Journal of Solid Mechanics Vol. 15, No. 4 (2023) pp. 428-439
DOI: 10.60664/jsm.2024.3011765

Research Paper

Fractional Cattaneo Heat Equation in A Multilayer Elliptic Ring Membrane and Its Thermal Stresses

G. Dhameja, L. Khalsa, V. Varghese *

Department of Mathematics, M.G. College, Armori, Gadchiroli, India

Received 27 January 2023; accepted 20 June 2023

ABSTRACT

A fractional Cattaneo model from the generalized Cattaneo model with two fractional derivatives of different orders is considered for studying the thermoelastic response for a multilayer elliptic ring membrane with source function. The solution is obtained by applying an integral transform technique analogous to Vodicka's approach considering series expansion functions in terms of an eigenfunction to the generalized fractional Cattaneo-type heat conduction equation within an elliptic coordinates system. The analytical expressions of displacement and stress components employing Airy's stress function approach are investigated. The results are obtained as a series solution in terms of Mathieu functions and hold convergence test. The effects of fractional parameters on the temperature fields and their thermal stresses are also discussed. The findings are depicted graphically for different kinds of surface temperature gradients, and it is distinguished that the higher the fractional-order parameter, the higher the thermal response.

Keywords: Fractional Cattaneo-type equation; Fractional calculus; Non-Fourier heat conduction; Thermal stress; Integral transform.

1 INTRODUCTION

MANY engineering applications necessitate a thorough understanding of the transient temperature distribution and heat flux inside elliptic structures comprising two or more rigid layers with internal source functions. Usually, these multilayer structural elements are subjected to a high-temperature environment, and due to the presence of interface boundary conditions, the layer coefficient follows a discontinuity of temperature at the corresponding interface. Hence, a precise evaluation of their thermal characteristic is of great interest for engineering design and manufacture due to their wide applications as structural members in modern industries such as mechanical, aerospace, nuclear engineering, and reactors.

Most heat conduction problems are described and analyzed using classical Fourier's law concept. In this context, many researchers have solved the transient heat conduction problem in composite layers with different analytical approaches, such as the orthogonal expansion technique [1-5], quasi-orthogonal expansion technique [6-8], Laplace

*Corresponding author. Tel.: +91 9665040845.
E-mail address: vino7997@gmail.com

transform method [9-11], Green's function approach [12-14], Galerkin procedure [15-16], finite integral transform technique [17-21], the line heat-source method [22,23], the method of separation of variables [24], and so on. This Fourier law is widely used and has been demonstrated to produce acceptable results in various contexts. This law's fundamental flaw is that it assumes that the heat flux is instantaneously produced (or vanished) when a temperature gradient is applied (or removed) [25]. This law needs to be amended because no procedure is instantaneous. To put it simply, the Fourier law implies that thermal disturbances propagate at an infinite speed, and hence it is unreasonable. Mathematically, this is caused by the parabolic nature of the heat equation. To prevent this, a relaxation of the flux is introduced, leading to the so-called Cattaneo's equation [26]. Such a hyperbolic equation characterizes heat conduction's combined diffusion and wave-like behavior and predicts a finite heat propagation speed [27].

Fractional calculus has recently become an increasingly growing field with applications in many areas, including physical sciences, engineering, biological sciences, and economics [28-33]. Compte and Metzler [34] examined four varieties of potential Cattaneo equation generalizations, three of which are validated by a different scheme: continuous-time random walks, nonlocal transport theory, and the delayed flux–force relationship. In this background, Povstenko [35] demonstrated that the time-nonlocal generalization of Fourier law with kernels of Mittag–Leffler would yield all fractional generalizations of the Cattaneo equations proposed by Compte and Metzler. Povstenko [36] also published highly cited literature reviews on recently published articles on fractional thermoelasticity. As the fractional-order Cattaneo model is compared to the classical Cattaneo-Vernotte and Fourier models, it is realized by Xu and Wang [37] that the heat flux expected by the fractional Cattaneo model always transports from high to low temperature, which is consistent with the second law of thermodynamics. Maillet [38] reviewed a series of previous experimental papers to validate whether the Cattaneo and Vernotte hyperbolic heat equation (non-Fourier models) could be applied to bioheat transfer simulation.

Recently, few researchers have obtained mathematical solutions to the fractional Cattaneo-Vernotte heat conduction problem with different boundary conditions on a finite or semi-infinite medium, which can be briefly summarized below to know the emerging trends in the current scenario. Qi et al. [39-40] used the time fractional Cattaneo model as the heat conduction model having a volumetric heat source and obtained the analytical solution for the temperature distribution using the Laplace transformation method. Choi et al. [41] brought a new analytical solution for the hyperbolic partial differential equations of the non-Fourier heat conduction in terms of a simple exponential function explicitly compared with the existing data for justification. Povstenko [35,36,42-44] has investigated the time-fractional heat conduction equation with Caputo derivative under mathematical and physical Robin-type boundary conditions. Jiang and Qi [45] studied the thermal wave model of bio-heat transfer with the modified Riemann Liouville fractional derivative and demonstrated that the fractional models could provide a unified approach to examining the heat transfer of biological tissues. Xu et al. [46] investigated the heat conduction process in metal materials irradiated by non-Gauss laser pulses using the transient temperature field based on the fractional heat conduction equation for laser heating.

Most studies on the fractional Cattaneo-Vernotte model have focused on heat conduction problems in semi-infinite spaces with the Neumann boundary condition [35,46–48] and finite thickness problems with the Dirac boundary condition [49]. Nevertheless, based on the Cattaneo-Vernotte fractional model, the multilayered elliptic membrane's heat conduction with convective-type mathematical boundary condition has yet to be studied. The heat conduction mechanism that differs from the fractional-order parameters is analyzed. The time-fractional thermoelastic analysis of the Cattaneo-type for an elliptic membrane under prescribed boundary conditions has not been investigated to the best of the author's knowledge.

The following is an outline of the remaining section of the paper: Section 2 presents the mathematical modeling of the generalized heat conduction equation in the fractional Cattaneo-type framework. Section 3 describes the mathematical formulation of the problem for a multilayer elliptic ring membrane and its associated thermal stresses. In Section 4, the solution of time-fractional Cattaneo analysis is obtained as a series solution in terms of Mathieu functions. Section 5 depicts the graphical outcomes. Finally, conclusive comments are summarized in Section 6.

