Journal of Solid Mechanics Vol. 9, No. 3 (2017) pp. 650-662

Elastic Analysis of Functionally Graded Variable Thickness Rotating Disk by Element Based Material Grading

A.K. Thawait [1,*], L. Sondhi [1] , Sh. Sanyal [2], Sh. Bhowmick [2]

[1]*Department of Mechanical Engineering, Shri Shankaracharya Technical Campus, SSGI,Bhilai, (C.G.), India*
[2]*Department of Mechanical Engineering, NIT Raipur, 492010, India*

Received 29 June 2017; accepted 29 August 2017

ABSTRACT

The present study deals with the elastic analysis of concave thickness rotating disks made of functionally graded materials (FGMs).The analysis is carried out using element based gradation of material properties in radial direction over the discretized domain. The resulting deformation and stresses are evaluated for free-free boundary condition and the effect of grading index on the deformation and stresses is investigated and presented. The results obtained show that there is a significant reduction of stresses in FGM disks as compared to homogeneous disks and the disks modeled by power law FGM have better strength.

Keywords: Functionally graded material; Elastic analysis; Annular rotating disk; Concave thickness profile rotating disk; Element based material gradation.

1 INTRODUCTION

MANY engineering components are modeled as rotating circular plates or disks in the field of marine, mechanical and aerospace industries such as gas turbines, gears, turbo-machinery, flywheel systems, centrifugal compressors, power transmission systems, machining devises, circular saws, microwave or baking ovens, and support tables, etc. The total stresses due to centrifugal load have important effects on their strength and safety. Thus, control and optimization of total stresses and displacement fields is an important task. Functionally gradation of the material properties and variable thickness profile optimize the component strength by controlling the changes of the local material properties. Depending on the function of the component, it is possible to utilize one, two or three-directional distributions of the material properties.

A few researchers have reported work on analysis of FGM disks, plates, shells, beams and bars by analytical and finite element method. Eraslan et al. [1] has obtained analytical solutions for the elastic plastic stress distribution in rotating variable thickness annular disks. Thickness of the disks has parabolic variation and the analysis is based on the tresca's yield criterion. Bayat et al. [2] reported work on analysis of a variable thickness FGM rotating disk. Material properties vary according to power law and the disk is subjected to both the mechanical and thermal loads. Afsar et al. [3] have analyzed a rotating FGM circular disk subjected to mechanical as well as thermal load by finite element method. The disk has exponentially varying material properties in radial direction. The disk is subjected to a thermal load along with centrifugal load due to non-uniform temperature distribution. An analytical thermoelasticity solution for a disk made of functionally graded materials (FGMs) is presented by Callioglu [4]. Bayat et al. [5] investigated displacement and stress behavior of a functionally graded rotating disk of variable thickness by semi analytical method. Radially varying one dimensional FGM is taken and material properties vary according to power

*Corresponding author.
E-mail address: amkthawait@gmail.com (A.K.Thawait).

law and Mori-Tanaka scheme. Disk subjected to centrifugal load is analyzed for fixed boundary condition at inner surface and free boundary condition at outer surface. Callioglu et al. [6] have analyzed thin FGM disks. Density and Modulus of elasticity varies according to power law in an FGM of aluminum ceramic and the effect of grading parameter on displacement and stresses are investigated. Callioglu et al. [7] has analyzed functionally graded rotating annular disk subjected to internal pressure and various temperature distributions such as uniform temperature, linearly increasing and decreasing temperature in radial direction. Sharma et al. [8] worked on the analysis of stresses, displacements and strains in a thin circular functionally graded material (FGM) disk by finite element method. The disk is subjected to mechanical as well as thermal loads. Ali et al. [9] reported study on the elastic analysis of two sigmoid FGM rotating disks. Metal-Ceramic-Metal disk is analyzed for both uniform and variable thickness disks and effect of grading index on the displacement and stresses are investigated. Nejad et al. [10] have found closed-form Analytical solution of an exponentially varying FGM disk which is subjected to internal and external pressure. Ghorbanpour et al. [11, 12] have worked on Thermo-piezo-magnetic behavior of a functionally graded piezo-magnetic (FGPM) rotating disk, under mechanical and thermal loads. A semi-analytical solution for magneto-thermo-elastic problem in functionally graded (FG) hollow rotating disks with variable thickness placed in uniform magnetic and thermal fields is also presented.

