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Variational Principle, Uniqueness and Reciprocity Theorems in Porous Piezothermoelastic with Mass Diffusion

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ABSTRACT

The basic governing equations in anisotropic elastic material under the effect of porous piezothermoelastic are presented. Biot [1], Lord & Shulman [4] and Sherief et al. [5] theories are used to develop the basic equations for porous piezothermoelastic with mass diffusion material. The variational principle, uniqueness theorem and theorem of reciprocity in this model are established under the assumption of positive definiteness of elastic, porousthermal, chemical potential and electric field.

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Keywords : Piezothermoelastic; Porous; Variational principle; Uniqueness ; Reciprocity.

1 INTRODUCTION

RECENT years have seen an ever-growing interest in the investigation of models of an elastic medium that take into account the influence of various physical fields such as thermal, electric and other fields. An impetus for such studies was the creation of many new materials possessing properties that are not characteristic of usual elastic bodies. Among these materials are piezoelectric bodies that form the core of modern structures and instruments. A stressed state of a piezoelectric body is produced mainly by its deformation, as well as by thermal and electric fields present in the body. Therefore a mathematical model piezothermoelasticity quite adequately reflects the properties of such bodies.

The theory of thermopiezoelectric material was first proposed by Mindlin [6] and derived governing equations of a thermopiezoelectric plate. The physical laws for the thermopiezoelectric material have been explored by Nowacki [7-8]. Chandrasekharaiah [9] used generalised Mindlin's theory of thermopiezoelectricity to account for the finite speed of propagation of thermal disturbances.

Rao and Sunar [10] pointed out the temperature variation in the piezoelectric media. Majhi [11] studied the transient thermal response of the semi-infinite piezoelectric rod subjected to the heat source. Chen et al. [12] derived the general solution for transversely isotropic piezothermoelastic media. In this general solution, all components of the coupled field are expressed by four harmonic functions. Sharma & Kumar [15] discussed the plane harmonic waves in piezothermoelastic material. Sharma et al. [16] studied the propagation characteristics of Rayliegh waves in transversely isotropic piezothermoelastic materials. Sharma & Walia [17] investigated Rayliegh waves in transversely isotropic piezothermoelastic materials. Sharma [18] discussed the propagation of inhomogeneous waves in anisotropic piezothermoelastic media. Alshaikh [19] presented the mathematical model for studying the influence of the initial stresses and relaxation waves in piezothermoelastic half-space.

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The dynamic behaviour of porous medium is important in the field of seismic exploration. The porosity and permeability are the basic and economic parameters for the field of oil production. Reservoir rocks also possess anisotropic behaviour in permeability of pores as a reservoir is a fluid-saturated porous solid medium pervaded by aligned cracks. Porosity is the geometrical property of the solid to hold the fluid.

Biot [13] developed the full dynamic theory for wave propagation in fluid-saturated porous media. Biot used Lagrange's equations to derive a set of coupled differential equations that govern the motions of solid and fluid phases. Biot [13] extended the acoustic propagation theory in the wider context of the mechanics of porous media. Biot [14] developed new features of the extended theory in more detail.

Sharma and Gogna [20] discussed wave propagation in porous solid with a viscoelastic frame filled with a viscous fluid. Sharma [21] used Biot's [2-3] theory to study the phase velocities and attenuations of quasi-waves in a general anisotropic porous solid with anisotropic permeability controlling the flow of viscous fluid in its pores. Sharma [22] studied velocities and polarisation in anisotropic porous solid saturated with non-viscous fluid. It is notified that several authors [21-22-] used Biot's theories to study the porous media. Sharma [23] studied the polarisations of quasi-waves in a general anisotropic porous solid saturated with viscous fluid. Sharma [24] investigated the wave propagation in thermoelastic saturated porous medium. The boundary conditions for porous solids saturated with viscous fluid are described by Sharma [25].

Porous piezoelectric materials are studied due to their applications such as low-frequency hydrophones, underwater sensing and actuation application [26-27]. It is high hydrostatic figures of merit and low sound velocity of these materials due to which the reduction of acoustic impedance and enhancement of coupling with water is possible. Some experimental studies [28-29] have been made for the characterization of properties of porous piezoelectric materials. A number of authors [30-31] developed theoretical models to study the effect of porosity on the elastic, piezoelectric and dielectric properties of porous piezoelectric materials. Vashishth and Gupta [32] described the vibrations of porous piezoelectric ceramic plates.

Diffusion can be defined as the movement of particles from an area of high concentration to an area of lower concentration until equilibrium is reached. It occurs as a result of second law of thermodynamic which states that the entropy or disorder of any system must always increase with time. Diffusion is important in many life processes. Now days, there is a great deal of interest in study of this phenomenon, due to its many application in geophysics and industrial applications. Until recently, thermodiffusion in solids, especially in metals, was considered as a quantity that is independent of body deformation. Practice however indicates that the process of thermodiffusion could have a very considerable influence on the deformation of the body. Thermodiffusion in elastic solid is due to the coupling of temperature, mass diffusion and strain in addition to the exchange of heat and mass with the environment.

Nowacki [33] developed the theory of thermoelastic diffusion by using coupled thermoelastic model. Sherief et al. [5] developed the theory of thermoelastic with mass diffusion by using Lord and Shulman [4] theory of thermoelasticity with one relaxation time. This implies infinite speed of propagation of thermoealstic waves. Kunag [37] discussed the variational principles for generalized thermodiffusion theory in pyroelectricity. Biot [1] discussed thermoelasticity and irreversible thermodynamics. Kumar and Kansal [38] studied the propagation of Lamb waves in transversely isotropic thermoelastic diffusion plate.

A comprehensive work has been done on uniqueness, reciprocity theorems and variational principle by different authors in different media notable among them are, Ezzat & Karamany [39], Li [40], Othman [41] , Aoudi [42], Vashishth and Gupta [43] and Karamany [39] ,Kumar and Tarun [44] and Kumar and Vandana [45].

Inspite of these studies not much work has been done in the elastic body under porous piezothermoelastic with mass diffusion. The main focus of the present investigation is to study the variational problem, reciprocity theorem and uniqueness of solutions in the considered model. These theorems will be helpful for the further investigation of the various problems.

2 BASIC EQUATIONS

Following Biot [1], Lord & Shulman [4], Sherief, Hamza and Saleh [5] and Kuang [37], the governing equations in a homogeneous, anisotropic elastic medium under the effect of porous piezothermoelastic with mass diffusion in the absence of thermal sources, mass diffusive sources and independent of free charge densities are:

Constitutive relations:

$$\sigma_{ij} = c_{ijkl}\,\varepsilon_{kl} - e_{ijk}E_k - \alpha_{ij}\theta + m_{ij}\varepsilon^* - \zeta_{kij}E_k^* - b_{ij}\mu, \tag{1}$$

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$$D_i = \xi_{ij} E_j + e_{ijk} \varepsilon_{jk} + \tau_i \theta + \zeta_i \varepsilon^* + A_{ij} E_j^* + b_i \mu, \tag{2}$$

$$E_i = -\phi_{,i}, \qquad (i, j, k, l = 1,2,3) \tag{3}$$

$$\sigma^* = m_{ij} \varepsilon_{ij} - \zeta_i E_i - \alpha_{ij}^f \theta + R \varepsilon^* - e_i^* E_i^* - b_{ij}^f \mu, \tag{4}$$

$$D_i^* = \zeta_{ijk} \varepsilon_{jk} + \tau_i^f \theta + A_{ij} E_j + \xi_{ij}^* E_j^* + e_i^* \varepsilon^* + b_i^f \mu, \tag{5}$$

$$E_i^* = -\phi^*_{,i}, \qquad (i, j, k, l = 1,2,3) \tag{6}$$

$$-q_{i,i} = T_0 \rho \dot{S}, \tag{7}$$

$$\rho S = \alpha_{ij} \varepsilon_{ij} + \tau_i E_i + r\theta + \alpha_{ij}^f \varepsilon^* + \tau_i^f E_i^* + a\mu, \tag{8}$$

$$-\eta_{i,i} = \dot{C}, \tag{9}$$

$$C = b_{ij} \varepsilon_{ij} + b_i E_i + b\mu + a\theta + b_{ij}^f \varepsilon^* + b_i^f E_i^*, \tag{10}$$

$$\mu = bC - b_{ij} \varepsilon_{ij} - b_i D_i + aS - b_{ij}^f \varepsilon^* - b_i^f E_i^*, \tag{11}$$

Equations of motion:

$$\sigma_{ij,j} + \rho_1 F_i - \rho_{11} \ddot{u}_i - \rho_{12} \ddot{u}_i^* = 0, \tag{12}$$

$$\sigma^*_{,i} + \rho_2 f_i - \rho_{12} \ddot{u}_i - \rho_{22} \ddot{u}_i^* = 0, \tag{13}$$

Equations of heat conduction:

$$-K_{ij} \theta_{,j} = \left(1 + \tau_0 \frac{\partial}{\partial t}\right) q_i \tag{14}$$

Equations of chemical potential:

$$-\alpha_{ij}^* \mu_{,j} = \left(1 + \tau^0 \frac{\partial}{\partial t}\right) \eta_i \tag{15}$$

Gauss equation:

$$D_{i,i} = 0 \tag{16}$$

$$D_i^*{}_{,i} = 0 \qquad (i, j = 1,2,3) \tag{17}$$

In the Eqs. (1)-(17) , $c_{ijkl}\ (= c_{klij} = c_{jikl} = c_{ijlk}), m_{ij}\ (= m_{ij})$ are the tensors of elastic constants. The elastic constant R measures the pressure to be exerted on fluid, ρ_{11} is the density for solids, ρ_{22} is the density for fluids, ρ_{12} is the mass coupling parameter and $\rho_1 = \rho_{11} + \rho_{12}, \rho_2 = \rho_{12} + \rho_{22}$ and $\rho = \rho_1 + \rho_2$, which is the density of combine phase, q_i and η_i are the components of heat and mass diffusion flux vectors q and η respectively, F_i and f_i are

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components of the external forces per unit mass for the solid and fluid phases, u_i and u_i* are the components of displacement vectors u and $u*$, $\sigma_{ij}(=\sigma_{ji})$ and $\sigma*$ are the components of the stress tensors for the solid and fluid phases, $\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i})$ and $\varepsilon*=u_i*_{,i}$ are the components of the strain tensors for the solid and fluid phases, $K_{ij}(=K_{ji}), \alpha_{ij}*(=\alpha_{ji}*)$ are respectively, the components of thermal conductivity and diffusion tensors, S,μ are entropy and chemical potential per unit mass respectively, E_i, E_i* are the electric field intensities, D_i, D_i* are the electric displacements, $\phi,\phi*$ are the electric potentials for the solid and fluid phases, θ is the absolute temperature of the medium, T_0 is the reference temperature of the body, C is the mass concentration of the diffusion material in the elastic body, *a, b, r* are respectively, coefficients describing the measure of thermal and mass diffusion effects, $\alpha_{ij}, \alpha_{ij}^f, \tau_i^f, \tau_i, e_{ijk}, \zeta_{ijk}, A_{ij}, \xi_{ij}, e_i*\zeta_i, \xi_{ij}*, b_{ij}, b_{ij}^f, b_i^f, b_i$ are tensors of porous piezothermal and diffusion moduli respectively, τ_0 is the thermal relaxation time, which will ensure that the heat conduction equation will predict finite speeds of heat propagation speeds and τ^0 is the diffusion relaxation time, which will ensure that the equation satisfied by the concentration will also predict finite speeds of propagation of matter from one medium to other.

