\|p\|\|u_n-p\|$$

$$+2\beta_n\epsilon_n\|B\|\|p\|\|x_n-z_n\|+2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle$$

$$\leq (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+(\beta_n\bar{\gamma}_2+\alpha_n\bar{\gamma}_1)^2\|x_n-p\|^2+(\beta_n)^2\|B\|^2\|x_n-z_n\|^2$$

$$+(\alpha_n)^2\|p\|^2+2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\beta_n\|B\|\|u_n-p\|\|x_n-z_n\|$$

$$+2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\alpha_n\|p\|\|u_n-p\|+2\beta_n\epsilon_n\|B\|\|p\|\|x_n-z_n\|$$

$$+2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle.$$

By $\alpha_n\to 0$, $\|x_{n+1}-x_n\|\to 0$ and $\|x_n-z_n\|\to 0$, then we obtain $\|Mx_n-Mp\|\to 0$ and $\|Nx_n-Np\|\to 0$ as $n\to\infty$.

P4: Since $p\in\Theta=\text{Fix}(T)\cap\text{GEPP}$, we can obtain

$$\|u_n-p\|^2\leq\|x_n-p\|^2-\|u_n-x_n\|^2+2r_n\|u_n-x_n\|(\|Mx_n-Mp\|+\|Nx_n-Np\|),$$

. It followes from (3.9) that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq (1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)^2\|u_n-p\|^2+(\beta_n)^2\|B\|^2\|x_n-z_n\|^2+(\epsilon_n)^2\|p\|^2\\
&\quad +2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\beta_n\|B\|\|u_n-p\|\|x_n-z_n\|\\
&\quad +2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\epsilon_n\|p\|\|u_n-p\|+2\beta_n\epsilon_n\|B\|\|p\|\|x_n-z_n\|\\
&\quad +2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle\\
&\leq (1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)^2(\|x_n-p\|^2-\|u_n-x_n\|^2\\
&\quad +2r_n\|u_n-x_n\|(\|Mx_n-Mp\|+\|Nx_n-Np\|))+(\beta_n)^2\|B\|^2\|x_n-z_n\|^2\\
&\quad +(\epsilon_n)^2\|p\|^2+2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\beta_n\|B\|\|u_n-p\|\|x_n-z_n\|\\
&\quad +2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\epsilon_n\|p\|\|u_n-p\|+2\beta_n\epsilon_n\|B\|\|p\|\|x_n-z_n\|\\
&\quad +2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle.\end{aligned}$$

Therefore

$$\begin{aligned}&(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)^2\|u_n-x_n\|^2\\
&\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+(\beta_n\bar{\gamma}_2+\alpha_n\bar{\gamma}_1)^2\|x_n-p\|^2\\
&\quad +2r_n(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)^2\|u_n-x_n\|(\|Mx_n-Mp\|+\|Nx_n-Np\|)+(\beta_n)^2\|B\|^2\|x_n-z_n\|^2\\
&\quad +(\epsilon_n)^2\|p\|^2+2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\beta_n\|B\|\|u_n-p\|\|x_n-z_n\|\\
&\quad +2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\epsilon_n\|p\|\|u_n-p\|+2\beta_n\epsilon_n\|B\|\|p\|\|x_n-z_n\|\\
&\quad +2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle\\
&\leq (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+(\beta_n\bar{\gamma}_2+\alpha_n\bar{\gamma}_1)^2\|x_n-p\|^2\\
&\quad +2r_n(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)^2\|u_n-x_n\|(\|Mx_n-Mp\|+\|Nx_n-Np\|)+(\beta_n)^2\|B\|^2\|x_n-z_n\|^2\\
&\quad +(\epsilon_n)^2\|p\|^2+2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\beta_n\|B\|\|u_n-p\|\|x_n-z_n\|\\
&\quad +2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\epsilon_n\|p\|\|u_n-p\|+2\beta_n\epsilon_n\|B\|\|p\|\|x_n-z_n\|\\
&\quad +2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle.\end{aligned}$$

Since $\alpha_n \to 0$, $\|x_{n+1}-x_n\| \to 0$, $\|Mx_n-Mp\| \to 0$, $\|Nx_n-Np\| \to 0$ and $\|x_n-z_n\| \to 0$ as $n \to \infty$ and we obtain

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.10}$$

Using (3.8) and (3.10), we obtain

$\|z_n - u_n\| \leq \|z_n - x_n\| + \|x_n - u_n\| \to 0$, as $n \to \infty$. Then $\lim_{n\to\infty} \|z_n - u_n\| = 0$.

