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Steady Thermal Stresses in a Thin Rotating Disc of Finitesimal Deformation with Mechanical Load

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ABSTRACT

Seth's transition theory is applied to the problems of thickness variation parameter in a thin rotating disc by finite deformation. Neither the yield criterion nor the associated flow rule is assumed here. The results obtained here are applicable to compressible materials. If the additional condition of incompressibility is imposed, then the expression for stresses corresponds to those arising from Tresca yield condition. It has observed that for rotating disc made of compressible material required higher angular speed to yield at the internal surface as compare to disc made of incompressible material and a much higher angular speed is required to yield with the increase in radii ratio. With the introduction of thermal effects, lesser angular speed is required to yield at the internal surface. Thermal effect in the disc increase the value of circumferential stress at the internal surface and radial stresses at the external surface for compressible as compare to incompressible material.

Keywords : Plastic; Transitional; Finitesimal; Stresses; Disc; Load; Temperature.

1 INTRODUCTION

AN annular disk mounted on a circular shaft rotating at high speed is widely used in engineering applications. In the case of elastic/plastic disks, the study of stress and strain fields in rotating disks has begun with models based on the Tresca yield criterion. Rotating discs are form an essential part of the design of rotating machinery, namely rotors, turbines, compressors, flywheel and computer's disc drive etc. The analysis of thin rotating discs made of isotropic material has been discussed extensively by Timoshenko and Goodier [1] in the elastic range and by Chakrabarty [2] and Heyman [3] for the plastic range. Their solution for the problem of fully plastic state does not involve the plane stress condition, that is to say, we can obtain the same stresses and angular velocity required by the disc to become fully plastic without using the plane stress condition. Parmaksigoglu, *et al.* [4] found the Plastic stress distribution in a rotating disc with rigid inclusion under a radial temperature gradient under the assumptions of Tresca's yield condition, its associated flow rule and linear strain hardening. To obtain the stress distribution, they matched the plastic stresses at the same radius $r = z$ of the disc. Perfect elasticity and ideal plasticity are two extreme properties of the material and the use of an ad-hoc rule like yield condition amounts to divide the two extreme properties by a sharp line which is not physically possible. When a material passes from one state to another qualitatively different state, transition takes place. Since this transition is non-linear in character and difficult to investigate, workers have taken certain ad-hoc assumptions like yield condition, incompressibility condition and a strain law, which may or may not valid for the problem. Seth's transition theory [5] does not

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required any assumptions like an yield condition, incompressibility condition and thus poses and solves a more general problem from which cases pertaining to the above assumptions can be worked out. This theory utilizes the concept of generalized strain measure and asymptotic solution at critical points or turning points of the differential equations defining the deformed field and has been successfully applied to a large number of problems [5, 6, 8 - 20].

The plastic stresses have been derived through the asymptotic solution of principal stress respectively. Results have been discussed and presented graphically.

2 FORMULATION OF THE PROBLEMS

Consider a circular disc of isotropic material with central bore of radius a and external radius b. The disc, produced of material of constant density, is mounted on a load. The disc is rotating with angular speed ω about a central axis perpendicular to its plane. The thickness of disc is assumed to be constant and is taken sufficiently small so that the disc is effectively in a state of plane stress, that is, the axial stress τ_{zz} is zero. The temperature at the central bore of the disc is Θ. The displacement components in cylindrical polar co-ordinate are given by [6]:

$$u = r(1-\beta), v = 0, w = dz \tag{1}$$

where β is position function, depending on $r = \sqrt{x^2 + y^2}$ only, and d is a constant. The generalized components of strain are given by Seth's [6]:

$$e_{rr} = \frac{1}{n}\left[1-\left(r\beta'+\beta\right)^n\right], e_{\theta\theta} = \frac{1}{n}\left[1-\beta^n\right], e_{zz} = \frac{1}{n}\left[1-(1-d)^n\right], e_{r\theta} = e_{\theta z} = e_{zr} = 0 \tag{2}$$

where $\beta' = d\beta/dr$. The thermal stress–strain relations are given by [7]:

$$\tau_{ij} = \lambda\delta_{ij}\Delta + 2\mu e_{ij} - \xi\Theta\delta_{ij}, \left(i, j = 1,\ 2,\ 3\right) \tag{3}$$

where τ_{ij} is the stress components, λ and μ are Lame's constants and $\Delta = e_{kk}$ is the first strain invariant, δ_{ij} is the Kronecker's delta and $\xi = \alpha\left(3\lambda + 2\mu\right), \alpha$ being the coefficient of thermal expansion and Θ is the temperature Further, Θ has to satisfy $\nabla^2\Theta = 0$.

