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Generalized Thermoelastic Problem of a Thick Circular Plate with Axisymmetric Heat Supply Due to Internal Heat Generation

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ABSTRACT

A two dimensional generalized thermoelastic problem of a thick circular plate of finite thickness and infinite extent subjected to continuous axisymmetric heat supply and an internal heat generation is studied within the context of generalized thermoelasticity. Unified system of equations for classical coupled thermoelasticity, Lord-Shulman and Green-Lindsay theory is considered. An exact solution of the problem is obtained in the transform domain. Inversion of Laplace transforms is done by employing numerical scheme. Mathematical model is prepared for Copper material plate and the numerical results are discussed and represented graphically.

Keywords : Thermoelasticity; Classical coupled; Lord-shulman; Green-lindsay; Internal heat generation; Axisymmetric heat supply.

1 INTRODUCTION

THE last few decades has seen a rapid development of generalized thermoelasticity, so as to overcome the shortcomings in the classical uncoupled and classical coupled thermoelasticity (CCTE) theories [1,2]. The CCTE predicts infinite speed of propagation of thermal disturbances which is contrary to the observed phenomena. It also gives unsatisfactory description of thermoelastic behavior of materials at low temperatures, inaccurate in studying the response of short laser pulse. Thus, the first modification to CCTE was put forth by Lord and Shulman [3] also known as extended thermoelasticity (ETE), wherein one relaxation time was introduced in the Fourier's law of heat conduction to obtain a hyperbolic heat conduction equation. The second modification was made by Green and Lindsay [4] also known as temperature rate dependent thermoelasticity (TRDTE), by the introduction of two different relaxation times into the constitutive relations of stress tensor and entropy equation. In TRDTE, Fourier's law of heat conduction is not violated if the body under consideration has a centre of symmetry. A detailed discussion on dynamic problems of thermoelasticity and theory of thermal stresses in generalized thermoelasticity can be found in [5,6]. Chandrasekariah [7] has referred to the wave like thermal disturbance as "The Second sound". Hetnarski and Ignaczak [8] examined five generalizations to the coupled theory and obtained many of important analytical results. Tripathi et al. [9] discussed a dynamic problem of generalized thermoelasticity in Lord-Shulman theory for a semi-infinite cylinder with heat sources. Maghraby and Abdel Halim [10] studied a problem of generalized thermoelasticity in Lord-Shulman theory for a half space subjected to a known axisymmetric temperature distribution. Aouadi [11] studied the discontinuities in an axisymmetric problem of thermoelasticity

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without heat source in the context of generalized thermoelasticity. Tripathi et al. [12] studied a problem of generalized thermoelastic diffusion interactions in a thick circular plate. Youssef [13] discussed a two dimensional generalized thermoelastic problem for a half space subjected to ramp type heating. Recently, Tripathi et al. [14, 15] discussed problems on generalized thermoelastic diffusion in a half space due to thermal and mechanical sources under axisymmetric distributions.

In the present paper, the work of Tripathi et al. [12] has been modified by considering a continuous axisymmetric heat supply with internal heat generation to a thick circular plate of infinite extent and finite thickness within the context of unified system of equations in classical coupled, Lord-Shulman and Green-Lindsay theory. The exact solutions for temperature distribution, displacement and the stress components are obtained in the Laplace transform domain. Numerical inversion of Laplace transform is performed based on Gaver-Stehfast [16-18] algorithm which is considerably more stable and computationally efficient than inversion using the discrete Fourier transform. All the integrals were calculated using Romberg's integration technique [19] with variable step size. The application of an internal heat generation is of particular interest in many engineering problems like thick-walled pressure vessels, such as a nuclear containment vessel, a cylindrical roller, etc.

2 FORMULATION OF THE PROBLEM

Consider a thick circular plate of thickness *2b* occupying the space *D* defined by $0 \le r \le \infty, -b \le z \le b$. Let the plate be subjected to a continuous axisymmetric temperature field dependent on the radial and axial directions of the cylindrical co-ordinate system. For time $t \succ 0$, heat is generated within the plate at the rate $Q(r,z,t)$. Under these conditions, the thermoelastic quantities in a semi-infinite thick circular plate are required to be determined.

We take the axis of symmetry as the *z* axis and the origin of the system of co-ordinates is at the middle plane between the upper and lower faces of the plate. The problem is studied using the cylindrical polar co-ordinates (r,φ,z). Due to the rotational symmetry about the *z* axis, all quantities are independent of the co-ordinate φ.