4 Strong Convergence Algorithm

Theorem 4.1. *The Algorithm defined by (3.1) convergence strongly to $z \in$ Fix$(T) \cap$ GEPP, which is a unique solution in of the variational inequality*

$$\langle(\gamma f - A)z, y - z\rangle \leq 0, \quad \forall y \in \Theta = \textit{Fix}(T) \cap \textit{GEPP}.$$

proof: Let $s = P_\Theta$. We get

$$\begin{aligned}\|s(I - A + \gamma f)(x) - s(I - A + \gamma f)(y)\| &\leq \|(I - A + \gamma f)(x) - (I - A + \gamma f)(y)\| \\ &\leq \|I - A\|\|x - y\| + \gamma\|f(x) - f(y)\| \\ &\leq (1 - \bar{\gamma}_1)\|x - y\| + \gamma\alpha\|x - y\| \\ &= (1 - (\bar{\gamma}_1 - \gamma\alpha))\|x - y\|.\end{aligned}$$

Then $s(I - A + \gamma f)$ is a contraction mapping from H into itself. Therefore by Banach contraction principle, there exists $z \in H$ such that $z = s(I - A + \gamma f)z = P_{\text{Fix}(T)\cap\text{EPP}}(I - A + \gamma f)z$.

We show that $\langle(\gamma f - A)z, x_n - z\rangle \leq 0$. To show this inequality, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle(\gamma f - A)z, x_n - z\rangle = \lim_{i\to\infty}\langle(\gamma f - A)z, x_{n_i} - z\rangle. \tag{4.1}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to some $w \in C$. Without loss of generality, we can assume that $x_{n_i} \rightharpoonup w$. Now, we prove that $w \in$ Fix$(S) \cap$ GEPP. Let us first show that $w \in$ Fix(S). From $\|x_n - u_n\| \to 0$, we obtain $u_{n_i} \rightharpoonup w$. On the other hand $\lim_{n\to\infty} \|z_n - u_n\| = 0$ and by Lemma 2.1, $I - T$ is demiclosed at 0. Thus, we obtain $w \in$ Fix(T). We show that $w \in$ GEPP. Since $u_n = T_{r_n}(x_n - r_n(M + N)x_n)$. we have

$$F(u_n, y) + \langle(M + N)x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \qquad \forall y \in C.$$

It follows from the monotonicity of F that

$$\langle(M + N)x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n), \qquad \forall y \in C$$

which implies that

$$\langle (M+N)x_{n_i}, y-u_{n_i}\rangle + \frac{1}{r_{n_i}}\langle y-u_{n_i}, u_{n_i}-x_{n_i}\rangle \geq F(y,u_{n_i}), \qquad \forall y\in C.$$

Let $u_t = ty+(1-t)w$ for all $t\in(0,1]$. Since $y\in C$ and $w\in C$, we get $u_t\in C$. It follows that

$$\begin{aligned}
\langle u_t-u_{n_i},(M+N)u_t\rangle &\geq \langle u_t-u_{n_i},(M+N)u_t\rangle - \langle u_t-u_{n_i},(M+N)x_{n_i}\rangle\\
&\quad -\langle u_t-u_{n_i},\tfrac{u_{n_i}-x_{n_i}}{r_{n_i}}\rangle + F(u_t,u_{n_i})\\
&= \langle u_t-u_{n_i},(M+N)u_t-(M+N)u_{n_i}\rangle\\
&\quad +\langle u_t-u_{n_i},(M+N)u_{n_i}-(M+N)x_{n_i}\rangle\\
&\quad -\langle u_t-u_{n_i},\tfrac{u_{n_i}-x_{n_i}}{r_{n_i}}\rangle + F(u_t,u_{n_i})\\
&= \langle u_t-u_{n_i},Mu_t-Mu_{n_i},\rangle + \langle u_t-u_{n_i},Nu_t-Nu_{n_i}\rangle\\
&\quad +\langle u_t-u_{n_i},Mu_{n_i}-Mx_{n_i},\rangle + \langle u_t-u_{n_i},Nu_{n_i}-Nx_{n_i}\rangle\\
&\quad -\langle u_t-u_{n_i},\tfrac{u_{n_i}-x_{n_i}}{r_{n_i}}\rangle + F(u_t,u_{n_i}).
\end{aligned}$$