3 VARIATIONAL PRINCIPLE

The principle of virtual work with variation of displacements for the elastic deformable body can be written as:

$$\int_V(\rho_1F_i-\rho_{11}\ddot{u}_i-\rho_{12}\ddot{u}_i*)\delta u_i\,dV+\int_V(\rho_2f_i-\rho_{12}\ddot{u}_i-\rho_{22}\ddot{u}_i*)\delta u_i*dV+\int_A(h_i\delta u_i+h_i*\delta u_i*)dA$$
$$+\int_A(c_0\delta\phi+c_0*\delta\phi*)dA=\int_A(\sigma_{ij}n_j\delta u_i+\sigma*n_i\delta u_i*)dA+\int_A(D_in_i\delta\phi+D_i*n_i\delta\phi*)dA, \tag{18}$$

where $h_i=\sigma_{ij}n_j, h_i*=\sigma*n_i$, $c_0=D_in_i$ and $c_0*=D_i*n_i$. On the left hand side, we have the virtual work of body forces F_i, f_i, internal forces $\rho_1\ddot{u}_i, \rho_2\ddot{u}_i*$, surface forces h_i, h_i*, whereas on the right hand side, we have the virtual work of internal forces. We denote by n_j or n_i the outward normal of ∂V. c_0, c_0* are the electric charge densities and $\phi,\phi*$ are the electric potentials for the solid and fluid phases.

Using the symmetry of the stress tensors, divergence theorem and the definition of the strain tensors, the Eq. (18) can be written in the alternative form as:

$$\int_V(\rho_1F_i-\rho_{11}\ddot{u}_i-\rho_{12}\ddot{u}_i*)\delta u_i\,dV+\int_V(\rho_2f_i-\rho_{12}\ddot{u}_i-\rho_{22}\ddot{u}_i*)\delta u_i*dV+\int_A(h_i\delta u_i+h_i*\delta u_i*)dA$$
$$+\int_A(c_0\delta\phi+c_0*\delta\phi*)dA=\int_V(\sigma_{ij}\delta u_{i,j}+\sigma*\delta u_i*_{,i})dV+\int_V(D_i\delta\phi_{,i}+D_i*\delta\phi*_{,i})dV \tag{19}$$

Substituting the value of σ_{ij} and $\sigma*$ from the relation (1) and (4) in the Eq. (19) and using Eq. (3) and (6), we obtain

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$$\begin{aligned}&\int_V(\rho_1 F_i-\rho_{11}\ddot{u}_i-\rho_{12}\ddot{u}_i{}^*)\delta u_i\,dV+\int_V(\rho_2 f_i-\rho_{12}\ddot{u}_i-\rho_{22}\ddot{u}_i{}^*)\delta u_i{}^*dV+\int_A(h_i\delta u_i+h_i{}^*\delta u_i{}^*)dA\\&+\int_A(c_0\delta\phi+c_0{}^*\delta\phi^*)dA=\int_V(c_{ijkl}\,\varepsilon_{kl}-e_{ijk}E_k-\alpha_{ij}\theta+m_{ij}\varepsilon^*-\zeta_{kij}E_k{}^*-b_{ij}\mu)\delta\varepsilon_{ij}dV-\int_V(D_i\delta E_i+D_i{}^*\delta E_i{}^*)dV\\&+\int_V(m_{ij}\,\varepsilon_{ij}-\zeta_i E_i-\alpha_{ij}{}^f\theta+R\varepsilon^*-e_i{}^*E_i{}^*-b_{ij}{}^f\mu)\delta\varepsilon^*dV=\delta W-\int_V e_{ijk}E_k\delta\varepsilon_{ij}dV-\int_V\alpha_{ij}\theta\delta\varepsilon_{ij}dV-\int_V\zeta_{kij}E_k{}^*\delta\varepsilon_{ij}\,dV\\&-\int_V b_{ij}\mu\delta\varepsilon_{ij}\,dV-\int_V e_i{}^*E_i{}^*\delta\varepsilon^*dV-\int_V\zeta_i E_i\delta\varepsilon^*dV-\int_V\alpha_{ij}{}^f\theta\delta\varepsilon^*dV-\int_V D_i\delta E_i dV-\int_V D_i{}^*\delta E_i{}^*dV-\int_V b_{ij}{}^f\mu\delta\varepsilon^*dV\end{aligned}\tag{20}$$

where

$$W=\frac{1}{2}\int_V(c_{ijkl}\,\varepsilon_{kl}\varepsilon_{ij}+R\varepsilon^*\varepsilon^*+2m_{ij}\varepsilon^*\varepsilon_{ij})dV\,,\ \delta u_{i,j}=\delta\varepsilon_{ij}\,,\ \delta u_i{}^*{}_{,i}=\delta\varepsilon^*,\ \delta\phi_{,j}=-\delta E_j \text{ and } \delta\phi^*{}_{,j}=-\delta E_j{}^*$$

The Eq. (20) would be complete for the uncoupled problem of porous piezothermoelastic diffusion where the temperature θ, the electric potential ϕ,ϕ^* and the concentration C are known functions. In the case, when we take into account the coupling of the deformation field with the temperature and concentration, there arises the necessity of considering two additional relations characterizing the phenomenon of the thermal conductivityand mass diffusion.

Following Biot [1] we define a vector J connected with the entropy through the relation

$$\rho S=-J_{i,i}\,.\tag{21}$$

Eqs. (7), (8), (14) and (21) combined together yield

$$T_0 L_{ij}\left(\frac{d}{dt}+\tau_0\frac{d^2}{dt^2}\right)J_i+\theta_{,j}=0,\tag{22}$$

$$-J_{i,i}=\alpha_{ij}\varepsilon_{ij}+\tau_i E_i+r\theta+\alpha_{ij}{}^f\varepsilon^*+\tau_i{}^f E_i{}^*+a\mu.\tag{23}$$

where L_{ij} the resistivity matrix, is the inverse of the thermal conductivity K_{ij}.

Multiplying both sides of the Eq. (22) by δJ_j and integrating over the region of the body, gives

$$\int_V\left[\theta_{,j}+T_0 L_{ij}\left(\frac{dJ_i}{dt}+\tau_0\frac{d^2J_i}{dt^2}\right)\right]\delta J_j\,dV=0\tag{24}$$

Now

$$\int_V\theta_{,j}\delta J_j dV=\int_V\left(\theta\delta J_j\right)_{,j}dV-\int_V\theta\delta J_{j,j}dV\tag{25}$$

Applying the divergence theorem defined by,

$$\int_V\left(\theta\delta J_j\right)_{,j}dV=\int_A\left(\theta\delta J_j\right)n_j dA.\tag{26}$$

In the Eq. (25), yields

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$$\int_V \theta_{,j}\,\delta J_j\,dV = \int_A \left(\theta\delta J_j\right) n_j\,dA - \int_V \theta\delta J_{j,j}\,dV\,. \tag{27}$$

Substituting Eq. (27) in the Eq. (24), we obtain

$$\int_A \left(\theta\delta J_j\right) n_j\,dA - \int_V \theta\delta J_{j,j}\,dV + T_0 \int_V L_{ij}\left(\frac{dJ_i}{dt} + \tau_0 \frac{d^2 J_i}{dt^2}\right)\delta J_j\,dV = 0. \tag{28}$$

Making use of Eq. (23) in the Eq. (28), yield the second variational equation

$$\int_A \theta\delta J_j n_j\,dA + \int_V \alpha_{ij}\theta\delta\varepsilon_{ij}\,dV + \int_V \theta\tau_j\,\delta E_j\,dV + \int_V \alpha_{ij}^{\,f}\,\delta\varepsilon^{*}\theta\,dV + \int_V \tau_i^{\,f}\,\delta E_i^{\,*}\theta\,dV + \int_V a\theta\delta\mu\,dV + \delta(M+H) = 0, \tag{29}$$

where the function of thermal potential M is defined by

$$M = \frac{r}{2}\int_V \theta^2\,dV\,,\quad \delta M = r\int_V \theta\delta\theta\,dV\,, \tag{30}$$

and the function of thermal dissipation H is defined by

$$H = \frac{T_0}{2}\int_V L_{ij}\left(\frac{dJ_i}{dt} + \tau_0\frac{d^2 J_i}{dt^2}\right) J_j\,dV\,,\quad \delta H = T_0\int_V L_{ij}\left(\frac{dJ_i}{dt} + \tau_0\frac{d^2 J_i}{dt^2}\right)\delta J_j\,dV\,. \tag{31}$$

In order to obtain the third of the variational equations, we now introduce the vector function N defined as follows

$$C = -N_{i,i}\,. \tag{32}$$

Eqs. (9),(10), (15) and (32), yield

$$a_{ij}\left(\frac{d}{dt} + \tau^0 \frac{d^2}{dt^2}\right) N_i + \mu_{,j} = 0\,, \tag{33}$$

$$-N_{i,i} = b_{ij}\varepsilon_{ij} + b_i E_i + b\mu + a\theta + b_{ij}^{\,f}\,\varepsilon^{*} + b_i^{\,f}\,E_i^{\,*}, \tag{34}$$

where a_{ij} is the inverse of the diffusion tensor $\alpha_{ij}^{\,*}$.