2 GENERALIZED FRACTIONAL CATTANEO-TYPE EQUATION

The classical Cattaneo model [26] as

$$q + \tau \frac{\partial q}{\partial t} = -k \nabla T \tag{1}$$

By combining Eq. (1) with the continuity equation

$$\rho C_v \frac{\partial T}{\partial t} = -k\,\nabla \cdot q + g \tag{2}$$

leads to the hyperbolic heat conduction equation

$$\frac{\partial T}{\partial t} + \tau \frac{\partial^2 T}{\partial t^2} = \kappa \nabla T \tag{3}$$

in which g is the rate of the heat generation per unit volume, q is the heat flux vector, τ is the relaxation time, k is the heat conductivity of a solid, k is the thermal diffusivity coefficient, ∇ is the gradient operator, T is the temperature, and t is the time, respectively.
The fractional generalization [40] of Eq. (1) is written as

$$\frac{\partial^{\alpha-1} q}{\partial t^{\alpha-1}} + \tau \frac{\partial^{\beta-1} q}{\partial t^{\beta-1}} = -k\,\nabla T \tag{4}$$

Taking the Laplace transform and its inversion, by neglecting the initial value, one obtains the time-nonlocal constitutive relationship of Eq. (4) as

$$q(t) = -\frac{k}{\tau}\int_0^t (t-\gamma)^{\alpha-2} E_{\alpha-\beta,\alpha-1}\left[-\frac{(t-\gamma)^{\alpha-\beta}}{\tau}\right] \nabla T(\gamma)\, d\gamma \tag{5}$$

where $E_{\alpha,\beta}$ is the generalized Mittag-Leffler functions in two parameters α and β [28].
By combining Eq. (4) with Eq. (2) leads to the fractional generalized Cattaneo equation as

$$\frac{\partial^{\alpha} T}{\partial t^{\alpha}} + \tau \frac{\partial^{\beta} T}{\partial t^{\beta}} = \kappa \nabla^2 T + \frac{\kappa}{k} g \tag{6}$$

in which we take $0 < \beta \le \alpha \le 2$, and the fractional Caputo derivative [28] of order α with lower limit zero in Eq. (6) is taken as

$$\frac{\partial^{\alpha} f}{\partial t^{\alpha}} = \frac{1}{\Gamma(m-\alpha)}\left\{\int_0^t \frac{\partial^m}{\partial \gamma^m}\left[\frac{f(r,\gamma)}{(t-\gamma)^{\alpha+1-m}}\right] d\gamma\right\},\ m-1 < \alpha < m,\ m \in N \tag{7}$$

whereas the Riemann-Liouville fractional derivative of order β is defined as

$$\frac{\partial^{\beta} f}{\partial t^{\beta}} = \frac{\partial^m}{\partial t^m}\left\{\frac{1}{\Gamma(m-\beta)}\int_0^t \left[\frac{f(r,\gamma)}{(t-\gamma)^{\alpha+1-m}}\right] d\gamma\right\},\ m-1 < \beta < m \tag{8}$$

where Γ is the Euler gamma function and m is an integer satisfying $m-1 < \beta < m$.
For the limiting case: (i) Taking $\tau = 0$, $\alpha = 0$, Eq. (6) reduces to classical Fourier heat conduction [50], (ii) Taking $\alpha = 2, \beta = 1$, Eq. (6) reduces to the telegraph equation [35], (iii) Taking $\alpha = \beta = 2$, Eq. (6) reduces the classical wave equation [35], and (iv) Otherwise stated, all four fractional generalizations of the Cattano constitutive relationship given by Povstenko [35] can be viewed as exceptional cases of Eq. (6).

3 FORMULATION OF THE PROBLEM

Consider a k-layer composite elliptic membrane occupying space $D=\{(\xi,\eta,z)\in R^3 : \xi_i<\xi<\xi_{i+1},\ 0<\eta<2\pi,\ 1\le i\le k\}$ in elliptical coordinates, related to the rectangular coordinate system by the relation $x=c\cosh\xi\cos\eta,\ y=c\sinh\xi\sin\eta,\ c=(a^2-b^2)^{1/2}$, where c is the semi-focal length of the ellipse, a and b are semi-major and semi-minor axis, as shown in Figure 1.

Let us consider the time-fractional heat conduction differential equation as

$$\frac{\partial^\alpha T_i}{\partial t^\alpha}+\tau\frac{\partial^\beta T_i}{\partial t^\beta}=\kappa_i\left[h^2\left(\frac{\partial^2 T_i}{\partial\xi^2}+\frac{\partial^2 T_i}{\partial\eta^2}\right)+\frac{g_i(\xi,\eta)f(t)}{\lambda_i}\right] \tag{9}$$

with the initial and boundary conditions as

$$T_i(\xi,\eta,z,t=0)=0,\frac{\partial}{\partial t}T_i(\xi,\eta,z,t=0)=0, \tag{10}$$

$$\begin{aligned}
&\alpha_1\frac{\partial T_1}{\partial\xi}(\xi=\xi_1,\eta,t)-h_0T_1(\xi=\xi_1,\eta,t)=f(t),\ h_0\ge 0,\\
&\alpha_k\frac{\partial T_k}{\partial\xi}(\xi=\xi_{k+1},\eta,t)+h_kT_k(\xi=\xi_{k+1},\eta,t)=f(t),\\
&\alpha_i\frac{\partial T_i}{\partial\xi}(\xi=\xi_{i+1},\eta,t)+h_i[T_i(\xi=\xi_{i+1},\eta,t)\\
&-T_{i+1}(\xi=\xi_{i+1},\eta,t)]=\alpha_{i+1}\frac{\partial T_{i+1}}{\partial\xi}(\xi=\xi_{i+1},\eta,t)\\
&+h_i[T_i(\xi=\xi_{i+1},\eta,t)-T_{i+1}(\xi=\xi_{i+1},\eta,t)]=0
\end{aligned} \tag{11}$$

in which the function $T_i=T_i(\xi,\eta,t)$ represents the temperature, κ_i be the thermal diffusivity, λ_i denotes the thermal conductivity, ρ_i for density, C_i as specific heat for the i^{th} layer, $f(t)=\delta(t)$ and $\delta()$ is the Dirac distribution, respectively.