The unified field equations governing the displacement, the thermal fields and the stress-strain-temperature relations in the context of TRDTE, ETE and CCTE for a homogeneous and isotropic medium are given by [11]

Equation of motion

$$(r,\mu\,\ddot{u}_{i,jj} + (\lambda+\mu)\ u_{j,ji} - \gamma\left(1+\tau_1\frac{\partial}{\partial t}\right)T_{,i} = \rho\,\ddot{u}_i \tag{1}$$

Equation of heat conduction

$$KT_{,ii} = \rho C_E\left(\frac{\partial}{\partial t} + \tau_0\frac{\partial^2}{\partial t^2}\right)\dot{T} + \left(1+\eta_0\tau_0\frac{\partial}{\partial t}\right)\left(T_0\gamma\dot{u}_{j,j} - \rho Q\right) \tag{2}$$

The constitutive relations

$$\sigma_{ij} = \mu\left(u_{i,j}+u_{j,i}\right) + \left[\lambda u_{i,i} - \gamma\left(T+\tau_1\dot{T}\right)\right]\delta_{ij} \tag{3}$$

where λ and μ are Lamé's constants, τ_0 and τ_1 are relaxation times, γ is a material constant given by $\gamma = (3\lambda+2\mu)\alpha_t, \alpha_t$ is the coefficient of linear thermal expansion, T_0 is the reference temperature chosen such that $\left|(T-T_0)/T_0\right| << 1$.

The use of symbol η_0 in Eq.(2) makes these fundamental equations possible for three different theories of thermoelasticity. For classical coupled thermoelasticity (CCTE) theory, $\eta_0 = 1, \tau_0 = \tau_1 = 0$; for the Lord-Shulman (ETE) theory, $\eta_0 = 1,\ \tau_1 = 0, \tau_0 > 0$; for Green-Lindsay (TRDTE) theory, $\eta_0 = 0$, $\tau_1 > \tau_0 > 0$.

The dilatation *e* is given by

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \text{ and } e = e_{ii} \tag{4}$$

The displacement vector, thus, has the form $\vec{u} = (u, 0, w)$. The equations of motion can be written as:

$$\mu \nabla^2 u - \frac{\mu}{r^2} u + (\lambda + \mu) \frac{\partial e}{\partial r} - \gamma \left(1 + \tau_1 \frac{\partial}{\partial t}\right) \frac{\partial T}{\partial r} = \rho \frac{\partial^2 u}{\partial t^2} \tag{5}$$

$$\mu \nabla^2 w + (\lambda + \mu) \frac{\partial e}{\partial z} - \gamma \left(1 + \tau_1 \frac{\partial}{\partial t}\right) \frac{\partial T}{\partial z} = \rho \frac{\partial^2 w}{\partial t^2} \tag{6}$$

The generalized equation of heat conduction has the form

$$K \nabla^2 T = \rho C_E \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2}\right) T + T_0 \gamma \left(\frac{\partial}{\partial t} + \eta_0 \tau_0 \frac{\partial^2}{\partial t^2}\right) e - \rho \left(1 + \eta_0 \tau_0 \frac{\partial}{\partial t}\right) Q \tag{7}$$

where T is the absolute temperature and e is the dilatation given by the relation

$$e = \frac{u}{r} + \frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} = \frac{1}{r} \frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} \tag{8}$$

where the Laplacian operator is given by $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$

The following constitutive relations supplement the above equations

$$\sigma_{\phi\phi} = 2\mu \frac{u}{r} + \lambda e - \gamma \left(T - T_0 + \tau_1 \frac{\partial T}{\partial t}\right) \tag{9}$$

$$\sigma_{rr} = 2\mu \frac{\partial u}{\partial r} + \lambda e - \gamma \left(T - T_0 + \tau_1 \frac{\partial T}{\partial t}\right) \tag{10}$$

$$\sigma_{zz} = 2\mu \frac{\partial w}{\partial z} + \lambda e - \gamma \left(T - T_0 + \tau_1 \frac{\partial T}{\partial t}\right) \tag{11}$$

$$\sigma_{rz} = \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) \tag{12}$$

$$\sigma_{r\phi} = \sigma_{z\phi} = 0 \tag{13}$$

We shall use the following non-dimensional variables

$$\mathrm{r}' = \eta_0 \mathrm{r}, \ \mathrm{z}' = \eta_0 \mathrm{z}, \ \mathrm{u}' = \eta_0 \mathrm{u}, \ \mathrm{w}' = \eta_0 \mathrm{w}, \ \mathrm{t}' = \mathrm{c}_1 \eta_0 \mathrm{t}, \ \tau_0' = \mathrm{c}_1 \eta_0 \tau_0, \ \sigma_{ij}' = \frac{\sigma_{ij}}{\mu},$$

$$\theta = \frac{\gamma (T - T_0)}{(\lambda + 2\mu)}, \ \tau_1' = \mathrm{c}_1 \eta_0 \tau_1, \ Q' = \frac{\rho \gamma Q}{K c_1^2 \eta_0^2 (\lambda + 2\mu)}$$

where $\eta_0 = \dfrac{\rho c_1 C_E}{K}$ is the dimensionless characteristic length, $c_1 = \sqrt{\dfrac{\lambda + 2\mu}{\rho}}$ is the speed of propagation of longitudinal wave.

