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Research Paper

Free Vibration Analysis of Sandwich Beams with FG Face Sheets Based on the High Order Sandwich Beam Theory

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Abstract
In this paper, the vibration behavior of the sandwich beams with functionally graded face-sheets is investigated based on the high order sandwich beam theory. The properties of the FGM are varied gradually across the thickness of the structures in accordant with the power-law rule. First-order shear deformation theory and polynomial patterns are used to model the displacements of the face-sheets and the core, respectively. The governing equations of the motion are obtained based on Hamilton's energy principle and solved by a Galerkin method. An algebraic method is used to reduce the number of equations. Boundary conditions are considered as simply supported and clamped. The effect of the power-law index and geometrical variations are surveyed on the fundamental frequency parameter for different sandwich beams in some numerical examples. In order to verify the results of the present study, they are compared with special cases of the literature.

Keywords
Sandwich Beam, FGM, High Order Sandwich Beam Theory, Vibration, Galerkin

1. Introduction
Due to the high flexural stiffness to weight ratio, sandwich structures have a wide application in modern industries such as aerospace, transportation, naval, and construction structures. Sandwiches include two thin and stiff faces that cover a thick and lightweight core. The core is usually flexible. Separation of the face sheets by a softcore increases the bending rigidity of the beam at expenses of small weight [1].
Application of the classical composite materials in high-temperature environments causes failure, delamination, and thermal stress concentration. Japanese researchers proposed the functionally graded materials (FGMs) to overcome this problem. FGMs are inhomogeneous microscopic materials that gradually graded from a metal surface to a ceramic one [2]. Investigation on these materials has been increased by material researchers. Rahmani et al. studied the buckling behavior of truncated conical sandwich shells with porous FG core. The materials were varied gradually in the thickness direction according to the power-law rule [3]. Rahmani et al. studied the vibration behavior of conical sandwich shells with both FG face sheets and FG core by using a power-law rule to model the material properties [4]. Fesharaki et al. studied the stress concentration factors in

FGM plate with central holes in different shapes [5]. Bouderba [6] studied the bending of FGM rectangular plates in the thermal condition. Properties varied in the thickness direction based on a power-law rule. Rahmani et al. [7] studied the buckling behavior of a conical sandwich shell with both porous FG face sheets and porous FG core. A power law rule was considered to model the material properties variation.

There are different approaches to investigate the mechanical behavior of panels such as shear deformation theory, 3D elastic theory, energy, and finite element method [8]. Hu et al. studied the local and global buckling of sandwich beams by using a finite element method [9]. Vo et al. investigated the vibration and buckling of the FG sandwich beams by a finite element model based on the refined shear deformation theory [10]. Adamek investigated the possibilities of the first-order shear deformation theory (FSDT) to three-layered elastic beams. He studied their modifications on their transient responses to a pulse of impact character [11]. The core is a flexible layer that is compressed transversely and the thickness of the sandwich panels is variable, but in the classical theories, the localized effects in the core can’t be calculated. Frostig et al. presented a high order theory to consider the variation of the thickness [12]. Mohammadi and Rahmani studied the buckling behavior of FG sandwich cylinders based on the high order sandwich shell theory [13]. Rahmani et al. studied the free vibration of FG conical sandwich shells based on an improved high order sandwich shell theory [14]. Rahmani et al. [16] investigated the vibration behavior of the porous FG circular sandwich plate based on a modified high order sandwich plate theory [15]. Salami discussed the bending of sandwich beams based on an extended high order sandwich panel theory. Salami [17] also studied the low-velocity impact response of sandwich beams based on a high order theory. Dariushi and Sadighi [18] investigated the nonlinear behavior of the orthotropic sandwich beam based on a high order sandwich beam theory. Canales and Mantari studied the buckling and free vibration of laminated beams by using higher-order shear deformation theory [19].

Many researchers have explored the vibration behavior of the sandwich beams. Khalili et al. studied the vibration of sandwich beams by using a dynamic stiffness method [20]. Arikoglu and Ozkol investigated the vibration of composite sandwich beams with a viscoelastic core based on the differential transform method [21]. Amirani et al. studied the vibration of sandwich beams with FG core by using the element free Galerkin method [22]. Tossapanon and Wattanasakulpong studied the stability and free vibration of FG sandwich beams resting on an elastic foundation by using the Chebyshev collocation method [23]. Khedir and Aldraihem [24] investigated the vibration of a sandwich beam with a softcore based on a zig-zag beam theory. Goncalves et al. studied the buckling and vibration of shear-flexible sandwich beams by using a couple-stress-based finite element [25]. Zhang et al. presented a vibration analysis of sandwich beams with honeycomb-corrugation hybrid cores [26]. Chen et al. studied the nonlinear free vibration of shear deformable sandwich beam with an FG porous core based on Timoshenko beam theory [27]. Xu et al. [28] studied the free vibration of a composite sandwich beam with graded corrugated lattice core based on a continuous homogeneous theory.

In this study, the vibration behavior of sandwich beams is investigated by using a high order sandwich beam theory which is modified by considering the flexibility of the core in the thickness direction. Sandwiches consist of two FG faces that cover a homogeneous core. FGM properties are location-dependent which graded in according to a power-law rule. Boundary conditions are clamped and simply supported. The equations are derived based on Hamilton's energy principle. To obtain the frequencies, a Galerkin method is applied. In order to validate the results of the present approach, they are compared with the results of the literature in special cases. Finally, the effects of the volume fraction distribution of FG face sheets and some geometrical effects on the vibration characteristics of defined sandwich beams are investigated.