The differential equation governing the Airy stress function $\chi(\xi,\eta,t)$ is given as [5]

$$h^2\nabla^2\,h^2\nabla^2\,\chi^{(i)}=-h^2\nabla^2 T_i \tag{12}$$

where

$$\nabla^2=h^2(\partial^2/\partial\xi^2+\partial^2/\partial\eta^2). \tag{13}$$

The stress distribution for each layer as [5]

$$\begin{aligned}
\sigma_{\xi\xi}^{(i)}&=h^4\left\{\frac{\partial^2}{h^2\partial\eta^2}+\frac{c^2}{2}\left(\sinh 2\xi\frac{\partial}{\partial\xi}-\sin 2\eta\frac{\partial}{\partial\eta}\right)\right\}\chi^{(i)},\\
\sigma_{\eta\eta}^{(i)}&=h^4\left\{\frac{\partial^2}{h^2\partial\xi^2}+\frac{c^2}{2}\left(-\sinh 2\xi\frac{\partial}{\partial\xi}+\sin 2\eta\frac{\partial}{\partial\eta}\right)\right\}\chi^{(i)},\\
\sigma_{\xi\eta}^{(i)}&=h^4\left\{-\frac{\partial^2}{h^2\partial\xi\partial\eta}+\frac{c^2}{2}\left(\sin 2\eta\frac{\partial}{\partial\xi}+\sinh 2\xi\frac{\partial}{\partial\eta}\right)\right\}\chi^{(i)}
\end{aligned} \tag{14}$$

where c is the semi-focal length of the ellipse.

The traction-free surfaces

$$\sigma_{\xi\xi}^{(1)}=\sigma_{\xi\eta}^{(1)}=\sigma_{\xi\xi}^{(k)}=\sigma_{\xi\eta}^{(k)}=\sigma_{\xi\xi}^{(i)}=\sigma_{\xi\eta}^{(i)}=0 \tag{15}$$

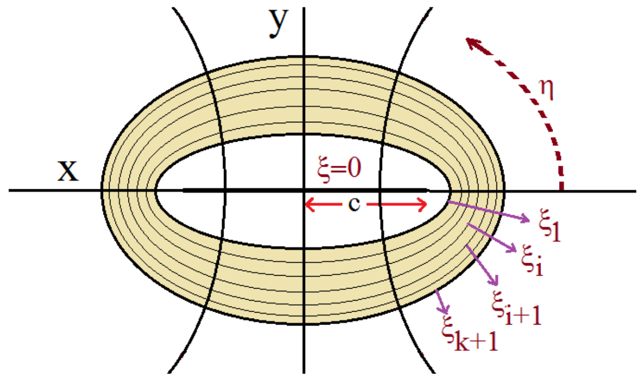

Fig.1
Schematic sketch of the k-layer confocal elliptic membrane.

4 THE SOLUTION OF THE PROBLEM

Initially, we recall an integral transform proposed by Dhakate et al. [5], which states that if $f_i(\xi,\eta)$ is continuous and has piecewise continuous first and second derivatives in the region $0 \le \xi \le \xi_0$, $0 \le \eta \le 2\pi$; satisfy interfacial and other boundary conditions, then the Sturm-Liouville transform for the composite region of order *n* and *m* over the variable ξ and η is defined as

$$\bar{f}_i(q_{2n,.m}) = \beta_i \int_0^{2\pi}\int_{\xi_i}^{\xi_{i+1}} f_i(\xi,\eta)(\cosh 2\xi - \cos 2\eta)\, \psi_{i,2n}\ (\xi,q_{2n,m}) ce_{2n}(\eta,q_{2n,m}) d\xi d\eta \tag{16}$$

and its inversion theorem is given by

$$f_i(\xi,\eta) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{i=1}^{k} \frac{\bar{f}_i(\xi,\eta)\psi_{i,2n}\ (\xi,q)\, ce_{2n}(\eta,q)}{\pi\int_{\xi_i}^{\xi_{i+1}} \psi_{i,2n}^2\ (\xi,q)(\cosh 2\xi - \Theta_{2n,m})\, d\xi} \tag{17}$$

in which $ce_{2n}(\eta,q)$ is the ordinary Mathieu function of the first kind of order *n* [51, pp. 21], $Ce_{2n}(\xi,q)$ is a modified Mathieu function of the first kind of order *n* [51, pp.27], $\psi_{i,2n}(\xi,q_{2n,m})$ is the kernel of the transform, $\bar{f}_i(q_{2n,.m})$ is Sturm-Liouville transform for the composite region, $\psi_{i,2n}(\xi)$ is the eigenfunction of the i^{th} layer, h_0 is the surface coefficient at $\xi = \xi_1$, $q_{2n,m}$ is the eigenvalue of $[A_{i2n}Ce_{2n}(\xi,q_{2n,m}) + B_{i2n}Fey(\xi,q_{2n,m})]ce_{2n}(\eta) = 0$, (α_i, β_i) is the characteristics of the i^{th} layer, h_i is the surface coefficient at $\xi = \xi_{i+1}$, h_k is the surface coefficient at $\xi = \xi_{k+1}$ and $L = \partial^2/\partial\xi^2 + \partial^2/\partial\eta^2$. The arbitrary constants A_{i2n} and B_{i2n} can be obtained using a system of $2k$ simultaneous equations. Here we consider the recurrence relations for the Bessel functions Y_{2r} (are identical to those for J_{2r}) that can be defined as

$$Fey_{2n}(\xi,q) = \frac{ce_{2n}(0,q)}{A_0^{(2n)}} = \sum_{r=0}^{\infty} A_{2r}^{(2n)}\, Y_{2r}(2k'\sinh\xi) \begin{pmatrix} |\sinh\xi| > 1 \\ R(\xi) > 0 \end{pmatrix} \tag{18}$$

with *y* in *Fey* [51, pp.159] signifies the *Y*-Bessel function and $q = k'^2 = \lambda c^2/4$.

Applying the Sturm–Liouville transform defined in Eq. (16), one obtains

$$\frac{\partial^\alpha \bar{T}_i}{\partial t^\alpha} + \tau\frac{\partial^\beta \bar{T}_i}{\partial t^\beta} + \alpha_{2n,m}^2 \bar{T}_i(q_{2n,m},t) = [\mu_i \bar{g}_i(q_{2n,m}) + \bar{F}(q_{2n,m})] f(t) \tag{19}$$

in which

$$\mu_i = \kappa_i / \lambda_i \ \beta_i , \ \alpha_{2n,m}^2 = 4\kappa_i \ q_{2n,m} /(\alpha_i c^2), \tag{20}$$

$$\bar{F}(q_{2n,m}) = \frac{4\pi\kappa_i}{c^2 \ \alpha_i} A_0^{(2n)} [\psi_{k,n,m}(\xi_{k+1}) - \psi_{1,n,m}(\xi_1)], \tag{21}$$

$$\bar{g}_i(q_{2n,m}) = \beta_i \int_0^{2\pi} \int_{\xi_i}^{\xi_{i+1}} g_i(\xi,\eta)(\cosh 2\xi - \cos 2\eta) \psi_{i,2n}(\xi, q_{2n,m}) ce_{i,2n}(\eta, q_{2n,m}) d\xi d\eta \tag{22}$$

and

$$\bar{H}(q_{2n,m}) = \mu_i \bar{g}_i(q_{2n,m}) + \bar{F}(q_{2n,m}) \tag{23}$$

Now, applying the Laplace transform to the Eq. (19), one obtains

$$\bar{\bar{T}}_i(q_{2n,m}, s) = \frac{\bar{\bar{H}}(q_{2n,m})}{s^\beta + \tau s^\alpha + \alpha_{2n,m}^2} F(s) \tag{24}$$

in which the double overbar is the transformed function, s is the Laplace parameter,

$$\bar{\bar{T}}_i(q_{2n,m}, s) = \int_0^\infty \bar{T}_i(q_{2n,m}, t) \exp(-st) dt, \ F(s) = \int_0^\infty f(t) \exp(-st) dt \ . \tag{25}$$

