Displacement Field Due to a Cylindrical Inclusion in a Thermoelastic Half-Space

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ABSTRACT

In this paper, the closed form analytical expressions for the displacement field due to a cylindrical inclusion in a thermoelastic half-space are obtained. These expressions are derived in the context of steady-state uncoupled thermoelasticity using thermoelastic displacement potential functions. The thermal displacement field is generated due to differences in the coefficients of linear thermal expansion between a subregion and the surrounding material. Further, comparison between displacement field in a half-space and in an infinite medium has been discussed. The variation of displacement field in a half-space and its comparison with an infinite medium is also shown graphically. © 2017 IAU, Arak Branch.All rights reserved.

Keywords: Displacement field; Thermoelastic half-space; Potential functions; Cylindrical inclusion, Uncoupled thermoelasticity.

1 INTRODUCTION

THERMOELASTICITY is the extension of theory of elasticity to include thermal effects. The theory of thermoelasticity is concerned with the interaction between thermal field and elastic bodies. The study has begun with Duhamel (1837) and Neumann (1885), who postulated the equations of linear thermoelasticity for isotropic bodies. Goodier [1] studied the static problem of uncoupled thermoelasticity by employing the method of superposition using displacement potential functions.

With the help of Goodier's method, Mindlin and Cheng [2] obtained the thermal stress in a semi-infinite solid with traction free surface for the centre of dilatation using Galerkin vector stress function. The result was then applied to the case of a spherical inclusion in the semi-infinite body. Yu and Sanday [3] derived the elastic solution for a centre of dilatation in a plate in terms of Galerkin vectors using method of images and applied it to the solution of thermally induced deformations and stresses. Hemayati and Karami [4] presented a formulation of two-dimensional boundary element method analysis for steady state, uncoupled thermoelastic problem. Nowinski [5] obtained the biharmonic solutions to the steady state thermoelastic problems in three dimensions. Wang and Huang [6] studied some thermoelastic problems in the half-space by using general solutions of the elasticity based on the Boussinesq solution.

Seremet et al. [7] has derived new Green's function and new Poisson's type integral formula for a boundary value problem in thermoelasticity for a half-space with mixed boundary conditions. Kedar et al. [8] obtained the expressions for the thermal stresses in a semi-infinite solid circular cylinder subjected to an arbitrary initial heat supply on the lower surface. The results were obtained in a series form in terms of Bessel's functions. Davies [9]

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derived the elastic field due to a non-uniform temperature or a coherently misfitting inclusion in a semi-infinite region from the corresponding field in an infinite region.

Boundary value problems in the uncoupled thermoelasticity have been discussed by several researchers (Sen [10]; Arpaci [11]; Rokne et al. [12] and Chao et al. [13] etc.). Researches in the field of thermoelastic state of structures and their elements are an urgent problem of solid mechanics. Thus the study of thermoelastic deformation has been of practical importance in a wide range of disciplines. The most important discipline is geosciences, where the static solution of thermoelastic deformation has been applied to geothermal fields, volcanoes and cracks.

In the present paper, we have studied the deformation of a thermoelastic half-space due to a cylindrical inclusion in context of steady-state uncoupled thermoelasticity as in Wang and Huang [6]. Following the method opted by Davies [9], we obtain the displacement field for an infinite region from which the corresponding fields can be derived in the semi-infinite region. The expressions for the displacement field are derived for the plane strain problem using thermoelastic displacement potential functions. These results are in good agreement with the results obtained by Wang and Huang [6] .The variation of displacement components are also shown graphically.

