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Dynamic Stability of Functionally Graded Beams with Piezoelectric Layers Located on a Continuous Elastic Foundation

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ABSTRACT
This paper studies dynamic stability of functionally graded beams with piezoelectric layers subjected to periodic axial compressive load that is simply supported at both ends lies on a continuous elastic foundation. The Young's modulus of beam is assumed to be graded continuously across the beam thickness. Applying the Hamilton's principle, the governing dynamic equation is established. The effects of the constituent volume fractions, the influences of applied voltage, foundation coefficient and piezoelectric thickness on the unstable regions are presented.

Keywords: Dynamic stability; Functionally graded beam; Elastic foundation; Piezoelectric layer

1 INTRODUCTION

THE dynamic stability of structures is a subject of considerable engineering importance and many investigations have been carried out in this regard. Development of new class of advanced materials such as functionally graded materials (FGMs), wherein the composition of each material constituent varies gradually with respect to spatial coordinates, and also piezoelectric materials as sensors and actuators have necessitated more research in this area. In 1985, Bailey and Hubbard [1] investigated the active vibration control of a cantilever beam using distributed piezoelectric polymer as an actuator. Crawley and de Luis [2] developed analytical models for the dynamic response of a cantilever beam with segmented piezoelectric actuators that are either bonded to an elastic substructure or embedded in a laminated composite. Shen [3] used the finite element method to study the free vibration problems of beams containing piezoelectric sensors and actuators.

Pierre and Dowell [4] reported the dynamic instability of plates using an extended incremental harmonic balance method. Liu et al. [5] used a finite element model to analyze the shape control and active vibration suppression of laminated composite plates with integrated piezoelectric sensors and actuators. By a feedback control loop, Tzou and Tseng [6] and Ha et al. [7] formulated three-dimensional incompatible finite elements for vibration control of structures containing piezoelectric sensors and actuators. The dynamic instability of a structure subjected to periodic axial compressive forces has attracted a lot of attention. The periodic axial forces may cause parametric vibration, a phenomenon that is characterized by unbounded growth of a small disturbance. It may eventually cause damages. Bolotin [8] summarized the results achieved in comprehensive studies for the dynamic stability of machine components and structural members. Briseghella et al. [9] used beam elements without axial deformability to solve the dynamic stability problem of beam structures. The load bending contribution was taken into account by means of a second-order approach. Takahashi et al. [10] investigated dynamically unstable regions of cantilever rectangular

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plates. They presented the numerical results obtained for various loading conditions that are applied along the edge. Recently, Zhu et al. [11] presented a three-dimensional theoretical analysis of the dynamic instability region of functionally graded piezoelectric circular cylindrical shells subjected to a combined loading of periodic axial compression and electric field in the radial direction.

To the author's knowledge, no research has been done on dynamic stability of functionally graded beams with piezoelectric actuators. In the present work, the dynamic stability of a functionally graded beam with piezoelectric actuators subjected to periodic axial compressive load that is simply supported at both ends lies on a continuous elastic foundation is studied. The elasticity modulus of functionally graded layer is assumed to vary as a power form of the thickness coordinate variable. Applying the Hamilton's principle, the dynamic equation of beam is derived. The effect of the applied voltages, piezoelectric thicknesses and functionally graded index on the unstable regions of beam are also discussed.

2 FORMULATIONS

Consider a functionally graded beam with piezoelectric actuators and rectangular cross-section as shown in Fig. 1. The thickness, length, and width of the beam are denoted, respectively, by $h, L,$ and b. Also, h_T and h_B are the thickness of top and bottom of piezoelectric actuators, respectively. The $x-y$ plane coincides with the mid-plane of the beam and the $z-$axis is located along the thickness direction. The Young's modulus E is assumed to vary as a power form of the thickness coordinate variable z $(-h/2 \le z \le h/2)$ as follow [12]

$$E(z) = (E_c - E_m)V + E_m$$
$$V = \left(\frac{2z+h}{2h}\right)^k \tag{1}$$

where k is the power law index and the subscripts m and c refer to the metal and ceramic constituents, respectively. The Poisson's ratio ν is assumed to be constant.