Using the above non-dimensional variables, the governing Eqs. (5)-(13) take the form,

$$\nabla^2 u - \frac{1}{r^2} u + (\xi^2 - 1)\frac{\partial e}{\partial r} - \xi^2 \left(1 + \tau_1 \frac{\partial}{\partial t}\right)\frac{\partial \theta}{\partial r} = \xi^2 \frac{\partial^2 u}{\partial t^2} \tag{14}$$

$$\nabla^2 w + (\xi^2 - 1)\frac{\partial e}{\partial z} - \xi^2 \left(1 + \tau_1 \frac{\partial}{\partial t}\right)\frac{\partial \theta}{\partial z} = \xi^2 \frac{\partial^2 w}{\partial t^2} \tag{15}$$

$$\nabla^2 \theta = \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2}\right)\theta + \varepsilon\left(\frac{\partial}{\partial t} + \eta_0 \tau_0 \frac{\partial^2}{\partial t^2}\right)e - \xi^2\left(1 + \eta_0 \tau_0 \frac{\partial}{\partial t}\right)Q \tag{16}$$

$$\sigma_{rr} = 2\frac{\partial u}{\partial r} + \left(\xi^2 - 2\right)e - \xi^2\left(1 + \tau_1 \frac{\partial}{\partial t}\right)\theta \tag{17}$$

$$\sigma_{\phi\phi} = 2\frac{u}{r} + \left(\xi^2 - 2\right)e - \xi^2\left(1 + \tau_1 \frac{\partial}{\partial t}\right)\theta \tag{18}$$

$$\sigma_{zz} = 2\frac{\partial w}{\partial z} + \left(\xi^2 - 2\right)e - \xi^2\left(1 + \tau_1 \frac{\partial}{\partial t}\right)\theta \tag{19}$$

$$\sigma_{rz} = 2\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) \tag{20}$$

where $\xi^2 = \dfrac{(\lambda + 2\mu)}{\mu}, \varepsilon = \dfrac{T_0 \gamma^2}{\rho C_E (\lambda + 2\mu)}$.

Using Helmholtz decomposition theorem, we seek the displacement components u and w in the form,

$$u = \frac{\partial \phi}{\partial r} + \frac{\partial^2 \psi}{\partial r \partial z} \tag{21}$$

$$w = \frac{\partial \phi}{\partial z} - \left(\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r}\right) \tag{22}$$

where the potential functions ϕ and ψ are the Lame's potentials representing irrotational and rotational parts of the displacement vector $\vec{u}$ respectively.

From Eqs. (4), (21) and (22), we obtain

$$e = \nabla^2 \phi \tag{23}$$

Using Eqs. (21)-(23) into Eqs. (14)-(16), we get,

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2}\right)\phi - \left(1 + \tau_1 \frac{\partial}{\partial t}\right)\theta = 0 \tag{24}$$

$$\left(\nabla^2 - \xi^2 \frac{\partial^2}{\partial t^2}\right)\psi = 0 \tag{25}$$

$$\left(\nabla^2 - \frac{\partial}{\partial t} - \tau_0 \frac{\partial^2}{\partial t^2}\right)\theta - \varepsilon\left(\frac{\partial}{\partial t} + \eta_0\tau_0 \frac{\partial^2}{\partial t^2}\right)\nabla^2\phi = -\,\xi^2\left(1 + \eta_0\tau_0 \frac{\partial}{\partial t}\right)Q \tag{26}$$

Eq.(25) for the function ψ represents a wave equation with wave velocity $v = 1/\xi$. This represents a transverse wave and it has no effect on temperature. Eq.(26) represents a longitudinal thermal wave moving with velocity $v_L = 1/\sqrt{\tau_0}$.