2   THEORY

In the linear theory of thermoelasticity, the total strain can be written as the sum of mechanical and thermal strains (Sadd [14]):

$$e_{ij} = e_{ij}^{(M)} + e_{ij}^{(T)} \tag{1}$$

in which for an isotropic material, the thermal strain takes the form $e_{ij}^{(T)} = \alpha T \delta_{ij}$,where α is the coefficient of linear thermal expansion, T is temperature difference, δ_{ij} is Kronecker delta. Then the generalized Hooke's law including the thermal effects for the plane strain problem can be written as,

$$\tau_{ij} = 2\mu e_{ij} + \frac{2\mu\sigma}{1-2\sigma} \delta_{ij} e_{kk} - \frac{2\mu(1+\sigma)}{1-2\sigma} \alpha \delta_{ij} T . \tag{2}$$

Substituting the above relation together with total strain-displacement relations in the equilibrium equations, without body force, one can write the Navier's equation as,

$$\nabla^2 \mathbf{u} + \frac{1}{1-2\sigma} \nabla(\nabla . \mathbf{u}) = \frac{2(1+\sigma)}{1-2\sigma} \alpha \nabla T , \tag{3}$$

where e_{ij} and τ_{ij} are the components of strain and stress tensor respectively; $\boldsymbol{u}$ is displacement vector; λ and μ are Lame's constants; σ is Poisson's ratio.

The uncoupled heat conduction equation for the steady state temperature field (T) with Q as heat supply and λ_0 as the thermal conductivity can be written as,

$$\nabla^2 T = \frac{-Q}{\lambda_0} . \tag{4}$$

The solution of inhomogeneous Eq. (3) can be expressed as,

$$\mathbf{u} = \mathbf{u}^c + \mathbf{u}^p , \tag{5}$$

where $\mathbf{u}^c$ is the complementary function satisfying the corresponding homogeneous Eq. (3) and $\mathbf{u}^p$ represents the particular solution of the displacement field generated by the temperature field (T).

According to Goodier's method (Timoshenko and Goodier [15]), the displacement $\mathbf{u}^{(\infty)}(\mathbf{r})$ for an infinite solid is given by $\mathbf{u}^{(\infty)} = \nabla\phi$, where the potential function ϕ satisfies the Poisson's equation:

$$\nabla^2\phi = \frac{1+\sigma}{1-\sigma}\alpha T(\mathbf{r}),$$ (6)

Then the function ϕ is obtained as,

$$\phi(\mathbf{r}) = \frac{-1}{4\pi}\frac{1+\sigma}{1-\sigma}\alpha\int\frac{T(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3(\mathbf{r}'),$$ (7)

where $|\mathbf{r}-\mathbf{r}'| = |(x,y,z)-(\xi,\eta,\varsigma)| = [(x-\xi)^2+(y-\eta)^2+(z-\varsigma)^2]^{1/2}$ is the distance between the points (x,y,z) and (ξ,η,ς).

Now according to Davies [9], the displacement within the semi-infinite region $z \geq 0$ with traction free surface in terms of displacement or strain components for an infinite region in Cartesian coordinates are given by:

$$\mathbf{u} = \mathbf{u}^{(\infty)} + (3-4\sigma)\overline{\mathbf{u}}^{(\infty)} + 2z\frac{\partial}{\partial z}(\overline{u}_x^{(\infty)}, \overline{u}_y^{(\infty)}, -\overline{u}_z^{(\infty)}),$$ (8)

or

$$\mathbf{u} = \mathbf{u}^{(\infty)} + (3-4\sigma)\overline{\mathbf{u}}^{(\infty)} - 2z(\overline{e}_{xz}^{(\infty)}, \overline{e}_{yz}^{(\infty)}, -\overline{e}_{zz}^{(\infty)}),$$ (9)

where bar represents that the sign of z is changed.

