Journal of Solid Mechanics Vol. 10, No. 4 (2018) pp. 874-893

Free Axisymmetric Bending Vibration Analysis of two Directional FGM Circular Nano-plate on the Elastic Foundation

M. Zarei *, Gh. Rahimi

Department of Mechanical Engineering, Tarbiat Modares University (TMU), Tehran, Iran

Received 3 August 2018; accepted 5 October 2018

ABSTRACT

In the following paper, free vibration analysis of two directional FGM circular nano-plate on the elastic medium is investigated. The elastic modulus of plate varies in both radial and thickness directions. Eringen's theory was employed to the analysis of circular nano-plate with variation in material properties. Simultaneous variations of the material properties in the radial and transverse directions are described by a general function. Ritz functions were utilized to obtain the frequency equations for simply supported and clamped boundary. Differential transform method also used to develop a semi-analytical solution the size-dependent natural frequencies of non-homogenous nano-plates. Both methods reported good results. The validity of solutions was performed by comparing present results with themselves and those of the literature for both classical plate and nano-plate. Effect of non-homogeneity on the nonlocal parameter, geometries, boundary conditions and elastic foundation parameters is examined the paper treats some interesting problems, for the first time.

Keywords : Eringen's theory; Free vibration; FGM nano-plate; Ritz method; Differential transform method.

1 INTRODUCTION

THE micro and nano-sized structures have engrossed widespread attention in modern technology fields as components in micro/nano electro mechanical systems (MEMS and NEMS) [1,2]. Due to small scale effects, these structures have outstanding mechanical, thermal and electrical properties in comparison with ordinary scale structures [3]. Thus, they can be employed in sensitive devices and high performance application such as gas detection graphene sensor, ultra capacitors and ultra-strength composite material [3-5]. The continuum mechanic based approaches offer the benefits of less computational cost in comparison with atomistic modeling and hybrid modeling methods [5]. However, there is a need for modifying classical continuum mechanic models to take into account micro/nanoscale effects. Various higher mode continuum or micro-continuum theories have received great attention to capturing size effects through continuum mechanics models such as strain gradient theory [6-10], couple stress theory [11-14], and nonlocal elasticity theory [10,15,16]. Among these theories, the theory of nonlocal elasticity presented by Eringen [15] assumes that stress at an individual point is a function of all other points in body

*Corresponding author.
E-mail address: Mehdi.zarei@modares.ac.ir (M. Zarei).

domain. Many researchers have utilized the nonlocal elasticity theory for buckling and free vibration analysis of nano-sized structures. As a pioneer, Peddieson [16,17] showed this theory could be employed in the analysis of nano structures. recently, this theory has incorporated in Functionally graded material (FGM) isotropic beams [18,19] and plate models [20,21]. Şimşek and Yurtcu [22] studied static and buckling behaviour of nonlocal functionally graded (FG) Timoshenko and Euler–Bernoulli beam. Murmu and Pradhan [23] investigated stability response of single-walled carbon nanotubes using Timoshenko beam theoryAksencer and Aydogdu [24] studied vibration and buckling of nano-plates for clamped and simply supported boundary condition. Narenda [25] presented buckling analysis of isotropic nano-plate using two-variable refined plate theory. Farajpour et al. [26] investigated buckling behavior of nano-scale circular plate subjected to in-plane forces using nonlocal elasticity. Hashemi-Hosseini et al. [20] studied free vibration behaviour of thick circular disks for various boundary conditions. Variable thickness is utilized to redistribute stress and reduce weight [27]. Various works related to variable thickness structures are found in several references ([28-31]). Danesh et al. [30] explored axial vibration behavior of variable thickness nano-rod using nonlocal elasticity theory. They reported result for various boundary conditions obtained by Differential Quadrature Method (DQM). Efraim and Eisenberger [32] provided an analytical solution for vibration analysis of variable thick annular FG plates employing first order shear deformation.

As can be observed in the literature, most studies presented solutions for problems with simple geometries. The Differential Transform Method (DTM) and Ritz methods can be used to model nano-sized structures with general boundary conditions. The Differential Transform Method (DTM) is utilized truncated Taylor series to solve mathematical models [33]. This approach offers highly accurate closed form solution for boundary value problems [34]. Furthermore, Plates resting on the elastic medium has great of the interest in versatile engineering applications such as in non-rigid boundary conditions modeling. Mohammadi et al. [35] investigated free vibration behavior of rectangular graphene sheet under in-plane shear load. The influence of boundary conditions and elastic medium parameters were studied using DQM and Galerkin methods. Pradhan et al. [36] studied nonlocal characteristic length's impact on the vibration of graphene sheet layer. The result showed the nonlocal effect is significant for graphene sheet. Behfar et al and Pradhan et al. [37] explored nano-scale vibration behavior of multilayered graphene on elastic foundation Mirzabeigy [38] considered free vibration of variable thickness beam under axial tensile forces on elastic foundation. Mohammadi et al. [39] free vibration of the circular graphene sheet studied under in-plane forces using nonlocal Kirchhoff plate theory. There many articles that investigated vibrational behavior variable thickness one directional and two-directional-functionally graded plates [40-48]. Zarei et al. [49] investigated the vibrational behavior of non-uniform circular nano-plate embedded in the elastic medium and resulted that varying thickness has a prominent effect on vibration of nano-plate. The Despite widespread use of variable thickness circular plate in all fields, buckling and free vibration of nonlocal variable thickness plates subjected to in-plane forces has not reported up to our knowledge.

In this study, free vibration and buckling analysis of variable thickness plate is conducted based on nonlocal elasticity theory to study size effect phenomena. The appropriate governing equations are obtained using energy methods based on nonlocal Eringen constitutive equation. Those are solved by applying Ritz method and DTM to acquire natural frequencies and buckling loads to study the impact of significant parameters.

2 THEORETICAL BACKGROUND

2.1 Nonlocal elasticity theory

Nonlocal elasticity theory assumes strain of every individual point as a function of all near particles. Therefore, size effect and atomic forces can be taken into account via a characteristic length e_0a [15]. The constitutive law, a differential equation, is as follows:

$$(1-(e_0a)^2\nabla^2)\sigma^{nl}=\sigma \tag{1}$$

where

$$\nabla^2=\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr} \tag{2}$$

where e_0 is an experimentally obtained material constant; is an internal characteristic length which depends on granular size or molecular diameter; parameter l defines an external characteristic length. Also, σ^{nl} is stress induced in a nano-scaled medium due to nonlocal effects.

2.2 Differential transform method
2.2.1 Deriving governing equations

A nano-sized plate assumed in polar coordinate are studied as can be seen in Fig. 1.

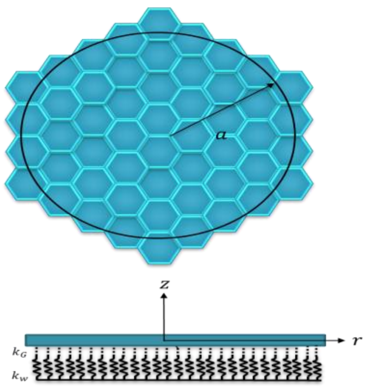


Fig.1
Two directional FGM nano-plate embedded in elastic medium.

According to classical plate theory, the strain-displacement relations neglecting nonlinear terms are as follow:

$$\varepsilon_{rr} = \frac{du}{dr} - z\,\frac{\partial^2 w}{\partial r^2}$$
$$\varepsilon_{\theta\theta} = \frac{u}{r} - z\,\frac{\partial^2 w}{\partial r^2} \tag{3}$$

Using Eq. (1), for a single layer nano-plate the strain-stress relationships are obtained as:

$$\begin{Bmatrix} \sigma_{rr}^{nl} \\ \sigma_{\theta\theta}^{nl} \end{Bmatrix} - \mu^2\nabla^2 \begin{Bmatrix} \sigma_{rr}^{nl} \\ \sigma_{\theta\theta}^{nl} \end{Bmatrix} = \begin{bmatrix} E/(1-\nu^2) & \nu E/(1-\nu^2) \\ \nu E/(1-\nu^2) & E/(1-\nu^2) \end{bmatrix} \begin{Bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \end{Bmatrix} \tag{4}$$

where $E_{11}, E_{22}, \nu_{12}, \nu_{21}$ are longitudinal elastic modulus, transverse elastic modulus, and in-plane Poisson's ratio in 1 and 2 directions, and $\mu = e_0 l_i$ is nonlocal parameter respectively. The plane stress condition is considered due to high radius to thickness ratio, therefore nonlocal stresses of $\sigma_{rr}^{nl}, \sigma_{\theta\theta}^{nl}$ and are only stresses induced in nano-plate.

