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# Inquisitive Analysis of the Point Source Effect on Propagation of SH Wave Through an Orthotropic Crustal Layer

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**ABSTRACT**

The occurrence of SH wave propagation under the effect of a point source in an orthotropic substratum lying over a heterogeneous orthotropic half space is deliberated in the prospect of a devastating earthquake. The quadratic alteration is acknowledged for density and shear modulus which is hypothesized to be a function of depth. The method of Green's function and transformation technique contributes to obtain the dispersion equation and dispersion curves. An effort has been accomplished to demonstrate the classical equation of Love wave followed from dispersion equation. "Mathematica" software is applied to depict the graphics. Graphics are designed to show the effect of heterogeneous parameters corresponding to density and shear modulus. Dispersion equation is obtained considering the case that the displacement and stress are continuous at the interface. The present work is an attempt to express the behavior of SH wave in an orthotropic medium under the effect of point source.



**Keywords:** Orthotropic; SH wave; Green's function; Transformation technique; Point source.

## 1 INTRODUCTION

PROPAGATION of SH wave in an orthotropic media is a proposition of substantial significance from the view point of acoustics, seismology and geophysics. In the field of seismology, one necessitates a complete understanding of wave propagation effects on the surface structure due to underground excavation as well as surface mining. The concepts of point source are widely applied in the source of light, electromagnetic radiation, sound, heat, fluid etc. The analysis of generating SH wave in an orthotropic layer is recognized because of an elementary characteristic of internal structure of the earth. A horizontal plane of symmetric orthotropy is some special type of anisotropic materials. Orthotropic symmetry is exhibited by olivine and orthorhombic, the principal rock forming minerals at the deep crust and upper mantle. Anisotropy is often orthotropic or transversely isotropic from the stand point of practical importance in engineering. The propagation of SH wave makes an extensive application in earthquake engineering and seismology on account of the occurrence of heterogeneities- in the crust as the earth is conjectured to be made up of dissimilar stratum. In observance of the non homogeneity phenomenon of an orthotropic medium, a numerous efforts have been initiated to comprehend the seismic wave propagation in

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different types of layered media. A classification is designed for body forces which is configured by impulsive force in respect of space and time in a particular orientation which may be symbolized through implementing the technique of Dirac Delta function. A lot of information about the seismology is available in a monograph by Ewing et al.[9]. The influence of point source and heterogeneities on the magnetoelastic shear wave propagation in a monoclinic medium is deliberated by Chattopadhyay et al.[3]. The propagation of Love wave in a heterogeneous medium over a heterogeneous half space under the effect of point source is deliberated by Kundu et al.[12]. Vlaar[16] represented the propagation of SH wave in a continuously layered heterogeneous half space because of the effect of point source. The propagation of SH wave in a viscoelastic layer lying over a viscoelastic half space under the effect of point source is discussed by Chattopadhyay et al.[4]. Kakar et al.[11] relinquished an apprehensive cogitation on the dispersion of Love wave in an isotropic layer reubened between orthotropic and heterogeneous prestressed half space. Kundu et al.[13] discussed the dispersion of SH wave in an isotropic medium sandwiched between an initially stressed orthotropic and heterogeneous semi infinite media. The technique of Green's function for SH wave in a cylindrically monoclinic material is applied by Watanabe and Payton[19].

The problems concerning the propagation of SH wave in an orthotropic media are not only advantageous in investigating the internal structure of the earth but also very helpful in exploration of natural possessions like oil, gases and other hydrocarbon and minerals etc. Being the highly anisotropic nature of the earth, elastic moduli, density, thermal conductivities are not homogeneous throughout the medium. Such characteristics motivate us to study the propagation of SH wave in an orthotropic medium. Colquitt et al.[6] studied the canonical form problem by extracting the dispersion relations analytically. An imprecise computation of Green's function for assembling bodies is made by Covert[5]. In the mean time, Deresiewich [8] developed an important inscription on Love wave in a homogeneous crust lying over a heterogeneous medium. Sevostianov and Kachanov[15] enlightened a theory on approximate symmetries of the elastic properties and elliptic orthotropy. Vaishnav et al.[17] analyzed the torsional surface wave propagation in an anisotropic layer sandwiched between two heterogenous half spaces.

Since the earth's crust and mantle are not homogeneous therefore, it is also interesting to know the propagation pattern of shear waves due to point source in a heterogeneous media. The motivation behind choosing quadratic depth is that heterogeneity exists in the earth's crust. It is verified from literature[3,4] that such type of heterogeneity may exist. The heterogeneity is taken in respect of shear modulus and density. Bullen[2] studied that the density inside the earth varies at different layers within the earth. It is noticed that the wave propagates in different manner because of occurrence of point source. An interesting problem is concerned with an initially undisturbed body which is in its interior and at a specified time $t = 0$ that is subjected to external disturbances. The external disturbances give rise to wave motion propagating away from the disturbed region. In seismology, the problem of source mechanism consists in relating observed seismic waves to the parameters that describe the source. For each medium, there is a different Green's function that defines how this medium reacts mechanically to an impulsive excitation force and is, therefore, a proper characteristic of each medium.The main aim of this paper is to analyze the propagation of SH wave due to point source in an orthotropic medium lying over a heterogeneous orthotropic half space. The variation in shear modulus and density are contemplated in the following approach

$$\begin{aligned} Q_3^{(2)} &= Q_3^{(2)} + \varepsilon\left(z - H\right)^2 \\ Q_1^{(2)} &= Q_1^{(2)} + \varepsilon'\left(z - H\right)^2 \\ \rho^{(2)} &= \rho^{(2)} + \varepsilon''\left(z - H\right)^2 \end{aligned} \tag{1}$$

where $\varepsilon, \varepsilon', \varepsilon''$ and are considered to be heterogeneous parameters associated with shear modulus and density with suitable dimension. Physically, the terms $\varepsilon, \varepsilon', \varepsilon''$ are considered in such a manner that it makes the dimensional quantity to dimensionless terms which is necessary to validate the problem. For example, the dimension of $\varepsilon''$ is supposed to be $kg/m^5$ to make the term $\frac{\varepsilon'' H^2}{\rho^{(2)}}$ as dimensionless. Similar perception is taken for $\varepsilon$ and $\varepsilon'$. Transformation technique and Green's function leads to obtaining the dispersion equation of SH wave in the assumed media.