In a recent work Zafarmand et al. [13] worked on elastic analysis of two-dimensional functionally graded rotating annular and solid disks with variable thickness. Axisymmetric conditions are assumed for the two-dimensional functionally graded disk and the graded finite element method (GFEM) has been applied to solve the equations. Rosyid et al. [14] worked on finite element analysis of nonhomogeneous rotating disk with arbitrarily variable thickness. Three types of grading law namely Power law, sigmoid and exponential distribution law is considered for the volume fraction distributions. Zafarmand et al. [15] investigated nonlinear elasticity solution of functionally graded nanocomposite rotating thick disks with variable thickness reinforced with single-walled carbon nanotubes (SWCNTs). The governing nonlinear equations derived are based on the axisymmetric theory of elasticity with the geometric nonlinearity in axisymmetric complete form and are solved by nonlinear graded finite element method (NGFEM).

In present research work rotating disks of parabolic concave thickness profile having free-free boundary condition are analyzed. Material properties of the disks are graded along the radial direction according to three types of distribution laws, namely power law, exponential law and Mori-Tanaka scheme by element based material grading. A finite element formulation based on principle of stationary total potential is presented for the rotating disks and finally the effect of grading parameter n on the deformation and stresses for different material gradation law is investigated.

2 PROBLEM FORMULATION

In this section geometric equation as well as different types of material property distributions are presented and the governing equations for the rotating disk are derived.

2.1 Geometric modeling

For an annular disk, the governing equation of radially varying thickness is assumed as:

$$h(r) = h_0\left[1 - q\left(\frac{r-a}{b-a}\right)^k\right] \tag{1}$$

where a and b are inner and outer radius, $h(r)$ and h_0 are half of the thickness at radius r and at the root of the disk, k and q are the constants that control the thickness profiles of the disk. For uniform thickness disk $q = 0$ and for variable thickness $q > 0$; $k < 1$ for concave thickness profile.

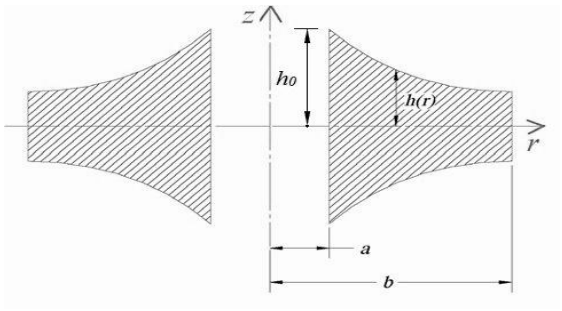

Fig.1
Geometric parameters of the rotating disk.

2.2 Material modeling

Three types of material models namely power law distribution [7], exponential distribution [3] and Distribution by Mori-Tanaka scheme [5] are considered in the present analysis. The effective Young's modulus *E(r)* and density *ρ(r)* of a disk having inner radius *a* and outer radius *b* can be obtained as:

Power Law

$$E\left(r\right)=E_o\left(\frac{r}{b}\right)^n \tag{2}$$

$$\rho\left(r\right)=\rho_o\left(\frac{r}{b}\right)^n \tag{3}$$

where E_o, ρ_o are modulus of elasticity and density at the outer radius.

Exponential law

$$E\left(r\right)=E_0 e^{\beta r} \tag{4}$$

$$\rho\left(r\right)=\rho_0 e^{\gamma r} \tag{5}$$

$$E_0=E_i e^{-\beta a} \tag{6}$$

$$\rho_0=\rho_i e^{-\gamma a} \tag{7}$$

$$\gamma=\frac{1}{a-b}ln\left(\frac{\rho_i}{\rho_o}\right) \tag{8}$$

$$\beta=\frac{1}{a-b}ln\left(\frac{E_i}{E_o}\right) \tag{9}$$

E_i, E_o and ρ_i, ρ_o are modulus of elasticity and density at the inner and outer radius.

Mori -Tanaka Scheme

The effective bulk modulus *B(r)*and shear modulus *G(r)* of the FGM disk, evaluated using the Mori-Tanaka scheme [5] are given by:

$$B(r)=(B_o-B_i)/V_o\left(1+(1-V_o)\frac{3(B_o-B_i)}{3B_i+4G_i}\right)+B_i \tag{10}$$

$$G(r)=(G_o-G_i)/V_o\left(1+(1-V_o)\frac{(G_o-G_i)}{G_i+f_1}\right)+G_i \tag{11}$$

$$f_1=\frac{G_i(9B_i+8G_i)}{6(B_i+2G_i)} \tag{12}$$

Here, V is the volume fraction of the phase material. The subscripts i and o refer to the inner and outer materials respectively. The volume fraction of the inner and outer phases are related by

$$V_i+V_o=1 \tag{13}$$

and V_o is expressed as:

