Rayleigh Wave in an Incompressible Fibre–Reinforced Elastic Solid Half–Space

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ABSTRACT

In this paper, the equation of motion for an incompressible transversely isotropic fibre-reinforced elastic solid is derived in terms of a scalar function. The general solution of the equation of motion is obtained, which satisfies the required radiation condition. The appropriate traction free boundary conditions are also satisfied by the solution to obtain the required secular equation for the Rayleigh wave speed. Iteration method is used to compute the numerical values of non-dimensional speed of Rayleigh wave. The dependence of the non-dimensional wave speed on non-dimensional material parameter is illustrated graphically. Effect of transverse isotropy is observed on the Rayleigh wave speed.

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Keywords : Rayleigh wave; Fibre-reinforced; Incompressibility; Transverse isotropy.

1 INTRODUCTION

BELFIELD et al. [2] gave the idea of introducing a continuous self-reinforcement at every point of an elastic solid. Fibre-reinforced composite concrete structures are significant due to their low weight and high strength. A reinforced composite has characteristic property where its components act together as a single anisotropic unit till they remain in the elastic condition. During an earthquake, the artificial structures on the surface of the earth are excited which gives rise to violent vibrations in some cases. The material structures which resist the oscillatory vibration are of much interest for engineers and architects. Hashin and Rosen [11] investigated the elastic moduli for fibre-reinforced materials.

Bose and Mal [3] studied the propagation of time-harmonic elastic waves in a fibre- reinforced composite. Scott and Hayes [23] discussed the small vibrations of a fibre reinforced composite. Scott [24-25] studied the waves in a fibre-reinforced elastic material. Sengupta and Nath [26] considered the surface waves in fibre-reinforced anisotropic elastic media. They expressed the plane strain displacement components in terms of two scalar potentials to decouple the plane motion into qP and qSV waves. Singh [28] showed that, for wave propagation in fibre-reinforced anisotropic media, this decoupling cannot be achieved by the introduction of the displacement potentials. The reflection of qP and qSV waves at the free surface of a fibre-reinforced anisotropic elastic half-space is studied by Singh and Singh [29] by using a direct method without the introduction of potentials. Singh [30] obtained the reflection coefficients from free surface of an incompressible transversely isotropic fibre-reinforced elastic half-space for the case when outer slowness section is re- entrant. Singh and Yadav [31] studied the reflection of plane waves from a free surface of a rotating fibre-reinforced elastic solid half-space with magnetic field. Surface waves are very important in the study of earthquake, geophysics and

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geodynamics. Rayleigh waves cause destruction to the structure due to its slower attenuation of the energy than that of the body waves. Surface waves in elastic solids were first studied by Lord Rayleigh [18] for an isotropic elastic solid. The extension of surface wave analysis and other wave propagation problems to anisotropic elastic materials has been the subject of many studies; see, for example, Musgrave [15]; Anderson [1]; Stoneley [32]; Crampin and Taylor [6]; Chadwick and Smith [4]; Royer and Dieulesaint [22]; Dowaikh and Ogden [10]; Mozhaev [14]; Nkemzi [17]; Nair and Sotiropoulos [16]; Malischewsky [12]; Destrade [7-8]; Ting [33]; Destrade [9]; Ogden and Vinh [19]; Ogden and Singh [20]; Vinh and Linh [34]; Shams and Ogden [27]; Ogden and Singh [21].

The aim of this paper is to obtain a secular equation for the Rayleigh wave in an incompressible transversely isotropic fibre-reinforced elastic medium. Using iteration method, non-dimensional wave speed of Rayleigh wave is plotted against non-dimensional material parameter to show the effect of transverse isotropy.

2 EQUATIONS OF MOTION

Let us consider an incompressible transversely isotropic fibre-reinforced elastic medium. The constitutive equation explaining the stress-strain response to small deformation of such a material is given by Chadwick [5] as:

$$\sigma = -pI + 2\mu_T \varepsilon + 2(\mu_L - \mu_T)\{e \otimes (\varepsilon e) + (\varepsilon e) \otimes e\} + 4(\mu_E - \mu_L)\{e(\varepsilon e)\}e \otimes e \tag{1}$$

where ε, σ and I denote the infinitesimal strain and stress tensors and the 3×3 identity tensors, respectively, and e is a unit vector defining the axis of transversely isotropy. In Eq. (1), μ_L and μ_T are longitudinal and transverse shear moduli and μ_E is a weighted shear modulus, given by $\mu_E = \dfrac{E_L}{E_T}\mu_T$ where E_L and E_T are longitudinal and transverse Young's moduli. p is the hydrostatic pressure required to maintain the incompressibility constraint.