Since $\|u_{n_i}-x_{n_i}\|\to 0$, we have $\|Mu_{n_i}-Mx_{n_i}\|\to 0$ and $\|Nu_{n_i}-Nx_{n_i}\|\to 0$. Further from monotonically of M and N, we obtain

$$\langle u_t-u_{n_i},Mu_t-Mu_{n_i}\rangle \geq 0, \qquad \langle u_t-u_{n_i},Nu_t-Nu_{n_i}\rangle \geq 0,$$

so as $i\to\infty$ from assumption 2, we have $\langle u_t-w,(M+N)u_t\rangle \geq F(u_t,w)$.
Therefore

$$\begin{aligned}
0=F(u_t,u_t) &\leq tF(u_t,y)+(1-t)F(u_t,w)\\
&\leq tF(u_t,y)+(1-t)\langle u_t-w,(M+N)u_t\rangle\\
&\leq tF(u_t,y)+(1-t)t\langle y-w,(M+N)u_t\rangle,
\end{aligned}$$

then $0\leq F(u_t,y)+(1-t)\langle y-w,(M+N)u_t\rangle$.

Letting $t\to 0$, we obtain $0\leq F(w,y)+\langle y-w,(M+N)w\rangle$. This implies that $w\in$ GEPP.
Now from Lemma 2.7, we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle(\gamma f-A)z,x_n-z\rangle &\leq \limsup_{i\to\infty}\langle(\gamma f-A)z,x_{n_i}-z\rangle\\
&= \langle(\gamma f-A)z,w-z\rangle \qquad (4.2)\\
&\leq 0.
\end{aligned}$$

Now we prove that x_n is strongly convergence to z.It follows from (3.3) that

$$\|x_{n+1}-z\|^2$$

$$= \alpha_n\langle \gamma f(x_n)-Az, x_{n+1}-z\rangle + \beta_n\langle Bx_n - Bz, x_{n+1}-z\rangle - \epsilon_n\langle z, x_{n+1}-z\rangle$$

$$+\langle((1-\epsilon_n)I-\beta_nB-\alpha_nA)(z_n-z), x_{n+1}-z\rangle$$

$$\leq \alpha_n(\gamma\langle f(x_n)-f(z), x_{n+1}-z\rangle + \langle \gamma f(z)-Az, x_{n+1}-z\rangle) + \beta_n\|B\|\|x_n-z\|\|x_{n+1}-z\|$$

$$-\epsilon_n\|z\|\|x_{n+1}-z\| + \|(1-\epsilon_n)I-\beta_nB-\alpha_nA\|\|z_n-z\|\|x_{n+1}-z\|$$

$$\leq \alpha_n(\gamma\langle f(x_n)-f(z), x_{n+1}-z\rangle + \langle \gamma f(z)-Az, x_{n+1}-z\rangle) + \beta_n\|B\|\|x_n-z\|\|x_{n+1}-z\|$$

$$-\epsilon_n\|z\|\|x_{n+1}-z\| + \|(1-\epsilon_n)I-\beta_nB-\alpha_nA\|d(z_n,Tz)\|x_{n+1}-z\|$$

$$\leq \alpha_n(\gamma\langle f(x_n)-f(z), x_{n+1}-z\rangle + \langle \gamma f(z)-Az, x_{n+1}-z\rangle) + \beta_n\|B\|\|x_n-z\|\|x_{n+1}-z\|$$

$$-\epsilon_n\|z\|\|x_{n+1}-z\| + \|(1-\epsilon_n)I-\beta_nB-\alpha_nA\|H(Tu_n,Tz)\|x_{n+1}-z\|$$

$$\leq \alpha_n\alpha\gamma\|x_n-z\|\|x_{n+1}-z\| + \alpha_n\langle \gamma f(z)-Az, x_{n+1}-z\rangle + \beta_n\bar{\beta}\|x_n-z\|\|x_{n+1}-z\|$$

$$-\epsilon_n\|z\|\|x_{n+1}-z\| + (1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\|x_n-z\|\|x_{n+1}-z\|$$