$$V_o=\left(\frac{r-r_i}{r_o-r_i}\right)^n \tag{14}$$

where $n(n \geq 0)$ is the volume fraction exponent. The elastic modulus E can be found as:

$$E(r)=\frac{9B(r)G(r)}{3B(r)+G(r)} \tag{15}$$

The mass density ρ can be given by the rule of mixtures as:

$$\rho(r)=(\rho_o-\rho_i)\left(\frac{r-r_i}{r_o-r_i}\right)^n+\rho_i \tag{16}$$

2.3 Finite element formulation

The rotating disk, being thin, is modeled as a plane stress axisymmetric problem. Using quadratic quadrilateral element, the displacement vector $\{u\}$ can be obtained as:

$$\{u\}=[N]\{\delta\} \tag{17}$$

where $\{u\}$ is element displacement vector, $[N]$ is matrix of quadratic shape functions and $\{\delta\}$ is nodal displacement vector.

$$[N]=[N_1 \quad N_2 \quad \ldots. \quad N_8]=\textit{matrix of shape functions}$$
$$\{\delta\}=\{u_1 \quad u_2 \quad \ldots \quad \ldots \quad u_8\}^T=\textit{nodal displacement vector}$$

In natural coordinates (ξ-η), the shape functions are given as [16]:

$$N_1=\left(\frac{1}{4}\right)(1-\xi)(1-\eta)(-1-\xi-\eta), \qquad N_5=\left(\frac{1}{2}\right)(1-\xi^2)(1-\eta)$$

$$N_2=\left(\frac{1}{4}\right)(1+\xi)(1-\eta)(-1+\xi-\eta), \qquad N_6=\left(\frac{1}{2}\right)(1+\xi)\left(1-\eta^2\right)$$
$$N_3=\left(\frac{1}{4}\right)(1+\xi)(1+\eta)(-1+\xi+\eta), \qquad N_7=\left(\frac{1}{2}\right)\left(1-\xi^2\right)(1+\eta)$$
$$N_4=\left(\frac{1}{4}\right)(1-\xi)(1+\eta)(-1-\xi+\eta), \qquad N_8=\left(\frac{1}{2}\right)(1-\xi)\left(1-\eta^2\right)$$

The strain components are related to elemental displacement components as:

$$\{\varepsilon\}=\{\varepsilon_r \quad \varepsilon_\theta\}^T=\left\{\frac{\partial u}{\partial r} \quad \frac{u}{r}\right\}^T \tag{18}$$

$$\left\{\frac{\partial u}{\partial r} \quad \frac{u}{r}\right\}^T=[B_1]\left\{\frac{\partial u}{\partial r} \quad \frac{\partial u}{\partial z} \quad \frac{u}{r}\right\}^T \tag{19}$$

where ε_r and $\varepsilon_{\theta,}$ are radial and tangential strain respectively. By transforming the global coordinates into natural coordinates (ξ-η),

$$\left\{\frac{\partial u}{\partial r} \quad \frac{\partial u}{\partial z} \quad \frac{u}{r}\right\}^T=[B_2]\left\{\frac{\partial u}{\partial \xi} \quad \frac{\partial u}{\partial \eta} \quad \frac{u}{r}\right\}^T \tag{20}$$

$$\left\{\frac{\partial u}{\partial \xi} \quad \frac{\partial u}{\partial \eta} \quad \frac{u}{r}\right\}^T=[B_3]\{u_1 \quad u_2 \quad \ldots \quad \ldots \quad u_8\}^T \tag{21}$$

The above elemental strain-displacement relationships can be written as:

$$\{\varepsilon\}=[B]\{\delta\}^e \tag{22}$$

where [B] is strain-displacement relationship matrix, which contains derivatives of shape functions. For a quadratic quadrilateral element it is calculated as:

$$[B]=[B_1]\times[B_2]\times[B_3]$$
$$[B_1]=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$$
$$[B_2]=\begin{bmatrix}\frac{J_{22}}{|J|} & \frac{-J_{12}}{|J|} & 0\\ \frac{-J_{21}}{|J|} & \frac{J_{11}}{|J|} & 0\\ 0 & 0 & 1\end{bmatrix} \tag{23}$$

where J is the Jacobian matrix, used to transform the global coordinates into natural coordinates. It is given as:

$$[J]=\begin{bmatrix}\sum_{i=1}^{8}\frac{\partial N_i}{\partial \xi}r_i & \sum_{i=1}^{8}\frac{\partial N_i}{\partial \xi}z_i \\ \sum_{i=1}^{8}\frac{\partial N_i}{\partial \eta}r_i & \sum_{i=1}^{8}\frac{\partial N_i}{\partial \eta}z_i\end{bmatrix}$$

$$[B_3]=\begin{bmatrix}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & .... & \frac{\partial N_8}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & .... & \frac{\partial N_8}{\partial \eta} \\ \frac{N_1}{r} & \frac{N_2}{r} & .... & \frac{N_8}{r}\end{bmatrix}$$