The beam is assumed to be slender, thus, the Euler-Bernoulli beam theory is adopted. The piezoelectric layers are also assumed to be polarized along the thickness direction. The axial stress and electrical displacement can be written as

$$\sigma_{xx} = \frac{E(z)}{1-\nu^2}\varepsilon_{xx} - e_{31}E_z$$
$$D_z = e_{31}\varepsilon_{xx} + \eta_{33}E_z \tag{2}$$

where

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \tag{3}$$

and

$$E_z = \frac{V}{h} \tag{4}$$

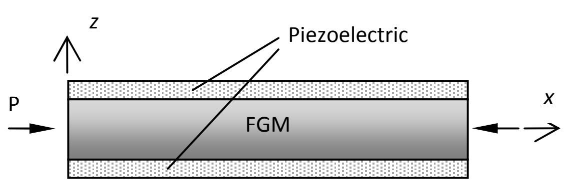

Fig. 1
Schematic view of the problem studied.

where σ_{xx}, D_z, e_{31}, and η_{33} are the normal stress, electrical displacement, piezoelectric elastic stiffness, and permittivity coefficient, respectively, and u and w are the displacement components in the $x-$ and $z-$directions, respectively. The potential energy can be expressed as

$$U=\frac{1}{2}\int_v(\sigma_{xx}\varepsilon_{xx}-D_zE_z)\,\mathrm{d}v \tag{5}$$

Substituting Eqs. (2)-(4) into Eq. (5) and neglecting the higher-order terms, we obtain

$$\begin{aligned}U=&\frac{1}{2}\int_v\frac{E(z)}{1-\nu^2}\left(\frac{\partial u}{\partial x}-z\frac{\partial^2 w}{\partial x^2}\right)^2\mathrm{d}v+\int_v\left[\frac{E(z)}{1-\nu^2}\left(\frac{\partial u}{\partial x}-z\frac{\partial^2 w}{\partial x^2}\right)-e_{31}\frac{V}{h}\right]\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\mathrm{d}v\\&-\int_v\left(\frac{\partial u}{\partial x}-z\frac{\partial^2 w}{\partial x^2}\right)e_{31}\frac{V}{h}\mathrm{d}v-\frac{1}{2}\int_v\left(\frac{V}{h}\right)^2\eta_{33}\mathrm{d}v\end{aligned} \tag{6}$$

The width of beam is assumed to be constant, which is obtained by integrating along y over v. Then Eq. (6) becomes

$$\begin{aligned}U=&\frac{b}{2}\int_0^L\left[A_{11}\left(\frac{\partial u}{\partial x}\right)^2-2B_{11}\frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial x^2}+D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2\right]\mathrm{d}x+\frac{b}{2}\int_0^L P'\left(\frac{\partial w}{\partial x}\right)^2\mathrm{d}x-\frac{b}{2}\int_0^L\eta_{33}\left(\frac{V_B^2}{h_B}+\frac{V_T^2}{h_T}\right)\mathrm{d}x\\&-b\int_0^L\left(\int_{-h_B-\frac{h}{2}}^{-\frac{h}{2}}\left(\frac{\partial u}{\partial x}-z\frac{\partial^2 w}{\partial x^2}\right)e_{31}\frac{V_B}{h_B}\mathrm{d}z+\int_{\frac{h}{2}}^{h_T+\frac{h}{2}}\left(\frac{\partial u}{\partial x}-z\frac{\partial^2 w}{\partial x^2}\right)e_{31}\frac{V_T}{h_T}\mathrm{d}z\right)\mathrm{d}x\end{aligned} \tag{7}$$

where

$$(A_{11},B_{11},D_{11})=\frac{1}{1-\nu^2}\int_{-h_B-\frac{h}{2}}^{h_T+\frac{h}{2}}(1,z,z^2)E(z)\mathrm{d}z \tag{8}$$

and

$$P'=\int_{-h_B-\frac{h}{2}}^{h_T+\frac{h}{2}}\left[\frac{E(z)}{1-\nu^2}\left(\frac{\partial u}{\partial x}-z\frac{\partial^2 w}{\partial x^2}\right)-e_{31}E_z\right]\mathrm{d}z=A_{11}\frac{\partial u}{\partial x}-B_{11}\frac{\partial^2 w}{\partial x^2}-\int_{-h_B-\frac{h}{2}}^{h_T+\frac{h}{2}}e_{31}\frac{V}{h}\mathrm{d}z \tag{9}$$

where $A_{11}, B_{11}, D_{11}, V_T, V_B$ and P' are the extensional stiffness, coupling stiffness, bending stiffness, applied voltages on the top and bottom actuators and piezoelectric force, respectively. When the applied voltage is negative, the piezoelectric force is tensile. Note that, no residual stresses due to the piezoelectric actuator are considered in the present study and the extensional displacement is neglected. Thus, the potential energy can be written as

