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Damping and Frequency Shift in Microscale Modified Couple Stress Thermoelastic Plate Resonators

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ABSTRACT

In this paper, the vibrations of thin plate in modified couple stress thermoelastic medium by using Kirchhoff- Love plate theory has been investigated. The governing equations of motion and heat conduction equation for Lord Shulman (L-S) [1] theory are written with the help of Kirchhoff- Love plate theory. The thermoelastic damping of micro-beam resonators is analyzed by using the normal mode analysis. The solutions for the free vibrations of plates under clamped-simply supported (CS) and clamped-free (CF) conditions are obtained. The analytical expressions for thermoelastic damping of vibration and frequency shift are obtained for couple stress generalized thermoelastic and coupled thermoelastic plates. A computer algorithm has been constructed to obtain the numerical results. The thermoelastic damping and frequency shift with varying values of length and thickness are shown graphically in the absence and presence of couple stress for (i) clamped-simply supported, (ii) clamped-free boundary conditions. Some particular cases are also presented.

Keywords : Modified couple stress theory; Thermoelasticity; Thermoelastic damping; Frequency shift.

1 INTRODUCTION

ON the basis of Cosserat continuum theory, the classical couple stress elasticity has been developed to describe the Cosserat size-dependent effects Mindlin [2,3], Toupin [4]. However, as it involves four material constants for isotropic elastic materials where two of them are separate material length scale parameters, it is a very difficult task to experimentally determine the micro-structural material length scale parameter. Yang et al. [5] presented couple stress based strain gradient theory for elasticity. The concept of representative volume element was introduced and the force and couple applied to a single material particle were defined. They developed a new set of equilibrium relations for a system of material particles to account for the rotations of these material particles and then results were generalized to the couple stress theory of continuum. By the introduction of a higher order equilibrium condition, the arbitrary nature of couples in the classical couple stress theory was resolved without the use of rigid vector attachment condition, as was used in the micropolar theory Eringen [6]. Tsiatas [7] developed a new Kirchhoff plate model in modified couple-stress theory. In this model, the static analysis of isotropic micro-plates with arbitrary shape containing only one internal material length scale parameter which can capture the size

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effect. Sun and Tohmyoh [8] investigated the thermoelastic damping of the vibrations in circular plate. The thermoelastic damping is also studied under the effects of environmental temperature, plate dimensions, clamped and simply supported boundary conditions. The basic equations of coupled thermoelastic theory are constructed by Sun and Saka [9] for out of plane vibration of a circular plate resonators. Thermoelastic damping have many applications in sensing, resonators and communications. Sharma and Sharma [10] studied the problem of thermoelastic circular plate resonators in the context of L-S theory. Ezzat et al. [11], El-Karamany and Ezzat [12], Ezzat and El-Karamany [13], Ezzat et al. [14] studied different problems in thermoviscoelasticity and two-temperature theory. Thermoelastic damping in circular plate with two dimensional heat conduction model with clamped and simply supported boundary conditions was studied by Fang et al. [15]. Shaat et al. [16] developed a new Kirchhoff plate model to study the effects of surface energy and microstructure on the plate rigidity and deflection using a modified couple-stress theory. Simsek et al. [17] studied the problem of forced vibration analysis of a microplate on the basis of modified couple stress theory and Kirchhoff plate theory. Darijani and Shahdadi [18] developed a new non-classical shear deformation plate model including two unknown functions using a modified couple stress theory. Gao and Zhang [19] constructed a non-classical Kirchhoff plate model by applying modified couple stress theory, surface elasticity theory and two –parameter elastic foundation. Reddy et al. [20] discussed the problem of functionally graded circular plates with modified couple stress theory by using finite element method. On the basis of global local theory, a model for a composite laminated Reddy plate of new modified couple-stress theory was developed by Chen and Wang [21]. Various problems related to micro polar bodies, micro stretch materials and dipolar thermoelastic bodies were investigated by Marin [22, 23], Sharma and Marin [24], Marin et al. [25].

In the present work, we studied the vibrations of thin plate in modified couple stress thermoelastic medium by applying Kirchhoff- Love plate theory. The theory of generalized (non-classical) thermoelasticity given by Lord and Shulman [1] has been employed to investigate the problem. The expressions for frequency shift and damping with varying values of length and thickness are shown graphically in the absence and presence of couple stress for clamped-simply supported and clamped-free boundary conditions. Special cases of interest are also deduced from the present problem.

2 BASIC EQUATIONS

Following Yang et al. [5] and Rao [26], the constitutive equation, the equations of motion and the equation of heat conduction for Lord-Shulman theory in a modified couple stress thermoelastic model in the absence of body forces and body couples are:

Constitutive relations

$$t_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} - \frac{1}{2}\varepsilon_{kij}m_{lk,l} - \beta T\delta_{ij}, \tag{1}$$

$$m_{ij} = 2\alpha\chi_{ij}, \tag{2}$$

$$\chi_{ij} = \frac{1}{2}\left(\omega_{i,j} + \omega_{j,i}\right), \tag{3}$$

$$\omega_i = \frac{1}{2}\varepsilon_{ipq}u_{q,p}, \qquad i, j, k = 1, 2, 3. \tag{4}$$

