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Research article

Convergence of Legendre and Chebyshev multiwavelets in Petrov-Galerkin method for solving Fredholm integro-differential equations of high orders

Saeed Akhavan*

Department of Mathematics, Khomeinishahr Branch, Islamic Azad University, Isfahan, 84175-119, Iran

*akhavan@iaukhsh.ac.ir

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Abstract

This work was intended as an attempt to motivate readers for a comparison study of constructions of Legendre multiwavelet and Chebyshev multiwavelet. It is also shown how to use them in Petrov-Galerkin approach for solving Fredholm integro-differential equation of high orders of the second kind. In fact, a numerical technique for the discretization method of Fredholm integro-differential equations is presented that yields linear system. The important point to note here is the convergence of presented methods. For the first time, two conditions are proved for convergence of Legendre and Chebyshev multiwavelets in Petrov-Galerkin method. The proof of these conditions with using linear algebra and matrix theory ensures that Petrov-Galerkin methods has a unique approximation. Finally, some relevent numerical examples, for which the exact solution is known, will indicate accuracy and applicability of the proposed method.

Keywords: Fredholm integro-differential equations; Petrov-Galerkin method; Legendre multiwavelet; Chebyshev multiwavelet.

1- Introduction and preliminaries

In a variety of scientific applications such as the theory of signal processing and neural networks arise an integro-differantial equation that is an equation involving one or more unknown functions, together both differential and integral operations [1,2,3]. The linear m th order ordinary Fredholm integro-differential equation of the second kind has the following general form

$$\sum_{i=0}^{m} a_i(t)x^{(i)}(t) - \int_a^b k(s,t)x(s)ds = f(t)\,,\ a \le t \le b \qquad (1.1)$$

where the function $f(t)$, the kernel $k(s,t)$ and $a_i(t)$ for each $i = 0,1,\dots,m$ are known and $x(t)$ the exact solution is unknown. It is necessary to define initial conditions $x(0), x'(0), \dots, x^{(m-1)}(0)$ for the determination of the particular solution $x(t)$ of eq.(1.1). The direct computation method and the Taylor series method are used for eq.(1.1) in [3] and numerical methods including quadrature, collocation and Galerkin methods for eq.(1.1) are used and their analysis may be found in [4,5].

Many approaches for the numerical solutions of these kinds of equations can be found in other literature. For example, spline collocation [22], analytical Lie group approach [21], fractional differential transform [25], least-squares [17], rationalized Haar function [20], exp-function method [16], and many others. In [23], a novel Legendre wavelet Petrov–Galerkin method was presented for fractional Volterra integro-differential equations. The Chebyshev wavelet method [18] has been used to nonlinear fractional Volterra-Fredholm integro-differential equations (FVFIDEs) with mixed boundary conditions. In [23], a novel Legendre wavelet Petrov–Galerkin method was presented for fractional Volterra integro-differential equations. The Chebyshev wavelet method [18] has been used to nonlinear fractional Volterra-Fredholm integro-differential equations (FVFIDEs) with mixed boundary conditions.

In this work, we restrict our attention to the following linear second order Fredholm integro-differential equation of the second kind with two boundary conditions

$$\sum_{i=0}^{m} a_i(t)x^{(i)}(t) - \int_a^b k(s,t)x(s)ds$$
$$= f(t),\ x(a) = \alpha\ ,\ x(b) = \beta \qquad (1.2)$$

Wavelets basis are already applied in order to solve various kinds of integral equation. In [12] Maleknejad and Sohrabi used Legendre wavelets, Shang and Han in [13] applied the Legendre multiwavelets. In [14] and [15] Lepik, Gu and Jiang proposed non-uniform Haar wavelets and Trigonometric Hermit wavelets too.

A class of like-wavelet basis for $L^2[0,1]$ are constructed and applied for approximating the solution of The Fredholm integral equation of the second kind [1]. Alpert, in [6] has employed Galerkin numerical method. In [7,8], the wavelet Petrov-Galerkin schemes based on discontinuous orthogonal multiwavelets were described. In this paper, we use Alperts multiwavelets based on Legendre polynomials and Chebyshev polynomials by using Petrov-Galerkin approach for solving eq.(1.2)

The Petrov-Galerkin method for Fredholm integral equations has been studied in [7]. By [2] we realize that in this method, we can choose two different spaces for the trial space and the test space unlike Galerkin method. This is an advantage because its order of convergence is similar to the Galerkin method without lacking computational cost. A various paper for applications of this method are published $[8-10]$.