$$\frac{d^2\Theta}{dr^2} + \frac{1}{r}\frac{d\Theta}{dr} \equiv \frac{1}{r}\frac{d}{dr}\left(r\frac{d\Theta}{dr}\right) = 0 \quad or \quad \frac{d\Theta}{dr} = \frac{k}{r}; \text{ which has solutions: } \Theta = k_1\left(\log r + k_2\right) \tag{4}$$

where k_1 and k_2 are constant of integration and can determined from the boundary condition.

Substituting Eq. (2) in Eq. (3), we get

$$\begin{aligned} \tau_{rr} &= \frac{2\mu}{n}\left[3-2c-\beta^n\left\{1-c+\left(2-c\right)\left(P+1\right)^n + \frac{nc\xi\Theta}{2\mu\beta^n}\right\}\right], \\ \tau_{\theta\theta} &= \frac{2\mu}{n}\left[3-2c-\beta^n\left\{2-c+\left(1-c\right)\left(P+1\right)^n + \frac{nc\xi\Theta}{2\mu\beta^n}\right\}\right], \tau_{r\theta} = \tau_{\theta z} = \tau_{zr} = \tau_{zz} = 0 \end{aligned} \tag{5}$$

where c is compressibility factor of the material in term of Lame's constant, there are given by $c = \dfrac{2\mu}{\lambda + 2\mu}$.

Equations of equilibrium are all satisfied except

$$\frac{d}{dr}\left(r\tau_{rr}\right)-\tau_{\theta\theta}+\rho\omega^2 r^2=0 \tag{6}$$

where ρ is the density of the material. The temperature satisfying Laplace Eq. (4) with boundary condition $\Theta=\Theta_0$ at $r=a, \Theta=0$ at $r=b$, where Θ_0 is constant, given by [7]; $k_1=\dfrac{\Theta}{\ln(a/b)}$ and $k_2=-\ln b$. Substituting k_1 and k_2 form Eq. (4), we get

$$\Theta=\frac{\Theta_0\ln(r/b)}{\ln(a/b)} \tag{7}$$

Using Eqs. (5) and (7) in Eq. (6), we get a non- linear differential equation in β as:

$$(2-c)n\beta^{n+1}P(P+1)^{n-1}\frac{dP}{d\beta}=\frac{n\rho\omega^2 r^2}{2\mu}-\frac{nc\xi\bar{\Theta}_0}{2\mu}+\beta^n\left[1-(P+1)^n-nP\left\{1-c+(2-c)(P+1)^n\right\}\right] \tag{8}$$

where $\bar{\Theta}_0=\dfrac{\Theta_0}{\ln(a/b)}$ and $r\beta'=\beta P$ (P is function of β and β is function of *r*). From Eq. (8), the critical points of β are $P=-1$ and $\pm\infty$. The boundary conditions are:

$$\tau_{rr}=0 \quad at \quad r=a, \quad \tau_{rr}=T_0 \quad at \quad r=b \tag{9}$$

3 SOLUTION THROUGH THE PRINCIPAL STRESS DIFFERENCE

It has been shown [5, 6, 8 - 20] that the asymptotic solution through the principal stress leads from elastic state to the plastic state at the transition point $P\to\pm\infty$. The transition function ζ is defined as:

$$\zeta=\frac{n}{2\mu}\left[T_{\theta\theta}-c\xi\Theta\right]=\left[(3-2c)-\beta^n\left\{2-c+(1-c)(P+1)^n\right\}-\frac{nc\xi\Theta}{\mu}\right] \tag{10}$$