$$U=\frac{b}{2}\int_\circ^L D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2\mathrm{d}x+\frac{P'b}{2}\int_\circ^L\left(\frac{\partial w}{\partial x}\right)^2\mathrm{d}x-\frac{b}{2}\int_\circ^L\eta_{33}\left(\frac{V_B^2}{h_B}+\frac{V_T^2}{h_T}\right)\mathrm{d}x+b\int_0^L\left[\int_{-h_B-\frac{h}{2}}^{-\frac{h}{2}}z\left(\frac{\partial^2 w}{\partial x^2}\right)e_{31}\frac{V_B}{h_B}\mathrm{d}z+\int_{\frac{h}{2}}^{h_T+\frac{h}{2}}z\left(\frac{\partial^2 w}{\partial x^2}\right)e_{31}\frac{V_T}{h_T}\mathrm{d}z\right]\mathrm{d}x \tag{10}$$

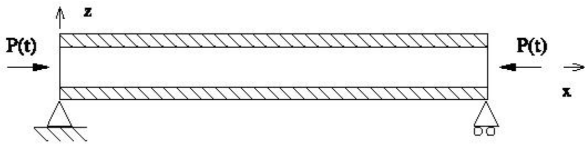

Fig. 2
Simply supported beam under periodic loads.

The beam is subjected to the periodic axial compressive loads, $P(t)$, as shown in Fig. 2.

$$P(t) = P_0 + P_t \cos\theta t = \alpha P^* + \beta P^* \cos\theta t \tag{11}$$

Here, α and β are the static and dynamic load factors, respectively. The work done by the periodic axial compressive load can be expressed as

$$W = \frac{1}{2}\int_0^L P(t)\left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x \tag{12}$$

The kinetic energy can be expressed as

$$T = \frac{1}{2}\int_\circ^L m\left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x \tag{13}$$

where m is the mass per unit length of the beam. The Hamilton's principle can be written as

$$\delta\int_0^t (T - U + W)\,\mathrm{d}t = 0 \tag{14}$$

Substitution from Eqs. (10), (12), and (13) into Eq. (14) leads to the following dynamic equation

$$m\frac{\partial^2 w}{\partial t^2} + bD_{11}\frac{\partial^4 w}{\partial x^4} + (P(t) - bP')\frac{\partial^2 w}{\partial x^2} = 0 \tag{15}$$

Assume that a functionally graded beam with piezoelectric actuators that is simply supported at both ends lies on a continuous elastic foundation, whose reaction at every point is proportional to the deflection (Winkler foundation). The dynamic equation of the functionally graded beams with piezoelectric layers located on a continuous elastic foundation subjected to a periodic axial compressive load is obtained from Eq. (15) by the addition of ηw for the foundation reaction as

$$m\frac{\partial^2 w}{\partial t^2} + bD_{11}\frac{\partial^4 w}{\partial x^4} + (P(t) - bP')\frac{\partial^2 w}{\partial x^2} + \eta w = 0 \tag{16}$$

where η is the foundation coefficient.

3 STABILITY ANALYSIS

For simply supported boundary condition, the solution of the dynamic equation is assumed as

$$w(x,t) = f_k(t)\sin\frac{k\pi x}{L}, \qquad k = 1,2,3,... \tag{17}$$

where $f_k(t)$ are as yet undetermined function of time, satisfies this equation. Substituting expression Eq. (17) into Eq. (16) leads to the following equation:

$$f_k'' + \omega_k^2 (1 - \frac{p(t) - bp'}{p_{*k}}) f_k = 0 \tag{18}$$

where

$$\omega_k^2 = \frac{1}{m}[bD_{11}(\frac{k\pi}{l})^4 + \eta] \tag{19}$$

$$p_{*k} = (\frac{k\pi}{l})^2 bD_{11} + \frac{l^2\eta}{k^2\pi^2} \tag{20}$$

where ω_k is the kth free vibration frequency of functionally graded beam with piezoelectric actuators loaded by a constant axial force P and p_{*k} is the critical buckling load. Analogous equations are obtained by considering the case of an infinitely long beam. In this case, Eq. (16) will be satisfied by assuming that

$$w(x,t) = f\ (t,\lambda)\sin\frac{\pi x}{\lambda} \tag{21}$$

where the length of the half-wave λ can take on arbitrary values from zero to infinity. Substitution leads to Eq. (18), where the parameter λ plays the part of the index k ; the coefficient of the equations depend on this parameter in the following manner:

$$\omega^2(\lambda) = \frac{1}{m}[\frac{bD_{11}\pi^4}{\lambda^4} + \eta] \tag{22}$$

$$p_*(\lambda) = \frac{\pi^2 bD_{11}}{\lambda^2} + \frac{\eta\lambda^2}{\pi^2} \tag{23}$$