Stress resultant are defined as:

$$N_r = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{rr}^{nl} dz\,,\; N_\theta = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\theta\theta}^{nl} dz\,, \tag{5}$$

$$M_r = \int_{-h/2}^{h/2} z\,\sigma_{rr}^{nl} dz\,,\; M_\theta = \int_{-h/2}^{h/2} z\,\sigma_{\theta\theta}^{nl} dz \tag{6}$$

Integrating Eq. (4) from $z = -h/2$ to $z = h/2$, we get nonlocal moments as:

$$N_r-\mu^2\nabla^2 N_r=A\left(\frac{du}{dr}+\nu\frac{u}{r}\right)$$
$$N_\theta-\mu^2\nabla^2 N_\theta=A\left(\frac{u}{r}+\nu\frac{du}{dr}\right) \tag{7}$$

Multiplying Eq. (4) in *z* and integrating from $z=-h/2$ to $z=h/2$, nonlocal moments are obtained as:

$$M_r-\mu^2\nabla^2 M_r=-D\left(\frac{d^2w}{dr^2}+\frac{\nu}{r}\frac{dw}{dr}\right)$$
$$M_\theta-\mu^2\nabla^2 M_\theta=-D\left(\nu\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}\right) \tag{8}$$

where nano-plate stiffnesses A, D is defined as:

$$\left(A\ ,D\right)=\left(h,\frac{h^3}{12}\right)\frac{E}{\left(1-\nu^2\right)} \tag{9}$$

By applying the principle of virtual work, we can gain following equations:

$$\frac{d\,(rN_r)}{dr}-N_\theta=0 \tag{10}$$

$$rQ_r=\frac{d\,(rM_r)}{dr}-M_\theta \tag{11}$$

$$\frac{1}{r}\frac{d\,(rQ_r)}{dr}+\frac{1}{r}\frac{d}{dr}\left(rN_r\frac{dw}{dr}\right)=0 \tag{12}$$

Inserting moment resultants in Eq. (12), the governing equation for isotropic type is given as:

$$\begin{aligned}&D\frac{\partial^4 w}{\partial R^4}+\frac{2}{R}\left(D+R\frac{\partial D}{\partial R}\right)\frac{\partial^3 w}{\partial R^3}+\frac{1}{R^2}\left(-D-2n^2D+R\left(2+\nu\right)\frac{\partial D}{\partial R}+R^2\frac{\partial^2 D}{\partial R^2}\right)\frac{\partial^2 w}{\partial R^2}\\&+\frac{1}{R^3}\left(D-R\frac{\partial D}{\partial R}+R^2\frac{\partial^2 D}{\partial R^2}+2n^2D-\frac{2R\,\partial D}{\partial R}\right)\frac{\partial w}{\partial R}+\left(-\frac{4D}{R^4}n^2-3\frac{\partial D}{R^3\partial R}n^2+\frac{\nu}{R^2}\frac{\partial^2 D}{\partial R^2}n^2+\frac{D}{R^4}n^4\right)\\&+k_w\left(w-\mu^2\left(\frac{\partial^2 w}{\partial R^2}+\frac{1}{R}\frac{\partial w}{\partial R}\right)\right)-k_G\left(\left(\frac{\partial^2 w}{\partial R^2}+\frac{1}{R}\frac{\partial w}{\partial R}\right)-\mu^2\left(\frac{\partial^4 w}{\partial R^4}+\frac{2}{R}\frac{\partial^3 w}{\partial R^3}-\frac{1}{R^2}\frac{\partial^2 w}{\partial R^2}+\frac{1}{R^3}\frac{\partial w}{\partial R}\right)\right)\\&-\rho\omega^2\left(1-\mu^2\nabla^2\right)\left(hw\right)=0\end{aligned} \tag{13}$$

Eq. (14) can be rewritten as follows by defining some non-dimensional parameter,

$$r=\frac{R}{a}\quad,\quad W=\frac{w}{a}\quad,\quad \tilde{h}=\frac{h}{a} \tag{14}$$

$$E\left(r,z\right)=E(z)\left(1+\alpha\left(\frac{r}{a}\right)^\beta\right) \tag{15}$$

$$E(z)=\left[\left(E_c-E_m\right)\left(\frac{z}{h}+\frac{1}{2}\right)^g+E_m\right] \tag{16}$$

$$\rho(z)=\left[\left(\rho_c-\rho_m\right)\left(\frac{z}{h}+\frac{1}{2}\right)^g+\rho_m\right] \tag{17}$$

$$\rho=\frac{1}{h}\int_{-h/2}^{h/2}\rho(z)dz=\frac{\rho_c+\rho_m g}{g+1} \tag{18}$$

$$D=\frac{1}{1-\nu^2}\int_{-h/2}^{h/2}E(z)z^2dz\left(1+\alpha\left(\frac{r}{a}\right)^\beta\right)=\frac{h^3}{1-\nu^2}\left[\left(E_c-E_m\right)\frac{g^2+g+2}{4(g+1)(g+2)(g+3)}+\frac{E_m}{12}\right]\left(1+\alpha\left(\frac{r}{a}\right)^\beta\right) \tag{19}$$

$$\begin{aligned}&D\frac{\partial^4 W}{\partial r^4}+\frac{2}{r}\left(D+r\frac{\partial D}{\partial r}\right)\frac{\partial^3 W}{\partial r^3}+\frac{1}{r^2}\left(-D+r\left(2+\nu\right)\frac{\partial D}{\partial r}+r^2\frac{\partial^2 D}{\partial r^2}\right)\frac{\partial^2 W}{\partial r^2}\\&+\frac{1}{r^3}\left(D-r\frac{\partial D}{\partial r}+r^2\frac{\partial^2 D}{\partial r^2}\right)\frac{\partial W}{\partial r}+k_w a^4\left(W-\frac{\mu^2}{a^2}\left(\frac{\partial^2 W}{\partial r^2}+\frac{1}{r}\frac{\partial W}{\partial r}\right)\right)\\&-k_G a^2\left(\left(\frac{\partial^2 W}{\partial r^2}+\frac{1}{r}\frac{\partial W}{\partial r}\right)-\frac{\mu^2}{a^2}\left(\frac{\partial^4 W}{\partial r^4}+\frac{2}{r}\frac{\partial^3 W}{\partial r^3}-\frac{1}{r^2}\frac{\partial^2 W}{\partial r^2}+\frac{1}{r^3}\frac{\partial W}{\partial r}\right)\right)\\&-\rho\omega^2 h a^4\left(W-\frac{\mu^2}{a^2}\left(\frac{\partial^2(W)}{\partial r^2}+\frac{1}{r}\frac{\partial(W)}{\partial r}\right)\right)=0\end{aligned} \tag{20}$$

Eq. (20) may be written as following by assuming linear variation through the thickness.

$$\begin{aligned}&Br^3\left[1+\alpha r^\beta+K_G\frac{\mu^2}{a^2}\right]\frac{\partial^4 W}{\partial r^4}+2Br^2\left(1+\alpha\left(1+\beta\right)r^\beta+K_G\frac{\mu^2}{a^2}\right)\frac{\partial^3 W}{\partial r^3}\\&+Br\left(-1+\left[\beta\left(\beta+1+\nu\right)-1\right]\alpha r^\beta-r^2K_G\left(1+\frac{\mu^2}{a^2}\right)-r^2K_w\frac{\mu^2}{a^2}\right)\frac{\partial^2 W}{\partial r^2}\\&+B\left(1+\left(1-\beta+\nu\beta\left(\beta-1\right)\right)\alpha r^\beta+K_G\left(r^2+\frac{\mu^2}{a^2}\right)-\frac{\mu^2}{a^2}r^2K_w\right)\frac{\partial W}{\partial r}\\&+r^3K_wW=Ar^3\Omega^2W-Ar^3\Omega^2\frac{\mu^2}{a^2}\left[\frac{\partial W}{r\partial r}+\frac{\partial^2 W}{\partial r^2}\right]\end{aligned} \tag{21}$$

where

$$\begin{aligned}&D_0=\frac{E_c h_0^3}{12\left(1-\nu^2\right)},\quad A=\frac{\rho_c+\rho_m g}{\rho_c(g+1)},\quad B=\left[3\left(1-\frac{E_m}{E_c}\right)\frac{g^2+g+2}{4(g+1)(g+2)(g+3)}+\frac{E_m}{E_c}\right]\\&D=D_0B,\quad D_0=\frac{E_c h_0^3}{12\left(1-\nu^2\right)},\quad \Omega^2=\frac{\rho_c h_0 a^4}{D_0},\ K_w=\frac{k_w a^4}{D_0},\ K_G=\frac{k_G a^2}{D_0},\ N_r=\frac{N_0 a^2}{D_0}\end{aligned} \tag{22}$$

Boundary conditions equations, which are needed to find unique solution to governing equation, are given as:
Simply-supported

$$f\left(1\right)=0\ ,\ M_r\ |_{r=1}=\left\{-D\left(\frac{\partial^2 f}{\partial r^2}+\nu\frac{1}{r}\frac{\partial f}{\partial r}\right)\right\}_{r=1}=0 \tag{23}$$