Extending the right side of (24) in a power series of s [28, pp 155], one gets

$$\bar{\bar{T}}_i(q_{2n,m}, s) = \bar{\bar{H}}(q_{2n,m}) \frac{1}{\alpha_{2n,m}^2} \frac{\alpha_{2n,m}^2 s^{-\alpha}}{s^{\beta-\alpha} + \tau} \frac{1}{1 + \dfrac{\alpha_{2n,m}^2 s^{-\alpha}}{s^{\beta-\alpha} + \tau}} \tag{26}$$

Assuming $\beta > \alpha$, one obtains

$$\frac{1}{1 + \dfrac{\alpha_{2n,m}^2 s^{-\alpha}}{s^{\beta-\alpha} + \tau}} = \sum_{j=0}^{\infty} (-1)^j \left(\frac{\alpha_{2n,m}^2 s^{-\alpha}}{s^{\beta-\alpha} + \tau} \right)^j \tag{27}$$

Substituting Eq. (27) into Eq. (26), one obtains

$$\bar{\bar{T}}_i(q_{2n,m}, s) = \bar{\bar{H}}(q_{2n,m}) \sum_{j=0}^{\infty} (-1)^j \frac{(\alpha_{2n,m}^2)^j s^{-(1+j\alpha)}}{(s^{\beta-\alpha} + \tau)^{j+1}} \tag{28}$$

The series in (27) is convergent [52] if we choose $s_0 > 0$, then

$$\left| \frac{\alpha_{2n,m}^2 s^{-\alpha}}{s^{\beta-\alpha} + \tau} \right| = \left| \frac{\alpha_{2n,m}^2}{s^\beta + s^\alpha \tau} \right| < 1 \tag{29}$$

hold if $\operatorname{Re} s > s_0$ and it determines $s_0 > 0$.

Thus, the term-by-term inversion, based on the general expansion theorem for the Laplace transform [28, pp 21], one obtains

$$\mathcal{L}^{-1} \left\{ \sum_{j=0}^{\infty} \frac{(-1)^j (\alpha_{2n,m}^2)^j}{j!} \frac{j! s^{-(1+j\alpha)}}{(s^{\beta-\alpha} + \tau)^{j+1}} \right\} = \sum_{j=0}^{\infty} (-1)^j (\alpha_{2n,m}^2)^j t^{\beta(j+1)-1} E_{\beta-\alpha, \beta+j\alpha}^{(j)} (-\tau t^{\beta-\alpha}) / j! \tag{30}$$

where $E_{\alpha,\beta}^{(l)}$ is the generalized Mittag-Leffler functions [28, pp 21] defined as

$$E_{\alpha,\beta}^{(l)}(z) = \frac{d^n}{dz^n} E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{(j+n)! z^j}{j! \Gamma(\alpha j + \alpha n + \beta)} \tag{31}$$

Finally, applying the inversion theorems, one obtains

$$T_i = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{\bar{\bar{H}}(q_{2n,m})}{C_{2n,m}} \psi_{i,2n}(\xi, q_{2n,m}) ce_{2n}(\eta, q_{2n,m}) [\int_0^t \bar{T}(u) f(t-u) du] \tag{32}$$

where

$$C_{2n,m} = \pi \int_{\xi_i}^{\xi_{i+1}} \psi_{i,2n}^2 \ (\xi, q_{2n,m})(\cosh 2\xi - \Theta_{2n,m})\, d\xi,$$
$$\Theta_{2n,m} = \frac{1}{\pi}\int_0^{2\pi} \cos 2\eta\, ce_{2n}^2(\eta, q_{2n,m})\, d\eta,$$
$$\overline{T}(u) = \sum_{j=0}^{\infty} (-1)^j (\alpha_{2n,m}^2)^j u^{\beta(j+1)-1} E_{\beta-\alpha,\beta+j\alpha}^{(j)}(-\tau u^{\beta-\alpha}) / j! \tag{33}$$

Now assume Airy's stress function, which satisfies condition (12) as

$$\chi^{(i)} = h^2 \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \psi_{i,2n} \ (\xi, q_{2n,m}) ce_{2n}(\eta, q_{2n,m}) \phi_1 / C_{2n,m} \tag{34}$$

in which $A_{2n,m}$ is the arbitrary constant that can be determined finally by using condition (9).

$$A_{2n,m} = \frac{ce_{2h} a(x) \text{ - } \cot 2x\, b(h)}{ce_{2h}\, g(x) \text{ - } b(h)} \tag{35}$$

in which

$$\alpha(\xi) = \cot 2\xi \sinh 2\xi\, \psi(\xi) / \psi(\xi)_{,\xi} - 2\sinh 2\xi,$$
$$\beta(\eta) = \sinh 2\eta\, ce(\eta, q_{2n,m})_{,\eta} - 2ce(\eta, q_{2n,m})_{,\eta\eta} / c^2 h^2,$$
$$\gamma(\xi) = 2\cot 2\xi \sinh 2\xi - \sinh 2\xi\, \psi(\xi) / \psi(\xi)_{,\xi}, \tag{36}$$
$$\phi_1 = \sum_{i=1}^{k} \left\{ \left[\int_0^t \overline{T}(u) f(t-u) du \right] \mu_i \overline{g}_i (q_{2n,m})(A_{2n,m} \sin 2\xi - \cos 2\xi) \right\}$$

Substituting $A_{2n,m}$ in Eq. (27), we get

$$\chi^{(i)} = h^2 \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{\psi_{i,2n}(\xi, q_{2n,m}) ce_{2n}(\eta, q_{2n,m}) \phi_2}{C_{2n,m}[\beta(\eta) - ce_{2n}(\eta, q_{2n,m})\gamma(\xi)]} \tag{37}$$

where

$$\phi_2 = \sum_{i=1}^{k} \left\{ [\int_0^t \overline{T}(u) f(t-u) du] \mu_i \overline{g}_i (q_{2n,m})[-\sin 2\xi\, \alpha(\xi) + \cos 2\xi\, \gamma(\xi)] \right\} \tag{38}$$