## 2 FORMULATION OF THE PROBLEM

An orthotropic stratum of thickness $H$ lying over a heterogeneous orthotropic half space is introduced to show the point source effect on SH wave propagation. The $x$ axis is considered along the propagation of waves and $z$ axis is assumed to be vertically downwards as illustrated in Fig. 1. The source of disturbance $S$ is acknowledged at the intersection of the interface lying between two dissimilar orthotropic medium. Let $Q_1^{(1)}, Q_3^{(1)}$ and $\rho^{(1)}$ denote shear moduli and density for the upper layer respectively. Similarly $Q_1^{(2)}, Q_3^{(2)}$ and $\rho^{(2)}$ denote the shear moduli and density for lower half space respectively.

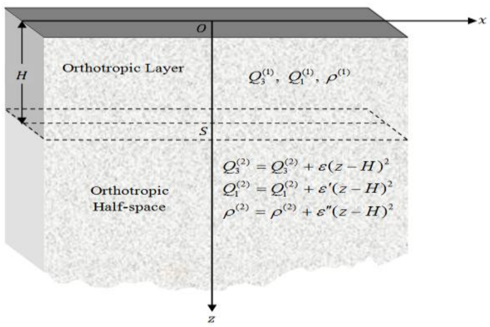


**Fig.1**
Geometry of the problem.

## 3 DYNAMICS FOR THE UPPER ORTHOTROPIC MEDIUM

The upper medium is contemplated to be made of an orthotropic material. The constitutive equation of motion neglecting the body force [1] is

$$\begin{aligned}
&\frac{\partial \kappa_{11}}{\partial x} + \frac{\partial \kappa_{12}}{\partial y} + \frac{\partial \kappa_{13}}{\partial z} = \rho^{(1)} \frac{\partial^2 u_1}{\partial t^2} \\
&\frac{\partial \kappa_{21}}{\partial x} + \frac{\partial \kappa_{22}}{\partial y} + \frac{\partial \kappa_{23}}{\partial z} = \rho^{(1)} \frac{\partial^2 v_1}{\partial t^2} \\
&\frac{\partial \kappa_{31}}{\partial x} + \frac{\partial \kappa_{32}}{\partial y} + \frac{\partial \kappa_{33}}{\partial z} = \rho^{(1)} \frac{\partial^2 w_1}{\partial t^2}
\end{aligned} \tag{2}$$

where $u_1, v_1$ and $w_1$ are the components of displacement. Here $\kappa_{ij}$ $(i, j = 1,2,3)$ and $\rho^{(1)}$ symbolizes the incremental stress and density of the upper medium respectively. Stress and strain components of this medium are related in such a manner that

$$\begin{aligned}
&\kappa_{11} = C_{11} f_{11} + C_{22} f_{22} + C_{13} f_{33}, \quad \kappa_{12} = 2Q_3 f_{12} \\
&\kappa_{22} = C_{21} f_{11} + C_{22} f_{22} + C_{23} f_{33}, \quad \kappa_{23} = 2Q_1 f_{23} \\
&\kappa_{33} = C_{31} f_{11} + C_{32} f_{22} + C_{33} f_{33}, \quad \kappa_{31} = 2Q_2 f_{31}
\end{aligned} \tag{3}$$

where $C_{ij}$ $(i, j = 1,2,3)$ and $Q_i$ $(i = 1,2,3)$ are used to denote the incremental normal elastic coefficients and shear modulus respectively. The components of strain $f_{ij}$ are expressed as:

$$f_{ij} = \frac{1}{2}\left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tag{4}$$

where $(u_1,u_2,u_3)=(u_1,v_1,w_1)$ and $(x_1,x_2,x_3)=(x,y,z)$. Applying conventional condition of SH-wave, $u_1=0$, $w_1=0$ and $v_1=v_1(x,z,t)$ in Eq.(2), it reduces to

$$\frac{\partial}{\partial x}\left(Q_3^{(1)}\frac{\partial v_1}{\partial x}\right)+\frac{\partial}{\partial z}\left(Q_1^{(1)}\frac{\partial v_1}{\partial z}\right)=\rho^{(1)}\frac{\partial^2 v_1}{\partial t^2} \tag{5}$$

Eq. (5) appears because the stress components $\kappa_{12}=2Q_1^{(1)}f_{12}$, $\kappa_{23}=2Q_3^{(1)}f_{23}$, and other stress components are zero. In consideration of point source, the equation of motion for upper orthotropic layer is expressed as:

$$\frac{\partial}{\partial x}\left(Q_3^{(1)}\frac{\partial v_1}{\partial x}\right)+\frac{\partial}{\partial z}\left(Q_1^{(1)}\frac{\partial v_1}{\partial z}\right)-\rho^{(1)}\frac{\partial^2 v_1}{\partial t^2}=4\pi\sigma_1(r,t) \tag{6}$$

where $\sigma_1(r,t)$ represents the force density distribution in the upper orthotropic medium under the influence of source of disturbance. The term containing $r$ and $t$ in $\sigma_1(r,t)$ denotes the distance $r$ from the origin at a time $t$. The substitutions for $v_1$ and $\sigma_1$ are considered as:

$$\begin{aligned} v_1(x,z,t)&=V_1(x,z)e^{i\omega t}\\ \sigma_1(r,t)&=\sigma_1(r)e^{i\omega t}\end{aligned} \tag{7}$$