Clamped

$$f\ (1)=0\ ,\ \frac{\partial f}{\partial r}\ |_{r=1}=0 \tag{24}$$

Also, satisfying following equations is needed to avoid infinite value at $r=0$ [54]

$$\frac{\partial f}{\partial r}\ |_{r=1}=0 \tag{25}$$

$$Q_r\ |_{r=0}=\left\{-D\left(\frac{\partial^3 f}{\partial r^3}+\frac{1}{r}\frac{\partial^2 f}{\partial r^2}-\frac{1}{r^2}\frac{\partial f}{\partial r}\right)+\frac{\partial D}{\partial r}\left(\frac{\partial^2 f}{\partial r^2}+\nu\frac{1}{r}\frac{\partial f}{\partial r}\right)\right\}_{r=0}=0 \tag{26}$$

2.2.2 Differential transform method (DTM) procedure

This method is utilized to find a semi-analytic solution for differential equations by transforming them to algebraic equations based on Taylor series. The Taylor series a real valued function $f\,(r)$ that is infinitely differentiable at $r=r_0$ are calculated as following

$$f\ (r)=\sum_0^n\left(r-r_0\right)^k\ \frac{1}{k\,!}\left[\frac{d^k f\ (r)}{dr^k}\right]_{r=r_0} \tag{27}$$

where r_0, $n!$, k are domain center, factorial of *n* and *k th* derivative of $f\,(r)$ obtained at $r=r_0$, respectively. From above Taylor series, Eq. (27), differential transform of the *K th* derivative *pf* $f\,(r)$ is introduced as:

$$F_k=\frac{1}{k\,!}\left[\frac{d^k f\ (r)}{dr^k}\right]_{r=r_0} \tag{28}$$

The differential inverse transform of the F_k is given as:

$$f\ (r)=\sum_0^n\left(r-r_0\right)^k\ F_k \tag{29}$$

The number of terms i.e. value of *n* depends on the convergence of natural frequencies and will be studied in next section. Some useful relations which are frequently used in the applying DTM are given in Table 1. Using this basic property of DTM (Table 1), we obtain transformed form of the Eq. (22) as:

$$\begin{aligned}&\left(k^2-1\right)^2\left(1+K_G\ \frac{\mu^2}{a^2}\right)F_{k+1}=\{\left(k-\beta-2\right)\left(k-\beta-1\right)\left(k-\beta\right)\left(k-\beta+1\right)\\&+2\left(1+\beta\right)\left(k-\beta-1\right)\left(k-\beta\right)\left(k-\beta+1\right)+(1-\beta+\nu_{12}+\beta\left(\beta-1\right)\left(k-\beta+1\right)\\&+\left(\beta\left(\beta+1+\nu_{12}\right)-1\right)\left(k-\beta\right)\left(k-\beta+1\right)\}F_{k-\beta+1}\end{aligned} \tag{30}$$

The recursive relation of Eq. (30) is used repetitively to obtain the coefficients F_i. It can be noted all coefficients are obtained in term of F_0 and F_2 after simplifying. Using above mentioned rules, transformed boundary conditions is obtained as follows:

Simply –supported

$$\sum_0^N F_k = 0 \ . \quad \sum_0^N \left[k\left(k-1\right) + \nu k \right] F_k = 0 \tag{31}$$

Clamped

$$\sum_0^N F_k = 0 \ . \quad \sum_0^N k F_k = 0 \tag{32}$$

Conditions at $r = 0$:

$$f\left(x\right) = \begin{cases} F_1 = 0 \ , F_3 = -\dfrac{2}{9}\beta\left(1+\nu\right)F_2 & n = 1 \\ F_1 = F_3 = F_5 \ldots = F_{2k-1} & n \neq 1 \end{cases} \tag{33}$$

Using boundary conditions above, Eqs. (31) -(33), a homogenous system of linear equations is obtained

$$\begin{aligned} \phi_{11}^{(m)}\left(\Omega\right)F_0 + \phi_{12}^{(m)}\left(\Omega\right)F_2 = 0 \\ \phi_{21}^{(m)}\left(\Omega\right)F_0 + \phi_{22}^{(m)}\left(\Omega\right)F_2 = 0 \end{aligned} \tag{34}$$

For non-trivial solution, determinant of coefficient matrix must vanish and thus

$$\begin{vmatrix} \phi_{11}^{(m)}\left(\Omega\right) & \phi_{12}^{(m)}\left(\Omega\right) \\ \phi_{21}^{(m)}\left(\Omega\right) & \phi_{22}^{(m)}\left(\Omega\right) \end{vmatrix} = 0 \tag{35}$$

It is can be observed $\phi_{ij}^{(m)}$ coefficients are a function of frequency Ω, therefore solving above nonlinear algebraic equations, Eq.(35), gives us natural frequencies for nano-plate varying thickness.

Table 1
Some basic theorems frequently used in the practical problems.

Original functions	Transformed functions
$f(r) = g(r) \pm h(r)$	$F_k = G_k + H_k$
$f(r) = \beta g(r)$	$F_k = \beta G_k$
$f(r) = g(\dot{r})h(r)$	$F_k = \sum_{l=0}^{k} G_l H_{k-l}$
$f(r) = \dfrac{d^n g(r)}{dr^n}$	$F_k = \dfrac{(k+n)!}{k!} G_{k+n}$
$f(r) = r^n$	$F_k = \delta(k-n) = \begin{cases} 1 & k = n \\ 0 & k \neq n \end{cases}$

2.3 Ritz method formulation

The linear elastic strain energy U_s for an orthotropic circular nano-plates may be written as follows [50]

$$U_s = \frac{1}{2}\int_0^a\int_0^{2\pi}\left[D\left\{\left(\frac{\partial^2 w}{\partial r^2}\right)^2 + 2\nu\frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial^2 w}{\partial r^2}\right)\right\} + \left(\frac{1}{r}\frac{\partial w}{\partial r}\right)^2\right] rdrd\theta$$
$$+\frac{1}{2}\int_0^a\int_0^{2\pi}k_w\left(w^2 + \mu^2\left(\frac{1}{r}\frac{\partial w}{\partial r}\right)^2\right) rdrd\theta + \frac{1}{2}\int_0^a\int_0^{2\pi}k_G\left[\left(\frac{1}{r}\frac{\partial w}{\partial r}\right)^2 + \mu^2\left\{\left(\frac{\partial^2 w}{\partial r^2}\right)^2 + \left(\frac{1}{r}\frac{\partial w}{\partial r}\right)^2\right\}\right] rdrd\theta \quad (36)$$

where D is flexural stiffness of plate. Also, k_w , k_G are winker modulus and shear modulus parameters of the elastic foundation. The kinetic energy of nano-plate T is given by [50]

$$T = \frac{1}{2}\rho\omega^2\int_0^a\int_0^{2\pi}h\left(w^2 + \mu^2\left(\frac{\partial w}{\partial r}\right)^2\right) rdrd\theta \quad (37)$$

By substituting $w(r,t) = w1(r)e^{i\omega t}$ into Eqs.(2)-(4), form of Rayleigh quotient for nonlocal ,an explicit equation for natural frequency can be expressed as:

$$\omega^{*2} = \frac{\rho a^4\omega^2 h_0}{D_0} = \frac{U_s}{T} = \frac{\frac{1}{2}B\int_0^1 f(\eta)\left\{\left(\frac{\partial^2 W}{\partial\eta^2}\right)^2 + 2\nu\frac{\partial^2 W}{\partial\eta^2}\left(\frac{1}{\eta}\frac{\partial W}{\partial\eta}\right) + \left(\frac{1}{\eta}\frac{\partial W}{\partial\eta}\right)^2\right\}\eta d\eta d\theta}{\frac{1}{2}A\left[\int_0^1\int_0^{2\pi}W^2\eta d\eta d\theta + \frac{\mu^2}{a^2}\int_0^1\int_0^{2\pi}\left(\frac{\partial W}{\partial\eta}\right)^2\eta d\eta d\theta\right]}$$
$$\frac{\frac{1}{2}B\int_0^1 K_w\left(W^2 + \frac{\mu^2}{a^2}\left(\frac{1}{\eta}\frac{\partial W}{\partial\eta}\right)^2\right)\eta d\eta d\theta + \frac{1}{2}\int_0^1 K_G\left[\left(\frac{1}{\eta}\frac{\partial W}{\partial\eta}\right)^2 + \frac{\mu^2}{a^2}\left\{\left(\frac{\partial^2 W}{\partial\eta^2}\right)^2 + \left(\frac{1}{\eta}\frac{\partial W}{\partial\eta}\right)^2\right\}\right]\eta d\eta d\theta}{\frac{1}{2}A\left[\int_0^1\int_0^{2\pi}W^2\eta d\eta d\theta + \frac{\mu^2}{a^2}\int_0^1\int_0^{2\pi}\left(\frac{\partial W}{\partial\eta}\right)^2\eta d\eta d\theta\right]} \quad (38)$$

where ω is vibration frequency and $W(\eta)$ denotes non-dimensional transverse deflection of nano-plate. Also non-dimensional parameters are defined

$$W = \frac{w1}{a}, \ \eta = \frac{r}{a}, \quad D_1 = \frac{E_c h^3}{12(1-\nu^2)}, \ \omega^{*2} = \frac{\rho_c a^4\omega^2 h_0}{D_1}, K_w = \frac{k_w a^4}{D_1}, \quad K_G = \frac{k_G a^2}{D_1} \quad (39)$$