An important point in this paper is convergence of presented methods which are based on properties of operational matrices [11].

This paper is organized as follows: Section 2 describe constructions of Legendre multiwavelet and Chebyshev multiwavelet. In third section, a brief review of the Petrov-Galerkin method and its convergence are given and convergence of this method with two different bases are proved. Section 4 exhibits a numerical method for transferring an integro-differential equation to a linear system by Petrov-Galerkin method. Section 5 illustrates some numerical examples to show the accuracy and applicably of Legendre and Chebyshev multiwavelets. Finally, section 6 concludes the paper.

2- Legendre and Chebyshev multiwavelets

In recent years, the various basic functions have been used to estimate the solution of integral equations. In this work, we review construction of two bases for $L^2[0,1]$ that each basis is comprised of dilates and translates of a finite set of functions $h_1, h_2, \ldots, h_k$. In particular, these bases consist of orthonormal systems

$$h_{j.m}^{n}(x) = 2^{m/2} h_j(2^m x - n)\ ,\ j = 1, \ldots, k; m, n \in Z$$

where the functions $h_{j,m}^{n}(x)$ are dilates and translates of the functions $h_1, h_2, \ldots, h_k$ that are piecewise polynomials. Two properties of them are vanishing outside the interval [0,1], and being orthogonal to low-order polynomials (have vanishing moments)

$$\int_0^1 h_j(x)x^i dx = 0\ ,\ j = 1,2,\dots,k\ ,\ i = 0,1,\dots,k-1$$

Suppose that $k \in N$ and $m = 0,1,2,\dots$, we can define a space S_m^k of piecewise polynomial functions,

$$S_m^k = \left\{ f : f(x) = \begin{cases} \text{a polynomial of degree} \le \text{k} & ; \ \frac{n}{2^m} \le x \le \frac{n+1}{2^m} \\ 0 & ; \ \text{otherwise} \end{cases} \right\}$$

where $n = 0,1,\dots,2^m - 1$. We can see $\dim S_m^k = 2^m k$ and

$$S_0^k \subset S_1^k \subset \cdots \subset S_m^k \subset \cdots$$

If the multi-resolution analysis is implied, the $2^m k$ dimensional space R_m^k can be defined such that being the orthogonal complement of S_m^k in S_{m+1}^k,

$$S_m^k \oplus R_m^k = S_{m+1}^k\ ,\ R_m^k \perp S_m^k$$

We can now composite S_m^k like follow:

$$S_m^k = S_0^k \oplus R_0^k \oplus R_1^k \oplus \cdots \oplus R_{m-1}^k$$

If we suppose functions $h_1, h_2, \dots, h_k : R \to R$ form an orthogonal basis for R_0^k, we could define the k functions $f_1, f_2, \dots, f_k : R \to R$, supported on $[-1,1]$, with the following form that help us for defining the functions h_i :

$$f_i(x) = \begin{cases} p_{k-1}(x) & 0 \le x \le 1 \\ (-1)^{i+k-1} p_{k-1}(-x) & -1 \le x \le 0 \end{cases},\ i = 1,2,\dots,k$$

where $p_{k-1}(x)$ is a polynomial of degree $k-1$ with indeterminate coefficients. These functions have the following properties:

1 The functions $f_1, f_2, \dots, f_k$ have the following orthogonality and normality conditions (orthonormality):

$$\int_{-1}^1 w(x) f_i(x) f_j(x) dx \equiv \langle f_i, f_j \rangle = \delta_{ij}\ ,\ i,j = 1,\dots,k$$

2 Moments of function f_j vanish

$$\int_{-1}^1 w(x) f_j(x) x^i dx = 0\ ,\ i = 0,1,\dots,j+k-2$$

We can now define $h_1, h_2, \dots, h_k$ by

$$h_i(x) = \sqrt{2} f_i(2x-1)\ ,\ i = 1,\dots,k$$

and obtain the equality

$$R_0^k = \text{ Linear span } \{h_i(x) : i = 1,\dots,k\}$$

and, more generally

$$R_m^k = \text{ Linear span } \left\{ h_{j,m}^n(x), j = 1,\dots,k; n = 0,\dots,2^m - 1 \right\}$$

we will perform continue of this process in two different next subsections.