2. Formulation

Consider a sandwich with the FG face sheets and a homogeneous core. Usually, it is considered that functionally graded materials are composed of metal and ceramic. Material properties are varied gradually across the thickness direction based on a power-law rule in terms of the volume fraction of the compositions. Material properties such as Young's modulus, density, Poisson's ratio can be expressed as:

$$P_j\left(z_j\right)=g\left(z_j\right)P_{ce}^j+\left[1-g\left(z_j\right)\right]P_m^j,\quad j=(t,b) \tag{1}$$

$$g\left(z_t\right)=\left(\frac{\frac{h_t}{2}-z_t}{h_t}\right)^N;\ g\left(z_b\right)=\left(\frac{\frac{h_b}{2}+z_b}{h_b}\right)^N \tag{2}$$

Where "P" is the material properties; "N" is the positive power-law index; "h" is the thickness; subscripts "m" and "ce" are metal and ceramic; and the superscript "t" and "b" are top and bottom face sheets, respectively.

To model the displacement fields of the face-sheets, First Order Shear Deformation Theory (FSDT) is employed as follows [29]:

$$u_j\left(x,z,t\right)=u_{0j}\left(x,t\right)+z_j\phi_j\ \ j=(t,b) \tag{3}$$

$$w_j\left(x,z,t\right)=w_{0j}\left(x,t\right) \tag{4}$$

where "0" denotes values with correspondence to the central plane of the layers; "u" and "w" are the in-plane deformation and the transverse deflections of the faces in the "x" and "z" directions, respectively. "Φ" is the rotation of the transverse normal line.

Also, the kinematic relations of the core are considered as polynomial patterns with the unknown coefficients, u_k (k= 0,1,2,3), for the in-plane and w_l (l = 0,1,2) for vertical displacement components which obtained by the variational principle [18]:

$$u_c\left(x,z_c,t\right)=u_0\left(x,t\right)+u_1\left(x,t\right)z_c+u_2\left(x,t\right)z_c^2+u_3\left(x,t\right)z_c^3 \tag{5}$$

$$w_c\left(x,z_c,t\right)=w_0\left(x,t\right)+w_1\left(x,t\right)z_c+w_2\left(x,t\right)z_c^2 \tag{6}$$

In this theory, the compatibility conditions assume that the faces are stuck to the core completely, and the interface displacements between the core and the face sheets can be obtained as follows:

$$u_c\left(z_c=-h_c/2\right)=u_t\left(z_t=h_t/2\right),\ w_c\left(z_c=-h_c/2\right)=w_t \tag{7}$$

$$u_b\left(z_b=-h_b/2\right)=u_c\left(z_c=h_c/2\right),\ w_b=w_c\left(z_c=h_c/2\right) \tag{8}$$

In order to investigate the vibration behavior of functionally graded sandwich beams and obtain the governing equations of the motion, Hamilton's energy principle is applied which consists of the variation of the kinetic energy and strain energy. The main equation is as follow [29]:

$$\int_{t1}^{t2} (-\delta K + \delta U)dt = 0 \qquad (9)$$

The variation of kinetic and the strain energy are "δK" and "δU", respectively; "t" is the time coordinate that varies between the times "t_1" and "t_2"; "δ" is the variation operator. The variation of the kinetic energy is calculated as follows:

$$\begin{aligned}
&\int_{t1}^{t2} \delta K dt \\
&= -\int_{t1}^{t2} \{ \int_0^L \int_{-\frac{h_t}{2}}^{\frac{h_t}{2}} \rho_t (z_t)(\ddot{u}_t \delta u_t + \ddot{w}_t \delta w_t) dx dz_t + \\
&\int_0^L \int_{-\frac{h_b}{2}}^{\frac{h_b}{2}} \rho_b (z_b)(\ddot{u}_b \delta u_b + \ddot{w}_b \delta w_b) dx dz_b + \\
&\int_0^L \int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} \rho_c (\ddot{u}_c \delta u_c + \ddot{w}_c \delta w_c) dx dz_c \} dt
\end{aligned} \qquad (10)$$

Where (··) indicates the second derivative with respect to time; the density is "ρ" which in the functionally graded layers is the function of the displacement; the core is indicated with "c".

The variation of the total strain energy in the face sheets and the core, also the compatibility conditions at the interfaces of the layers which are the constraints and attended in the Hamilton's principle in terms of Lagrange multipliers, is expressed as follows:

$$\begin{aligned}
&\delta U_p = \int_{A_t} (\sigma_{xx}^t \delta\varepsilon_{xx}^t + \tau_{xz}^t \delta\gamma_{zz}^t) dA + \\
&\int_{A_b} (\sigma_{xx}^b \delta\varepsilon_{xx}^b + \tau_{xz}^b \delta\gamma_{xz}^b) dv + \\
&\int_{A_{core}} (\sigma_{xx}^c \delta\varepsilon_{xx}^c + \sigma_{zz}^c \delta\varepsilon_{zz}^c + \tau_{xz}^c \delta\gamma_{xz}^c) dv + \\
&\delta\int_0^L [\lambda_{xt} \left(\mathrm{u_t}\left(z_t = \frac{h_t}{2}\right) - u_c\left(z_c = -\frac{h_c}{2}\right)\right) + \\
&\lambda_{zt}\left(\mathrm{w_t} - \mathrm{w_c}\left(z_c = -\frac{h_c}{2}\right)\right) + \\
&\lambda_{xb}(\mathrm{u_c}\left(z_c = \frac{h_c}{2}\right) - u_b\left(z_b = -\frac{h_b}{2}\right) + \\
&\lambda_{zb}\left(\mathrm{w_c}\left(z_c = \frac{h_c}{2}\right) - w_b\right)] dx
\end{aligned} \qquad (11)$$

"σ_{xx}" and "τ_{xz}" display the normal and shear stresses; "ε_{xx}" and "γ_{xz}" are the normal and shear strains of the layers; "σ_{zz}^c" and "ε_{zz}^c" present the lateral normal stress and strain in the core; "τ_{xz}^c" and "γ_{xz}^c" declare the shear stresses and shear strains in the thickness direction of the core; "λ_x" and "λ_z" are the Lagrange multipliers at the face sheet-core interfaces.