As for classical macro-sized plate, Eq. (38) is utilized for free vibration analysis of nonlocal nano-plates varying thickness. An approximate solution is assumed a linear combination of N known basis functions based on Ritz method

$$W = \sum_{k=1}^{N} C_k\chi_k(\eta) \quad (40)$$

where the C_k are unknown coefficient and the χ_k are approperiate basis functions satisfying geometric boundary conditions [51]. Here, the selected basis functions are

$$\chi_i = \left(1-\eta^2\right)^{t+i}, \quad i = 1,2,\ldots, \quad (41)$$

and

$$t = \begin{cases} 0 & \textit{simply supported} \\ 1 & \textit{clamped} \end{cases} \quad (42)$$

Inserting Eq. (40)-(41) in Eq. (38) and minimizing ω as a function of coefficients C_i gives us following algebraic equations

$$\sum_{k=1}^{N}\left(a_{mn}-\Omega^2 b_{mn}\right)C_k=0 \tag{43}$$

where

$$a_{mn}=\int_0^1 f_k(\eta)\left\{\chi_i''\chi_j''+\frac{\nu_\theta}{\eta}\left(\chi_i''\chi_j'+\chi_i'\chi_j''\right)+p^2\frac{\chi_i'}{\eta}\frac{\chi_j'}{\eta}\right\}\eta d\eta+$$
$$\int_0^1\left\{K_W\left(\chi_i\chi_j+\frac{\mu^2}{a^2}\frac{\chi_i'}{\eta}\frac{\chi_j'}{\eta}\right)+K_G\left(\frac{\chi_i'}{\eta}\frac{\chi_j'}{\eta}+\frac{\mu^2}{a^2}\left[\frac{\chi_i''}{\eta}\frac{\chi_j''}{\eta}+\frac{\chi_i'}{\eta^2}\frac{\chi_j'}{\eta^2}\right]\right)\right\}\eta d\eta \tag{44}$$

$$b_{mn}=\int_0^1\left(\chi_i\chi_j+\frac{\mu^2}{a^2}\chi_i'\chi_j'\right)\eta d\eta \tag{45}$$

$$f(\eta)=1+\alpha\eta^\beta \tag{46}$$

In Eq.(43), a_{mn} and b_{mn} denote stiffness and mass matrixes, respectively. The Eigen value problem, Eq.(43), is solved to obtain the natural frequency of non-uniform nano-plate using MATLAB procedures.

3 NUMERICAL RESULTS AND DISCUSSIONS

In the present circular nano-plate, the upper and lower layers are considered ceramic and metal [54]. The Poisson coefficient is equal to a constant value of 0.3. Table 2. shows convergence study of both the Ritz and DTM methods for the values $b=2,\ \alpha=0.5,\ g=2,\ \mu=2,\ K_w=100,\ K_G=10$ and number of three modes. As can be seen, the table reveals 8, 52 terms need for attaining convergence for Ritz and DTM.

Table 2
Convergence of frequency parameter for Clamp and Simply FGM nano plates for $\mu=2,\ \beta=2,\ g=2,\ \alpha=0.5,\ K_w=100,\ K_G=10$.

	Clamp			Simply		
	mode			mode		
N	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3
			DTM			
20	17.2180	35.2442	-	14.3648	28.5300	-
25	17.2181	33.7127	-	14.3649	27.9965	-
30	17.2181	33.7543	-	14.3649	28.0078	-
35	17.2181	33.7586	51.6161	14.3649	28.0089	45.1664
40	17.2181	33.7585	51.9253	14.3649	28.0089	45.2544
45	17.2181	33.7585	51.9681	14.3649	28.0089	45.2654
50	17.2181	33.7585	51.9667	14.3649	28.0089	45.2651
51	17.2181	33.7585	51.9668	14.3649	28.0089	45.2651
52	17.2181	33.7585	51.9668	14.3649	28.0089	45.2651
			Ritz			
4	17.2181	33.7677	53.9976	14.3054	27.9574	49.3000
5	17.2181	33.7587	52.1677	14.3054	27.9335	45.6288
6	17.2181	33.7585	51.9780	14.3054	27.9331	45.2301
7	17.2181	33.7585	51.9672	14.3054	27.9330	45.2060
8	17.2181	33.7585	51.9668	14.3054	27.9330	45.2052
9	17.2181	33.7585	51.9668	14.3054	27.9330	45.2052

Figs. 2 and 3 show the effect of the radial variations of the modulus of elasticity according to the nonlocal parameter. It can be observed that by increasing the nonlocal parameter, the natural frequency always decreases. The Fig. 4 shows the effect of the parameter of *g* (power law index) on the natural frequency for different values of the nonlocal parameter.it can be seen, the difference between the graphs, increases by increasing the parameter of *g* for the classic plate and the nano-plate. In addition, for larger values of the nonlocal parameter, the difference between the graphs increases. Fig. 5 shows the natural frequency versus power law parameter of FGM for different values of the radial parameter of FGM and different values of the nonlocal parameter, the figure reveals that the frequency difference for larger values of α is greatest. Fig. 6 shows that with increasing the radius of the nano-plate non-dimensional frequency increases , and if α is greater than 0, the speed of the increasing of the frequency is higher and also for higher values of β , great non-dimensional frequencies achieve, but when the value of α is less than 0, for smaller values of β , a higher non-dimensional frequencies obtain. Fig. 7 shows the variation of the natural frequency versus radius for different power law parameters and for the two classic plate and nano-plate. It can be seen, for more values of *g*, the frequency difference between the classical and the nano-plate decreases. Fig. 8, depicts the natural frequency variations with α for the two radii and the first three modes. It can be observed, the most speed of frequency occurs in the higher modes and larger radii, and frequency difference between two different radii increases with increasing in number of modes. Fig. 9 depicts the frequency variation versus parameter of β . It can be seen that with increasing parameter of β , for values of α lower than zero, the natural frequency decreases and for values of α greater than 0, the natural frequency increases. Also, the most rate of increasing or decreasing of the frequency occurs in higher modes. Fig. 10 shows that with increasing parameter of α , higher frequencies obtain, also for minus values of α higher frequency achieve for lower values of β and for positive values of the most frequency obtain for lower values of β . The effect of the radial parameters of α and β on the shape of the classical and nano-plate are shown in Figs. 11 and 12. According to the figures, it is clear that the with increasing in nonlocal parameter, non-dimensional defection increases. For values of α larger than 0, the variation of deflection parameter is larger, and the greater value of deflection parameter achieves for higher of β . A two-dimensional and three-dimensional graph of the dimensionless deflection for clamp boundary conditions presented in Figs. 13-16.

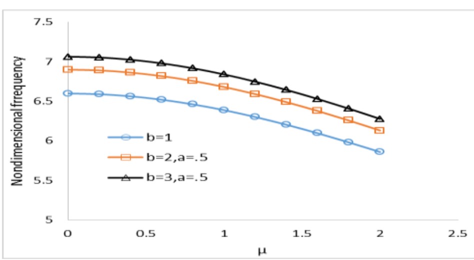


Fig.2
Effect of the radial change of the modulus of elasticity for different values of *b*.

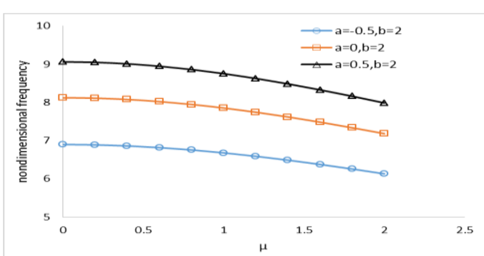


Fig.3
The effect of the radial change of the modulus of elasticity for different values of *b*.

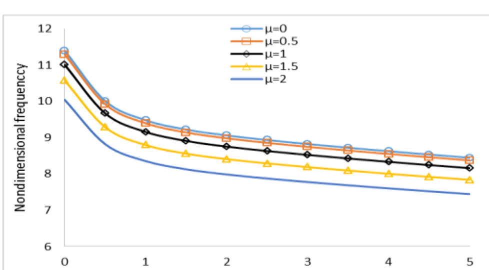


Fig.4
The effect of the parameter power *g* (power law index) on the natural frequency for different values of the nanlocal parameter.

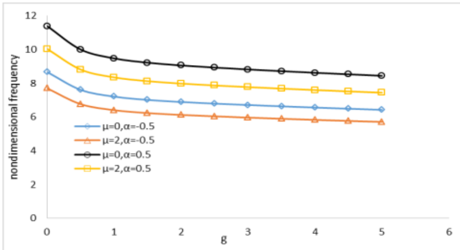


Fig.5
The natural frequency charts in terms of the parameter *g* for a classic and nano plate.