Equation of motion

$$\left(\lambda + \mu + \frac{\alpha}{4}\Delta\right)\nabla(\nabla.\boldsymbol{u}) + \left(\mu - \frac{\alpha}{4}\Delta\right)\nabla^2\boldsymbol{u} - \beta\nabla T = \rho\ddot{\boldsymbol{u}}, \tag{5}$$

Equation of heat conduction

$$K\,\Delta T-\left(\frac{\partial}{\partial t}+\tau_0\,\frac{\partial^2}{\partial t^2}\right)\left(\rho c_e T+T_0\beta\left(\nabla.\boldsymbol{u}\right)\right)=0, \tag{6}$$

where t_{ij} are the components of stress tensor, λ and μ are Lame constants, δ_{ij} is Kronecker's delta, ε_{ij} are the components of strain tensor, ε_{ijk} is alternate tensor, m_{ij} are the components of couple-stress, $\beta=\left(3\lambda+2\mu\right)\alpha_t$. Here α_t are the coefficients of linear thermal expansion respectively, *T* is the temperature change, α is the couple stress parameter, χ_{ij} is symmetric curvature, ω_i is the rotational vector. $\boldsymbol{u}=\left(u_1,u_2,u_3\right)$ is the components of displacement vector, ρ is the density, Δ is the Laplacian operator, ∇ is del operator. *K* is the coefficient of the thermal conductivity, c_e is the specific heat at constant strain, T_0 is the reference temperature assumed to be such that $T\,/T_0<<1$. Here τ_0 is the thermal relaxation time.

3 FORMULATION OF THE PROBLEM

We consider a thin couple stress thermoelastic plate with uniform thickness *h*. The origin of the Cartesian coordinate system $\left(x\,,y\,,z\,\right)$ is taken at the centre of the plate. In equilibrium conditions, the plate is unstrained, unstressed and continues at uniform environmental temperature T_0 everywhere. We define the displacement components $u\left(x\,,y\,,z\,,t\right)$, $v\left(x\,,y\,,z\,,t\right)$, $w\left(x\,,y\,,z\,,t\right)$ and temperature $T\left(x\,,y\,,z\,,t\right)$. According to Kirchhoff's-Love Plate theory, the displacement components are given by

$$u\left(x\,,y\,,z\,,t\right)=-z\,\frac{\partial w}{\partial x},\;v\left(x\,,y\,,z\,,t\right)=-z\,\frac{\partial w}{\partial y},\;w\left(x\,,y\,,z\,,t\right)=w\left(x\,,y\,,t\right), \tag{7}$$

Here *t* denotes the time.

The strain and stress components are taken Rao [26] as:

$$\varepsilon_{xx}=-z\,\frac{\partial^2 w}{\partial x^2}, \tag{8}$$

$$\varepsilon_{yy}=-z\,\frac{\partial^2 w}{\partial y^2}, \tag{9}$$

$$\gamma_{xy}=-2\mu z\,\frac{\partial^2 w}{\partial x\,\partial y}=-\frac{Ez}{\left(1+\nu\right)}\frac{\partial^2 w}{\partial x\,\partial y}. \tag{10}$$

$$t_{xx}=\frac{E}{1-\nu^2}\left[\varepsilon_{xx}+\nu\varepsilon_{yy}-(1+\nu)\alpha_T T\right], \tag{11}$$

$$t_{yy}=\frac{E}{1-\nu^2}\left[\varepsilon_{yy}+\nu\varepsilon_{xx}-(1+\nu)\alpha_T T\right], \tag{12}$$

$$t_{xy}=\mu\gamma_{xy}\,. \tag{13}$$

Following Rao [26], the bending and torsion moments are defined as:

$$M_x = \int_{-h/2}^{h/2} t_{xx} z dz + \int_{-h/2}^{h/2} m_{yx} dz, \tag{14}$$

$$M_y = \int_{-h/2}^{h/2} t_{yy} z dz - \int_{-h/2}^{h/2} m_{xy} dz, \tag{15}$$

$$M_{xy} = \int_{-h/2}^{h/2} t_{xy} z dz + \frac{1}{2} \int_{-h/2}^{h/2} \left(m_{yy} - m_{xx}\right) dz, \tag{16}$$

Using Eqs. (7)-(13) in Eqs. (14)-(16), yield

$$M_x = -D\left[\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} + \alpha_T M_T \left(1+\nu\right)\right] + \frac{\alpha h}{2}\left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x^2}\right), \tag{17}$$

$$M_y = -D\left[\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} + \alpha_T M_T \left(1+\nu\right)\right] - \frac{\alpha h}{2}\left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x^2}\right), \tag{18}$$

$$M_{xy} = -\frac{\partial^2 w}{\partial x \, \partial y}\left[D\left(1-\nu\right) + \alpha h\right]. \tag{19}$$

The equations for shear force resultants are

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}, \quad Q_y = \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x}. \tag{20}$$

The equation of motion (force equilibrium z in the direction) is given by

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} - \rho h \frac{\partial^2 w}{\partial t^2} = q_0(x,t). \tag{21}$$

Substituting Eqs. (17) -(19) in Eqs. (20) and (21) then the equation of motion for micro plate with symmetry about y- axes is taken as:

$$D\frac{\partial^4 w}{\partial x^4} + \frac{E\alpha_T}{\left(1-\nu\right)\beta d}\frac{\partial^2 M_T}{\partial x^2} - \frac{\alpha h}{2}\frac{\partial^4 w}{\partial x^4} + \rho h \frac{\partial^2 w}{\partial t^2} = q_0(x,t), \tag{22}$$

where $D = \frac{Eh^3}{12\left(1-\nu^2\right)}$ is the flexural rigidity of the plate, $E = \frac{\mu\left(3\lambda + 2\mu\right)}{\left(\lambda+\mu\right)}$ is Young's modulus, $\nu = \frac{\lambda}{2\left(\lambda+\mu\right)}$ is the Poisson ratio and $q_0\left(x,t\right)$ represents the load acting along the thickness direction.