From hooks law, components of stresses in radial and circumferential direction are related to components of total strain as:

$$\varepsilon_r=\frac{1}{E}\left(\sigma_r-\nu\sigma_\theta\right),\qquad \varepsilon_\theta=\frac{1}{E}\left(\sigma_\theta-\nu\sigma_r\right) \tag{24}$$

By solving above equations, stress strain relationship can be obtained as follows:

$$\sigma_r=\frac{E(r)}{(1-\nu)^2}\left[\varepsilon_r+\nu\varepsilon_\theta\right] \tag{25}$$

$$\sigma_\theta=\frac{E(r)}{(1-\nu)^2}\left[\varepsilon_\theta+\nu\varepsilon_r\right] \tag{26}$$

In standard finite element matrix notation above stress strain relations can be written as [16]:

$$\{\sigma\}=\left[D(r)\right]\{\varepsilon\} \tag{27}$$

where

$$\{\sigma\}=\{\sigma_r \quad \sigma_\theta\}^T,\qquad \{\varepsilon\}=\{\varepsilon_r \quad \varepsilon_\theta\}^T,\qquad D(r)=\frac{E(r)}{(1-\nu)^2}\begin{bmatrix}1 & \nu \\ \nu & 1\end{bmatrix}$$

In FEM, the functional grading is popularly carried out by assigning the average material properties over a given geometry followed by adhering the geometries thus resulting into layered functional grading of material properties. The downside of this approach is that it yields singular field variable values at the boundaries of the glued geometries. To get better results, it is an established practice to divide the total geometry into very fine geometries. However, a better approach is to assign the average material properties to the elements of mesh of the single geometry. This is, in other words, better described as assigning material properties to the finite elements instead of geometry. In Eq. (27), the [*D*(*r*)] matrix, being a function of *r*, is calculated numerically at each node and this results into continuous material property variation throughout the geometry.

The element based grading of material property yields an appropriate approach of functional grading as the shape functions in the elemental formulations being co-ordinate functions make it easier to implement the same [13].

$$\varnothing^e=\sum_{i=1}^{8}\varnothing_i N_i \tag{28}$$

where ϕ^e is element material property, ϕ_i is material property at node i and N_i is the Shape function.

Upon rotation, the disk experiences a body force which under constrained boundary results in deformation and stores internal strain energy U [16].

$$U = \frac{1}{2}\int_V \{\varepsilon\}^T \{\sigma\}\, dv \tag{29}$$

The work potential due to body force resulting from centrifugal action is given by

$$V = -\int_V \{\delta\}^T \{q_v\} dv \tag{30}$$

Upon substituting Eqs. (22) and (27) in Eqs. (29) and (30), the elemental strain energy and work potential are obtained as:

$$U^e = \int_V \pi r h_r \{\delta\}^{eT} [B]^T [D(r)][B]\{\delta\}^e\, dr \tag{31}$$

$$V^e = -2\int_V \pi r h_r \{\delta\}^{eT} [N]^T \{q_v\} dr \tag{32}$$

For a disk rotating at ω rad/sec, the body force vector for each element is given by

$$\{q_v\} = \begin{Bmatrix} \rho(r)\omega^2 r \\ 0 \end{Bmatrix}$$

The total potential of the element is obtained from Eqs. (31) and (32) as:

$$\pi_p{}^e = \frac{1}{2}\{\delta\}^{eT} [K]^e \{\delta\}^e - \{\delta\}^{eT} \{f\}^e \tag{33}$$

Here, defining element stiffness matrix $[K]^e$ and element load vector $\{f\}^e$ as:

$$[K]^e = 2\int_V \pi r h_r [B]^T [D(r)][B] dr \tag{34}$$

$$\{f\}^e = 2\int_V \pi r h_r [N]^T \{q_v\} dr \tag{35}$$

By transforming the global coordinates into natural coordinates

$$[K]^e = 2\pi \int_{-1}^{1}\int_{-1}^{1} [B]^T [D(r)][B] r |J| d\xi d\eta \tag{36}$$

$$\{f\}^e = 2\pi \int_{-1}^{1}\int_{-1}^{1} [N]^T \{q_v\} r |J| d\xi d\eta \tag{37}$$