Considering small deflection, the strain components for the faces can be declared as follows [30]:

$$\varepsilon_{xx}^j (x, z_j, t) = u_{0j,x}(x,t) + z_j \phi_{j,x}(x,t) \qquad (12)$$

$$\gamma_{xz}^{j}\left(x,z_j,t\right)=\phi_j(x,t)+w_{j0,x}(x,t) \tag{13}$$

The"(),i" expresses derivation with respect to i. The strain of the core can be defined as [30]:

$$\varepsilon_{xx}^{c}\left(x,z_c,t\right)=u_{c,x}\left(x,z_c,t\right) \tag{14}$$

$$\gamma_{xz}^{c}\left(x,z_c,t\right)=u_{c,z}(x,z_c,t)+w_{c,x}(x,z_c,t) \tag{15}$$

$$\varepsilon_{zz}^{c}\left(x,z_c,t\right)=w_{c,z}\left(x,z_c,t\right) \tag{16}$$

In this model by substituting the expressions of the Equations 10 and 11 according to the kinematic relations of the layers and using the interfaces relations, and after some algebraic operations, the thirteen equations of motion are obtained. These equations are not independent and by using the compatibility conditions and based on a reduction method the number of equations is reduced to nine. These equations include two unknowns of the faces and seven unknowns of the core which are presented in the follows:

$$\begin{aligned}&+I_{0t}\ddot{u}_{0c}\,h_t/2-I_{1t}\ddot{u}_{0c}-I_{0t}\ddot{\phi}_{0c}\,h_th_c/4+I_{1t}\ddot{\phi}_{0c}\,h_c/2+I_{0t}\ddot{u}_{2c}\,h_th_c^2/8\\&-I_{1t}\ddot{u}_{2c}\,h_c^2/4-I_{0t}\ddot{u}_{3c}\,h_th_c^3/16+I_{1t}\ddot{u}_{3c}\,h_c^3/8-I_{0t}\ddot{\phi}_t\,h_t^2/4+I_{1t}\ddot{\phi}_th_t-I_{2t}\ddot{\phi}_t+\\&h_t/2\,N_{xx,x}^t-M_{xx,x}^t+N_{xz}^t=0\end{aligned} \tag{17}$$

$$\begin{aligned}&-I_{0b}\ddot{u}_{0c}\,h_b/2-I_{1b}\ddot{u}_{0c}-I_{0b}\ddot{\phi}_{0c}\,h_bh_c/4-I_{1b}\ddot{\phi}_{0c}\,h_c/2-I_{0b}\ddot{u}_{2c}\,h_bh_c^2/8\\&-I_{1b}\ddot{u}_{2c}\,h_c^2/4-I_{0b}\ddot{u}_{3c}\,h_bh_c^3/16-I_{1b}\ddot{u}_{3c}\,h_c^3/8-I_{0b}\ddot{\phi}_b\,h_b^2/4-I_{1b}\ddot{\phi}_bh_b-I_{2b}\ddot{\phi}_b-\\&h_b/2\,N_{xx,x}^b-M_{xx,x}^b+N_{xz}^b=0\end{aligned} \tag{18}$$

$$\begin{aligned}&-I_{0t}\ddot{u}_{0c}+I_{0t}\ddot{\phi}_{0c}\,h_c/2-I_{0t}\ddot{u}_{2c}\,h_c^2/4+I_{0t}\ddot{u}_{3c}\,h_c^3/8+I_{0t}\ddot{\phi}_t\,h_t/2-I_{1t}\ddot{\phi}_t\\&-I_{0b}\ddot{u}_{0c}-I_{0b}\ddot{\phi}_{0c}\,h_c/2-I_{0b}\ddot{u}_{2c}\,h_c^2/4-I_{0b}\ddot{u}_{3c}\,h_c^3/8-I_{0b}\ddot{\phi}_b\,h_b/2-I_{1b}\ddot{\phi}_t\\&-I_{0c}\ddot{u}_{0c}-I_{1c}\ddot{\phi}_{0c}-I_{2c}\ddot{u}_{2c}-I_{3c}\ddot{u}_{3c}-N_{xx,x}^t-N_{xx,x}^b-R_{x,x}^c=0\end{aligned} \tag{19}$$

$$\begin{aligned}&+I_{0t}\ddot{u}_{0c}\,h_c/2-I_{0t}\ddot{\phi}_{0c}\,h_c^2/4+I_{0t}\ddot{u}_{2c}\,h_c^3/8-I_{0t}\ddot{u}_{3c}\,h_c^4/16-I_{0t}\ddot{\phi}_t\,h_th_c/4+I_{1t}\ddot{\phi}_t\,h_c/2\\&-I_{0b}\ddot{u}_{0c}\,h_c/2-I_{0b}\ddot{\phi}_{0c}\,h_c^2/4-I_{0b}\ddot{u}_{2c}\,h_c^3/8-I_{0b}\ddot{u}_{3c}\,h_c^4/16-I_{0b}\ddot{\phi}_b\,h_bh_c/4-I_{1b}\ddot{\phi}_b\,h_c/2\\&-I_{1c}\ddot{u}_{0c}-I_{2c}\ddot{\phi}_{0c}-I_{3c}\ddot{u}_{2c}-I_{4c}\ddot{u}_{3c}+h_c/2\,N_{xx,x}^t-h_c/2\,N_{xx,x}^b-M_{x1,x}^c+Q_{xz}^c=0\end{aligned} \tag{20}$$