The equation of heat conduction for L-S theory is given by

$$K\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial z^2}\right)-\left(\frac{\partial}{\partial t}+\tau_0\frac{\partial^2}{\partial t^2}\right)\left(\rho c_e T - T_0\beta z\frac{\partial^2 w}{\partial x^2}\right)=0. \tag{23}$$

For convenience, we define the non-dimensional parameters as:

$$\left(x',z',u',w'\right)=\frac{\left(x,z,u,w\right)}{L},\ \tau_0'=\frac{\tau_0 v}{L},\ t'=\frac{tv}{L},\ T'=\frac{T}{T_0}, M_T'=\frac{M_T}{d\beta T_0 h^2},\ q_0'=\frac{L^2 q_0}{\left(\lambda+2\mu\right)dh},\ v^2=\frac{E}{\rho}. \tag{24}$$

From Eqs. (22) and (23), with the aid of non-dimensional quantities given by (24) after surpassing the primes, we obtain

$$\frac{\partial^4 w}{\partial x^4}+\frac{E\alpha_T T_0 h^2 L}{\left(1-\nu\right)D}\left(\frac{\partial^2 M_T}{\partial x^2}\right)-\frac{\alpha h}{2D}\left(\frac{\partial^4 w}{\partial x^4}\right)+\frac{\rho h v^2 L^2}{D}\frac{\partial^2 w}{\partial t^2}=\frac{\left(\lambda+2\mu\right)dhLq_0\left(x,t\right)}{D}, \tag{25}$$

$$\frac{K}{Lv\beta}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial z^2}\right)-\left(\frac{\partial}{\partial t}+\tau_0\frac{\partial^2}{\partial t^2}\right)\left(\frac{\rho c_e}{\beta}T - z\frac{\partial^2 w}{\partial x^2}\right)=0. \tag{26}$$

4 PROBLEM SOLUTION

Assuming the time harmonic vibrations as:

$$w\left(x,t\right)=W_1\left(x\right)e^{i\omega t},\ T\left(x,t\right)=T_1\left(x\right)e^{i\omega t},\ q_0\left(x,t\right)=q\,\delta\left(x-a\right)e^{i\omega t}, \tag{27}$$

where ω is the frequency of the beam, q is the magnitude of the load applied and $\delta(\ )$ represents the Direc-delta function. Making use of (27) in the Eqs. (25) and (26), yield

$$\frac{\partial^4 W_1}{\partial x^4}+\frac{E\alpha_T T_0 h^2 L}{\left(1-\nu\right)D}\left(\frac{\partial^2 M_T}{\partial x^2}\right)-\frac{\alpha h}{2D}\left(\frac{\partial^4 W_1}{\partial x^4}\right)-\frac{\rho h v^2 L^2\omega^2}{D}W_1=\frac{\left(\lambda+2\mu\right)dhLq\,\delta\left(x-a\right)}{D}, \tag{28}$$

$$\frac{K}{Lv\beta}\left(\frac{\partial^2 T_1}{\partial x^2}+\frac{\partial^2 T_1}{\partial z^2}\right)=i\,\omega\left(1+\tau_0 i\,\omega\right)\left(\frac{\rho c_e}{\beta}T_1 - z\frac{\partial^2 W_1}{\partial x^2}\right). \tag{29}$$

5 THERMAL FIELD ON THE THICKNESS DIRECTION

We assume that the thermal gradient of the beam is very small as compared to that along its thickness direction $\left(i.e.\left|\frac{\partial T_1}{\partial x}\right|\ll\left|\frac{\partial T_1}{\partial z}\right|\right)$, are taken from Sharma and kaur $[27]$. Therefore, the Eq. (29) take the form

$$\frac{\partial^2 T_1}{\partial z^2}+p^2T_1+\frac{zi\,\omega\beta L v\delta_t^1}{K}\frac{\partial^2 W_1}{\partial x^2}=0, \tag{30}$$

where

$$p^2=-\frac{\rho c_e i\,\omega L v\delta_t^1}{K},\ \delta_t^1=\left(1+\tau_0 i\,\omega\right),\ M_{T_1}=\beta d\int_{\frac{-h}{2}}^{\frac{h}{2}}T_1\left(x,z\right)z\ dz. \tag{31}$$

Let us assumed that there is no heat of flow across the upper and lower surfaces of the beam, then

$$\frac{\partial T_1}{\partial z}=0 \quad \text{at} \quad z=\pm\frac{h}{2}. \tag{32}$$

with the use of this condition, the general solution of equation (30) is written as:

$$T_1\left(x,z\right)=-\frac{i\,\omega L\,\beta v\delta_t^1}{Kp^2\tau_T^1}\left[z-\frac{\sin\left(pz\right)}{p\cos\left(p/2\right)}\right]\frac{\partial^2 W_1}{\partial x^2}. \tag{33}$$

