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Analysis of Five Parameter Viscoelastic Model Under Dynamic Loading

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ABSTRACT

The purpose of this paper is to analysis the viscoelastic models under dynamic loading. A five-parameter model is chosen for study exhibits elastic, viscous, and retarded elastic response to shearing stress. The viscoelastic specimen is chosen which closely approximates the actual behavior of a polymer. The module of elasticity and viscosity coefficients are assumed to be space dependent i.e. functions of $'x'$ in non-homogeneous case and stress-strain are harmonic functions of time $'t'$ The expression for relaxation time for five parameter viscoelastic model is obtained by using constitutive equation. The dispersion equation is obtained by using Ray techniques. The model is justified with the help of cyclic loading for maxima or minima.

Keywords: Shear waves; Viscoelastic media; Asymptotic method; Dynamic loading

1 INTRODUCTION

THE viscoelasticity theory is used in the field of solid mechanics, seismology, exploration geophysics, acoustics and engineering. The solutions of many problems related with wave-propagation for homogeneous media are available in many literatures of continuum mechanics of solids. However, in the recent years, the interest has arisen to solve the problems connected with non-homogeneous bodies. These problems are useful to understand the properties of polymeric materials and industrial related applications. The vibrations in earthquakes are due to differences in dynamic characteristics therefore the cyclic stress-strain behavior of material play a vital role for reliable prediction of the seismic response. Many researchers studied structural pounding during earthquakes. The lack of information concerns multi-dimensional waves in viscoelastic-media, and in particular for non-homogenous media, therefore, a formal study of non-homogeneous viscoelastic models under dynamic loading is presented.

Modeling and model parameter estimation is of great importance for a correct prediction of the foundation behavior. Many researchers like Alfrey [1], Barberan [2], Achenbach [3], Bhattacharya [4], and Acharya [5] formulated and developed this theory. Further, Bert [6], Abd-Alla [7], Batra [8] successfully applied this theory to wave-propagation in homogeneous, elastic media. Murayama [9] and Schiffman et al. [10] have proposed higher order viscoelastic models of five and seven parameters to represent the soil behavior. Jankowski [11] discussed the linear viscoelastic model and the nonlinear viscoelastic model. Anagnostopoulos [12] studied the linear viscoelastic model of collision to simulate structural pounding. Jankowski et al. [13] studied the pounding of superstructure segments in bridges with the help of linear viscoelastic model. Muthukumar and DesRoches [14] made a comparative study using two single degree of freedom (SDOF) systems for capturing pounding. Westermo [15] suggested linking buildings with beams which can transmit the forces between them eliminating dynamic contacts.

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In this paper, the module of elasticity and viscosity coefficients are assumed to be space dependent i.e. functions of $'x'$. Further, shearing strain and stress are taken as harmonic functions of time $'t'$ i.e. $\gamma = \gamma_0 e^{i\omega t}$ and $\sigma = G_0\gamma = G_0\gamma e^{i\omega t} = \sigma_0 e^{i\omega t}$. The expression for relaxation time for five parameter viscoelastic model is obtained by using constitutive equation. The dispersion equation is obtained by using Ray techniques. The model is justified with the help of cyclic loading for maxima or minima.

2 RESEARCH METHODOLOGY

The assumptions chosen are such that the conclusions drawn on the basis of which, agree quite reasonably and closely with the observed results of experimental tests. Following are the principal assumptions and hypothesis on which the problem has been constructed.

I. Homogeneity: - The material of a structure to be considered should be homogeneous in structure and continuous at all points of the body. A homogeneous structure means that any how so ever small particle/portion of the body under consideration must possess the same properties. Among the materials that are considered to be homogeneous are metals, alloys, such as steel, aluminum, copper etc.
II. Absolutely Elastic: - The bodies considered being absolutely elastic with respect to deformation, when their deformations which appear due to external force, completely disappear upon removal of the load. Actually this holds true up to a definite value of load.
III. Isotropy: - Material considered is taken to be isotropic, when it possesses the same characteristic in all directions. Isotropic materials include metals, concrete and some plastics. Materials possessing different properties in various directions are called an isotropic. Examples are wood, reinforced plastic etc.
IV. Infinite small Deformations: - When deformations of elastic bodies, under the action of external loads, are small as compared with the dimensions of the bodies, i.e. the dimensions/shape are not changed substantially on elastic deformations. This assumption simplifies substantially the calculation, since it makes possible to neglect changes in the arrangement of the forces on deformation.
V. Super-position-principle: - Since the deformation considered to be small, it can be assumed that external forces act independently from one-another, i.e. the deformations and internal forces appearing inelastic bodies do not depend on the order in which the external forces are applied. Besides, it is assumed that the total effect of the whole system of forces acting on the body is the sum of the effects produced by individual forces.

3 ABOUT THE MODEL

It is a five parameter model with two springs $S_1(G_1), S_2(G_2)$ with module of elasticity G_1, G_2 and three dash pots $D_2(\eta_2), D_2^{/}(\eta_2^{/}), D_3(\eta_3)$ with viscosities η_2, $\eta_2^{/}$ and η_3. It has three sections. Section I, Contains one spring $S_1(G_1)$, section II contains three elements one spring $S_2(G_2)$ and two dash pots $D_2(\eta_2), D_2^{/}(\eta_2^{/})$ where the spring $S_2(G_2)$ and dash-pot $D_2(\eta_2)$ are in series forming Maxwell-model and the dash pot $D_2^{/}(\eta_2^{/})$ is parallel to the Maxwell element. The section III contains only one dash-pot $D_3(\eta_3)$. The spring represents recoverable elastic response and dash pot represents elements in structure giving rise to the viscous drag/ dissipative response (where the viscosity of the oil/fluid in the dash–pot decreases with the increase in temperature).