$$\begin{aligned}&-I_{0t}\ddot{u}_{0c}\,h_c^2/4+I_{0t}\ddot{\phi}_{0c}\,h_c^3/8-I_{0t}\ddot{u}_{2c}\,h_c^4/16+I_{0t}\ddot{u}_{3c}\,h_c^5/32+I_{0t}\ddot{\phi}_t\,h_th_c^2/8-I_{1t}\ddot{\phi}_t\,h_c^2/4\\&-I_{0b}\ddot{u}_{0c}\,h_c^2/4-I_{0b}\ddot{\phi}_{0c}\,h_c^3/8-I_{0b}\ddot{u}_{2c}\,h_c^4/16-I_{0b}\ddot{u}_{3c}\,h_c^5/32-I_{0b}\ddot{\phi}_b\,h_bh_c^2/8\\&-I_{1b}\ddot{\phi}_b\,h_c^2/4-I_{2c}\ddot{u}_{0c}-I_{3c}\ddot{\phi}_{0c}-I_{4c}\ddot{u}_{2c}-I_{5c}\ddot{u}_{3c}-h_c^2/4\,N_{xx,x}^t-h_c^2/4\,N_{xx,x}^b\\&-M_{x2,x}^c+2M_{Q1xc}^c=0\end{aligned} \tag{21}$$

$$\begin{aligned}&+I_{0t}\ddot{u}_{0c}\,h_c^3/8-I_{0t}\ddot{\phi}_{0c}\,h_c^4/16+I_{0t}\ddot{u}_{2c}\,h_c^5/32-I_{0t}\ddot{u}_{3c}\,h_c^6/64-I_{0t}\ddot{\phi}_t\,h_th_c^3/16\\&+I_{1t}\ddot{\phi}_t\,h_c^3/8-I_{0b}\ddot{u}_{0c}\,h_c^3/8-I_{0b}\ddot{\phi}_{0c}\,h_c^4/16-I_{0b}\ddot{u}_{2c}\,h_c^5/32-I_{0b}\ddot{u}_{3c}\,h_c^6/64\\&-I_{0b}\ddot{\phi}_b\,h_bh_c^3/16-I_{1b}\ddot{\phi}_b\,h_c^3/8-I_{3c}\ddot{u}_{0c}-I_{4c}\ddot{\phi}_{0c}-I_{5c}\ddot{u}_{2c}-I_{6c}\ddot{u}_{3c}+h_c^3/8\,N_{xx,x}^t\\&-h_c^3/8\,N_{xx,x}^b-M_{x3,x}^c+3M_{Q2xc}^c=0\end{aligned} \tag{22}$$

$$\begin{aligned}&-I_{0t}\ddot{w}_{0c}+I_{0t}\ddot{w}_{1c}\,h_c/2-I_{0t}\ddot{w}_{2c}\,h_c^2/4-I_{0b}\ddot{w}_{0c}-I_{0b}\ddot{w}_{1c}\,h_c/2-I_{0b}\ddot{w}_{2c}\,h_c^2/4-\\&I_{0c}\ddot{w}_{0c}-I_{1c}\ddot{w}_{0c}-I_{2c}\ddot{w}_{2c}-N_{xz,x}^t-N_{xz,x}^b-Q_{xz,x}^c=0\end{aligned} \tag{23}$$

$$\begin{aligned}&+I_{0t}\ddot{w}_{0c}\,h_c/2-I_{0t}\ddot{w}_{1c}\,h_c^2/4+I_{0t}\ddot{w}_{2c}\,h_c^3/8-I_{0b}\ddot{w}_{0c}\,h_c/2-I_{0b}\ddot{w}_{1c}\,h_c^2/4-I_{0b}\ddot{w}_{2c}\,h_c^3/8\\&-I_{1c}\ddot{w}_{0c}-I_{2c}\ddot{w}_{0c}-I_{3c}\ddot{w}_{2c}+h_c/2\,N_{xz,x}^t-h_c/2\,N_{xz,x}^b-M_{Q1xc,x}^c+R_z^c=0\end{aligned} \tag{24}$$

$$-I_{0t}\ddot{w}_{0c}\,h_c^2/4+I_{0t}\ddot{w}_{1c}\,h_c^3/8-I_{0t}\ddot{w}_{2c}\,h_c^4/16-I_{0b}\ddot{w}_{0c}\,h_c^2/4-I_{0b}\ddot{w}_{1c}\,h_c^3/8$$
$$-I_{0b}\ddot{w}_{2c}\,h_c^4/16-I_{2c}\ddot{w}_{0c}-I_{3c}\ddot{w}_{0c}-I_{4c}\ddot{w}_{2c}-h_c^2/4\,N^t_{xz,x}-h_c^2/4\,N^b_{xz,x}-M^c_{Q2xc,x} \quad (25)$$
$$+2M^c_z=0$$

$$u_{0t}+\frac{h_t}{2}\phi^t=u_{0c}-\frac{h_c}{2}\phi_0^c+\frac{h_c^{\,2}}{4}u_{2c}-\frac{h_c^{\,3}}{8}u_{3c}=0 \quad (26)$$

$$w_{0t}=+w_{0c}-h_c/2\,w_{1c}+h_c^{\,2}/4\,w_{2c} \quad (27)$$

$$u_{0b}-\frac{h_b}{2}\phi^b=u_{0c}+\frac{h_c}{2}\phi_0^c+\frac{h_c^{\,2}}{4}u_{2c}+\frac{h_c^{\,3}}{8}u_{3c}=0 \quad (28)$$

$$w_{0b}=+w_{0c}+h_c/2\,w_{1c}+h_c^{\,2}/4\,w_{2c} \quad (29)$$

Stress resultants, moment resultants, inertia terms of the faces, and high order stress resultants of the core have been presented in Appendix 1. Finally, by substituting the high order stress resultants in the equations of the face sheets and the core in terms of the displacement components, the governing equations of motion are derived in terms of the nine unknowns. However, for a sandwich beam, Galerkin method solution can be established.