Substituting Eq. (33) in Eq. (31), the thermal moment is given by

$$\frac{d^2 M_T}{dx^2}=-\frac{i\,\omega\beta^2 dh^3 L v\delta_t^1}{12Kp^2}\frac{d^4 W_1}{dx^4}\left[1+f\ \left(p\right)\right], \tag{34}$$

where

$$f\ \left(p\right)=\frac{24}{\left(p_0^* h\right)^3}\left(\frac{ph}{2}-tan\,\frac{ph}{2}\right). \tag{35}$$

Using Eq. (34) in Eq. (28), yield

$$D_\omega\frac{d^4 W_1}{dx^4}-a_1 W_1=0, \tag{36}$$

where

$$\varepsilon_1=-\frac{i\,\omega E\,\alpha_T T_0 h^5 L^2\beta^2 d\,v\delta_t^1}{12Kp^2 D\left(1-v\right)},\ D_\omega=\left[1-\varepsilon_1\left(1+f\ \left(p\right)\right)-\frac{\alpha h}{2D}\right],\ a_1=\frac{\rho h v^2 L^2\omega^2}{D}. \tag{37}$$

6 BOUNDARY CONDITIONS

Let us consider a micro plate whose ends are either clamped-clamped (CC) or simply supported (SS), so we have following the two set of boundary conditions:

Clamped-simply supported (CS) plate

$$W_1 = 0, \frac{dW_1}{dx} = 0, \quad \text{at} \quad x = 0,$$
$$W_1 = 0, \frac{d^2W_1}{dx^2} = 0, \quad \text{at} \quad x = L. \tag{38}$$

Clamped-free (CF) plate

$$W_1 = 0, \quad \frac{dW_1}{dx} = 0, \qquad \text{at} \quad x = 0,$$
$$\frac{d^2W_1}{dx^2} = 0, \quad \frac{d^3W_1}{dx^3} = 0, \quad \text{at} \quad x = L. \tag{39}$$

7 SOLUTION OF THE PROBLEM

The Laplace transformation is applied to solve the Eq. (36) under the boundary conditions (38) and (39). The Laplace transform with respect to x is defined as:

$$W_1(s) = \int_0^\infty e^{-sx} W_1(x) dx, \tag{40}$$

where S is the Laplace transform parameter. Making use of Laplace transform defined by Eq. (40) on Eq. (36), yield

$$\left[s^4\bar{W}_1(s) - s^3\bar{W}_1(0) - s^2\bar{W}_1(0) - sW_1''(0) - W_1'''(0)\right] - \xi^4\bar{W}_1(s) = \frac{a_2 q^*}{D_\omega} e^{-as}, \tag{41}$$

where

$$\xi^4 = \frac{a_1\omega^2}{D_\omega}. \tag{42}$$

Applying boundary conditions (38) and (39) in Eq. (41) at $x = 0$, we obtain
Case (i)

$$\bar{W}_1(s) = \frac{b_1}{2\xi^2}\left(\frac{s}{s^2 - \xi^2} - \frac{s}{s^2 + \xi^2}\right) + \frac{b_2}{2\xi^2}\left(\frac{1}{s^2 - \xi^2} - \frac{1}{s^2 + \xi^2}\right) + \frac{a_2 q^*}{2\xi^2 D_\omega} e^{-as}\left(\frac{1}{s^2 - \xi^2} - \frac{1}{s^2 + \xi^2}\right), \tag{43}$$

Here,

$$b_1 = W_1''(0), \ b_2 = W_1'''(0). \tag{44}$$

Case (ii)

$$\bar{W}_1(s) = \frac{b_3}{2}\left(\frac{1}{s^2 + \xi^2} + \frac{1}{s^2 - \xi^2}\right) + \frac{b_4}{2\xi^2}\left(\frac{1}{s^2 - \xi^2} - \frac{1}{s^2 + \xi^2}\right) + \frac{a_2 q^*}{2\xi^2 D_\omega} e^{-as}\left(\frac{1}{s^2 - \xi^2} - \frac{1}{s^2 + \xi^2}\right), \tag{45}$$

Here,

$$b_3 = W_1(0), \ b_4 = W_1'''(0). \tag{46}$$

Appling inverse Laplace transform on Eqs. (43) and (45), gives
Case (i)

$$W_1(x)=\frac{b_1}{2\xi^2}C(\xi x)+\frac{b_2}{2\xi^3}S(\xi x)+\frac{a_2q^*}{2\xi^3 D_\omega}S_a(\xi x)H_a(x). \tag{47}$$

Case (ii)

$$W_1(x)=\frac{b_3}{2\xi}S^*(\xi x)+\frac{b_4}{2\xi^3}S(\xi x)+\frac{a_2q^*}{2\xi^3 D_\omega}S_a(\xi x)H_a(x), \tag{48}$$

Here

$$C(\xi x)=\cosh(\xi x)-\cos(\xi x),\ S(\xi x)=\sinh(\xi x)-\sin(\xi x), S^*(\xi x)=\sin(\xi x)+\sinh(\xi x).$$

The symbol $S_a(\xi x)$ used in Eqs. (47) and (48) can be written from $S(\xi x)$ by replacing x by $x-a$ and $H_a(x)=H_a(x-a)$ denotes the Heaviside unit function.