2-1- Construction of Legendre multiwavelet

In this subsection, we study construction of Legendre multiwavelet. We let S_0^k the trial space be the space of polynomials of degree less than k on $[0,1]$ for each $k \in N$ and them vanish elsewhere. In this case, we suppose

$$S_0^k = \text{ Linearspan } \{L_0(x), L_1(x), \dots, L_{k-1}(x)\}$$

where $L_i(x)$ are the orthonormal Legendre polynomials and implemented in Mathematica as $l_i[t] =$ LegendreP $[i,t]; L_{i+1}[t] = \sqrt{2i} + 1$ Expand $[l_i[2t-1]]$. For making R_0^k the test space, we have to derive all $f_i(x)$ for each k. For example, suppose $k = 3$ then

$$f_1(x) = \begin{cases} ax^2 + bx + c & 0 \le x \le 1 \\ -ax^2 + bx - c & -1 \le x \le 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f_2(x) = \begin{cases} dx^2 + ex + f & 0 \le x \le 1 \\ dx^2 - ex + f & -1 \le x \le 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f_3(x) = \begin{cases} gx^2 + hx + i & 0 \le x \le 1 \\ -gx^2 + hx - i & -1 \le x \le 0 \\ 0 & \text{otherwise} \end{cases}$$

Properties 1 and 2 in section 2 yield a nonlinear system which its solutions are all unknown coefficients. Although this system do not have unique solution, you can uniquely see all $f_i(x)$ for each k in [6]. After that, $h_1(x), h_2(x), h_3(x)$ can be derived from them and a basis can be formed for R_m^k for each m, k.

3- Construction of Chebyshev multiwavelet

This construction is similar to previous construction but it have two differencese in respect to previous.

1. V_m^k the trial space consist of Chebyshev polynomials that implemented in Mathematica as

$$T_0[t] = \sqrt{1/\pi}, w_i[t] = \text{ Chebyshev } T[i,t]; T_i[t] = \sqrt{2/\pi}\text{Expand}[w_i[2t-1]] i = 1, \dots, k$$

therefore

$$V_0^k = \text{ Linearspan } \{T_0(x), T_1(x), \dots, T_{k-1}(x)\}$$

2. In this case, inner product define as follow

$$\langle f_i, f_j \rangle = \int_{-1}^{1} \frac{f_i(x) f_j(x)}{\sqrt{1-x^2}} dx = \delta_{ij}$$

where the weight function for interval [0,1] is $\frac{1}{\sqrt{x-x^2}}$.

4- The Petrov-Galerkin method and its convergence

In this section, we present a brief review of the Petrov-Galerkin method and conditions of its convergence. We follow the notations of [1]. If X is a Banach space with the norm $\|.\|$ and X^* is its dual space, two different sequences of finite dimensional subspaces $X_n \subseteq \mathbf{X}$ and $Y_n \subseteq X^*$ can be chosen such that satisfying the condition (H) :

(H): For each $x \in \mathbf{X}$ and $y \in X^*$, there exist $x_n \in X_n$ and $y_n \in Y_n$ such that

- $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$ as $n \to \infty$
- $\dim X_n = \dim Y_n \; n = 1,2, \dots$

In Petrov-Galerkin method that is a numerical method, we seek $x_n \in X_n$ such that each $y_n \in Y_n$ be orthogonal on both side of eq.(1.1).

$$\left\langle \left(\sum_{i=0}^{m} a_i(t) D^{(i)} - K \right) x_n, y_n \right\rangle = \langle f, y_n \rangle \text{ for all } \mathrm{y_n} \in \mathrm{Y_n}$$

On the other hand, for $x \in X$, an element $p_n x \in X$ is called a generalized best approximation from X_n to x with respect to Y_n if it satisfies the equation

$$\langle x - p_n x, y_n \rangle = 0 \text{ for all } y_n \in Y_n$$

Thereupon, the Petrov-Galerkin method is a projection method with a generalized best approximation projection. For existence and uniqueness of the generalized best approximation, the following proposition exists:

Proposition 3.1. For each $x \in X$, the generalized best approximation from X_n to x with respect to Y_n exists uniquely if and only if

$$Y_n \cap X_n^{\perp} = \{0\}$$

where $X_n^{\perp}$ denotes the annihilator of X_n in X^* that is the set of all functions satisfying a given set of conditions which is zero on every member of a given set and say that $X_n \perp Y_n$ if $Y_n \cap X_n^{\perp} \neq \{0\}$. By this condition p_n is a projection.