Multiplying Eq. (33) by δN_j and integrating over the region of the body, gives

$$\int_V \left[a_{ij}\left(\frac{dN_i}{dt} + \tau^0\frac{d^2 N_i}{dt^2}\right) + \mu_{,j}\right]\delta N_j\,dV = 0, \tag{35}$$

Consider

$$\int_V \mu_{,j}\,\delta N_j\,dV = \int_V \left(\mu\delta N_j\right)_{,j} dV - \int_V \mu\delta N_{j,j}\,dV \tag{36}$$

We know that

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$$\int_V \left(\mu\delta N_j\right)_{,j} dV = \int_A \left(\mu\delta N_j\right) n_j dA. \tag{37}$$

Substituting the value of $\int_V \left(\mu\delta N_j\right)_{,j} dV$ from Eq. (37) in the Eq. (36),we obtain

$$\int_V \mu_{,j}\delta N_j dV = \int_A \left(\mu\delta N_j\right) n_j dA - \int_V \mu\delta N_{j,j} dV. \tag{38}$$

Making use of Eq. (38) in the Eq. (35) yields

$$\int_A \left(\mu\delta N_j\right) n_j dA - \int_V \mu\delta N_{j,j} dV + \int_V a_{ij}\left(\frac{dN_j}{dt} + \tau^0 \frac{d^2 N_j}{dt^2}\right)\delta N_j dV = 0 \tag{39}$$

Substituting the value of $N_{i,i}$ from Eq. (34) in the Eq. (39), we obtain the third variational equation

$$\int_A \left(\mu\delta N_j\right) n_j dA + \int_V b_{ij}\,\mu\delta\varepsilon_{ij} dV + \int_V \mu b_j \delta E_j dV + \int_V \mu b_{ij}^f\, \delta\varepsilon^* + a\int_V \mu\delta\theta dV + \int_V \mu b_i^f\, \delta E_i{}^* dV + \delta(F+G) = 0, \tag{40}$$

where the function of diffusion potential F is defined by

$$F = \frac{b}{2}\int_V \mu^2 dV, \quad \delta F = b\int_V \mu\delta\mu dV, \tag{41}$$

and the function of diffusion dissipation G is defined by

$$G = \frac{1}{2}\int_V a_{ij}\left(\frac{dN_i}{dt} + \tau^0 \frac{d^2 N_i}{dt^2}\right) N_j dV, \quad \delta G = \int_V a_{ij}\left(\frac{dN_i}{dt} + \tau^0 \frac{d^2 N_i}{dt^2}\right)\delta N_j dV. \tag{42}$$

Eliminating integrals $\int_V \alpha_{ij}^f\, \delta\varepsilon^*\theta dV$, $\int_V \alpha_{ij}\,\theta\delta\varepsilon_{ij} dV$, $\int_V b_{ij}^f\, \mu\delta\varepsilon^* dV$ and $\int_V b_{ij}\,\mu\delta\varepsilon_{ij} dV$ from Eqs. (20), (29) and (40) with the aid of Eqs. (3) and (6), we obtain the variational principle in the following form

$$\begin{aligned}
\delta\left(W + M + H + F + G + \int_V a\mu\theta dV\right) &= \int_V (\rho_1 F_i - \rho_{11}\,\ddot{u}_i - \rho_{12}\,\ddot{u}_i{}^*)\delta u_i\, dV + \int_V (\rho_2 f_i - \rho_{12}\,\ddot{u}_i - \rho_{22}\,\ddot{u}_i{}^*)\delta u_i{}^* dV \\
&+ \int_A (h_i\,\delta u_i + h_i{}^*\,\delta u_i{}^*) dA + \int_A (c_0\delta\phi + c_0{}^*\delta\phi^*) dA - \int_A \theta\delta J_j n_j dA + \int_V e_{ijk}\, E_k\, \delta\varepsilon_{ij} dV - \int_V \theta\tau_j\,\delta E_j dV \\
&+ \int_V \zeta_{kij} E_k{}^*\delta\varepsilon_{ij}\, dV + \int_V D_i{}^*\delta E_i{}^* dV + \int_V D_i\,\delta E_i dV - \int_A \mu\delta N_j n_j dA - \int_V \mu b_i^f\, \delta E_i{}^* dV \\
&+ \int_V (\zeta_i E_i + e_i{}^* E_i{}^*)\delta\varepsilon^* dV - \int_V \tau_i^f\, \delta E_i{}^*\theta dV - \int_V \mu b_j\,\delta E_j dV.
\end{aligned} \tag{43}$$

On the right-hand side of Eq. (43), we find all the causes, the mass forces, inertial forces, the surface forces, the heating and the electric potential and the chemical potential on the surface A bounding the body.

Particular Case:

1. In the absence of diffusion effect, our results for the variational principle are similar as proved by Kuang [37].

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2. In absence of porous piezoelectric effect, our results for variational principle are similar as obtained by Kumar and Kansal [44].

4 UNIQUENESS THEOREM

We assume that the virtual displacements $\delta u_i, \delta u_i*$, the virtual increment of the temperature $\delta\theta$, etc. correspond to the increments occurring in the body. Then

$$\delta u_i = \frac{\partial u_i}{\partial t}dt = \dot{u}_i\,dt \quad, \delta u_i * = \frac{\partial u_i *}{\partial t}dt = \dot{u}_i *dt,\ \delta\theta = \frac{\partial\theta}{\partial t}dt = \dot{\theta}dt \text{, etc.} \tag{44}$$

and Eq. (43) reduces to the following relation

$$\begin{aligned}\frac{d}{dt}\left(W + M + H + F + G + \int_V a\mu\theta dV\right) &= \int_V (\rho_1 F_i - \rho_{11}\ddot{u}_i - \rho_{12}\ddot{u}_i*)\dot{u}_i\,dV + \int_V (\rho_2 f_i - \rho_{12}\ddot{u}_i - \rho_{22}\ddot{u}_i*)\dot{u}_i*dV \\ &+ \int_A (h_i\delta\dot{u}_i + h_i*\delta\dot{u}_i*)dA + \int_A (c_0\dot{\phi} + c_0*\dot{\phi}*)dA - \int_A \theta\dot{J}_j n_j dA + \int_V e_{ijk}E_k\dot{\varepsilon}_{ij}dV - \int_V \theta\tau_j\dot{E}_j dV \\ &+ \int_V \zeta_{kij}E_k*\dot{\varepsilon}_{ij}\,dV + \int_V D_i*\dot{E}_i*dV + \int_V D_i\dot{E}_i dV - \int_A \mu\dot{N}_j n_j dA - \int_V \mu b_i^f\dot{E}_i*dV \\ &+ \int_V (\zeta_i\dot{E}_i + e_i*\dot{E}_i*)\dot{\varepsilon}*dV - \int_V \tau_i^f\dot{E}_i*\theta dV - \int_V \mu b_j\dot{E}_j dV.\end{aligned} \tag{45}$$

Now

$$\int_V (\rho_{11}\ddot{u}_i\dot{u}_i + \rho_{12}\ddot{u}_i*\dot{u}_i + \rho_{12}\ddot{u}_i\dot{u}_i* + \rho_{22}\ddot{u}_i*\dot{u}_i*)dV = \frac{\partial K}{\partial t}, \tag{46}$$

where $K = \frac{1}{2}\int_V (\rho_{11}\dot{u}_i\dot{u}_i + 2\rho_{12}\dot{u}_i*\dot{u}_i + \rho_{22}\dot{u}_i*\dot{u}_i*)dV$, is the kinetic energy of the body enclosed by the volume V. We also have

$$M + F + a\int_V \mu\theta dV = \frac{1}{2}\int_V (r\theta^2 + b\mu^2 + 2a\theta\mu)dV. \tag{47}$$

Using Eqs. (46) and (47) in the Eq. (45), we obtain

$$\begin{aligned}\frac{d}{dt}\left(W + H + K + G + \frac{1}{2}\int_V (r\theta^2 + b\mu^2 + 2a\theta\mu)dV\right) &= \int_V \rho_1 F_i\,\dot{u}_i\,dV + \int_V \rho_2 f_i\,\dot{u}_i*dV \\ &+ \int_A (h_i\delta\dot{u}_i + h_i*\delta\dot{u}_i*)dA + \int_A (c_0\dot{\phi} + c_0*\dot{\phi}*)dA - \int_A \theta\dot{J}_j n_j dA + \int_V e_{ijk}E_k\dot{\varepsilon}_{ij}dV - \int_V \theta\tau_j\dot{E}_j dV \\ &+ \int_V \zeta_{kij}E_k*\dot{\varepsilon}_{ij}\,dV + \int_V D_i*\dot{E}_i*dV + \int_V D_i\dot{E}_i dV - \int_A \mu\dot{N}_j n_j dA - \int_V \mu b_i^f\dot{E}_i*dV \\ &+ \int_V (\zeta_i\dot{E}_i + e_i*\dot{E}_i*)\dot{\varepsilon}*dV - \int_V \tau_i^f\dot{E}_i*\theta dV - \int_V \mu b_j\dot{E}_j dV.\end{aligned} \tag{48}$$

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The above equation is the basis for the proof of the following uniqueness theorem.

Theorem: There is only one solution of the Eqs. (12)-(17), subject to the boundary conditions on the surface A

$$h_i = \sigma_{ij} n_j = h_{i1}\,,\ \theta = \theta_1\,, c_0 = D_i n_i = c_{01},\, h_i{}^* = \sigma^* n_i = h_{i1}{}^*,\, c_0{}^* = D_i{}^* n_i = c_{01}{}^*,\, \mu = \mu_1$$

and the initial conditions on the surface at $t = 0$

$$\mathrm{u}_i = \mathrm{u}_i{}^0\,,\ \dot{\mathrm{u}}_i = \dot{\mathrm{u}}_i{}^0\,, \mathrm{u}_i{}^* = \mathrm{u}_i{}^{*0}\,,\ \dot{\mathrm{u}}_i{}^* = \dot{\mathrm{u}}_i{}^{*0}, \theta = \theta^0\,, \dot{\theta} = \dot{\theta}^0\,, \phi = \phi^0,\ \dot{\phi} = \dot{\phi}^0, \phi^* = \phi^{*0},\ \dot{\phi}^* = \dot{\phi}^{*0}, \mu = \mu^0\,, \mu = \dot{\mu}^0$$

where $h_{i1}, h_{i1}{}^*, \theta_1, c_{01}, c_{01}{}^*, \mathrm{u}_i{}^0, \dot{\mathrm{u}}_i{}^0, \mathrm{u}_i{}^{*0}, \dot{\mathrm{u}}_i{}^{*0}, \theta^0, \dot{\theta}^0, \phi^0, \dot{\phi}^0, \phi^{*0}, \dot{\phi}^{*0}, \mu^0, \dot{\mu}^0$ are known functions. We assume that the material parameters satisfy the inequalities

$$T_0 > 0,\ \tau_0 > 0,\ \tau^0 > 0, \rho_{11} > 0, \rho_{12} > 0, \rho_{22} > 0, \tag{49}$$

$c_{ijkl}\,, L_{ij}\,, R\,, a_{ij}$ and m_{ij} are positive definite.