Section I has been represented by only one spring $S_1(G_1)$ represents the elastic region (glassy), which is dominant at low temperatures. In this range of behavior of the material, an applied stress (load) produces a strain, which is reversible upon the release of the stress under the elastic limits (instantaneous deformation). In case of polymer materials, the strain is due to the stretching of bonds within and between molecular chains. The chains, which are frozen to-gather initially, cannot flow past each other and may only be separated by fracture, which in our case does not happen as we are considering small-deformations. Thus, the spring $S_1(G_1)$ represents the behavior of polymer in glass region.

Section II represents leathery and rubbery region. In the leathery region, the modulus of elasticity drops rapidly with load (temperature) and reversible, sliding becomes possible in short segments of the chains of macromolecules.

Small sections move and then cause the neighboring sections to move co-operatively. Here a transition appears between the elastic behaviors of section I and viscoelastic behavior of section II. In section II, the dash-pot $D_2^{/}\left(\eta_2^{/}\right)$ is free to flow as is not restricted by any spring so the model exhibits long term viscous flow. The viscous-element $D_2^{/}\left(\eta_2^{/}\right)$, the Maxwell-element $D_2\left(\eta_2\right), S_2(G_2)$ possesses the property of long term viscous flow. During relaxation the dash-pot $D_2^{/}\left(\eta_2^{/}\right)$, which is free from the restrictions of a spring will eventually take up the whole extension and stress will drop to zero slowly and ultimately. The reversibility of the movements of the short chain segments is expressed by the spring $S_2(G_2)$ in section II and the resistance to this movement by the dash-pots $D_2\left(\eta_2\right)$ and $D_2^{/}\left(\eta_2^{/}\right)$. In the rubbery phase, the viscoelastic behavior in section-II dominates the deformation. As the load increases, the molecular segments slide reversibly past one another and tend to straighten out in the direction of the load.

Section III is represented by a single dashpot $D_3\left(\eta_3\right)$. At the higher loading, the viscosity decreases, due to internal fractions, which give rise to temperature increase and apparent modulus also drops, even to such an extent the material behaves as fluid as in the case of glaciers or melts, gels etc.

Middle section II, which is a series combination with a spring $S_1(G_1)$ of section I and a dash-pot $D_3\left(\eta_3\right)$ section III can be generated from Voigt Model by adding one more dash-pot to the spring side, so that it becomes a Maxwell-model or it can be degenerated from a parallel combination of two Maxwell elements by detaching one spring from one of the Maxwell element i.e. taking the modulus of elasticity in this Maxwell element as infinitely greater i.e. $G \to \infty$. It is further added that the combination / network of elastic elements (Spring $S(G)$) and viscous element (dashpots $D\left(\eta\right)$), Maxwell-model, Voigt-model is unidirectional i.e. all the elements lie in the same direction and all concerned forces and deformations act in this direction and are in the same plane.

4 CONSTITUTIVE RELATION FOR FIVE PARAMETER MODEL

The five parameter model consists of two springs $S_1(G_1), S_2(G_2)$ where G_1 and G_2 are the modulli of elasticity associated to them and three dash-pots $D_2\left(\eta_2\right), D_2^{/}\left(\eta_2^{/}\right), D_3\left(\eta_3\right)$ where η_2, $\eta_2^{/}$ and η_3 are the Newtonian viscosity coefficients associates to these elements. The module of elasticity and viscosity coefficients are assumed to be space dependent i.e. functions of $'x'$ in inhomogeneous case taken into consideration. Unidirectional problem is formed by taking the material in the form of filament of non-homogeneous viscoelastic material by taking one end at x = 0. The co-ordinate x is measured positive in the direction of the axis of the filament. Time is specified by t, and σ, γ and u respectively specify the only non-zero components of stress, shearing strain and particle displacement. The model has be divided into three sections, I, II, III. In Fig.1, the section I, section II and section III has one spring $S_1(G_1)$, two dash-pots $D_2\left(\eta_2\right), D_2^{/}\left(\eta_2^{/}\right)$ one spring $S_2(G_2)$ and one dash-pot $D_3\left(\eta_3\right)$ respectively.

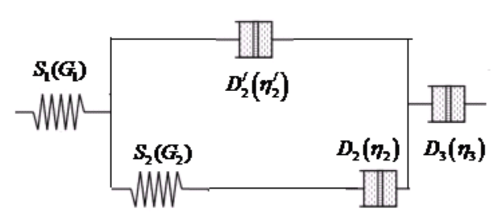

Fig. 1
Five parameter viscoelastic model.

Under the supper- supposition principle strains are added in the case of series connections and stresses are added when they are in parallel. Now if $\gamma_1, \gamma_2, \gamma_3$ be the three shearing strains elongations in respective sections connected in series, then total elongation is $\gamma = \gamma_1 + \gamma_2 + \gamma_3$. The total stress in the network remains the same. In each section but in the case of section II which is sub-divided into two sections is added i.e. $\sigma = \sigma_1 + \sigma_2$, where σ_1 and σ_2 are the stresses in the sub-sections. Relation for stress and strain for $D_2^{/}\left(\eta_2^{/}\right)$ for section II (represented by single dash-pot) is:

$$\sigma_1 = \eta_2^{/}\gamma_2 \tag{1}$$

Since the sub-section II is represented by a Maxwell- element, then the relation is expressed as:

$$\left(D\left(G_2\right) + \frac{1}{\eta_2}\right)\sigma_2 = \left(D\left(\gamma_2\right)\right) \tag{2}$$