8 ANALYSIS OF FREE VIBRATION

In case of free vibration, the load is absent i.e. $q^*=0$, and Eqs. (47) and (48) are written as:
Case (i)

$$W_1(x)=\frac{b_1}{2\xi^2}C(\xi x)+\frac{b_2}{2\xi^3}S(\xi x). \tag{49}$$

Case (ii)

$$W_1(x)=\frac{b_3}{2\xi}S^*(\xi x)+\frac{b_4}{2\xi^3}S(\xi x). \tag{50}$$

Using boundary conditions (38) and (39) at $x=1$ of the plate, we obtain a homogeneous system of two equations in pairs of b_1,b_2,b_3 and b_4 . These system of equations have non-trivial solution if and if the determinant of the coefficients of unknown variables vanish. This requirement leads to the characteristic equations of the plate vibrations in the respective cases:
Case (i)

$$\tan\xi L-\tanh\xi L=0. \tag{51}$$

Case (ii)

$$\cos\xi L\cosh\xi L=-1. \tag{52}$$

The characteristic roots of the Eqs. (51) and (52) are taken as:
Case (i)

$$\xi=(4m+1)\frac{\pi}{4L}, \qquad m\ \varepsilon\ \mathrm{I} \tag{53}$$

Case (ii)

$$\xi = (2m-1)\frac{\pi}{2L}, \qquad m \ \varepsilon \ \mathrm{I} \tag{54}$$

The solutions for the defection and thermal moment are written as:
Case (i)

$$W_1(x,t) = \frac{1}{2}\sum_n \frac{A_n}{\xi_n^2 S(\xi_n)}\left[S(\xi_n)C(\xi_n x) - C(\xi_n)S(\xi_n x)\right]e^{i\omega_n t}, \tag{55}$$

$$M_T(x,z,t) = \frac{-i\,\omega\delta_t^1 \beta L\nu\left(1+f(p)\right)}{24Kp^2}\sum_n \frac{A_n}{S(\xi_n)}\left[S(\xi_n)C^*(\xi_n x) - C(\xi_n)S^*(\xi_n x)\right]e^{i\omega_n t}. \tag{56}$$

Case (ii)

$$W_1(x,t) = \frac{1}{2}\sum_n \frac{B_n}{\xi_n S^*(\xi_n)}\left[S^*(\xi_n)S^*(\xi_n x) - S(\xi_n)S(\xi_n x)\right]e^{i\omega_n t}, \tag{57}$$

$$M_T(x,z,t) = \frac{-i\,\omega\delta_t^1 \beta L\nu\left(1+f(p)\right)}{24Kp^2}\sum_n \frac{B_n\xi_n}{S^*(\xi_n)}\left[S^*(\xi_n)S(\xi_n x) - S(\xi_n)S^*(\xi_n x)\right]e^{i\omega_n t}. \tag{58}$$

9 FREQUENCY SHIFT AND DAMPING

Now the vibration frequency of the plate in the presence of thermoelastic coupling and thermal relaxation time is given by

$$\xi^4 = \frac{a_1\omega^2}{D_\omega}. \tag{59}$$

$$\omega_n = \frac{\xi_n^2}{\sqrt{a_1}}\sqrt{D_\omega} = \omega_0\left[1 - \frac{\alpha h}{2D} - \left(\frac{i\,\omega E\,\alpha_T T_0 h^2 L^2 \beta\nu\delta_t^1}{12Kp^2 D(1-\nu)}\right)\left(1+f(p)\right)\right]^{\frac{1}{2}}, \tag{60}$$

where

$$\omega_0 = \frac{\xi_n^2}{\sqrt{a_1}}. \tag{61}$$

and following Sharma [28], we can replace $f(p)$ with $f(\omega_0)$ and expand Eq. (60) upto first order, we obtain

$$\omega_n = \omega_0\left[1 - \frac{\alpha h}{4D} - \left(\frac{i\,\omega E\,\alpha_T T_0 h^2 L^2 \beta\nu\delta_t^1}{24Kp^2 D(1-\nu)}\right)\left(1+f(p)\right)\right], \tag{62}$$

The thermal gradients in the plane of cross-section along the thickness direction of the plate are much larger than those along its length and hence $\frac{\partial^2 T}{\partial x^2} \cong 0$ so that

$$p^2 + \frac{\rho c_e i\,\omega L \nu \delta_t^1}{K} = 0, \tag{63}$$

This implies that

$$p = p_0 e^{\frac{-i\alpha_1}{2}},\ p_0 = \sqrt{\frac{\rho c_e \omega_n L \nu s^*}{K}},\ s^* = \sqrt{1+\tau_0 \omega_n^2},\ \alpha_1 = \tan^{-1}\left(-\frac{1}{\tau_0 \omega_n}\right). \tag{64}$$

Replacing ω_n with ω_0 in Eqs. (63) and (64), we obtain

$$p = \sqrt{2} p_0^* \left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right), \tag{65}$$

and

$$p_0^* = \sqrt{\frac{\rho c_e \omega_0 L \nu s_0^*}{2K}},\ s_0^* = \sqrt{1+\tau_0 \omega_0^2},\ \theta = \tan^{-1}\left(-\frac{1}{\tau_0 \omega_0}\right). \tag{66}$$