For the proof we refer the reader to [1].

But this condition is not sufficient for insurance every $x \in X$ has a unique Petrov-Galerkin approximation. Therefore, the new concept of the regular pair should introduce right here.

Definition 3.2. If there exists a linear operator $\Pi_n: X_n \to Y_n$ with $\Pi_n X_n = Y_n$ such that satisfying the condition

$$\|x_n\| \le c_1 \langle x_n, \Pi_n x_n \rangle^{1/2} \text{ for all } x_n \in X_n$$

$$\|\Pi_n x_n\| \le c_2 \|x_n\| \text{ for all } x_n \in X_n$$

where c_1 and c_2 are positive constants independent of n. The $\{X_n, Y_n\}$ is called a regular pair.

On the other hand, if X_n and Y_n satisfy the condition (H) and $\{X_n, Y_n\}$ be a regular pair, the following statements drive:

1 $\|P_n x - x\| \to 0$ as $n \to \infty$, for all $x \in X$.

2 $\|P_n x - x\| \le C\|Q_n x - x\|$ for some constant $C > 0$ independent of n.

It means, for ensuring existence and uniqueness of approximation solution for every $x \in X$, we have to consider the condition (H), and the conditions $(H-1)$,$(H-2)$ for each construction separately. If S_m^k and $S_{m'}^{k'}$ are chosen such that $\dim S_m^k = \dim S_{m'}^{k'}$ and V_m^k and $V_{m'}^{k'}$ such that $\dim V_m^k = \dim V_{m'}^{k'}$, the condition (H) will satisfy and by assumption linear operation $\Pi_n: S_m^k \to S_{m'}^{k'}$ as follow:

$$\Pi_n(x_n(t)) = \Pi_n\left(\sum_{j=1}^{2^m k} c_j b_j(t)\right) = \sum_{j=1}^{2^{m'} k'} \left(c_j d_j(t)\right) \qquad (4.1)$$

the conditions $(H-1), (H-2)$ can be proved in two subsections.

4-1- Convergence of Legendre multiwavelet and Chebyshev multiwavelet

$(H-1)$: By definition $\Pi_n X_n = Y_n$ and the norm $\|.\|$, the following relation conclude that

$$\langle x_n, \Pi_n x_n \rangle = \int_0^1 x_n(t) \Pi_n(x_n(t)) dt \qquad (4.2)$$

Now, $x_n(t)$ and $\Pi_n(x_n(t))$ are approximated in S_m^k and $S_{m'}^{k'}$

$$= \int_0^1 \left(\sum_{j=1}^{2^m k} c_j b_j(t)\right)\left(\sum_{j=1}^{2^{m'} k'} \left(c_j d_j(t)\right)\right) dt$$

By assumption $\dim S_m^k = \dim S_{m'}^{k'}$, we can write

$$= \int_0^1 \left(\sum_{j=1}^{2^m k} c_j b_j(t)\right)\left(\sum_{j=1}^{2^m k} \left(c_j d_j(t)\right)\right) dt$$

$$= \int_0^1 C^T \Phi(t) \Psi^T(t) C dt = C^T B C \qquad (4.3)$$

$$= c_1^2 b_{11} + c_2^2 b_{22} + \cdots + c_N^2 b_{NN} \qquad (4.4)$$

where $[B]_{i,j} = \int_0^1 b_i(t) d_j(t) dt$. Matrices B are diagonal with N positive integer as its diagonal entries that they are eigenvalues of B. Therefore

$$\ge \lambda_{(\mathrm{m}\ B)}(c_1^2 + c_2^2 + \cdots + c_N^2) = \lambda_{(\mathrm{m}\ B)} \|x_n\|$$

With choosing $c_1 = \frac{1}{\sqrt{\lambda_{(\min B)}}}$, this relation can be rewrite as follow

$$\|x_n\| \le \frac{1}{\sqrt{\lambda_{(\mathrm{m}\ B)}}} \langle x_n, \Pi_n x_n \rangle^{1/2}$$