$$tr\ \varepsilon = 0 \tag{2}$$

Consider a Cartesian coordinate system $Ox_1x_2x_3$, such that Ox_1 is parallel to the direction of transversely isotropy, the constitutive relation (1) is written in component form as:

$$\sigma_{ij} = -p\delta_{ij} + 2\mu_T \varepsilon_{ij} + 2(\mu_L - \mu_T)(\delta_{i1}\varepsilon_{j1} + \varepsilon_{i1}\delta_{j1}) + 4(\mu_E - \mu_L)\varepsilon_{11}\delta_{i1}\delta_{j1} \tag{3}$$

The Eq. (3) may be written in matrix form (stiffness matrix) as:

$$
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} =
\begin{pmatrix}
-p + (4\frac{E_L}{E_T} - 1)\mu_T & 0 & 0 & 0 & 0 & 0 \\
0 & -p + 2\mu_T & 0 & 0 & 0 & 0 \\
0 & 0 & -p + 2\mu_T & 0 & 0 & 0 \\
0 & 0 & 0 & 2\mu_L & 0 & 0 \\
0 & 0 & 0 & 0 & 2\mu_L & 2\mu_L \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{bmatrix} \tag{4}
$$

Let $(x_1x_2x_3)$ be Cartesian coordinates and consider a transversely isotropic fibre-reinforced elastic material occupying the half-space $x_2 \prec 0$, with traction-free boundary $x_2 = 0$. We consider a plane motion in the (x_1, x_2) plane with displacement components (u_1, u_2, u_3) such that

$$u_1 = u_1(x_1, x_2, t), \quad u_2 = u_2(x_1, x_2, t), \quad u_3 = 0 \tag{5}$$

where t is time. For an incompressible material, we have

$$u_{1,1} + u_{2,2} = 0 \tag{6}$$

From which we deduce the existence of a scalar function, denoted $\psi(x_1, x_2, t)$, such that

$$u_1 = \psi_{,2}, \quad u_2 = -\psi_{,1} \tag{7}$$

Using the strain-displacement relation $2\varepsilon_{ij} = u_{i,j} + u_{j,i}$ in Eq. (4), the stress components are written in terms of displacement components and pressure as:

$$\sigma_{11} = (c_1^2 - c_2^2)u_{1,1} - p \tag{8}$$

$$\sigma_{12} = c_3^2(u_{1,2} + u_{2,1}) \tag{9}$$

$$\sigma_{22} = 2c_2^2 u_{2,2} - p \tag{10}$$

where $p = p(x_1, x_2, t)$ is the hydrostatic pressure associated with the incompressibility constraint, and $c_1^2 = 4\mu_E - \mu_T$, $c_2^2 = \mu_T$, $c_3^2 = \mu_L$. The requirements of a physically reasonable response ensure that c_1^2, c_2^2 and c_3^2 are all positive. In absence of body forces, the equations of motion are

$$\sigma_{ij,j} = \rho \ddot{u}_i \tag{11}$$

which upon using Eqs. (8) to (10) become

$$(c_1^2 - c_2^2)u_{1,11} + c_3^2(u_{1,22} + u_{2,12}) - p_{,1} = \rho \ddot{u}_1 \tag{12}$$

$$c_3^2(u_{1,12} + u_{2,11}) + 2c_2^2 u_{2,22} - p_{,2} = \rho \ddot{u}_2 \tag{13}$$

with the help of Eq. (7), the Eqs. (12) and (13) become

$$(c_1^2 - c_2^2)\psi_{,211} + c_3^2(\psi_{,222} - \psi_{,112}) - p_{,1} = \rho \ddot{\psi}_{,2} \tag{14}$$

$$c_3^2(\psi_{,212} - \psi_{,111}) - 2c_2^2 \psi_{,122} - p_{,2} = -\rho \ddot{\psi}_{,1} \tag{15}$$

Elimination of p by cross differentiation leads to the following equation of motion

$$c_3^2 \psi_{,1111} + (c_1^2 + c_2^2 - 2c_3^2)\psi_{,1122} + c_3^2 \psi_{,2222} = \rho(\ddot{\psi}_{,11} + \ddot{\psi}_{,22}) \tag{16}$$