Following the above conditions as given by Eq.(7), Eq.(6) is transformed as:

$$Q_3^{(1)}\frac{\partial^2 V_1}{\partial x^2}+Q_1^{(1)}\frac{\partial^2 V_1}{\partial z^2}+\rho^{(1)}\omega^2 V_1=4\pi\sigma_1(r) \tag{8}$$

where $\omega=kc$ denotes the angular frequency, $k$ and $c$ represent for wave number and wave velocity respectively. At the source point $t=0$, the disturbance occurs because of impulsive force $\sigma_1(r)$ which is expressed in the form of Dirac Delta function

$$\sigma_1(r)=\delta(x)\delta(z-H) \tag{9}$$

Applying Eq.(9) in Eq.(8), it becomes

$$Q_3^{(1)}\frac{\partial^2 V_1}{\partial x^2}+Q_1^{(1)}\frac{\partial^2 V_1}{\partial z^2}+\rho^{(1)}\omega^2 V_1=4\pi\delta(x)\delta(z-H) \tag{10}$$

The Fourier transform of $V_r(x,z)$ is taken as $\bar{V}_r(\eta,z)$ which is defined as:

$$\bar{V}_r(\eta,z)=\int_{-\infty}^{\infty}V_r(x,z)e^{i\eta x}dx \tag{11}$$

Now, the expression for inverse Fourier transform is

$$V_r(x,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{V}_r(\eta,z)e^{-i\eta x}d\eta \quad for \;\; r=1,2 \tag{12}$$

with the help of Eq.(12), Eq.(10) is converted to the following differential equation

$$\frac{d^2\bar{V}_1}{dz^2}-\left(\frac{Q_3^{(1)}}{Q_1^{(1)}}\eta^2-\frac{\rho^{(1)}\omega^2}{Q_1^{(1)}}\right)\bar{V}_1=\frac{2}{Q_1^{(1)}}\delta\left(z-H\right) \tag{13}$$

Eq.(13) can be rewritten as:

$$\frac{d^2\bar{V}_1}{dz^2}-\alpha^2\bar{V}_1=\frac{2}{Q_1^{(1)}}\delta\left(z-H\right) \tag{14}$$

where $\alpha^2=\frac{Q_3^{(1)}}{Q_1^{(1)}}\left(\eta^2-k_1^2\right),\quad k_1^2=\frac{\rho^{(1)}}{Q_3^{(1)}}\omega^2$.

## 4 DYNAMICS FOR THE HETEROGENEOUS ORTHOTROPIC HALF SPACE

The lower half space is assumed to be heterogeneous orthotropic half space. In lower half space, the equation of motion is defined as: [1]

$$\frac{\partial}{\partial x}\left(Q_3^{(2)}\frac{\partial v_2}{\partial x}\right)+\frac{\partial}{\partial z}\left(Q_1^{(2)}\frac{\partial v_2}{\partial z}\right)-\rho^{(2)}\frac{\partial^2 v_2}{\partial t^2}=4\pi\sigma_2\left(r,t\right) \tag{15}$$

where $Q_3^{(2)}$, $Q_1^{(2)}$ and $\rho^{(2)}$ denote the shear modulus and density for the lower orthotropic half space respectively. The symbolic term $\sigma_2\left(r,t\right)$ represents the force density distribution in the lower heterogeneous half space because of point source. The term containing $r$ and $t$ in $\sigma_2\left(r,t\right)$ denotes the distance *r* from the origin at a time *t*. In similar manner, to solve Eq.(15) substitutions are assumed for lower half space which is as follow:

$$\begin{aligned} v_2\left(x,z,t\right)&=V_2\left(x,z\right)e^{i\omega t}\\ \sigma_2\left(r,t\right)&=\sigma_2\left(r\right)e^{i\omega t}\end{aligned} \tag{16}$$

Considering the Eqs.(1),(11) and (16), the Eq.(15) is reduced to the following differential equation

$$\frac{d^2\bar{V}_2}{dz^2}-\beta^2\bar{V}_2=4\pi\sigma_2\left(z\right) \tag{17}$$

where $\beta^2=\frac{Q_3^{(2)}}{Q_1^{(2)}}\left(\eta^2-k_2^2\right)$, $k_2^2=\frac{\rho^{(2)}}{Q_3^{(2)}}\omega^2$ and $4\pi\sigma_2\left(z\right)=-\frac{1}{Q_1^{(2)}}\left(\varepsilon'\left(z-H\right)^2\frac{d^2\bar{V}_2}{dz^2}+2\varepsilon'\left(z-H\right)\frac{d\bar{V}_2}{dz}+\left(\omega^2\varepsilon''\left(z-H\right)^2-\varepsilon\eta^2\left(z-H\right)^2\right)\bar{V}_2\right)$

## 5 CONVENTIONAL BOUNDARY CONDITIONS

The upper surface is stress free at $z=0$ which facilitates that

$$\frac{d\bar{V}_1}{dz}=0 \tag{18}$$

The displacement component is continuous at $z=H$ which stands that

$$\bar{V}_1\left(z\right)=\bar{V}_2\left(z\right) \tag{19}$$

The stress component is continuous at $z = H$ which requires that

$$Q_1^{(1)} \frac{d\bar{V}_1}{dz} = Q_1^{(2)} \frac{d\bar{V}_2}{dz} \tag{20}$$

The technique of Green's function is manifested to compute the differential Eqs.(14) and (17) under the proposed boundary conditions as represented by Eqs.(18)-(20). Now, let $G^{(1)}(z/z_0)$ be the Green's function which satisfies the equation for upper orthotropic layer. The Green's function $G^{(1)}(z/z_0)$ is considered in such a manner that