$(H-2)$: Clearly, we have

$$\|\Pi_n x_n\|_2^2 = \int_0^1 \left(\sum_{j=1}^{2^{m'} k'} \left(c_j d_j(t)\right)\right)^2 dt$$

$$= \int_0^1 \left(\sum_{j=1}^{2^m k} \left(c_j d_j(t)\right)\right)^2 dt$$

$$= \int_0^1 \left(C^T \Psi(t)\right)^2 \mathrm{dt} \qquad (4.5)$$

$$= \int_0^1 (C^T \Psi(t)\Psi^T(t)C)dt$$

$$= C^T \left\{ \int_0^1 \left(\Psi(t)\Psi^T(t)\right)dt \right\}$$

$$= C^T I C = \|x_n\|_2^2 \qquad (4.6)$$

This relation shows that the choice of an integer for $c_2 \geq 1$ yields $(H-2)$ condition.

Convergence of Chebyshev multiwavelet have similar characteristics with ago discussion.

5- Numerical method

This section concerns the discretization method of Fredholm integro-differential equation. Operational matrices are used for transferring an integro-differential equation to a linear system by Petrov-Galerkin approach. At first, we know $x_n \in X_n$ and S_m^k forms a basis for the trial space X_n. Further, let $x_n(t)$ be an approximation of exact solution $x(t)$. We can write

$$x_n(t) = \sum_{i=1}^{N} c_i b_i(t) = C^T \Phi(t) \qquad (5.1)$$

where vectors $\mathrm{C} = (\mathrm{c}_1, \mathrm{c}_2, \dots, \mathrm{c}_N)^T$ and $\Phi(t) = (b_1(t), b_2(t), \dots, b_N(t))^T$. If we substitute $x_n(t)$ instead of $x(t)$ in eq. (1.1), we derive

$$\sum_{j=0}^{m} a_j(t) \sum_{q=1}^{N} c_q b_q{}^{(j)}(t) - \int_0^1 k(s,t)\left[\sum_{q=1}^{N} c_q b_q(s)\right] ds = f(t)\,,\; 0 \leq t \leq 1 \qquad (5.2)$$

Or

$$\sum_{j=0}^{m} a_j(t) C^T \Phi^{(j)}(t) - \int_0^1 k(s,t) C^T \Phi(s) ds = f(t) \qquad (5.3)$$

This relation can be simplified as follow:

$$C^T W G - C^T K = f(t) \qquad (5.4)$$

where

$$K = \begin{bmatrix} \int_0^1 k(s,t) b_1(s) ds \\ \int_0^1 k(s,t) b_2(s) ds \\ \vdots \\ \int_0^1 k(s,t) b_N(s) ds \end{bmatrix}$$

$$W = \begin{bmatrix} b_1(t) & b_1^{(1)}(t) & \cdots & b_1^{(m)}(t) \\ b_2(t) & b_2^{(1)}(t) & \cdots & b_2^{(m)}(t) \\ \vdots & \vdots & \vdots & \vdots \\ b_N(t) & b_N^{(1)}(t) & \cdots & b_N^{(m)}(t) \end{bmatrix}$$

$$,\ G = \begin{bmatrix} a_0(t) \\ a_1(t) \\ \vdots \\ a_m(t) \end{bmatrix}$$

We now inner multiply both side in each element of Y_n basis, where $S_{m'}^{k'}$ forms a basis for Y_n (where $2^m k = \dim S_m^k = \dim S_{m'}^{k'} = 2^{m'} k'$ and $k \geq \begin{cases} m+1 & m = 2k_1 - 1 \\ m+2 & m = 2k_1 \end{cases}\Big)$.

$$C^T \int_0^1 W G \Psi^T(t) dt - C^T \int_0^1 K \Psi^T(t) dt = \int_0^1 f(t)\Psi(t) dt \qquad (5.5)$$

where $\Psi(t) = (d_1(t), d_2(t), \dots, d_N(t))^T$. The above system have the following matrix form

$$C^T [R - M] = F \qquad (5.6)$$

or

$$[R - M]^T C = F$$

where

$$[R]_{i,j} = \int_0^1 [WG]_i d_j(t)dt,$$

$$[M]_{i,j} = \int_0^1 [K]_i d_j(t)dt$$

$$= \int_0^1 \int_0^1 k(s,t)b_i(s)d_j(t)dsdt$$

In the (5.6) system, we could use two exact equations instead of some two row of approximation equations. These two additional equations derive from boundary conditions.

$$\begin{cases} \sum_{i=1}^{N} b_i(0) = \alpha \\ \sum_{i=1}^{N} b_i(1) = \beta \end{cases}$$

Solution of new system will derive the approximation solution.