Since, $\sigma = \sigma_1 + \sigma_2$ for Section II, therefore:

$$\left(D\left(G_2\right) + \frac{1}{\eta_2}\right) = \sigma\left(1 + (\ \eta'_2\left(\frac{D}{G_2} + \frac{1}{\eta_2}\right)\right)\ \sigma = \eta'_2\left[D^2 + G_2\left(\frac{1}{\eta'_2} + \frac{1}{\eta_2}\right)D\right]\gamma_2 \tag{3}$$

For section I, for the spring $S_1(G_1)$, the stress-strain relation is given by:

$$\sigma = G_1\gamma_1 \tag{4}$$

For Section III; for the dash-pot $D_3\left(\eta_3\right)$, the stress –strain relation is given by:

$$\sigma = \eta_3\dot{\gamma}_3 \tag{5}$$

The Stress-strain relation for the model representing the viscoelastic body for total stress (σ) and strain$\left(\gamma\right)$ can be obtained from Eq. (3), Eq. (4) and Eq. (5) as:

$$\left[D^2 + \left\{\frac{G_1}{\eta'_2} + \frac{G_1}{\eta_3} + \frac{G_2}{\eta'_2} + \frac{G_2}{\eta_2}\right\}D + \left\{\frac{G_1}{\eta'_2}\frac{G_2}{\eta_2} + \frac{G_1}{\eta_3}\left(\frac{G_2}{\eta'_2} + \frac{G_2}{\eta_2}\right)\right\}\right]\sigma = G_1\left\{D^2 + \left(\frac{G_2}{\eta'_2} + \frac{G_2}{\eta_2}\right)D\right\}\gamma \tag{6}$$

Now we take

$$\tau_{ij}^{-1} = \theta_{ij} = \frac{G_i}{\eta_j} = \frac{S_i(G_i)}{Dj(\eta_j)} \tag{7}$$

where $S_i(G_i)$ elastic modulus of spring and $Dj(\eta_j)$ = viscosity of dash-pot, $\tau_{ij} = \dfrac{\eta_j}{G_i}, (i = 1,2\ ;\ j = 2', 2, 3)$ Using, Eq. (6) and Eq. (7), we get

$$\begin{bmatrix} D^2 + \left\{\left(\theta'_{12} + \theta_{13}\right) + \left(\theta_{22} + \theta'_{22}\right)\right\}D \\ + \left(\theta'_{12}.\theta_{22} + \theta_{13}\left(\theta_{22} + \theta'_{22}\right)\right) \end{bmatrix}\sigma = G_1\left\{D^2 + \left(\theta_{22} + \theta'_{22}\right)D\right\}\gamma \tag{8}$$

Put $R_1 = \theta'_{12} + \theta_{13}$, $R_2 = \theta_{22} + \theta'_{22}$, $R_3 = \theta'_{12}.\theta_{22}$ in Eq. (8), we get

$$\left[D^2+\left(R_1+R_2\right)D+\left(R_3+\theta_{13}R_2\right)\right]\sigma=G_1\left\{D^2+R_2\,D\right\}\gamma=\left(G_1D^2+G_1\;R_2\,D\right)\gamma \tag{9}$$

The Eq. (9) can be written in terms of differential operator form as:

$$\sum_{n=1}^{2}\alpha_n b^n\gamma\left(x,t\right)=\sum_{m=0}^{2}\beta_m D^m\sigma\left(x,t\right) \tag{10}$$

where the order m and n of sums on right hand side and left hand side in the Eq. (11) depends upon the structure of the mechanical model representing the viscoelastic body. α_n and β_m are the combinations of the material constants and $\alpha_2=G_1$, $\alpha_1=G_1R_2$, $\beta_2=1$, $\beta_1=R_1+R_2$, $\beta_0=R_3+\theta_{13}R_2$, $D\equiv\dfrac{d}{dt}$.

Eq. (10) is the required differential operator form of constitutive relation for the model for viscoelastic material to be studied.

5 GOVERNING EQUATIONS FOR VISCOELASTIC MODEL

One of the governing equation for the viscoelastic model is constitutive relation and is [16]

$$f\left(\sigma,\sigma_{,t,}\,\sigma_{,tt},\;\gamma,\gamma_{,t}\right)=0 \tag{11}$$

$$\beta_2\sigma_{,tt}+\beta_1\sigma_{,t}+\beta_0\sigma=\alpha_2\,\gamma_{,tt}+\alpha_1\gamma_{,t} \tag{12}$$

The equation of motion is:

$$\sigma_{,x}=\rho u_{,tt}$$
or
$$\left(\frac{1}{\rho}\sigma_{,x}\right)_{,x}=\frac{1}{\rho}\left[\sigma_{,xx}-\left(log\,\rho\right)_{,x}\sigma_{,x}\right]=u_{,tt} \tag{13}$$

The displacement-strain relation is:

$$\gamma=u_{,x} \tag{14}$$

The shearing stress field is:

$$\beta_2\sigma_{,ttt}+\beta_1\sigma_{,tt}+\beta_0\sigma_{,t}=\frac{\alpha_2}{\rho}\left\{\sigma_{,xxt}-\left(log\,\rho\right)_{,x}\sigma_{,xt}\right\}+\frac{\alpha_1}{\rho}\left\{\sigma_{,xx}-\left(log\,\rho\right)_{,x}\sigma_{,x}\right\} \tag{15}$$