$$= (1-\alpha_n(\bar{\gamma}_1-\alpha\gamma))\|x_n-z\|\|x_{n+1}-z\| - \epsilon_n\|z\|\|x_{n+1}-z\| + \alpha_n\langle \gamma f(z)-Az, x_{n+1}-z\rangle$$

$$\leq \tfrac{1-\alpha_n(\bar{\gamma}_1-\alpha\gamma)}{2}(\|x_n-z\|^2+\|x_{n+1}-z\|^2) - \epsilon_n\|z\|\|x_{n+1}-z\| + \alpha_n\langle \gamma f(z)-Az, x_{n+1}-z\rangle$$

$$\leq \tfrac{1-\alpha_n(\bar{\delta}-\alpha\gamma)}{2}\|x_n-z\|^2 + \tfrac{1}{2}\|x_{n+1}-z\|^2 - \epsilon_n\|z\|\|x_{n+1}-z\| + \alpha_n\langle \gamma f(z)-Az, x_{n+1}-z\rangle.$$

This implies that

$$\begin{aligned} 2\|x_{n+1}-z\|^2 &\leq (1-\alpha_n(\bar{\gamma}_1-\alpha\gamma))\|x_n-z\|^2 + \|x_{n+1}-z\|^2 \\ &\quad -2\alpha_n\|z\|\|x_{n+1}-z\| + 2\alpha_n\langle \gamma f(z)-Az, x_{n+1}-z\rangle. \end{aligned}$$

Then

$$\begin{aligned} \|x_{n+1}-z\|^2 &\leq (1-\alpha_n(\bar{\gamma}_1-\alpha\gamma))\|x_n-z\|^2 - 2\alpha_n\|z\|\|x_{n+1}-z\| \\ &\quad +2\alpha_n\langle \gamma f(z)-Az, x_{n+1}-z\rangle \qquad (4.3) \\ &= (1-k_n)\|x_n-z\|^2 + 2\alpha_n l_n, \end{aligned}$$

where $k_n = \alpha_n(\bar{\gamma}_1-\alpha\gamma)$ and $l_n = \langle \gamma f(z)-Az, x_{n+1}-z\rangle - \|z\|\|x_{n+1}-z\|$.
Since $\lim_{n\to\infty}\alpha_n = 0$ and $\Sigma_{n=0}^{\infty}\alpha_n = \infty$, it is easy to see that $\lim_{n\to\infty} k_n = 0$, $\Sigma_{n=0}^{\infty}k_n = \infty$ and $\limsup_{n\to\infty} l_n \leq 0$.

Hence, from (4.2) and (4.3) and Lemma 2.9, we deduce that $x_n \to z$, where $z = P_\Theta(I - A + \gamma f)z$.

Remark 4.1. *Putting $A = B = M = N = 0, \gamma = 1$, we obtain methods introduced in Theorem 3.1 [8].*

5 Numerical Examples

In this section, we give some examples and numerical results for supporting our main theorem. All the numerical results have been produced in Matlab 2017 on a Linux workstation with a 3.8 GHZ Intel annex processor and 8 Gb of memory.

Example 5.1. Let $H = \mathbb{R}$, the set of all real numbers, with the inner product defined by $\langle x, y\rangle = xy, \ \forall x, y \in \mathbb{R}$, and induced usual norm $|\,.\,|$. Let $C = [0, 2]$; let $F : C \times C \to \mathbb{R}$ be defined by $F(x, y) = (x - 4)(y - x), \ \forall x, y \in C$; let $M, N : C \to H$ be defined by $M(x) = x$ and $N(x) = 2x, \ \forall x \in C$, such that $\bar{\alpha} = \frac{1}{2}$ and $\bar{\beta} = \frac{1}{3}$ respectively, and let for each $x \in \mathbb{R}$, we define $f(x) = \frac{1}{8}x, \ A(x) = 2x, \ B(x) = \frac{1}{3}x$ and

$$Tx = \begin{cases} \{x\}, & 0 \le x \le 1 \\ \{\frac{1}{2}\}, & 1 < x \le 2 \end{cases}$$