3. Verification and Numerical Results

In order to solve the equations of the free vibration of the FG sandwich beam, a Galerkin method with nine trigonometric shape functions, which satisfy the boundary conditions, is established. The shape functions of the simply supported boundary condition can be expressed as:

$$\phi_j=\left[C_{\phi j}\cos a_m x\right]e^{i\omega t}\quad ,j=(t,b,c) \quad (30)$$

$$u_{ck}=\left[C_{uck}\cos a_m x\right]e^{i\omega t}\qquad k=(0,1,2) \quad (31)$$

$$w_l=\left[C_{wl}\sin a_m x\right]e^{i\omega t}\qquad l=(0,1,2) \quad (32)$$

The shape functions of the clamped boundary condition can be expressed as:

$$\phi_j=\left[C_{\phi j}\cos a_m x\right]e^{i\omega t}\quad ,j=(t,b,c) \quad (33)$$

$$u_{ck}=\left[C_{uck}\sin a_m x\right]e^{i\omega t}\qquad k=(0,1,2) \quad (34)$$

$$w_l=C_{wl}(\sinh(\frac{\lambda_m x}{L})-\sin(\frac{\lambda_m x}{L})+\gamma_m(\cosh(\frac{\lambda_m x}{L})-\cos(\frac{\lambda_m x}{L})))e^{i\omega t} \quad (35)$$

$$\cos\lambda_m.\cosh\lambda_m=1 \quad (36)$$

$$\gamma_m=\frac{\sinh\lambda_m-\sin\lambda_m}{\cos\lambda_m-\cosh\lambda_m}\quad m=(1,2,3,...) \quad (37)$$

Where $a_m=m\pi/L$; m is the wave number and $C_{uk},C_{wk},C_{\phi j}$ are the nine unknown constants of the shape functions. These nine equations can be written in a 9*9 matrix which includes the mass, “M”, and stiffness, “K”, matrices as follows:

$$(k_m-\omega_m^2 M_m)C_m=0 \quad (38)$$

Equation (38), ω_m is the natural frequency; and C_m is the Eigen vector which contains nine unknown constants.

In order to validate the results of the present approach, they are compared with the results of works of literature [10], [31] and [32] in a special case, which are shown in Table 1, for the simply supported (S-S) and clamped (C-C) boundary conditions. Consider different FG sandwich beams which are assumed to be made from a mixture of Alumina (Al_2O_3) as ceramic phases and Aluminum (Al) as metal phases. The mechanical properties of such materials are available in reference [23]. In general, h_t-h_c-h_b sandwich beam is a structure with the indices of outer face sheet thickness, core thickness and inner face sheet thickness equal to h_t, h_c and h_b, respectively. Therefore, in 2-1-2 sandwich, every face sheet thicknesses is two times of the core thickness and the structure is symmetric and in 1-8-1 sandwich, the core thickness is eight times of the every face sheet thickness.

Some geometrical effects on the fundamental frequency of FG sandwich beams are investigated. Table 2 and Table 3 show the effect of length to thickness ratio on the fundamental frequency parameter for 2-1-2, 1-1-1 and 1-8-1 FG sandwich beams in the simply supported and clamped boundary conditions, respectively.

Table1. Fundamental frequency parameters of present results and literatures [10], [31] and [32] (L/h=5)

B.C	reference	N=0	N=0.5	N=1	N=2
S-S	[31]	5.1525	4.4083	3.9902	3.6344
	[10]	5.1526	4.3990	3.9711	3.6050
	[32]	5.1525	4.4075	3.9902	3.6344
	Present method	5.0789	4.3312	3.8618	3.5487
C-C	[31]	10.0344	8.7005	7.9253	7.2113
	[10]	9.9984	8.6717	7.9015	7.1901
	Present method	9.9151	8.5887	7.8080	7.1088

For simplicity, the non-dimensional fundamental frequency parameter is defined as follows:

$$\bar{\omega} = \frac{\omega L^2}{h}\sqrt{\frac{\rho_0}{E_0}} \tag{39}$$

Where "L" is the length of the sandwich beam; "h" is the total thickness of the sandwich beam; ρ_0 is density equal to 1kg/m^3 and E_0 is the young module equal to 1 GPa.

When ratios are increased in a constant “N”, the fundamental frequency parameter increase, but the natural frequencies decrease. Based on Tables 2 and 3, the values of 2-1-2 sandwiches are more than the others. The fundamental frequency parameters of the 1-8-1 are the lowest. By increasing of ratio, the stability of the structure reduces. It is important to consider that long length is not proper for the FG sandwich beams. By increasing the power-law index, “N”, the fundamental frequency parameters decrease. For example, in the simply supported boundary condition, for L/h=20, by increasing “N”, the fundamental frequency parameter decreases 13.42% in 1-8-1 sandwiches, 37.63% in 2-1-2 sandwiches and 32.20% in 1-1-1 sandwiches. And for the clamped one, with the same parameters, the non-dimensional frequency decreases 9.98% in 1-8-1 sandwiches, 40.18% in 2-1-2 sandwiches, and 33.12% in 1-1-1 sandwiches. Also, it should be noted that the values of the clamped sandwiches are more than simply supported ones.

