Bending Analysis of Laminated Composite Plates with Arbitrary Boundary Conditions

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ABSTRACT

It is well known that for laminated composite plates a Levy-type solution exists only for cross-ply and antisymmetric angle-ply laminates. Numerous investigators have used the Levy method to solve the governing equations of various equivalent single-layer plate theories. It is the intension of the present study to introduce a method for analytical solutions of laminated composite plates with arbitrary lamination and boundary conditions subjected to transverse loads. The method is based on separation of spatial variables of displacement field components. Within the displacement field of a first-order shear deformation theory (FSDT), a laminated plate theory is developed. Two systems of coupled ordinary differential equations with constant coefficients are obtained by using the principle of minimum total potential energy. Since the procedure used is simple and straightforward it can, therefore, be adopted in developing higher-order shear deformation and layerwise laminated plate theories. The obtained equations are solved analytically using the state-space approach. The results obtained from the present method are compared with the Levy-type solutions of cross-ply and antisymmetric angle-ply laminates with various admissible boundary conditions to verify the validity and accuracy of the present theory. Also for other laminations and boundary conditions that there exist no Levy-type solutions the present results may be compared with those obtained from finite element method. It is seen that the present results have excellent agreements with those obtained by Levy-type method.

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Keywords: Laminated plates; Analytical solution; Arbitrary boundary conditions; First-order shear deformation theory

1   INTRODUCTION

SUPERIOR mechanical properties of composite materials such as high stiffness and strength to weight ratios, corrosive resistance and low coefficients of thermal expansion, caused it to be used increasingly in many areas of technology including marine [1], aerospace [2], automotive [3], civil [4], medicine [5] and others. Numerous investigators have employed analytical methods for bending analysis of laminated composite plates (for example, see [6-19]). However, the number of papers in which laminated plates with non-simply supported edges are studied, is not considerable. It can be said that the most popular analytical method for analysis of non-simply supported laminated plates is the Levy-type solution which is able to analyze cross-ply and antisymmetric angle-ply laminates with two simply supported opposite edges. Furthermore, nearly all of the previous works are restricted to special laminations such as cross-ply or antisymmetric angle-ply.

Only few theories are presented that can analyze plates with more general laminations or boundary conditions. Taking the idea of Timoshenko [20], Bhaskar and Kaushik [21] presented an exact solution for symmetric cross-ply thin plates with any combination of simply supported and clamped edges. Their methodology was based on superposition of the Navier solution corresponding to the applied transverse load and a number of double sine series solutions, equal to the number of clamped edges, each corresponding to the appropriate edge moment. Khalili et al.

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[22] conducted an analytical method for static and dynamic analysis of symmetric cross-ply laminated plates with different boundary conditions. They assumed that the functions describing the plate displacement in the *z* direction and shear rotations in the *x* and *y* directions are in the form of double Fourier series and then exploited Stokes's transformation [23] to legitimize the derivatives of this Fourier series for different sets of boundary conditions. They had to fulfill an elaborate mathematical procedure to obtain the unknown due to the every set of boundary conditions on the edges of the plate. Kabir and Chaudhuri [24] reported a minor variant of Green's approach [23] wherein the assumed displacement functions satisfy the clamped boundary conditions a priori; expansion of cosine functions in a sine series, or vice versa, as suggested by Green and Hearmon [25]. Chaudhuri and Kabir [26] extended their earlier work to derive a boundary-continuous-displacement solution for an arbitrarily laminated clamped plate. They used the first-order shear deformation theory (FSDT) and illustrated their results for a general laminate of [0°/60°] construction. A disadvantage with Green's approach, besides the uncertain nature of convergence of the series employed, is the larger number of unknown variables that one has to solve for – namely, the Fourier coefficients of the double series assumed for the displacements as against the coefficients of the single series assumed for the edge moments in the superposition approach. Vel and Batra [27] generalized the Eshelby–Stroh formalism [28] to study the three-dimensional deformations of anisotropic laminated rectangular plates subjected to arbitrary boundary conditions at the edges. They satisfied the interface continuity and the boundary conditions in the sense of Fourier series which results in an infinite system of equations in infinite unknowns. The truncation of this set of equations inevitably involves some errors which can be minimized by increasing the number of terms in the series. However, Vel and Batra [27] presented the numerical results, only, for a cross-ply plate simply supported on two opposite edges and subjected to different sets of boundary conditions on the other edges and a clamped plate with [0°/90°/0°] and [45°/-45°/45°] laminations.

The purpose of the present work is to develop an analytical method for bending analysis of laminated composite plates with arbitrary lamination and boundary conditions. Also in order to demonstrate the accuracy of the proposed method, a Levy-type solution is employed. As the numerical result, two problems are examined: the first, contains an anisymmetric angle-ply plate that there is Levy's solution for it and the latter contains a laminated plate with boundary conditions and lamination that there exist no Levy-type solutions. The comparison of the results with those obtained from the Levy-type solution shows an excellent agreement.