We shall assume that the initial state of the medium is quiescent. The boundary conditions of the problem are taken as:

$$\frac{\partial\theta}{\partial z} = \pm g_0 F(r,z), \quad z = \pm b \tag{27}$$

and the traction free surface stress functions,

$$\sigma_{zz} = \sigma_{rz} = 0, \quad z = \pm b \tag{28}$$

3 ANALYTICAL SOLUTION

Applying the Laplace transform defined by the relation,

$$\bar{f}(r,z,p) = L\left[f(r,z,t)\right] = \int_0^\infty e^{-pt} f(r,z,t)dt$$

and the Hankel transform of order zero with respect to r of a function $\bar{f}(r,z,p)$ defined by the relation,

$$\bar{f}^*(\alpha,z,p) = H\left[\bar{f}(r,z,p)\right] = \int_0^\infty \bar{f}(r,z,p)\, r\, J_0(\alpha r)dr$$

where J_0 is the Bessel function of the first kind of order zero. The inverse Hankel transform is given by the relation

$$\bar{f}(r,z,p) = H^{-1}\left[\bar{f}^*(\alpha,z,p)\right] = \int_0^\infty \bar{f}^*(\alpha,z,p)\ \alpha J_0(\alpha r)d\alpha$$

Applying Laplace and Hankel transform to Eqs. (24)-(26), we get,

$$\left(D^2 - \alpha^2 - p^2\right)\bar{\phi}^* - \left(1 + \tau_1 p\right)\bar{\theta}^* = 0 \tag{29}$$

$$\left(D^2 - \alpha^2 - \xi^2 p^2\right)\bar{\psi}^* = 0 \tag{30}$$

$$\left(D^2 - \alpha^2 - p - \tau_0 p^2\right)\bar{\theta}^* - \varepsilon\, p\left(1 + \eta_0\tau_0 p\right)\left(D^2 - \alpha^2\right)\bar{\phi}^* = -(1 + \eta_0\tau_0\, p)\bar{Q} \tag{31}$$

where $D = \partial/\partial z$. On eliminating $\bar{\theta}^*$ between Eqs. (29) and (31), we get,

$$\left(D^2-k_1^{\,2}\right)\left(D^2-k_2^{\,2}\right)\bar{\phi}^* = -\left(1+\tau_1 p\right)(1+\eta_0\tau_0\, p)\bar{Q} \tag{32}$$

where $\pm k_1$ and $\pm k_2$ are the roots of the characteristic equation given by,

$$k^4-\left(f(p)+2\alpha^2\right)k^2+\alpha^2(f(p)+\alpha^2)+p^3(1+\tau_0\, p)=0 \tag{33}$$

where $f(p)=p^2+p(1+\tau_0\, p)+\varepsilon\, p(1+\eta_0\tau_0 p)(1+\tau_1 p)$. The solution of Eq. (32) can be written in the form,

$$\bar{\phi}^* = \sum_{i=1}^{2}\bar{\phi}_i^* + \bar{\phi}_p^* \tag{34}$$

where $\bar{\phi}_i^*$ is a general solution of the homogeneous differential equation given by

$$\left(D^2-\alpha_i^{\,2}\right)\bar{\phi}_i^* = 0 \quad ,i=1,2. \tag{35}$$

The solution of Eq. (35) can be written in the form

$$\bar{\phi}_i^* = C_i(\alpha,p)\cosh(k_i z) \ \ , \ i=1,2 \tag{36}$$

and $\bar{\phi}_p^*$ is the particular integral satisfying the equation

$$\left(D^2-k_1^{\,2}\right)\left(D^2-k_2^{\,2}\right)\bar{\phi}_p^* = -\left(1+\tau_1 p\right)(1+\eta_0\tau_0\, p)\bar{Q} \tag{37}$$

We take the internal heat generation $Q(r,z,t)$ in the following form

$$Q(r,z,t)=\frac{q_0\delta(t)\,\delta(r)\cosh z}{2\pi r} \tag{38}$$

This is a cylindrical shell heat source releasing heat instantaneously at $t=0$ and situated at the centre $r=0$ varying in the axial direction where q_0 denotes the strength of the internal heat generation. Let $Q_0=\dfrac{q_0}{2\pi}$.

On applying the Laplace and Hankel transform to Eq. (38), we get,

$$\bar{Q}^* = Q_0\cosh z \tag{39}$$

The solution of the Eq. (37) can be represented as,

$$\bar{\phi}_p^* = -\frac{Q_0\left(1+\tau_1 p\right)\left(1+\eta_0\tau_0 p\right)}{\left(1-k_1^{\,2}\right)\left(1-k_2^{\,2}\right)}\cosh z \tag{40}$$

The complete solution ofEq. (32) is given by,

$$\bar{\phi}^* = \sum_{i=1}^{2} C_i(\alpha,p)\cosh(k_i z) - \frac{Q_0\left(1+\tau_1 p\right)\left(1+\eta_0\tau_0 p\right)}{\left(1-k_1^{\,2}\right)\left(1-k_2^{\,2}\right)}\cosh z \tag{41}$$

On using Eq. (41) into Eq. (29), we get,

$$\bar{\theta}^* = \sum_{i=1}^{2} \frac{k_i^2 - \alpha^2 - p^2}{1+\tau_1 p} C_i(\alpha, p)\cosh(k_i z) - \frac{Q_0\left(1+\eta_0\tau_0 p\right)\left(1-\alpha^2-p^2\right)}{\left(1-k_1^2\right)\left(1-k_2^2\right)}\cosh z \tag{42}$$