3 FORMULATION AND SOLUTION OF PROBLEM

We consider the plane strain problem of a cylindrical inclusion in the upper half-space ($x \geq 0$) having different coefficient of linear thermal expansion to that of the half-space but have same elastic constants as in Wang and Huang [6]. Due to this difference in the coefficients of linear thermal expansion between a sub region and its surrounding material, say η_0, the thermoelastic displacement field is generated. The axis of the cylinder is taken parallel to the surface of half-space and the center of axis is located on the line $x = h$ and $y = 0$. The radius of cylinder is 'a', where $h > a$ and the surface $x = 0$ is traction free surface as shown in Fig.1.
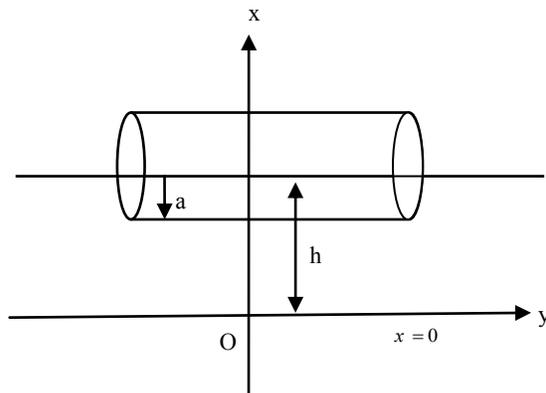
Fig.1
A cylindrical inclusion in a half-space.

Then according to Wang and Huang [6], the thermoelastic potential function ϕ satisfying the following Poisson's equation, when the temperature of the semi-infinite region increases T_0 is given by,

$$\nabla^2\phi = \frac{1+\sigma}{1-\sigma}\alpha T = \frac{1+\sigma}{1-\sigma}\eta_0 T_0 \qquad \text{for} \quad R_1 \leq a, \tag{10}$$

and

$$\nabla^2\phi = 0 \qquad \text{for} \quad R_1 > a, \tag{11}$$

where $R_1^2 = (x-h)^2 + y^2$ is the distance of the point (x,y) from $(h,0)$. Then, function ϕ for this problem is taken as (Wang and Huang [6]),

$$\phi = \frac{1}{4}KR_1^2 \qquad \text{for} \quad R_1 \leq a, \tag{12}$$

and

$$\phi = \frac{1}{2}Ka^2\left[\ln\frac{R_1}{a} + \frac{1}{2}\right] \qquad \text{for} \quad R_1 > a, \tag{13}$$

where

$$K = \frac{1+\sigma}{1-\sigma}\eta_0 T_0. \tag{14}$$

Now Eqs. (8) and (9) for the plane strain problem in xy-plane are written in the form:

$$\mathbf{u} = \mathbf{u}^{(\infty)} + (3-4\sigma)\bar{\mathbf{u}}^{(\infty)} - 2x\frac{\partial}{\partial x}(-\bar{u}_x^{(\infty)},\ \bar{u}_y^{(\infty)}), \tag{15}$$

or

$$\mathbf{u} = \mathbf{u}^{(\infty)} + (3-4\sigma)\bar{\mathbf{u}}^{(\infty)} - 2x\,(-\bar{e}_{xx}^{(\infty)},\ \bar{e}_{yx}^{(\infty)}). \tag{16}$$

The displacement field in an infinite region and that at the image point for exterior points $(R_1 > a)$ of the cylindrical inclusion is obtained from $\mathbf{u}^{(\infty)} = \nabla\phi$ on using Eq. (13),

$$\mathbf{u}^{(\infty)} = \frac{1}{2}Ka^2\frac{(x-h,\ y)}{R_1^2}, \tag{17}$$

$$\bar{\mathbf{u}}^{(\infty)} = \frac{1}{2}Ka^2\frac{(-x-h,\ y)}{R_2^2}, \tag{18}$$

where $(-h, 0)$ is the image of point $(h, 0)$ and $R_2^2 = (x+h)^2 + y^2$ is the distance of the point (x,y) from $(-h, 0)$.

The corresponding strain components in the infinite region and those at the image point are,

$$e_{xx}^{(\infty)} = \frac{1}{2} K a^2 \left[\frac{1}{R_1^2} - \frac{2(x-h)^2}{R_1^4} \right],$$ (19)

$$e_{yx}^{(\infty)} = \frac{1}{2} K a^2 \frac{(-2)(x-h)y}{R_1^4},$$ (20)

$$\bar{e}_{xx}^{(\infty)} = \frac{1}{2} K a^2 \left[\frac{1}{R_2^2} - \frac{2(x+h)^2}{R_2^4} \right],$$ (21)

$$\bar{e}_{yx}^{(\infty)} = \frac{1}{2} K a^2 \frac{2(x+h)y}{R_2^4}.$$ (22)