$$\frac{dG^{(1)}}{dz} = 0 \tag{21}$$

at $z = 0$ and $z = H$. Employing Eq.(21) in Eq.(14), it is expressed as:

$$\frac{d^2 G^{(1)}(z/z_0)}{dz^2} - \alpha^2 G^{(1)}(z/z_0) = \delta(z - z_0) \tag{22}$$

where $z_0$ is considered as a point in the orthotropic layer and $z$ is known as the field point. Multiplying Eq.(14) by $G^{(1)}(z/z_0)$ and Eq.(22) by $\bar{V}_1(z)$, then subtracting and integrating with respect to $z$ from $z = 0$ to $z = H$, it is reduced to the following integral equation

$$\begin{aligned} &\int_0^H \left( G^{(1)}(z/z_0) \frac{d^2\bar{V}_1(z)}{dz^2} - \bar{V}_1(z) \frac{d^2 G^{(1)}(z/z_0)}{dz^2} \right) dz = \\ &\int_0^H \frac{2}{Q_1^{(1)}} \delta(z - H) G^{(1)}(z/z_0) dz - \int_0^H \bar{V}_1(z) \delta(z - z_0) dz \end{aligned} \tag{23}$$

Integrating Eq.(23) and applying Eq.(21), it can be expressed as:

$$G^{(1)}(H/z_0) \left[ \frac{d\bar{V}_1}{dz} \right]_{z=H} = \frac{2}{Q_1^{(1)}} G^{(1)}(H/z_0) - \bar{V}_1(z_0) \tag{24}$$

From the symmetric property of Green's function, it is well known that

$$G^{(1)}(H/z) = G^{(1)}(z/H) \tag{25}$$

Applying Eq.(25) in (24) and replacing $z_0$ by $z$, it becomes

$$\bar{V}_1(z) = \frac{2}{Q_1^{(1)}} G^{(1)}(H/z) - G^{(1)}(H/z) \left[ \frac{d\bar{V}_1}{dz} \right]_{z=H} \tag{26}$$

Now, let us suppose that $G^{(2)}(z/z_0)$ be the Green's function which satisfies the solution of lower half space. Considering the case $\frac{dG^{(2)}}{dz} = 0$ at $z = 0$ and $z = H$, and using Eq.(17), we have

$$\frac{d^2 G^{(2)}(z/z_0)}{dz^2} - \beta^2 G^{(2)}(z/z_0) = \delta(z - z_0) \tag{27}$$

Multiplying Eq.(27) by $\bar{V}_2$ and (17) by $G^{(2)}(z/z_0)$ then subtracting and integrating with respect to $z$ from $H$ to $\infty$, it becomes

$$\int_H^\infty \left( G^{(2)}(z/z_0)\frac{d^2\bar{V}_2(z)}{dz^2} - \bar{V}_2(z)\frac{d^2 G^{(2)}(z/z_0)}{dz^2} \right) dz = \int_H^\infty \left( 4\pi\sigma_2(z) G^{(2)}(z/z_0) - \delta(z-z_0)\bar{V}_2(z) \right) dz \tag{28}$$

Integrating Eq.(28), it transforms to

$$-G^{(2)}(H/z_0)\left[\frac{d\bar{V}_2}{dz}\right]_{z=H} = \int_H^\infty \left(4\pi\sigma_2(z) G^{(2)}(z/z_0)\right) dz - \bar{V}_2(z_0) \tag{29}$$

Interchanging $z$ and $z_0$ in Eq.(29) and simplifying, it converts to

$$\bar{V}_2(z) = G^{(2)}(z/H)\left[\frac{d\bar{V}_2}{dz}\right] + \int_H^\infty 4\pi\sigma_2(z) G^{(2)}(z/z_0) dz \tag{30}$$

Employing the boundary conditions as represented by Eq.(19) and (20) and replacing $z$ by $H$, it becomes as:

$$\frac{2}{Q_1^{(1)}} G^{(1)}(H/H) - G^{(1)}(H/H)\left[\frac{d\bar{V}_1}{dz}\right]_{z=H} = \frac{Q_1^{(1)}}{Q_1^{(2)}} G^{(2)}(H/H)\left[\frac{d\bar{V}_1}{dz}\right]_{z=H} + \int_H^\infty 4\pi\sigma_2(z) G^{(2)}(H/z_0) dz_0 \tag{31}$$

Eq.(31) is expressed in the following form

$$\left(\frac{Q_1^{(1)}}{Q_1^{(2)}} G^{(2)}(H/H) + G^{(1)}(H/H)\right)\left[\frac{d\bar{V}_1}{dz}\right]_{z=H} = \frac{2}{Q_1^{(1)}} G^{(1)}(H/H) - \int_H^\infty 4\pi\sigma_2(z) G^{(2)}(H/z_0) dz_0 \tag{32}$$

Eq.(32) signifies that

$$\left[\frac{d\bar{V}_1}{dz}\right]_{z=H} = \frac{\dfrac{2}{Q_1^{(1)}} G^{(1)}(H/H) - \displaystyle\int_H^\infty 4\pi\sigma_2(z) G^{(2)}(H/z_0) dz_0}{G^{(1)}(H/H) + \dfrac{Q_1^{(1)}}{Q_1^{(2)}} G^{(2)}(H/H)} \tag{33}$$

with the help of Eq.(33), Eq.(26) is converted as:

$$\bar{V}_1(z) = \frac{2}{Q_1^{(1)}} G^{(1)}(H/z) - \frac{2}{Q_1^{(1)}} \frac{G^{(1)}(H/z) G^{(1)}(H/H)}{A} + \frac{G^{(1)}(H/z)}{A}\int_H^\infty 4\pi\sigma_2(z) G^{(2)}(H/z_0) dz_0 \tag{34}$$

where $A = G^{(1)}(H/H) + \dfrac{Q_1^{(1)}}{Q_1^{(2)}} G^{(2)}(H/H)$.