Taking the logarithmic differentiating of Eq. (10) with respect to *r*, and using Eq. (8), we gets:

$$\frac{d(\log\zeta)}{dr}=-\left[\frac{\beta^n\left(\frac{1-c}{2-c}\right)\left[1-(P+1)^n-n(1-c)P+\frac{n\rho\omega^2r^2}{2\mu\beta^n}+\frac{nc\xi\bar{\Theta}_0(3-2c)}{\mu(4-2C)\beta^n}\right]+(2-c)nP\beta^n}{r\left[3-2c-\beta^n\left\{2-c+(1-c)(P+1)^n\right\}-\frac{nc\xi\Theta}{\mu}\right]}\right] \tag{11}$$

Taking the asymptotic value $P\to\pm\infty$ from Eq. (11) and integrating, we get

$$\frac{d(\log\zeta)}{dr}=-\frac{1}{(2-c)r} \tag{12}$$

Integrating Eq. (12), we get:

$$\zeta=k_3 r^{\nu-1} \tag{13}$$

where $\nu = 1 - c/2 - c$ be Poisson's ratio in terms of compressibility factor and k_3 is a constant of integration, which can be determine by boundary condition. From Eqs. (10) and (13), we have

$$\tau_{\theta\theta} = \left(\frac{2\mu}{n}\right) k_3 r^{\nu-1} + \frac{c\xi\Theta_0 \log(r/b)}{\log(a/b)} \tag{14}$$

Substituting Eq. (14) in Eq. (6) and integrating, we get

$$\tau_{rr} = \left(\frac{2\mu}{n\nu} k_3 r^{\nu-1} + \frac{c\xi\Theta_0 \log(r/b)}{\log(a/b)} - \frac{c\xi\Theta_0}{\log(a/b)} - \frac{\rho\omega^2 r^2}{3} + \frac{k_4}{r}\right) \tag{15}$$

where k_4 is a constant of integration, which can be determined by boundary condition. By applying boundary condition (9) in Eq. (15), we get

$$k_3 = \frac{n\nu}{2\mu\left(b^\nu - a^\nu\right)}\left(bT_0 + \frac{\rho\omega^2}{3}\left[b^3 - a^3\right] + \frac{c\xi\Theta_0}{\ln(a/b)}\left[a\ln(a/b) - a + b\right]\right);$$

$$k_4 = \frac{-a^\nu}{\left(b^\nu - a^\nu\right)}\left(bT_0 + \frac{\rho\omega^2}{3}\left[b^3 - a^3\right] + \frac{c\xi\Theta_0}{\ln(a/b)}\left[a\ln(a/b) - a + b\right]\right) - c\xi\Theta_0 a\left[\frac{\ln(a/b) - 1}{\ln(a/b)}\right] + \frac{\rho\omega^2 a^3}{3};$$

By substituting the value of k_3 and k_4 into Eqs. (14) and (15), we get:

$$\tau_{rr} = \left[\begin{array}{l} \dfrac{r^\nu - a^\nu}{r\left(b^\nu - a^\nu\right)}\left[bT_0 + \dfrac{\rho\omega^2}{3}\left(b^3 - a^3\right) + \dfrac{\alpha E(2-c)\Theta_0}{\ln(a/b)}\left(a\ln(a/b) - a + b\right)\right] + \dfrac{\rho\omega^2}{3r}\left(a^3 - r^3\right) \\ + \dfrac{\alpha E(2-c)\Theta_0}{\log(a/b)}\left[\left\{\ln(r/b) - 1\right\} - \dfrac{a}{r}\left\{\ln(a/b) - 1\right\}\right] \end{array}\right] \tag{16}$$

$$\tau_{\theta\theta} = \frac{r^{\nu-1}\nu}{\left(b^\nu - a^\nu\right)}\left(bT_0 + \frac{\rho\omega^2}{3}\left[b^3 - a^3\right] + \frac{\alpha E(2-c)\Theta_0}{\ln(a/b)}\left[\begin{array}{l} a\ln(a/b) \\ -a + b \end{array}\right]\right) + \frac{\alpha E(2-c)\Theta_0 \log(r/b)}{\log(a/b)} \tag{17}$$

where $c\xi = \alpha E(2-c)$. It is seen from Eq. (17) that $T_{\theta\theta}$ is maximum at the internal surface, therefore, yielding will take place at the internal surface and Eq. (17) become:

$$\left|\tau_{\theta\theta}\right|_{r=a} = \left|\frac{a^{\nu-1}\nu}{\left(b^\nu - a^\nu\right)}\left(bT_0 + \frac{\rho\omega^2}{3}\left[b^3 - a^3\right] + \frac{\alpha E(2-c)\Theta_0}{\ln(a/b)}\left[a\ln(a/b) - a + b\right]\right) + \alpha E(2-c)\Theta_0\right| \equiv Y\ (say)$$

and angular speed ω_i necessary for initial yielding is given by:

$$\Omega_i^2 = \frac{\rho\omega_i^2 b^2}{Y} = \left|\frac{3b^2.\left(b^\nu - a^\nu\right)}{\left(b^3 - a^3\right).a^{\nu-1}\nu}\right| + \left|\left(\frac{3b^2}{\left(b^3 - a^3\right)}\right)\left[b\sigma_0 + \frac{\alpha E\Theta_0(2-c)}{Y}\left\{\begin{array}{l} a - \dfrac{a}{\ln(a/b)} + \dfrac{b}{\ln(a/b)} \\ + \dfrac{\left(b^\nu - a^\nu\right)}{\nu.a^{\nu-1}} \end{array}\right\}\right]\right| \tag{18}$$

where $\sigma_0 = T_0/Y$ and $\omega_i = \frac{1}{b}\Omega_i (Y/\rho)^{\frac{1}{2}}$. We introduce the following non-dimensional components are $R = r/b, R_0 = a/b, \Omega^2 = \rho\omega^2 b^2/Y, \sigma_r = \tau_{rr}/Y, \sigma_\theta = \tau_{\theta\theta}/Y, \alpha E\Theta_0/Y = \Theta_1$ and $\sigma_0 = T_0/Y$. Eqs. (16), (17) and (18) become:

$$\sigma_r = \left[\left(\frac{R^\nu - R_0^\nu}{R\left(1-R_0^\nu\right)}\right)k_5 - \frac{\Omega_i^2}{3R}\left(R^3 - R_0^3\right) + \frac{\Theta_1(2-c)\left[R(\ln R - 1) + R_0(1-\ln R_0)\right]}{R\ln(R_0)}\right] \tag{19}$$

$$\sigma_\theta = \frac{\nu . R^{\nu-1}}{\left(1-R_0^\nu\right)}k_5 + \frac{\Theta_1(2-c)\ln R}{\ln(R_0)} \tag{20}$$

$$\Omega_i^2 = \left|\frac{3.\left(1-R_0^\nu\right)}{\nu\left(1-R_0^3\right).R_0^{\nu-1}}\right| + \left|\left(\frac{3}{1-R_0^3}\right).\left[\sigma_0 + \Theta_1(2-c)\left\{R_0 - \frac{R_0}{\ln R_0} + \frac{1}{\ln R_0} + \frac{\left(1-R_0^\nu\right)}{\nu R_0^{\nu-1}}\right\}\right]\right| \tag{21}$$

where $k_5 = \sigma_0 + \frac{\Omega_i^2}{3}\left(1-R_0^3\right) + \frac{\Theta_1(2-C)}{\ln(R_0)}\{R_0\ln(R_0) - R_0 + 1\}$. Eqs. (19), (20) and (21) give thermo elastic-plastic transitional stresses and angular speed for thin rotating disc with loading edge. Stresses and angular speed given by Eqs. (19), (20) and (17) for fully plastic state $\nu = 1/2 \; or \; c = 0$ becomes

$$\sigma_r = \left[\left(\frac{R^{1/2} - R_0^{1/2}}{R\left(1-R_0^{1/2}\right)}\right)k_6 - \frac{\Omega_i^2}{3R}\left(R^3 - R_0^3\right) + \frac{2\Theta_1\{\ln(R_0) - 1\}}{R\ln(R_0)}\left[R - R_0\right]\right] \tag{22}$$

$$\sigma_\theta = \frac{R^{-1/2}}{2\left(1-R_0^{1/2}\right)}k_6 + \frac{2\Theta_1\ln R}{\ln(R_0)} \tag{23}$$