Substituting Eqs. (17)-(18) and (21)-(22) into (16), the displacement components in the thermoelastic half-space for exterior points $(R_1 > a)$ can be expressed as:

$$u_x = \frac{1}{2} K a^2 \left[\frac{x-h}{R_1^2} - \frac{(3-4\sigma)(x+h)}{R_2^2} + 2x \left\{ \frac{1}{R_2^2} - \frac{2(x+h)^2}{R_2^4} \right\} \right],$$ (23)

$$u_y = \frac{1}{2} K a^2 y \left[\frac{1}{R_1^2} + \frac{(3-4\sigma)}{R_2^2} - \frac{4x\,(x+h)}{R_2^4} \right].$$ (24)

Also for the interior points $(R_1 \le a)$,

$$\mathbf{u}_{\text{int}} = \mathbf{u}_{ext} + \frac{a^2 K \,\mathbf{R}_1}{2} \left(\frac{1}{a^2} - \frac{1}{R_1^2} \right).$$ (25)

Eq. (25) is in a similar form for a cylindrical inclusion as in Mindlin and Cheng [2] for the interior points $(R_1 \le a)$ of a spherical inclusion.

Substituting Eqs. (23) and (24) into (25), the displacement components in the thermoelastic half-space for interior points $(R_1 \le a)$ are obtained as under:

$$u_x = \frac{1}{2} K \left[x - h - (3-4\sigma)(x+h) \frac{a^2}{R_2^2} + 2x \frac{a^2}{R_2^2} - 4x\,(x+h)^2 \frac{a^2}{R_2^4} \right],$$ (26)

$$u_y = \frac{1}{2} K y \left[1 + (3-4\sigma) \frac{a^2}{R_2^2} - 4x\,(x+h) \frac{a^2}{R_2^4} \right].$$ (27)

The results obtained above are in good agreement to those of Wang and Huang [6] for a dissimilar cylinder in the interior of a thermoelastic half-space using Boussinesq solution. Now we consider some particular cases of the displacement field due to a cylindrical inclusion in the thermoelastic half-space.

3.1 Surface displacement at the free surface $x = 0$

Considering traction free boundary i.e. at $x = 0$, $\tau_{xx} = \tau_{xy} = 0$. This implies, we have $R_1^2 = R_2^2 = h^2 + y^2 = R^2$ (say), then from Eqs. (23), (24), (26) and (27), the surface displacements in a thermoelastic half-space is obtained as:

$$\mathbf{u}(0,y) = 2(1-\sigma)Ka^2 \frac{(-h,y)}{R^2} \quad \text{for} \quad (R > a),$$ (28)

$$\mathbf{u}(0,y) = \frac{1}{2}K\left[1 + (3-4\sigma)\frac{a^2}{R^2}\right](-h,y) \quad \text{for} \quad (R \le a).$$ (29)

3.2 Displacement field at point $(0,0)$

Further, the displacement components in the thermoelastic half-space at point $(0,0)$ for exterior points $(R_1 > a)$ are derived using Eq. (28),

$$u_x(0,0) = -2(1-\sigma)K\frac{a^2}{h},$$ (30)

$$u_y(0,0) = 0,$$ (31)

and using Eq.(29), for interior points $(R_1 \le a)$, we have

$$u_x(0,0) = \frac{-1}{2}K\left[h + (3-4\sigma)\frac{a^2}{h}\right],$$ (32)

$$u_y(0,0) = 0.$$ (33)

For a Poissonian half-space $\sigma = 0.25$, the Eqs. (30) and (32) can be reduced into the following form:

$$\frac{-3}{5a\eta_0 T_0}u_x(0,0) = \frac{3}{2}\left[\frac{1}{(h/a)}\right],$$ (34)

$$\frac{-3}{5a\eta_0 T_0}u_x(0,0) = \frac{1}{2}\left[\frac{h}{a} + \frac{2}{(h/a)}\right].$$ (35)