Proof: Let $\mathrm{u}_i{}^{(1)},\ \theta^{(1)}, \mathrm{u}_i{}^{*(1)}, \phi^{(1)}, \phi^{*(1)}, \mu^{(1)}, \ldots$ and $\mathrm{u}_i{}^{(2)},\ \theta^{(2)}, \mathrm{u}_i{}^{*(2)}, \phi^{(2)}, \phi^{*(2)}, \mu^{(2)}, \ldots$ be two solutions sets of Eqs. (1)-(13). Let us take

$$\mathrm{u}_i = \mathrm{u}_i{}^{(1)} - \mathrm{u}_i{}^{(2)},\ \mathrm{u}_i{}^* = \mathrm{u}_i{}^{*(1)} - \mathrm{u}_i{}^{*(2)}, \theta = \theta^{(1)} - \theta^{(2)},\ \phi^* = \phi^{*(1)} - \phi^{*(2)},\ \mu = \mu^{(1)} - \mu^{(2)} \text{ and } \phi = \phi^{(1)} - \phi^{(2)}. \tag{50}$$

The functions $\mathrm{u}_i\,, \mathrm{u}_i{}^*, \theta, \phi$ and ϕ^* satisfy the governing equations with zero body forces and homogeneous initial and boundary conditions. Thus, these functions satisfy an equation similar to the Eq. (48) with zero right hand side, that is,

$$\frac{d}{dt}\left(W + H + K + G + \frac{1}{2}\int_V (r\theta^2 + b\mu^2 + 2a\theta\mu)dV \right) = 0. \tag{51}$$

Since, we have $L_{ij} = L_{ji}$ and $a_{ij} = a_{ji}$ .Therefore, from Eqs. (31) and (42), we obtain

$$\frac{dH}{dt} = T_0 \int_V L_{ij} \dot{J}_i \dot{J}_j dV + \frac{d}{dt}\left[\frac{T_0 \tau_0}{2} \int_V L_{ij} \dot{J}_i \dot{J}_j dV \right], \tag{52}$$

and

$$\frac{dG}{dt} = \int_V a_{ij} \dot{N}_i \dot{N}_j dV + \frac{d}{dt}\left[\frac{\tau^0}{2} \int_V a_{ij} \dot{N}_i \dot{N}_j dV \right]. \tag{53}$$

Substitution of Eqs. (52) and (53) in the Eq. (51), yield

$$\frac{d}{dt}\left(W + K + \frac{1}{2}\int_V (r\theta^2 + b\mu^2 + 2a\theta\mu)dV + \frac{T_0\tau_0}{2}\int_V L_{ij}\dot{J}_i\dot{J}_j dV + \frac{\tau^0}{2}\int_V a_{ij}\dot{N}_i\dot{N}_j dV \right) + T_0\int_V L_{ij}\dot{J}_i\dot{J}_j dV + \int_V a_{ij}\dot{N}_i\dot{N}_j dV = 0. \tag{54}$$

Using the inequalities (49) in Eq. (54), we obtain

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$$\frac{d}{dt}\left(W+K+\frac{1}{2}\int_V(r\theta^2+b\mu^2+2a\theta\mu)dV+\frac{T_0\tau_0}{2}\int_V L_{ij}\dot{J}_i\dot{J}_j dV+\frac{\tau^0}{2}\int_V a_{ij}\dot{N}_i\dot{N}_j dV\right)\leq 0. \tag{55}$$

We thus see that the expression

$$W+K+\frac{1}{2}\int_V(r\theta^2+b\mu^2+2a\theta\mu)dV+\frac{T_0\tau_0}{2}\int_V L_{ij}\dot{J}_i\dot{J}_j dV+\frac{\tau^0}{2}\int_V a_{ij}\dot{N}_i\dot{N}_j dV\ , \tag{56}$$

Is a decreasing function of time. We also note that the expression $\int_V(r\theta^2+b\mu^2+2a\theta\mu)dV$ occurring in the expression (56)is always positive, since by the laws of thermodynamics Nowacki [33-36].

$$0<a^2<r\theta. \tag{57}$$

Thus, the expression (56) vanishes for $t=0$, due to the homogeneous initial conditions, and it must be always non-positive for $t>0$. Using inequalities (49) and (55), it follows immediately that the expression (56) must be identically zero for $t>0$. We thus have $\phi=\phi^*=\mathrm{u}_i=\mathrm{u}_i{}^*=\theta=\mu=\varepsilon_{ij}=\varepsilon^*=\sigma_{ij}=\sigma^*=0.$

This proves the uniqueness of the solution to the complete system of field equations subjected to the electric potential-displacement-temperature-chemical potential initial and boundary conditions.

Particular Case: In absence of porous piezoelectric effect, the results obtained are similar as derived by Kumar and Kansal [44].

5 RECIPROCITY THEOREM

We shall consider a homogeneous anisotropic elastic body under the effect of porous piezothermoelastic occupying the region V and bounded by the surface A. We assume that the stresses σ_{ij},σ^* and the strains $\varepsilon_{ij},\varepsilon^*$ are continuous together with their first order derivatives whereas the displacements $\mathrm{u}_i,\mathrm{u}_i{}^*$, temperature θ, concentration C, chemical potential μ and the electrical potentials ϕ,ϕ^* are continuous and have continuous derivatives up to second order, for $x\in V+A$, $t>0$. The components of surface tractions, the normal component of the heat flux, the normal component of the chemical flux and electric displacements at regular points of ∂V, are given by respectively.

$$h_i=\sigma_{ij}n_j\ ,\ h_i{}^*=\sigma^*n_i\ ,\ q=q_i\cdot n_j\ ,\ p=\eta_i\cdot n_j\ ,\ c_0{}^*=D_i{}^*n_i\ ,\ c_0=D_in_i\ , \tag{58}$$

To the system of field equations, we must adjoin boundary conditions and initial conditions. We consider the following boundary conditions:

$$\begin{aligned}&\mathrm{u}_i(x,t)=\mathrm{U}_i(x,t),\ \theta(x,t)=\eta(x,t),\ \phi(x,t)=e_0(x,t)\ ,\ \mu(x,t)=\varsigma(x,t)\\&\phi^*(x,t)=e_0{}^*(x,t),\mathrm{u}_i{}^*(x,t)=\mathrm{U}_i{}^*(x,t)\end{aligned} \tag{59}$$

For all $x\in A$, $t>0$, and the homogeneous initial conditions

$$\mathrm{u}_i(x,0)=\dot{\mathrm{u}}_i(x,0)=0\ \ ,\theta(x,0)=\dot{\theta}(x,0)=0\ \ ,\mathrm{u}_i{}^*(x,0)=\dot{\mathrm{u}}_i{}^*(x,0)=0, \tag{60}$$

and $\phi(x,0)=\dot{\phi}(x,0)=0$, $\phi^*(x,0)=\dot{\phi}^*(x,0)=0$, $\mu(x,0)=\dot{\mu}(x,0)=0$, for all $x\in V,t=0$.

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We derive the dynamic reciprocity relationship for a generalised porous piezothermoelastic diffusion bounded body V, which satisfies Eqs (1)-(17), the boundary conditions (59) and the homogeneous initial conditions (60), and are subjected to the action of body forces $F_i(x,t), f_i(x,t)$, surface tractions $h_i(x,t), h_i{}^*(x,t)$, the heat flux $q(x,t)$, the chemical flux $p(x,t)$ and the surface charge densities $c_0(x,t), c_0{}^*(x,t)$. We define the Laplace transform as:

$$\bar{f}(x,s) = L(f(x,t)) = \int_0^\infty f(x,t)e^{-st}dt, \tag{61}$$

Applying the Laplace transform defined by the Eq. (61) on the Eqs. (1)-(17) and omitting the bars for simplicity, we obtain

Constitutive relations:

$$\sigma_{ij} = c_{ijkl}\,\varepsilon_{kl} - e_{ijk}E_k - \alpha_{ij}\theta + m_{ij}\varepsilon^* - \zeta_{kij}E_k{}^* - b_{ij}\mu, \tag{62}$$

$$D_i = \xi_{ij}E_j + e_{ijk}\varepsilon_{jk} + \tau_i\theta + \zeta_i\varepsilon^* + A_{ij}E_j{}^* + b_i\mu, \tag{63}$$

$$E_i = -\phi_{,i}, \qquad (i,j,k,l = 1,2,3) \tag{64}$$

$$\sigma^* = m_{ij}\,\varepsilon_{ij} - \zeta_i E_i - \alpha_{ij}^f\theta + R\varepsilon^* - e_i{}^*E_i{}^* - b_{ij}^f\mu, \tag{65}$$

$$D_i{}^* = \zeta_{ijk}\varepsilon_{jk} + \tau_i^f\theta + A_{ij}E_j + \xi_{ij}{}^*E_j{}^* + e_i{}^*\varepsilon^* + b_i^f\mu, \tag{66}$$

$$E_i{}^* = -\phi^*_{,i}, \qquad (i,j,k,l = 1,2,3) \tag{67}$$

$$-q_{i,i} = T_0 s\rho S, \tag{68}$$

$$\rho S = \alpha_{ij}\varepsilon_{ij} + \tau_i E_i + r\theta + \alpha_{ij}^f\varepsilon^* + \tau_i^f E_i{}^* + a\mu, -\eta_{i,i} = sC, \tag{69}$$

$$-\eta_{i,i} = sC, \tag{70}$$

$$C = b_{ij}\varepsilon_{ij} + b_i E_i + b\mu + a\theta + b_{ij}^f\varepsilon^* + b_i^f E_i{}^*, \tag{71}$$

Equations of motion:

$$\sigma_{ij,j} + \rho_1 F_i - \rho_{11}s^2\,\mathrm{u}_i - \rho_{12}s^2\,\mathrm{u}_i{}^* = 0, \tag{72}$$

$$\sigma^*_{,i} + \rho_2 f_i - \rho_{12}s^2\,\mathrm{u}_i - \rho_{22}s^2\,\mathrm{u}_i{}^* = 0, \tag{73}$$

Equations of heat conduction:

$$-K_{ij}\,\theta_{,j} = (1+\tau_0 s)q_i \tag{74}$$

Equations of chemical potential:

$$-\alpha_{ij}{}^*\mu_{,j} = (1+\tau^0 s)\eta_i \tag{75}$$

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Gauss equation:

$$D_{i,i} = 0 \tag{76}$$

$$D_i{}^*{}_{,i} = 0 \qquad (i, j = 1,2,3) \tag{77}$$

We now consider two problems where applied body forces, electric potential and the surface temperature are specified differently. Let the variables involved in these two problems be distinguished by superscripts in parentheses. Thus, we have $\mathrm{u}_i{}^{(1)}, \mathrm{u}_i{}^{*(1)}, \varepsilon_{ij}{}^{(1)}, \varepsilon^{*(1)}, \sigma_{ij}{}^{(1)}, \sigma^{*(1)}, \theta^{(1)}, \phi^{(1)}, \phi^{*(1)}, \mu^{(1)}$ for the first problem and $\mathrm{u}_i{}^{(2)}, \mathrm{u}_i{}^{*(2)}, \varepsilon_{ij}{}^{(2)}, \varepsilon^{*(2)}, \sigma_{ij}{}^{(2)}, \sigma^{*(2)}, \theta^{(2)}, \phi^{(2)}, \phi^{*(2)}, \mu^{(2)}$ for the second problem. Each set of variables satisfies the Eqs. (62)-(77).