But for Chebyshev multiwavelet, with substituting $x_n(t) = \sum_{i=1}^{N} c_i b_i(t)$ instead of $x(t)$ in (1.2), the equation (5.2) is yielded.

We have to inner multiply both side of (5.2) in each element of Y_n basis that the weight function of them is $w(t) = \frac{1}{\sqrt{t^2-t}}$. On the other hand $V_{m'}^{k'}$ forms a basis for Y_n, where $2^m k = \dim V_m^k = \dim V_{m'}^{k'} = 2^{m'}k'$ and

$$k \geq \begin{cases} m+1 & m = 2k_1 - 1 \\ m+2 & m = 2k_1 \end{cases}\Bigg)$$

therefore

$$C^T \left[\int_0^1 \frac{WG\Psi^T(t)}{\sqrt{t^2-t}} dt - C^T \int_0^1 \frac{K\Psi^T(t)}{\sqrt{t^2-t}} dt \right] = \int_0^1 \frac{f(t)\Psi(t)}{\sqrt{t^2-t}} dt \qquad (5.7)$$

The system (5.7) have the following matrix form

$$[R-M]^T C = F \qquad (5.8)$$

where

$$[R]_{i,j} = \int_0^1 \frac{[WG]_i d_j(t)}{\sqrt{t^2-t}} dt\,,\ [M]_{i,j} = \int_0^1 \int_0^1 \frac{k(s,t)b_i(s)d_j(t)}{\sqrt{t^2-t}} dsdt$$

In the (5.8) system, we can use some two exact equations instead of two approximation equations. Two additional equations will derive from boundary conditions.

$$\begin{cases} \sum_{i=1}^{N} b_i(0) = \alpha \\ \sum_{i=1}^{N} b_i(1) = \beta \end{cases}$$

Solution of (5.8) system derive the approximation solution.

6- Numerical results

In the follow examples, we use both Legendre multiwavelet and Chebyshev multiwavelet bases for Petrov-Galerkin method with different value of k, n. The computations associated with the examples were performed using Mathematica 8 software on a personal computer.

Example 6.1

$$t^2x''(t) - tx'(t) - 3x(t) - \int_0^1 \frac{e^t \sin t}{1+s^2} x(s)\mathrm{d}s = \frac{1}{2}e^t(-1+\ln 2)\sin t, 0 \leq t \leq 1$$

$$x(0) = 0$$

$$x(1) = 1$$

with exact solution $x(t) = t^3$. In Tables 1,2 $\left\| x_n\left(t_j^{(i)}\right) - x\left(t_j^{(i)}\right) \right\|_\infty$ are computed respectively with Legendre multiwavelet and Chebyshev multiwavelet.

Table 1: Legendre multiwavelet

X_n, Y_n	$\left\Vert x_n\left(t_j^{(i)}\right) - x\left(t_j^{(i)}\right)\right\Vert_\infty$
S_0^4, S_1^2	$4.82916 * 10^{-11}$
S_0^6, S_1^3	$5.23078 * 10^{-11}$
S_0^8, S_1^4	$4.86447 * 10^{-11}$
S_1^4, S_2^2	$1.406651 * 10^{-10}$

Table 2: Chebyshev multiwavelet

X_n, Y_n	$\left\Vert x_n\left(t_j^{(i)}\right) - x\left(t_j^{(i)}\right)\right\Vert_\infty$
V_0^4, V_1^2	$7.75844 * 10^{-10}$
V_0^6, V_1^3	$7.36472 * 10^{-10}$
V_0^8, V_1^4	$1.1147 * 10^{-8}$

Example 6.2

$$\frac{1}{2}e^{\frac{1}{3}}x'(t) + x(t) - \int_0^1 \left(-\frac{1}{3}e^{2t-\frac{5}{3}s}\right)x(s)\mathrm{d}s$$

$$= 2e^{2t+\frac{1}{3}},\ 0 \le t \le 1$$

$x(0) = 1$

$x(1) = e^2$

with exact solution $x(t) = e^{2t}$. In Tables 3,4 $\left\Vert x_n\left(t_j^{(i)}\right) - x\left(t_j^{(i)}\right)\right\Vert_\infty$ are computed respectively with Legendre multiwavelet and Chebyshev multiwavelet