6 SOLUTION FOR FIVE PARAMETER VISCOELASTIC MODEL

We assume that the solution σ (*x, t)* of Eq. (15) may be represented by the series

$$\sigma\left(x,t\right)=\sum_{n=0}^{\infty}A_nF_n\left(t-h\left(x\right)\right)=\sum_{n=0}^{\infty}A_n\,F_n\left(x\right)\;,\;\;x=t-h\left(x\right),A_n=0\;,n<0 \tag{16}$$

with $F'_n=F_{n-1}$, $F_{n,x}=-h_{,x}\,F_{n-1}$, $F_{n,t}=F_{n-1}$, $\sigma=A_nF_n$, $\sigma_{,t}=A_nF_{n-1,}\sigma_{,tt}=A_nF_{n-2,}\sigma_{,ttt}=A_nF_{n-3,}$

The various derivatives stress with respect to x and t are:

$$\begin{aligned}
&\sigma_{,x} = A'_n F_n - h_{,x} A_n F_{n-1}, \\
&\sigma_{,xx} = A''_n F_n - \left(2h_{,x} A'_n + h_{,xx} A_n\right) F_{n-1} + A_n h_{,x}^2 F_{n-2} \\
&\sigma_{,xxt} = A''_n F_{n-1} - \left(2h_{,x} A'_n + h_{,xx} A_n\right) F_{n-2} + A_n h_{,x}^2 F_{n-3} \\
&\sigma_{,xt} = A'_n F_{n-1} - h_{,x} A_n F_{n-2},
\end{aligned} \tag{17}$$

From Eq. (16) and Eq. (17)

$$\beta_0 A_n F_{n-1} + \beta_1 A_n F_{n-2} + \beta_2 A_n F_{n-3} = \frac{\alpha_1}{\rho}\left\{A''_n - (\log\rho),x\, A'_n\right\} F_n + \begin{bmatrix} \dfrac{\alpha_1}{\rho}\begin{Bmatrix} h_{,x}(\log\rho)_{,x} A_n \\ -\begin{pmatrix} 2h_{,x} A'_n \\ +h_{,xx} A_n \end{pmatrix} \end{Bmatrix} \\ +\dfrac{\alpha_2}{\rho}\begin{Bmatrix} A''_n \\ -(\log\rho)_{,x} A'_n \end{Bmatrix} \end{bmatrix} F_{n-1} + \begin{bmatrix} \dfrac{\alpha_1}{\rho} h_{,x}^2 A_n \\ +\dfrac{\alpha_2}{\rho}\begin{Bmatrix} (\log\rho)_{,x} h_{,x} A_n \\ -\begin{pmatrix} 2h_{,x} A'_n \\ +h_{,xx} A_n \end{pmatrix} \end{Bmatrix} \end{bmatrix} F_{n-2} + \frac{\alpha_2}{\rho} A_n h_{,x}^2 F_{n-3} \tag{18}$$

Comparing the coefficient of F_n , we get

$$\begin{aligned}
&\frac{\alpha_1}{\rho}\left\{A''_n - (\log\rho), x\ A'_n\right\} = 0 \\
&A''_n = (\log\rho)_{,x} A'_n
\end{aligned} \tag{19}$$

Comparing the coefficient of F_{n-1}, we get

$$\beta_0 A_n = \left[\frac{\alpha_1}{\rho}\left\{h_{,x}(\log\rho)_{,x} A_n - \left(2h_{,x} A'_n + h_{,xx} A_n\right)\right\} + \frac{\alpha_2}{\rho}\left\{A''_n - (\log\rho)_{,x} A'_n\right\}\right] \tag{20}$$

Comparing the coefficient of F_{n-2}, we get

$$\beta_1 A_n = \frac{\alpha_1}{\rho} h_{,x}^2 A_n + \frac{\alpha_2}{\rho}\left\{(\log\rho)_{,x} h_{,x} A_n - \left(2h_{,x} A'_n + h_{,xx} A_n\right)\right\} \tag{21}$$

Comparing the coefficient of F_{n-3}, we get

$$\beta_2 A_n = \frac{\alpha_2}{\rho} A_n h_{,xx}^2 \tag{22}$$

Let, $\beta_2 = 1$ and $h_{,xx}^2 = \dfrac{\rho}{G_1}$ then Eq. (22) reduces to:

$$h_{,x}^2 = \frac{\rho}{\alpha_2} = \frac{\rho}{G_1} \tag{23}$$

From Eq. (18) and Eq. (21), we get

$$A_n \beta_1 = \frac{\alpha_1}{\rho}\left\{h_{,x}(\log\rho)_{,x} A_n - \left(2h_{,x} A'_n + h_{,xx} A_n\right)\right\} \tag{24}$$

From Eq. (21) and Eq. (24)

$$\left(\beta_1 - \frac{\alpha_1}{\rho} h_{,x}^2\right)\alpha_1 = \beta_1\alpha_2$$
$$\alpha_1\beta_1 - \beta_1\alpha_2 = \frac{\alpha_1}{\rho} h_{,x}^2$$
$$\rho\left(\frac{\beta_1}{\alpha_1} - \frac{\alpha_2}{\alpha_1}.\frac{\beta_1}{\alpha_1}\right) = h_{,x}^2 = \frac{\rho}{G_1} \tag{25}$$
$$\frac{\beta_1}{\alpha_1} - \frac{\alpha_2}{\alpha_1}.\frac{\beta_1}{\alpha_1} = \frac{1}{G_1}$$

From Eq. (9), we get

$$\beta_0 A_n = \frac{\alpha_1}{\rho}\ \mathrm{S} \tag{26}$$

where, $S = h_{,x}(log\rho)_{,x} A_n - \left(2h_{,x} A'_n + h_{,xx} A_n\right)$.