On solving Eq. (30), we get,

$$\bar{\psi}^* = D(\alpha, p)\sinh(az) \tag{43}$$

where $a^2 = \alpha^2 + \xi^2 p^2$. In Eqs. (41)-(43), the constants $C_1(\alpha,p), C_2(\alpha,p)$ and $D(\alpha,p)$ depend on both α and p. Using Eqs. (21) and (22) in Eqs. (19) and (20), the stress components $\bar{\sigma}^*_{zz}$, $\bar{\sigma}^*_{rz}$ take the form,

$$\bar{\sigma}^*_{zz} = \left(\xi^2 p^2 + 2\alpha^2\right)\bar{\phi}^* + 2\alpha^2 \frac{\partial \bar{\psi}^*}{\partial z} \tag{44}$$

$$\bar{\sigma}^*_{rz} = H\left\{\frac{\partial}{\partial r}\left[2D\bar{\phi} + \left(2D^2 - \xi^2 p^2\right)\bar{\psi}\right]\right\} \tag{45}$$

where H represents the Hankel transform.

Now applying the Laplace and Hankel transform to the boundary conditions (27) and (28), we get,

$$\frac{\partial \bar{\theta}^*}{\partial z} = \pm g_0 \bar{F}^*(\alpha, z) \quad , \quad z = \pm b \tag{46}$$

$$\bar{\sigma}^*_{zz} = \bar{\sigma}^*_{rz} = 0 \quad , \quad z = \pm b \tag{47}$$

Here, we consider the function $F(r,z)$ which falls off exponentially as one moves away from the centre of the plate in the radial direction and increases symmetrically along the axial direction given by, $F(r,z)=z^2 e^{-\omega r}$, $\omega > 0$

On applying Laplace and Hankel transforms to the above function, we get,

$$\bar{F}^*(\alpha, z) = \frac{z^2 \omega}{p\left(\omega^2 + \alpha^2\right)^{3/2}} \tag{48}$$

Making use of the values of $\bar{\theta}, \bar{\sigma}_{zz}$ and $\bar{\sigma}_{rz}$ in the boundary conditions (46)-(47) and with the aid of Eq. (48), we get,

$$\sum_{i=1}^{2} \frac{C_i\left(k_i^2 - \alpha^2 - p^2\right)k_i \sinh(k_i b)}{(1+\tau_1 p)} - \frac{Q_0\left(1+\eta_0\tau_0 p\right)\left(1-\alpha^2-p^2\right)}{\left(1-k_1^2\right)\left(1-k_2^2\right)}\sinh(b) = \frac{g_0 b^2 \omega}{p\left(\omega^2+\alpha^2\right)^{3/2}} \tag{49}$$

$$\left(\xi^2 p^2 + 2\alpha^2\right)\left\{\sum_{i=1}^{2} C_i \cosh(k_i b) - \frac{Q_0\left(1+\tau_1 p\right)\left(1+\eta_0\tau_0 p\right)}{\left(1-k_1^2\right)\left(1-k_2^2\right)}\cosh(b)\right\} + 2\alpha^2 a D\cosh(ab) = 0 \tag{50}$$

$$2\sum_{i=1}^{2} k_i C_i \sinh(k_i b) - \frac{2Q_0\left(1+\tau_1 p\right)\left(1+\eta_0\tau_0 p\right)}{\left(1-k_1^2\right)\left(1-k_2^2\right)}\sinh(b) + \left(2a^2 - \xi^2 p^2\right)D\sinh(ab) = 0 \tag{51}$$

On solving Eqs.(49)-(51) which is a system of linear equations with $C_1(\alpha,p), C_2(\alpha,p)$ and $D(\alpha,p)$ as unknown parameters, we get the complete solution of the problem in the Laplace transform domain.

3.1 Inversion of Hankel transform

On applying inversion of Hankel transform to equation (42), we obtain,

$$\bar{\theta}(r,z,p)=\int_0^\infty\left\{\sum_{i=1}^{2}\frac{k_i^2-\alpha^2-p^2}{1+\tau_1 p}C_i(\alpha,p)\cosh(k_i z)-\frac{Q_0\left(1+\eta_0\tau_0 p\right)\left(1-\alpha^2-p^2\right)}{\left(1-k_1^2\right)\left(1-k_2^2\right)}\cosh z\right\}\alpha J_0(\alpha r)\,d\alpha \tag{52}$$