2 FORMULATION

2.1 Displacement field and strains

Consider a generally laminated plate as shown in Fig. 1 with a total thickness *h*, width *b* in the lateral (*y*-) direction, and length *a* in the longitudinal (*x*-) direction. It is also assumed that the middle plane of plate lies on the *xy* plane of a Cartesian coordinate system. Here, the theory will be developed within the framework of the FSDT [29]. To this end, it is assumed that the displacement field of the plate may be presented as:

$$
\begin{aligned}
u(x,y,z) &= u_i(x)\,\overline{u}_i(y) + z\,\psi_i(x)\,\overline{\psi}_i(y) \\
v(x,y,z) &= v_i(x)\,\overline{v}_i(y) + z\,\phi_i(x)\,\overline{\phi}_i(y), \qquad i = 1,2,\dots,n \\
w(x,y) &= w_i(x)\,\overline{w}_i(y)
\end{aligned}
\tag{1}
$$

where for the sake of brevity, the Einstein summation convention has been introduced –a repeated index indicates summation over all values of that index. In Eqs. (1), $u(x,y,z)$, $v(x,y,z)$, and $w(x,y,z)$ are, respectively, the displacements in *x*, *y*, and *z* directions, and $u_i(x)$, $\overline{u}_i(y)$, $v_i(x)$, $\overline{v}_i(y)$, $\psi_i(x)$, $\overline{\psi}_i(y)$, $\phi_i(x)$, $\overline{\phi}_i(y)$, $w_i(x)$, and $\overline{w}_i(y)$ are unknown functions. Also *n* is the total number of terms considered in the summation.
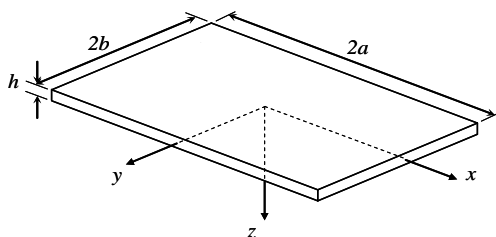

Fig. 1
The plate geometry and coordinate system.

Upon substitution of the displacement field (1) into the linear strain-displacement relations of elasticity the following strain-displacement relations will be obtained:

$$
\begin{aligned}
\varepsilon_x &= u_i \bar{u}_i' + z\,\psi_i \bar{\psi}_i' = \varepsilon_x^0 + z\,\kappa_x \\
\varepsilon_y &= v_i \bar{v}_i' + z\,\phi_i \bar{\phi}_i' = \varepsilon_y^0 + z\,\kappa_y \\
\gamma_{yz} &= \varphi_i \bar{\varphi}_i + w_i \bar{w}_i' = \gamma_{yz}^0 \\
\gamma_{xz} &= \psi_i \bar{\psi}_i + w_i' \bar{w}_i = \gamma_{xz}^0 \\
\gamma_{xy} &= u_i \bar{u}_i' + v_i' \bar{v}_i + z\,(\psi_i \bar{\psi}_i' + \phi_i' \bar{\phi}_i) = \gamma_{xy}^0 + z\,\kappa_{xy} \\
\varepsilon_z &= 0
\end{aligned}
\tag{2}
$$

2.2 Equilibrium equations

Next, using the principle of minimum total potential energy [30], two sets of equilibrium equations and boundary conditions corresponding to the independent variables can be shown to be:

$$
\begin{aligned}
\delta u_i &: \quad \frac{dN_x^i}{dx} - N_{xy1}^i = 0 \\
\delta v_i &: \quad \frac{dN_{xy2}^i}{dx} - N_y^i = 0 \\
\delta \psi_i &: \quad \frac{dM_x^i}{dx} - M_{xy1}^i - Q_{x1}^i = 0 \\
\delta \varphi_i &: \quad \frac{dM_{xy2}^i}{dx} - M_y^i - Q_{y1}^i = 0 \\
\delta w_i &: \quad \frac{dQ_{x2}^i}{dx} - Q_{y2}^i + q_i(x) = 0
\end{aligned}
\tag{3}
$$

and

$$
\begin{aligned}
\delta \bar{u}_i &: \quad \frac{d\bar{N}_{xy1}^i}{dy} - \bar{N}_x^i = 0 \\
\delta \bar{v}_i &: \quad \frac{d\bar{N}_y^i}{dy} - \bar{N}_{xy2}^i = 0 \\
\delta \bar{\psi}_i &: \quad \frac{d\bar{M}_{xy1}^i}{dy} - \bar{M}_x^i - \bar{Q}_{x1}^i = 0 \\
\delta \bar{\varphi}_i &: \quad \frac{d\bar{M}_y^i}{dy} - \bar{M}_{xy2}^i - \bar{Q}_{y1}^i = 0 \\
\delta \bar{w}_i &: \quad \frac{d\bar{Q}_{y2}^i}{dy} - \bar{Q}_{x2}^i + \bar{q}_i(y) = 0
\end{aligned}
\tag{4}
$$

In the above equations the generalized stress resultants, $q_i(x)$, and $\bar{q}_i(y)$ are defined as

$$\begin{bmatrix} \{N^i\}^T \\ \{M^i\}^T \\ \{\Theta^i\}^T \end{bmatrix} = \begin{bmatrix} N_x^i & N_y^i & N_{xy1}^i & N_{xy2}^i \\ M_x^i & M_y^i & M_{xy1}^i & M_{xy2}^i \\ Q_{y1}^i & Q_{y2}^i & Q_{x1}^i & Q_{x2}^i \end{bmatrix} = \int_0^b \begin{bmatrix} N_x\bar{u}_i & N_y\bar{v}_i' & N_{xy}\bar{u}_i' & N_{xy}\bar{v}_i \\ M_x\bar{\psi}_i & M_y\bar{\phi}_i' & M_{xy}\bar{\psi}_i' & M_{xy}\bar{\phi}_i \\ Q_y\bar{\phi}_i & Q_y\bar{w}_i' & Q_x\bar{\psi}_i & Q_x\bar{w}_i \end{bmatrix} \mathrm{d}y \tag{5}$$