where $k_6 = \sigma_0 + \frac{\Omega_i^2}{3}\left(1-R_0^3\right) + \frac{2\Theta_1}{\ln(R_0)}\{R_0\ln(R_0) - R_0 + 1\}$. From Eq. (17) the angular speed required for fully plastic state ($\nu = 0.5$ or $c = 0$) at the external surface is given by

$$\Omega_f^2 = \frac{\rho\omega_f^2 b^2}{Y} = \frac{3}{\left(1-R_0^3\right)}\left[2\left(1-\sqrt{R_0}\right) - \sigma_0 - \frac{2\Theta_1}{\ln R_0}\left[R_0\ln R_0 - R_0 + 1\right]\right] \tag{24}$$

where $\omega_f = \frac{1}{b}\Omega_f (Y/\rho)^{\frac{1}{2}}$.

4 NUMERICAL ILLUSTRATION AND DISCUSSION

For calculating the stresses and angular speed based on the above analysis, the following values have been taken $c = 0$ (incompressible material), 0.25 (compressible material), 0.5 (compressible material); $\sigma_0 = 0, 1, 1.5$ and $\Theta_0 = 0, 5000$ and 7000°F, $\alpha = 5.0\times10^{-5}$ deg F^{-1} (for methyl methacrylate) [21], $\Theta_1 = \alpha E\Theta_0/Y = 0, 0.25$ and 0.35, respectively. In Fig. 1, curves have been drawn between angular speed Ω_i^2 required for initial yielding of the rotating disc with loads $\sigma_0 = 0,1,1.5$ for different values of temperature along the radii ratios $R_0 = a/b$. It has been observed that for rotating disc made of compressible material required higher angular speed to yield at the internal surface as compare to disc made of incompressible material and a much higher angular speed is required to yield with the increase in radii ratio. With the introduction of thermal effects, lesser angular speed is required to yield at the internal surface. Figs. 2, 3, curves have been drawn for thermal stresses distribution at elastic-plastic transitional state and fully plastic state of rotating disc with respect to radius ratio $R = r/b$. It has been seen that the circumferential stresses has maximum value at the internal surface and radial stresses is maximum at the external surface of the rotating disc made of compressible material as compare to incompressible material. With thermal effect it, increases the value of circumferential stress at the internal surface and radial stresses at the external surface for compressible as compare to incompressible material. Whereas from Fig. 3, it can be seen that thermal effect increases the value of circumferential and radial stresses at the internal and external surface for fully-plastic state.