From Eq. (12), we get

$$\beta_1 A_n - \frac{\alpha_1}{\rho} h_{,x}^2 A_n = \frac{\alpha_2}{\rho} S \tag{27}$$

From Eq. (26) and Eq. (27)

$$\beta_0 A_n \alpha_2 = \frac{\alpha_1\alpha_2}{\rho}\ S \ \text{ and } \left(\beta_1 A_n - \frac{\alpha_1}{\rho} h_{,x}^2 A_n\right)\alpha_1 = \frac{\alpha_1\alpha_2}{\rho}\ S$$

Taking, $\beta_2 = 1, \alpha_2 = G_1\ , \alpha_1 = G_1 R_2$, $\beta_1 = R_1 + R_2$ and $R_3 + \theta_{13} R_2 = \beta_0$

$$\beta_0\ \ \alpha_2 = \beta_1\ \ \alpha_1 - \frac{\alpha_1^2}{G_1} \ \text{ or } \ \ ((R_3 + \theta_{13} R_2\ )G_1 = G_1.R_2\ (R_1 + R_2) - \frac{G_1^2}{G_1} R_2^2 \ \text{ and finally we get,}$$

$$R_1 R_2 = R_3 + \theta_{13} R_2 \tag{28}$$

Eq. (28) is the expression for relaxation time for five parameter viscoelastic model.

7 DYNAMIC LOADING

The time parameter $'t'$ is introduced into an experimental scheme in dynamic experiments by cyclic deformation of the specimen, frequency ω of the oscillations plays the role of the time factor. The cyclic deformation is the fundamental process of determining the mentioned characteristics. The greatest preference is given to harmonic oscillations. A Harmonic action of the stress/strain produces a corresponding harmonic response in the strain/stress. Let us consider that shearing strain (γ), induced in elastic body which can be expressed by a harmonic action as:

$$\gamma = \gamma_0 e^{iwt} \tag{29}$$

where γ_0 is the amplitude, ω is the frequency of oscillations and $'t'$ is the time. According to Hooke's law, stress $'\sigma'$ is

$$\sigma = \sigma_0 e^{iwt} \tag{30}$$

where, $\sigma_0 = G_0\gamma_0$, at t=0

For an elastic body, the strain and stress vary harmonically and there is no lack in the harmonic motion in phase as both have e^{iwt} as a factor. Thus an elastic body responds instantaneously to the external action (strain/stress). The phase shift angle between strain and stress is zero. For an ideal viscous body, the Newton's law of flow to a fluid body is as:

$$\sigma = \eta_0\gamma_0 i\omega e^{iwt} \tag{31}$$

where, η_0 is the viscosity of the body and $\eta_0\gamma_0\omega = \sigma_0$ at t=0

Thus, for a viscous deformation, stress advances by the strain by a phase angle $\frac{\pi}{2}$. Thus the phase shift angle for the stress-strain under periodic harmonic deformation for elastic body is $\frac{\pi}{2}$, also for the viscous body, it is $\frac{\pi}{2}$. Therefore the phase shift angle δ' for the viscoelastic body must be between zero and $\frac{\pi}{2}$ i.e. $0 < \delta < \frac{\pi}{2}$. The lagging in phase of the strain behind the stress is due to the presence of relaxation processes in the case of viscoelastic body, as phase shift angle δ, is given by $0 < \delta < \frac{\pi}{2}$. Hence,

$$\sigma = \sigma_0 e^{i(wt+\delta)} \tag{32}$$

If we represent the projection of the stress vector on axis of co-ordinates by taking $\sigma' \leftrightarrow x$ and $\sigma'' \leftrightarrow y$, where σ' and σ'' represent that real and imaginary parts respectively. If the strain is initially set harmonically. Then the modulus of viscoelastic body with harmonic loading can be written as:

$$\frac{\sigma^*}{\gamma^*} = \frac{\sigma^*}{\gamma'} = \frac{\sigma'}{\gamma'} + i\frac{\sigma''}{\gamma'} = G' + iG'' = G^* \tag{33}$$

The phase angle δ is given as , tan $\delta = \frac{G^*}{G'}$

In the case of present model (Five-Parameter; two springs $S_1(G_1), S_2(G_2)$; three dash-pots $D_2(\eta_2), D_2^/(\eta_2^/), D_3(\eta_3)$) which represents a linear viscoelastic behavior under a given action of loading, the stress is directly proportional to strain. This is also true for time dependent stress and strain relation i.e. for viscoelastic body, the stress is:

$$\sigma(t) = G^*\gamma_0 e^{iwt} \tag{34}$$

where $\gamma = \gamma_0 e^{iwt}$. Using, the relation $\sigma = G\gamma$ for an elastic body, the constitutive relation for the physical state representing the five parameter model is given by:

$$D^2\sigma + (\alpha_1+\alpha_2)D\sigma + (\alpha_3+\alpha_4)\sigma = G_1\left(D^2 + \alpha_1 D\right)\gamma \tag{35}$$

where $\alpha_1 = \theta_{13} + \theta'_{12}$, $\alpha_2 = \theta_{22} + \theta'_{22}$, $\alpha_3 = \alpha_2\theta_{13}$, $\alpha_4 = \theta'_{12}\,\theta_{22}$.