$$\begin{bmatrix} \{\bar{N}^i\}^T \\ \{\bar{M}^i\}^T \\ \{\bar{\Theta}^i\}^T \end{bmatrix} = \begin{bmatrix} \bar{N}_x^i & \bar{N}_y^i & \bar{N}_{xy1}^i & \bar{N}_{xy2}^i \\ \bar{M}_x^i & \bar{M}_y^i & \bar{M}_{xy1}^i & \bar{M}_{xy2}^i \\ \bar{Q}_{y1}^i & \bar{Q}_{y2}^i & \bar{Q}_{x1}^i & \bar{Q}_{x2}^i \end{bmatrix} = \int_0^a \begin{bmatrix} N_x u_i' & N_y v_i & N_{xy}u_i & N_{xy}v_i' \\ M_x \psi_i' & M_y \phi_i & M_{xy}\psi_i & M_{xy}\phi_i' \\ Q_y \phi_i & Q_y w_i & Q_x \psi_i & Q_x w_i' \end{bmatrix} \mathrm{d}x \tag{6}$$

$$q_i(x) = \int_0^b q(x,y)\bar{w}_i \, \mathrm{d}y \tag{7}$$

$$\bar{q}_i(y) = \int_0^a q(x,y)w_i \, \mathrm{d}x \tag{8}$$

Also the stress resultants are

$$(N_x, N_y, N_{xy}, Q_y, Q_x) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \sigma_{xy}, \sigma_{yz}, \sigma_{xz}) \, \mathrm{d}z$$

$$(M_x, M_y, M_{xy}) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \sigma_{xy})z \, \mathrm{d}z \tag{9}$$

The boundary conditions consist of specifying the following quantities at the edges of the plate. For edges parallel to *y*-axis (i.e., *x*=0,*a*):

Geometric (essential)		Force (natural)		
u_i	or	N_x^i		
v_i	or	N_{xy2}^i		
ψ_i	or	M_x^i	$i=1,2,\dots,n$	(10)
φ_i	or	M_{xy2}^i		
w_i	or	Q_{x2}^i		

and for edges parallel to *x*-axis (i.e., *y*=0,*b*):

Geometric (essential)		Force (natural)		
$\bar{u}_i$	or	$\bar{N}_{xy1}^i$		
$\bar{v}_i$	or	$\bar{N}_y^i$		
$\bar{\psi}_i$	or	$\bar{M}_{xy1}^i$	$i=1,2,\dots,n$	(11)
$\bar{\varphi}_i$	or	$\bar{M}_y^i$		
$\bar{w}_i$	or	$\bar{Q}_{y2}^i$		

2.3 Laminate constitutive relations

The linear constitutive relations for the *k*th orthotropic lamina in the laminate coordinates (*x,y,z*) are given

$$\{\sigma\}^{(k)} = [\bar{Q}]^{(k)}\{\varepsilon\}^{(k)} \tag{12}$$

where $[\bar{Q}]^{(k)}$ denotes the transformed reduced plane-stress stiffness matrix of the kth lamina. Upon substitution of Eqs. (2) into Eqs. (12) and the subsequent results into Eqs. (9), the stress resultants are obtained which can be presented as follows:

$$
\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ & & A_{66} & B_{16} & B_{26} & B_{66} \\ & & & D_{11} & D_{12} & D_{16} \\ & & & & D_{22} & D_{26} \\ \text{symm.} & & & & & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}, \qquad \begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = k^2 \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^0 \\ \gamma_{xz}^0 \end{Bmatrix} \tag{13}
$$

Here, k^2 ($=5/6$) is the shear correction factor of FSDT. Also A_{ij}, B_{ij}, and D_{ij} denote the extensional stiffnesses, the bending-extensional coupling stiffnesses, and the bending stiffnesses, respectively.

$$
(A_{ij}, B_{ij}, D_{ij}) = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \bar{Q}_{ij}^{(k)} (1, z, z^2) \, dz \tag{14}
$$

where N is the total number of layers. Upon substitution of Eqs. (2) into (13) and the subsequent results into Eqs. (5) and (6) the generalized stress resultants are obtained which can be represented as follows:

$$
\begin{Bmatrix} \{N^i\} \\ \{M^i\} \end{Bmatrix} = \begin{bmatrix} A^{ij} \end{bmatrix} \{\xi_j\}, \qquad \{\Theta^i\} = \begin{bmatrix} B^{ij} \end{bmatrix} \{\eta_j\} \tag{15}
$$

$$
\begin{Bmatrix} \{\bar{N}^i\} \\ \{\bar{M}^i\} \end{Bmatrix} = \begin{bmatrix} \bar{A}^{ij} \end{bmatrix} \{\bar{\xi}_j\}, \qquad \{\bar{\Theta}^i\} = \begin{bmatrix} \bar{B}^{ij} \end{bmatrix} \{\bar{\eta}_j\} \tag{16}
$$

where

$$
\{\xi_j\} = \begin{bmatrix} u_j' & v_j & u_j & v_j' & \psi_j' & \varphi_j & \psi_j & \varphi_j' \end{bmatrix}^T, \qquad \{\eta_j\} = \begin{bmatrix} \varphi_j & w_j & \psi_j & w_j' \end{bmatrix}^T \tag{17}
$$

$$
\{\bar{\xi}_j\} = \begin{bmatrix} \bar{u}_j & \bar{v}_j' & \bar{u}_j' & \bar{v}_j & \bar{\psi}_j & \bar{\varphi}_j' & \bar{\psi}_j' & \bar{\varphi}_j \end{bmatrix}^T, \qquad \{\bar{\eta}_j\} = \begin{bmatrix} \bar{\varphi}_j & \bar{w}_j' & \bar{\psi}_j & \bar{w}_j \end{bmatrix}^T \tag{18}
$$

and the stiffness coefficients A_{mn}^{ij}, B_{mn}^{ij}, $\bar{A}_{mn}^{ij}$, and $\bar{B}_{mn}^{ij}$ are defined by