Using Eqs.(41)-(43) in Eqs. (21)-(22) and taking inverse Hankel transform, it yields the solution for displacement components in Laplace transform domain,

$$\bar{u}(r,z,p)=\int_0^\infty-\alpha^2\left[\sum_{i=1}^{2}C_i(\alpha,p)\cosh(k_i z)-\frac{Q_0\left(1+\tau_1 p\right)\left(1+\eta_0\tau_0 p\right)}{\left(1-k_1^2\right)\left(1-k_2^2\right)}\cosh z+D(\alpha,p)\cosh(az)\right]J_1(\alpha r)\,d\alpha \tag{53}$$

$$\bar{w}(r,z,p)=\int_0^\infty\left[\sum_{i=1}^{2}k_i\,C_i(\alpha,p)\sinh(k_i z)-\frac{Q_0\left(1+\tau_1 p\right)\left(1+\eta_0\tau_0 p\right)}{\left(1-k_1^2\right)\left(1-k_2^2\right)}\sinh z+D(\alpha,p)\,\alpha^2\sinh(az)\right]\alpha J_0(\alpha r)\,d\alpha \tag{54}$$

Applying Laplace transform to Eqs. (17)-(19) and using the solutions given in Eqs. (52)-(54), we obtain the stress components in the Laplace transform domain,

$$\bar{\sigma}_{rr}(r,z,p)=\int_0^\infty\left\{\begin{array}{l}\alpha J_0(\alpha r)\sum_{i=1}^{2}(\xi^2p^2+2\alpha^2-2k_i^2)C_i(\alpha,p)\cosh(k_i z)+\left(2-2\alpha^2-\xi^2p^2\right)\dfrac{Q_0\left(1+\tau_1 p\right)\left(1+\eta_0\tau_0 p\right)}{\left(1-k_1^2\right)\left(1-k_2^2\right)}\cosh z\\[2ex] +2\alpha^3\left(\dfrac{1}{\alpha r}J_1(\alpha r)-J_0(\alpha r)\right)\left[\begin{array}{l}C_i(\alpha,p)\cosh k_i z+D(\alpha,p)a\,\cosh(az)\\ \sum_{i=1}^{2}-\dfrac{Q_0\left(1+\tau_1 p\right)\left(1+\eta_0\tau_0 p\right)}{\left(1-k_1^2\right)\left(1-k_2^2\right)}\cosh z\end{array}\right]\end{array}\right\}d\alpha \tag{55}$$

$$\bar{\sigma}_{\varphi\varphi}(r,z,p)=\int_0^\infty\left\{\begin{array}{l}\alpha J_0(\alpha r)\left[\sum_{i=1}^{2}(\xi^2p^2+2\alpha^2-2k_i^2)C_i(\alpha,p)\cosh_i(k_i z)+\left(2-2\alpha^2-\xi^2p^2\right)\dfrac{Q_0\left(1+\tau_1 p\right)\left(1+\eta_0\tau_0 p\right)}{\left(1-k_1^2\right)\left(1-k_2^2\right)}\cosh z\right]\\[2ex] -\dfrac{2}{r}\left[\sum_{i=1}^{2}C_i(\alpha,p)\cosh(k_i z)+D(\alpha,p)a\,\cosh(az)-\dfrac{Q_0\left(1+\tau_1 p\right)\left(1+\eta_0\tau_0 p\right)}{\left(1-k_1^2\right)\left(1-k_2^2\right)}\cosh z\right]\alpha^2J_1(\alpha r)\end{array}\right\}d\alpha \tag{56}$$

$$\bar{\sigma}_{zz}(r,z,p)=\int_0^\infty(\xi^2p^2+2\alpha^2)\left[\begin{array}{l}\sum_{i=1}^{2}C_i(\alpha,p)\cosh(k_i z)-\dfrac{Q_0\left(1+\tau_1 p\right)\left(1+\eta_0\tau_0 p\right)\left(1-\alpha^2-p^2\right)}{\left(1-k_1^2\right)\left(1-k_2^2\right)}\cosh z\\ +2\alpha^2D(\alpha,p)\cosh(az)\end{array}\right]\alpha J_0(\alpha r)\,d\alpha \tag{57}$$

Eqs. (52)-(57) present the complete solution of the problem in the Laplace transform domain.