Substituting Eq. (30) into Eq. (14), one obtains the stress components as

$$\sigma_{\xi\xi}^{(i)} = h^6 \left\{ \frac{\partial^2}{h^2 \partial \eta^2} + \frac{c^2}{2}\left(\sinh 2\xi \frac{\partial}{\partial \xi} - \sin 2\eta \frac{\partial}{\partial \eta} \right) \right\} \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{\psi_{i,2n}(\xi, q_{2n,m}) ce_{2n}(\eta, q_{2n,m}) \phi_2}{C_{2n,m}[\beta(\eta) - ce_{2n}(\eta, q_{2n,m})\gamma(\xi)]} \tag{39}$$

$$\sigma_{\eta\eta}^{(i)} = h^6 \left\{ \frac{\partial^2}{h^2 \partial \xi^2} + \frac{c^2}{2}\left(-\sinh 2\xi \frac{\partial}{\partial \xi} + \sin 2\eta \frac{\partial}{\partial \eta} \right) \right\} \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{\psi_{i,2n}(\xi, q_{2n,m}) ce_{2n}(\eta, q_{2n,m}) \phi_2}{C_{2n,m}[\beta(\eta) - ce_{2n}(\eta, q_{2n,m})\gamma(\xi)]} \tag{40}$$

$$\sigma_{\xi\eta}^{(i)} = h^6 \left\{ -\frac{\partial^2}{h^2 \partial \xi \partial \eta} + \frac{c^2}{2}\left(\sin 2\eta \frac{\partial}{\partial \xi} + \sinh 2\xi \frac{\partial}{\partial \eta} \right) \right\} \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{\psi_{i,2n}(\xi, q_{2n,m}) ce_{2n}(\eta, q_{2n,m}) \phi_2}{C_{2n,m}[\beta(\eta) - ce_{2n}(\eta, q_{2n,m})\gamma(\xi)]} \tag{41}$$

5 NUMERICAL RESULTS, DISCUSSION, AND REMARKS

We introduce the following dimensionless values

$$\overline{\xi_i} = \xi_i / b,\ \overline{a} = a / b,\ \overline{b} = b / b,\ e = c / b, \tau = \kappa\, t / b^2,$$
$$T = T / T_0,\ \overline{\sigma}_{ij} = \sigma_{ij} / E\, \alpha_t\, \theta_0, (i, j = \xi, \eta) \tag{42}$$

For the sake of simplicity of numerical calculations, we consider a three-layered composite elliptic annulus membrane. The numerical computations have been carried out for Aluminium and Tin metal with initial temperature and for ($t > 0$), the temperature raised to a finite value. The physical parameters are considered as $\xi_1 = a$ = 0.1 cm, ξ_2=0.5cm, $\xi_3 = b$ = 1cm, and a case of $\alpha = \beta$. The mechanical material properties are considered as specific heat at constant pressure in the i^{th} section, C_{v1}=0.181 cal/g^{0}C and C_{v2} = 0.054 cal/g^{0}C; Modulus of Elasticity, E_1= 6.9 × 10^6 N/cm^2 and E_2 = 4.7× 10^6 N/cm^2; Thermal expansion coefficient, α_1 = 24.8×10^{-6} cm/cm-^{0}C, α_2 = 23.0×10^{-6} cm/cm-^{0}C; Thermal conductivity λ_1 = 0.52 cal sec^{-1}/cm ^{0}C and λ_2 = 0.15 cal sec^{-1}/cm^{0}C; $f_1(t) = f_2(t) = 20$ cm-^{0}C. To examine the influence of heating on the membrane, we performed the numerical calculation for all variables, and numerical calculations are depicted in the following figures with the help of MATHEMATICA software. Figs. 2-3 illustrate the numerical results of temperature distribution and stresses of the elliptical membrane due to internal heat generation within the solid. It can be seen that Fig. 2(a) and Fig. 2(c) both exhibit relatively similar patterns. Fig. 2(a) and Fig. 2(c) show the temperature distribution along $\bar{\xi}$ - direction for α=β and a different time and η, respectively. In both the figures, temperature increases gradually towards the outer end of each layer due to the combined effect of sectional heat supply and internal heat energy, as shown in Fig. 2(b). Initially, the temperature approaches a maximum, whereas it attains a minimum as time increases. Fig. 2(d) shows that the temperature approaches a minimum at both extreme ends, i.e., at $\eta = 0$ and $\eta = \pi$ due to more compressive force. In contrast, due to a tensile force, the temperature is high at the central, i.e., at $\eta = \pi / 2$, which gives an overall bell-shaped curve for all three layers of different materials. The given result in Eq. (32) was in good agreement with the result [53] with a three-layer plate of finite dimension. Fig. 3(a) illustrates the axial stress decreases with time. Initially, the axial stress attains maximum expansion due to the accumulation of thermal energy dissipated by sectional heat supply and internal heat energy, further decreasing time. Fig. 3(b) indicated that the axial stress along the ξ- direction is initially on the negative side due to more compressive stress occurring at the inner edge, which goes on increasing along ξ- direction and attains a maximum at the outer edge. Fig. 3(c) depicts tangential stress along the radial direction achieving minimum at the inner and maximum at the outer edge. In Fig. 3(d), shear stress is of negative magnitude. Initially, it reaches a minimum, decreasing at the middle core and suddenly increasing towards the outer region. Fig. 3(e) observed the minimum tangential stress with a negative magnitude which increases gradually with time.

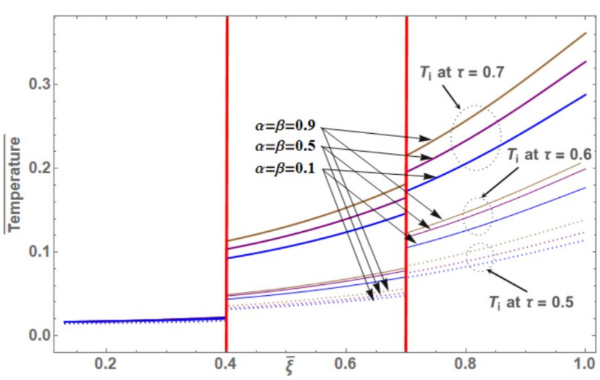

Fig.2a

Temperature distribution along $\bar{\xi}$-direction for α =β and different τ. .