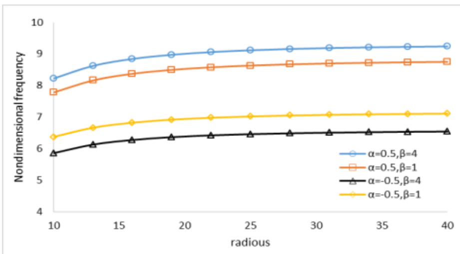


Fig.6
The effects of the FGM parameters on the dimensionless frequency.

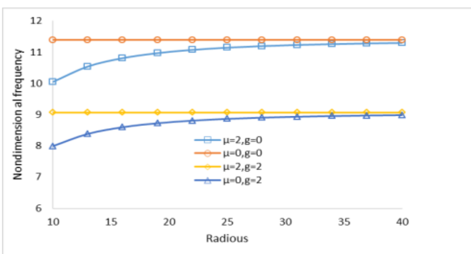


Fig.7
The variation of the natural frequency with radius this time for the *g* parameter and for the two classic plates and the nano plate.

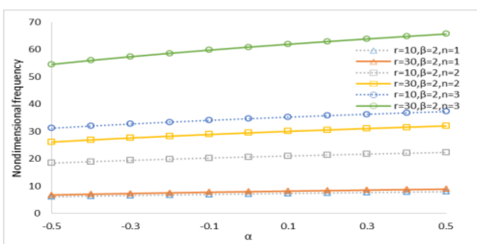


Fig.8
The natural frequency variations with α are given for the two radii and the first three modes.

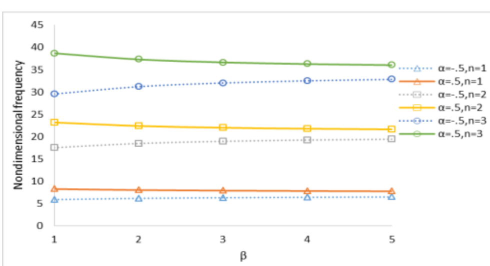


Fig.9
The natural frequency variations with β are given for the two radii and the first three modes.

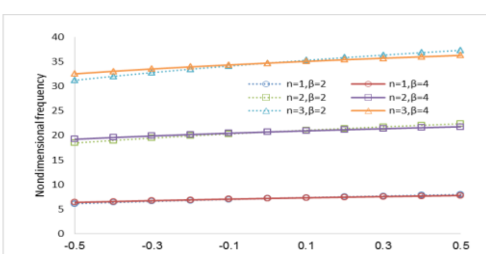


Fig.10
The effect of the FGM parameter on the dimensionless frequency for different modes.

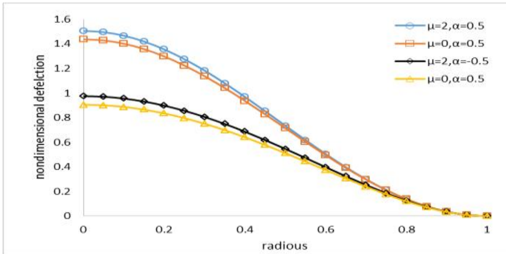


Fig.11
Effect of the FGM parameter on the deflection for classic and nano plate.

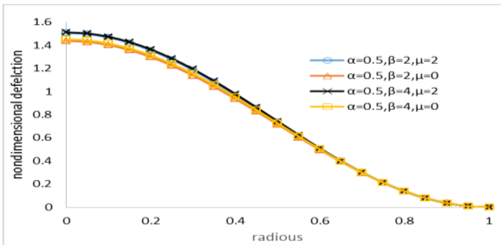


Fig.12
Effect of the FGM parameters on the deflection for classic and nano plate.

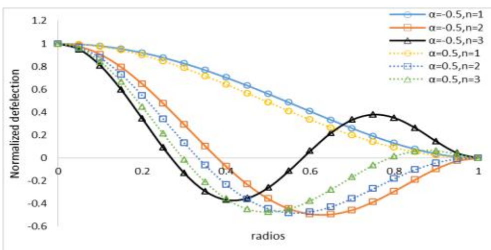


Fig.13
Two-dimensional graph of the dimensionless deflection for a clamp boundary conditions of the nano plate for $g = 2$ and $\mu = 2$.

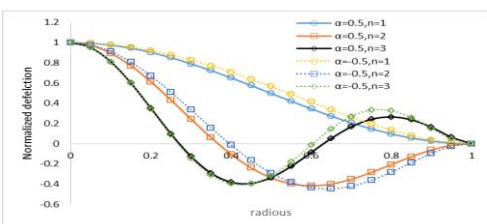


Fig.14
Two-dimensional graph of the dimensionless deflection for a clamp boundary conditions of the classic plate for $g = 2$ and $\mu = 2$.

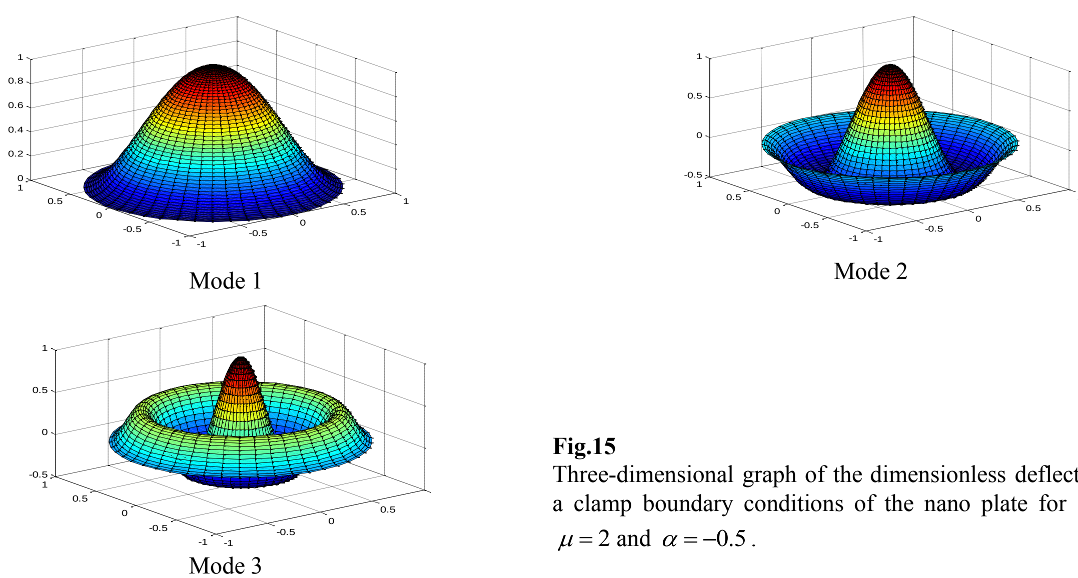
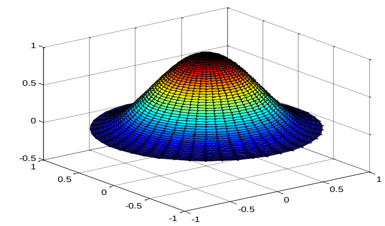
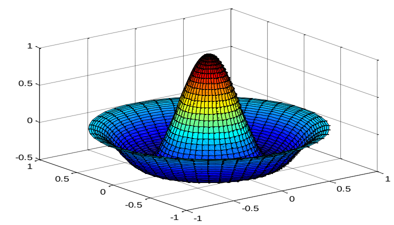


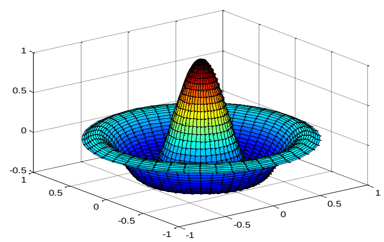
Fig.15
Three-dimensional graph of the dimensionless deflection for a clamp boundary conditions of the nano plate for $g = 2$, $\mu = 2$ and $\alpha = -0.5$.



Mode 1



Mode 2



Mode 3

Fig.16
Three-dimensional graph of the dimensionless deflection for a clamp boundary conditions of the nano plate for $g = 2$, $\mu = 2$ and $\alpha = 0.5$.

Table 3. shows a comparison for a circular nano-plate. As can be seen, the results are good agreement with those of various references. Table 4. also shows a comparison for a classic plate with two-directional FGM, which, as can be seen, results of both methods are in agreement with the reference results.

Table 3
Comparison of non-dimensional frequencies for square and circular nano-plate with constant thickness and zero in-plane forces.

Results	$\mu(nm)$										
	0	.2	.4	.6	.8	1	1.2	1.4	1.6	1.8	2
Square nano-plate [52]	2.000	1.998	1.992	1.982	1.969	1.952	1.932	1.909	1.884	1.857	1.827
Circle nano-plate [53]	2.000	1.997	1.990	1.974	1.964	1.944	1.921	1.895	1.866	1.835	1.802
Mohammadi et al. [39]	2.000	1.997	1.990	1.974	1.964	1.944	1.921	1.895	1.866	1.835	1.802
DTM	2.0001	1.9978	1.9909	1.9796	1.9640	1.9445	1.9215	1.8953	1.8664	1.8351	1.8020

Table 4
Comparison of non-dimensional frequencies for classic plate with different FGM parameter.