The element matrices are then assembled to yield the global stiffness matrix and global load vector respectively. Total potential energy of the disk is given by

$$\pi_p = \sum \pi_p{}^e = \frac{1}{2}\{\delta\}^T [K]\{\delta\} - \{\delta\}^T \{F\} \tag{38}$$

where

$$[K] = \sum_{n=1}^{N} [K]^e = Global\ Stiffness\ matrix$$

$$\{F\} = \sum_{n=1}^{N} [f]^e = Global\ load\ vector$$

N = no. of elements. Using the Principle of Stationary Total Potential (PSTP), the total potential is set to be stationary with respect to small variation in the nodal degree of freedom that is

$$\frac{\partial \pi_p}{\partial \{\delta\}^T} = 0 \tag{39}$$

From above, the system of simultaneous equations is obtained as follows:

$$[K]\{\delta\} = \{F\} \tag{40}$$

3 RESULTS AND DISCUSSION

3.1 Validation

A uniform thickness rotating disk having power law distribution of material properties (grading parameter n = 0, 0.5 and 1) of reference [7] is reconsidered and analyzed again and results are presented in Fig 2. The results obtained are in good agreement with the pre-established results of reference.

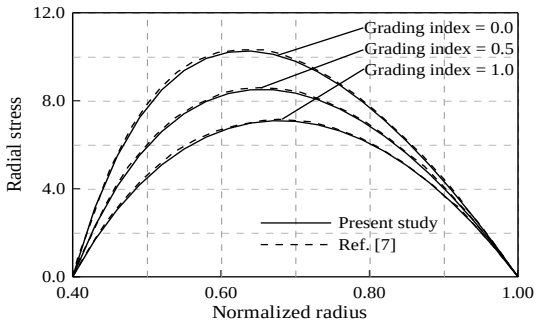

Fig.2
Comparison of the results of current work with reference [7].

3.2 Numerical results and discussion

In this section rotating annular disks of parabolic concave thickness profiles having free-free boundary condition are analyzed. Disks are made of aluminum and zirconia ceramic as well as their different FGM's. Finally the effects of grading parameter n on stresses and deformation states are investigated.

The material properties of aluminum and zirconia are given as [5]: E_{Al} = 70 *GPa*, E_{cer} = 151 *GPa*, ρ_{Al} = 2700 kg/m^3, ρ_{cer} = 5700 kg/m^3, B_{Al} = 58.3333 *GPa*, B_{cer} = 128.8333 *GPa*, G_{Al} = 26.9231*GPa*, G_{cer} = 58.0769 *GPa*, υ = 0.3. The disks have geometric parameters as $k = 0.5$, inner diameter = 0.2 *m*, outer diameter = 1 *m*, $q = 0.96$ and h_0 *is 0.075 m*. Disks are rotating with unit angular velocity that is 1 rad/sec. Grading index n = 0 indicates that the disk is made of outer material completely means the disk is homogeneous in composition. For ceramic-metal FGM, n = 0 indicates homogeneous metallic (aluminum) disk while for metal-ceramic FGM it indicates homogeneous ceramic

(zirconia) disk. For values of n other then 0, volume fraction varies with the radius according to different equations and gives different types of FGMs.

It can be observed that ceramic-metal FGM disks have less radial deformation and radial stress while higher circumferential and von Mises stress as compared to metal-ceramic FGM disks in exponential FGM. Radial stress is zero at the inner and outer radius for all cases, which confirms the free-free boundary condition applied on the disks. In metal-ceramic FGM disks modeled by Mori-Tanaka scheme radial deformation increases and stresses decreases with increasing value of grading parameter n while in ceramic-metal FGM radial deformation decreases and stresses increases with increasing value of n. Increasing n means volume fraction of the outer material is decreasing and inner material is increasing. In case of metal-ceramic FGM, increasing metallic content and decreasing ceramic content results in higher deformation and lesser stresses while in case of ceramic-metal FGM, increasing ceramic and decreasing metallic content results higher stresses and lesser deformation. Metal-ceramic FGM having $n = 1.5$ has highest radial deformation while ceramic-metal FGM having $n = 0.5$ has the lowest radial deformation. Ceramic-metal FGM disk having $n = 0$ has lowest stresses and metal-ceramic FGM having $n = 0$ has highest stresses among all the FGMs modeled by Mori-Tanaka scheme.

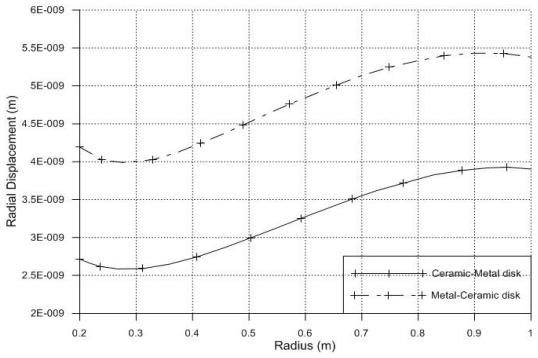

Fig.3
Distribution of radial displacements for exponential FGM disks.