Thus, for a given length of the half-wave, the boundaries of the principal regions of dynamic instability can be determined by the harmonic balance method [8]. Therefore, the boundary frequency of the instability region obtained as follow

$$\theta_*^2(\lambda) = \frac{4}{m}[\frac{\pi^2 bD_{11}}{\lambda^4} - \frac{\pi^2(p_0 \pm \frac{p_t}{2} - bp')}{\lambda^2} + \eta] \tag{24}$$

By setting the power law index equal to zero $(k = 0)$ and neglecting the piezoelectric effect, Eq. (16) is reduced to the parametric resonance of homogeneous beams

$$m\frac{\partial^2 w}{\partial t^2} + bD\frac{\partial^4 w}{\partial x^4} + P(t)\frac{\partial^2 w}{\partial x^2} + \eta w = 0 \tag{25}$$

where

$$D = \frac{Eh^3}{12} \tag{26}$$

Eq. (25) has been reported by Bolotin [8].

4 NUMERICAL RESULTS

The dynamic stability of a functionally graded beam with piezoelectric actuators subjected to periodic axial compressive load that is simply supported at both ends lies on a continuous elastic foundation is studied in this paper. It is assumed that both the top and bottom piezoelectric layers have the same thickness; $h_T = h_B$ and the same voltages are applied to both actuators. The material properties of the beam are listed in Table 1. The effect of power law index k and applied voltage (V) on the frequency of the instability region $\theta_*^2(\lambda)$ is shown in Fig. 3. It is found that, as k increases, the boundary frequency of the instability region decreases. Fig. 4 illustrates the effect of the foundation coefficient (η) on the frequency of the instability region. As the foundation coefficient increases, the frequency of the instability region increases. Also, comparisons of the frequency of the instability region for the functionally graded beam and isotropic beam are shown in Fig. 4. It is evident that the frequency of the instability region decreases when the beam is made of functionally graded materials.

Table 1
Material properties

Property	Piezoelectric layer	FGM layer	
		Stainless steel	Nickel
Young's modulus E (GPa)	63	221.04	223.95
Poisson's ratio ν	0.3	0.3	0.3
Length L (m)	0.3	0.3	0.3
Thickness h (m)	0.00005	0.01	0.01
Density ρ (Kgm^{-3})	7600	8166	8900
Piezoelectric constant e_{31}, e_{32} (Cm^{-2})	17.6	-	-

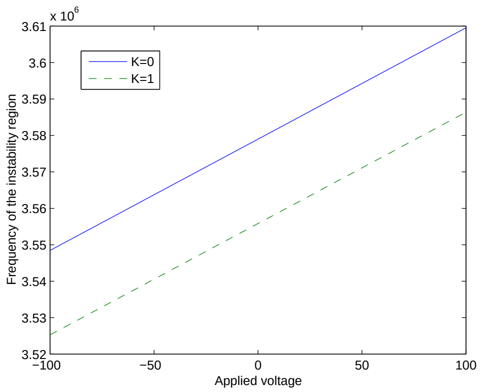

Fig. 3
Effect of power law index on the free vibration frequency.

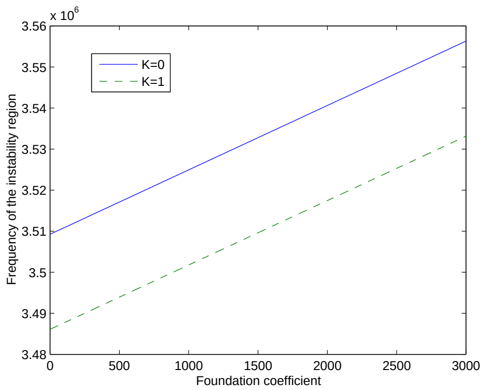

Fig. 4
Effect of foundation coefficient on the free vibration frequency.

5 CONCLSIONS

The dynamic stability of a functionally graded beam with piezoelectric actuators subjected to periodic axial compressive load that is simply supported at both ends lies on a continuous elastic foundation has been presented. It was shown that the piezoelectric actuators induce tensile piezoelectric force produced by applying negative voltages that significantly affect the frequency of the instability region of the functionally graded beam with piezoelectric actuators. The frequency of the instability region decreases when the applied voltage is negative. The functionally graded beam with a smaller foundation coefficient is more stable. The comparison of the stability for the functionally graded beam and isotropic beam shows that the functionally graded beam is more stable.

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