Applying the boundary conditions as represented by Eq.(19) and (20) in Eq.(26) and (30), it reduces to

$$\frac{2}{Q_1^{(1)}} G^{(1)}(H/H) - G^{(1)}(H/H)\frac{Q_1^{(2)}}{Q_1^{(1)}}\left[\frac{d\bar{V}_2}{dz}\right]_{z=H} = G^{(2)}(H/H)\left[\frac{d\bar{V}_2}{dz}\right]_{z=H} + \int_H^\infty 4\pi\sigma_2(z) G^{(2)}(H/z_0) dz_0 \tag{35}$$

Eq.(35) implies that

$$\left[\frac{d\bar{V}_2}{dz}\right]_{z=H}=\frac{\frac{2}{Q_1^{(1)}}G^{(1)}\left(H/H\right)-\int_H^{\infty}4\pi\sigma_2\left(z\right)G^{(2)}\left(H/z_0\right)dz_0}{G^{(2)}\left(H/H\right)+\frac{Q_1^{(2)}}{Q_1^{(1)}}G^{(1)}\left(H/H\right)} \tag{36}$$

with the help of Eq.(30) and (36), and applying the method of successive approximation which leads to the following solution

$$\bar{V}_2\left(z\right)=\frac{\frac{2}{Q_1^{(1)}}G^{(1)}\left(H/H\right)G^{(2)}\left(z/H\right)}{B} \tag{37}$$

where

$$B=\frac{Q_1^{(2)}}{Q_1^{(1)}}G^{(1)}\left(H/H\right)+G^{(2)}\left(H/H\right) \tag{38}$$

Switching the value of $4\pi\sigma_2\left(z\right)$ in Eq.(34), it transforms to

$$\begin{aligned}\bar{V}_1\left(z\right)=&\frac{G^{(1)}\left(H/z\right)G^{(2)}\left(H/H\right)}{Q_1^{(1)}G^{(2)}\left(H/H\right)+Q_1^{(2)}G^{(1)}\left(H/H\right)}-\frac{G^{(1)}\left(H/z\right)G^{(1)}\left(H/H\right)}{\left(Q_1^{(1)}G^{(2)}\left(H/H\right)+Q_1^{(2)}G^{(1)}\left(H/H\right)\right)^2}\times\\&\int_H^{\infty}\left[\begin{array}{l}\varepsilon'\left(z_0-H\right)^2\frac{d^2G^{(2)}}{dz^2}+2\varepsilon'\left(z_0-H\right)\frac{dG^{(2)}}{dz}+\\\left(\omega^2\left(z_0-H\right)^2\varepsilon''-\varepsilon\eta^2\left(z_0-H\right)^2\right)G^{(2)}\left(z_0/H\right)\end{array}\right]G^{(2)}\left(H/z_0\right)dz_0\end{aligned} \tag{39}$$

Two independent solutions of Eq.(14) vanishing at $z=-\infty$ and $z=\infty$ are $\psi_1\left(z\right)=e^{\alpha z}$ and $\psi_2\left(z\right)=e^{-\alpha z}$. Therefore, the clarification of the Eq.(14) for a medium which is considered as:

$$\frac{\psi_1\left(z\right)\psi_2\left(z_0\right)}{W}\quad for\qquad z<z_0 \tag{40}$$

$$\frac{\psi_1\left(z_0\right)\psi_2\left(z\right)}{W}\quad for\qquad z>z_0 \tag{41}$$

where $W=\psi_1\left(z_0\right)\psi_2{}'\left(z\right)-\psi_1{}'\left(z_0\right)\psi_2\left(z\right)=-2\alpha\neq 0$. $W$ being the Wronskian of solution of Eq.(14). Therefore the required solution is

$$-\frac{e^{-\alpha|z-z_0|}}{2\alpha} \tag{42}$$

Since $G^{(1)}\left(z/z_0\right)$ is to satisfy the condition $\frac{dG^{(1)}}{dz}=0$ at $z=0$ and $z=H$, Therefore, we can assume that

$$G^{(1)}\left(z/z_0\right)=C_1e^{\alpha z}+C_2e^{-\alpha z}-\frac{e^{-\alpha|z-z_0|}}{2\alpha} \tag{43}$$

where $C_1$ and $C_2$ are arbitrary constants which can be evaluated using condition $\frac{dG^{(1)}}{dz}=0$ at $z=0$ and $z=H$. Therefore, we get

$$G^{(1)}\left(z/z_0\right)=\frac{1}{2\alpha}\left[e^{-\alpha|z-z_0|}+\frac{e^{\alpha z\left(e^{-\alpha\left(H+z_0\right)}+e^{-\alpha\left(H-z_0\right)}\right)}}{e^{\alpha H}-e^{-\alpha H}}+\frac{e^{-\alpha z\left(e^{\alpha\left(H-z_0\right)}+e^{-\alpha\left(H-z_0\right)}\right)}}{e^{\alpha H}-e^{-\alpha H}}\right] \tag{44}$$

Therefore, we have

$$G^{(1)}\left(z/H\right)=-\frac{1}{\alpha}\left[\frac{e^{\alpha z}+e^{-\alpha z}}{e^{\alpha H}-e^{-\alpha H}}\right] \tag{45}$$

$$G^{(1)}\left(H/H\right)=-\frac{1}{\alpha}\left[\frac{e^{\alpha H}+e^{-\alpha H}}{e^{\alpha H}-e^{-\alpha H}}\right] \tag{46}$$