Table2. Fundamental frequency parameters of different kinds of simply supported FG sandwich beams

		The fundamental frequency parameter			
	L/h	N=0	N=0.5	N=1	N=4
1-8-1	10	0.55214	0.52162	0.50640	0.47847
	20	0.55996	0.52880	0.51327	0.48479
	30	0.56145	0.53018	0.51458	0.48599
	40	0.56198	0.53066	0.51504	0.48642
	50	0.56222	0.53088	0.51525	0.48662
2-1-2	10	0.86949	0.70332	0.64128	0.54203
	20	0.88139	0.71326	0.65042	0.54970
	30	0.88366	0.71516	0.65216	0.55116
	40	0.88446	0.71583	0.65278	0.55168
	50	0.88483	0.71614	0.65306	0.55191
1-1-1	10	0.78224	0.66307	0.61373	0.53053
	20	0.79353	0.67265	0.62255	0.53797
	30	0.79569	0.67448	0.62423	0.53939
	40	0.79645	0.67513	0.62483	0.53989
	50	0.79680	0.67542	0.62510	0.54012

Table3. Fundamental frequency parameters of different kinds of clamped FG sandwich beams

		The fundamental frequency parameter			
	L/h	N=0	N=0.5	N=1	N=4
1-8-1	10	1.88035	1.79661	1.75690	1.68823
	20	3.64903	3.48983	3.41465	3.28531
	30	5.44098	5.20465	5.09313	4.90149
	40	7.23928	6.92534	6.77725	6.52285
	50	9.04018	8.64844	8.46366	8.14632
2-1-2	10	2.90712	2.28712	2.06482	1.74266
	20	5.63357	4.42474	3.99217	3.36960
	30	8.39762	6.59337	5.94803	5.02055
	40	11.17194	8.77053	7.91171	6.67808
	50	13.95047	10.95117	9.87860	8.33832
1-1-1	10	2.58753	2.15746	1.98843	1.73085
	20	5.00941	4.17334	3.84574	3.35024
	30	7.46564	6.21856	5.73023	4.99278
	40	9.93131	8.27185	7.62219	6.64167
	50	12.40084	10.32844	9.51722	8.29316

Figures 1 and 2 depict the effect of the variation of the core to face sheet thickness ratio, h_c/h_t, on the fundamental frequency parameter in various power-law indices, and constant total thickness. When $h_c/h_t=0.5$, it means the thickness of the faces are two times of the core thickness, so it shows the results of the 2-1-2 sandwich. And, when $h_c/h_t=8$, it shows the results of the 1-8-1 sandwich. For all indices, by increasing the ratio in a constant total thickness, the number of metal increases and the structure will be softer, so the fundamental frequency parameters decrease. Also, when the power-law index is increased in a constant thickness, ceramic quantity decrease, and for all values of h_c/h_t, by increasing the ratio, the fundamental frequency parameters decrease.

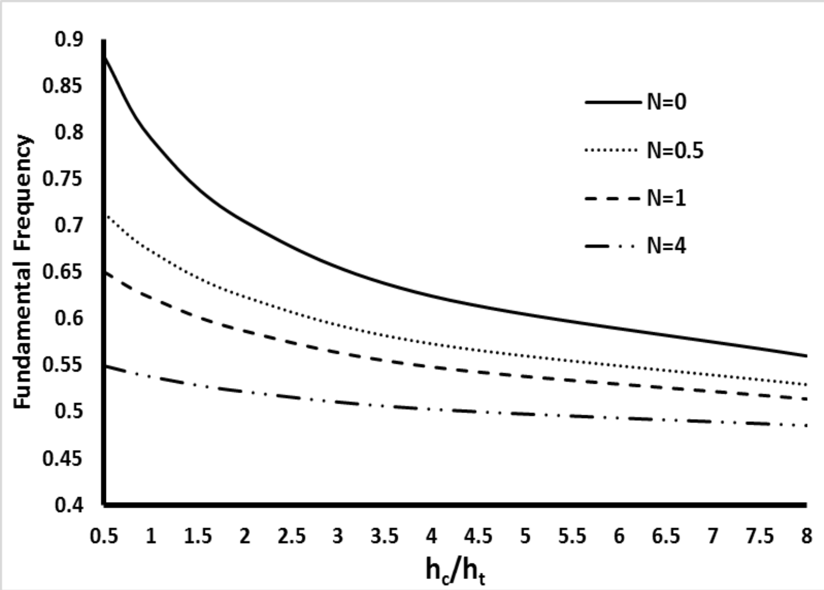

Figure1. Effect of variation of the core to face sheets thickness ratio on the fundamental frequency parameter for simply supported FG sandwich beam

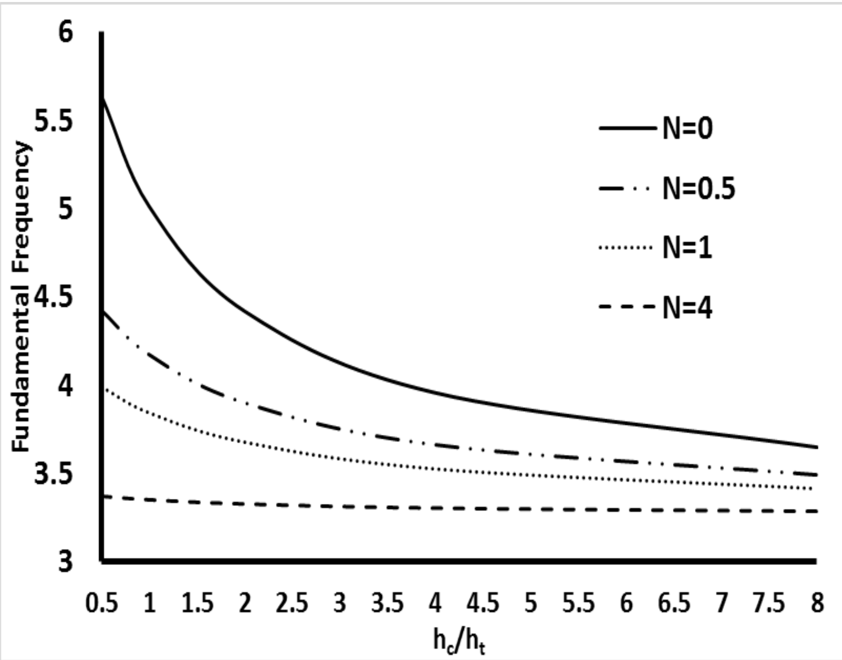

Figure 2. Effect of variation of the core to face sheets thickness ratio on the fundamental frequency parameter for clamped FG sandwich beam

Effect of the variation of the wave number, “m”, on the fundamental frequency parameter for various power law indices and constant total thickness is depicted in the Figure 3 and Table 4. By increasing the wave number, the fundamental frequency parameters increase.