Let

$$G^* = G^{'} + iG^{"} \tag{36}$$

Using Eq. (36), Eq. (37) and Eq. (38), we get

$$\left\{-\omega^2 + \left(\alpha_1 + \alpha_2\right) i\omega + (\alpha_3 + \alpha_4)\right\} G^* \gamma_0 e^{iwt} = G_1 \left(-\omega^2 + \alpha_2 \omega i\right) \gamma_0 e^{i} \tag{37}$$

On solving, we get

$$G^*(i\omega) = \frac{G_1\left(-\omega^2 + \alpha_2\omega i\right)}{\left(\left(\alpha_3 + \alpha_4 - \omega^2\right) + \left(\alpha_1 + \alpha_2\right) i\omega\right)}$$
$$G^*(i\omega) = \frac{G_1\left(-\omega^2 + \alpha_2\omega i\right)\left[\left(\left(\alpha_3 + \alpha_4 - \omega^2\right) - i\left(\alpha_1 + \alpha_2\right) i\omega\right)\right.}{A_1} \tag{38}$$

$$G^{'} + iG^{"} = G^{"}\left(i\omega\right) = \frac{G_1\left[\left\{\alpha_2\left(\alpha_1 + \alpha_2\right) - \left(\alpha_3 + \alpha_4\right) + \omega^2\right\}\omega^2 + \left\{\alpha_2\left(\alpha_3 + \alpha_4\right) + \alpha_1\omega^2\right\} i\right.}{A_1} \tag{39}$$

Separate Eq. (41) into real and imaginary parts, we get

$$G' = \frac{G_1\left[\left\{\alpha_2\left(\alpha_1 + \alpha_2\right) - \left(\alpha_3 + \alpha_4\right) + \omega^2\right\}\omega^2\right]}{A_1}$$
$$G" = \frac{G_1\left[\left\{\alpha_2\left(\alpha_3 + \alpha_4\right)\omega + \alpha_1\omega^2\right\}\omega\right]}{A_1} \tag{40}$$

and loss tangent is given by:

tan

$$\delta = \frac{G"}{G'} = \frac{\alpha_2\left(\alpha_3 + \alpha_4\right) + \alpha_1\omega^2}{\omega\left\{\alpha_2\left(\alpha_1 + \alpha_2\right) - \left(\alpha_3 + \alpha_4\right) + \omega^2\right\}} = \frac{\alpha_2\left(\alpha_3 + \alpha_4\right) + \alpha_1\omega^2}{\omega\left[\omega^2 - \left\{\left(\alpha_3 + \alpha_4\right) - \alpha_{2(\alpha_1 + \alpha_2)}\right\}\right]} \tag{41}$$

To find the values of $G"$, we put

$$G" = G_1 \frac{A\omega + B\omega^2}{\left(c - \omega^2\right)^2 + D^2\omega^2} \tag{42}$$

where $A = \alpha_2\left(\alpha_3 + \alpha_4\right)$, $B = \alpha_1$, $C = \alpha_3 + \alpha_4$, $D = \alpha_1 + \alpha_2$, $A_1 = \left(c - \omega^2\right)^2 + D^2\omega^2$.

Now

$$D'(G"(i\omega)) = 0 \tag{43}$$

With the help of Eq. (45), the dispersion relation can be derived (calculations are shown in the Appendix)

$$\omega^6 - \left\{ D^2 - (3\frac{A}{B} + 2C) \right\} \omega^4 + \left\{ \frac{A}{B} D^2 - \left(2\frac{A}{B} + 3C \right) C \right\} \omega^2 - \frac{AC^2}{B} = 0 \tag{44}$$

where

$$A = \alpha_2 \left(\alpha_3 + \alpha_4 \right), B = \alpha_1, C = \alpha_3 + \alpha_4, D = \alpha_1 + \alpha_2, A_1 = \left(c - \omega^2 \right)^2 + D^2 \omega^2 \tag{45}$$

Eq. (46) gives the dispersion equation for wave propagation. it is a cubic in ω^2, giving three roots, it must have one real root as complex roots always occur in conjugate pairs or all three roots are real, for $G"$ has either a maximal value or minimum value. Therefore, taking roots as $\omega_1^2, \omega_2^2, \omega_3^2$, we get

Sum of roots,

$$\omega_1^2 + \omega_2^2 + \omega_3^2 = D^2 - (3\frac{A}{B} + 2C) \tag{46}$$

Product of roots taken two at a time,

$$\omega_1^2 \omega_2^2 + \omega_2^2 \omega_3^2 + \omega_3^2 \omega_1^2 = \frac{A}{B} D^2 - \left(2\frac{A}{B} + 3C \right) C \tag{47}$$

Products of roots,

$$\omega_1^2 \omega_2^2 \omega_3^2 = \frac{AC^2}{B} \tag{48}$$

To determine $\omega_1^2, \omega_2^2, \omega_3^2$ through Eq. (47) seems not to be so easy, but if we observe carefully the value of $G"$, we can conclude about the roots $\omega_1^2, \omega_2^2, \omega_3^2$, as follows: (see Appendix)

$$G" = G_1 \frac{\left\{ \alpha_2 \left(\alpha_3 + \alpha_4 \right) \omega + \alpha_1 \omega^2 \right\} \omega}{\left(\alpha_3 + \alpha_4 - \omega^2 \right)^2 + (\alpha_1 + \alpha_2)^2 \omega^2} \tag{49}$$

To find the other two roots ω_2^2, ω_3^2 for the Eq. (46) from Eq. (48), Eq. (49) and Eq. (50), such that (Taking one of the values for $\omega_1^2, \omega_2^2, \omega_3^2$ for the extreme values of $G"$ as $\omega_1^2 = C$)

$$\omega_2^2 + \omega_3^2 = D^2 - 3\left(\frac{A}{B} + C \right), \quad \omega_2^2 . \omega_3^2 = \frac{AC}{B}$$

Then Eq. (46) can be expressed as:

$x^2 - \left(D^2 - 3(\frac{A}{B} + C) \right) x + \frac{AC}{B} = 0$ We get the roots,

$$\omega_2^2 = \frac{AC}{B\left(D^2 - 3\left(\frac{A}{B} + C \right) \right)} = \frac{AC}{BD^2 - 3\left(A + BC \right)} \tag{50}$$

$$\omega_3^2 = \frac{AC}{B\omega_2^2} = \frac{BD^2 - 3\left(A + BC\right)}{B} \tag{51}$$

where

$$\begin{aligned}
A &= \alpha_2\left(\alpha_{3+}\alpha_4\right) = \left(\theta_{22} + \theta'_{22}\right)\left(\theta_{13}\alpha_2 + \theta'_{12}\,\theta_{22}\right) = \left[\frac{G_2}{\eta_2} + \frac{G_2}{\eta'_2}\right]\left[\frac{G_1}{\eta'_2}.\frac{G_2}{\eta_2} + \frac{G_1}{\eta_3}\left(\frac{G_2}{\eta_2} + \frac{G_2}{\eta'_2}\right)\right] \\
B &= \alpha_{1=}\ \theta_{13} + \theta'_{12} = \frac{G_1}{\eta_3} + \frac{G_1}{\eta'_2} \\
C &= \alpha_{3+}\alpha_4 = \theta_{13}\alpha_2 + \theta'_{12}\,\theta_{22} = \frac{G_1}{\eta_3}\left(\frac{G_2}{\eta_2} + \frac{G_2}{\eta'_2}\right) + \frac{G_1}{\eta'_2}.\frac{G_2}{\eta_2} \\
D &= \alpha_1 + \alpha_2 = \theta_{13} + \theta'_{12} + \left(\theta_{22} + \theta'_{22}\right) = \frac{G_1}{\eta_3} + \frac{G_1}{\eta'_2} + \frac{G_2}{\eta_2} + \frac{G_2}{\eta'_2}
\end{aligned} \tag{52}$$

Error due to approximation is:

$$4AC << B\left(D^2 - 3\left(\frac{A}{B} + C\right)\right) \tag{53}$$

Using Eq. (49), Eq. (50) and Eq. (48), one can find

$$\omega_1^2 = C\ ,\ \omega_2^2 = \frac{AC}{\left\{D^2B - 3\left(A + BC\right)\right\}}\ ,\ \omega_3^2 = \left\{BD^2 - 3\left(A + BC\right)\right\}/B\ .$$

From, Eq. (49)

$$\omega_1^2\omega_2^2 + \omega_2^2\omega_3^2 + \omega_3^2\,\omega_1^2 = \frac{A}{B}D^2 - \left(2\frac{A}{B} + 3C\right)C = \frac{AD^2 - \left(2A + 3BC\right)C}{B} \tag{54}$$

Approximate value

$$\omega_1^2\omega_2^2 + \omega_2^2\omega_3^2 + \omega_3^2\,\omega_1^2 = \frac{AC^2}{\left\{D^2B - 3\left(A + BC\right)\right\}} + \frac{AC}{B} + \frac{C}{B}\left\{D^2B - 3\left(A + BC\right)\right\} \tag{55}$$

Error can be calculated by subtracting Eq. (58) and Eq. (59)

$$\text{Error }(\xi) = \frac{A}{B}D^2 - \frac{3\left(A + BC\right)C}{B} - \frac{AC^2}{\left\{D^2B - 3\left(A + BC\right)\right\}} - \frac{C}{B}\left\{D^2B - 3\left(A + BC\right)\right\}$$

$$(\xi) = \frac{1}{B}\left\{AD^2 - 3\left(A + BC\right)C\right\} - \frac{C}{B}\frac{\left[ABC + \left\{D^2B - 3(A + BC)\right\}^2\right]}{D^2B - 3\left(A + BC\right)}$$

Taking the +ve sign, we get

$$\omega_3^2 = \frac{1}{B}\left\{D^2B - 3\left(A + BC\right)\right\}\left[1 - \frac{ACB}{\left\{D^2B - 3(A + BC)\right\}^2}\right]$$

$$\omega_3^2 = \frac{1}{B}\left\{D^2B - 3\left(A + BC\right)\right\} - \frac{AC}{D^2B - 3\left(A + BC\right)}$$

Case-1
At very low frequencies, $\omega = 0$, (from Eq. (51))

$$G''(\omega = 0) = 0, G''\left(\omega^2 = -\frac{\alpha_2\left(\alpha_3 + \alpha_4\right)}{\alpha_1}\right) = 0 \tag{56}$$

Then it is to be inferred that during the cyclic loading initially $\omega = 0$ *i.e.* $G''\left(0\right) = 0,$ there must be a point of maxima or minima between $\omega = 0$ and $\omega = -\frac{\alpha_2\left(\alpha_3 + \alpha_4\right)}{\alpha_1}$,

Case-2
At very high frequencies, $\omega = \infty$ (from Eq. (51))

$$G''(\infty) = 0 \tag{57}$$

Then it is to be inferred that during the cyclic loading initially $\omega = 0$ *i.e.* $G''\left(0\right) = 0,$ there must be a point of maxima or minima between $\omega = 0$ and $\omega = -\frac{\alpha_2\left(\alpha_3 + \alpha_4\right)}{\alpha_1}$, but for $\omega^2 = \alpha_3 + \alpha_4$ it is observed that for $\omega^2 = C$ there must be a point of maxima as when initially $G''(0)$ increases from zero to maximum value $G''_m = \frac{G_1\sqrt{(\alpha_3 + \alpha_4)}}{\left(\alpha_1 + \alpha_2\right)}$ and again states that diminishing and reaches zero at $\omega^2 = -\frac{\alpha_2\left(\alpha_3 + \alpha_4\right)}{\alpha_1}$, which justifies for the model for the Viscoelastic materials.