The frequency ω_n is complex in nature and hence we take

$$\omega_n = \omega_R^n + i\,\omega_I^n,\quad \omega_R^n = \mathrm{Re}(\omega_n),\quad \omega_I^n = \mathrm{Im}(\omega_n), \tag{67}$$

and

$$\omega_R^n = \omega_0\left[1 - \frac{\alpha h}{4D} - \left(\frac{i\,\omega_0 E\,\alpha_T T_0 h^2 L^2 \beta \nu \delta_t^1}{24Kp^2 D(1-\nu)}\right)(1+f(R))\right],\quad \omega_I^n = \left(\frac{i\,\omega_0^2 E\,\alpha_T T_0 h^2 L^2 \beta \nu \delta_t^1}{24Kp^2 D(1-\nu)}\right) f(I). \tag{68}$$

where

$$f(R) = \frac{6\cos\theta}{(p_0^* h)^2} - \frac{6\sqrt{2}\cos\frac{3\theta}{2}}{(p_0^* h)^3}\left[\frac{\sin\eta_1 + \tan\frac{3\theta}{2}\sinh\eta_1\eta_2}{\cos\eta_1 + \cosh\eta_1\eta_2}\right], \tag{69}$$

and

$$f(I) = \frac{6\sin\theta}{(p_0^* h)^2} - \frac{6\sqrt{2}\cos\frac{3\theta}{2}}{(p_0^* h)^3}\left[\frac{\tan\frac{3\theta}{2}\sin\eta_1 - \sinh\eta_1\eta_2}{\cos\eta_1 + \cosh\eta_1\eta_2}\right], \tag{70}$$

Here, $\eta_1 = \sqrt{2} p_0^* h \cos\left(\frac{\theta}{2}\right)$, $\eta_2 = \tan\left(\frac{\theta}{2}\right)$ and ω_0 is taken from Eq. (60).

The frequency shift and damping in a thermoelastic plate are taken from Sharma [28]

$$\omega_s = \left|\frac{\omega_R - \omega_0}{\omega_0}\right|. \tag{71}$$

and

$$Q^{-1} = 2\left|\frac{\omega_n^I}{\omega_n^R}\right|. \tag{72}$$

10 PARTICULAR CASES

Coupled thermoelastic (CT) plate

In the absence of thermal relaxation time $(\tau_0 = 0)$, we obtain

$$p = p_0(1-i),\ p_0^* = \sqrt{\frac{\rho c_e \omega_0 L\nu}{2K}},\ s_0^* = 1,\ \theta = \frac{\pi}{2},\ \eta_2 = 1,\ \eta_1 = p_0^* h.$$

Accordingly, Eqs. (69) and (70) became

$$f(R) = \frac{6}{\left(p_0^* h\right)^3}\left(\frac{\sin\eta_1 - \sinh\eta_1}{\cos\eta_1 + \cosh\eta_1}\right), \tag{73}$$

and

$$f(I) = \frac{6}{\left(p_0^* h\right)^2} - \frac{6}{\left(p_0^* h\right)^3}\left(\frac{\sin\eta_1 + \sinh\eta_1}{\cos\eta_1 + \cosh\eta_1}\right). \tag{74}$$

If couple stress parameter $(\alpha = 0)$, Eq. (68) reduces to

$$\omega_R^n = \omega_0\left[1 - \left(\frac{i\,\omega_0 E\,\alpha_T T_0 h^2 L^2 \beta\nu\delta_t^1}{24Kp^2 D(1-\nu)}\right)\left(1 + f(R)\right)\right],\quad \omega_I^n = \left(\frac{i\,\omega_0^2 E\,\alpha_T T_0 h^2 L^2 \beta\nu\delta_t^1}{24Kp^2 D(1-\nu)}\right) f(I), \tag{75}$$

where

$$f(R) = \frac{6\cos\theta}{\left(p_0^* h\right)^2} - \frac{6\sqrt{2}\cos\frac{3\theta}{2}}{\left(p_0^* h\right)^3}\left(\frac{\sin\eta_1 + \tan\frac{3\theta}{2}\sinh\eta_1\eta_2}{\cos\eta_1 + \cosh\eta_1\eta_2}\right),\ f(I) = \frac{6\sin\theta}{\left(p_0^* h\right)^2} - \frac{6\sqrt{2}\cos\frac{3\theta}{2}}{\left(p_0^* h\right)^3}\left(\frac{\tan\frac{3\theta}{2}\sin\eta_1 - \sinh\eta_1\eta_2}{\cos\eta_1 + \cosh\eta_1\eta_2}\right). \tag{76}$$

11 NUMERICAL RESULTS AND DISCUSSION

The mathematical model is prepared with magnesium material for the purpose of numerical computations. The material constants of the problem are taken from Daliwal and Singh [29]. The values of damping factor and frequency shift of first two vibration modes have been computed from Eqs. (71) and (72) in the absence and presence of couple stress. The numerical computations have been carried out with the help of MATLAB software for magnesium material. The computed simulated results have been presented graphically in Figs. 1-8 for clamped-simply supported and clamped-free plate. The different cases of plate with respect to dimensions considered as:

(i) Fixed length $L = 50$ and varying thickness (h).
(ii) Fixed thickness $H = 10$ and varying length (L).