In agreement of Green's function property, the value of $G^{(2)}\left(z/z_0\right)$ can be written as:

$$G^{(2)}\left(z/z_0\right)=-\frac{1}{2\beta}\left[e^{-\beta|z-z_0|+e^{-\beta\left(z+z_0-2H\right)}}\right] \tag{47}$$

and

$$G^{(2)}\left(H/z_0\right)=\frac{e^{-\beta\left(z_0-H\right)}}{\beta} \tag{48}$$

and

$$G^{(2)}\left(H/H\right)=-\frac{1}{\beta} \tag{49}$$

Substituting all these values in Eq.(39), it turns to the following form

$$\bar{V_1}(z)=\frac{-2\left(e^{\alpha z}+e^{-\alpha z}\right)}{\beta Q_1^{(2)}\left(e^{\alpha H}+e^{-\alpha H}\right)+\alpha Q_1^{(1)}\left(e^{\alpha H}-e^{-\alpha H}\right)}\begin{bmatrix}1+\dfrac{e^{\alpha H}+e^{-\alpha H}}{Q_1^{(2)}\beta\left(e^{\alpha H}+e^{-\alpha H}\right)+Q_1^{(1)}\alpha\left(e^{\alpha H}-e^{-\alpha H}\right)}\\ \times\dfrac{1}{4\beta^3}\left(\omega^2\varepsilon''-\varepsilon'\beta^2-\varepsilon\eta^2\right)\end{bmatrix} \tag{50}$$

Neglecting the higher powers of $\varepsilon,\varepsilon'$ and $\varepsilon''$ and then approximating Eq.(50), the value of $\bar{V_1}(z)$ is expressed as:

$$\bar{V_1}(z)=\frac{\dfrac{-2\left(e^{\alpha z}+e^{-\alpha z}\right)}{\beta Q_1^{(2)}\left(e^{\alpha H}+e^{-\alpha H}\right)+\alpha Q_1^{(1)}\left(e^{\alpha H}-e^{-\alpha H}\right)}}{[1-\dfrac{e^{\alpha H}+e^{-\alpha H}}{Q_1^{(2)}\beta\left(e^{\alpha H}+e^{-\alpha H}\right)+Q_1^{(1)}\alpha\left(e^{\alpha H}-e^{-\alpha H}\right)}\times\dfrac{1}{4\beta^3}\left(\omega^2\varepsilon''-\varepsilon'\beta^2-\varepsilon\eta^2\right)]} \tag{51}$$

Applying the technique of inverse Fourier transform in Eq.(51), the displacement component of the upper orthotropic layer is observed as:

$$\bar{V}_1(z) = \frac{-2}{\pi}\int_{-\infty}^{\infty} \frac{\dfrac{\left(e^{\alpha z}+e^{-\alpha z}\right)}{\beta Q_1^{(2)}\left(e^{\alpha H}+e^{-\alpha H}\right)+\alpha Q_1^{(1)}\left(e^{\alpha H}-e^{-\alpha H}\right)} e^{-i\eta x}}{[1-\dfrac{e^{\alpha H}+e^{-\alpha H}}{Q_1^{(2)}\beta\left(e^{\alpha H}+e^{-\alpha H}\right)+Q_1^{(1)}\alpha\left(e^{\alpha H}-e^{-\alpha H}\right)}\times\dfrac{1}{4\beta^3}\left(\omega^2\varepsilon''-\varepsilon'\beta^2-\varepsilon\eta^2\right)]} d\eta \tag{52}$$

The dispersion equation of SH wave is computed equating the denominator to be 0 which is represented as:

$$\left[1-\frac{e^{\alpha H}+e^{-\alpha H}}{Q_1^{(2)}\beta\left(e^{\alpha H}+e^{-\alpha H}\right)+Q_1^{(1)}\alpha\left(e^{\alpha H}-e^{-\alpha H}\right)}\times\frac{1}{4\beta^3}\left(\omega^2\varepsilon''-\varepsilon'\beta^2-\varepsilon\eta^2\right)\right]=0 \tag{53}$$

Eq.(53) provides the dispersion equation of SH wave in an orthotropic layer due to point source which is expressed as follows:

$$\tan\left(\sqrt{\frac{Q_3^{(1)}}{Q_1^{(1)}}}\left(\sqrt{\frac{c^2}{\beta_1^2}-1}\right)kH\right)=\frac{Q_1^{(2)}}{Q_1^{(1)}}\frac{\sqrt{\frac{Q_3^{(2)}}{Q_1^{(2)}}}\sqrt{1-\frac{c^2}{\beta_2^2}}}{\sqrt{\frac{Q_3^{(1)}}{Q_1^{(1)}}}\sqrt{\frac{c^2}{\beta_1^2}-1}}-\frac{\frac{\varepsilon''H^2}{\rho^{(2)}}\frac{Q_3^{(2)}}{Q_1^{(1)}}\frac{c^2}{\beta_2^2}-\frac{\varepsilon'H^2}{Q_1^{(1)}}\frac{Q_3^{(2)}}{Q_1^{(2)}}\left(1-\frac{c^2}{\beta_2^2}\right)-\frac{\varepsilon H^2}{Q_3^{(2)}}\frac{Q_3^{(2)}}{Q_1^{(1)}}}{2(kH)^2\sqrt{\frac{Q_3^{(1)}}{Q_1^{(1)}}}\sqrt{\frac{c^2}{\beta_1^2}-1}\left(\frac{Q_3^{(2)}}{Q_1^{(2)}}\left(1-\frac{c^2}{\beta_2^2}\right)\right)^{\frac{3}{2}}} \tag{54}$$

where $\beta_i=\sqrt{\dfrac{Q_3^{(i)}}{\rho^{(i)}}}$, $i=1,2$. Eq.(54) represents the dispersion equation of SH wave in an orthotropic media because of source of disturbance.