Table 3: Legendre multiwavelet

X_n, Y_n	$\left\Vert x_n\left(t_j^{(i)}\right) - x\left(t_j^{(i)}\right)\right\Vert_\infty$
S_0^4, S_1^2	$2.48764 * 10^{-2}$
S_0^6, S_1^3	$6.27907 * 10^{-4}$
S_0^8, S_1^4	$1.99096 * 10^{-6}$
S_1^4, S_2^2	$2.62349 * 10^{-1}$

Table 4: Chebyshev multiwavelet

X_n, Y_n	$\left\Vert x_n\left(t_j^{(i)}\right) - x\left(t_j^{(i)}\right)\right\Vert_\infty$
V_0^2, V_1^1	1.51332
V_0^4, V_1^2	$7.31826 * 10^{-2}$
V_0^6, V_1^3	$5.21438 * 10^{-4}$
V_0^8, V_1^4	$2.81471 * 10^{-6}$

Example 6.3

$$-t(t+1)^2x''(t) - \frac{23}{18}t(t+1)x'(t) + x(t)$$
$$-\int_0^1 \left(s^2t - \frac{3}{2}st^2\right)x(s)ds = \ \le 1$$
$$\frac{3}{4}t^2 - \frac{4}{3}\mathrm{Ln}(2)t + 2\mathrm{Ln}(t+1),\ 0 \le t \le 1$$

$x(0) = 0$

$x(1) = 2\mathrm{Ln}(2)$

with exact solution $x(t) = 2\mathrm{Ln}(t+1)$. In Tables 5,6 $\left\Vert x_n\left(t_j^{(i)}\right) - x\left(t_j^{(i)}\right)\right\Vert_\infty$ are computed respectively with Leqendre multiwavelet and Chebyshev multiwavelet.

Table 5: Legendre multiwavelet

X_n, Y_n	$\left\Vert x_n\left(t_j^{(i)}\right) - x\left(t_j^{(i)}\right)\right\Vert_\infty$
S_0^4, S_1^2	$3.80754 * 10^{-3}$
S_0^6, S_1^3	$4.54463 * 10^{-5}$
S_0^8, S_1^4	$7.21549 * 10^{-7}$
S_1^4, S_2^2	$3.35981 * 10^{-2}$

Table 6: Chebyshev multiwavelet

X_n, Y_n	$\left\| x_n\left(t_j^{(i)}\right) - x\left(t_j^{(i)}\right) \right\|_\infty$
V_0^4, V_1^2	$3.48248 * 10^{-3}$
V_0^6, V_1^3	$9.17848 * 10^{-5}$
V_0^8, V_1^4	$1.79712 * 10^{-6}$

7- Conclusion

In this work, convergence of Petrov-Galerkin method with Legendre and Chebyshev multiwavelets bases are investigated. For the first time, conditions of them are proved with using linear algebra and matrix theory. After that, Fredholm integro-differential equations of the second kind are solved by using Legendre and Chebyshev multiwavelets via Petrov-Galerkin approach. The discretization method of Fredholm integro-differential equations is performed with using operational matrices that yields linear system. Under numerical results, this matter is realized that Legendre and Chebyshev multiwavelets in solving the Fredholm integro-differential equations of the second kind have approximately the same accurate.

In the end, this method can be easily extended and applied to multi-dimensional integral equations or systems of FVFIDEs easily with some modifications. We also believe that it shall not be difficult to extend this approach to nonlinear equations of general form, which will be the subject of future researches.

References

[1] Abdul-Majid, W. A. Z. W. A. Z. (2011). Linear and Nonlinear Integral Equations: Methods and Applications.

[2] Davis, H. T. (1962). Introduction to Nonlinear Differential Integral Equations, US Atomic Energy Commission, Washington, DC, 1960. 592 p. *Reprint: Dover, NewYork*.

[3] Kanwal, R. P. (2013). *Linear integral equations*. Springer Science & Business Media.