4 INVERSION OF DOUBLE TRANSFORMS

Due to the complexity of the solution in the Laplace transform domain, the inverse of the Laplace transform is obtained using the Gaver-Stehfast algorithm. Gaver [16] and Stehfast [17,18] derived the formula given below. By this method the inverse $f(t)$ of the Laplace transform $\bar{f}(p)$ is approximated by,

$$f(t)=\frac{\ln 2}{t}\sum_{j=1}^{K}D(j,K)F\left(j\frac{\ln 2}{t}\right) \tag{58}$$

with

$$D(j,K)=(-1)^{j+M}\sum_{n=m}^{\min(j,M)}\frac{n^{M}(2n)!}{(M-n)!\,n!(n-1)!(j-n)!(2n-j)!} \tag{59}$$

where K is an even integer, whose value depends on the word length of the computer used. $M=K/2$ and m is the integer part of the $(j+1)/2$. The optimal value of K was chosen as described in Gaver-Stehfast algorithm, for the fast convergence of results with the desired accuracy. The Romberg numerical integration technique [19] with variable step size was used to evaluate the integrals involved. All the programs were made in mathematical software Matlab.

5 NUMERICAL RESULTS AND DISCUSSION

Mathematical model is prepared with Copper material for purposes of numerical computations. The material constants of the problem are given below [11]

$K=386N/K\,s$, $\alpha_t=1.78\times10^{-5}K^{-1}$, $C_E=383.1m^2/K$, $b=1$, $\eta=8886.73\,m.s^{-2}$

$\mu=3.86\times10^{10}N.m^{-2}$, $\lambda=7.76\times10^{10}N.m^{-2}$, $\rho=8954\ kg.m^{-3}$, $c_1=4.158\times10^{3}m.s^{-1}$

$\tau_0=0.02$, $\tau_1=0.08$, $T_0=293\ K$, $\varepsilon=0.0168$, $\beta^2=4$, $\omega=5$, $Q=1$

The numerical values for temperature θ, the radial displacement component u, the axial stress component σ_{zz} have been calculated at the middle of the plane ($z=0$) for different time instants $t=0.1$, 1.2 along the radial direction and are displayed graphically for CCTE, ETE and TRDTE theories of thermoelasticity as shown in Figs. 1, 2 and 3 respectively. The displacement component w vanishes at the middle plane of the plate and the differences among the different models for the other stress components are similar to those for the axial stress component σ_{zz}.

From Figs. 1, 2, and 3, it is observed that at time $t=0.1$, the CCTE, ETE and TRDTE theories show different results and for time $t=1.2$, the ETE and TRDTE models are in somewhat agreement. This is due to the arrival of the elastic wave at the middle plane at this time. From Fig. 1, it is clearly seen that the non-dimensional temperature θ drops gradually along the radial direction. It is also observed that at time $t=0.1$ the non-dimensional temperature θ and axial stress σ_{zz} exhibit non-zero values for all the three models of thermoelasticity. In addition, temperature θ and radial displacement u do not become identically zero due to presence of the continuous axisymmetric heat supply and internal heat generation within it. It is further observed from Fig.3 that at time $t=0.1$, the axial stress σ_{zz} is initially tensile in nature and becomes compressive near $r=5$. For time $t=1.2$, the axial stress σ_{zz} is compressive in nature and becomes tensile after $r=5$.

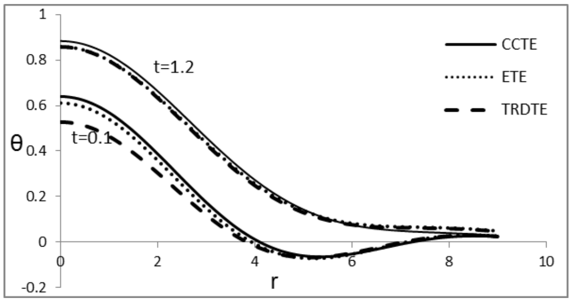

Fig.1
Temperature distribution θ in the middle plane.

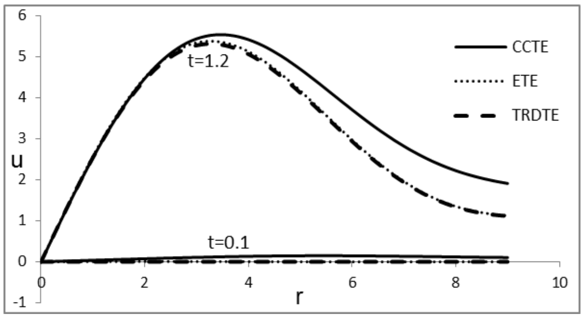

Fig.2
Radial displacement u distribution in the middle plane.

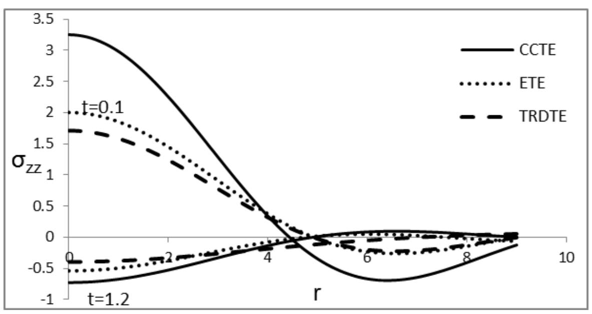

Fig.3
Axial stress component σ_{zz} in the middle plane.