Then there exist unique sequences $\{x_n\} \subset \mathbb{R}$ and $\{u_n\} \subset C$ generated by the iterative schemes

$$u_n = T^F_{r_n}(x_n - r_n(M + N)x_n); \tag{5.1}$$

$$x_{n+1} = (\frac{1}{8n} + \frac{1}{3n^2})x_n + ((1 - \frac{1}{2n^2 - 3})I - \frac{1}{n^2}B - \frac{1}{n}A)z_n \tag{5.2}$$

where $\alpha_n = \frac{1}{n}, \ \beta_n = \frac{1}{n^2}, \ \epsilon_n = \frac{1}{2n^2-3}$ and $r_n = 1$. Then $\{x_n\}$ converges to $\{1\} \in \text{Fix}(T) \cap$ GEPP. *It is easy to prove that the bifunction F satisfy the Assumption 2. Further, f is contraction mapping with constant $\alpha = \frac{1}{5}$ and A is a strongly positive bounded linear operator with constant $\bar{\gamma}_1 = 1$ on $\mathbb{R}$. Therefore, we can choose $\gamma = 1$ which satisfies $0 < \gamma < \frac{\bar{\gamma}_1}{\alpha} < \gamma + \frac{1}{\alpha}$. Furthermore, it is easy to observe that Fix$(T) = [0, 1]$ and GEPP $= \{1\}$. Hence Fix$(T) \cap$ GEPP $= \{1\} \neq \emptyset$. After simplification, schemes (5.3) and (5.4) reduce to*

$$u_n = 2 - x_n$$

$$Tu_n = \begin{cases} \{2 - x_n\}, & 0 \le u_n \le 1 \quad or (1 \le x_n \le 2) \\ \{\frac{1}{2}\}, & 1 < u_n \le 2 \quad or (0 \le x_n < 1) \end{cases}$$

If $z_n = 2 - x_n$ for $x_n \in [1, 2]$, we have

$$x_{n+1} = (-1 + \frac{17}{8n} + \frac{2}{3n^2} + \frac{1}{2n^2 - 3})x_n + 2(1 - \frac{1}{2n^2 - 3} - \frac{1}{3n^2} - \frac{2}{n}).$$

If $z_n = \frac{1}{2}$ for $x_n \in [0, 1)$, we have

$$x_{n+1} = (\frac{1}{8n} + \frac{1}{3n^2})x_n + \frac{1}{2}(1 - \frac{1}{2n^2 - 3} - \frac{1}{3n^2} - \frac{2}{n}).$$

Following the proof of Theorem 4.1, we obtain that $\{x_n\}, \ \{u_n\}$ converges strongly to $w = \{1\} \in$ Fix$(T) \cap$ GEPP.

Figure 1 indicates the behavior of x_n with initial point $x_1 = 0.5$, which converges to the same solution, that is, $w = \{1\} \in$ Fix$(S) \cap$ GEPP as a solution of this example.

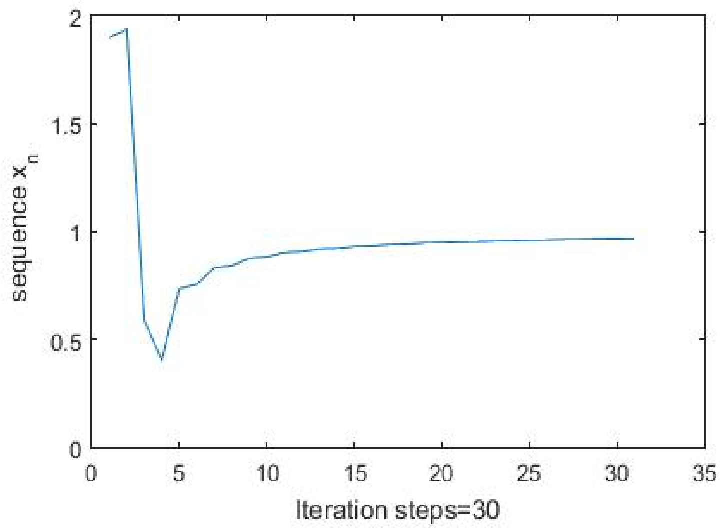

Figure 1: The graph of $\{x_n\}$ with initial value $x_1 = 0.5$.