Using the assumption $\sigma_{ij} = \sigma_{ji}$, we obtain

$$\begin{aligned}\int_V \sigma_{ij}{}^{(1)}\varepsilon_{ij}{}^{(2)} dV + \int_V \sigma^{*(1)}\varepsilon^{*(2)}\, dV &= \int_V \sigma_{ij}{}^{(1)}\mathrm{u}_{i,j}{}^{(2)} dV + \int_V \sigma^{*(1)}\,\mathrm{u}_i{}^*{}_{,i}{}^{(2)}\, dV \\ &= \int_V \left(\sigma_{ij}{}^{(1)}\mathrm{u}_i{}^{(2)}\right)_{,j} dV + \int_V \left(\sigma^{*(1)}\,\mathrm{u}_i{}^{*(2)}\right)_{,i} dV - \int_V \sigma_{ij.j}{}^{(1)}\mathrm{u}_i{}^{(2)} dV - \int_V \sigma^*{}_{,i}{}^{(1)}\,\mathrm{u}_i{}^{*(2)}\, dV\ .\end{aligned} \tag{78}$$

Using the divergence theorem in the first term of the right hand side of Eq. (78) yields

$$\begin{aligned}\int_V \sigma_{ij}{}^{(1)}\varepsilon_{ij}{}^{(2)} dV + \int_V \sigma^{*(1)}\varepsilon^{*(2)}\, dV &= \int_A \sigma_{ij}{}^{(1)}\mathrm{u}_i{}^{(2)} n_j\, dA + \int_A \sigma^{*(1)}\,\mathrm{u}_i{}^{*(2)}\, n_i\, dA - \int_V \sigma_{ij.j}{}^{(1)}\mathrm{u}_i{}^{(2)} dV \\ &- \int_V \sigma^*{}_{,i}{}^{(1)}\,\mathrm{u}_i{}^{*(2)}\, dV\end{aligned} \tag{79}$$

Eq. (79) with the aid of Eqs. (58), (72) and (73) gives

$$\begin{aligned}\int_V (\sigma_{ij}{}^{(1)}\varepsilon_{ij}{}^{(2)} + \sigma^{*(1)}\,\varepsilon^{*(2)}) dV &= \int_A (h_i{}^{(1)}\mathrm{u}_i{}^{(2)} + h_i{}^{*(1)}\,\mathrm{u}_i{}^{*(2)}) dA + \int_V (\rho_1 F_i{}^{(1)}\mathrm{u}_i{}^{(2)} - \rho_{11} s^2 \mathrm{u}_i{}^{(1)}\mathrm{u}_i{}^{(2)} - \rho_{12} s^2 \mathrm{u}_i{}^{*(1)}\,\mathrm{u}_i{}^{(2)}) dV \\ &+ \int_V (\rho_2 f_i{}^{(1)}\mathrm{u}_i{}^{*(2)} - \rho_{12} s^2 \mathrm{u}_i{}^{(1)}\mathrm{u}_i{}^{*(2)} - \rho_{22} s^2 \mathrm{u}_i{}^{*(1)}\,\mathrm{u}_i{}^{*(2)}) dV\end{aligned} \tag{80}$$

A similar expression is obtained for the integral $\int_V (\sigma_{ij}{}^{(2)}\varepsilon_{ij}{}^{(1)} + \sigma^{*(2)}\,\varepsilon^{*(1)}) dV,$ from which together with the Eq. (80), it follows that

$$\begin{aligned}\int_V (\sigma_{ij}{}^{(1)}\varepsilon_{ij}{}^{(2)} - \sigma_{ij}{}^{(2)}\varepsilon_{ij}{}^{(1)} + \sigma^{*(1)}\,\varepsilon^{*(2)} - \sigma^{*(2)}\,\varepsilon^{*(1)}) dV &= \int_A (h_i{}^{(1)}\mathrm{u}_i{}^{(2)} - h_i{}^{(2)}\mathrm{u}_i{}^{(1)} + h_i{}^{*(1)}\,\mathrm{u}_i{}^{*(2)} - h_i{}^{*(2)}\,\mathrm{u}_i{}^{*(1)}) dA \\ &+ \int_V \rho_1 (F_i{}^{(1)}\mathrm{u}_i{}^{(2)} - F_i{}^{(2)}\mathrm{u}_i{}^{(1)}) dV + \int_V \rho_2 (f_i{}^{(1)}\mathrm{u}_i{}^{*(2)} - f_i{}^{(2)}\mathrm{u}_i{}^{*(1)}) dV\end{aligned} \tag{81}$$

Now multiplying Eqs. (62), (65) by $\varepsilon_{ij}{}^{(2)}, \varepsilon^{*(2)}$ and $\varepsilon_{ij}{}^{(1)}, \varepsilon^{*(1)}$ for the first and second problems respectively, subtracting and integrating over the region *V*, we obtain

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$$\int_V (\sigma_{ij}{}^{(1)}\varepsilon_{ij}{}^{(2)} - \sigma_{ij}{}^{(2)}\varepsilon_{ij}{}^{(1)} + \sigma^{*(1)}\,\varepsilon^{*(2)} - \sigma^{*(2)}\,\varepsilon^{*(1)})dV = \int_V c_{ijkl}\,(\varepsilon_{kl}{}^{(1)}\varepsilon_{ij}{}^{(2)} - \varepsilon_{kl}{}^{(2)}\varepsilon_{ij}{}^{(1)})dV$$
$$-\int_V \alpha_{ij}\,(\theta^{(1)}\varepsilon_{ij}{}^{(2)} - \theta^{(2)}\varepsilon_{ij}{}^{(1)})dV - \int_V \zeta_i\,(\phi_{,i}{}^{(2)}\varepsilon^{*(1)} - \phi_{,i}{}^{(1)}\varepsilon^{*(2)})dV - \int_V \zeta_{ijk}\,(\phi^*{}_{,k}{}^{(2)}\,\varepsilon_{ij}{}^{(1)} - \phi^*{}_{,k}{}^{(1)}\,\varepsilon_{ij}{}^{(2)})dV$$
$$-\int_V e_i{}^*(\phi^*{}_{,i}{}^{(2)}\,\varepsilon^{*(1)} - \phi^*{}_{,i}{}^{(1)}\,\varepsilon^{*(2)})dV - \int_V \alpha_{ij}{}^f\,(\theta^{(1)}\varepsilon^{*(2)} - \theta^{(2)}\varepsilon^{*(1)})dV - \int_V e_{ijk}\,(\phi_{,k}{}^{(2)}\varepsilon_{ij}{}^{(1)} - \phi_{,k}{}^{(1)}\varepsilon_{ij}{}^{(2)})dV$$
$$-\int_V \left\{b_{ij}\,(\mu^{(1)}\varepsilon_{ij}{}^{(2)} - \mu^{(2)}\varepsilon_{ij}{}^{(1)}) + b_{ij}{}^f\,(\mu^{(1)}\varepsilon^{*(2)} - \mu^{(2)}\varepsilon^{*(1)})\right\}dV \;. \tag{82}$$

Using the symmetry properties of c_{ijkl}, we obtain

$$\int_V (\sigma_{ij}{}^{(1)}\varepsilon_{ij}{}^{(2)} - \sigma_{ij}{}^{(2)}\varepsilon_{ij}{}^{(1)} + \sigma^{*(1)}\,\varepsilon^{*(2)} - \sigma^{*(2)}\,\varepsilon^{*(1)})dV = -\int_V e_{ijk}\,(\phi_{,k}{}^{(2)}\varepsilon_{ij}{}^{(1)} - \phi_{,k}{}^{(1)}\varepsilon_{ij}{}^{(2)})dV$$
$$-\int_V \alpha_{ij}\,(\theta^{(1)}\varepsilon_{ij}{}^{(2)} - \theta^{(2)}\varepsilon_{ij}{}^{(1)})dV - \int_V \zeta_i\,(\phi_{,i}{}^{(2)}\varepsilon^{*(1)} - \phi_{,i}{}^{(1)}\varepsilon^{*(2)})dV - \int_V \zeta_{ijk}\,(\phi^*{}_{,k}{}^{(2)}\,\varepsilon_{ij}{}^{(1)} - \phi^*{}_{,k}{}^{(1)}\,\varepsilon_{ij}{}^{(2)})dV$$
$$-\int_V e_i{}^*(\phi^*{}_{,i}{}^{(2)}\,\varepsilon^{*(1)} - \phi^*{}_{,i}{}^{(1)}\,\varepsilon^{*(2)})dV - \int_V \alpha_{ij}{}^f\,(\theta^{(1)}\varepsilon^{*(2)} - \theta^{(2)}\varepsilon^{*(1)})dV$$
$$-\int_V \left\{b_{ij}\,(\mu^{(1)}\varepsilon_{ij}{}^{(2)} - \mu^{(2)}\varepsilon_{ij}{}^{(1)}) + b_{ij}{}^f\,(\mu^{(1)}\varepsilon^{*(2)} - \mu^{(2)}\varepsilon^{*(1)})\right\}dV \,. \tag{83}$$