6 CONCLUSIONS

In this study we have used the generalized theories of thermoelasticity i.e. ETE and TRDTE models to solve the problem for thick circular plate of infinite extent with continuous axisymmetric heat supply and internal heat generation within it and compared the models with CCTE. The generalized theories ETE and TRDTE models involve a hyperbolic wave equation, thus predicting finite speeds of heat propagation whereas the CCTE model involves a parabolic heat equation, thus predicting infinite speeds of heat wave propagation. Thus for small time the differences in results between the three theories are more visible as the heat wave takes time to reach the middle plane in ETE and TRDTE models. Thus for studying realistic engineering problems ETE and TRDTE models of generalized thermoelasticity must be studied instead of CCTE theory. As a special case, we have constructed a mathematical model for copper plate with axisymmetric heat supply and internal heat generation. Note that our numerical procedure successfully evaluated the solutions in the time domain. We may also conclude that the system of equations in this paper may prove to be useful in studying the thermal characteristics of various bodies in important engineering problems using the more realistic generalized models of thermoelasticity instead of CCTE.

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REFERENCES

[1] Biot M. A., 1956, Thermoelasticity and irreversible thermodynamics, *Journal of Applied Physics* **27**: 240-253.
[2] Nowacki W., 1966, Couple stresses in the theory of thermoelasticity , *Bulletin L'Academie Polonaise des Science, Serie des Sciences Technology* **14**: 129-138.
[3] Lord H., Shulman Y., 1967, A generalized dynamical theory of thermo-elasticity, *Journal of the Mechanics and Physics of Solids* **15**: 299-309.
[4] Green A. E., Lindsay K. A, 1972, Thermoelasticity, *Journal of Elasticity* **2**: 1-7.
[5] Nowacki W., 1975, *Dynamic Problems of Thermoelasticity*, Noordhoff International Publishing, Leyden, The Netherlands.
[6] Hetnarski R. B., Eslami M. R., 2009, *Thermal Stresses-Advanced Theory and Applications*, Springer.
[7] Chandrasekariah D. S., 1986, Thermoelasticity with second sound: a review, *Applied Mechanics Review* **39**: 355-376.
[8] Hetnarski R. B., Ignaczak J., 1999, Generalized thermoelasticity, *Journal of Thermal Stresses* **22**: 451-476.
[9] Tripathi J. J., Kedar G. D., Deshmukh K. C., 2014, Dynamic problem of generalized thermoelasticity for a semi-infinite cylinder with heat sources, *Journal of Thermoelasticity* **2**(1): 01-08.
[10] Maghraby N. M., Abdel Halim A. A., 2010, A generalized thermoelastic problem for a half space with heat sources under axisymmetric distribution, *Australian Journal of Basic and Applied Science* **4**(8): 3803-3814.
[11] Aouadi M., 2005, Discontinuities in a axisymmetric generalized thermoelastic problem, *International Journal of Mathematics and Mathematical Sciences* **7**: 1015-1029
[12] Tripathi J. J., Kedar G. D., Deshmukh K. C., 2015, Generalized thermoelastic diffusion problem in a thick circular plate with axisymmetric heat supply, *Acta Mechanica* **226**(7): 2121-2134.
[13] Youssef H. M., 2006, Two-dimensional generalized thermoelasticity problem for a half- space subjected to ramp-type heating, *European Journal of Mechanics A/Solids* **25**: 745-763.
[14] Tripathi J. J., Kedar, G. D., Deshmukh K. C., 2015, Two dimensional generalized thermoelastic diffusion in a half space under axisymmetric distributions, *Acta Mechanica* **226**: 3263-3274.
[15] Tripathi J. J., Kedar G. D., Deshmukh K. C., 2015, Theoretical study of disturbances due to mechanical source in a generalized thermoelastic diffusive half space, *Proceeding of the Third International Conference on Advances in Applied Science and Environmental Engineering - ASEE* **2015**: 57-61.
[16] Gaver D. P., 1966, Observing stochastic processes and approximate transform inversion, *Operations Research* **14**: 444-459.
[17] Stehfast H., 1970, Algorithm 368: Numerical inversion of Laplace transforms, *Communications of the ACM* **13**: 47-49.
[18] Stehfast H., 1970, Remark on algorithm 368, Numerical inversion of Laplace transforms, *Communications of the ACM* **13**: 624.
[19] Press W. H., Flannery B. P., Teukolsky S. A., Vetterling W. T., 1986, *Numerical Recipes*, Cambridge University Press, Cambridge, The Art of Scientific Computing.