$$
\begin{bmatrix} A^{ij} \end{bmatrix} = \int_0^b \begin{bmatrix} [\alpha] \otimes \{\bar{\xi}_i\}\{\bar{\xi}_j\}^T \end{bmatrix} dy, \qquad \begin{bmatrix} B^{ij} \end{bmatrix} = \int_0^b \begin{bmatrix} [\beta] \otimes \{\bar{\eta}_i\}\{\bar{\eta}_j\}^T \end{bmatrix} dy \tag{19}
$$

$$
\begin{bmatrix} \bar{A}^{ij} \end{bmatrix} = \int_0^a \begin{bmatrix} [\alpha] \otimes \{\xi_i\}\{\xi_j\}^T \end{bmatrix} dx, \qquad \begin{bmatrix} \bar{B}^{ij} \end{bmatrix} = \int_0^a \begin{bmatrix} [\beta] \otimes \{\eta_i\}\{\eta_j\}^T \end{bmatrix} dx \tag{20}
$$

where $[\alpha]$ and $[\beta]$ are

$$[\alpha] = \begin{bmatrix} A_{11} & A_{12} & A_{16} & A_{16} & B_{11} & B_{12} & B_{16} & B_{16} \\ & A_{22} & A_{26} & A_{26} & B_{12} & B_{22} & B_{26} & B_{26} \\ & & A_{66} & A_{66} & B_{16} & B_{26} & B_{66} & B_{66} \\ & & & A_{66} & B_{16} & B_{26} & B_{66} & B_{66} \\ & & & & D_{11} & D_{12} & D_{16} & D_{16} \\ & & & & & D_{22} & D_{26} & D_{26} \\ & & & & & & D_{66} & D_{66} \\ \text{symm.} & & & & & & & D_{66} \end{bmatrix} \tag{21}$$

$$[\beta] = k^2 \begin{bmatrix} A_{44} & A_{44} & A_{45} & A_{45} \\ & A_{44} & A_{45} & A_{45} \\ & & A_{55} & A_{55} \\ \text{symm.} & & & A_{55} \end{bmatrix} \tag{22}$$

It must be noted that the sign $\otimes$ used in Eqs. (19) and (20) is referred to *array multiplication* of two matrices.

2.4 Governing equations of equilibrium

The equilibrium Eqs. (3) and (4) can be expressed in terms of displacements by substituting the generalized stress resultants from (15) and (16). Hence, two sets of ordinary differential equations will be obtained as follows:

$$\delta u_i : A_{11}^{ij} u_j'' + (A_{13}^{ij} - A_{31}^{ij})u_j' - A_{33}^{ij} u_j + A_{14}^{ij} v_j'' + (A_{12}^{ij} - A_{34}^{ij})v_j' - A_{32}^{ij} v_j + A_{15}^{ij}\psi_j'' + (A_{17}^{ij} - A_{35}^{ij})\psi_j' - A_{37}^{ij}\psi_j$$
$$+ A_{18}^{ij}\varphi_j'' + (A_{16}^{ij} - A_{38}^{ij})\varphi_j' - A_{36}^{ij}\varphi_j = 0$$

$$\delta v_i : A_{41}^{ij} u_j'' + (A_{43}^{ij} - A_{21}^{ij})u_j' - A_{23}^{ij} u_j + A_{44}^{ij} v_j'' + (A_{42}^{ij} - A_{24}^{ij})v_j' - A_{22}^{ij} v_j + A_{45}^{ij}\psi_j'' + (A_{47}^{ij} - A_{25}^{ij})\psi_j' - A_{27}^{ij}\psi_j$$
$$+ A_{48}^{ij}\varphi_j'' + (A_{46}^{ij} - A_{28}^{ij})\varphi_j' - A_{26}^{ij}\varphi_j = 0$$

$$\delta\psi_i : A_{51}^{ij} u_j'' + (A_{53}^{ij} - A_{71}^{ij})u_j' - A_{73}^{ij} u_j + A_{54}^{ij} v_j'' + (A_{52}^{ij} - A_{74}^{ij})v_j' - A_{72}^{ij} v_j + A_{55}^{ij}\psi_j'' + (A_{57}^{ij} - A_{75}^{ij})\psi_j'$$
$$- (A_{77}^{ij} + B_{33}^{ij})\psi_j + A_{58}^{ij}\varphi_j'' + (A_{56}^{ij} - A_{78}^{ij})\varphi_j' - (A_{76}^{ij} + B_{31}^{ij})\varphi_j - B_{34}^{ij} w_j' - B_{32}^{ij} w_j = 0$$

$$\delta\varphi_i : A_{81}^{ij} u_j'' + (A_{83}^{ij} - A_{61}^{ij})u_j' - A_{63}^{ij} u_j + A_{84}^{ij} v_j'' + (A_{82}^{ij} - A_{64}^{ij})v_j' - A_{62}^{ij} v_j + A_{85}^{ij}\psi_j'' + (A_{87}^{ij} - A_{65}^{ij})\psi_j'$$
$$- (A_{67}^{ij} + B_{13}^{ij})\psi_j + A_{88}^{ij}\varphi_j'' + (A_{86}^{ij} - A_{68}^{ij})\varphi_j' - (A_{66}^{ij} + B_{11}^{ij})\varphi_j - B_{14}^{ij} w_j' - B_{12}^{ij} w_j = 0$$

$$\delta w_i : B_{43}^{ij}\psi_j' - B_{23}^{ij}\psi_j + B_{41}^{ij}\varphi_j' - B_{21}^{ij}\varphi_j + B_{44}^{ij} w_j'' + (B_{42}^{ij} - B_{24}^{ij})w_j' - B_{22}^{ij} w_j = -q_i(x) \tag{23}$$