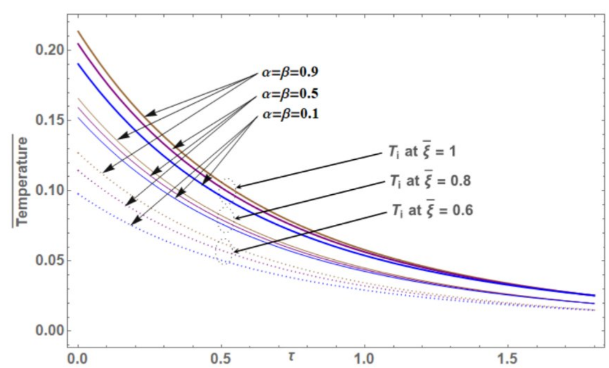

Fig.2b

Temperature distribution along τ for α =β and different $\bar{\xi}$. .

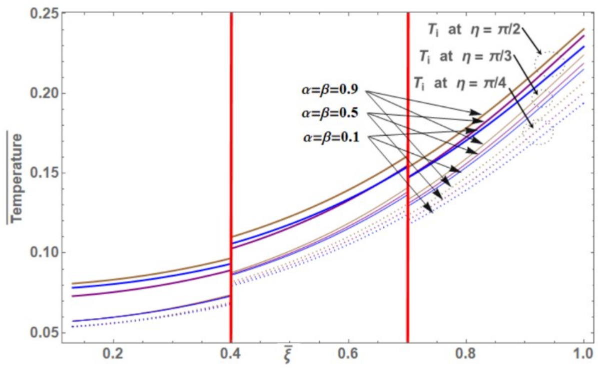

Fig.2c
Temperature distribution along $\bar{\xi}$-direction for α =β and different $\eta_\circ$.

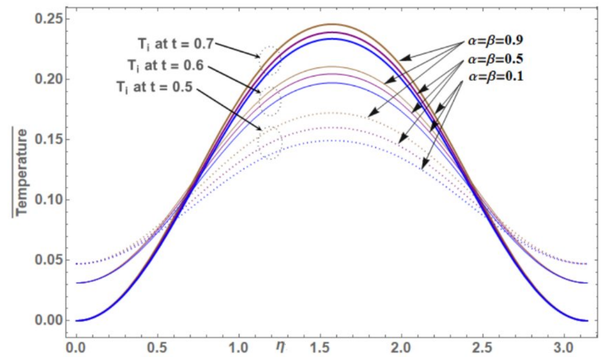

Fig.2d
Temperature distribution along η-direction for α =β and different $\tau_\circ$.

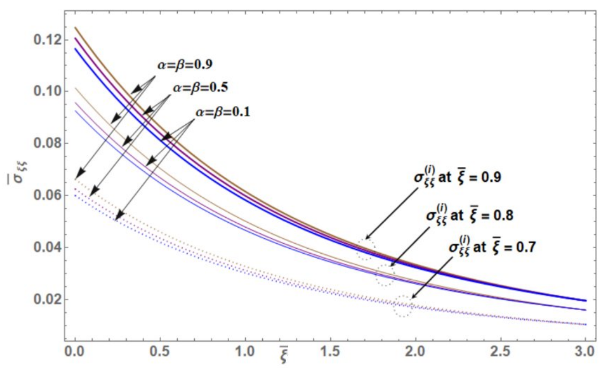

Fig.3a
Radial stress along τ for α =β and different $\bar{\xi}_\circ$.

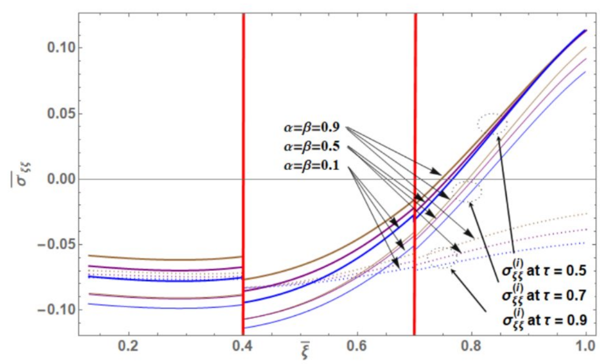

Fig.3b
Radial stress along $\bar{\xi}$ - direction for α =β and different $\tau_\circ$.

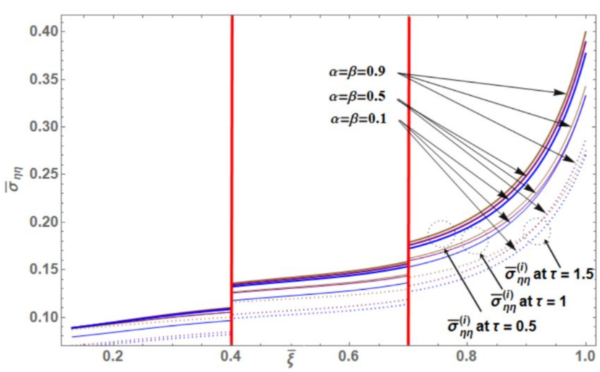

Fig.3c
Tangential stress along $\bar{\xi}$- direction for α =β and different $\tau_\circ$.

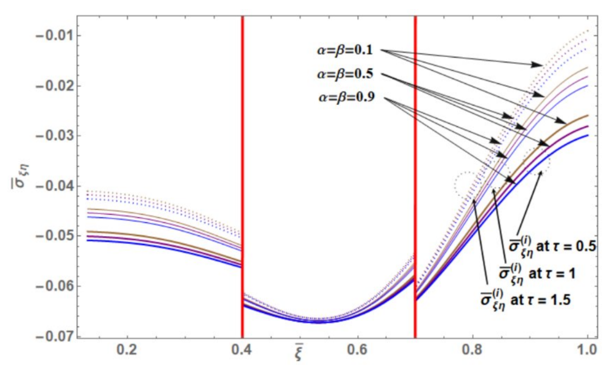

Fig.3d
Shear stress along $\bar{\xi}$ - direction for α =β and different $\tau_\circ$.

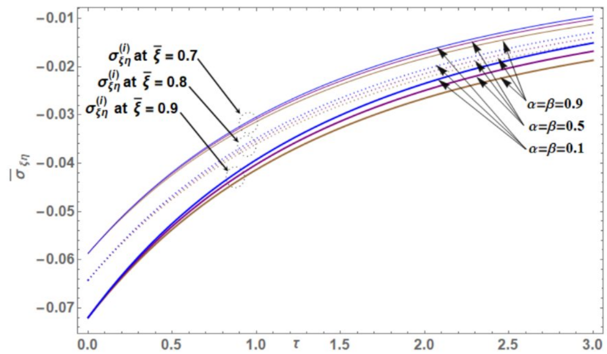

Fig.3e
Shear stress along τ for different $\alpha = \beta$ and different $\bar{\xi}$.