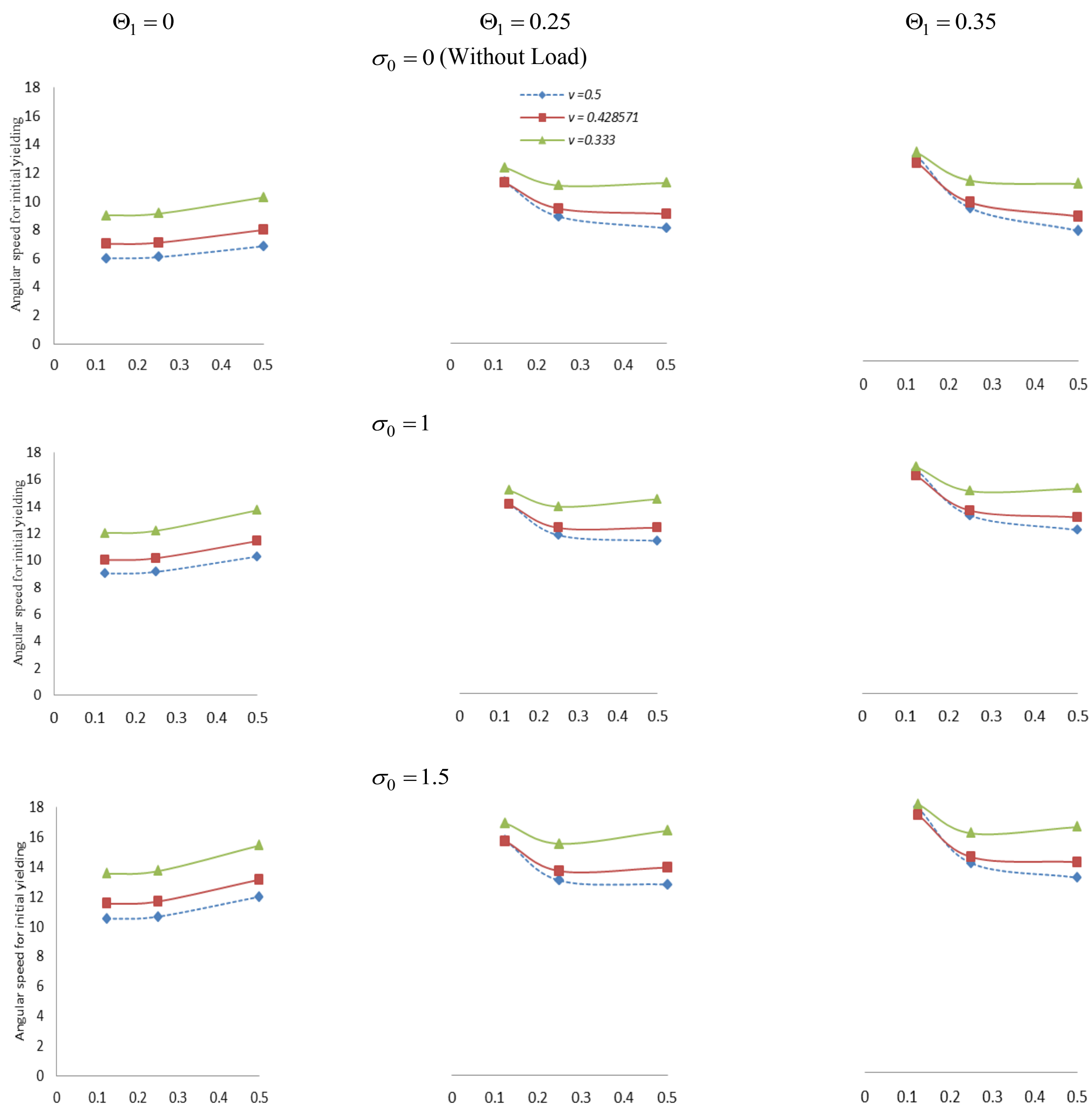

Fig.1
Angular speed required for initial yielding of the rotating disc with load having different temperature along the radii ratio $R_0 = a/b$.