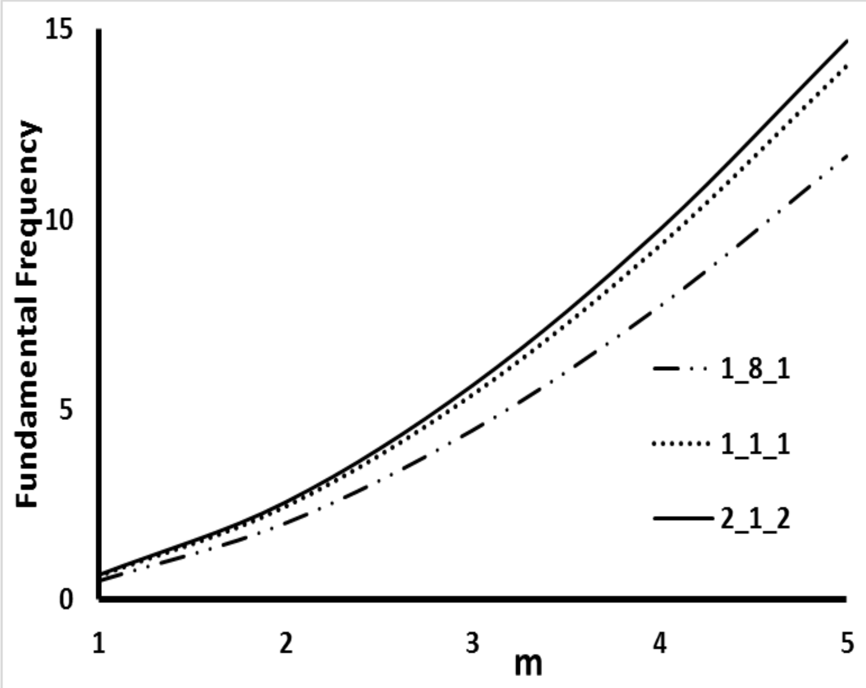

Figure 3. Effect of variation of the wave number on the fundamental frequency parameter for clamped FG sandwich beam

Table 4. Variation of the frequency parameter with wave number change for clamped sandwich beams

		The fundamental frequency parameter			
	m	N=0	N=0.2	N=1	N=2
	1	3.64903	3.48983	3.41465	3.28531
	2	5.93308	5.60491	5.44926	5.18004
1-8-1	3	6.98279	6.67958	6.53647	6.29046
	4	7.68987	7.35673	7.19962	6.92984
	5	7.62867	7.29839	7.14261	6.87504
	1	5.63357	4.42474	3.99217	3.36960
	2	11.36682	8.86812	7.97068	6.67594
2-1-2	3	10.77754	8.46289	7.63486	6.44457
	4	11.87949	9.32416	8.41042	7.09875
	5	11.77198	9.24104	8.33596	7.03633
	1	5.00941	4.17334	3.84574	3.35024
	2	9.75776	8.04889	7.37556	6.35222
1-1-1	3	9.58114	7.98151	7.35501	6.40865
	4	10.55677	8.79173	8.10088	7.05886
	5	10.46374	8.71535	8.03093	6.99831

Effect of the variation of the total thickness of the sandwiches, “h”, on the fundamental frequency parameter in various power-law indices for different simply supported and clamped FG sandwich beams is depicted in Table 5 and Table 6. For example, in the simply supported boundary condition, for L/h=20 and N=1, by increasing h, the fundamental frequency parameter decreases 79.65% in 1-8-1 sandwiches, 79.63% in 2-1-2 sandwiches and 79.62% in 1-1-1 sandwiches. And for the clamped one, with the same parameters, the non-dimensional frequency decreases 3.65% in 1-8-1 sandwiches, 4.31% in 2-1-2 sandwiches, and 4.27% in 1-1-1 sandwiches. It is seen that the rate of variation in both boundary conditions is constant in the different sandwiches. Also, the simply supported boundary condition is so sensitive via the variation of the total thickness than the clamped one.

Table5. Variation of the frequency parameter with a total thickness of simply supported sandwich beams

	The fundamental frequency parameter				
	h	N=0	N=0.5	N=1	N=4
	0.01	1.10429	1.04325	1.01280	0.95695
	0.02	0.55996	0.52880	0.51327	0.48479
1-8-1	0.03	0.37430	0.35345	0.34305	0.32399
	0.04	0.28099	0.26533	0.25752	0.24321
	0.05	0.22489	0.21235	0.20610	0.19464
	0.01	1.73899	1.40664	1.28256	1.08406
	0.02	0.88139	0.71326	0.65042	0.54970
2-1-2	0.03	0.58911	0.47677	0.43477	0.36744
	0.04	0.44223	0.35791	0.32639	0.27584
	0.05	0.35393	0.28645	0.26122	0.22076
	0.01	1.56449	1.32615	1.22747	1.06107
	0.02	0.79353	0.67265	0.62255	0.53797
1-1-1	0.03	0.53046	0.44965	0.41615	0.35959
	0.04	0.39822	0.33756	0.312416	0.26994
	0.05	0.31872	0.27017	0.25004	0.21605