The key that was derived by Dhakate et al. [5] for an isotropic, homogeneous, elastic multilayer elliptic annulus is consistent with the determined thermoelastic solutions. This piece of research brings a fractional-order constitutive model and the standard continuity equation together. Recent research [54,55] shows that it is possible for a non-Fourier constitutive model and a non-trivial continuity equation based on the Boltzmann transport theory to coexist. The findings demonstrate that the constitutive model and the continuity equation are not independent of one another, which this work does not consider.

6 CONCLUSIONS

We introduced the fractional Cattaneo constitutive model and the corresponding fractional Cattaneo equation with two fractional derivatives of different orders to investigate the thermoelastic response of a multilayer elliptic ring membrane with a source function. The solution is investigated using the integral transform method establishing the Sturm-Liouville integral transform, which is analogous to Vodicka's approach considering series expansion function in terms of an eigenfunction to solve the heat conduction partial differential equation in elliptical coordinates. The temperature and thermal stresses for the three-layer elliptic region have been computed numerically and exhibited graphically. The following results were obtained during our research:

- The advantage of this method is its generality and its mathematical power to handle different types of mechanical and thermal boundary conditions.
- The maximum tensile stress shifting from the central core to the outer region may be due to heat, stress, concentration, or internal heat sources under-considered temperature field.
- Finally, the maximum tensile stress occurring in the circular core on the major axis compared to the elliptical central part indicates the distribution of weak heating. It might be due to insufficient heat penetration through the elliptical inner surface.

ACKNOWLEDGMENT

The author(s) are thankful to the reviewers and editors for their helpful feedback and positive comments that contributed to the revision of the paper in its present form.

REFERENCES

[1] V. Vodicka, Warmeleitung in geschichteten Kugel-und Zylinderkorpern, *Schweizer Archiv*, Vol. 10, pp. 297-304, 1950.
[2] V. Vodicka, Eindimensionale Wärmeleitung in geschichteten. Körpern, *Math. Nachr.*, Vol. 14, pp. 47–55, 1955.
[3] R. Chiba, An analytical solution for transient heat conduction in a composite slab with time-dependent heat transfer coefficient, *Math. Probl. Eng.*, Vol. 2018, Article ID 4707860, 2018. DOI: 10.1155/2018/4707860

[4] S.M. Moghimi, M. Hosseini, and M. Ghanbarpour, Temperature distribution definition in one-dimensional transient cooling in a three-layer slab using orthogonal expansion technique, *International Journal of Nonlinear Dynamics in Engineering and Sciences*, Vol. 11, No. 2, pp. 21-30, 2019.

[5] T. Dhakate, V. Varghese and L. Khalsa, An analytical solution for transient asymmetric heat conduction in a multilayer elliptic annulus and its associated thermal stresses, *Int. J. Math. And Appl.*, Vol. 6, No. 1, pp. 29-42, 2018.

[6] C.W. Tittle, Boundary value problems in composite media: quasi-orthogonal functions, *J. Applied Physics*, Vol. 36, No. 4, pp. 1486-1488, 1965.

[7] P.E. Bulavin and V.M. Kascheev, Solution of the non-homogeneous heat conduction equation for multilayered bodies, *Int. Chemical Engineering*, Vol. 1, No. 5, pp. 112-115, 1965.

[8] M.D. Mikhailov and M.N. Ozisik, Transient conduction in a three-dimensional composite slab*, Int. J. Heat Mass Transfer*, Vol. 29, pp. 340–342, 1986.

[9] X. Lu, P. Tervola, M. Viljanen, A new analytical method to solve heat equation for multi-dimensional composite slab, *J. Phys. A: Math. Gen.*, Vol. 38, pp. 2873–2890, 2005.

[10] X. Lu, P. Tervola, M. Viljanen, Transient analytical solution to heat conduction in multi-dimensional composite cylinder slab, *Int. J. Heat Mass Transfer*, Vol. 49, pp. 1107–1114, 2006.

[11] H.S. Carslaw, J.C. Jaeger*, Conduction of Heat in Solids*, second ed., Oxford University Press, Oxford, 1959.

[12] Kevin D. Cole, A. Haji-Sheikh, James V. Beck, Bahman Litkouhi, *Heat conduction using Green's function*, Taylor and Francis Group, LLC, 2011.

[13] Haji-Sheikh, J.V. Beck, Temperature solution in multidimensional multilayer bodies, *Int. J. Heat Mass Transfer*, Vol. 45, pp. 1865–1877, 2002.

[14] T. Dhakate, V. Varghese, L. Khalsa, A Green's function approach for the thermoelastic analysis of an elliptical cylinder, *Int. J. Adv. Appl. Math. and Mech.*, Vol. 5, no. 2, pp. 30-40, 2017.

[15] W. Heidemann, H. Mandel, E. Hahne, *Computer aided determination of closed form solutions for linear transient heat conduction problems in inhomogeneous bodies*, in: L.C. Wrobel, C.A. Brebbia, A.J. Nowak (Eds.), Advanced Computational Methods in Heat Transfer III, Computational Mechanics Publications, Boston, 1994, pp. 19-26.

[16] S. Verma, V.S. Kulkarni, K.C. Deshmukh, Finite element solution to transient asymmetric heat conduction in multilayer annulus, *Int. J. Adv. Appl. Math. and Mech.*, Vol. 2, No. 3, pp. 119-125, 2015.

[17] P.P. Bhad, V. Varghese, and L. Khalsa, Heat source problem of thermoelasticity in an elliptic plate with thermal bending moments, *J. Therm. Stresses*, Vol. 40, no. 1, pp. 96-107, 2016.

[18] P.P. Bhad, V. Varghese, and L. Khalsa, A modified approach for the thermoelastic large deflection in the elliptical plate, *Arch. Appl. Mech.*, Vol. 87, no. 4, pp. 767–781, 2016.

[19] I. Khan, L. Khalsa, and V. Varghese, Inverse quasi-static unsteady-state thermal stresses in a thick circular plate, *Cogent Math.*, Vol. 4, Article ID 1283763, 2017.

[20] P.P. Bhad, V. Varghese, and L. Khalsa, Thermoelastic-induced vibrations on an elliptical disk with internal heat sources, *J. Therm. Stresses*, Vol. 40, no. 4, pp. 502-516, 2017.

[21] T. Dhakate, V. Varghese, and L. Khalsa, Integral transform approach for solving dynamic thermal vibrations in the elliptical disk, *J. Therm. Stresses*, Vol. 40, no. 9, pp. 1093-1110, 2017.