Table 1
Following Daliwal and Singh [29], for taking the physical data for Magnesium material.

Quantity	Magnesium material	Unit
λ	2.696×10^{10}	$Kg\ m^{-1}\ s^{-2}$
μ	1.639×10^{10}	$Kg\ m^{-1}\ s^{-2}$
ρ	1.74×10^{3}	$Kg\ m^{-3}$
T_0	2.696×10^{10}	K
c_e	1.04×10^{3}	$J\ Kg^{-1}\ K^{-1}$
K	1.7×10^{2}	$W\ m^{-1}\ K^{-1}$
α_t	1.78×10^{-5}	K^{-1}
α	15	N
ω	10	Sec^{-1}
τ_0	0.2	Sec

Fig. 1 represents the thermoelastic damping of first two modes in case of clamped-simply supported plate in the absence and presence of couple stress with fixed length and varying thickness. It is observed that the damping factor of vibration modes decreases smoothly with increase in the value of thickness in the assumed range. The value of damping factor of vibration modes is observed to have greater value in the absence of couple stress then that of presence of couple stress.

Fig. 2 depicts the thermoelastic damping of first two modes in case of clamped-free plate in the absence and presence of couple stress with fixed length and varying thickness. It is noticed that the damping factor of vibration modes initially decreases rapidly and then stable in the considered range of thickness. Moreover, it is observed that the damping factor of first mode of vibration has more value than that of second mode of vibration for both cases of couple stress.

Fig. 3 shows the thermoelastic damping of first two modes in case of clamped-simply supported plate in the absence and presence of couple stress with fixed thickness and varying length. It is observed that the damping factor increases monotonically with increase in the value of length. Also, it is clear from the figure that in the presence of couple stress, the damping factor has smaller value in the range $0 \le L \le 0.2$ and remains opposite in the remaining range.

Fig. 4 represents the thermoelastic damping of first two modes in case of clamped-free plate with fixed thickness and varying length in the context of absence and presence of couple stress. It is observed that the damping factor of vibration modes increases slowly for smaller value of length and also increases rapidly for higher values of length in the considered range of thickness. In addition, the value of damping factor of first vibration mode has greater value than that of second vibration mode for both cases of couple stress.

Fig.5 shows the frequency shift of first two modes in case of clamped-simply supported plate with fixed length and varying thickness in the context of absence and presence of couple stress. It is observed that the frequency shift increases smoothly in the range $0 \le h \le 0.05$ and then remains stable in the assumed region of thickness.

Fig. 6 shows the frequency shift of first two modes in case of clamped-free plate with fixed length and varying thickness in the absence and presence of couple stress. It is noticed from the figure that the frequency shift decreases slowly in the range $0 \le h \le 0.07$ and then remains stable in the remaining region of thickness. Moreover, the value of frequency shift for first and second modes is observed to have more value in the presence of couple stress than that for absence of couple stress.

Fig. 7 depicts the frequency shift of first two modes in case of clamped- simply supported plate with fixed thickness and varying length under the effects of couple stress. It is clear from the figure that the frequency shift increases monotonically for first mode of vibration and decreases very slowly in case of second mode of vibration.

Fig. 8 shows the frequency shift of first two modes in case of clamped-free plate with fixed thickness and varying length in the absence and presence of couple stress. It is observed that the value of frequency shift increases very slowly in the range $0 \le h \le 0.07$ and then increases rapidly in the considered region of length for both cases of couple stress and both modes of vibration.

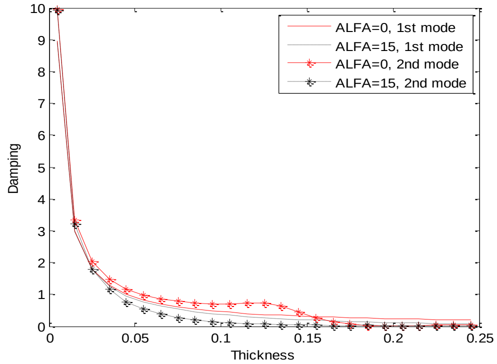

Fig.1
Damping of different modes with thickness (h) in a clamped-simply supported plate of fixed length.

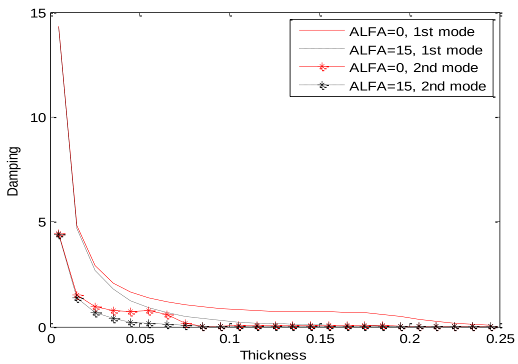

Fig.2
Damping of different modes with thickness (h) in a clamped-free plate of fixed length.

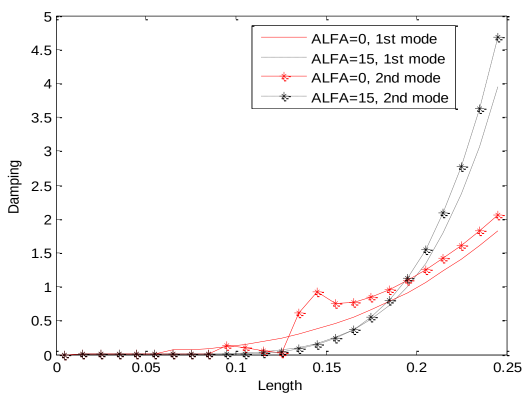

Fig.3
Damping of different modes with length (L) in a clamped-simply supported plate of fixed thickness.