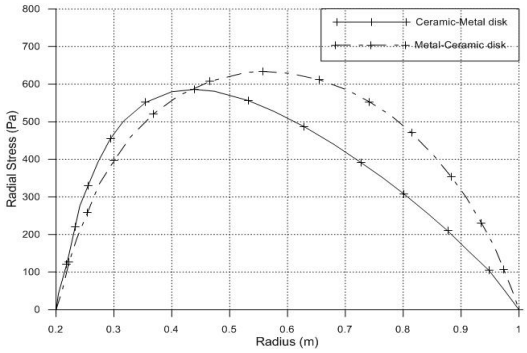

Fig.4
Distribution of radial stress for exponential FGM disks.

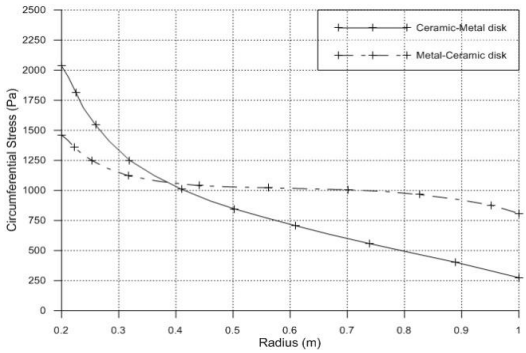

Fig.5
Distribution of circumferential stresses for exponential FGM disks.

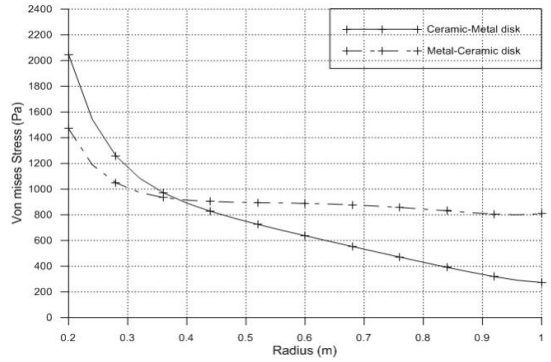

Fig.6
Distribution of von mises stresses for exponential FGM disks.

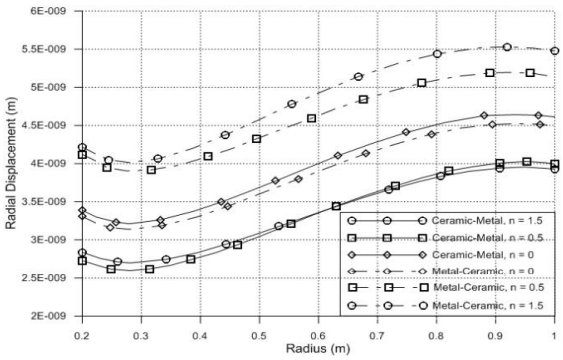

Fig.7
Distribution of radial displacements for Mori-Tanaka FGM disks.

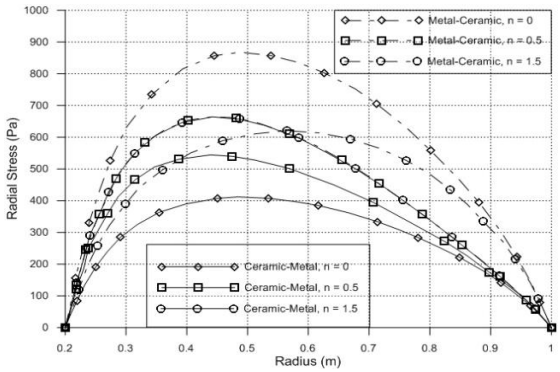

Fig.8
Distribution of radial stresses for Mori-Tanaka FGM disks.

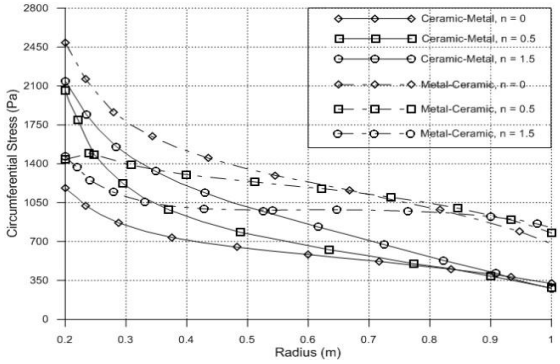

Fig.9
Distribution of circumferential stresses for Mori-Tanaka FGM disks.