and

$$\delta\overline{u}_i : \overline{A}_{33}^{ij}\overline{u}_j'' + (\overline{A}_{31}^{ij} - \overline{A}_{13}^{ij})\overline{u}_j' - \overline{A}_{11}^{ij}\overline{u}_j + \overline{A}_{32}^{ij}\overline{v}_j'' + (\overline{A}_{34}^{ij} - \overline{A}_{12}^{ij})\overline{v}_j' - \overline{A}_{14}^{ij}\overline{v}_j + \overline{A}_{37}^{ij}\overline{\psi}_j'' + (\overline{A}_{35}^{ij} - \overline{A}_{17}^{ij})\overline{\psi}_j' - \overline{A}_{15}^{ij}\overline{\psi}_j + \overline{A}_{36}^{ij}\overline{\varphi}_j''$$
$$+ (\overline{A}_{38}^{ij} - \overline{A}_{16}^{ij})\overline{\varphi}_j' - \overline{A}_{18}^{ij}\overline{\varphi}_j = 0$$

$$\delta\overline{v}_i : \overline{A}_{23}^{ij}\overline{u}_j'' + (\overline{A}_{21}^{ij} - \overline{A}_{43}^{ij})\overline{u}_j' - \overline{A}_{41}^{ij}\overline{u}_j + \overline{A}_{22}^{ij}\overline{v}_j'' + (\overline{A}_{24}^{ij} - \overline{A}_{42}^{ij})\overline{v}_j' - \overline{A}_{44}^{ij}\overline{v}_j + \overline{A}_{27}^{ij}\overline{\psi}_j'' + (\overline{A}_{25}^{ij} - \overline{A}_{47}^{ij})\overline{\psi}_j' - \overline{A}_{45}^{ij}\overline{\psi}_j + \overline{A}_{26}^{ij}\overline{\varphi}_j''$$
$$+ (\overline{A}_{28}^{ij} - \overline{A}_{46}^{ij})\overline{\varphi}_j' - \overline{A}_{48}^{ij}\overline{\varphi}_j = 0$$

$$\delta\overline{\psi}_i : \overline{A}_{73}^{ij}\overline{u}_j'' + (\overline{A}_{71}^{ij} - \overline{A}_{53}^{ij})\overline{u}_j' - \overline{A}_{51}^{ij}\overline{u}_j + \overline{A}_{72}^{ij}\overline{v}_j'' + (\overline{A}_{74}^{ij} - \overline{A}_{52}^{ij})\overline{v}_j' - \overline{A}_{54}^{ij}\overline{v}_j + \overline{A}_{77}^{ij}\overline{\psi}_j'' + (\overline{A}_{75}^{ij} - \overline{A}_{57}^{ij})\overline{\psi}_j' - (\overline{A}_{55}^{ij} + \overline{B}_{33}^{ij})\overline{\psi}_j$$
$$+ \overline{A}_{76}^{ij}\overline{\varphi}_j'' + (\overline{A}_{78}^{ij} - \overline{A}_{56}^{ij})\overline{\varphi}_j' - (\overline{A}_{58}^{ij} + \overline{B}_{31}^{ij})\overline{\varphi}_j - \overline{B}_{32}^{ij}\overline{w}_j' - \overline{B}_{34}^{ij}\overline{w}_j = 0$$

$$\delta\overline{\varphi}_i : \overline{A}_{63}^{ij}\overline{u}_j'' + (\overline{A}_{61}^{ij} - \overline{A}_{83}^{ij})\overline{u}_j' - \overline{A}_{81}^{ij}\overline{u}_j + \overline{A}_{62}^{ij}\overline{v}_j'' + (\overline{A}_{64}^{ij} - \overline{A}_{82}^{ij})\overline{v}_j' - \overline{A}_{84}^{ij}\overline{v}_j + \overline{A}_{67}^{ij}\overline{\psi}_j'' + (\overline{A}_{65}^{ij} - \overline{A}_{87}^{ij})\overline{\psi}_j' - (\overline{A}_{85}^{ij} + \overline{B}_{13}^{ij})\overline{\psi}_j$$
$$+ \overline{A}_{66}^{ij}\overline{\varphi}_j'' + (\overline{A}_{68}^{ij} - \overline{A}_{86}^{ij})\overline{\varphi}_j' - (\overline{A}_{88}^{ij} + \overline{B}_{11}^{ij})\overline{\varphi}_j - \overline{B}_{12}^{ij}\overline{w}_j' - \overline{B}_{14}^{ij}\overline{w}_j = 0$$

$$\delta \overline{w}_i : \overline{B}_{23}^{ij}\overline{\psi}_j' - \overline{B}_{43}^{ij}\overline{\psi}_j + \overline{B}_{21}^{ij}\overline{\varphi}_j' - \overline{B}_{41}^{ij}\overline{\varphi}_j + \overline{B}_{22}^{ij}\overline{w}'' + (\overline{B}_{24}^{ij} - \overline{B}_{42}^{ij})\overline{w}_j' - \overline{B}_{44}^{ij}\overline{w}_j = -\overline{q}_i(y) \tag{24}$$