BC			$\beta = 2$			$\beta = 4$		
			[54]	DTM	Ritz	[54]	DTM	Ritz
simply	$\alpha = 0.5$	Ω_1	7.2994	7.2994	7.2995	7.2143	7.2144	7.2143
		Ω_2	29.8048	29.8048	29.8048	28.9789	28.9792	28.9789
		Ω_3	69.6276	69.6276	69.6278	67.6622	67.6691	67.6622
	$\alpha = -0.5$	Ω_1	6.9328	6.9328	6.9328	7.02565	7.0257	7.02565
		Ω_2	26.0317	26.0317	26.0317	27.0248	27.0237	27.0248
		Ω_3	59.5052	59.5052	59.5053	61.8419	61.9380	61.8419
clamped	$\alpha = 0.5$	Ω_1	11.4543	11.4543	11.4543	11.25689	11.2569	11.25689
		Ω_2	38.6604	38.6604	38.6604	37.7438	37.7441	37.7438
		Ω_3	83.0237	83.0237	83.0237	80.8935	80.9057	80.8935
	$\alpha = -0.5$	Ω_1	9.5109	9.5109	9.5109	9.7237	9.7237	9.7237
		Ω_2	32.3670	32.3670	32.3670	33.4495	33.4490	33.4495
		Ω_3	69.6425	69.6425	69.6425	72.1518	72.2070	72.1518

Tables 5-8. show The effect of the Winkler medium for the simply support and clamped boundary conditions using both the Ritz method and DTM. It can be seen, with increasing Winkler coefficient, the frequency parameter increased, and special for low nonlocal parameters. The effect of the Winkler parameter is more for minus values of α .with increasing the Pasternak parameter, the natural frequency will increase and the effect of the Pasternak parameter is more when values of α are negative.

Table 5
Frequency parameter for Clamped FGM nano plates for different values of Pasternak parameter $K_G = 10$.

clamped			μ							
			1				2			
			β				β			
			1		3		1		3	
			g							
			1	5	1	5	1	5	1	5
			DTM							
α	K_w	Ω_1	7.5074	6.9749	7.9297	7.3367	7.0201	6.5595	7.4046	6.8870
	`10	Ω_2	24.3872	21.8222	26.3155	23.5342	18.6357	16.7241	20.1081	18.0278
-0.5		Ω_3	47.1702	42.0826	51.0484	45.5351	31.1099	27.7940	33.6888	30.0869
	100	Ω_1	12.7106	12.9303	12.9645	13.1290	12.4290	12.7111	12.6501	12.8831
		Ω_2	26.4562	24.3875	28.2437	25.9307	21.2717	19.9559	22.5729	21.0605
		Ω_3	48.2724	43.4682	52.0685	46.8187	32.7571	29.8505	35.2155	31.9963
0.5	10	Ω_1	10.0241	9.1482	9.6444	8.8181	9.2565	8.4815	8.9080	8.1798
		Ω_2	32.3436	28.8908	30.7191	27.4467	24.4850	21.9090	23.2483	20.8116
		v	61.8455	55.1497	58.6355	52.2910	40.6242	36.2564	38.4997	34.3661
	100	Ω_1	14.3416	14.2209	14.0788	14.0108	13.8160	13.8014	13.5850	13.6181
		Ω_2	33.9309	30.8743	32.3862	29.5273	26.5464	24.4653	25.4103	23.4876
		Ω_3	62.6902	56.2141	59.5258	53.4125	41.8990	37.8559	39.8425	36.0496
			Ritz							
α		Ω_1	7.5074	6.9749	7.9297	7.3367	7.0201	6.5595	7.4046	6.8870
	`10	Ω_2	24.3872	21.8222	26.3155	23.5342	18.6357	16.7241	20.1081	18.0278
-0.5		Ω_3	47.1702	42.0826	51.0484	45.5351	31.1099	27.7940	33.6888	30.0869
	100	Ω_1	12.7106	12.9303	12.9645	13.1290	12.4290	12.7111	12.6501	12.8831
		Ω_2	26.4562	24.3875	28.2437	25.9307	21.2717	19.9559	22.5729	21.0605
		Ω_3	48.2724	43.4682	52.0685	46.8187	32.7571	29.8505	35.2155	31.9963
0.5	10	Ω_1	10.0241	9.1482	9.6444	8.8181	9.2565	8.4815	8.9080	8.1798
		Ω_2	32.3436	28.8908	30.7191	27.4467	24.4850	21.9090	23.2483	20.8116
		Ω_3	61.8455	55.1497	58.6355	52.2910	40.6242	36.2564	38.4997	34.3661
	100	Ω_1	14.3416	14.2209	14.0788	14.0108	13.8160	13.8014	13.5850	13.6181
		Ω_2	33.9309	30.8743	32.3862	29.5273	26.5464	24.4653	25.4103	23.4876
		Ω_3	62.6902	56.2141	59.5258	53.4125	41.8990	37.8559	39.8425	36.0496

Table 6
Frequency parameter for Clamped FGM nano plates for different values of Winkler parameter $K_w = 100$.

clamped		μ							
		1				2			
		β				β			
		1		3		1		3	
		g							
		1	5	1	5	1	5	1	5
α	K_G	9.3228	9.1110	9.6565	9.3808	9.3831	9.3113	9.6735	9.5444
	5	35.0474	26.5592	29.9259	27.9767	24.0750	23.2631	25.2359	24.2213
-0.5		52.2986	48.4307	55.8165	51.4531	38.9935	37.3380	41.0807	39.0751
	15	13.0671	13.3147	13.3003	13.4950	13.7444	14.1548	13.9441	14.3090
		35.0474	34.4991	36.4050	35.5959	32.6330	32.9766	33.5009	33.6612
		61.4505	59.3088	64.4646	61.7954	51.3723	51.5012	52.9742	52.7743
0.5	5	11.5308	10.9607	11.2161	10.7018	11.2003	10.7997	10.9252	10.5759
		35.3912	32.6683	33.9226	31.4096	28.8269	27.2162	27.7898	26.3470
		65.8330	60.1258	62.8332	57.5222	46.8807	43.9449	45.0544	42.4011
	15	14.8226	14.7397	14.5915	14.5593	15.0773	15.2035	14.8812	15.0511
		41.0828	39.4608	39.8374	38.4384	36.2781	27.2162	35.4644	35.2238
		73.3117	69.1896	70.6390	66.9489	57.5600	56.4542	56.0845	55.2627
					Ritz				
α		9.3228	9.1110	9.6565	9.3808	9.3831	9.3113	9.6735	9.5444
	5	35.0474	26.5592	29.9259	27.9767	24.0750	23.2631	25.2359	24.2213
-0.5		52.2986	48.4307	55.8165	51.4531	38.9935	37.3380	41.0807	39.0751
	15	13.0671	13.3147	13.3003	13.4950	13.7444	14.1548	13.9441	14.3090
		35.0474	34.4991	36.4050	35.5959	32.6330	32.9766	33.5009	33.6612
		61.4505	59.3088	64.4646	61.7954	51.3723	51.5012	52.9742	52.7743
0.5	5	11.5308	10.9607	11.2161	10.7018	11.2003	10.7997	10.9252	10.5759
		35.3912	32.6683	33.9226	31.4096	28.8269	27.2162	27.7898	26.3470
		65.8330	60.1258	62.8332	57.5222	46.8807	43.9449	45.0544	42.4011
	15	14.8226	14.7397	14.5915	14.5593	15.0773	15.2035	14.8812	15.0511
		41.0828	39.4608	39.8374	38.4384	36.2781	27.2162	35.4644	35.2238
		73.3117	69.1896	70.6390	66.9489	57.5600	56.4542	56.0845	55.2627

Table7
Frequency parameter for simply supported FGM Nano plates for different values of Pasternak parameter $K_G = 10$.