3.3 Displacement field at a boundary point $(h-a,0)$

From Eqs. (23) and (24) (or Eqs. (26) and (27)), we find that the displacement components in the thermoelastic half-space at a boundary point $(h-a,0)$ can be expressed as:

$$u_x(h-a,\,0) = \frac{-1}{2}Ka\left[1 + (3-4\sigma)\frac{a}{2h-a} + \frac{2a(h-a)}{(2h-a)^2}\right],$$ (36)

$$u_y(h-a,\,0) = 0.$$ (37)

For $\sigma = 0.25$, the Eq. (36) can be reduced into the equation given below:

$$\frac{-3}{5a\eta_0 T_0}u_x(h-a,\,0) = \frac{1}{2}\left[1 + \frac{2\left(3\frac{h}{a}-2\right)}{\left(2\frac{h}{a}-1\right)^2}\right].$$ (38)

4   COMPARISON OF THE DISPLACEMENT COMPONENTS IN A HALF-SPACE AND IN AN INFINITE MEDIUM

Substituting Eq. (17) into (25), the displacement in an infinite medium for interior points $(R_1 \leq a)$ can be written as:

$$\mathbf{u}^{(\infty)}(x,y) = \frac{1}{2} K (x - h, y). \tag{39}$$

Then Eqs. (17) and (39) at the free surface $x = 0$ can be written as:

$$\mathbf{u}^{(\infty)}(0,y) = \frac{1}{2} K a^2 \frac{(-h, y)}{R^2} \quad \text{for} \quad (R > a), \tag{40}$$

$$\mathbf{u}^{(\infty)}(0,y) = \frac{1}{2} K (-h, y) \quad \text{for} \quad (R \leq a). \tag{41}$$

From the equations given above, it is noted that the following relations hold between the displacement in the thermoelastic half-space and an infinite medium at the free surface $x = 0$,

$$\mathbf{u}(0,y) = 4(1 - \sigma) \mathbf{u}^{(\infty)}(0,y) \quad \text{for} \quad (R > a), \tag{42}$$

$$\mathbf{u}(0,y) = \left[1 + (3 - 4\sigma) \frac{a^2}{R^2} \right] \mathbf{u}^{(\infty)}(0,y) \quad \text{for} \quad (R \leq a). \tag{43}$$

Further from Eqs. (40) and (41), the displacement fields in the thermoelastic infinite medium at point $(0,0)$ for exterior points $(R_1 > a)$ and interior points $(R_1 \leq a)$ can be expressed as:

$$\mathbf{u}^{(\infty)}(0,0) = \left(\frac{-1}{2} \frac{Ka^2}{h}, 0 \right) \quad \text{for} \quad (R_1 > a), \tag{44}$$

$$\mathbf{u}^{(\infty)}(0,0) = \left(\frac{-1}{2} K h, 0 \right) \quad \text{for} \quad (R_1 \leq a). \tag{45}$$

For $\sigma = 0.25$, the Eqs. (44) and (45) can be reduced into the equations given below:

$$\frac{-3}{5 a \eta_0 T_0} u_x^{(\infty)}(0,0) = \frac{1}{2} \left[\frac{1}{(h/a)} \right], \tag{46}$$

$$\frac{-3}{5 a \eta_0 T_0} u_x^{(\infty)}(0,0) = \frac{1}{2} \left(\frac{h}{a} \right). \tag{47}$$

Now using Eq. (17) or Eq. (39), the displacement field in the thermoelastic infinite medium at a boundary point $(h - a, 0)$ can be expressed as:

$$\mathbf{u}^{(\infty)}(h - a, 0) = \left(\frac{-1}{2} K a, 0 \right). \tag{48}$$

For $\sigma = 0.25$, the Eq. (48) can be reduced into the equation given below:

$$\frac{-3}{5a\eta_0 T_0} u_x^{(\infty)}(h-a,\,0) = \frac{1}{2}.$$ (49)