3   ANALYTICAL SOLUTIONS

Here, we employ the state-space approach [31] to solve the equilibrium equations obtained in the previous section. The linear system of ordinary differential equations in (23) can be expressed in the form of single, first-order, matrix differential equation

$$\{X'\} = [C]\{X\} + \{F\} \tag{25}$$

where the state vector $\{X\}$ is defined as

$$\{X_1\} = \{u_j'\}, \ \{X_2\} = \{v_j\}, \ \{X_3\} = \{u_j\}, \ \ \{X_4\} = \{v_j'\}, \ \{X_5\} = \{\psi_j'\}, \ \{X_6\} = \{\phi_j\}$$
$$\{X_7\} = \{\psi_j\}, \ \{X_8\} = \{\phi_j'\}, \ \{X_9\} = \{w_j'\}, \ \ \{X_{10}\} = \{w_j\}. \tag{26}$$

In order to solve Eq. (25), let us assume that $\overline{u}_i(y)$, $\overline{u}_i'(y)$,…, $\overline{w}_i'(y)$ are chosen so that the boundary conditions at $y=0,b$ are identically satisfied. Next, the coefficients A_{mn}^{ij} and B_{mn}^{ij} are found. Since these coefficients are constant, Eq. (25) will be five linear ordinary differential equations with constant coefficients. The general solution of Eq. (25) is given by [32]:

$$\{X\} = [U][Q]\{K\} + [U][Q]\int_x [Q]^{-1}[U]^{-1}\{F\}\,dx \tag{27}$$

where $[U]$ is the matrix of distinct eigenvectors of matrix $[C]$ and $\{K\}$ is a vector of unknown constants to be found by imposing the boundary conditions at edges $x=0,a$. Also the diagonal matrix $[Q]$ is defined as

$$[Q] = diag(e^{\lambda_1 x}, e^{\lambda_2 x}, \dots, e^{\lambda_{10n} x}) \tag{28}$$

where λ_k $(k=1,2,…,10n)$ are the eigenvalues associated with matrix $[C]$. Next, we can substitute the general solution of $u_i(y)$, $u_i'(y)$,…, $w_i'(y)$ into Eqs. (20) to find $\overline{A}_{mn}^{ij}$ and $\overline{B}_{mn}^{ij}$ which, here, will be constant. The solution procedure for Eqs. (24) is analogous to the one presented for Eqs. (23) and therefore, for the sake of brevity will not be taken up here. This procedure (solving the coupled systems of ordinary differential equations) will be continued until the solution is converged.

4   NUMERICAL RESULTS

Two numerical examples including various sets of boundary conditions are studied in this section. The aim of the first example is to demonstrate the accuracy and validity of the present method while the second, attempts to show the capability of the method to analyze laminated plates for which there exist no Levy-type solutions. To this end, a Levy-type solution based on FSDT is also developed. In the first example, results are compared with those obtained by the Levy solution. It is worth to recall that Levy's solution exists only for cross-ply and antisymmetric angle-ply laminates with two opposite simply supported edges. In the both examples, each lamina is assumed to be of the same thickness and has the following orthotropic material properties in the principal material coordinate system [29]:

$$E_1 = 25E_2, \ G_{12} = G_{13} = 0.5E_2, \ G_{23} = 0.2E_2, \ \upsilon_{12} = 0.25 \tag{29}$$

where 1,2, and 3 indicate the on-axis material coordinate. Denoting simply supported, clamped, and free boundary conditions by S, C, and F, a 4-word notation such as SFSC is employed to show the boundary conditions on the four edges of the plate. The 1-4th word indicates the boundary conditions on edges $x=0$, $y=0$, $x=a$, and $y=b$, respectively.

Example 1

Consider an eight-layer antisymmetric angle-ply square laminate [-45°/30°/-45°/0°/0°/45°/-30°/45°] with width-to-thickness ratio b/h=10. The plate is subjected to a uniform distributed transverse load of magnitude q_0 or a sinusoidal one:

$$q = q_0 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} \tag{30}$$

It is to be noted that the boundary conditions used in this example are restricted to Levy's admissible boundary conditions, therefore, the simple support applied at the edge of the laminate is defined as:
S2:

$$u_i = w_i = \varphi_i = N_{xy}^i = M_x^i = 0 \quad \text{at} \quad x=0,a \tag{31a}$$

$$v_i = w_i = \psi_i = N_{xy}^i = M_y^i = 0 \quad \text{at} \quad y=0,b \tag{31b}$$

All the numerical results for deflections and stresses shown in what follows are nondimensionalized as below:

$$\bar{w} = w\left(\frac{E_2 h^3}{b^4 q_0}\right) \times 10^2$$

$$(\bar{\sigma}_x, \bar{\sigma}_y, \bar{\sigma}_{xy}) = (\sigma_x, \sigma_y, \sigma_{xy})\left(\frac{h^2}{b^2 q_0}\right) \tag{32}$$

$$(\bar{\sigma}_{xz}, \bar{\sigma}_{xz}) = (-\sigma_{xz}, -\sigma_{xz})\left(\frac{h}{bq_0}\right)$$

The results achieved from the present method and the Levy method are compared by Figs. 2-4. The variation of nondimensionalized deflection versus x/a at $y=b/2$ is shown in Fig. 2 corresponding to three sets of SSSS, SCSC, and SFSC boundary conditions.