simply			μ							
			1				2			
			β				β			
			1		3		1		3	
			g							
			1	5	1	5	1	5	1	5
						DTM				
α	K_w	Ω_1	4.8397	4.7424	5.0783	4.9366	4.6646	4.6009	4.8777	4.7732
	`10	Ω_2	18.7073	16.7875	20.4376	18.3197	14.4952	13.0672	15.8090	14.2257
-0.5		Ω_3	39.8547	35.5717	43.3900	38.7178	26.4660	23.6678	28.7943	25.7361
	100	Ω_1	11.3411	11.8758	11.4450	11.9547	11.2675	11.8200	11.3574	11.8881
		Ω_2	21.3345	20.0091	22.8669	21.3109	17.7569	17.0087	18.8447	17.9141
		Ω_3	41.1534	37.2007	44.5858	40.2196	28.3839	26.0520	30.5665	27.9444
0.5	10	Ω_1	5.6227	5.3849	5.4148	5.2129	5.3670	5.1734	5.1801	5.0198
		Ω_2	24.4890	21.9126	23.0851	20.6668	18.9045	16.9620	17.8367	16.0173
		Ω_3	52.2095	46.5689	49.3447	44.0183	34.6244	30.9189	32.7361	29.2397
	100	Ω_1	11.6967	12.1466	11.5982	12.0714	11.5760	12.0544	11.4905	11.9893
		Ω_2	26.5501	24.4685	25.2610	23.3594	21.5076	20.1557	20.5754	19.3674
		Ω_3	53.2074	47.8247	50.3994	45.3449	36.1116	32.7799	34.3052	31.2011
						Ritz				
		Ω_1	4.8397	4.7424	5.0783	4.9366	4.6646	4.6009	4.8777	4.7732
	`10	Ω_2	18.7073	16.7875	20.4376	18.3197	14.4952	13.0672	15.8090	14.2257
-0.5		Ω_3	39.8547	35.5717	43.3900	38.7178	26.4660	23.6678	28.7943	25.7361
	100	Ω_1	11.3411	11.8758	11.4450	11.9547	11.2675	11.8200	11.3574	11.8881
		Ω_2	21.3345	20.0091	22.8669	21.3109	17.7569	17.0087	18.8447	17.9141
		Ω_3	41.1534	37.2007	44.5858	40.2196	28.3839	26.0520	30.5665	27.9444
0.5	10	Ω_1	5.6227	5.3849	5.4148	5.2129	5.3670	5.1734	5.1801	5.0198
		Ω_2	24.4890	21.9126	23.0851	20.6668	18.9045	16.9620	17.8367	16.0173
		Ω_3	52.2095	46.5689	49.3447	44.0183	34.6244	30.9189	32.7361	29.2397
	100	Ω_1	11.6967	12.1466	11.5982	12.0714	11.5760	12.0544	11.4905	11.9893
		Ω_2	26.5501	24.4685	25.2610	23.3594	21.5076	20.1557	20.5754	19.3674
		Ω_3	53.2074	47.8247	50.3994	45.3449	36.1116	32.7799	34.3052	31.2011

Table 8
Frequency parameter for Clamped FGM nano plates for different values of Winkler parameter $K_w = 100$.

simply			μ							
			1				2			
			β				β			
			1		3		1		3	
			g							
			1	5	1	5	1	5	1	5
			DTM							
α	K_G	Ω_1	6.7058	6.8376	6.8777	6.9715	6.4363	6.5819	6.5903	6.7015
	`5	Ω_2	22.6853	21.6167	24.1272	22.8218	19.2561	18.7522	20.2572	19.5707
-0.5		Ω_3	44.8750	41.7662	48.0360	44.4691	33.5082	32.1767	35.3673	33.7183
	15	Ω_1	10.5493	11.0204	10.6586	11.1035	10.2006	10.6713	10.2977	10.7450
		Ω_2	29.4353	29.3837	30.5564	30.2779	26.7208	27.1670	27.4478	27.7360
		Ω_3	53.7128	52.2025	56.3738	54.3829	44.5453	44.7728	45.9554	45.8896
0.5	5	Ω_1	7.2827	7.2894	7.1233	7.1633	6.9593	6.9914	6.8161	6.8784
		Ω_2	27.6325	25.7832	26.3961	24.7330	22.7506	21.6372	21.8721	20.9054
		Ω_3	56.1111	51.4279	53.4561	49.1306	40.2282	37.7942	38.6156	36.4340
	15	Ω_1	10.9204	11.3027	10.8147	11.2217	10.5376	10.9282	10.4435	10.8562
		Ω_2	33.3829	32.5586	32.3668	31.7332	29.3367	29.2321	28.6610	28.6949
		Ω_3	63.3743	60.1864	61.0363	58.2358	49.7813	48.9550	48.4879	47.9131
			Ritz							
α	K_G	Ω_1	6.7058	6.8376	6.8777	6.9715	6.4363	6.5819	6.5903	6.7015
	`5	Ω_2	22.6853	21.6167	24.1272	22.8218	19.2561	18.7522	20.2572	19.5707
-0.5		Ω_3	44.8750	41.7662	48.0360	44.4691	33.5082	32.1767	35.3673	33.7183
	15	Ω_1	10.5493	11.0204	10.6586	11.1035	10.2006	10.6713	10.2977	10.7450
		Ω_2	29.4353	29.3837	30.5564	30.2779	26.7208	27.1670	27.4478	27.7360
		Ω_3	53.7128	52.2025	56.3738	54.3829	44.5453	44.7728	45.9554	45.8896
0.5	5	Ω_1	7.2827	7.2894	7.1233	7.1633	6.9593	6.9914	6.8161	6.8784
		Ω_2	27.6325	25.7832	26.3961	24.7330	22.7506	21.6372	21.8721	20.9054
		Ω_3	56.1111	51.4279	53.4561	49.1306	40.2282	37.7942	38.6156	36.4340
	15	Ω_1	10.9204	11.3027	10.8147	11.2217	10.5376	10.9282	10.4435	10.8562
		Ω_2	33.3829	32.5586	32.3668	31.7332	29.3367	29.2321	28.6610	28.6949
		Ω_3	63.3743	60.1864	61.0363	58.2358	49.7813	48.9550	48.4879	47.9131

4 CONCLUSIONS

In this study, the free vibration of circular nano-plate was studied on the Winkler and Pasternak foundation. The appropriate governing equations were obtained for Ritz method and DTM. In both methods, the convergence study was carried out and the number of series for convergence was discovered. A parametric study including the effect of nonlocal and radial parameters and FGM thickness, nano-plate radius and elastic foundation on a non-dimensional natural frequency was carried out, and following results obtained:

- By increasing the parameter of the nonlocal, the natural frequency always decreases
- With increasing parameter of g for conventional plates and nano- plates, natural frequency decreases
- For values of α greater than 0, the rate of increase of frequency is higher and for higher values of β the higher frequencies obtained
- The frequency parameter is greater in higher modes and larger radii

- For values of α greater than 0, the deflection parameter is larger, and also for more large values β higher deflection obtained.
- By increasing the elastic factor, the natural frequency parameter increases, this increase is higher for lower values of the nonlocal parameters.

REFERENCES

[1] Sari M.S., Al-Kouz W.G., 2016, Vibration analysis of non-uniform orthotropic Kirchhoff plates resting on elastic foundation based on nonlocal elasticity theory, *International Journal of Mechanical Sciences* **114**: 1-11.

[2] Sakhaee-Pour A., Ahmadian M.T., Vafai A., 2008, Applications of single-layered graphene sheets as mass sensors and atomistic dust detectors, *Solid State Communications* **145**: 168-172.

[3] Arash B., Wang Q., 2012, A review on the application of nonlocal elastic models in modeling of carbon nanotubes and graphenes, *Computational Materials Science* **51**: 303-313.

[4] Murmu T., Pradhan S.C., 2009, Vibration analysis of nano-single-layered graphene sheets embedded in elastic medium based on nonlocal elasticity theory, *Journal of Applied Physics* **105**: 64319.

[5] Arash B., Wang Q., 2014, A review on the application of nonlocal elastic models in modeling of carbon nanotubes and graphenes, *Modeling of Carbon Nanotubes, Graphene and their Composites* **2014**: 57-82.

[6] Mindlin R.D., Eshel N.N., 1968, On first strain-gradient theories in linear elasticity, *International Journal of Solids and Structures* **4**: 109-124.

[7] Mindlin R.D.,1965, Second gradient of strain and surface-tension in linear elasticity, *International Journal of Solids and Structures* **1**: 417-438.

[8] Lam D.C.C., Yang F., Chong A.C.M., Wang J., Tong P., 2003, Experiments and theory in strain gradient elasticity, *Journal of the Mechanics and Physics of Solids* **51**: 1477-1508.

[9] Ramezani S., 2012, A micro scale geometrically non-linear Timoshenko beam model based on strain gradient elasticity theory, *International Journal of Non-Linear Mechanics* **47**: 863-873.

[10] Alibeigloo A., 2011, Free vibration analysis of nano-plate using three-dimensional theory of elasticity, *Acta Mechanica* **222**: 149.

[11] Şimşek M.,2010, Dynamic analysis of an embedded micro-beam carrying a moving micro-particle based on the modified couple stress theory, *International Journal of Engineering Science* **48**: 1721-1732.

[12] Sahmani S., Ansari R., Gholami R., Darvizeh A., 2013, Dynamic stability analysis of functionally graded higher-order shear deformable micro-shells based on the modified couple stress elasticity theory, *Composites Part B: Engineering* **51**: 44-53.

[13] Toupin R.A., 1964, Theories of elasticity with couple-stress, *Archive for Rational Mechanics and Analysis* **17**: 85-112.