[22] J.C. Jaeger, Some problems involving line sources in conduction of heat, *London Edinburgh Dub. Phil. Mag. J. Sci.*, Vol. 242, pp. 169–179, 1944.

[23] M. Li, A.C.K. Lai, Analytical model for short-time responses of borehole ground heat exchangers: model development and validation, *Appl. Energy*, Vol. 104, pp. 510–516, 2013.

[24] P.E. Bulavin and V.M. Kashcheev, Solution of nonhomogenous heat-conduction equation for multilayer bodies, *Int. Chem. Eng.*, Vol. 5, no. 1, pp. 112–115, 1965.

[25] A. Agrawal, Higher-order continuum equation based heat conduction law, *INAE Lett.*, pp. 1-5, 2016. DOI 10.1007/s41403-016-0007-3

[26] C. Cattaneo, Sulla conduzione del calore, *Atti Sem. Mat. Fis. Univ. Modena*, Vol. 3, pp. 83–101, 1948.

[27] L.Q. Wang, X.S. Zhou, and X.H. Wei, *Heat conduction*, Springer, Berlin, 2008.

[28] I. Podlubny, *Fractional differential equations*, Academic Press, New York, 1999.

[29] R. Hilfer, *Applications of fractional calculus in physics*, World Scientific, Singapore, 2000.

[30] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, 2006.

[31] R.L. Magin, *Fractional calculus in bioengineering*, Begell House Publishers, Connecticut, 2006.

[32] W. Chen, H.G. Sun, and X.C. Li, *Fractional derivative modelling of mechanical and engineering problems*, Science Press, Beijing, 2010.

[33] R. Herrmann, *Fractional calculus: An introduction for physicists*, World Scientific, Singapore, 2011.

[34] A. Compte and R. Metzler, The generalized Cattaneo equation for the description of anomalous transport processes, *J. Phys. A: Math. Gen.*, Vol. 30, pp. 7277-7289, 1997.

[35] Y. Povstenko, Fractional Cattaneo-type equations and generalized thermoelasticity, *J. Therm. Stresses*, Vol. 34, No. 2, pp. 97-114, 2011. DOI: 10.1080/01495739.2010.511931

[36] Y. Povstenko, Fractional thermoelasticty, Springer, New York, 2015.

[37] G. Xu, and J. Wang, Analytical solution of time fractional Cattaneo heat equation for finite slab under pulse heat flux, *Appl. Math. Mech.*, Vol. 39, pp. 1465–1476, 2018. DOI 10.1007/s10483-018-2375-8

[38] D. Maillet, A review of the models using the Cattaneo and Vernotte hyperbolic heat equation and their experimental validation, *Int. J. Therm. Sci.*, Vol. 139, pp. 424-432, 2019.DOI 10.1016/j.ijthermalsci.2019.02.021

[39] H.T. Qi, H.Y. Xu, and X.W. Guo, The Cattaneo-type time fractional heat conduction equation for laser heating, *Comput. Math. with Appl.*, Vol. 66, No. 5, pp. 824-831, 2013. DOI 10.1016/j.camwa.2012.11.021

[40] H.Y. Xu, H.T. Qi, and X.Y. Jiang, Fractional Cattaneo heat equation in a semi-infinite medium, *Chin. Phys. B*, Vol. 22, No. 1, 014401, 2013. DOI: 10.1088/1674-1056/22/1/014401

[41] J.H. Choi, S.H. Yoon, and S.G. Park, and S.H. Choi, Analytical solution of the Cattaneo-Vernotte equation (non-Fourier heat conduction), *J. Korean Soc. of Marine Engineering*, Vol. 40, No. 5 pp. 389~396, 2016. DOI 10.5916/jkosme.2016.40.5.389

[42] Y. Povstenko, Axisymmetric Solutions to Time-fractional heat conduction equation in a half-space under Robin boundary conditions, *Int. J. Differ. Equ.*, Vol. 2012, pp. 1–13, 2012. DOI: 10.1155/2012/154085.

[43] Y. Povstenko, Axisymmetric solutions to fractional diffusion-wave equation in a cylinder under Robin boundary condition, *Eur. Phys. J. Spec. Top.*, Vol. 222, No. 8, pp. 1767–1777, Sep. 2013. DOI: 10.1140/epjst/e2013-01962-4.

[44] Y. Povstenko, Fundamental solutions to the fractional heat conduction equation in a ball under Robin boundary condition, *Centr. Eur. J. Math.*, Vol. 12, No. 4, pp. 611–622, 2014. DOI: 10.2478/s11533-013-0368-8.

[45] X.Y. Jiang and H.T. Qi, Thermal wave model of bioheat transfer with modified Riemann Liouville fractional derivative, *J. Phys. A-Math. Theor.*, Vol. 45, No. 48, 485101, 2012.

[46] G.Y. Xu, J.B. Wang, and Z. Han, Study on the transient temperature field based on the fractional heat conduction equation for laser heating, *Appl. Math. Mech.*, Vol. 36, pp. 844–849, 2015.

[47] H.Y. Xu, H.T. Qi, and X.Y. Jiang, Fractional Cattaneo heat equation on a semi-infinite medium, *Chin. Phys. B*, Vol. 22, 014401, 2013.

[48] H.T. Qi and X.W. Guo, Transient fractional heat conduction with generalized Cattaneo model, *Int. J. Heat Mass Transfer*, Vol. 76, pp. 535–539, 2014.

[49] T.N. Mishra and K.N. Rai, Numerical solution of FSPL heat conduction equation for analysis of thermal propagation, *Appl. Math. Comput.*, Vol. 273, pp. 1006–1017, 2016.

[50] M. N. Özisik, *Heat Conduction*, John Wiley & Sons, New York, 1993.

[51] N.W. McLachlan, *Theory and application of Mathieu functions*, Oxford University Press, Oxford, 1947.

[52] T.M. Atanackovic, S. Pilipovic, and D. Zorica, A diffusion wave equation with two fractional derivatives of different order, *J. Phys. A: Math. Theor.*, Vol. 40, pp. 5319–5333, 2007. DOI 10.1088/1751-8113/40/20/006

[53] V. Vodicka, Heat exchange in a three-layer plate of finite dimension, *Arch. budowy maszyn*, Vol. 3, No. 4, pp. 319-331, 1956.

[54] S. N. Li and B.Y. Cao, Fractional Boltzmann transport equation for anomalous heat transport and divergent thermal conductivity, *Int. J. Heat Mass Transf.*, 137, 84-89, 2019.

[55] S. N. Li and B.Y. Cao, Fractional-order heat conduction models from generalized Boltzmann transport equation, *Philos. Trans. R. Soc*. A, 378, 20190280, 2020.