Also Figs. 3 and 4 illustrate the through-thickness distributions of normal stress $\bar{\sigma}_y(a/2, b/2, z/h)$ and transverse shear stress $\bar{\sigma}_{xz}(a/4, b/4, z/h)$ of described laminated plate under various boundary conditions. It is to be noted that the numerical values of interlaminar stresses are obtained by integrating the local equilibrium equations of elasticity.
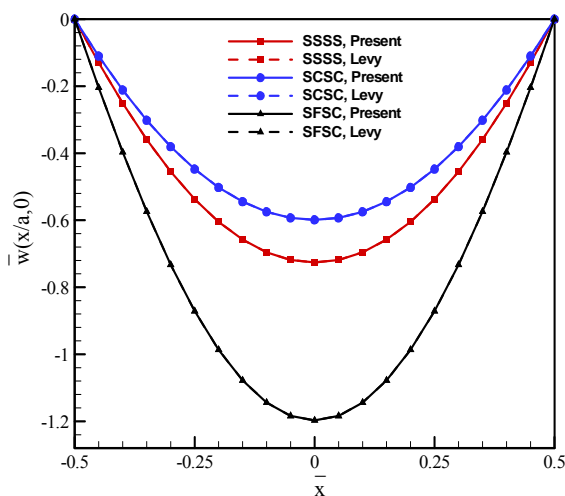
Fig. 2
Variations of deflection versus x/a for [-45°/30°/-45°/0°/0°/45°/-30°/45°] square laminate subjected to the uniform transverse load.

Above mentioned figures indicate that there is an excellent agreement between the present results and those obtained by Levy's solution. However, it can be said that the magnitude of errors depends on the type of boundary conditions imposed on the edges of the plate and the maximum error has occurred in stress values for SCSF boundary conditions. The through-thickness variations of shear stresses $\bar{\sigma}_{xy}(a/2, b/2, z/h)$ and $\bar{\sigma}_{yz}(a/4, b/4, z/h)$ due to the sinusoidal loading are shown, respectively, in Figs. 5 and 6. In this case there is no difference between Levy's solution and the solution presented here.

Example 2

The applicability of the proposed method to analyze laminated plates with arbitrary lamination and boundary conditions is demonstrated, using [45°/90°/0°/45°] laminated plate under several sets of boundary conditions. The plate has length-to-width ratio a/b=2 and width-to-thickness ratio b/h=10 and is subjected to sinusoidally distributed transverse load as defined in Eq. (30). It should be noted that the types of simple supports used in this example, is defined as follows:
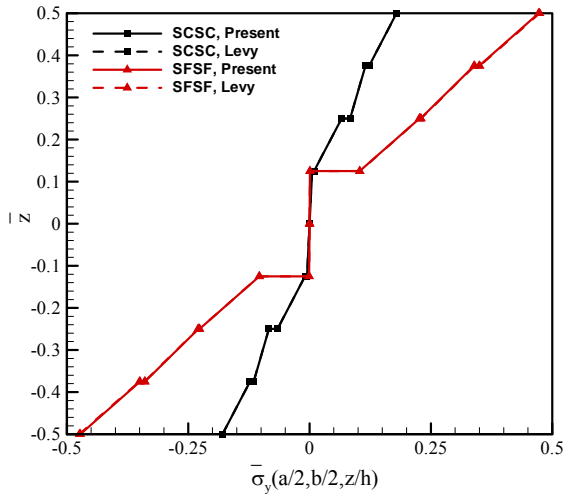
Fig. 3
Variations of normal stress $\bar{\sigma}_y(a/2, b/2, z/h)$ through the thickness of [-45°/30°/-45°/0°/0°/45°/-30°/45°] square laminate subjected to the uniform transverse load.
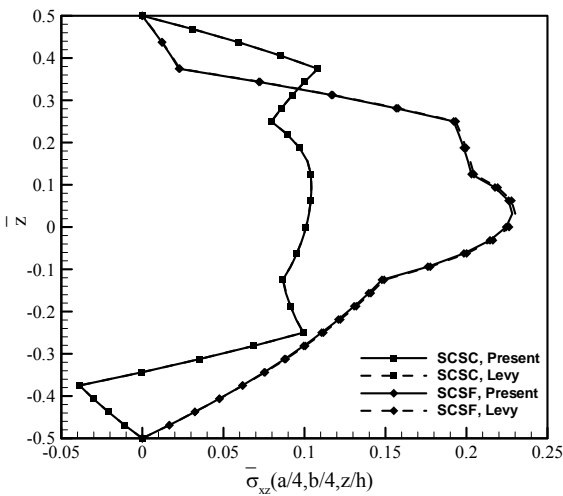

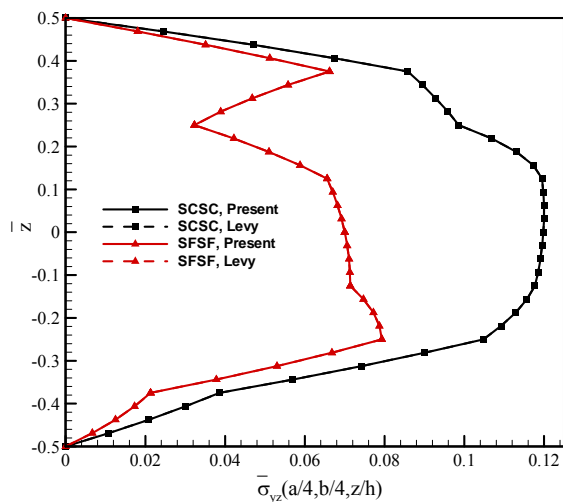**Fig. 4**
Distributions of transverse shear stress $\bar{\sigma}_{xz}(a/4, b/4, z/h)$ through the thickness of [-45°/30°/-45°/0°/0°/45°/-30°/45°] laminate subjected to the uniform transverse load.

S1:

$$v_i = w_i = \phi_i = N_x^i = M_x^i = 0 \qquad \text{at} \qquad x = 0, a \qquad (33a)$$

$$u_i = w_i = \psi_i = N_y^i = M_y^i = 0 \qquad \text{at} \qquad y = 0, b \qquad (33b)$$

The variation of deflection at $y=0$, along the length of the plate with four sets of boundary conditions: CCFF, SSSS, CCCC, and FSCS, is presented in Fig. 7. As expected, the curve corresponding to boundary conditions CCCC is located above the other curves.