[4] Maleknejad, K., Rabbani, M., Aghazadeh, N., & Karami, M. (2009). A wavelet Petrov–Galerkin method for solving integro-differential equations. *International Journal of Computer Mathematics*, *86*(9), 1572-1590.

[5] Mustapha, K. (2008). A Petrov--Galerkin method for integro-differential equations with a memory term. *ANZIAM Journal*, *50*, C610-C624.

[6] Alpert, B. K. (1993). A class of bases in L^2 for the sparse representation of integral operators. *SIAM journal on Mathematical Analysis*, *24*(1), 246-262.

[7] Chen, Z., & Xu, Y. (1998). The Petrov--Galerkin and Iterated Petrov--Galerkin Methods for Second-Kind Integral Equations. *SIAM Journal on Numerical Analysis*, *35*(1), 406-434.

[8] Chen, Z., Micchelli, C. A., & Xu, Y. (1997). The Petrov–Galerkin method for second kind integral equations II: multiwavelet schemes. *Advances in Computational Mathematics*, *7*(3), 199-233.

[9] Maleknejad, K., & Karami, M. (2003, August). The Petrov—Galerkin method for solving second kind Fredholm integral equations. In *34th Iranian Mathematics Conference* (Vol. 30).

[10] Bialecki, B., Ganesh, M., & Mustapha, K. (2004). A Petrov–Galerkin method with quadrature for elliptic boundary value problems. *IMA journal of numerical analysis*, *24*(1), 157-177.

[11] Jiang, Z. H. (1992). W. Schaufelberger, block Pulse Functions and Their Applications in Control Systems.

[12] Maleknejad, K., & Sohrabi, S. (2007). Numerical solution of Fredholm integral equations of the first kind by using Legendre wavelets. *Applied Mathematics and Computation*, *186*(1), 836-843.

[13] Shang, X., & Han, D. (2007). Numerical solution of Fredholm integral equations of the first kind by using linear Legendre multi-wavelets. *Applied Mathematics and Computation*, *191*(2), 440-444.

[14] Guf, J. S., & Jiang, W. S. (1996). The Haar wavelets operational matrix of integration. *International Journal of Systems Science*, *27*(7), 623-628.

[15] Lepik, Ü. (2008). Solving integral and differential equations by the aid of non-uniform Haar wavelets. *Applied Mathematics and Computation*, *198*(1), 326-332.

[16] Guner, O., & Bekir, A. (2017). Exp-function method for nonlinear fractional differential equations. *Nonlinear Sci. Lett. A*, *8*(1), 41-49.

[17] Mahdy, A. M., & Mohamed, E. M. (2016). Numerical studies for solving system of linear fractional integro-differential equations by using least squares method and shifted Chebyshev polynomials. *Journal of Abstract and Computational Mathematics*, *1*(1), 24-32.

[18] Mohyud-Din, S. T., Khan, H., Arif, M., & Rafiq, M. (2017). Chebyshev wavelet method to nonlinear fractional Volterra–Fredholm integro-differential equations with mixed boundary conditions. *Advances in Mechanical Engineering*, *9*(3), 1687814017694802.

[19] Nazari, D., & Shahmorad, S. (2010). Application of the fractional differential transform method to fractional-order integro-differential equations with nonlocal boundary conditions. *Journal of Computational and Applied Mathematics*, *234*(3), 883-891.

[20] Ordokhani, Y., & Rahimi, N. (2014). Numerical solution of fractional Volterra integro-differential equations via the rationalized Haar functions. *J. Sci. Kharazmi Univ*, *14*(3), 211-224.

[21] Pashayi, S., Hashemi, M. S., & Shahmorad, S. (2017). Analytical lie group approach for solving fractional integro-differential equations. *Communications in Nonlinear Science and Numerical Simulation*, *51*, 66-77.

[22] Pedas, A., & Vikerpuur, M. (2021). Spline Collocation for Multi-Term Fractional Integro-Differential Equations with Weakly Singular Kernels. *Fractal and Fractional*, *5*(3), 90.

[23] Sahu, P. K., & Ray, S. S. (2016). WITHDRAWN: A novel Legendre wavelet Petrov–Galerkin method for fractional Volterra integro-differential equations.

[24] Riahi Beni, M. (2021). Legendre wavelet method combined with the Gauss quadrature rule for numerical solution of fractional integro-differential equations. *Iranian Journal of Numerical Analysis and Optimization*.