Example 5.2. Let $H = \mathbb{R}$, the set of all real numbers, with the inner product defined by $\langle x, y\rangle = xy,\ \forall x, y \in \mathbb{R}$, and induced usual norm $|\ .\ |$. Let $C = [-1, 3]$; let $F : C \times C \to \mathbb{R}$ be defined by $F(x, y) = x(y - x),\ \forall x, y \in C$; let $M, N : C \to H$ be defined by $M(x) = 2x$ and $N(x) = 3x,\ \forall x \in C$, such that $\bar{\alpha} = \frac{1}{3}$ and $\bar{\beta} = \frac{1}{4}$ respectively, and let for each $x \in \mathbb{R}$, we define $f(x) = \frac{1}{6}x,\ A(x) = \frac{x}{2},\ B(x) = \frac{1}{10}x$ and

$$Tx = \begin{cases} \{\frac{x}{2}\}, & 0 < x \leq 3 \\ \{0\}, & -1 \leq x \leq 0 \end{cases}$$

Then there exist unique sequences $\{x_n\} \subset \mathbb{R}$ and $\{u_n\} \subset C$ generated by the iterative schemes

$$u_n = T^F_{r_n}(x_n - r_n(M + N)x_n); \tag{5.3}$$

$$x_{n+1} = (\frac{1}{3\sqrt{n}} + \frac{1}{10(n+1)^2})x_n + ((1 - \frac{2}{n^2})I - \frac{1}{(n+1)^2}B - \frac{1}{\sqrt{n}}A)z_n \tag{5.4}$$

where $\alpha_n = \frac{1}{\sqrt{n}},\ \beta_n = \frac{1}{(n+1)^2},\ \epsilon_n = \frac{2}{n^2}$ and $r_n = 1 + \frac{1}{n}$. Then $\{x_n\}$ converges to $\{0\} \in \text{Fix}(T) \cap \text{GEPP}$. *It is easy to prove that the bifunction F satisfy the Assumption 2. Further, f is contraction mapping with constant $\alpha = \frac{1}{5}$ and A is a strongly positive bounded linear operator with constant $\bar{\gamma}_1 = 1$ on $\mathbb{R}$. Therefore, we can choose $\gamma = 2$ which satisfies $0 < \gamma < \frac{\bar{\gamma}_1}{\alpha} < \gamma + \frac{1}{\alpha}$. Furthermore, it is easy to observe that Fix$(T) = \{0\}$ and GEPP $= \{0\}$. Hence Fix$(T) \cap$ GEPP $= \{0\} \neq \emptyset$. After simplification, schemes (5.3) and (5.4) reduce to*

$$u_n = (\frac{-4n - 5}{2n + 1})x_n$$

$$Tu_n = \begin{cases} \{0\}, & -15 \leq u_n < 0 \quad or(0 < x_n \leq 3) \\ \{(\frac{-4n-5}{4n+2})x_n\}, & 0 \leq u_n \leq 2 \quad or(-1 \leq x_n \leq 0) \end{cases}$$

If $z_n = \frac{-4n-5}{4n+2}x_n$ *for* $x_n \in [-1, 0]$*, we have*

$$x_{n+1} = (\frac{1}{3\sqrt{n}} + \frac{1}{10(n+1)^2})x_n + (1 - \frac{2}{n^2} - \frac{1}{10(n+1)^2} - \frac{1}{2\sqrt{n}})(\frac{-4n-5}{4n+2})x_n.$$

If $z_n = 0$ *for* $x_n \in (0, 3]$*, we have*

$$x_{n+1} = (\frac{1}{3\sqrt{n}} + \frac{1}{10(n+1)^2})x_n.$$

Following the proof of Theorem 4.1, we obtain that $\{x_n\},\ \{u_n\}$ *converges strongly to* $w = \{0\} \in$ *Fix*$(T) \cap$ *GEPP . Figure 2 indicates the behavior of* x_n *with initial point* $x_1 = 0.5$*, which converges to the same solution, that is,* $w = \{1\} \in$ *Fix*$(S) \cap$ *GEPP as a solution of this example.*

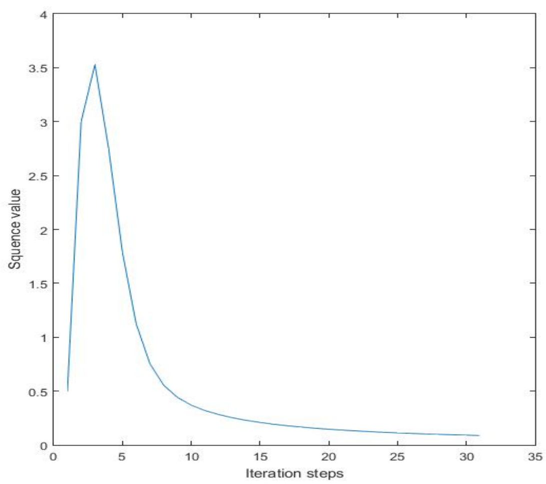

Figure 2: The graph of $\{x_n\}$ with initial value $x_1 = 0.5$.