3 RAYLEIGH WAVES

We now consider a half-space occupying the region $x_2 \langle 0$ in the reference configuration with boundary $x_2 = 0$ and Rayleigh surface waves propagating along the direction x_1 and we write ψ in the form

$$\psi(x_1 + x_2, t) = \varphi(y)\exp\left[ik(x_1 - ct)\right] \tag{17}$$

where $y = kx_2$ and k is wave number. Using Eq. (17) into Eq. (16), we obtain

$$c_3^2\varphi''''(y) - (c_1^2 + c_2^2 - 2c_3^2 - \rho c^2)\varphi''(y) + (c_3^2 - \rho c^2)\varphi(y) = 0 \tag{18}$$

The boundary conditions on $x_2 = 0$ are

$$\sigma_{21} = 0, \qquad \sigma_{22} = 0 \tag{19}$$

which are written as:

$$c_3^2(u_{1,2} + u_{2,1}) = 0 \tag{20}$$

$$2c_2^2 u_{2,2} - p = 0 \tag{21}$$

with the help of Eq.(7), the Eq. (20) is written as:

$$c_3^2(\psi_{22} - \psi_{11}) = 0 \tag{22}$$

Differentiating Eq.(21) with respect to x_1 and using Eq.(7) and Eq.(14), we have

$$c_3^2(\psi_{222} - \psi_{112}) + (c_1^2 + c_2^2)\psi_{112} - \rho\ddot{\psi}_2 = 0 \tag{23}$$

In addition to conditions (22) and (23), we need also the following condition on ψ

$$\psi(x_1, x_2, t) \to 0, \quad as \quad x_2 \to -\infty \tag{24}$$

In terms of φ, the conditions (22) to (24) become

$$\varphi''(0) + \varphi(0) = 0 \tag{25}$$

$$c_3^2\varphi'''(0) + (c_3^2 - c_1^2 - c_2^2 + \rho c^2)\varphi'(0) = 0 \tag{26}$$

$$\varphi(x_2) \to 0, \quad as \quad x_2 \to -\infty \tag{27}$$

The general solution $\varphi(y)$ of Eq. (18) that satisfies *r*adiation condition (27) is:

$$\varphi(y) = A\exp(s_1 y) + B\exp(s_2 y) \tag{28}$$

where A, B are constants and s_1, s_2 are solutions of following quadratic equation in s^2

$$c_3^2 s^4 - (c_1^2 + c_2^2 - 2c_3^2 - \rho c^2)s^2 + (c_3^2 - \rho c^2) = 0 \tag{29}$$

If the roots s_1^2 and s_2^2 of the quadratic Eq. (29) are real, the they must be positive to ensure that s_1 and s_2 can have a positive real part. If they are complex then they are conjugate. In either case, the product $s_1^2 s_2^2$ must be positive and hence a real (surface) wave speed satisfies the inequalities

$$0 \langle \rho c^2 \langle c_3^2 \tag{30}$$

Using solutions (28) into boundary conditions (25) and (26), we have

$$(1 + s_1^2)A + (1 + s_2^2)B = 0 \tag{31}$$

$$\left[c_3^2 s_1^3 + (c_3^2 - c_1^2 - c_2^2 + \rho c^2)s_1 \right]A + \left[c_3^2 s_2^3 + (c_3^2 - c_1^2 - c_2^2 + \rho c^2)s_2 \right]B = 0 \tag{32}$$

Following Ogden and Vinh [19], the non-trivial solution of Eqs. (31) and (32) requires

$$(c_1^2 + c_2^2 - \rho c^2)\sqrt{1 - \frac{\rho c^2}{c_3^2}} - \rho c^2 = 0 \tag{33}$$

which is the secular equation for wave speed. Eq. (33) may also be written as:

$$(4\mu_E - \rho c^2)\sqrt{1 - \frac{\rho c^2}{\mu_L}} - \rho c^2 = 0 \tag{34}$$

For a real solution for c of Eq. (34), the inequality $\rho c^2 \langle 4\mu_E$ must also hold along with inequality (30). With the help of Eqs. (7), (17), (28), (29), (31) and (32), it can be shown that the particles in a solid move in elliptical paths with the major axis of the ellipse perpendicular to the surface of the solid. The width of the elliptical path decreases with the increase in depth into the solid.