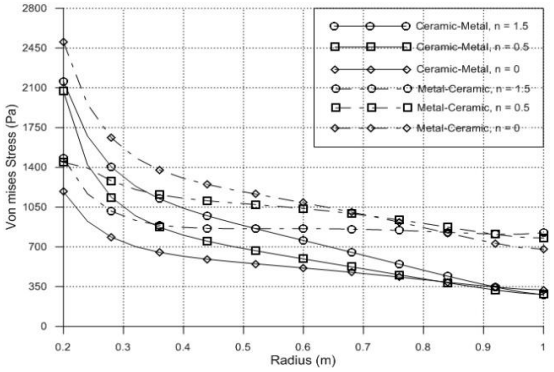

Fig.10
Distribution of von mises stresses for Mori-Tanaka FGM disks.

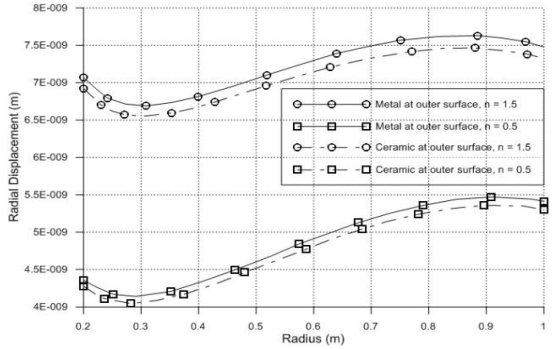

Fig.11
Distribution of radial displacements for Power law FGM disks.

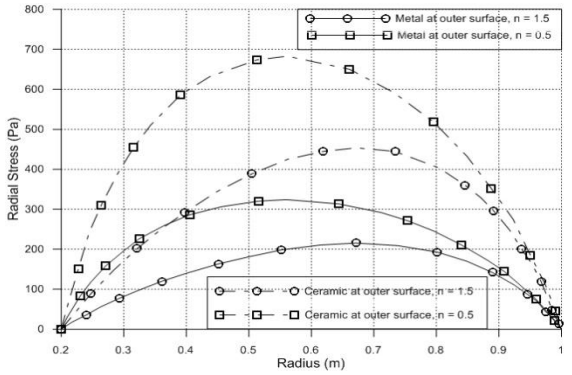

Fig.12
Distribution of radial stresses for Power law FGM disks.

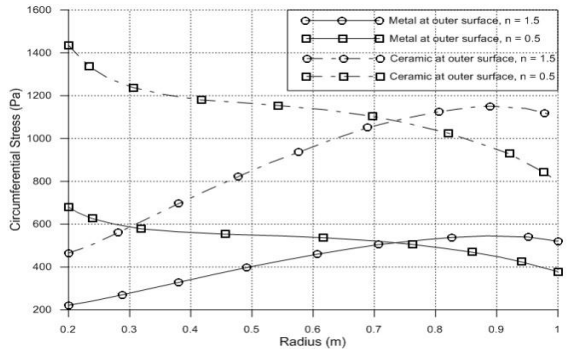

Fig.13
Distribution of circumferential stresses for Power law FGM disks.

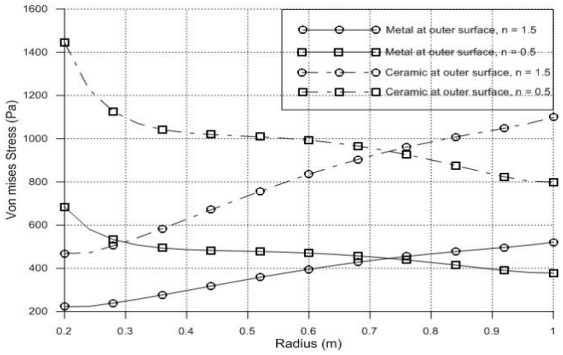

Fig.14
Distribution of von mises stresses for Power law FGM disks.

In power law FGM disk radial deformation increases and stresses decreases with increasing grading parameter n. Increasing n means decreasing (r/b) ratio, which decreases $E(r)$ and hence deformation increases and stresses decreases. FGM disk having metal at outer surface and n = 1.5 has maximum radial deformation and minimum radial, circumferential and von mises stresses while disk having n = 0.5 and ceramic at outer radius has minimum radial deformation and maximum stresses. Further it is also observed that hoop or tangential stress is higher as compared to radial and von mises stress for all cases. Therefore for designing the rotating disks, hoop stress should be taken as limit working stress criteria. By comparing all types of distribution law it is observed that power law FGM disk having metal at outer radius and n = 1.5, has the highest radial deformation and least hoop stress while exponential law (Ceramic-Metal) disk has the lowest radial deformation and disk of full ceramic has the highest hoop stress. Therefore it suggested that FGM modeled by power law having metal at outer radius and n = 1.5 can be most effectively employed for rotating disk.