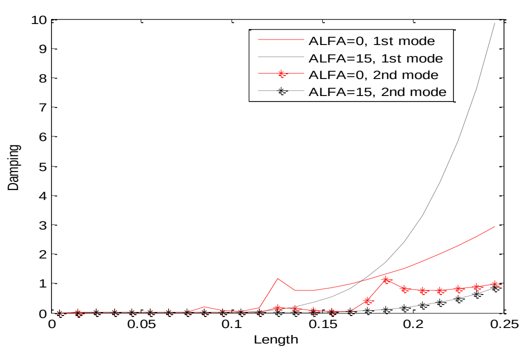

Fig.4
Damping of different modes with length (L) in a clamped-free plate of fixed thickness.

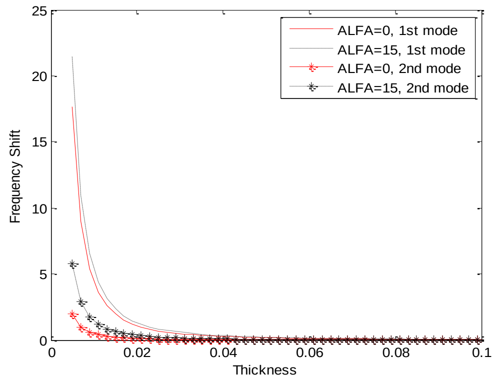

Fig.5
Frequency shift of different modes with thickness (h) in a clamped-simply supported plate of fixed length.

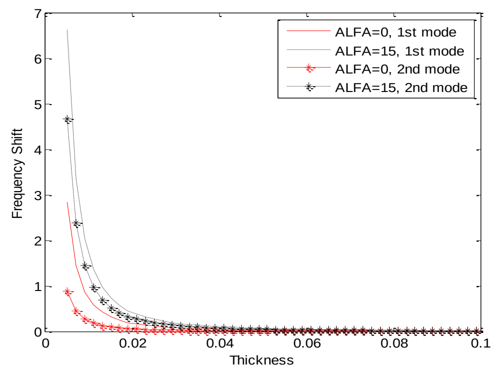

Fig.6
Frequency shift of different modes with thickness (h) in a clamped-free plate of fixed length.

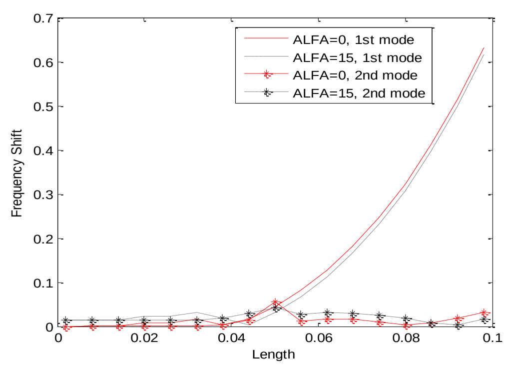

Fig.7
Frequency shift of different modes with length (L) in a clamped-simply supported plate of fixed thickness.

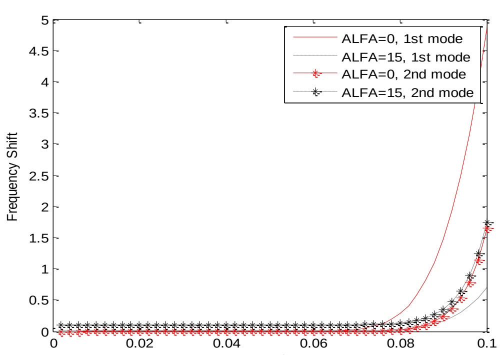

Fig.8
Frequency shift of different modes with length (L) in a clamped-free plate of fixed thickness.

12 CONCLUSIONS

In this work the vibrations of thin plate in modified couple stress thermoelastic medium has been discussed in the context of Kirchhoff-Love plate theory and Lord-Shulman thermoelasticity theory. The mathematical expressions for thermoelastic damping of vibration and frequency shift are obtained for couple stress generalized thermoelastic and coupled thermoelastic plates. Damping factor and frequency shift with varying values of length and thickness are shown graphically to show the effect of couple stress for first two vibration modes with clamped-simply supported and clamped-free boundary conditions. It is concluded from the figures that the damping factor and frequency shift decreases with increasing value of thickness, whereas its value increases with increasing in length for both cases of couple stress and also both modes of vibration. In case of clamped-simply supported plate, the damping factor has larger value in the absence of couple stress and smaller in the presence of couple stress. Also, the damping factor has greater value for first mode of vibration than that of second mode of vibration in case of clamped-free plate. Similarly, frequency shift has more value for first mode of vibration than that of second mode of vibration for both clamped-simply supported plate and clamped-free plate. Moreover, frequency shift has larger value in the presence of couple stress as compared with the absence of couple stress for both clamped-simply supported plate and clamped-free plate. The results of this problem may be useful for Infra-Red (IR) detections and imaging in addition to chemical and biological agent sensing.

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