4 NUMERICAL RESULTS

Taking $x = \dfrac{\rho c^2}{\mu_L}$ and $R = \dfrac{4\mu_E}{\mu_L}$ into the Eq. (34) and squaring both sides, we can obtain

$$x = \frac{R^2 + 2Rx^2 - x^3}{2R + R^2} \tag{35}$$

Using iteration method, the Eq. (35) is solved for x in the range $0 \le R \le 10$. The non-dimensional speed $x = \dfrac{\rho c^2}{\mu_L}$ is plotted against the ratio $R = \dfrac{4\mu_E}{\mu_L}$ of material constants in Fig. 1. For small values of ratio R of material constants, the wave speed is very small. It increases sharply with the increase in value of ratio R. For $\mu_E = \mu_L (R = 4)$, the wave speed corresponds to isotropic case and is approximately 0.9126. For very large value of R, the non-dimensional speed $x = \dfrac{\rho c^2}{\mu_L}$ tends to one. It is necessary to mention here that the Fig. 1 is similar to that plotted by Ogden and Vinh [19] using an explicit formula for wave speed.

Taking $x = \dfrac{\rho c^2}{4\mu_E}$ and $R = \dfrac{4\mu_E}{\mu_L}$ into the Eq. (34) and squaring both sides, we can obtain

$$x = \frac{1}{R} + 2x^2 - x^3 \tag{36}$$

Using iteration method, the Eq. (36) is solved for x in the range $0.2 \leq R \leq 10$. The non-dimensional speed $x = \dfrac{\rho c^2}{4\mu_E}$ is plotted against the ratio $R = \dfrac{4\mu_E}{\mu_L}$ of material constants in Fig. 2. For small values of ratio R of material constants, the wave speed is very l a r g e . It decreases very sharply with the increase in value of ratio R. For $\mu_E = \mu_L (R = 4)$, the wave speed corresponds to isotropic case and is 0.25. At higher values of R , the non-dimensional speed $x = \dfrac{\rho c^2}{4\mu_E}$ decreases slowly.
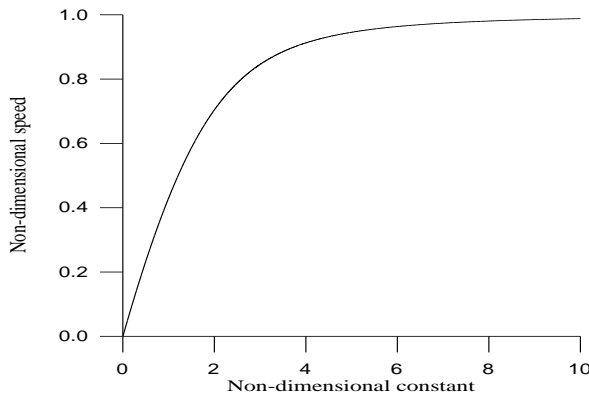

Fig.1

Variation of non-dimensional speed $\dfrac{\rho c^2}{\mu_L}$ of Rayleigh wave against non-dimensional constant $R = \dfrac{4\mu_E}{\mu_L}$.
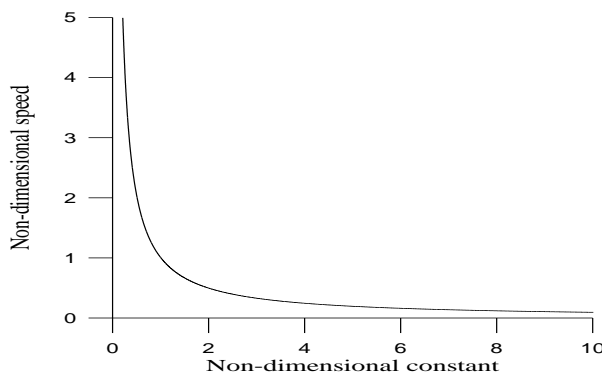

Fig.2

Variation of non-dimensional speed $\dfrac{\rho c^2}{4\mu_E}$ of Rayleigh wave against non-dimensional constant $R = \dfrac{4\mu_E}{\mu_L}$.

5   CONCLUSIONS

A problem on Rayeligh wave in an incompressible transversely isotropic fibre-reinforced elastic solid half-space is considered. The secular equation for Rayleigh wave speed is obtained. The numerical results show the dependence of wave speed on material parameters. The present theoretical and numerical results may provide important information to experimental seismologists for their further research on the subject.

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