Equating Eqs. (82) and (83), we get the first part of the reciprocity theorem

$$\int_A (h_i{}^{(1)}\mathrm{u}_i{}^{(2)} - h_i{}^{(2)}\mathrm{u}_i{}^{(1)} + h_i{}^{*(1)}\,\mathrm{u}_i{}^{*(2)} - h_i{}^{*(2)}\,\mathrm{u}_i{}^{*(1)})dA + \int_V \rho_1(F_i{}^{(1)}\mathrm{u}_i{}^{(2)} - F_i{}^{(2)}\mathrm{u}_i{}^{(1)})dV$$
$$+\int_V \rho_2(f_i{}^{(1)}\mathrm{u}_i{}^{*(2)} - f_i{}^{(2)}\mathrm{u}_i{}^{*(1)})dV = -\int_V e_{ijk}\,(\phi_{,k}{}^{(2)}\varepsilon_{ij}{}^{(1)} - \phi_{,k}{}^{(1)}\varepsilon_{ij}{}^{(2)})dV - \int_V \alpha_{ij}\,(\theta^{(1)}\varepsilon_{ij}{}^{(2)} - \theta^{(2)}\varepsilon_{ij}{}^{(1)})dV$$
$$-\int_V \zeta_i\,(\phi_{,i}{}^{(2)}\varepsilon^{*(1)} - \phi_{,i}{}^{(1)}\varepsilon^{*(2)})dV - \int_V \zeta_{ijk}\,(\phi^*{}_{,k}{}^{(2)}\,\varepsilon_{ij}{}^{(1)} - \phi^*{}_{,k}{}^{(1)}\,\varepsilon_{ij}{}^{(2)})dV$$
$$-\int_V e_i{}^*(\phi^*{}_{,i}{}^{(2)}\,\varepsilon^{*(1)} - \phi^*{}_{,i}{}^{(1)}\,\varepsilon^{*(2)})dV - \int_V \alpha_{ij}{}^f\,(\theta^{(1)}\varepsilon^{*(2)} - \theta^{(2)}\varepsilon^{*(1)})dV$$
$$-\int_V \left\{b_{ij}\,(\mu^{(1)}\varepsilon_{ij}{}^{(2)} - \mu^{(2)}\varepsilon_{ij}{}^{(1)}) + b_{ij}{}^f\,(\mu^{(1)}\varepsilon^{*(2)} - \mu^{(2)}\varepsilon^{*(1)})\right\}dV \,. \tag{84}$$

Eq. (84) contains the mechanical causes of motion F_i, f_i and $h_i, h_i{}^*$.
Using Eq. (69), Eq. (68) reduces to

$$-q_{i,i} = T_0 s\,(\alpha_{ij}\varepsilon_{ij} + \tau_i E_i + r\theta + \alpha_{ij}{}^f\,\varepsilon^* + \tau_i{}^f\,E_i{}^* + a\mu). \tag{85}$$

Now, taking the divergence on both sides of Eq. (74) and using Eq. (85), we arrive at the equation of heat conduction, namely

$$\frac{\partial}{\partial x_i}\left(K_{ij}\theta_{,j}\right) = (s + \tau_0 s^2)T_0(\alpha_{ij}\varepsilon_{ij} + \tau_i E_i + r\theta + \alpha_{ij}{}^f\,\varepsilon^* + \tau_i{}^f\,E_i{}^* + a\mu), \tag{86}$$

To derive the second part, multiplying Eq. (86) by $\theta^{(2)}$ and $\theta^{(1)}$ for the first and the second problems respectively, subtracting and integrating over *V*, we get

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$$\int_V \left(\left(K_{ij}\theta_{,j}{}^{(1)}\right)_{,i} \theta^{(2)} - \left(K_{ij}\theta_{,j}{}^{(2)}\right)_{,i} \theta^{(1)} \right) dV = (s+\tau_0 s^2)T_0 \int_V \alpha_{ij} \left(\varepsilon_{ij}{}^{(1)}\theta^{(2)} - \varepsilon_{ij}{}^{(2)}\theta^{(1)}\right) dV$$
$$+(s+\tau_0 s^2)T_0 \int_V \tau_i \left(E_i{}^{(1)}\theta^{(2)} - E_i{}^{(2)}\theta^{(1)}\right) dV + (s+\tau_0 s^2)T_0 \int_V \alpha_{ij}{}^f \left(\varepsilon^{*(1)}\theta^{(2)} - \varepsilon^{*(2)}\theta^{(1)}\right) dV \tag{87}$$
$$+(s+\tau_0 s^2)T_0 \int_V \tau_i{}^f \left(E_i{}^{*(1)}\theta^{(2)} - E_i{}^{*(2)}\theta^{(1)}\right) dV + (s+\tau_0 s^2)T_0 \int_V a\left(\mu^{(1)}\theta^{(2)} - \mu^{(2)}\theta^{(1)}\right) dV$$

Now

$$\left(K_{ij}\theta_{,j}{}^{(1)}\right)_{,i}\theta^{(2)} = \left(K_{ij}\theta_{,j}{}^{(1)}\theta^{(2)}\right)_{,i} - K_{ij}\theta_{,j}{}^{(1)}\theta_{,i}{}^{(2)} \text{ and } \left(K_{ij}\theta_{,j}{}^{(2)}\right)_{,i}\theta^{(1)} = \left(K_{ij}\theta_{,j}{}^{(2)}\theta^{(1)}\right)_{,i} - K_{ij}\theta_{,j}{}^{(2)}\theta_{,i}{}^{(1)} \tag{88}$$

Eq. (87) with the help of Eqs. (58), (59), (88) and the divergence theorem can be written as:

$$\int_A \left(q^{(1)}\eta^{(2)} - q^{(2)}\eta^{(1)}\right) dA = -(s+\tau_0 s^2)T_0 \int_V \alpha_{ij} \left(\varepsilon_{ij}{}^{(1)}\theta^{(2)} - \varepsilon_{ij}{}^{(2)}\theta^{(1)}\right) dV$$
$$-(s+\tau_0 s^2)T_0 \int_V \tau_i \left(E_i{}^{(1)}\theta^{(2)} - E_i{}^{(2)}\theta^{(1)}\right) dV - (s+\tau_0 s^2)T_0 \int_V \alpha_{ij}{}^f \left(\varepsilon^{*(1)}\theta^{(2)} - \varepsilon^{*(2)}\theta^{(1)}\right) dV \tag{89}$$
$$-(s+\tau_0 s^2)T_0 \int_V \tau_i{}^f \left(E_i{}^{*(1)}\theta^{(2)} - E_i{}^{*(2)}\theta^{(1)}\right) dV - (s+\tau_0 s^2)T_0 \int_V a\left(\mu^{(1)}\theta^{(2)} - \mu^{(2)}\theta^{(1)}\right) dV$$

The Eq. (89) constitutes the second part of reciprocity theorem which contains the thermal causes of motion η and q. From Eqs. (70), (71) and (72), we obtain the equation of chemical potential

$$\frac{\partial}{\partial x_i}\left(\alpha_{ij}{}^*\mu_{,j}\right) = (s+\tau^0 s^2)(b_{ij}\varepsilon_{ij} + b_i E_i + a\theta + b\mu + b_{ij}{}^f \varepsilon^* + b_i{}^f E_i{}^*). \tag{90}$$

To derive the third part, multiplying Eq. (90) by $\mu^{(2)}$ and $\mu^{(1)}$ for the first and second problems respectively, subtracting and integrating over V, we obtain

$$\int_V \left(\left(\alpha_{ij}{}^*\mu_{,j}{}^{(1)}\right)_{,i} \mu^{(2)} - \left(\alpha_{ij}{}^*\mu_{,j}{}^{(2)}\right)_{,i} \mu^{(1)} \right) dV = (s+\tau^0 s^2) \int_V b_{ij} \left(\varepsilon_{ij}{}^{(1)}\mu^{(2)} - \varepsilon_{ij}{}^{(2)}\mu^{(1)}\right) dV$$
$$+(s+\tau^0 s^2) \int_V \left\{ b_i \left(E_i{}^{(1)}\mu^{(2)} - E_i{}^{(2)}\mu^{(1)}\right) \right\} dV + (s+\tau^0 s^2) \int_V a\left(\theta^{(1)}\mu^{(2)} - \theta^{(2)}\mu^{(1)}\right) dV \tag{91}$$
$$+(s+\tau^0 s^2) \int_V \left\{ b_{ij}{}^f \left(\varepsilon^{*(1)}\mu^{(2)} - \varepsilon^{*(2)}\mu^{(1)}\right) + b_i{}^f \left(E_i{}^{*(1)}\mu^{(2)} - E_i{}^{*(2)}\mu^{(1)}\right) \right\} dV.$$

Consider

$$\left(\alpha_{ij}{}^*\mu_{,j}{}^{(1)}\right)_{,i}\mu^{(2)} = \left(\alpha_{ij}{}^*\mu_{,j}{}^{(1)}\mu^{(2)}\right)_{,i} - \alpha_{ij}{}^*\mu_{,j}{}^{(1)}\mu_{,i}{}^{(2)} \text{ and } \left(\alpha_{ij}{}^*\mu_{,j}{}^{(2)}\right)_{,i}\mu^{(1)} = \left(\alpha_{ij}{}^*\mu_{,j}{}^{(2)}\mu^{(1)}\right)_{,i} - \alpha_{ij}{}^*\mu_{,j}{}^{(2)}\mu_{,i}{}^{(1)}. \tag{92}$$

Eq. (91) with the aid of Eqs .(58), (59), (92) and the divergence theorem yields

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$$\int_A \left(p^{(1)}\varsigma^{(2)} - p^{(2)}\varsigma^{(1)}\right)dA = -(s+\tau^0 s^2)\int_V b_{ij}\left(\varepsilon_{ij}{}^{(1)}\mu^{(2)} - \varepsilon_{ij}{}^{(2)}\mu^{(1)}\right)dV$$
$$-(s+\tau^0 s^2)\int_V b_i\left(\phi_{,i}{}^{(2)}\mu^{(1)} - \phi_{,i}{}^{(1)}\mu^{(2)}\right)dV - (s+\tau^0 s^2)\int_V a\left(\theta^{(1)}\mu^{(2)} - \theta^{(2)}\mu^{(1)}\right)dV \tag{93}$$
$$-(s+\tau^0 s^2)\int_V \left\{b_{ij}{}^f\left(\varepsilon^{*(1)}\mu^{(2)} - \varepsilon^{*(2)}\mu^{(1)}\right) + b_i{}^f\left(\phi^*{}_{,i}{}^{(1)}\mu^{(2)} - \phi^*{}_{,i}{}^{(2)}\mu^{(1)}\right)\right\}dV .$$

The Eq. (93) constitutes the third part of reciprocity theorem which contains the chemical causes of motion ς and p.