References

[1] R. P. Agarwal, D. O'Regan, D. R. Sahu, *Fixed Point Theory for Lipschitzian-type Mappings with Applications*, Springer, 2009.

[2] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Stud., 63 (1994), 123-145.

[3] L. C. Ceng and J. C. Yao, *Hybrid viscosity approximation schemes for equilibrium problems and fixed point problems of infinitely many nonexpansive mappings*, Appl. Math. Comput., 198 (2008), 729-741.

[4] L. C. Ceng, S. Al-Homidan, Q. H. Ansari and J. C. Yao, *An iterative scheme for equilibrium problems and fixed point problems of strict pseudo-contraction mappings*, J. Comput. Appl. Math., 223 (2009), 967-974.

[5] L. C. Ceng, Hui-Ying Hu, M. M. Wong, *Strong and weak convergence theorems for generalized mixed equilibrium problem with perturbation and fixed point problem of infinitely many nonexpansive mappings*, Taiwanese. J. Math. vol. 15, no. 3, (2011) 1341-1367.

[6] S. S. Chang, J. Lee, H. W. Chan, *An new method for solving equilibrium problem, fixed point problem and variational inequality problem with application to optimization*, Nonlinear Analysis. 70(2009)3307-3319.

[7] W. Cholamjiak, S. Suantai, *A hybrid method for a countable family of multivalued maps, equilibrium problems, and variational inequality problems*, Discrete Dyn. Nat. Soc. Volume 2010 (2010), Article ID 349158, 14 pages.

[8] P. Cholamjiak, W. Cholamjiak, S. Suantai, *Viscosity approximation methods for nonexpansive multi-valued nonself mapping and equilibrium problems*, Demonstratio Mathematica (2014).

[9] F. Cianciaruso, G. Marino, L. Muglia, Y. Yao, *A hybrid projection algorithm for finding solution of mixed equilibrium problem and variational inequality problem*, Fixed Point Theory Appl. 2010(2010)383740.

[10] P. L. Combettes, A. Hirstoaga, *Equilibrium programming in Hilbert space*, J. Nonlinear Convex Anal. 6(2005)117-136.

[11] G. Marino, HK, Xu, *A general iterative method for nonexpansive mappings in Hilbert spaces*, Math. Appl. 318(2006)43-52.

[12] G. Marino, H. K. Xu, *Weak and strong convergence theorems for strict pseudocontractions in Hilbert spaces*, J. Math. Anal. Appl. 329 (2007), 336-346.

[13] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Am. Math. Soc. 73(4)(1967)595-597.

[14] T. Suzuki, *Strong convergence of Krasnoselskii and Mann's type sequences for one parameter nonexpansive semigroups without Bochner integrals*, J. Math. Anal. Appl. 305(2005)227-239.

[15] A. Tada and W. Takahashi, *Strong convergence theorem for an equilibrium problem and a nonexpansive mapping*, J. Optim. Theory Appl., 133 (2007), 359-370.

[16] M. Taherian, M. Azhini, *Strong convergence theorems for fixed point problem of infinite family of non self mapping and generalized equilibrium problems with perturbation in Hilbert spaces*, Advances and Applications in Mathematical Sciences 15(2)(2016)25-51.

[17] W. Takahashi, *Nonlinear Functional Analysis, Fixed Point Theory and Its Application*, Yokohama-Publishers, Yokohama, Japan 2000.

[18] S. Takahashi and W. Takahashi, *Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces*, J. Math. Anal. Appl., 331 (2007), 506-515.

[19] S. Takahashi, W. Takahashi, *Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space*, Nonlinear Anal. 69(2008)1025-1033.

[20] H. K. Xu, *Viscosity approximation method for nonexpansive semigroups*, J. Math. Anal. Appl. 298(2004)279-291.