## 6 SPECIAL CASES

Case I: When $Q_3^{(2)}\to Q_1^{(2)}\to\mu_2,\ Q_1^{(1)}\to Q_3^{(1)}\to\mu_1$, when the uppermost and undermost medium is homogeneous with rigidity $\mu_1$ and $\mu_2$ respectively, it becomes

$$\tan\left(\sqrt{\left(\frac{c^2}{\beta_1^2}-1\right)}kH\right)=\frac{\mu_2}{\mu_1}\frac{\sqrt{1-\frac{c^2}{\beta_2^2}}}{\sqrt{\frac{c^2}{\beta_1^2}-1}}-\frac{1}{2(kH)^2\sqrt{\frac{c^2}{\beta_1^2}-1}}\left[\frac{\frac{\varepsilon''H^2}{\rho^{(2)}}\frac{\mu_2}{\mu_1}\frac{c^2}{\beta_2^2}-\frac{\varepsilon'H^2}{\mu_1}\left(1-\frac{c^2}{\beta_2^2}\right)-\frac{\varepsilon H^2}{\mu_2}\frac{\mu_2}{\mu_1}}{\left(1-\frac{c^2}{\beta_2^2}\right)^{\frac{3}{2}}}\right] \tag{55}$$

This represents the dispersion equation of SH wave in an isotropic layer lying over an isotropic half space in the presence of point source.
Case II: When $\varepsilon\to 0,\ \varepsilon'\to 0,\ \varepsilon''\to 0$, when all the point source parameter becomes 0, then

$$\tan\left(\sqrt{\left(\frac{c^2}{\beta_1^2}-1\right)}kH\right)=\frac{\mu_2}{\mu_1}\frac{\sqrt{1-\frac{c^2}{\beta_2^2}}}{\sqrt{\frac{c^2}{\beta_1^2}-1}} \tag{56}$$

This represents the classical Love wave equation in an isotropic layer lying over an isotropic half space in absence of point source parameter.

## 7 NUMERICAL CALCULATION AND GRAPHICAL OBSERVATION

The propagation of SH wave in an orthotropic stratum lying over a heterogeneous ortotropic half space is described graphically to deliberate the influence of heterogenous parameter associated with shear modulus and density. For the graphical issue, the data has been proceeded from Gubbins[10] as $Q_1^{(1)} = 5.65\times10^{10} N/m^2$, $Q_3^{(1)} = 2.46\times10^{10} N/m^2$, $\rho^{(1)} = 7800 Kg/m^3$, $Q_1^{(2)} = 5.82\times10^{10} N/m^2$, $Q_3^{(2)} = 3.99\times10^{10} N/m^2$, $\rho^{(2)} = 4500 Kg/m^3$. All the figures are drawn in respect of dimensionless wave number $kH$ and phase velocity $\frac{c}{\beta_1}$.

In Fig. 2 , curves are sketched to analyze the effect of heterogeneous parameter associated with density. Curve 1 is stalemated to manifest the case when heterogeneous parameter $\frac{\varepsilon'' H^2}{\rho^{(2)}}$ tends to 0. All other curves are drawn by considering the heterogeneous parameter in increasing order. As the value of $\frac{\varepsilon'' H^2}{\rho^{(2)}}$ increases, phase velocity increases.

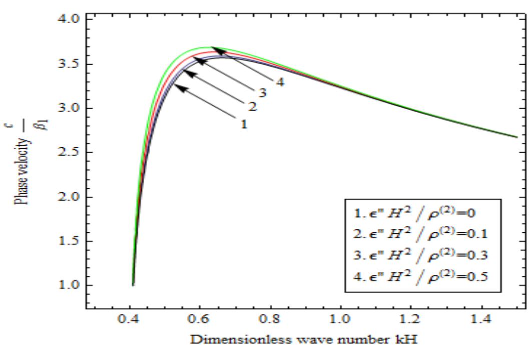


**Fig.2**
Variation of $\frac{\varepsilon'' H^2}{\rho^{(2)}}$ with respect to phase velocity and dimensionless wave number *kH*.

Figs.3 and 4 are configured to express the effect of heterogeneous parameter accomplished to transverse and longitudinal shear modulus respectively. In account of considering the respective heterogeneous parameter related to shear modulus to be 0, the curve 1 is sketched in both the Fig.3 and 4. In both figures, Phase velocity decreases as the respective heterogeneous parameter increases.

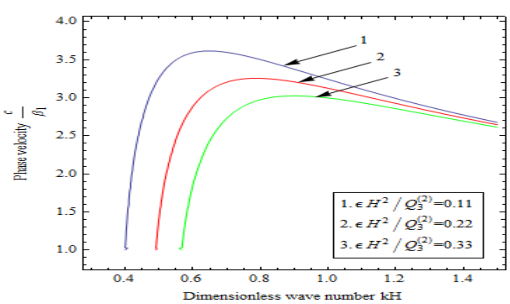


**Fig.3**
Variation of $\frac{\varepsilon H^2}{Q_3^{(2)}}$ with respect to phase velocity and dimensionless wave number *kH*.

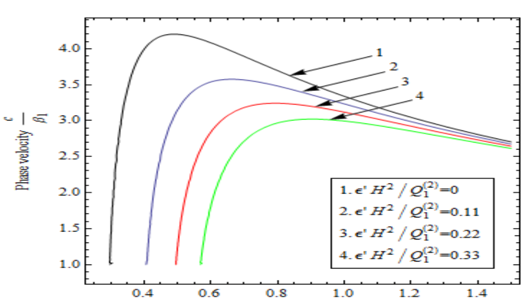


**Fig.4**
Variation of $\frac{\varepsilon' H^2}{Q_1^{(2)}}$ with respect to phase velocity and dimensionless wave number *kH*.

Fig.5 is sketched to show the particular cases separately in respect of general case. Case I and II expresses the case in presence and absence of point source respectively. It is worthy to mention that curve 1 in Figs. 2,3 and 4 corresponds to the case of regular orthotropic medium over a semi infinite medium without respective heterogeneities due to point source.