5 NUMERICAL RESULTS AND DISCUSSION

In this section, the graphical representations of the displacement components at the points $(0,0)$ and $(h-a,0)$ for exterior points $(R_1 > a)$ and interior points $(R_1 \leq a)$ of a cylindrical inclusion are obtained by using MATLAB software programming. The numerical computations are carried out for $\sigma = 0.25$. Figs. 2 and 3 respectively, show the variation of displacement normal to the surface of a thermoelastic half-space at point (0,0) for exterior points and interior points. From Fig. 2, it can be seen that displacement decreases rapidly with increasing value of distance h/a and it vanishes for infinitely large values of h/a. From Fig. 3, we observe that the displacement first decreases slowly when $1 \leq h/a \leq \sqrt{2}$ and attains a minimum value $\sqrt{2}$ at $h/a = \sqrt{2}$. It then increases rapidly with the increasing values of h/a and becomes infinitely large for infinitely large values of h/a. Fig. 4 shows the variation of displacement normal to the surface of a thermoelastic half-space at a boundary point $(h-a,\,0)$. From this figure, we observe that displacement decreases less rapidly with the increasing value of distance h/a and it approaches the value 0.5 for infinitely large values of h/a. Fig. 5 shows the comparison of displacements normal to the surface of a thermoelastic half-space at points (0,0) and $(h-a,\,0)$ for exterior points.
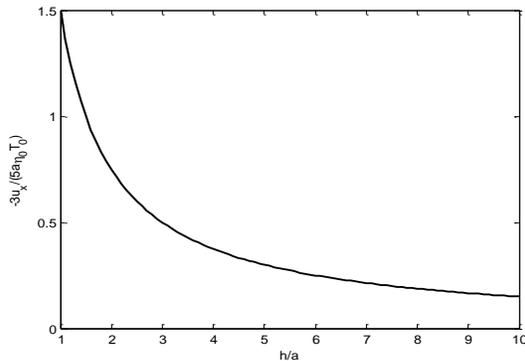
Fig.2
Displacement normal to surface of a thermoelastic half-space at point (0,0) for exterior points.
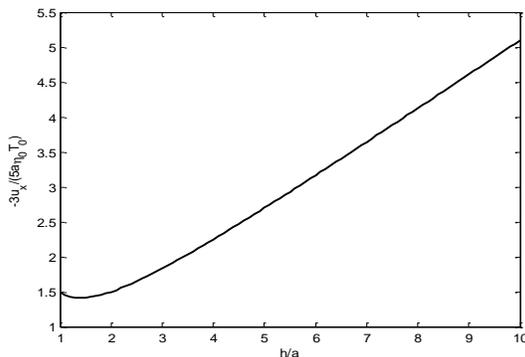
Fig.3
Displacement normal to surface of a thermoelastic half-space at point (0,0) for interior points.
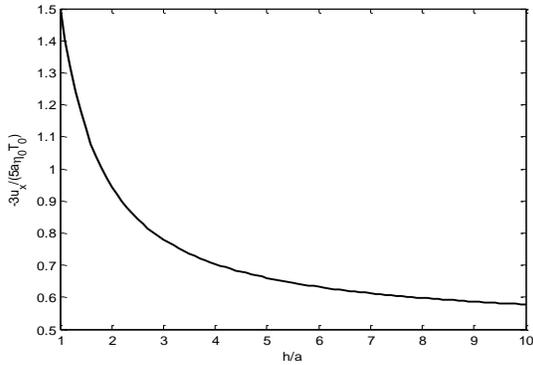
Fig.4
Displacement normal to surface of a thermoelastic half-space at boundary point $(h-a, 0)$.
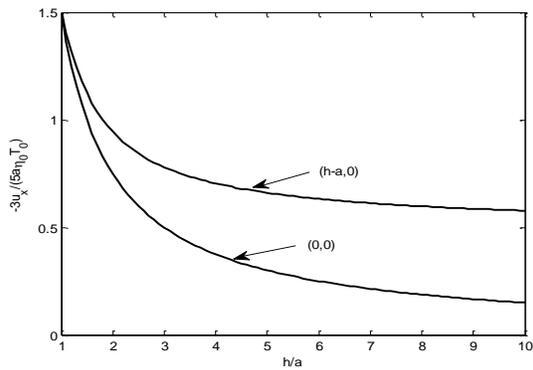
Fig.5
Comparison of displacement components normal to surface of a thermoelastic half-space at points (0, 0) and $(h-a, 0)$ for exterior points.