4 CONCLUSIONS

In present study stress and deformation analysis of FGM rotating disks of variable thickness is done. Material properties are modeled by three different distribution law, which is achieved by element based material grading. The disks are subjected to free-free boundary condition and analysis is carried out for metal-ceramic as well as ceramic-metal both the type of FGM. The governing equations are derived using principle of stationary total potential. It is observed that there is a significant reduction in stresses and deformation behavior of the FGM disks compared to homogeneous disks. Further it is observed that metal-ceramic FGM disk having n = 1.5 and modeled by power law possesses better strength than all other FGMs investigated and therefore is most economical for the purpose of rotating disk.

REFERENCES

[1] Eraslan A.N., 2003, Elastic–plastic deformations of rotating variable thickness annular disks with free, pressurized and radially constrained boundary conditions, *International Journal of Mechanical Sciences* **45**: 643-667.

[2] Bayat M., Saleem M., Sahari B.B., Hamouda A.M.S., Mahdi E., 2009, Mechanical and thermal stresses in a functionally graded rotating disk with variable thickness due to radially symmetry loads, *International Journal of Pressure Vessels and Piping* **86**: 357-372.

[3] Afsar A.M., Go J., 2010, Finite element analysis of thermoelastic field in a rotating FGM circular disk, *Applied Mathematical Modelling* **34**: 3309-3320.

[4] Callioglu H., 2011, Stress analysis in a functionally graded disc under mechanical loads and a steady state temperature distribution, *Sadhana* **36**: 53-64.

[5] Bayat M., Sahari B.B., Saleem M., Dezvareh E., Mohazzab A.H., 2011, Analysis of functionally graded rotating disks with parabolic concave thickness applying an exponential function and the Mori-Tanaka scheme, *IOP Conference Series: Materials Science and Engineering* 17:1-11.

[6] Callioglu H., Sayer M., Demir E., 2011, Stress analysis of functionally graded discs under mechanical and thermal loads, *Indian Journal of Engineering & Material Sciences* **18**: 111-118.

[7] Callioglu H., Bektas N.B., Sayer M., 2011, Stress analysis of functionally graded rotating discs: analytical and numerical solutions, *Acta Mechanica Sinica* **27**: 950-955.

[8] Sharma J.N., Sharma D., Kumar S., 2012, Stress and strain analysis of rotating FGM thermoelastic circular disk by using FEM, *International Journal of Pure and Applied Mathematics* **74**: 339-352.

[9] Ali A., Bayat M., Sahari B.B., Saleem M., Zaroog O.S., 2012, The effect of ceramic in combinations of two sigmoid functionally graded rotating disks with variable thickness, *Scientific Research and Essays* **7**: 2174-2188.

[10] Nejad A., Abedi M., Hassan M., Ghannad M., 2013, Elastic analysis of exponential FGM disks subjected to internal and external pressure, *Central European Journal of Engineering* **3**: 459-465.

[11] Ghorbanpour Arani A. , Loghman A. , Shajari A. R. , Amir S. , 2010, Semi-analytical solution of magneto-thermo-elastic stresses for functionally graded variable thickness rotating disks, *Journal of Mechanical Science and Technology* **24**: 2107-2118.

[12] Ghorbanpour Arani A. , Khoddami Maraghi Z. , Mozdianfard M. R. , Shajari A. R. ,2010, Thermo-piezo-magneto-mechanical stresses analysis of FGPM hollow rotating thin disk, *International Journal of Mechanics and Materials in Design* **6**: 341-349.

[13] Zafarmand H., Hassani B., 2014, Analysis of two-dimensional functionally graded rotating thick disks with variable thickness, *Acta Mechanica* **225**: 453-464.

[14] Rosyid A., Saheb M.E., Yahia F.B., 2014, Stress analysis of nonhomogeneous rotating disc with arbitrarily variable thickness using finite element method, *Research Journal of Applied Sciences, Engineering and Technology* **7**: 3114-3125.

[15] Zafarmand H., Kadkhodayan M., 2015, Nonlinear analysis of functionally graded nanocomposite rotating thick disks with variable thickness reinforced with carbon nanotubes, *Aerospace Science and Technology* **41**: 47-54, 2015.

[16] Seshu P., 2003, *A Text Book of Finite Element Analysis*, PHI Learning Pvt.