Figs. 8 and 9 depict through the thickness distributions of normal stress $\bar{\sigma}_x(0,0,\bar{z})$ and transverse shear stress $\bar{\sigma}_{yz}(a/4, b/4, z/h)$ for different sets of boundary conditions.
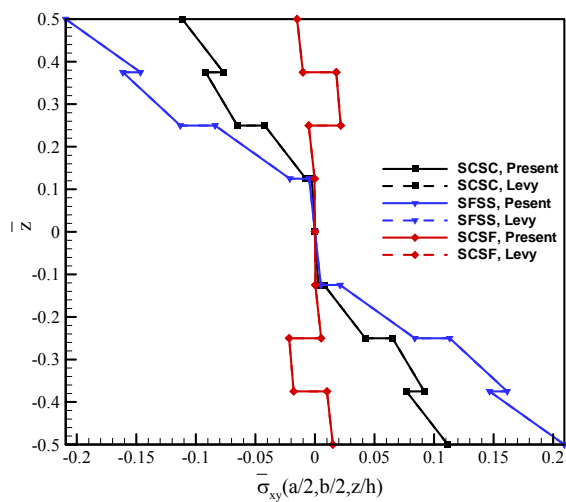
Fig. 5
Variations of shear stress $\bar{\sigma}_{xy}(a/2, b/2, z/h)$ through the thickness of [-45°/30°/-45°/0°/0°/45°/-30°/45°] square laminate subjected to the sinusoidal transverse load.

Fig. 6
Distributions of transverse shear stress $\bar{\sigma}_{yz}(a/4, b/4, z/h)$ through the thickness of [-45°/30°/-45°/0°/0°/45°/-30°/45°] laminate subjected to the sinusoidal transverse load.
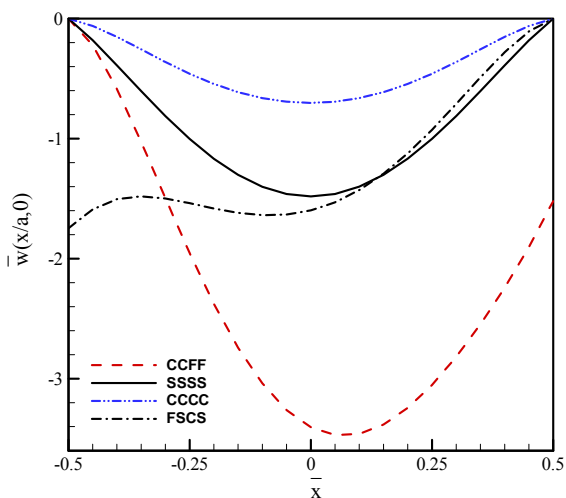
Fig. 7
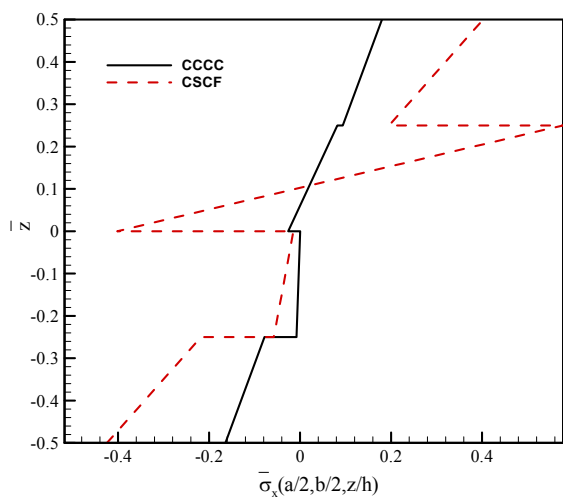Variations of deflection versus *x/a* for [45°/90°/0°/45°] laminate subjected to the sinusoidal load.

Fig. 8
Variations of normal stress $\bar{\sigma}_x(0,0,\bar{z})$ through the thickness of [45°/90°/0°/45°] laminated plate subjected to the sinusoidal transverse load.

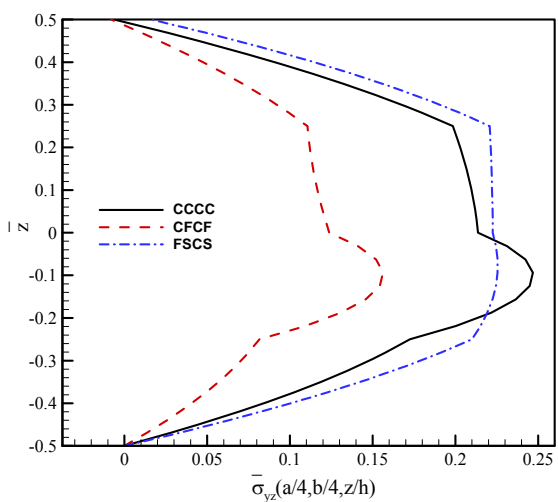**Fig. 9**
Distributions of transverse shear stress $\bar{\sigma}_{yz}(a/4, b/4, z/h)$ through the thickness of [45°/90°/0°/45°] laminated plate subjected to the sinusoidal transverse load.

5   CONCLUSION

In this study an analytical method based on an idea is developed to study the bending behavior of laminated composite plates. The method is capable to analyze laminated plates with arbitrary lamination and boundary conditions. A Levy-type solution based on FSDT is used as a benchmark. The numerical results are compared with the Levy-type solutions. All the numerical results, especially those for plates subjected to double-sinusoid transverse loading, have excellent agreements between the present method and the exact Levy-type method. Finally, several numerical results are presented for laminated plates which have no Levy-type solutions.

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