Table 6. Variation of the frequency parameter with a total thickness of clamped sandwich beams

	The fundamental frequency parameter				
	h	N=0	N=0.5	N=1	N=4
	0.01	3.76071	3.59323	3.51381	3.37647
	0.02	3.64903	3.48983	3.41465	3.28531
1-8-1	0.03	3.62732	3.46977	3.39542	3.26766
	0.04	3.61964	3.46267	3.38862	3.26142
	0.05	3.61607	3.45937	3.38546	3.25852
	0.01	5.81425	4.57425	4.12965	3.48532
	0.02	5.63357	4.42474	3.99217	3.36960
2-1-2	0.03	5.59841	4.39558	3.96535	3.34703
	0.04	5.58597	4.38526	3.95585	3.33904
	0.05	5.58018	4.38046	3.95144	3.33533
	0.01	5.17506	4.31492	3.97686	3.46170
	0.02	5.00941	4.17334	3.84574	3.35024
1-1-1	0.03	4.97709	4.14570	3.82015	3.32852
	0.04	4.96565	4.135926	3.81109	3.32083
	0.05	4.96033	4.13137	3.80688	3.31726

4. Conclusion

In this study for three kinds of sandwich beams, 1-8-1, 2-1-2, and 1-1-1, according to a high order sandwich beam theory, the displacement fields of the face-sheets and the core were considered based on the first-order shear deformation theory and the polynomial distributions, respectively. High order stress resultants were considered in the core. A power law distribution was used to model the material properties of the FG face sheets. The equations of the motion were obtained by Hamilton's principal and solved by using the Galerkin method. Also, an approach was used to reduce the equations of motion from 13 to 9 equations. In order to survey the capabilities of this model for free vibration analysis of simply supported and clamped sandwich beams with FG face sheets, the results were verified by literature results in a special case. Based on the results, there was a good agreement between them and the following conclusion can be drawn:

- While the power-law index is increased, the amount of ceramic reduces, so the fundamental frequency parameter decreases.
- In a constant power-law index, the fundamental frequency parameter increases when the length to thickness ratio is increased.
- In a constant total thickness, by increasing the core to face-sheet thickness ratio in different power-law indices, the fundamental frequency parameters decrease. For example, in the value of h_c/h_t=0.5, 2-1-2 type, FG faces sandwiches due to the more quantity of ceramic have stiffer structure than the value of h_c/h_t=8, 1-8-1 type, so the fundamental frequency parameter in 2-1-2 type is higher.
- By increasing the wave number, the fundamental frequency parameter increases.
- By increasing the total thickness of the sandwich beams, the fundamental frequency parameter decreases. The simple support boundary condition is more sensitive than the clamped one.
- The values of the frequencies in the clamped boundary condition are more than simply supported boundary conditions.

5. Appendix 1

In the relations of the face sheets, The "N"s depict the stress resultants and the "M"s refer to the moment resultants which calculated as follows [23]:

$$N_{xx}^{j} = A_{11}u_{0,x}^{j} + B_{11}\phi_{,x}^{j}\,,\; j=(t,b) \qquad \text{(a)}$$

$$M_{xx}^{j} = B_{11}u_{0,x}^{j} + D_{11}\phi_{,x}^{j} \qquad \text{(b)}$$

$$M_{xx}^{j} = \pi^2/12\, A_{55}(\phi^{j} + w_{0,x}^{j}) \qquad \text{(c)}$$

The constant coefficients A1lj and A55j, B11j and D11j indicate the stretching, bending-stretching, and bending stiffnesses, respectively, which are obtained by:

$$\begin{Bmatrix} A_{11}^{j} \\ B_{11}^{j} \\ D_{11}^{j} \end{Bmatrix} = \int_{-hj/2}^{hj/2} \left(\frac{E_j}{1-\nu_j^{\,2}} \right) \begin{Bmatrix} 1 \\ z_j \\ z_j^2 \end{Bmatrix} dz_j$$

$$\left\{ A_{55}^{j} \right\} = \int_{-hj/2}^{hj/2} \left(\frac{E_j}{1+2\nu_j} \right) dz_j \qquad \text{(d)}$$

Where E, ν and α are Young's modulus, the Poisson’s ratio, and the thermal expansion coefficient, respectively, which in the functionally graded layers are the function of the displacement.

The inertia terms of the face sheets and the core are calculated as follows:

$$\left(I_{0j},I_{1j},I_{2j}\right)=\int_{-hj/2}^{hj/2}\rho_j\left(1,z_j,z_j^2\right)dz_j,\qquad \left(j=t,b\right) \tag{e}$$

$$\left(I_{0c},I_{1c},I_{2c},I_{3c},I_{4c},I_{5c},I_{6c}\right)=\int_{-hc/2}^{hc/2}\rho_c\left(1,z_c,z_c^2,z_c^3,z_c^4,z_c^5,z_c^6\right)dz_c \tag{f}$$

The out-of-plane and in-plane stresses in the core leads to the high order resultants:

$$Q_{xc},M_{Q1xc},M_{Q2xc}=\int_{-hc/2}^{hc/2}\left(1,z_c,z_c^2\right)\sigma_{xz}^c dz_c \tag{g}$$

$$R_{zc},M_{zc}=\int_{-hc/2}^{hc/2}\left(1,z_c\right)\sigma_{zz}^c dz_c \tag{h}$$

$$R_x^c,M_{x1}^c,M_{x2}^c,M_{x3}^c=\int_{-hc/2}^{hc/2}\left(1,z_c,z_c^2,z_c^3\right)\sigma_{xx}^c dz_c \tag{i}$$

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