[14] Yang F., Chong A.C.M., Lam D.C.C., Tong P., 2002, Couple stress based strain gradient theory for elasticity, *International Journal of Solids and Structures* **39**: 2731-2743.

[15] Eringen A.C.,1983, On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves, *Journal of Applied Physics* **54**: 4703-4710.

[16] Peddieson J., Buchanan G.R., McNitt R.P., 2003, Application of nonlocal continuum models to nanotechnology, *International Journal of Engineering Science* **41**: 305-312.

[17] Lu P., Lee H.P., Lu C., Zhang P.Q., 2007, Application of nonlocal beam models for carbon nanotubes, *International Journal of Solids and Structures* **44**: 5289-5300.

[18] Rahmani O., Pedram O., 2014, Analysis and modeling the size effect on vibration of functionally graded nanobeams based on nonlocal Timoshenko beam theory, *International Journal of Engineering Science* **77**: 55-70.

[19] Şimşek M., 2016, Nonlinear free vibration of a functionally graded nano-beam using nonlocal strain gradient theory and a novel Hamiltonian approach, *International Journal of Engineering Science* **105**: 12-27.

[20] Hosseini-Hashemi S., Bedroud M., Nazemnezhad R., 2013, An exact analytical solution for free vibration of functionally graded circular/annular Mindlin nano-plate s via nonlocal elasticity, *Composite Structures* **103**: 108-118.

[21] Belkorissat I., Houari M.S.A., Tounsi A., Bedia E.A.A., Mahmoud S.R., 2015, On vibration properties of functionally graded nano-plate using a new nonlocal refined four variable model, *Steel and Composite Structures* **18**: 1063-1081.

[22] Şimşek M., Yurtcu H.H., 2013, Analytical solutions for bending and buckling of functionally graded nano-beams based on the nonlocal Timoshenko beam theory, *Composite Structures* **97**: 378-386.

[23] Murmu T., Pradhan S.C., 2009, Buckling analysis of a single-walled carbon nanotube embedded in an elastic medium based on nonlocal elasticity and Timoshenko beam theory and using DQM, *Physica E: Low-Dimensional Systems and Nanostructures* **41**: 1232-1239.

[24] Aksencer T., Aydogdu M., 2011, Levy type solution method for vibration and buckling of nano-plate s using nonlocal elasticity theory, *Physica E: Low-Dimensional Systems and Nanostructures* **43**: 954-959.

[25] Narendar S., 2011, Buckling analysis of micro-/nano-scale plates based on two-variable refined plate theory incorporating nonlocal scale effects, *Composite Structures* **93**: 3093-3103.

[26] Farajpour A., Mohammadi M., Shahidi A.R., Mahzoon M., 2011, Axisymmetric buckling of the circular graphene sheets with the nonlocal continuum plate model, *Physica E: Low-Dimensional Systems and Nanostructures* **43**: 1820-1825.

[27] Tornabene F., Fantuzzi N., Bacciocchi M., 2016, The local GDQ method for the natural frequencies of doubly-curved shells with variable thickness: A general formulation, *Composites Part B: Engineering* **92**: 265-289.

[28] Farajpour A., Shahidi A.R., Mohammadi M., Mahzoon M.,2012, Buckling of orthotropic micro/nanoscale plates under linearly varying in-plane load via nonlocal continuum mechanics, *Composite Structures* **94**: 1605-1615.

[29] Farajpour A., Danesh M., Mohammadi M., 2011, Buckling analysis of variable thickness nano-plate s using nonlocal continuum mechanics, *Physica E: Low-Dimensional Systems and Nanostructures* **44**: 719-727.

[30] Danesh M., Farajpour A., Mohammadi M., 2012, Axial vibration analysis of a tapered nano-rod based on nonlocal elasticity theory and differential quadrature method, *Mechanics Research Communications* **39**: 23-27.

[31] Şimşek M., 2012, Nonlocal effects in the free longitudinal vibration of axially functionally graded tapered nano-rods, *Computational Materials Science* **61**: 257-265.

[32] Efraim E., Eisenberger M., 2007, Exact vibration analysis of variable thickness thick annular isotropic and FGM plates, *Journal of Sound and Vibration* **299**: 720-738.

[33] Zhou J.K., 1986, *Differential Transformation and its Applications for Electrical Circuits*, Huazhong University Press, Wuhan, China.

[34] Arikoglu A., Ozkol I., 2010, Vibration analysis of composite sandwich beams with viscoelastic core by using differential transform method, *Composite Structures* **92**: 3031-3039.

[35] Mohammadi M., Farajpour A., Goodarzi M., Shehni nezhad pour H., 2014, Numerical study of the effect of shear in-plane load on the vibration analysis of graphene sheet embedded in an elastic medium, *Computational Materials Science* **82**: 510-520.

[36] Pradhan S.C., Phadikar J.K., 2009, Small scale effect on vibration of embedded multilayered graphene sheets based on nonlocal continuum models, *Physics Letters A* **373**: 1062-1069.

[37] Behfar K., Naghdabadi R., 2005, Nanoscale vibrational analysis of a multi-layered graphene sheet embedded in an elastic medium, *Composites Science and Technology* **65**: 1159-1164.

[38] Mirzabeigy A., 2013, Semi-analytical approach for free vibration analysis of variable cross-section beams resting on elastic foundation and under axial force, *International Journal of Engineering - Transactions C: Aspects* **27**: 385.

[39] Mohammadi M., Goodarzi M., Ghayour M., Farajpour A., 2013, Influence of in-plane pre-load on the vibration frequency of circular graphene sheet via nonlocal continuum theory, *Composites Part B: Engineering* **51**: 121-129.

[40] Alipour M.M., Shariyat M., Shaban M., 2010, A semi-analytical solution for free vibration and modal stress analyses of circular plates resting on two-parameter elastic foundations, *Journal of Solid Mechanics* **2**(1): 63-78.

[41] Shariyat M., Jafari A.A., Alipour M.M., 2013, Investigation of the thickness variability and material heterogeneity effects on free vibration of the viscoelastic circular plates, *Acta Mechanica Solida Sinica* **26**(1): 83-98.

[42] Alipour M.M., Shariyat M., Shaban M., 2010, A semi-analytical solution for free vibration of variable thickness two-directional-functionally graded plates on elastic foundations, *International Journal of Mechanics and Materials in Design* 6(4): 293-304.

[43] Alipour M.M., Shariyat M., 2010, Stress analysis of two-directional FGM moderately thick constrained circular plates with non-uniform load and substrate stiffness distributions, *Journal of Solid Mechanics* **2**(4): 316-331.

[44] Alipour M.M., Shariyat M., 2011, A power series solution for free vibration of variable thickness Mindlin circular plates with two-directional material heterogeneity and elastic foundations, *Journal of Solid Mechanics* **3**(2): 183-197.

[45] Shariyat M., Alipour M.M., 2013, A power series solution for vibration and complex modal stress analyses of variable thickness viscoelastic two-directional FGM circular plates on elastic foundations, *Applied Mathematical Modelling* **37**(5): 3063-3076.

[46] Alipour M.M., Shariyat M.,2013, Semianalytical solution for buckling analysis of variable thickness two-directional functionally graded circular plates with nonuniform elastic foundations, *ASCE Journal of Engineering Mechanics* **139**(5): 664-676.

[47] Shariyat M., Alipour M.M., 2012, A zigzag theory with local shear correction factors for semi-analytical bending modal analysis of functionally graded viscoelastic circular sandwich plates, *Journal of Solid Mechanics* **4**(1): 84-105.

[48] Neha A., Roshan L., 2015, Buckling and vibration of functionally graded circular plates resting on elastic foundation, *Mathematical Analysis and its Applications* **2015**: 545-555.

[49] Zarei M., Ghalami-Choobari M., Rahimi G.H., Faghani G.R., 2018, Axisymmetric free vibration anlysis of non-uniform circular nano-plate resting on elastic medium, *Journal of Solid Mechanics* **10**(2): 400-415.

[50] Anjomshoa A., 2013, Application of Ritz functions in buckling analysis of embedded orthotropic circular and elliptical micro/nano-plates based on nonlocal elasticity theory, *Meccanica* **48**: 1337-1353.

[51] Singh B., Saxena V., 1995, Axisymmetric vibration of a circular plate with double linear variable thickness, *Journal of Sound and Vibration* **179**: 879-897.

[52] Liew K.M., He X.Q., Kitipornchai S., 2006, Predicting nano-vibration of multi-layered graphene sheets embedded in an elastic matrix, *Acta Materialia* **54**: 4229-4236.

[53] Mohammadi M., Ghayour M., Farajpour A., 2013, Free transverse vibration analysis of circular and annular graphene sheets with various boundary conditions using the nonlocal continuum plate model, *Composites Part B: Engineering* **45**: 32-42.

[54] Shariyat M., Alipour M.M., 2011, Differential transform vibration and modal stress analyses of circular plates made of two-directional functionally graded materials resting on elastic foundations, *Archive of Applied Mechanics* **81**(9): 1289-1306.