In Figs. 6-7 respectively, comparison of displacements normal to the surface of a thermoelastic half-space and an infinite medium at point (0,0) for exterior points and interior points are presented. From Fig. 6, we conclude that the displacement in the thermoelastic half-space is three times to that of the infinite medium. Also both the displacements approach the value zero for infinitely large values of h/a. From Fig.7, it is obvious that displacement in the infinite medium increases linearly with the increasing values of h/a and becomes infinite for infinitely large values of h/a. Fig. 8 shows the comparison of displacements normal to the surface of a thermoelastic half-space and an infinite medium at a boundary point $(h-a, 0)$. From here, it is found that the displacement in the infinite medium always remains the same with value 0.5 for all values of h/a and this value is approached by the thermoelastic half-space for infinitely large values of h/a.
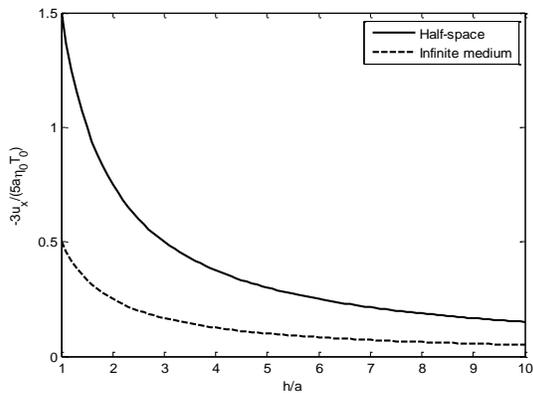
Fig.6
Comparison of displacement components normal to surface of a thermoelastic half-space and an infinite medium at point (0,0) for exterior points.
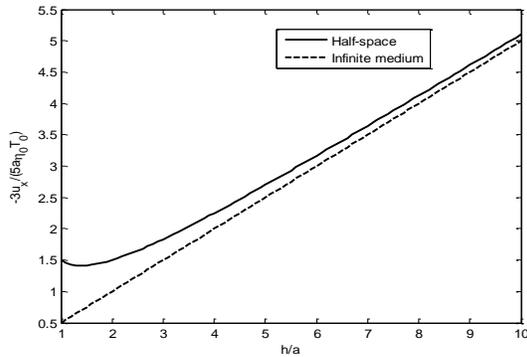
Fig.7
Comparison of displacement components normal to surface of a thermoelastic half-space and an infinite medium at point (0,0) for interior points.
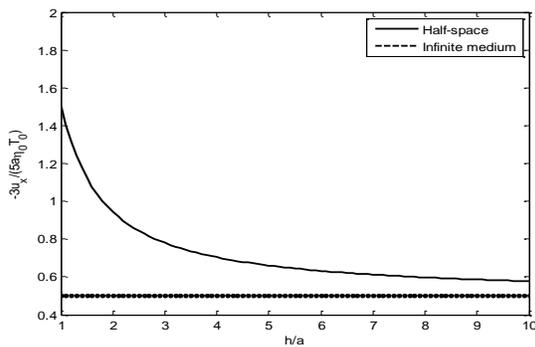
Fig.8
Comparison of displacement components normal to surface of a thermoelastic half-space and an infinite medium at boundary point $(h-a, 0)$.

6   CONCLUSIONS

In this paper, we have studied the plane strain deformation of a thermoelastic half-space due to a cylindrical inclusion by using thermoelastic displacement potential functions. A complete solution of displacement field due to difference in the coefficients of linear thermal expansion between a sub region and its surrounding material is obtained. The results thus obtained are in good agreement to the results obtained by Wang and Huang [6] using Boussinesq solution. Further, comparison between displacement field in a half-space and in an infinite medium has been discussed. The numerical results are also presented in graphical form.

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