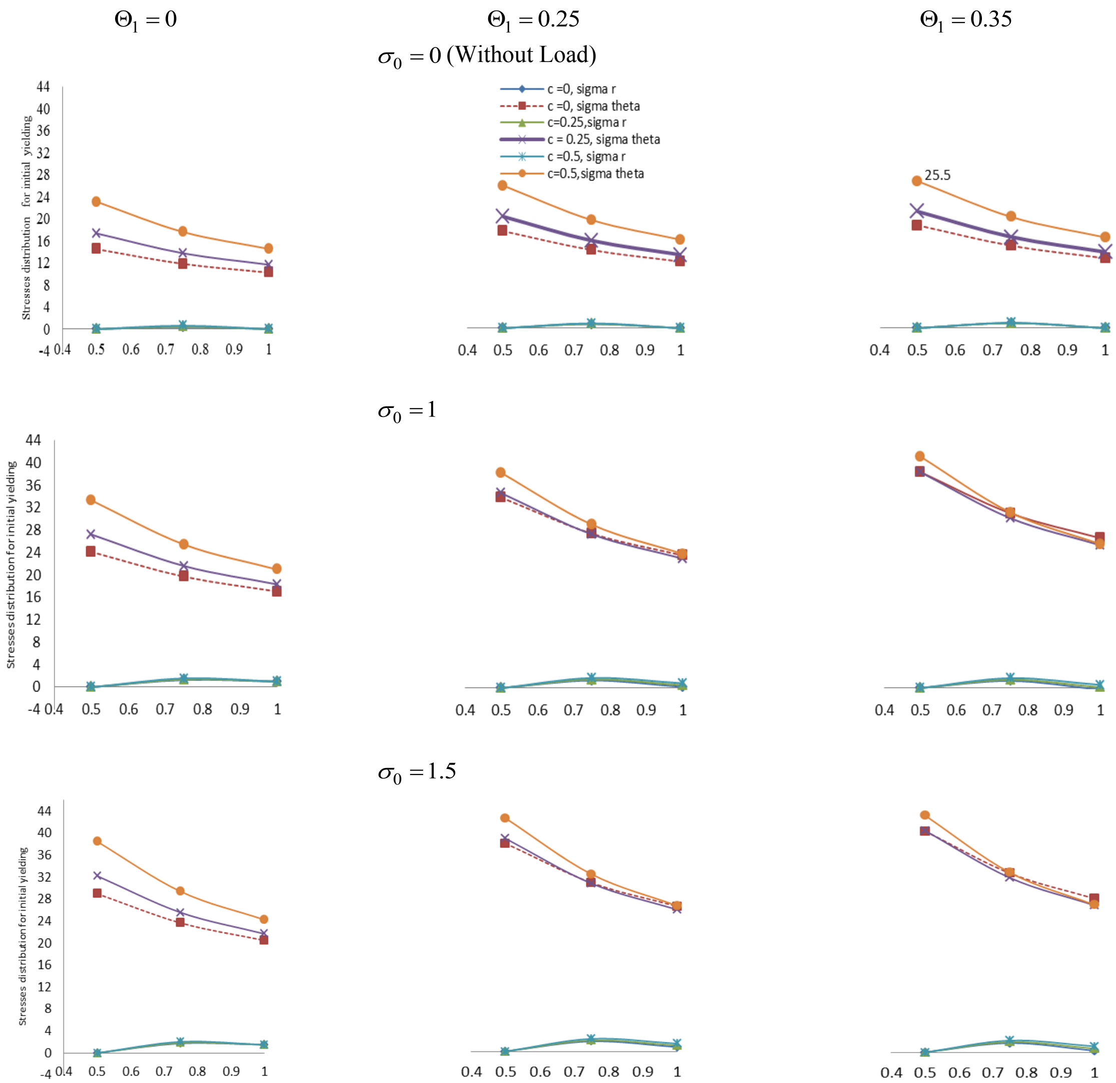

Fig.2
Stresses at the elastic-plastic transition state along the radius $R = r/b$ for different temperature.

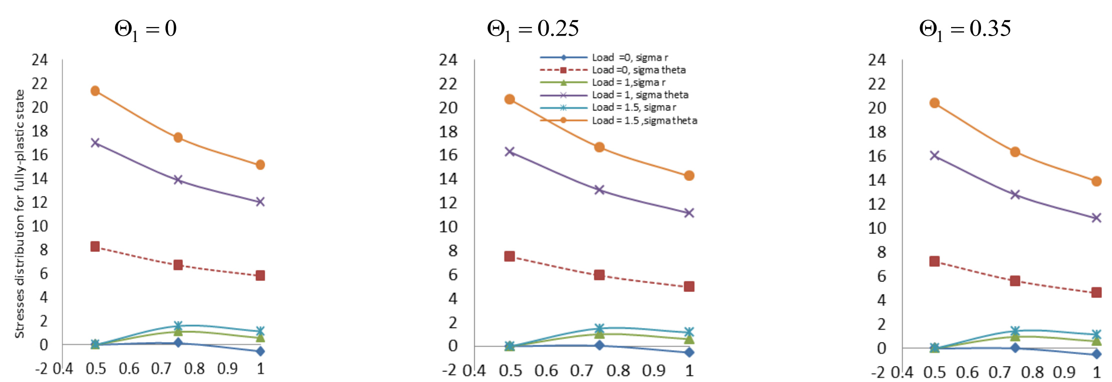

Fig.3
Stresses for fully-plastic state along the radius $R = r/b$ for different temperature.

5 CONCLUSIONS

It has been observed that for rotating disc made of compressible material required higher angular speed to yield at the internal surface as compare to disc made of incompressible material and a much higher angular speed is required to yield with the increase in radii ratio. Thermal effect in the disc increase the value of circumferential stress at the internal surface and radial stresses at the external surface for compressible as compare to incompressible material.

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