To derive the last part, multiplying Eqs. (63), (66) by $E_i{}^{(2)}, E_i{}^{*(2)}$ and $E_i{}^{(1)}, E_i{}^{*(1)}$ for the first and the second problems respectively, subtracting and integrating over V, we get

$$\int_V \left(D_i{}^{(1)}E_i{}^{(2)} - D_i{}^{(2)}E_i{}^{(1)} + D_i{}^{*(1)}E_i{}^{*(2)} - D_i{}^{*(2)}E_i{}^{*(1)}\right)dV = \int_V e_{ijk}(\varepsilon_{jk}{}^{(1)}E_i{}^{(2)} - \varepsilon_{jk}{}^{(2)}E_i{}^{(1)})dV$$
$$+\int_V \tau_i{}^f(\theta^{(1)}E_i{}^{*(2)} - \theta^{(2)}E_i{}^{*(1)})dV + \int_V \tau_i(\theta^{(1)}E_i{}^{(2)} - \theta^{(2)}E_i{}^{(1)})dV + \int_V \zeta_i(\varepsilon^{*(1)}E_i{}^{(2)} - \varepsilon^{*(2)}E_i{}^{(1)})dV \tag{94}$$
$$+\int_V e_i{}^*(\varepsilon^{*(1)}E_i{}^{*(2)} - \varepsilon^{*(2)}E_i{}^{*(1)})dV + \int_V \zeta_{ijk}(\varepsilon_{jk}{}^{(1)}E_i{}^{*(2)} - \varepsilon_{jk}{}^{(2)}E_i{}^{*(1)})dV$$
$$+\int_V b_i(\mu^{(1)}E_i{}^{(2)} - \mu^{(2)}E_i{}^{(1)})dV + \int_V b_i{}^f(\mu^{(1)}E_i{}^{*(2)} - \mu^{(2)}E_i{}^{*(1)})dV .$$

Eq. (94) with the aid of Eq. (64) and (67) yields

$$\int_V \left(D_i{}^{(1)}E_i{}^{(2)} - D_i{}^{(2)}E_i{}^{(1)} + D_i{}^{*(1)}E_i{}^{*(2)} - D_i{}^{*(2)}E_i{}^{*(1)}\right)dV = -\int_V e_{ijk}(\varepsilon_{jk}{}^{(1)}\phi_{,i}{}^{(2)} - \varepsilon_{jk}{}^{(2)}\phi_{,i}{}^{(1)})dV$$
$$-\int_V \tau_i{}^f(\theta^{(1)}\phi^*{}_{,i}{}^{(2)} - \theta^{(2)}\phi^*{}_{,i}{}^{(1)})dV - \int_V \tau_i(\theta^{(1)}\phi_{,i}{}^{(2)} - \theta^{(2)}\phi_{,i}{}^{(1)})dV - \int_V \zeta_i(\varepsilon^{*(1)}\phi_{,i}{}^{(2)} - \varepsilon^{*(2)}\phi_{,i}{}^{(1)})dV \tag{95}$$
$$-\int_V e_i{}^*(\varepsilon^{*(1)}\phi^*{}_{,i}{}^{(2)} - \varepsilon^{*(2)}\phi^*{}_{,i}{}^{(1)})dV - \int_V \zeta_{ijk}(\varepsilon_{jk}{}^{(1)}\phi^*{}_{,i}{}^{(2)} - \varepsilon_{jk}{}^{(2)}\phi^*{}_{,i}{}^{(1)})dV$$
$$-\int_V b_i(\mu^{(1)}\phi_{,i}{}^{(2)} - \mu^{(2)}\phi_{,i}{}^{(1)})dV + \int_V b_i{}^f(\mu^{(1)}\phi^*{}_{,i}{}^{(2)} - \mu^{(2)}\phi^*{}_{,i}{}^{(1)})dV .$$

Also, using (64) and (67), we have

$$\int_V \left(D_i{}^{(1)}E_i{}^{(2)} - D_i{}^{(2)}E_i{}^{(1)} + D_i{}^{*(1)}E_i{}^{*(2)} - D_i{}^{*(2)}E_i{}^{*(1)}\right)dV = \int_V \left(D_i{}^{(2)}\phi_{,i}{}^{(1)} - D_i{}^{(1)}\phi_{,i}{}^{(2)}\right)dV \tag{96}$$
$$+\int_V (D_i{}^{*(2)}\phi^*{}_{,i}{}^{(1)} - D_i{}^{*(1)}\phi^*{}_{,i}{}^{(2)})dV .$$

Now

$$D_i{}^{(2)}\phi_{,i}{}^{(1)} = \left(D_i{}^{(2)}\phi^{(1)}\right)_{,i} - D_{i,i}{}^{(2)}\phi^{(1)} \quad , \quad D_i{}^{(1)}\phi_{,i}{}^{(2)} = \left(D_i{}^{(1)}\phi^{(2)}\right)_{,i} - D_{i,i}{}^{(1)}\phi^{(2)},$$
$$D_i{}^{*(2)}\phi^*{}_{,i}{}^{(1)} = \left(D_i{}^{*(2)}\phi^{*(1)}\right)_{,i} - D_i{}^*{}_{,i}{}^{(2)}\phi^{*(1)} \quad , \quad D_i{}^{*(1)}\phi^*{}_{,i}{}^{(2)} = \left(D_i{}^{*(1)}\phi^{*(2)}\right)_{,i} - D_i{}^*{}_{,i}{}^{(1)}\phi^{*(2)} . \tag{97}$$

Using Eqs. (76), (77), (97) and divergence theorem in Eq. (96), we obtain

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$$\int_V \left(D_i^{(1)}E_i^{(2)} - D_i^{(2)}E_i^{(1)} + D_i^{*(1)} E_i^{*(2)} - D_i^{*(2)} E_i^{*(1)}\right)dV = \int_V \left(\left(D_i^{(2)}\phi^{(1)}\right)_{,i} - \left(D_i^{(1)}\phi^{(2)}\right)_{,i}\right)dV$$
$$+\int_V \left(D_{i,i}^{(1)}\phi^{(2)} - D_{i,i}^{(2)}\phi^{(1)}\right)dV + \int_V \left(\left(D_i^{(2)}\phi^{*(1)}\right)_{,i} - \left(D_i^{(1)}\phi^{*(2)}\right)_{,i}\right)dV + \int_V (D_i{}^*{}_{,i}^{(1)} \phi^{*(2)} - D_i{}^*{}_{,i}^{(2)} \phi^{*(1)})dV \tag{98}$$
$$= \int_A \left(D_i^{(2)}\phi^{(1)}n_i - D_i^{(1)}\phi^{(2)}n_i + D_i^{*(2)} \phi^{*(1)} n_i - D_i^{*(1)} \phi^{*(2)} n_i\right)dA.$$

Eq. (98) with the aid of Eq. (58), gives

$$\int_V \left(D_i^{(1)}E_i^{(2)} - D_i^{(2)}E_i^{(1)} + D_i^{*(1)} E_i^{*(2)} - D_i^{*(2)} E_i^{*(1)}\right)dV = \int_A \left(c_0^{(2)}\phi^{(1)} - c_0^{(1)}\phi^{(2)} + c_0^{*(2)} \phi^{*(1)} - c_0^{*(1)} \phi^{*(2)}\right)dA. \tag{99}$$

From Eqs. (95) and (99), we have

$$\int_A \left(c_0^{(2)}\phi^{(1)} - c_0^{(1)}\phi^{(2)} + c_0^{*(2)} \phi^{*(1)} - c_0^{*(1)} \phi^{*(2)}\right)dA = -\int_V e_{ijk} (\varepsilon_{jk}^{(1)}\phi_{,i}^{(2)} - \varepsilon_{jk}^{(2)}\phi_{,i}^{(1)})dV$$
$$-\int_V \tau_i^f (\theta^{(1)}\phi^*{}_{,i}^{(2)} - \theta^{(2)}\phi^*{}_{,i}^{(1)})dV - \int_V \tau_i (\theta^{(1)}\phi_{,i}^{(2)} - \theta^{(2)}\phi_{,i}^{(1)})dV - \int_V \zeta_i (\varepsilon^{*(1)} \phi_{,i}^{(2)} - \varepsilon^{*(2)} \phi_{,i}^{(1)})dV \tag{100}$$
$$-\int_V e_i^* (\varepsilon^{*(1)} \phi^*{}_{,i}^{(2)} - \varepsilon^{*(2)} \phi^*{}_{,i}^{(1)})dV - \int_V \zeta_{ijk} (\varepsilon_{jk}^{(1)}\phi^*{}_{,i}^{(2)} - \varepsilon_{jk}^{(2)}\phi^*{}_{,i}^{(1)})dV$$
$$-\int_V b_i (\mu^{(1)}\phi_{,i}^{(2)} - \mu^{(2)}\phi_{,i}^{(1)})dV + \int_V b_i^f (\mu^{(1)}\phi^*{}_{,i}^{(2)} - \mu^{(2)}\phi^*{}_{,i}^{(1)})dV.$$

The Eq. (100) constitutes the last part of reciprocity theorem which contains the electric potentials ϕ, ϕ^* and surface charge densities c_0, c_0^*. Eliminating the integrals

$$\int_V \alpha_{ij}^f \left(\varepsilon^{*(1)} \theta^{(2)} - \varepsilon^{*(2)} \theta^{(1)}\right)dV, \int_V \alpha_{ij} \left(\varepsilon_{ij}^{(1)}\theta^{(2)} - \varepsilon_{ij}^{(2)}\theta^{(1)}\right)dV, \int_V e_{ijk} (\varepsilon_{jk}^{(1)}\phi_{,i}^{(2)} - \varepsilon_{jk}^{(2)}\phi_{,i}^{(1)})dV,$$
$$\int_V \tau_i^f (\theta^{(1)}\phi^*{}_{,i}^{(2)} - \theta^{(2)}\phi^*{}_{,i}^{(1)})dV, \int_V \tau_i (\theta^{(1)}\phi_{,i}^{(2)} - \theta^{(2)}\phi_{,i}^{(1)})dV, \int_V e_i^* (\varepsilon^{*(1)} \phi^*{}_{,i}^{(2)} - \varepsilon^{*(2)} \phi^*{}_{,i}^{(1)})dV,$$
$$\int_V \zeta_{ijk} (\varepsilon_{jk}^{(1)}\phi^*{}_{,i}^{(2)} - \varepsilon_{jk}^{(2)}\phi^*{}_{,i}^{(1)})dV, \int_V \zeta_i (\varepsilon^{*(1)} \phi_{,i}^{(2)} - \varepsilon^{*(2)} \phi_{,i}^{(1)})dV.$$

From Eqs. (83), (88), (92) and (100) with the aid of Eq. (64) and (67), we obtain

$$s(1+\tau_0 s)(1+\tau^0 s)T_0\left[\int_A (h_i^{(1)}\mathrm{u}_i^{(2)} - h_i^{(2)}\mathrm{u}_i^{(1)} + h_i^{*(1)} \mathrm{u}_i^{*(2)} - h_i^{*(2)} \mathrm{u}_i^{*(1)})dA + \int_V \rho_1(F_i^{(1)}\mathrm{u}_i^{(2)} - F_i^{(2)}\mathrm{u}_i^{(1)})dV\right]$$
$$+s(1+\tau_0 s)(1+\tau^0 s)T_0\left[\int_V \rho_2(f_i^{(1)}\mathrm{u}_i^{*(2)} - f_i^{(2)}\mathrm{u}_i^{*(1)})dV + \int_A \left(c_0^{(1)}\phi^{(2)} - c_0^{(2)}\phi^{(1)} + c_0^{*(1)} \phi^{*(2)} - c_0^{*(2)} \phi^{*(1)}\right)dA\right] \tag{101}$$
$$+(1+\tau^0 s)\int_A \left(q^{(1)}\eta^{(2)} - q^{(2)}\eta^{(1)}\right)dA + (1+\tau_0 s)T_0\int_A \left(p^{(1)}\varsigma^{(2)} - p^{(2)}\varsigma^{(1)}\right)dA = 0.$$

This is the general reciprocity theorem in the Laplace transform domain.