In Fig.5, curve 1 (Case I) and curve 2 (Case II) are drawn for isotropic case in presence and absence of point source respectively. In case of isotropy, phase velocity lies in the range of real time data having some error. The theory lies in the fact that glass is an example of an isotropic material. Glass is a hard, brittle substance that is usually transparent or translucent. It is made by melting together sand (Silicon dioxide), soda (Sodium Carbonate), limestone (Calcium Carbonate) and other ingredients. From the study of Standard Rock Physics Laboratory, [14] it is analyzed that propagation velocity in an isotropic medium varies between $100m/s$ to $6000m/s$ which also validates the obtained theoretical results. The estimated error is calculated using $C$ programming with a graphical approach.

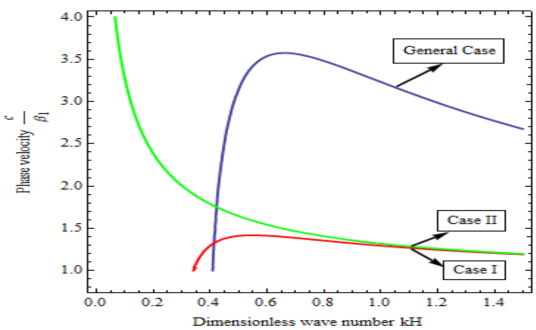


**Fig.5**
Variation of special cases with respect to phase velocity and dimensionless wave number *kH*.

Finally, it is concluded that:

i. In respect of phase velocity and wave number , phase velocity increases for some range of wave number, then after it behaves reversely.

ii. The phase velocity of all the graphs lies between $1-4.3$ $km/sec$ which match with the seismological lab.

## 8 ERROR ANALYSIS

A graphical approach is concerned to demonstrate the accuracy of propagation velocity of SH wave in an orthotropic media. Before comparing the theoretical results with real time data, it is to be mentioned that olivin is an example of orthotropic material [7]. Also olivin is the name of a group of rock forming minerals that are typically found in mafic and ultramafic igneous rocks such as basalt, gabbro, dunite, diabase and peridotite. These rocks are typically made of $Mg_2Sio_4$ and $Fe_2Sio_4$. Olivin has a very high crystallization temperature compared to other minerals. The propagation velocity of seismic wave through orthotropic media goes to $2.8$ $m/s$ to $3.4$ $m/s$ [14]. An approximate error is calculated by the formula $\frac{\text{Exact Value} - \text{Approximate Value}}{\text{ExactValue}} \times 100$ [18]. Approximate value of phase velocity is taken for the curve which is to be compared and exact value is taken from the results of seismological lab. Error analysis helps us to compare the theoretical work with real time data. Thus it makes an important application in real life problem. The estimated error is calculated graphically using $C$ programming. The errors are summarized in Table 1. and Table 2.

Among all the Figs. 2, 3 and 4, the propagation velocity lies between $0.847$ $km/s$ to $3.718$ $km/s$ which lies in the range of real time data. Table 1. expresses an analogy between graphically obtained result and seismological lab results for Figs. 2,3 and 4.

In Figs. 5, 3 curves are drawn to show the particular cases (Case I and Case II) and general case. A curve (General case) which is drawn for fixed value, as taken in numerical computation section, is supposed to be exact. Applying the above technique, we have computed the Table 2. to estimate the error in graph for particular cases.

**Table 1**
Comparison with theoretical and seismological lab result.

| Graphs | Minimum phase velocity through seismological Lab | Maximum phase velocity through seismological Lab | Minimum phase velocity through theoretical result | Maximum phase velocity through theoretical result | Minimum Error in percentage | Maximum Error in percentage |
|---|---|---|---|---|---|---|
| Graph 2 | 2.8 | 3.4 | 0.969 | 3.718 | 65 | 9.352 |
| Graph 3 | 2.8 | 3.4 | 0.971 | 4.241 | 65.321 | 22.131 |
| Graph 4 | 2.8 | 3.4 | 0.847 | 3.619 | 69.725 | 6.441 |

**Table 2**
Comarison with theoretical and seismological lab result.

| | Phase Velocity | Error in percentage | Phase Velocity | Error in percentage | Phase Velocity | Error in percentage | Phase Velocity | Error in percentage |
|---|---|---|---|---|---|---|---|---|
| Wave number | 0.2 | | 0.6 | | 1.0 | | 1.4 | |
| General Case | - | | 3.553 | | 3.211 | | 2.769 | |
| Case I | - | Not Found | 1.422 | 59.97 | 1.289 | 59.85 | 1.201 | 56.62 |
| Case II | 2.294 | Not Found | 1.554 | 56.26 | 1.323 | 58.79 | 1.201 | 56.62 |
| Average Error | - | - | | 59.615 | - | 58.055 | | 56.62 |

## 9 CONCLUSIONS

The conclusive investigation approaches to the dispersion of propagation of SH wave in an orthotropic layer lying over a heterogeneous orthotropic half space under the influence of point source. It is observed that the technique of Green's function and Fourier transformation leads to the computation of dispersion equation. The present study arrives at the following conclusions:

1. The phase velocity of SH wave behaves as a monotonic curve. Sometimes, the phase velocity increases with the increment of wave number and then decreases eventually with the increment of wave number.
2. The effect of non homogeneity parameter is very prominent to the propagation of SH wave.
3. The classical equation of Love wave is deduced in a homogeneous isotropic layer lying over an isotropic half space.

Analysis of this study has its special characteristic to the problem of wave motion and vibrations where the wave signals have to travel through dissimilar layers of various material properties and containing irregularities due to continental margin, mountain roots and salts. The present study may be effectively utilized to artificial explosion and material structure during non destructive testing.

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