8 LOADING OF THE MODEL

When relaxation is applied to the model i.e. the model is under the influence the constant deformation, the specimen representing the model is deformed to the given strain γ_0 and after which it is maintained constant, where as the stress required to maintained these strains (γ_0) diminishes at $e = \gamma_0$ constant, under thermal conditions. The Constitutive equation under constant deformation (strain) $e = \gamma_0$ constant reduces to:

$$D^2\sigma + B_1 D\sigma + B_0\sigma = 0 \tag{58}$$

where

$$B_1 = \left(\theta_{13} + \theta'_{12}\right) + (\theta_{22} + \theta'_{22}) \ , \ B_{0=}\theta'_{12}\,\theta_{22} + \theta_{13}(\theta_{22} + \theta'_{22})$$

Eq. (60) can be solved, we taking the roots of the auxiliary equation as:

$$m_1 = \frac{1}{\tau_1} \qquad and \qquad m_2 = \frac{1}{\tau_2} \tag{59}$$

where τ_1 *and* τ_2 are relaxation times of the specimen.

$$m_1 m_2 = \frac{1}{\tau_1}.\frac{1}{\tau_2} = \theta'_{12}\,\theta_{22} + \theta_{13}(\theta_{22} + \theta'_{22}) \tag{60}$$

From, Eq. (62), the Eq. (60) becomes

$$\left\{D^2 + \left(m_1 + m_2\right)D + (m_1 m_2)\right\}\sigma = 0 \tag{61}$$

The Solution of Eq. (63) is:

$$\sigma\left(t\right) = A_1 e^{-m_1 t} + A_2 e^{-m_2 t} \tag{62}$$

To eliminate A_1 and A_2 At $t = 0$, Eq. (64) reduces to $\sigma_0 = \gamma_0 G, \ \frac{d\gamma_0}{dt} = 0$

Hence,

$$A_1 + A_2 = \sigma_0 \tag{63}$$

$$m_1 A_1 + m_2 A_2 = 0 \tag{64}$$

where

$$\begin{gathered} A_1 = \frac{m_2\sigma_0}{m_2 - m_1} \quad , A_2 == \frac{m_1\sigma_0}{m_1 - m_2} \\ \sigma\left(t\right) = \frac{\sigma_0}{m_1 - m_2}\left\{m_1 e^{-m_2 t} - m_2 e^{-m_1 t}\right\} = \frac{G_0\gamma_0}{m_1 - m_2}\left\{m_1 e^{-m_2 t} - m_2 e^{-m_1 t}\right\} \end{gathered} \tag{65}$$

For sufficiently large time, $t > \tau_{1,}\tau_2$ so that $\sigma \to 0$. Hence with longer periods of observation, the stress in the specimen will drop to zero, i.e. equilibrium state will be achieved.

9 CONCLUSIONS

1. During the cyclic loading initially there must be a point of maxima or minima between $\omega = 0$ and $\omega = -\frac{\alpha_2\left(\alpha_3 + \alpha_4\right)}{\alpha_1}$.
2. For sufficiently large relaxation time, the stress in the specimen will drop to zero.
3. The phase shift angle 'δ' for the viscoelastic body must be between zero and $\frac{\pi}{2}$.

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APPENDIX

$$D'(G''(i\omega)) = \frac{G_1\left[\left(A+3B\omega^2\right)A_1-\left(A\omega+B\omega^3\right)\left(A_1\right)\left\{-2.2\omega(c-\omega^2+2D^2\omega\right\}\right]}{A_1^2}$$

$$\left(A+3B\omega^2\right)A_1-2\omega^2\left(A+B\omega^2\right)\left\{\left(D^2-2c\right)+2\omega^2\right\}=0$$

$$\left(A+3B\omega^2\right)\left(\left(c-\omega^2\right)^2+D^2\omega^2\right)-2\omega^2\left(A+B\omega^2\right)\left\{\left(D^2-2c\right)+2\omega^2\right\}=0$$

$$\left(A+3B\omega^2\right)\left\{C^2+\omega^4+\left(D^2-2c\right)\omega^2\right\}-2\omega^2\left[A\left(D^2-2c\right)+\left\{2A+B\left(D^2-2c\right)\right\}\omega^2+2B\omega^4\right]=0$$

$$AC^2+\left\{3BC^2-A\left(D^2-2c\right)\right\}\omega^2+\left\{-3A+BCD^2-2C\right\}\omega^4-B\omega^6=0$$

$$G''(\omega^2=\alpha_3+\alpha_4)=\frac{G_1\ \left\{\alpha_2\left(\alpha_3+\alpha_4\right)+\alpha_1(\alpha_3+\alpha_4)\right\}\sqrt{(\alpha_3+\alpha_4)}}{\left(\alpha_1+\alpha_2\right)^2(\alpha_3+\alpha_4)}$$

$$G''(\omega^2=\alpha_3+\alpha_4)=\frac{G_1\sqrt{(\alpha_3+\alpha_4)}}{\left(\alpha_1+\alpha_2\right)}$$

m_1, m_2 are the roots of the equation

$$x^2-\left(\theta_{13}+\theta'_{12}+\theta_{22}+\theta'_{22}\right)x+(\theta'_{12}\theta_{22}+\theta_{13}\left(\theta_{22}+\theta'_{22}\right)=0$$

$$x=\frac{1}{2}\left(D^2-3\left(\frac{A}{B}+C\right)\right)\pm\sqrt{D^2-3\left(\frac{A}{B}+C\right)^2-4\frac{AC}{B}}$$

$$\omega_2^2\omega_3^2=\frac{1}{2}\left(D^2-3\left(\frac{A}{B}+C\right)\right)\left[1\pm\left\{1-\frac{\frac{2AC}{B}}{\left(D^2-3\left(\frac{A}{B}+C\right)^2\right)^2}\right\}\right]$$

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