For applying inverse Laplace transform on the Eqs. (83), (88), (92), (100) and (101), we shall use the convolution theorem

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$$L^{-1}(F(s)G(s))=\int_0^t f(t-\xi)g(\xi)d\xi=\int_0^t g(t-\xi)f(\xi)d\xi, \tag{102}$$

and the symbolic notation

$$\begin{aligned}&B(f)=1+\tau_0\frac{\partial f(x,\xi)}{\partial\xi},\quad Q(f)=1+\tau^0\frac{\partial f(x,\xi)}{\partial\xi}\\&\wedge(f)=1+(\tau_0+\tau^0)\frac{\partial f(x,\xi)}{\partial\xi}+\tau_0\tau^0\frac{\partial^2 f(x,\xi)}{\partial\xi^2},\end{aligned} \tag{103}$$

Eqs. (83), (88), (92) and (100) with the aid of Eq. (102) yield the first, second, third and last parts of the reciprocity theorem in the final form

$$\begin{aligned}&\int_A\int_0^t (h_i^{(1)}(x,t-\xi)\mathrm{u}_i^{(2)}(x,\xi)+h_i{*}^{(1)}(x,t-\xi)\mathrm{u}_i{*}^{(2)}(x,\xi))d\xi dA+\int_V\int_0^t\rho_1(F_i^{(1)}(x,t-\xi)\mathrm{u}_i^{(2)}(x,\xi))d\xi dV\\&+\int_V\int_0^t\rho_2(f_i^{(1)}(x,t-\xi)\mathrm{u}_i{*}^{(2)}(x,\xi))d\xi dV-\int_V\int_0^t e_{ijk}(\phi_{,k}^{(1)}(x,t-\xi)\varepsilon_{ij}^{(2)}(x,\xi))d\xi dV\\&+\int_V\int_0^t\alpha_{ij}(\theta^{(1)}(x,t-\xi)\varepsilon_{ij}^{(2)}(x,\xi))d\xi dV+\int_V\int_0^t\alpha_{ij}^f(\theta^{(1)}(x,t-\xi)\varepsilon{*}^{(2)}(x,\xi))d\xi dV\\&-\int_V\int_0^t\zeta_{ijk}\phi{*}_{,k}^{(1)}(x,t-\xi)\varepsilon_{ij}^{(2)}(x,\xi)d\xi dV-\int_V\int_0^t e_i{*}\phi{*}_{,i}^{(1)}(x,t-\xi)\varepsilon{*}^{(2)}(x,\xi)d\xi dV\\&+\int_V\int_0^t b_{ij}\mu^{(1)}(x,t-\xi)\varepsilon_{ij}^{(2)}(x,\xi)d\xi dV+\int_V\int_0^t b_{ij}^f\mu^{(1)}(x,t-\xi)\varepsilon{*}^{(2)}(x,\xi)d\xi dV=S_{21}^{12},\end{aligned} \tag{104}$$

$$\begin{aligned}&\int_A\int_0^t q^{(1)}(x,t-\xi)\eta^{(2)}(x,\xi)d\xi dA-T_0\int_V\int_0^t\alpha_{ij}\theta^{(1)}(x,t-\xi)\frac{\partial B(\varepsilon_{ij}^{(2)}(x,\xi))}{\partial\xi}d\xi dV\\&-T_0\int_V\int_0^t\alpha_{ij}^f\theta^{(1)}(x,t-\xi)\frac{\partial B(\varepsilon{*}^{(2)}(x,\xi))}{\partial\xi}d\xi dV+T_0\int_V\int_0^t\tau_i\theta^{(1)}(x,t-\xi)\frac{\partial B(\phi_{,i}^{(2)}(x,\xi))}{\partial\xi}d\xi dV\\&+T_0\int_V\int_0^t\tau_i^f\theta^{(1)}(x,t-\xi)\frac{\partial B(\phi{*}_{,i}^{(2)}(x,\xi))}{\partial\xi}d\xi dV-T_0\int_V\int_0^t a\left(\theta^{(1)}(x,t-\xi)\frac{\partial B(\mu^{(2)}(x,\xi))}{\partial\xi}\right)d\xi dV=S_{21}^{12},\end{aligned} \tag{105}$$

$$\begin{aligned}&\int_A\int_0^t\left(p^{(1)}(x,t-\xi)\varsigma^{(2)}(x,\xi)\right)d\xi dA-\int_V\int_0^t b_{ij}\mu^{(1)}(x,t-\xi)\frac{\partial Q(\varepsilon_{ij}^{(2)}(x,\xi))}{\partial\xi}d\xi dV\\&+\int_V\int_0^t b_i\mu^{(1)}(x,t-\xi)\frac{\partial Q(\phi_{,i}^{(2)}(x,\xi))}{\partial\xi}d\xi dV+\int_V\int_0^t b_i^f\mu^{(1)}(x,t-\xi)\frac{\partial Q(\phi{*}_{,i}^{(2)}(x,\xi))}{\partial\xi}d\xi dV\\&-\int_V\int_0^t a\mu^{(1)}(x,t-\xi)\frac{\partial Q(\theta^{(2)}(x,\xi))}{\partial\xi}d\xi dV-\int_V\int_0^t b_{ij}^f\mu^{(1)}(x,t-\xi)\frac{\partial Q(\varepsilon{*}^{(2)}(x,\xi))}{\partial\xi}d\xi dV=S_{21}^{12},\end{aligned} \tag{106}$$

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and

$$\int_A\int_0^t (c_0{}^{(1)}(x,t-\xi)\phi^{(2)}(x,\xi)+c_0{}^{*(1)}(x,t-\xi)\phi^{*(2)}(x,\xi))d\xi dA+\int_V\int_0^t e_{ijk}\,\phi_{,i}{}^{(1)}(x,t-\xi)\varepsilon_{jk}{}^{(2)}(x,\xi)d\xi dV$$
$$+\int_V\int_0^t \tau_i\,\phi_{,i}{}^{(1)}(x,t-\xi)\theta^{(2)}(x,\xi)d\xi dV+\int_V\int_0^t \tau_i^f\,\phi^*{}_{,i}{}^{(1)}(x,t-\xi)\theta^{(2)}(x,\xi)d\xi dV$$
$$+\int_V\int_0^t e_i{}^*\phi^*{}_{,i}{}^{(1)}(x,t-\xi)\varepsilon^{*(2)}(x,\xi)d\xi dV+\int_V\int_0^t \zeta_{ijk}\,\phi^*{}_{,i}{}^{(1)}(x,t-\xi)\varepsilon_{jk}{}^{(2)}(x,\xi)d\xi dV \tag{107}$$
$$+\int_V\int_0^t \zeta_i\,\phi_{,i}{}^{(1)}(x,t-\xi)\varepsilon^{*(2)}(x,\xi)d\xi dV=S_{21}^{12}.$$

Here S_{21}^{12} indicates the same expression as on the left-hand side except that the superscripts (1) and (2) are interchanged. Finally, Eq. (101) with the aid of Eq. (102) gives the general reciprocity theorem in the final form

$$\int_A\int_0^t h_i{}^{(1)}(x,t-\xi)\frac{\partial\wedge(\mathrm{u}_i{}^{(2)}(x,\xi))}{\partial\xi}d\xi dA+\int_A\int_0^t h_i{}^{*(1)}(x,t-\xi)\frac{\partial\wedge(\mathrm{u}_i{}^{*(2)}(x,\xi))}{\partial\xi}d\xi dA$$
$$+\int_V\int_0^t\left(\rho_1 F_i{}^{(1)}(x,t-\xi)\frac{\partial\wedge(\mathrm{u}_i{}^{(2)}(x,\xi))}{\partial\xi}+\rho_2 f_i{}^{(1)}(x,t-\xi)\frac{\partial\wedge(\mathrm{u}_i{}^{*(2)}(x,\xi))}{\partial\xi}\right)d\xi dV \tag{108}$$
$$+\int_A\int_0^t\left(c_0{}^{(1)}(x,t-\xi)\frac{\partial\wedge(\phi^{(2)}(x,\xi))}{\partial\xi}+c_0{}^{*(1)}(x,t-\xi)\frac{\partial\wedge(\phi^{*(2)}(x,\xi))}{\partial\xi}\right)d\xi dA$$
$$+\frac{1}{T_0}\int_A\int_0^t\left(q^{(1)}(x,t-\xi)Q(\eta^{(2)}(x,\xi))\right)d\xi dA+\int_A\int_0^t\left(p^{(1)}(x,t-\xi)B(\varsigma^{(2)}(x,\xi))\right)d\xi dA=S_{21}^{12}.$$

Particular Case: If porous piezoelectric effects are neglected, the results obtained are similar as obtained by Kumar and Kansal [44].

6 CONCLUSIONS

In this paper, the governing equations for porous piezothermoelastic model are presented in the context of Biot [1] theory of porousealsticity, thermoelastic theory with one relaxation time and thermoelastic theory with mass diffusion developed by Biot [2], Lord and Shulman [4] and Sherief et al. [5] respectively. The variational principle, reciprocity and uniqueness theorems are proved in the above proposed model. The results proved in the above model are verified from the known results.

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